Relative sectional curvature in compact angled 2-complexes

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Abstract

We define the notion of relative sectional curvature for 2-complexes, and prove that a compact angled 2-complex that has negative sectional curvature relative to planar sections has coherent fundamental group. We analyze a certain type of 1-complex that we call flattenable graphs $\Gamma \to X$ for an compact angled 2-complex X, and show that if X has nonpositive sectional curvature, and if for every flattenable graph $\pi_1(\Gamma) \to \pi_1(X)$ is finitely presented, then X has coherent fundamental group. Finally we show that if X is a compact angled 2-complex with negative sectional curvature relative to π-gons and planar sections then $\pi_1(X)$ is coherent. Some results are provided which are useful for creating examples of 2-complexes with these properties, or to test a 2-complex for these properties.

Résumé

Nous définissons la notion de la courbure sectionelle relative sur les 2-complexes et prouvons qu'un 2-complexe compacte avec angles qui a une courbure sectionelle négative relativement aux sections planaires a un groupe fondamental cohrent. Nous analysons un certain type de 1-complexe qu'on appelle un "flattenable graph" $\Gamma \to X$ pour un 2-complexe compacte avec angles X et démontrons que si X a une courbure sectionelle non-positive et si pour tout "flattenable graphe" $\pi_1(\Gamma) \to \pi_1(X)$ est de présentation finie alors X a un groupe fondamental cohérent. Finalement nous montrons que si X est un 2-complexe compacte avec angles et avec une courbure sectionelle négative relative aux π-gones et les sections planaires alors $\pi_1(X)$ est cohérent. Quelques résultats sont fournis qui sont utiles pour créer des exemples de 2-complexes avec ces propriétés ou pour tester si un 2-complexe a ces propriétés.

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To my parents Maria and Bert and to the rest of my family, thank you for your encouragement and support. This paper is dedicated to you.

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Introduction

The concept of sectional curvature for 2-complexes was introduced by D. Wise in [11] by considering more general sections as apposed to the usual planar sections. To require that an angled 2-complex X has negatively or nonpositively curved general sections is a stronger condition than negative or nonpositive planar sectional curvature. These stronger conditions, however, allowed him to prove many theorems about these 2-complexes and their fundamental groups.

It was independently shown by Scott and Shalen that fundamental groups of 3-manifolds are coherent (Definition 4.1.1) [10]. Freighn-Handel proved that ascending HNN extensions of free groups are coherent [2] and McCammond-Wise showed that what they called perimeter groups were coherent [7].

Wise showed in [11] that any cover $\widehat{X}$ of a compact angled 2-complex X with negative sectional curvature had what he referred to as the compact core property (for any compact subcomplex C of $\widehat{X}$ there exists a compact 2-complex T containing C such that the inclusion $T \hookrightarrow X$ is a π_1-isomorphism). As a consequence we know that $\pi_1(X)$ is coherent. He showed that the compact core conclusion can fail for a compact angled 2-complex that has nonpositive sectional curvature. The counterexample he provided had an angle assignment that contained negative angles. He postulated that the compact core property will hold for a compact nonnegatively angled 2-complex that has nonpositive sectional curvature.

In this paper we focus on coherence and the fundamental group of a 2-complex with nonpositive sectional curvature. We begin by defining the notion of relative negative sectional curvature, allowing 2-cells to have nonpositive curvature and requiring that general sections have negative curvature except for certain sections which we allow to have nonpositive curvature. It is an assumption that is stronger than nonpositive sectional curvature but weaker that negative sectional curvature. It allows us to prove Theorem 4.1.2 which says that an angled 2-complex that has negative sectional curvature relative to planar sections has coherent fundamental group.

By analyzing a sequence of 2-complexes that is constructed in the proof of Theorem 4.1.2 we identify a special class of 1-complexes which we refer to as flattenable graphs. Corollary 4.3.2 states that if flattenable graphs have finitely presented image in a compact angled 2-complex X with nonpositive sectional curvature then $\pi_1(X)$ is coherent.

Finally we introduce a type of graph that we call a π-gon and show, in Theorem 4.4.1, that an compact angled 2-complex that has negative sectional curvature relative to π-gons and planar sections has coherent fundamental group.

Chapter 1

Angled 2-Complexes and Sectional Curvature

1.1 Angled 2-Complexes

We will be concerned with the category of 2-complexes. We use the notation X^n to denote the set of n-cells of X and $|X^n|$ to denote the number of n-cells of X.

A map $f : X \to Y$ between 2-complexes is a *combinatorial map* if when f is restricted to a cell $C \in X$ it is a homeomorphism onto $f(C)$. We say that a 2-complex is *combinatorial* if (possibly after a suitable subdivision) the attaching maps of each of the cells is combinatorial. A map between 2-complexes $Y \to X$ is an *immersion* if it is locally injective. A map between 2-complexes $Y \to X$ is an *near-immersion* if $(Y - Y^0) \to X$ is an immersion. The spaces discussed will be connected combinatorial 2-complexes and the maps between them will be combinatorial immersions unless otherwise stated.

Definition 1.1.1. We say that two immersions $Y_1 \to X$ and $Y_2 \to X$ are *equivalent* if there exists and isomorphism $Y_1 \to Y_2$ such that the following diagram commutes:

$$
\begin{tikzcd}
Y_1 \arrow[r] \arrow[dr] & Y_2 \arrow[d] \\
& X
\end{tikzcd}
$$

If this is not the case we say that the immersions are *distinct.*

Lemma 1.1.2 (Finitely many immersions). Let X be a compact 2-complex and $n \in \mathbb{N}$. There are finitely many distinct immersions $Y \to X$ such that $|Y^0| \leq n$ and Y is compact.

Proof. Consider a collection of $\leq n$ 0-cells to be the 0-skeleton Y^0 of a 2-complex Y. Since X is compact there are only exponentially many ways of completing Y^0 to a complex Y which immerses in X and exponentially many different immersions from Y to X. $\square$

Definition 1.1.3 (Angled 2-Complex). Let X be a 2-complex. X is called an *angled 2-complex* if for each $v \in X^0$ there is an assigned real number $\measuredangle c$ for every every corner c of X at v. We say that X is *positively* or *nonnegatively* angled if all such angles are positive or nonnegative respectfully.

Definition 1.1.4 (Link). Let X be a 2-complex. The link of a 0-cell $v \in X^0$ is the graph link(v) that corresponds the the "epsilon sphere" about v in X. The 0-cells in X are in one-to-one correspondence with the vertices in link(v) and the 1-cells in X are in one-to-one correspondence with the edges in link(v). If X is an angled 2-complex then each edge in link(v) will have an angle associated with it.

Definition 1.1.5 (Curvature). Let X be an angled 2-complex, let f be a 2-cell in X and let $|\partial f|$ denote the length of the attaching map of f. The *curvature of f* is defined by

$$\kappa(f) = \left(\sum_{c \in \text{corners}(f)} \measuredangle c \right) - (|\partial f| - 2)\pi$$

or in other words the actual angle sum of the 2-cell f minus the expected Euclidian angle sum.

Let v be a 0-cell in X. The curvature of v is defined by

$$\kappa(v) = 2\pi - \pi \cdot \chi(\text{link}(v)) - \left(\sum_{c \in \text{corners}(v)} \measuredangle c \right).$$

Let V denote the number of vertices in link(v) and def(c) $= \pi - \measuredangle c$. Then this definition is equivalent to

$$\kappa(v) = (2 - V)\pi + \left(\sum_{c \in \text{corners}(v)} \text{def}(c) \right).$$

The following theorem was first proved in [1] and then observed in [8].

Theorem 1.1.6 (Combinatorial Gauss-Bonnet.). Let X be an angled 2-complex. Then the sum of the 2-cell curvatures and the 0-cell curvatures is equal to the Euler characteristic of X times 2π. In symbols:

$$\sum_{f \in 2\text{-cells}(X)} \kappa(f) + \sum_{v \in 0\text{-cells}(X)} \kappa(v) = 2\pi \cdot \chi(X)$$

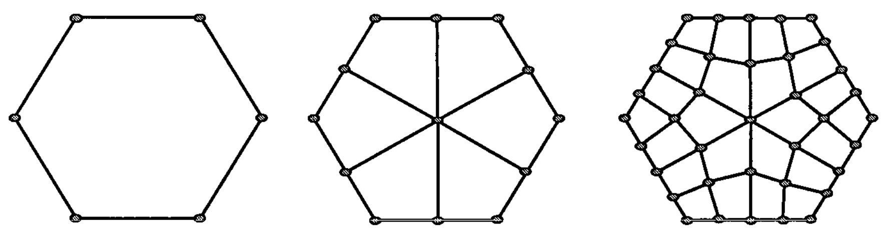
It will be also useful to talk about the curvature of a graph or link instead of the curvature of a 0-cell.

Definition 1.1.7. Let Γ be a graph with angles assigned to the edges. We call Γ an *angled graph.* Let V, and E denote the number of vertices and edges in Γ respectively and let A denote the set of edges in Γ. The curvature of Γ is defined by

$$\kappa(\Gamma) = 2\pi - \pi(V - E) - \left(\sum_{e \in A} \measuredangle(e) \right).$$

Given a 2-complex X it will be useful to consider a subdivided 2-complex where we subdivide certain 2-cells in X by adding new 0-cells and 1-cells. A useful subdivision of a 2-cell c will be the *square subdivision* in which a 0-cell is added to the interior (usually the center) of c and added to the center of each bounding 1-cell. We then add a 1-cell connecting each new boundary 0-cell to the new interior 1-cell. The resulting 2-cells are all squares (as long as c has more than one side). The square subdivision of the resulting cells of a square subdivision is called the *second square subdivision.* If we perform the square subdivision to every 2-cell in X we call the resulting space the *square subdivision of* X and if we perform the square subdivision to every 2-cell in the square subdivision of X we call this space the *second square subdivision of* X.

Example 1.1.8. The following is the square subdivision and second square subdivision of a six sided 2-cell c:



c — Square Subdivision of c — Second Square Subdivision of c

Lemma 1.1.9. The total number of 0-cells obtained from a n-sided 2-cell after the second square subdivision is $6n + 1$.

Proof. Let c be an n-sided 2-cell, $\bar{c}$ be the first square subdivision of c, and $\bar{\bar{c}}$ be the second square subdivision. Then c has n 0-cells. After the first square subdivision of c a new 0-cell is added for each boundary 1-cell and one 0-cell is added to the centre of c resulting in $n+1$ new 0-cells and $2n+1$ 0-cells in total. After the first subdivision c is divided into n 2-cells to each of which we perform the square subdivision to obtain the second square subdivision of c. There are $2n$ 1-cells along the boundary of $\bar{c}$ to each of which we add a 0 cell, n 1-cells connecting the boundary to the centre 0-cell to each of which we add a 0-cell, and n 2-cells in $\bar{c}$ to each of which we add a 0 cell in the centre. This results in $4n$ new 0-cells being added when going from $\bar{c}$ to $\bar{\bar{c}}$ and so $\bar{\bar{c}}$ has $6n+1$ 0-cells. $\square$

1.2 Sectional Curvature

Definition 1.2.1 (Sections). We say that a graph Γ is *regular* if it is compact, connected, spurless (does not contain an edge ending in a valence 1 vertex), and contains at least one edge. We note that, in this paper, regular refers to the above property and not the graph theory definition of a regular graph stating that all vertex degrees are equal.

Let X be an angled 2-complex. A section of X at the 0-cell $x \in X^0$ is a based immersion $(S,s) \to (X,x)$. A section is *regular* if $\text{link}(s)$ is a regular graph.

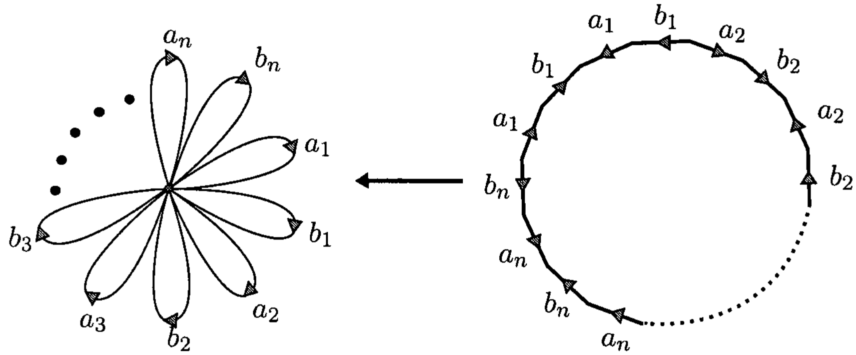
We consider S to be an angled 2-complex by assigning a corner in S the angle of the corner to which it is mapped, pulling back the angle assignment of X to an angle assignment of S. The *curvature of the section* $(S,s) \to (X,x)$ is defined to be $\kappa(s)$.

We say that X has *sectional curvature* $\leq r$ *at* x if all regular sections $(S,s) \to (X,x)$ have curvature $\leq r$. We say that X has *sectional curvature* $\leq r$ if all regular sections $(S,s) \to (X,x)$ have curvature $\leq r$ and and each 2-cell $f \in X$ has curvature $\kappa(f) \leq r$. We use the term nonpositive sectional curvature when $r=0$ and the term negative sectional curvature when $r=0$ and the inequality is strict.

A section $(S,s) \to (X,x)$ is called *planar* if $\text{link}(s)$ is a circle. We say that X has *planar sectional curvature* $\leq r$ if all planar sections $(S,s) \to (X,x)$ have curvature $\leq r$ and each 2-cell $f \in X$ has curvature $\kappa(f) \leq r$.

It is clear that there is a correspondence between sections and subgraphs of links so often we may refer to an angled graph as having sectional curvature $\leq r$ if every regular subgraph has curvature $\leq r$.

Example 1.2.2. The 2-complex construction of the connected sum of n-tori has negative sectional curvature for $n \geq 2$. Let X be the 2-complex constructed from a bouquet of $2n$ circles which we call B_{2n}, labeled $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$, and one 2-cell W_1 with edge attaching map given by $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots\ a_n b_n a_n^{-1} b_n^{-1}$. We assign an angle of π/n to each corner of W_1.



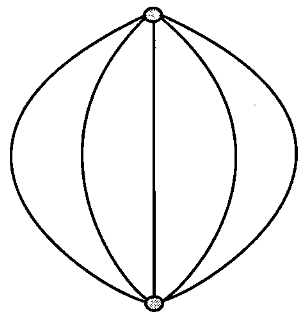
W_1 $\qquad$ B_{2n}

Then $\kappa(W_1) = 4n(\pi/n) - (4n-2)\pi = 6\pi - 4n\pi < 0$. The link of the one 1-cell x is a cycle with $4n$ edges. Since there are no proper subgraphs of link(x) that correspond to regular sections we need only check the curvature of x. We have that

$$\kappa(x) = 2\pi - \pi \cdot 0 - 4n(\pi/n) < 0.$$

This proves that X has negative sectional curvature.

Definition 1.2.3. We say that a graph is a π-gon if it contains two vertices, at least two edges and each edges is adjacent to both vertices. We say that a graph is a bigon if it is a subdivided (possibly trivially) π-gon.



A π-gon With 5 Edges

If X is an angled two complex with nonpositive sectional curvature and $(S, s) \to (X, x)$ is a section such that link(s) is a π-gon then each edge in link(s) must be assigned an angle of at least π.

Definition 1.2.4 (Relative Sectional Curvature). We will be concerned with special cases of angled 2-complexes that have nonpositive sectional curvature. We say that X has *negative sectional curvature relative to planar sections* if X has nonpositive sectional curvature and has the property that any regular section $(S, s) \to (X, x)$ with $\kappa(s) = 0$ is planar. We say that X has *negative sectional curvature relative to π-gons* if X has nonpositive sectional curvature and has the property that if $(S, s) \to (X, x)$ is a regular section with $\kappa(s) = 0$ then link(s) is a π-gon. We define *negative sectional curvature relative to bigons* and *negative sectional curvature relative to π-gons and planar sections* similarly.

We now mention a remark originally from [11].

Remark 1.2.5. If X is compact, all regular sections at 0-cells of X have negative sectional curvature and all 2-cells have nonpositive curvature then we may assume that X has negative sectional curvature since we may decrease the angles slightly so that all 0-cells have negative sectional curvature and all 2-cells have negative sectional curvature. If X is compact with nonpositive sectional curvature and all 2-cells have negative curvature then we may increase the angles slightly so that all 0-cells have negative sectional curvature and all 2-cells have negative sectional curvature.

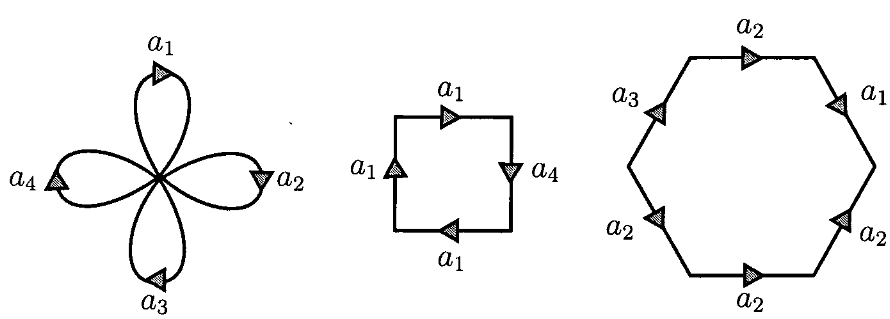
Definition 1.2.6 (Standard 2-complex). Let G be a group with presentation

$$\langle a_1, a_2, \ldots, a_n | W_1, W_2, \ldots, W_k \rangle.$$

We construct the standard 2-complex for G as follows: Let B_n be the bouquet of n circles. Label the circles a_1 to a_n and give each circle a direction. For each relator W_i we define a 2-cell with $|W_i|$ sides which we will call C_i. Label and direct the boundary 1-cells of C_i via the word W_i. For example, if $W_i = a_2 a_3 a_1^{-2} a_4$ then the boundary of C_i would read a_2, a_3, a_1, a_1, a_4 but the direction of the a_1 1-cells would be reversed. We now attach the 2-cells $C_1, \ldots, C_k$ to B_n using the labeling as the attaching map and call the resulting space X. The Seifert-van Kampen Theorem [6] tells us that $\pi_1(X) = G$.

When unambiguous we may refer to the 2-cell C_i as W_i.

Example 1.2.7. Let $G = \langle a_1, a_2, a_3, a_4 \,|\, a_1 a_4 a_1^2,\ a_2 a_1 a_2^{-3} a_3 \rangle$. Let B_4 be the bouquet of 4 circles, C_1 be a four sided 2-cell and C_2 be a six sided 2-cell. Label and direct the edges of B_4 as a_1, a_2, a_3, a_4, the edges of C_1 as a_1, a_4, a_1, a_1, and the edges of C_2 as $a_2, a_1, a_2^{-1}, a_2^{-1}, a_2^{-1}, a_3$. The spaces B_4, C_1 and C_2 look like the following:



B_4 C_1 C_2

We glue the spaces together with the attaching map given by the labeling to obtain X, the standard 2-complex for G, with $\pi_1(X) = G$.

We present the following generalization of Theorem 9.1 from [11]. We will state

Theorem 9.1 as Corollary 1.2.10 of the following theorem:

Theorem 1.2.8. Let $G = \langle a_1, a_2, \ldots | W_1, W_2, \ldots W_n \rangle$ and let X be the standard 2-complex for the G. Let X be an angled 2-complex with all angles $\leq \pi$ and $\kappa(W_i) \leq 0$ for $i = 1, \ldots, n$.

Suppose all regular sections $(S, s) \to (X, x)$ with the property that link(s) has $< 2(n+1)$ verticies have nonpositive sectional curvature. Then X has nonpositive sectional curvature.

If instead all regular sections $(S, s) \to (X, x)$ such that link(s) has $< 2(n+1)+1$ verticies have negative sectional curvature then X has negative sectional curvature.

Proof. First we will prove the nonpositive case. By assumption we need only concern ourselves with regular sections with $\geq 2(n+1)$ 0-cells. Let $(S, s) \to (X, x)$ be such a section and let V denote the number of vertices in link(s). If $\kappa(W_i) < 0$ then we may increase the angles in W_i so that we may assume that $\kappa(W_i) = 0$ for all $i = 1, \ldots n$ and all regular sections with $< 2(n+1)$ 0-cells will still have nonpositive sectional curvature.

Since each edge $e \in$ link(s) comes from a corner in one of $W_1, \ldots, W_n$ we have that

$$\sum_{e \in \text{link}(s)} \text{def}(e) \leq \left(\sum_{c \in \text{Corners}(W_1)} \text{def}(c) \right) + \ldots + \left(\sum_{c \in \text{Corners}(W_n)} \text{def}(c) \right).$$

For each W_i we have that

$$
\begin{aligned}
\kappa(W_i) &= \left(\sum_{c \in \mathrm{Corners}(W_i)} \measuredangle(c) \right) - (|W_i| - 2)\,\pi \\
&= 2\pi - |W_i|\,\pi + \left(\sum_{c \in \mathrm{Corners}(W_i)} \measuredangle(c) \right) \\
&= 2\pi - \left(\sum_{c \in \mathrm{Corners}(W_i)} \pi - \measuredangle(c) \right) \\
&= 2\pi - \left(\sum_{c \in \mathrm{Corners}(W_i)} \mathrm{def}(c) \right)
\end{aligned}
$$

giving us that

$$
\sum_{c \in \mathrm{Corners}(W_i)} \mathrm{def}(c) = 2\pi - \kappa(W_i).
$$

Therefore, since $\kappa(W_i) = 0$ for each i,

$$
\begin{aligned}
\sum_{e \in \mathrm{link}(s)} \mathrm{def}(e) &\leq (2\pi - \kappa(W_1)) + \ldots + (2\pi - \kappa(W_n)) \\
&= 2n\pi
\end{aligned}
$$

and so since $V \geq 2(n+1)$,

$$
\begin{aligned}
\kappa(s) &= \pi(2-V) + \left(\sum_{e \in \mathrm{link}(s)} \mathrm{def}(e)\right) \\
&\leq 2\pi - V\pi + 2n\pi \\
&\leq 2(n+1)\pi - V\pi \\
&\leq 0.
\end{aligned}
$$

This implies that X has nonpositive sectional curvature.

We will now discuss the negative case. As above we may increase the angles in W_i so that we may assume that $\kappa(W_i) = 0$ for all $i = 1, \ldots n$ and all regular sections with $< 2(n+2)$ 0-cells will still have negative sectional curvature. This implies that all of the above inequalities hold except since $V > 2(n+1)$ the last inequality is strict implying that all sections have negative curvature. Again we may decrease the angles in W_i for $i = 1, \ldots, n$ slightly retaining strict negative curvature on the sections but also obtaining $\kappa(W_i) < 0$. This implies that X has negative sectional curvature.

□

The following lemma was proven by wise in [11].

Lemma 1.2.9. Let Γ be obtained from the finite angled graph Λ by adding edges $e_1, \ldots, e_r$ such that $\measuredangle(e_i) \geq \pi$ for each i. Then $\kappa(\Gamma) \leq \kappa(\Lambda)$.

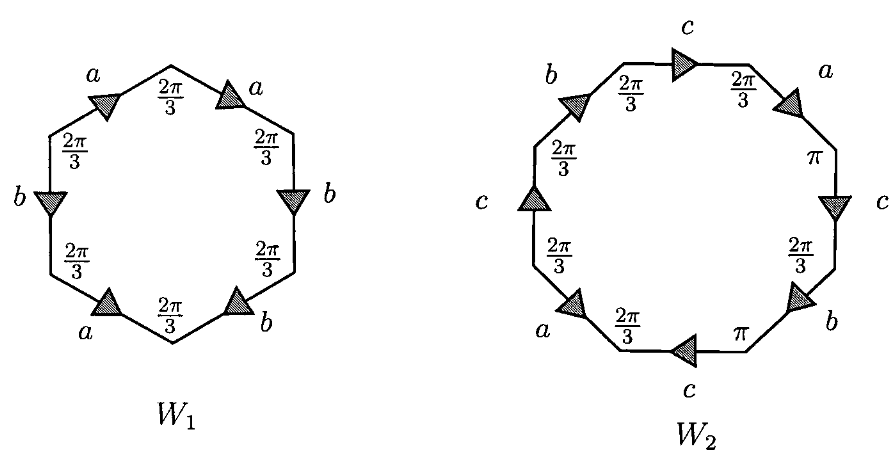
Corollary 1.2.10. Let X be the standard 2-complex of a one-relator group $\langle a_1, a_2, \ldots | W \rangle$. Suppose that X is an angled 2-complex with nonpositive planar sectional curvature, and each angle is $\leq \pi$. Then X has nonpositive sectional curvature.

Proof. First we prove the nonpositive case. Suppose $(S, s) \to (X, x)$ is a section such that $\mathrm{link}(s)$ has < 4 vertices. Any pair of edges with the same endpoints form a cyclic subgraph representing a cyclic section and so must each have angle π by our assumptions. By removing duplicate edges of angle π we obtain a graph that is either a bigon, a triangle, or a pair of bigon meeting at a vertex. A loop is ruled out since it would require an edge to have an angle of $\geq 2\pi$. The first two cases have angle sum $\geq 2\pi$ ensuring curvature of ≤ 0 and the third case has curvature $\leq -\pi$. Therefore $\kappa(s) \leq 0$ by Lemma 1.2.9. The corollary follows from Theorem 1.2.8.

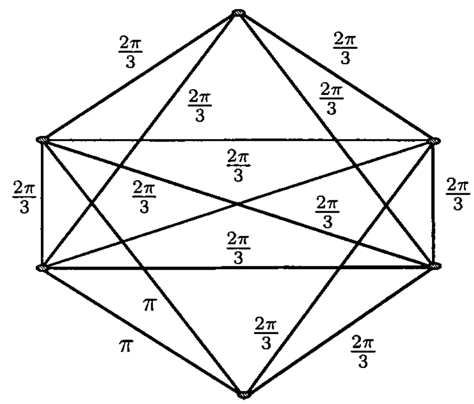
The negative case is similar. All graphs are already reduced since multiple edges do not occure since they would need an angle sum $> 2\pi$. We also need to examine graphs with 4 vertices. All graphs have curvature < 0 and so $\kappa(s) \leq 0$ by Lemma 1.2.9. $\square$

We will see that the assumptions of angles $\leq \pi$ and nonpositive sectional curvature are not enough in the more general case of a group with more than one relator.

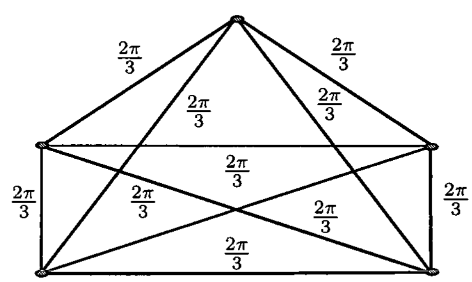
Example 1.2.11. Let $G = \langle\, a, b, c \,|\, a^2b^2c^{-1}b^{-1},\ cacbca^{-1}cb^{-1} \rangle$. Let X be the standard 2-complex for G. Then X has one 1-cell x and two 2-cells W_1 and W_2. We assign angles to X by assigning angles to W_1 and W_2 as follows:



Then $\kappa(W_1) = 4\pi - (6-2)\pi = 0$ and $\kappa(W_2) = 6\pi - (8-2)\pi = 0$. We observe that link$(x)$ is the following graph:



Notice that all angles are $\leq \pi$ and that every cycle in link(x) has curvature ≤ 0 meaning that X has nonpositive sectional curvature. However, curvature of the section $(S, s) \longrightarrow (X, x)$ corresponding to the following subgraph has curvature $\kappa(s) = 2\pi - \pi(5 - 10) - 10(2\pi/3)\pi > 0$:



1.3 Weight Test and Embedded Links

For further backround on the nonpositve and negative weight tests we refer the reader to [9] and [3]. We present these definitions and strengthen them slightly.

Definition 1.3.1 (Weight test). Let X be a an angled 2-complex. We say that X satisfies the α *weight test* if for each $x \in X^0$ the sum of the angles in any immersed cycle $\sigma \to$ link(x) is $\geq \alpha$. Similarly, X satisfies the *strict α weight test* if this inequality is strict.

We say that X satisfies the *nonpositive weight test* if for each $x \in X^0$ the sum of the angles in any immersed cycle $\sigma \to \text{link}(x)$ is $\geq 2\pi$. X satisfies the *negative weight test* if this inequality is strict.

This concept is similar to planar sectional curvature. Wise showed that if an angled 2-complex has nonpositive angles and nonpositive (negative) sectional curvature then it satisfies the nonpositive (negative) weight test but that the converse doesn't hold in general ([11], Lemma 2.11, Example 2.12).

Wise also showed that a 2-complex X that satisfies the nonpositive (negative) weight test and has the property that for each $x \in X$ link(x) that can be embedded in the 2-sphere (which he calls having *spherical links*) has nonpositive (negative) sectional curvature ([11], Theorem 10.2). We present the following generalization:

Theorem 1.3.2. Let X be an angled 2-complex with links that can be embedded into compact surfaces. Let S_x be a surface such that link(x) embeds into S_x. Suppose that X satisfies the (strict) $3(2-\chi(S_x))\pi$ weight test. Then X has nonpositive (negative) sectional curvature.

We note that the condition that there exists a surface S_x such that link(x) embeds into S_x is satisfied if X is compact.

Proof. Let $k = \chi(S_x)$. Let (S, s) be a regular section and let $\Gamma = \text{link}(s)$. Then Γ is a connected spurless nontrivial subgraph of link$(x) \subseteq S_x$. Notice that Γ is the 1-skeleton of a cell structure of S_x. Let V, E, and F denote the number of 0-cells, 1-cells and 2-cells in this cell structure. Each edge e in Γ is labeled by the angle $\measuredangle(e)$ inherited from the corresponding corner in X at x. Notice that each 2-cell in S_x determines a cycle in Γ and each edge in Γ appears twice amongst the cycles. Therefore

$$2\sum_{e\in\Gamma}\measuredangle(e) = \sum_{f}\sum_{e\in f}\measuredangle(e) \geq \sum_{f} 2(3-k)\pi = 2(3-k)\pi \cdot F$$

where f is a 2-cell in S_x. Therefore $\sum_{e\in\Gamma} \measuredangle(e) \geq (3-k)\pi \cdot F$ and so we have that

$$\begin{aligned}
\kappa(s) &= 2\pi - \pi\chi(\Gamma) - \sum_{e\in\Gamma} \measuredangle(e) \\
&\leq 2\pi - \pi(V-E) - (3-k)\pi \cdot F \\
&= 2\pi - \pi(V-E+F) - 2\pi F + k\pi F \\
&\leq k\pi F - \pi\chi(S_x) \\
&\leq k\pi - k\pi \\
&= 0
\end{aligned}$$

as required.

The negative case is similar with strict inequalities.

□

1.4 Torsion Free

Definition 1.4.1. A 2-complex is *aspherical* if it is not possible to embed a sphere $S^2 \to X$.

Lemma 1.4.2. Let X be an compact angled 2-complex with nonpositive sectional curvature. Then X is aspherical.

Proof. Suppose that X is not aspherical. Let $S \to X$ be an embedded sphere. Give S the induced angled 2-complex structure. Since $\chi(S) = 2$ Theorem 1.1.6 tells us that S has a 2-cell or 0-cell that has positive curvature. But both of these cases contradict that X has negative sectional curvature. □

Lemma 1.4.3. Let X be a connected aspherical 2-complex. Then $\pi_1(X)$ is torsion free.

Proof. Let $\widehat{X}$ be the universal cover of X. We know that $\pi_0(\widehat{X}) = \pi_1(\widehat{X}) = 0$ and since X is aspherical so is $\widehat{X}$. This means that any sphere embedded in $\widehat{X}$ is contractable to a point in $\widehat{X}$. Since $\pi_1(\widehat{X}) = \pi_2(\widehat{X}) = \pi_3(\widehat{X}) = 0$ we have that $\pi_4(\widehat{X}) = \mathsf{H}_4(\widehat{X}) = 0$ but $\mathsf{H}_4(\overline{X}) = 0$ since X is a 2-complex. Inductively we have that $\pi_n(\widehat{X}) = 1$ for every $n \in \mathbb{N}$ since

$$\pi_{i-1}(\widehat{X}) = \pi_i(\widehat{X}) = \pi_{i+1}(\widehat{X}) = 0 \implies \pi_{i+2}(\widehat{X}) = \mathsf{H}_{i+2}(\widehat{X}) = 0.$$

This implies that $\widehat{X}$ is contractable and so the group $\mathbb{Z}_n$ cannot act freely on $\widehat{X}$. Therefore $\pi_1(X)$ is torsion free. □

Corollary 1.4.4. Let X be an compact angled 2-complex with nonpositive sectional curvature. Then $\pi_1(X)$ is torsion free.

Chapter 2

Boundary and Disc Diagrams

2.1 The Boundary of a Complex

The following definitions are originally from [11].

Definition 2.1.1. Let X be a combinatorial complex. We call an n-cell f of X a *free face* of an $(n+1)$-cell c if it is adjacent to c in only one way and no other cell of dimension $\geq n+1$. We say that an n-cell is *isolated* if it is not adjacent to any $(n+1)$-cells.

We define ∂X to be the closure of the set of free faces of X. We call the set ∂X the *boundary* of X. We define ξX to be the closure of the set of isolated 1-cells in X. We use the notation $|\partial X|$ and $|\xi X|$ to refer to the number of 1-cells in ∂X and ξX respectfully.

2.2 Disc Diagrams

Definition 2.2.1. A *Disc Diagram* D is a simply connected planar 2-complex. Often, for a 2-complex X, we may also refer to a map $D \to X$ as a disc diagram.

Let $P \to X$ be a be a closed path that factors as $P \to D \to X$. If P maps onto ∂D and the preimage of each 1-cell e in ∂D is one 1-cell in P (or two 1-cells in P if e is not adjacent to a 2-cell) then P is a *boundary path* for D. If P is a boundary path for D we say the disc diagram $D \to X$ is a disc diagram for $P \to X$.

If a pair of 2-cells C_1, C_2 in D meet along a 1-cell e and have boundary cycles that, beginning with e, are sent to identical paths in X by $D \to X$ then we call C_1 and C_2 a *cancellable pair*. A disc diagram is called *reduced* if it has no cancellable pairs.

It is clear that a path $P \to X$ that has a disc diagram is null-homotopic. It is a theorem of Van Kampen that a reduced disc diagram exists for any null-homotopic closed path [5]. For more backround on disc diagrams we refer the reacher to [7] and [5].

The following lemma is from lemma 6.4 in [11].

Lemma 2.2.2. Let $A \to X$ be an immersion. Let $D \to X$ be a near-immersion of a disc diagram with boundary path P, and suppose the following disc diagram commutes:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr] & D \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

Form the 2-complex $A \cup_P D$ by attaching D to A along P. Let $A \cup_P D \to C$ be a surjective combinatorial map such that we have the following commutative diagram:

$$\begin{tikzcd}
A \cup_P D \arrow[r] \arrow[dr] & C \arrow[d] \\
& X
\end{tikzcd}$$

Then $|\xi C| \leq |\xi A|$ and $|\xi C| + |\partial C| \leq |\xi A| + |\partial A|$.

Chapter 3

Factoring Through Immersions

3.1 Tower Lifts

In this section we will examine towers and present some backround due to Howie [4]. Originally towers were presented as maps between CW-complexes. In our definitions they will be maps between 2-complexes.

Definition 3.1.1 (Tower, tower lift). A map $T \to X$ of connected 2-complexes is a *tower* if it can be expressed as a decomposition

$$T = B_n \hookrightarrow \widehat{B}_{n-1} \to B_{n-1} \hookrightarrow \ldots \hookrightarrow \widehat{B}_2 \to B_2 \hookrightarrow \widehat{B}_1 \to B_1 \hookrightarrow X$$

where the maps $B_i \hookrightarrow \widehat{B}_{i-1}$ are inclusion maps and the maps $\widehat{B}_i \to B_i$ are covering maps.

Let $Y \to X$ be a map between connected 2-complexes. A map $Y \to T$ is a *tower lift* of $Y \to X$ if there is a tower $T \to X$ such that the following diagram commutes:

$$\begin{tikzcd} & T \arrow[d] \\ Y \arrow[ur] \arrow[r] & X \end{tikzcd}$$

We call a tower lift $Y \to T$ *maximal* if for any tower lift $Y \to T'$ of $Y \to T$, the map $T' \to T$ is an isomorphism.

The following lemma was proven by Howie in [4] with respect to CW-complexes.

Lemma 3.1.2. Let S be a compact connected 2-complex and $S \to K$ be a combinatorial map. Then $S \to K$ has a maximal tower lift.

Lemma 3.1.3. Let $f : Y \to X$ be a combinatorial map between 2-complexes with Y compact. Then there exist a combinatorial immersion $Y' \to X$ such that the map $Y \to X$ factors as $Y \to Y' \to X$, and $Y \to Y'$ is surjective and π_1-surjective.

Proof. Let $Y \to Y'$ be a maximal tower lift of $Y \to X$. Then $Y \to X$ factors as $Y \to Y' \to X$. We know that $Y' \to X$ can be decomposed as a composition of alternating inclusion maps and covering maps both of which are immersions. Therefore $Y' \to X$ is an immersion as required. □

3.2 Folding

We now present an alternate proof to Lemma 3.1.3. This method of factoring called *folding* will later be utilized in Theorem 4.1.2. The tower lift method is useful when dealing with the compact core property.

Proof. If f is an immersion setting $Y' = Y$ completes the proof, otherwise there is a vertex $v \in Y^0$ such that the induced map $\text{link}(v) \to \text{link}(f(v))$ is not injective.

Specifically, there is a pair of edges or vertices in link(v) that map to one edge or vertex in link$(f(v))$, respectively. We select a vertex v of the second type. The pair of vertices in link(v) correspond to a pair 1-cells, e_1,e_2, in Y adjacent to v that are mapped to the same 1-cell in X. Let Y_1 be the quotient space $Y/(e_1 = e_2)$.

It is clear that the map $Y \longrightarrow Y_1$ is continuous and surjective and we claim that it is π_1-surjective. Let $\overline{v}$ be the image of v in Y_1 and $\overline{P}$ be a path in ${Y_1}^1$. We want to show that there is a path P in Y^1 whose image is path homotopic to $\overline{P}$. Let v_1 and v_2 be the endpoints of e_1 and e_2 respectively. If the image of v_1 and v_2 is the same 0-cell in $\overline{Y}$ then there is nothing to prove since we may let P be the pre-image of $\overline{P}$ in Y, possibly without e_2. Indeed, P is a path and $P \longrightarrow \overline{P}$. If v_1 and v_2 are mapped to different 0-cells then the pre-image of $\overline{P}$ may be a path that is separated between v_1 and v_2 for every instance of which we add to the preimage the edges $e_1e_2^{-1}$ or $e_2e_1^{-1}$ to obtain a path P whose image is path homotopic to $\overline{P}$. Indeed, for each $e_1e_2^{-1}$ or $e_2e_1^{-1}$ in P contributes one backtrack in the image, $\overline{e}\overline{e}^{-1}$ or $\overline{e}^{-1}\overline{e}$ where $\overline{e}$ is the image of e_1 and e_2.

We repeat this process for every instance of a vertex v that has a pair of vertices in link(v) that map to the same vertex in link$(f(v))$ and obtain the space Y_i.

If $Y_i \longrightarrow X$ is not and immersion then there must be is a vertex $v \in Y^0$ such that a pair of edges in link(v) that map to one edge in link$(f(v))$ which means there is a pair of 2-cells, c_1 and c_2, adjacent to v and are mapped to the same 2-cell via f. Let $Y_{i+1} = Y_i/(c_1 = c_2)$. Then $Y_i \longrightarrow Y_{i+1}$ is continuous, surjective and π_1-surjective since it is a homeomorphism of 1-skeletons. Repeating this process we obtain a space Y' with the desired properties.

$\square$

Chapter 4

Relative Sectional Curvature and Coherence

4.1 Negative Sectional Curvature Relative to Planar Sections

Definition 4.1.1. A group G is called *coherent* if every finitely generated subgroup is finitely presented.

Theorem 4.1.2. Let X be a compact angled 2-complex with negative sectional curvature relative to planar sections. Then $\pi_1(X)$ is coherent.

We apologize for the length of the proof of the Theorem 4.1.2, although the ideas are not complicated there are many details. We will give an outline of the important points to simplify the reading.

We begin with a finitely generated subgroup of $\pi_1(X)$ and a finite immersed graph $B \to X$ such that $\pi_1(B) \twoheadrightarrow G$. We see that the map $B \to X$ π_1-surjects but most likely does not π_1-inject, if this were the case we would be done. We move closer to this goal by attaching a disc diagram to B for a path in B whose image in X is null homotopic. For now call this space T_2. We see that the kernel of the map $T_2 \to X$ is smaller than that of $T_1 = B \to X$ in fact if we construct a sequence $T_1, T_2, T_3, \ldots$, where each space is obtained from the previous by attaching a disc diagram for a path whose image in X is null homotopic, then the kernel of the maps $T_i \to X$ are decreasing as i increases. If at some T_n we were unable to attach a disc diagram we would have that $T_n \to X$ is π_1-injective, $\pi_1(T_n) = G$, and T_n would be compact thus completing the proof. Unfortunately there is no guarantee the sequence must terminate. However, if the maps $T_i \to X$ were immersions and we had an upperbound on the number of 0-cells in any T_i Lemma 1.1.2 would tell us that the sequence would have to terminate.

We can insure that the maps are immersions by folding at each stage of the construction. The euler characteristic of B provides us with an upperbound on the number of all the 0-cells in any T_i except the regular ones with zero curvature. If there are no such 0-cells, as in the negative sectional curvature case, then we would be done. In our case we deal with the zero curvature 0-cells by adjusting our construction of the sequence $\{T_i\}$ slightly.

Step 1. Adding a disc: Let $T_1 = B$. We begin in the same way, adding a disc diagram to T_i for a path whose image is null homotopic. We then fold and call the resulting complex Z_i

Analysis of Flat Subcomplexes. We identify F, the subcomplex of Z_i that contains only regular zero curvature 0-cells, which we call the "flat part" of Z_i. By subdividing Z_i to $\overline{Z}_i$ we ensure that each component $\overline{S}_i$ of $F - \{\text{image of } B\}$ is a surface and that their boundary components are circles. It will be shown that each component of $\partial\overline{S}_j$ lies in a distinct component of $\overline{Z}_i - int(\overline{S}_j)$.

Step 2. Adjustment: We call L the component of $\overline{Z}_i - \cup_i S_i$ that contains the image of B. It turns out that L contains all "relavent" information about the fundamental group of $\overline{Z}_i$ and that we are able to remove the subspaces $K_i = \overline{Z}_i - L$

which contain all of the zero curvature 0-cells and replace them with subspaces that have upper bounds on the number of 0-cells and, in particular, the number of zero curvature 0-cells. We either remove these subspaces from $\overline{Z}_i$ entirely (**Chopping**) if $\pi_1(K_i) = (Z)$ or replace them with disc diagrams (**Capping**) if $\pi_1(K_i) = 1$. We fold the resulting space to obtain T_{i+1} such that $\pi_1(T_{i+1}) = \pi_1(Z_i)$, we have an upper bound on the 0-cells of T_{i+1}, and $T_{i+1} \to X$ is an immersion. This guarantees that the sequence $\{T_i\}$ terminates at some T_n such that T_n is compact and $\pi_1(T_n) = G$ as required.

Proof. Let G be a finitely generated subgroup of $\pi_1(X)$. Let $B \to X$ be a finite immersed graph such that $\pi_1(B) \twoheadrightarrow G$.

Let $T_1 = B$. We shall construct a sequence of 2-complexes

$$T_1, T_2, T_3, \ldots$$

of immersed 2-complexes $T_i \to X$ such that $\pi_1(T_i) \twoheadrightarrow G$, $ker(\pi_1(T_i) \to \pi_1(X))$ is decreasing, and there is a number N such that $|T_i^0| \leq N$ for every i. We will use the properties listed above and the finiteness of X to show that the sequence will terminate at some $T_n \to X$ where $\pi_1(T_n) \to \pi_1(X)$ is injective. Assuming $\pi_1(T_i) \to \pi_1(X)$ is not injective we will demonstrate how to produce $T_{i+1} \to X$ using the following two steps.

Step 1. Adding a disc: Let $P_i \to T_i$ be an essential path whose projection $P_i \to X$ is null-homotopic. Let $D_i \to X$ be a reduced disc diagram whose boundary path is $P_i \to X$. Let $Y_i = T_i \cup_{P_i} D_i$ and $Y_i \to X$ be the induced map. Fold $Y_i \to X$ to obtain an immersion $Z_i \to X$. Note that the map from $T_i \to Z_i$ is π_1-surjective but not π_1-injective.

Analysis of Flat Subcomplexes. Let V be the set of all 0-cells in Z_i with zero curvature, let E be the set of all 1-cells in Z_i with both adjacent 0-cells in V, and let

C be the set of 2-cells in Z_i whose adjacent 1-cells lie in E. Let $F = V \cup E \cup C$ be the "flat part" of Z_i.

Let N be the smallest open cellular neighbourhood of the image of B in Z_i. If $F - N = \varnothing$ then define $T_{i+1} = Z_i$ and skip step 2. Otherwise let $A_1, ..., A_k$ be the components of $F - N$, and we examine these components more closely.

Let $j \in \{1, ..., k\}$. For each $a \in A_j^0$ we have that $\text{link}_{A_j}(a) \subset \text{link}_{z_i}(a) \cong S^1$. Thus while A_j is a "singular surface" it may not actually be a surface. We may correct this by subdividing Z_i and thickening A_j to a surface $\overline{S}_j$.

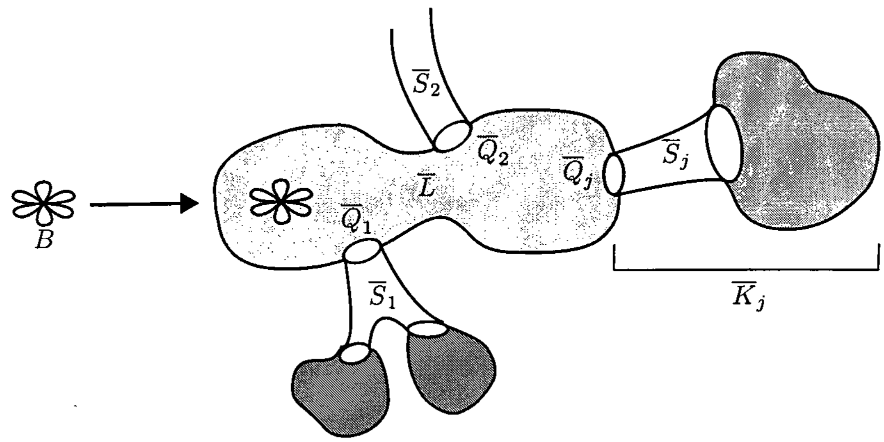
Let $\overline{Z}_i$ be the second square subdivision of Z_i. Let $\overline{A}_j$ be A_j after the subdivision and let $\overline{S}_j$ be the smallest closed cellular neighbourhood of $\overline{A}_j$ (including $\overline{A}_j$). Note that both $\overline{S}_j$ and its smallest closed cellular neighbourhood are surfaces. Since $\overline{S}_j$ is both a surface and a subset of a surface its boundary components are circles.

Let $\overline{U}_j = \overline{Z}_i - int(\overline{S}_j)$. A homological argument will show that distinct components of $\partial\overline{S}_j$ lie in distinct components of $\overline{U}_j$. Indeed, we will show that the induced inclusion homomorphism $\mathsf{H}_0(\partial\overline{S}_j) \to \mathsf{H}_0(\overline{U}_j)$ is injective. Note that $\overline{U}_j \cup \overline{S}_j = \overline{Z}_i$ and $\overline{U}_j \cap \overline{S}_j = \partial\overline{S}_j$. Consider the Mayer-Vietoris sequence for complexes:

$$\ldots \to \mathsf{H}_1(\overline{U}_j) \oplus \mathsf{H}_1(\overline{S}_j) \xrightarrow{g} \mathsf{H}_1(\overline{Z}_i) \xrightarrow{h} \mathsf{H}_0(\partial\overline{S}_j) \xrightarrow{\ell} \mathsf{H}_0(\overline{U}_j) \oplus \mathsf{H}_0(\overline{S}_j) \to \mathsf{H}_0(\overline{Z}_i) \to 0$$

Since the image of $B \to \overline{Z}_i$ factors as $B \to \overline{U}_j \to \overline{Z}_i$ we see that $\pi_1(\overline{U}_j) \twoheadrightarrow \pi_1(\overline{Z}_i)$ and since $\overline{Z}_i$ is connected $\mathsf{H}_1(\overline{U}_j) \twoheadrightarrow \mathsf{H}_1(\overline{Z}_i)$. This implies that g is surjective and so $im(g) = \mathsf{H}_1(\overline{Z}_i) = ker(h)$ and so $im(h) = 0 = ker(\ell)$ by exactness. Therefore $\mathsf{H}_0(\partial\overline{S}_j) \to \mathsf{H}_0(\overline{U}_j)$ is injective as claimed.

We have shown that each circle in $\partial\overline{S}_j$ lies in a distinct component of $\overline{U}_j$. Since N and $\overline{S}_j$ are disjoint, the image of B must lie in one component of $\overline{U}_j$. In fact, the image of B must lie in one component $\overline{L}$ of $\overline{Z}_i - \cup\overline{S}_j$. By relabeling, let $\{\overline{S}_1, ..., \overline{S}_m\}$ be the relabeled subset of $\{\overline{S}_1, ..., \overline{S}_k\}$ whose elements intersect with $\overline{L}$. For $j = 1, ..., m$ let $\overline{Q}_j$ be the circle in $\overline{S}_j \cap \overline{L}$, and let $\overline{K}_j$ be the component of $\overline{Z}_i - \overline{Q}_j$ that does not contain the image of B union $\overline{Q}_j$.



Step 2. Adjustment: Let $\overline{Z}_{i_1} = \overline{Z}_i$. We will now construct a sequence $\overline{Z}_{i_1}, ..., \overline{Z}_{i_m}$ at each step, $j = 1, ..., m$ replacing $\overline{K}_j$ with a complex R_j in such a way that we will be able to bound the number of 0-cells in R_j. This will allow us to bound the number of 0-cells in T_i. There will be two cases to consider. If $\pi_1(\overline{Q}_j) \to \pi_1(\overline{Z}_i)$ is injective we will be able to replace $\overline{K}_j$ with $\overline{Q}_j$ itself (chopping). If $\pi_1(\overline{Q}_j) \to \pi_1(\overline{Z}_i)$ is not injective we will replace $\overline{K}_j$ with a disc diagram (capping).

We begin with $j = 1$.

Chopping. If $\pi_1(\overline{Q}_j) \to \pi_1(\overline{Z}_{i_j})$ is injective then $\pi_1(\overline{Z}_{i_j} - \overline{K}_j) \to \pi_1(\overline{Z}_{i_j})$ is an isomorphism. Indeed,

$$\pi_1(\overline{Z}_{i_j}) \cong \pi_1(\overline{Z}_{i_j} - \overline{K}_j) \underset{\pi_1(\overline{Q}_j)}{*} \pi_1(\overline{K}_j).$$

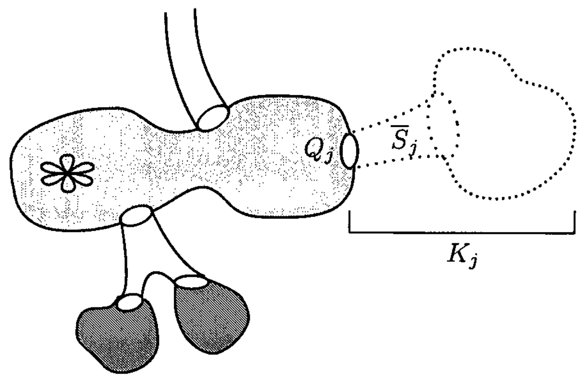
Since B factors through $\overline{Z}_{i_j} - \overline{K}_j$ we see that $\pi_1(\overline{Z}_{i_j} - \overline{K}_j) \twoheadrightarrow \pi_1(\overline{Z}_{i_j})$ which is impossible unless $\pi_1(\overline{Q}_j) \twoheadrightarrow \pi_1(\overline{K}_j)$.

Since 0-cells in $\overline{Q}_j$ are new 0-cells obtained from the subdivisions there is a natural way to push $\overline{Q}_j$ onto original 0-cells defining a deformation retraction of $\overline{Z}_{i_j} - \overline{K}_j$

onto a subspace, $\overline{Z}_{i_{j+1}}$. Indeed $\overline{Q}_j$ can be pushed along a cylinder to a (possibly degenerate) boundary circle Q_j of $\overline{Z}_{i_{j+1}}$ where Q_j contains only original 0-cells not obtained from the subdivisions. Then $\pi_1(\overline{Z}_{i_{j+1}}) \cong \pi_1(\overline{Z}_{i_j} - \overline{K}_j)$. This means that

$$\pi_1(\overline{Z}_{i_{j+1}}) \cong \pi_1(Z_i).$$

We note that $\overline{Z}_{i_{j+1}}$ is essentially $\overline{Z}_{i_j} - \overline{K}_j$.



The 2-complex $\overline{Z}_{i_{j+1}}$ After Chopping

Capping. If $\pi_1(\overline{Q}_j) \to \pi_1(\overline{Z}_{i_j})$ is not injective we choose a base point to lie on $\overline{Q}_j$. We regard $\overline{Q}_j \to \overline{Z}_{i_j}$ as a closed based path we see that $\overline{Q}_j^{\,n} = 1$ for some $n > 0$. By Corollary 1.4.4, $\pi_1(\overline{Z}_{i_j})$ is torsion free and so $\overline{Q}_j$ is null-homotopic in $\overline{Z}_{i_j}$. Attaching a 2-cell to $\overline{Z}_{i_j}$ by gluing its boundary to $\overline{Q}_j$ we obtain a space M such that $\pi_1(M) \approx \pi_1(\overline{Z}_{i_j})$. Calling this new 2-cell $\overline{D}$ we see that

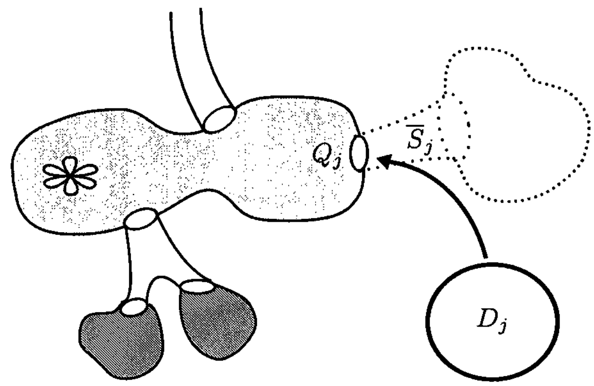
$$\pi_1(\overline{Z}_{i_j}) \cong \pi_1(\overline{Z}_{i_j} - \overline{K}_j) \cup \overline{D}) * \pi_1(\overline{K}_j \cup \overline{D}).$$

Now, $\overline{Z}_{i_j} - \overline{K}_j$ contains the image of B therefore $\pi_1(\overline{K}_j \cup \overline{D}) = 1$ and $\pi_1((\overline{Z}_{i_j} - \overline{K}_j) \cup \overline{D}) \approx \pi_1(M) \approx \pi_1(\overline{Z}_i)$.

If instead of a 2-cell $\overline{D}$ we attached a reduced disc diagram $\overline{D}_j$ for $\overline{Q}_j$ we still have that $\pi_1((\overline{Z}_{i_j} - \overline{K}_j) \cup \overline{D}_j) \approx \pi_1(\overline{Z}_i)$. Unfortunately, $(\overline{Z}_{i_j} - \overline{K}_j) \cup \overline{D}_j$ contains new

0-cells obtained from the subdivisions. As was the case in the chopping procedure, we may obtain a subspace where the induced boundary circle (possibly degenerate) Q_j contains only old 0-cells, and attach a reduced disc diagram D_j for Q_j to this subspace. We call the resulting space $\overline{Z}_{i_{j+1}}$. Then

$$\pi_1(\overline{Z}_{i_{j+1}}) \approx \pi_1(Z_{i_j}).$$



The 2-complex $\overline{Z}_{i_{j+1}}$ After Capping

We have now ensured that after both the capping or chopping procedures only original 0-cell remain in Q_j.

We move on to $\overline{Q}_{j+1}$ and perform the capping or chopping procedure again to obtain the space $\overline{Z}_{i_{j_2}}$. We repeat this process, obtaining spaces $\overline{Z}_{i_1}, \ldots \overline{Z}_{i_m}$ until we exhaust the set $\{\overline{Q}_1, \ldots, \overline{Q}_m\}$. Since all chopping or capping was preformed on original 0-cells we may now disregard these new 0-cells obtained in the subdivision passing to a space Z_i'. Folding Z_i' we obtain T_{i+1}.

If $T_{i+1} \to X$ is π_1-injective then we the procedure terminates, otherwise we return to step one.

Upper Bound on 0-cells. We now return to the discussion of an global upper bound for the number of 0-cells in T_{i+1}. We will show that the number of 0-cells in Z_i' are bounded and since folding does not increase, and possibly decreases, the number of 0-cells this will imply that the number of 0-cells in T_{i+1} is bounded. There are going to be three classes of 0-cells $v \in Z_i'$ that we have to bound: **Nonregular:** 0-cells such that $(Z_i', v) \to (X, x)$ is not a regular section, **Negative:** 0-cells such that $(Z_i', v) \to (X, x)$ is a regular section and $\kappa(v) < 0$, and **Zero:** 0-cells such that $(Z_i', v) \to (X, x)$ is a regular section and $\kappa(v) = 0$.

Nonregular: Suppose $(Z_i', v) \to (X, x)$ is not a regular section. This means that $v \in \delta Z_i'$ or $v \in \xi Z_i'$. But $|\delta Z_i'| + |\xi Z_i'| \leq |\delta T_1| + |\xi T_1|$ by Lemma 2.2.2 and because folding reduces the number of vertices. Therefore there is an upper bound for the number of 0-cells in Z_i' that yield non-regular sections.

Negative: Since $\pi_1(B) \twoheadrightarrow \pi_1(Z_i')$ we have that $\beta_1(B) \geq \beta_1(Z_i')$ where β_i is the i-th betti number. The Euler characteristic is given by

$$\chi(Z_i') = 1 - \beta_1(Z_i') + \beta_2(Z_i') \geq -\beta(Z_i') \geq -\beta(B).$$

The Combinatorial Gauss-Bonnet Theorem yields the following inequality:

$$-2\pi\beta_1(B) \leq \pi_1\chi(Z_i') - \sum k(f) + \sum k(v) \leq \sum k(v)$$

with the last inequality holding because $k(f) \leq 0$ for all faces, f, of Z_i'. X is compact so there exists an $M \in R$ such that $k(v) < M < 0$ for all $v \in Z_i'$ with negative curvature. Therefore the number of vertices $v \in Z_i'$ with negative curvature is bounded above for all i.

Zero. Suppose $v \in Z_i'$ such that $\kappa(v) = 0$ and $(Z_i', v) \to (X, x)$ is a regular section. Then v must be in the image of B, a 0-cell in a disc diagram added in the capping procedure onto a circle Q_j, or be on a circle Q_j.

The number of 0-cells in the image of B are bounded above since B is finite.

Each vertex in Q_j is at most one edge away from a 0-cell from the set of 0-cell already bounded in one of the above cases since it lies on the boundary of a set S_j.

Since X is finite and $Z_i' \to X$ is an immersion the valence of each 0-cell is bounded above by a real number n_1. This bounds the number of 0-cells in Q_j. Let n_2 be this upper bound.

By construction the set $\{\overline{Q}_1, ..., \overline{Q}_m\}$ embeds into $\overline{Z}_i$. In particular, it embeds into the set of 0-cells obtained in the subdivisions of Z_i which we will call V_{sub}. Since X is finite and $Z_i' \to X$ each 2-cell in Z_i' has at most n_3 sides which means that each 2-cell contains at most $6n_3 + 1$ after the by Lemma 1.1.9. We observe that each subdivided 2-cell is adjacent to at least one 0-cell in $\{Q_1, ..., Q_m\}$ and so

$$|V_{sub}| \leq n_1 \cdot n_2 \cdot (6n_3 + 1) = n_4.$$

Therefore, the number of 0-cells in the set $\{\overline{Q}_1, ..., \overline{Q}_m\}$ is bounded above by n_4. The length of Q_j can at most be n_3 times the length of $\overline{Q}_j$ and so the number of 0-cells in the set $\{Q_{i_1}, ..., Q_m\}$ is bounded above by $n_3 \cdot n_4$.

X is compact and so it has an isoperimetric function. Since the length of all of Q_j are bounded the number of 2-cells in all of the D_j is bounded. Consequently the number of 0-cells in all of the D_j are bounded since each 2-cell has at most n_3 0-cells.

Terminating Sequence. We have constructed the sequence

$$T_1, T_2, T_3, \ldots$$

each of which immerses into X and the total number of vertices in T_i is bounded above for all i. This means there are finitely many distinct T_i that immerse into X by Lemma 1.1.2. The spaces are distinct since step one in our procedure guarantees that the the kernel of the maps $T_i \to X$ are a strictly increasing sequence of sets. This implies that the sequence terminates at some T_n with the property $T_n \to X$ is π_1-injective which completes the proof.

□

We notice that Example 1.2.2 showed that the direct product of n tori had negative sectional curvature for $n > 2$ meaning that it has negative sectional curvature relative to planar sections. Theorem 4.1.2 allows us to concluded that

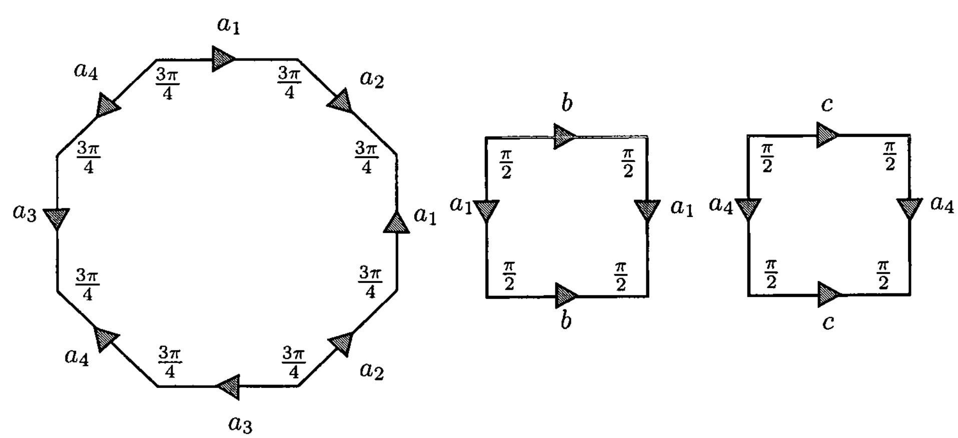
$$\pi_1(T_n) = \langle a_1, b_1, a_2, b_2, \dots a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_n b_n a_n^{-1} b_n^{-1} \rangle$$

is coherent. We will construct a more interesting example that has negative sectional curvature relative to planar sections but does not have negative sectional curvature.

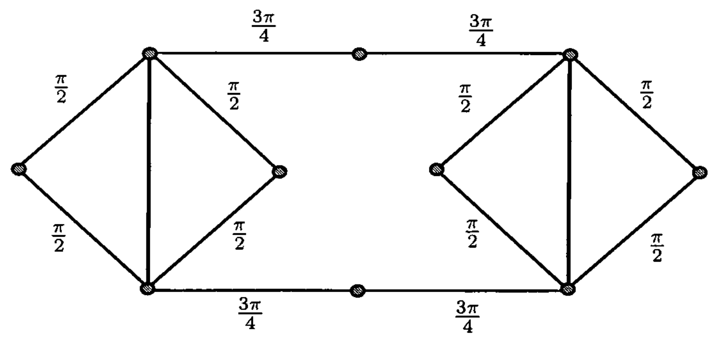
Example 4.1.3. Let G be the group given by the presentation

$$G = \left\langle \begin{array}{c} a_1, a_2, a_3, a_4, \\ b, c \end{array} \middle| \begin{array}{c} a_1 a_2 a_1^{-1} a_2^{-1} a_3 a_4 a_3^{-1} a_4^{-1}, \\ b a_1 b^{-1} a_1^{-1},\ c a_4 c^{-1} a_4^{-1} \end{array} \right\rangle.$$

Let X be the standard 2-complex for G. We give the 2-cells the following angle assignment and observe that they have the following attaching maps:



The link of the one 0-cell $x \in X$ is the following graph:



By exhaustion it is evident that any subgraph of link(x) that corresponds to a regular section has negative curvature unless it is one of the two 4-cycles containing all $\pi/2$ edges obtained from the relators $ba_1b^{-1}a_1{-}1$ and $ca_4c^{-1}a_4^{-1}$ in which case it has zero curvature. This shows that X has negative sectional curvature relative to planar sections.

For some examples of spaces that have negative sectional curvature relative to planar sections it is possible to changes the angle assignment slightly by redistributing the angles in 2-cells so that the resulting angle assignment yields negative sectional curvature, or at least only negative regular sections. This is not possible to do with Example 4.1.3, however, since the two cycles with zero curvature in link(x) that form the zero curvature subsections each inherit their angles from one 2-cell. Redistributing the angles in these 2-cells does not change the angle sum in these cycles and so X does not have negative sectional curvature.

The method of embedding links from Theorem 1.3.2 can be used to identify another class of examples.

Lemma 4.1.4. Let Γ be a regular angled graph such that Γ embeds in S^2 and all cycles have angle sum $> 2\pi$ except one cycle with angle sum $= 2\pi$. Then Γ has negative sectional curvature with isolated flats.

Proof. Let γ be a regular subgraph of Γ. Then γ is the 1-skeleton of a cell structure of S^2. Let V, E, and F denote the number of 0-cells, 1-cells and 2-cells in this cell structure respectively. Each 2-cell corresponds to a cycle in γ and each edge appears in exactly two 2-cells. Each edge e in γ is labeled by the angle $\measuredangle(e)$. If γ is the cycle with angle sum $= 2\pi$ then $\kappa(\gamma) = 0$. Otherwise we have that

$$2\sum_{e\in\gamma}\measuredangle(e) = \sum_{f}\sum_{e\in f}\measuredangle(e) > \sum_{f} 2\pi = 2\pi \cdot F$$

where f is a 2-cell in the cell structure. Therefore,

$$\begin{aligned}\kappa(\gamma) &= 2\pi - \pi(V-E) - \sum_{e}\measuredangle(e)\\ &< 2\pi - \pi(V-E) - \pi F\\ &= 2\pi - \pi(V-E+F)\\ &= 2\pi - \pi\cdot\chi(S^2)\\ &= 0\end{aligned}$$

as required. □

It is possible to strengthen Lemma 4.1.4. We can assume that the graph has as many cycles with angle sum $= 2\pi$ as we wish as long as for each subgraph γ the induced cell structure on S^2 has at least one 2-cell whose boundary has angle sum $> 2\pi$.

The following lemma is also useful for constructing examples of angled 2-complexes with negative sectional curvature relative to planar sections.

Lemma 4.1.5 (Adding a $\geq \pi$ edge.). Suppose Γ is an finite angled graph with negative sectional curvature and positive angles. Let Γ' be a graph obtained from Γ by adding an edge e with an assigned angle $\geq \pi$ and let α and β be initial and terminal vertices of e. Then Γ' has negative sectional curvature relative to planar sections if all paths in Γ from α to β have angle sum $\geq \pi$ (and has negative sectional curvature if the inequality is sharp or if $\alpha = \beta$).

Proof. We need to show that all regular subgraphs of Γ' have negative curvature or zero curvature in the case of a cycle. Let γ' be a subgraph of Γ'. If $e \notin \gamma'$ then we are done, otherwise let $\gamma = \gamma' - e$. If $\gamma \subset \Gamma$ is regular then $\kappa(\gamma) < 0$ and so the effect of adding e on the curvature is

$$\kappa(\gamma') = \kappa(\gamma) + \pi - \measuredangle(e) < \kappa(\gamma) < 0.$$

This is the case if $\alpha = \beta$ unless γ is a single point in which case the result still holds.

If γ is not regular then it must contain spurs to which e is attached, or it is not connected and becomes connected when e is added, or both.

Suppose γ only contains spurs but is connected. A spur contributes negatively to the curvature of the graph, equal to the angle assigned to the edge since a spur is one edge and one vertex. If we remove all spurs from γ we are left with a graph that is regular, in which case it has negative curvature, or it is a single vertex. If it is regular graph, then with the negatively contributing spurs we have that $\kappa(\gamma) < 0$ and so $\kappa(\gamma') < 0$ after adding e, as we saw above. If it was a single vertex then γ was a path with angle sum $\geq \pi$ ($> \pi$) and so γ' is a cycle with curvature $\kappa(\gamma') \leq 0$ (< 0).

Suppose γ is disconnected but does not contain spurs. Then γ must contain exactly two connected components γ_1 and γ_2 both of which are regular. This means that $\kappa(\gamma_1) < 0$ and $\kappa(\gamma_2) < 0$ implying that

$$\kappa(\gamma') = -\pi + \kappa(\gamma_1) + \kappa(\gamma_1) - \measuredangle(e) < 0.$$

The case where γ contains spurs and is disconnected is handled by combining the above two cases ensuring that $\kappa(\gamma') < 0$.

$\square$

Notice that the finiteness of X allows us to bound the number of 0-cells within a bounded distance from nonregular 0-cells and positively curved 0-cells due to the upper bound on number of these two types of 0-cells. Suppose X is a compact angled 2-complex with nonpositive sectional curvature and let $T \to X$ be an immersion. If every 0-cell in T is no more that n 1-cells from a 0-cell that is not regular or has negative curvature then we know the total number of 0-cells in T is bounded. In fact, if this is the case for every immersion then the global bound on the number of nonregular 0-cells and negatively curved 0-cells in the sequence T_i bounds the total number of 0-cells in T_i ensuring that the sequence must terminate. This leads us to the following theorem:

Theorem 4.1.6. Let X be a compact angled 2-complex with nonpositive sectional curvature. Suppose that for every immersion $T \to X$ every 0-cell is at most n 1-cells away from a nonregular or negatively curved 0-cell. Then $\pi_1(X)$ is coherent.

4.2 Transition Graphs

Let us examine the procedure described in the proof of Theorem 4.1.2 more closely. The key to the proof was the chopping or capping of the subspaces $\overline{K}_j$ which relied on the circles Q_j being "manageable" or more precisely the fundamental group of each circle was trivial or $\mathbb{Z}$. More generally we will have the notion of a transition graph and if this graph is again "manageable" we will have coherence.

Let X be a compact angled 2-complex with nonpositive sectional curvature. Let G be a finitely generated subgroup of $\pi_1(X)$. Let $B \to X$ be a finite immersed graph such that $\pi_1(B) \twoheadrightarrow G$.

Much of the material presented here is almost identical to that of the proof of Theorem 4.1.2 with the main exception that the subdivision used to ensure components of the flat part of Z_i are surfaces is not used which simplifies the procedure slightly.

Let us begin to generate the sequence of spaces T_i as before. Let $T_1 = B$. Assume we have constructed the space T_i and perform step 1.

Step 1. Adding a disc: Let $P_i \to T_i$ be an essential path whose projection $P_i \to X$ is null-homotopic. Let $D_i \to X$ be a reduced disc diagram whose boundary path is $P_i \to X$. Let $Y_i = T_i \cup_{P_i} D_i$ and $Y_i \to X$ be the induced map. Fold $Y_i \to X$ to obtain an immersion $Z_i \to X$.

Analysis of Flat Subcomplexes. Let V be the set of all 0-cells in Z_i with zero curvature, let E be the set of all 1-cells in Z_i with both adjacent 0-cells in V, and let C be the set of 2-cells in Z_i whose adjacent 1-cells lie in E. Let $F = V \cup E \cup C$ be the "flat part" of Z_i.

Let N be the smallest open cellular neighbourhood of the image of B in Z_i. If $F - N = \varnothing$ then we are able to bound the number of 0 cells in T_i we will define $T_{i+1} = Z_i$. Otherwise let $S_1, ..., S_k$ be the components of $F - N$.

Let $j \in \{1, ..., k\}$. As stated before, S_j is a singular surface but may not be a surface. This means that the boundary components are not circles but instead graphs made up of boundary 0-cells and 1-cells.

Let $U_j = Z_i - int(S_j)$. A homological argument will show that distinct components of ∂S_j lie in distinct components of U_j. Indeed, we will show that the induced inclusion homomorphism $H_0(\partial S_j) \to H_0(U_j)$ is injective. Note that $U_j \cup S_j = Z_i$ and $U_j \cap S_j = \partial S_j$. Consider the Mayer-Vietoris sequence for complexes:

$$\ldots \to \mathsf{H}_1(U_j) \oplus \mathsf{H}_1(S_j) \xrightarrow{g} \mathsf{H}_1(Z_i) \xrightarrow{h} \mathsf{H}_0(\partial S_j) \xrightarrow{\ell} \mathsf{H}_0(U_j) \oplus \mathsf{H}_0(S_j) \to \mathsf{H}_0(Z_i) \to 0$$

Since the image of $B \to Z_i$ factors as $B \to U_j \to Z_i$ we see that $\pi_1(U_j) \twoheadrightarrow \pi_1(Z_i)$ and since Z_i is connected $\mathsf{H}_1(U_j) \twoheadrightarrow \mathsf{H}_1(Z_i)$. This implies that g is surjective and so $im(g) = \mathsf{H}_1(Z_i) = ker(h)$ and so $im(h) = 0 = ker(\ell)$ by exactness. Therefore $H_0(\partial S_j) \to H_0(U_j)$ is injective as claimed.

We have shown that each graph in ∂S_j lies in a distinct component of U_j. Since N and S_j are disjoint, the image of B must lie in one component of U_j. In fact, the image of B must lie in one component L of $Z_i - \cup S_j$. By relabeling, let $\{S_1, ..., S_m\}$ be the relabeled subset of $\{S_1, ..., S_k\}$ whose elements intersect with L. For $j = 1, ..., m$ let Γ_j be the graph in $S_j \cap L$. We are now ready for the following definition.

Definition 4.2.1 (Transition Graph). We call the graph Γ_j obtained by the above process a transition graph.

Step 2. Adjustment: Let K_j be the component of $(Z_i - \Gamma_j)$ that does not contain the image of B union Q_j. In order to cap or chop K_j with a space R_j as before we must insure that the chop or cap preserves or at improves the fundamental group of Z_i relative to G. In other words we need the the π_1-image of R_j to be equal to the π_1-image of K_j and

$$\ker(\pi_1(R_j) \to G) \subset \ker(\pi_1(K_j) \to G).$$

In most cases it may be easier for us to examine $\pi_1(\Gamma_j)$ instead of $\pi(K_j)$. The following allows us to so.

Same π_1-image. The image of $\pi_1(\Gamma_j) \to \pi_1(X)$ is the same as the image of $\pi(K_j) \to \pi_1(X)$.

Indeed, let x be a base point on Γ and P_1 a path in K_j starting at a and ending at b. Since the maps $B \to L \to Z_i$ are π_1-surjective there is a path P_2 in L that is path homotopic to P_1 in Z_i. Therefore the path P_1P_2 is null homotopic and so there is a reduced disc diagram D such that $P_1P_2 \to Z_i$ factors as $P_1P_2 \to D \to Z_i$.

Let a' be the preimage of a in D and b' be the preimage of b in D. Then a' and b' are in the same component of the preimage of Γ in D. If this were not the case there would be a path (possibly passing through 2-cells) from the preimage of P_1 to P_2 that does not intersect the preimage of Γ. The image of this path in Z_i would be a path intersecting both L and K_j but not intersecting Γ_j, a contradiction.

This situation ensures that there is a path P_3' in D that starts at the initial point a' and ends at terminal point b' and contains only 1-cells mapped to Γ_j. Let P_3 be the image of $P_3' \to Z_i$. Then P_3 is path homotopic to P_1 and is a path in Γ_j.

Chopping/Capping Let us suppose for the moment that the image of $\Gamma_j \to X$ is finitely presented for $j = 1, ..., k$. Then the preceding discussion implies that K_j is finitely presented and we may proceed with the capping and chopping procedure as before because each disconnecting graph Γ_j requires only finitely many disc diagrams to replace K_j.

For $j = 1, ..., k$ glue to Γ_j the finitely many reduced disc diagrams, $D_1, ..., D_n$, required to make the fundamental group of the resulting space $\pi_1(\Gamma_j \bigcup_{m=1}^{n} D_m)$ isomorphic to the image of $\pi_1(\Gamma_j) \to \pi_1(X)$. We will call the resulting space R_j.

Let

$$Z_{i_2} = L \bigcup_{\Gamma_1} R_1 \ldots \bigcup_{\Gamma_k} R_k.$$

Increasing Kernel. The map $B \to X$ factors as $B \to Z_i \to X$ and $B \to Z_{i_2} \to X$. The adjustments made to Z_i are only beneficial if

$$\ker(B \to Z_i) \subset \ker(B \to Z_{i_2}).$$

We will show that this is indeed the case.

Let P be a path in B such that $P \to T_i$ is nullhomotopic. There exists a reduced disc diagram, D, for $P \to T_i$. There is a set of maximal cycles, $P_{1_j}, ..., P_{r_j}$, each in a distinct component of the preimage of Γ_j, that bound all cells mapped to Γ_j. For

$q_j = 1_j, ..., r_j$ let J_{q_j} be the image of P_{q_j} in Γ_j. Then the image of $P_{q_j} \to J_{q_j} \to X$ is nullhomotopic which implies that J_{q_j} is nullhomotopic in Z_{i_2} by construction and so there is a disc diagram D_{q_j} for $J_{q_j} \to Z_{i_2}$. If we replace the interior of P_{q_j} in D with D_{q_j} for $j = 1, ...m$ and $q_j = 1_j, ..., r_j$ we get a disc diagram for $P \to Z_{i_2}$ proving that the image of P in Z_{i_2} is nullhomotopic and that $\ker(B \to Z_i) \subset \ker(B \to Z_{i+1})$.

B **π_1-surjects.** To bound the number of 0-cells it will be important that $\pi_1(B) \twoheadrightarrow \pi_1(Z_{i_2})$. Let P_1 be a (directed) path in Z_{i_2}. In Z_{i_2} the path P_1 is homotopic to a path P_2 which lies in the subspace L since $\pi_1(\Gamma_j) \twoheadrightarrow \pi_1(\Gamma_j \cup R_j)$. This path P_2 can be viewed as a path in Z_i which contains L as a subspace. We know already that $\pi_1(B) \twoheadrightarrow \pi_1(Z_i)$ so there is a path P_3 in the image of $B \to Z_i$ that is homotopic to P_2 in Z_i. Let D be the disc diagram for the null homotopic path $P_2P_3^{-1}$. Let $j \in \{1, ..., k\}$ and look at the preimage of Γ_j in D. As above, here is a set of maximal cycles, $J_{1_j}, ..., J_{r_j}$, each in a distinct component of the preimage of Γ_j, that bound all cells mapped to Γ_j. Each of these cycles are trivial in Z_i since they are inside of a disc diagram and so their image $J_{q_j} \to X$ are trivial implying that their image in Z_{i_2} is trivial. There exist a disc diagram D_{q_j} for $J_{q_j} \to Z_{i_2}$. If we replace the interior of J_{q_j} in D with D_{q_j} for $j = 1, ...m$ and $q_j = 1_j, ..., r_j$ we get a disc diagram for $P_2P_3^{-1} \to Z_{i_2}$. This implies that P_3 is homotopic to P_2 and is therefore homotopic to P_1 in Z_{i_2} proving that $\pi_1(B) \twoheadrightarrow \pi_1(Z_{i_2})$.

We fold $Z_{i_2} \to X$ to obtain T_{i+1}.

Upper Bound on 0-cells. As before, we would like to have an upper bound on the number of 0-cells in T_i for and $i \in \mathbb{Z}$ and we will show this by showing there is an upperbound on Z_{i_2}. There are going to be three classes of 0-cells $v \in Z_{i_2}$ that we have to bound: **Nonregular:** 0-cells such that $(Z_{i_2}, v) \to (X, x)$ is not a regular section, **Negative:** 0-cells such that $(Z_{i_2}, v) \to (X, x)$ is a regular section and $\kappa(v) < 0$, and **Zero:** 0-cells such that $(Z_{i_2}, v) \to (X, x)$ is a regular section and $\kappa(v) = 0$. The nonregular and negative cases are exactly the same as above.

Zero. Suppose $v \in Z_{i_2}$ such that $\kappa(v) = 0$ and $(Z_{i_2}, v) \to (X, x)$ is a regular section. Then v must be in the image of B, a 0-cell in a disc diagram added in the capping procedure onto a circle Γ_{i_j} or be on a circle Γ_{i_j}.

The number of 0-cells in the image of B are bounded above since B is finite.

Each vertex in Γ_j is at most one edge away from a 0-cell from the set of 0-cell already bounded in one of the above cases since it lies on the boundary of a set S_j. Since X is finite and $T_i \to X$ is an immersion the valence of each 0-cell is bounded above by a real number, n_1. This bounds the number of 0-cells in $\bigcup_{j=1}^{k} \Gamma_j$. Let n_2 be this upper bound.

This means that each transition graph Γ_j can have at most n_2 0-cells each of which has bounded valence so its image in X has a bounded number of generators and is by assumption finitely presented. The finiteness of X ensures that there is an upper bound, n_3, on the number of disc diagrams (relators) required to attach to Γ_j and an upper bound n_4 on the maximum number of 0-cells in each disc diagram. There are also finitely many graphs that can be made up of n_2 0-cells with maximum valence n_1 and we will call this number n_5. This means that the number of 0-cells added by attaching the $\bigcup_{j=1}^{k} R_j$ is at most $n_5 \cdot n_3 \cdot n_4$.

Terminating Sequence. We use the described method above to construct the sequence of spaces

$$T_1, T_2, T_3, \ldots$$

each of which immerses into X and the total number of vertices in T_i is bounded above for all i. This means there are finitely many distinct T_i that immerse into X by Lemma 1.1.2. The spaces are distinct since step one in our procedure guarantees that the the kernels of the maps $T_i \to X$ are a strictly increasing sequence of sets. This implies that the sequence terminates at some T_n with the property $T_n \to X$ is π_1-injective. This implies the following result:

Theorem 4.2.2. Let X be a compact angled 2-complex with nonpositive sectional curvature. Suppose that the π_1-image of transitions graphs in the sequence $Z_1, Z_2, Z_3, \ldots$ are finitely presented. Then $\pi_1(X)$ is coherent.

4.3 Flattenable Graphs

The definition of a transition graph is not very convenient since one would have to construct the sequence of spaces T_i and check to see if each transition graph had finitely presented image.

Let ε be an ϵ-neighbourhood of Γ_j in L. Essentially the same proof we used to show that the graph Γ_j has the same π_1-image as K_j in X shows that the π_1-image of ε in X is that same as the π_1-image as Γ_j in X.

This leads us to the following definition:

Definition 4.3.1 (Flattenable Graph). Let X be an angled 2-complex and let $\psi : \Gamma \to X$ be an 1-complex. We say that Γ is *flattenable* if there exists a an immersed 2-complex $\phi : E \to X$ such that

1. $\Gamma \to X$ factors as $\Gamma \to E \to X$,
2. $(E, \phi(a))$ is regular for each $a \in \Gamma^0$,
3. $\kappa_E(\phi(a)) = 0$ for each $a \in \Gamma^0$ and
4. There is an ϵ-neighbourhood of $\psi(\Gamma)$ lying in one of the components of $E - \psi(\Gamma)$ that has the same π_1-image as Γ in X.

The discussion above showed us that transition graphs satisfy the definition of a flattenable graph and so the following corollary to Theorem 4.2.2 is immediate.

Corollary 4.3.2. Let X be a compact angled 2-complex with nonpositive sectional curvature. Suppose that the π_1-image of all flattenable graphs in X are finitely presented. Then $\pi_1(X)$ is coherent.

We present a corollary which provides a situations which ensures that the π_1-image of flattenable graphs are finitely presented. First we will need the following Lemma.

Lemma 4.3.3. Let X be an angled 2-complex with nonpositive sectional curvature. Let $F \to X$ be such that for each 0-cell $y \in F$, $(F, y) \to (X, x)$ is a regular section and $\kappa_F(y) = 0$. Then $\pi_1(F) \to \pi_1(X)$ is injective.

Proof. Suppose $\pi_1(F) \to \pi_1(X)$ is not injective. Let $P \subset F$ be an essential path whose image in X is trivial. Let $D \to F$ be a reduced disc diagram for P. Since $\chi(D) > 0$ and since 2-cells in D with inherited angles from X have curvature ≤ 0, Theorem 1.1.6 gives us that there is a 0-cell $v \in D$ such that $\kappa(v) > 0$. Each 2-cell in the interior of D corresponds to a regular section mapped to X so v must lie on the boundary of v that is mapped to P. Attach v and it's adjacent boundary 1-cells to F via the map $D \to F$ and call the resulting space F'. Let v' denote the image of v in $D \to F'$ and let a be the image of v in $D \to F$. Then $(F', v') \to (X, x)$ is a regular section, where x is the image of v' in X. Let V, V', and A denote the number of vertices in $\text{link}(v)$, $\text{link}(v')$ and $\text{link}(a)$ respectively. Then $V' = V + A - 2$. The curvature of v' is given by

$$
\begin{aligned}
\kappa(v') &= (2-V')\pi + \sum_{c\in \text{Corners}(v')} \text{def}(c) \\
&= (2-(V+A-2))\pi + \left(\sum_{c\in \text{Corners}(v)\cup \text{Corners}(a)} \text{def}(c) \right) \\
&= \left((2-V)\pi + \sum_{c\in \text{Corners}(v)} \text{def}(c) \right) + \left((2-A)\pi + \sum_{c\in \text{Corners}(a)} \text{def}(c) \right) \\
&= \kappa(v) + \kappa(a) \\
&= \kappa(v) + 0 \\
&> 0.
\end{aligned}
$$

But this is a contradiction since X has nonpositive sectional curvature.

□

At first glance Lemma 4.3.3 seems to prove that nonpositive sectional curvature in X implies coherence of $\pi_1(X)$ since for each $y \in \Gamma_j$ of a transition graph $\kappa_{Z_i}(y) = 0$ and so it would seem like $\Gamma_j \to X$ is π_1-injective which would mean that we may chop K_j and no capping is required. This is not necessarily the case since in order to apply Lemma 4.3.3 we would require all 0-cells in Z_i to have zero curvature or each 0-cell in Γ_j to have zero curvature in some subspace of Z_i containing only zero curvature 0-cells. Even K_j is not necessarily a good candidate for such a subspace since there is no guarantee boundary 0-cells have zero curvature when removed from Z_i.

Corollary 4.3.4. Let X be a compact angled 2-complex with nonpositive sectional curvature. Suppose there is a family of immersions $F_i \to X$, $i \in I$, satisfying the following properties:

1. For each 0-cell $y \in F_i$, $(F_i, y) \to (X, x)$ is a regular section and $\kappa(y) = 0$.

2. For each regular section $(S, s) \to X$ the zero cell s factors locally (zero cell and adjacent 1-cells) through F_i to X.

3. For each regular section $(S, s) \to X$ each 2-cell $e \in S$ such that the adjacent 0-cells have zero curvature factor through a unique F_i to X.

4. For each i, $\pi_1(F_i)$ is coherent.

Then $\pi_1(X)$ is coherent.

Proof. Let Γ be a flattenable graph. The second property ensures that each 0-cell from Γ factors through an F_i on it's way to X while the third property ensures that all of the 0-cells and 1-cells from Γ factor through the same F_i. This means that the π-image of Γ is a subgroup of the π_1-image of F_i in X. Since all 0-cells in F_i have zero curvature Lemma 4.3.3 tells us that the π_1-image of F_i in X is coherent implying that the π_1-image of Γ in X is finitely presented. The result now follows from Corollary 4.2.2. □

Surfaces and spaces with trivial fundamental groups would work as members F_i of this family since both have coherent fundamental group.

4.4 Negative Sectional Curvature Relative to π-gons

In the proof of Theorem 4.1.2 the transition graphs turned out to have finitely presented π_1-image in X since they were circles. If the complements of flattenable graphs were circles, or at least ϵ-neighbourhoods of flattenable graphs in the complement were circles, we would know that their π_1-images in X were finitely presented as well.

We use the idea of locally analyzing compliments of flattenable graphs to prove the following result.

Theorem 4.4.1. Let X be a compact angled 2-complex with negative sectional curvature relative to π-gons and planar sections. Then $\pi_1(X)$ is coherent.

Proof. Let G be a finitely generated subgroup of $\pi_1(X)$. Let $B \to X$ be a finite immersed graph such that $\pi_1(B) \twoheadrightarrow G$.

Let $T_1 = B$. We will again construction a sequence

$$T_1, T_2, T_3, \ldots$$

of immersed 2-complexes $T_i \to X$ as in Theorem 4.2.2. We perform **Step 1: Adding a Disc** and construct the space Z_i. Define L, K_j and Γ_j for $j = 1, ..., k$ as in Theorem 4.2.2.

We would like to ensure that Γ_j does not contain any spurs which are inconsequential since spurs do not contribute to the π_1-image of Γ_j anyway. We make the following adjustment to the definition of Γ_j.

Let L_j be the union of all closed cells intersecting $Z_i - K_j$ and let ξ_j be the component of the boundary graph of L_j that is a subset of Γ_j. This ensures that any spur in ξ_j comes from an isolated 1-cell in Z_i but since all 0-cells in Γ_j yield regular sections this does not occur.

We recall from the proof of Theorem 4.2.2 that it is enough to show that the π_1-image of ξ_j in X is finitely presented. This is because we may perform the adjustment process of Theorem 4.2.2 by chopping along ξ_j, $j = 1, ..., k$ and we will require only finitely many disc diagrams for capping ξ_j. The **Upper Bound on 0-cells** and **Terminating Sequence** parts of the proof still hold. We will now show that the π_1-image of ξ_j in X is finitely presented.

We examine the link of each 0-cell $s \in \xi_j$ (as a 0-cell in Z_i). Since $\kappa(s) = 0$

we know that link(s) is either a circle or a π-gon. Suppose link(s) is a π-gon. The fact that ξ_j is spurless ensures that it contains the two edges associated with the two vertices in the link. Removing these two vertices disconnects the link into single edges which means that locally at s the complements of ξ_j have the same homotopy type as lines. This implies that for a small enough ϵ all components of the ϵ-neighbourhood of ξ_j in the complement of ξ_j locally have the same homotopy type as a line implying that all components of the ϵ-neighbourhood of ξ_j in the complement of ξ_j have the same homotopy type as a line or a circle. We know that one of the components of the ϵ-neighbourhood of ξ_j in the complement of ξ_j has the same π_1-image in X as ξ_j and since they all have same homotopy type as a line or a circle we know that the π_1-image of it is finitely presented (since we know that X is torsion free).

If link(s) is a circle then removing the vertices in the link associated with the edges of ξ_j disconnects link(s) into line segments implying, as above, that the π_1-image of ξ_j in X is finitely presented.

□

We make use of Theorem 4.4.1 by presenting the following class of examples.

Example 4.4.2. Let G= $\langle a_1, ..., a_n | W_1, ..., W_n \rangle$ and let X be the standard 2 complex for G. Suppose X is an angled 2-complex with an angle assignment that has π assigned to the corner between each repeated edge in the words $W_1, ..., W_n$. Suppose that X has nonpositive sectional curvature. We define a new group G' by replacing each instance of a_i^2 with a_i^n for any $n \geq 2$ we choose (n need not be the same for each instance of a_i^2) in the words $W_1, ..., W_n$. We will show that G' is coherent.

Let X' be the standard 2-complex for G'. We give X' the same angle assignment as X and assign an angle of π to the newly created corners. This process has the effect of increasing multiple edges in the link of vertices creating π-gons as subgraphs. We give X' the same angle assignment as X and assign an angle of π to the newly created corners which means that the newly created edges in the links have angle π assigned to them. We see that regular sections for X' have negative sectional curvature except sections whith π-gon links which have nonpositive sectional curvature. Let $W_1', ..., W_n'$ denote the new two cells in X'. Increasing a_i^2 to a_i^n in W_j adds $n-2$ edges and adds and $(n-2)\pi$ to the angle sum in W_j and so these changes do not affect the curvature of the two cell, that is $\kappa(W_i') = \kappa(W_i)$. Therefore X' has negative sectional curvature relative to π-gons thus G' is coherent by Theorem 4.4.1.

Conclusion

Our method for proving Theorem 4.4.1 does not allow us to conclude the compact core condition for a compact angled 2-complex X with relative nonpositive sectional curvature. Indeed, Wise showed in [11] that if the angle assignment contains negative angles the conclusion may not hold.

If we attempted to prove the compact core condition using this method we would begin with the union of the compact subcomplex C of $\widehat{X}$ and a compact subcomplex that generates the finitely presented subgroup G. We attach disc diagrams for each essential path with nullhomotopic image in X to this space and form the sequence T_i. At each stage in the proof where we fold to obtain immersions we would instead take tower lifts so that theses immersions were covering maps. The method fails at the adjustment step in our proof: when we chop or cap, the resulting space need no longer be a cover. If instead of folding we then take a tower lift we no longer have a bound on the number of zero curvature regular 0-cells which we use to conclude that the sequence T_i terminates. An adjustment in this method however may yield fruitful results toward proving the conjecture put forth by Wise in [11]:

Conjecture 4.4.3. Let X be a compact nonnegatively angled 2-complex with nonpositive sectional curvature. Let $\widehat{X}$ be a based cover with $\pi_1(\widehat{X})$ finitely generated. Then every compact subcomplex of $\widehat{X}$ is contained in a compact core of $\widehat{X}$.

We have shown that $\pi_1(X)$ is coherent when X has negative sectional curvature relative to π-gons or planar sectionals. I suspect that this result can also be improved to allow bigons to have 0-curvature and that this can be shown by analyzing compliments of 0-cells locally as in Theorem 4.4.1, however this requires the examination of many cases. If it is possible to classify the links of 0-cells which occur in transition graphs for 2-complexes with nonpositive sectional curvature we may be able to handle these 0-cells in the same manner. This leads us to the following conjecture:

Conjecture 4.4.4. Let X be a compact angled 2-complex with nonpositive sectional curvature. Then $\pi_1(X)$ is coherent.

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