SCATTERING MATRIX CALCULATIONS IN NUCLEON-NUCLEON

AND

PION-NUCLEON SCATTERING UP TO 310 MEV.

by

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INTRODUCTION

The Nucleon-Nucleon Interaction

One of the central problems in nuclear physics is the determination of the forces which bind nuclei together. A few facts were obtained in the beginning from the vast collection of data on the binding energies of nuclei. In order to account for saturation, Heisenberg (1) and Majorana (2) introduced the idea of exchange forces, somewhat akin to the exchange forces which occur in molecular binding. These forces give rise to spin and parity dependence in the interaction. From the large ratio of the binding energy of the alpha particle to that of the deuteron, Wigner (3) deduced that the forces were short ranged and very strong within that range. In 1935 Yukawa (4) developed a meson theory of the force, along the same lines as quantum electromagnetic field theory. The force~~s~~ between charged particles is assumed to be due to an exchange of massless photons. In order to account for the short range of the nuclear force, Yukawa postulated the existence of a particle of mass around 300 electron masses. The discovery of the pion twelve years later was a triumph of this idea. In the years following the proposal of the pi meson, field theoretical work on the nature of the interaction continued. During this period, progress in understanding the interaction also was being made from a completely different approach. The so called phenomenological approach.

not a sentence

In order to explain the discrepancy between the observed low energy total cross section, and that predicted from the binding energy of the deuteron and effective range theory, Wigner (5) suggested that the forces might be spin dependent, so that the deuteron data, which is associated with a triplet spin state, is not adequate to calculate the total low energy cross section which is made up of an incoherent mixture of triplet and singlet parts. Schwinger and Teller (6) showed that the coherent scattering of very slow neutrons from ortho-and para-hydrogen would furnish additional information, with which it would be possible to calculate the singlet and triplet scattering lengths. The concept of spin dependence was further developed by Wigner (7), who derived the most general form of potential consistent with certain invariance properties which we hope are satisfied by the physical world, at least for the strong interactions. Experimental determination of the magnetic moment of the deuteron indicated that it was not simply the sum of the intrinsic moments of the neutron and proton, thus suggesting that the deuteron ground state was not a pure S-state, and thus indicating the existence of a noncentral force within the two nucleon potential. This led authors to postulate the existence of a tensor force within the potential. The discovery of the quadrupole moment of the deuteron firmly established this idea. Treatment of the data up to 1940 was given by Rarita and Schwinger (8) in terms of phenomenological potentials including tensor forces. Between 1945 and

1957, a few attempts at fitting the existing data with phenomenological models were made. None of these attempts ~~were~~ was completely successful, but they were nevertheless important, in that they contained many interesting ideas and served as a taking off point for the later work of Gammel and Thaler and also Signell and Marshak. Jastrow (9) attempted to fit the data by using central and tensor potentials with hard cores. Feshbach and Lomon (10) tried to fit the data by representing the many-meson exchange region of the interaction by an energy independent boundary condition on the logarithmic derivative of the wave function at the boundary of this region. Also Case and Pais (11) suggested that the inclusion of a spin orbit force might allow a fit of the p-p polarization data. What was established during this period is the following. The introduction of an effective range expansion for the low energy phase shifts made it clear that the n-p data at low energies was determined by four quantities viz. the singlet and triplet effective ranges and scattering lengths. The singlet values were in agreement with the values obtained from p-p scattering, after the removal of coulomb effects, except for a small difference in the scattering lengths which Schwinger (12) showed could be explained by the inclusion of non-coulombic electro-magnetic effects. These facts lend support to the idea of charge independence of the forces which had previously been suggested by Breit.(13). Charge symmetry, i.e.,n-n equals p-p, had even previously been inferred from the ground state binding energy data of mirror nuclei. The lack of quantitative success in fitting the data was a direct result of the complicated nature of the interaction on the one hand, and the availability of mainly angular distribution data on the other. Wolfenstein (14) had already pointed out that angular distribution data alone were was not adequate to predict the scattering matrix. In 1957 the situation changed, there were two reasons for this. 1) Double and triple scattering experiments were being performed, and 2) high speed electronic computing became available. Stapp *et al.* (15) performed a complete set of experiments at 310 Mev. and made phase shift analyses of these results, yielding 5 acceptable solutions. It is easy to see how a few more experiments at this energy could make the solution unique. Since then a number of experimental groups have been active, performing polarization experiments at several energies. At Harvard and Harwell work is in progress at 150 Mev. At the University of Rochester, a group is working at 240 Mev., and at Liverpool between 320 - 380 Mev., while at Berkley work continues at 310 Mev. Meanwhile the field theoretic approach to the problem continues. A number of interesting attempts to calculate the force have been made, one of the most interesting of these by S. Gartenhaus, (16), the so called Gartenhaus potential, which is based on the non-relativistic Chew cut-off theory. Gartenhaus has shown that the low energy ($<$ 30 Mev.) data is adequately explained by this form of potential. Work by J. L. Gammel and

R. Thaler (17) has shown that this potential however, does not fit the data at high (300 Mev.) or even moderate energies. They conclude that this potential only adequately describes the force at distances > 0.5 fermis. For smaller distances, this meson theoretic approach is of dubious validity. Gammel and Thaler (18) again attempted to fit the data up to 310 Mev. This time they used Yukawa type potentials with hard cores, including both central and tensor forces. They obtained good fits to most of the data, but could not reproduce the p-p polarization at 170 and 310 Mev. They (19) repeated their work, adding a short ranged spin orbit force with a hard core. This type of force is consistent with the form of potential as predicted by Wigner, and its existence was shown to be implied by meson theory by Marshak as early as 1947. With this type of potential, they were able to obtain good agreement with the data up to 310 Mev. About the same time, Signell and Marshak (20) also obtained good agreement with the data up to 150 Mev. by using the Gartenhaus potential with the addition of a one paramater spin orbit force.

The early work of Feshbach and Lomon (10) yielded phase shifts which are different in nature from those of Gammel and Thaler, (19), and also Signell and Marshak (20). $^3P_0$ (?) They are characterized by a large negative $^3P_0$ phase shift. Solutions of this type exist both in the work of Stapp (15) (solution #6), and also in the analyses by Stabler (21) of data from Harvard and Harwell. In section II-5, we use this type of phase shift solution to calculate certain polarization parameters, which when compared with experiment should shed some light on the validity of this type of solution. It is to be hoped that with the work progressing as it is at present, we will have a unique picture of the scattering matrix within the next few years.

Pion Nucleon Scattering

The early work on the scattering of pions on protons was carried out by Fermi and Anderson (22) at Chicago in 1953. For energies less than 300 Mev. they assumed that the scattering was entirely due to S and P waves. The contributing angular momentum states were therefore $S_{1/2}$, $P_{1/2}$ and $P_{3/2}$. If we consider the meson Compton wave length as giving the approximate range of the interaction, then the usual centrifugal barrier arguments justify this approximation for energies less than 200 Mev., however at about 300 Mev. it looks as though D waves might be contributing. As it turns out, the approximation is satisfactory even at 300 Mev. With this assumption and that of charge independence, they found that they were able to fit the existing data. From the expression for the π^+ on p cross section, they were able to deduce several qualitative facts about the various phase shifts. The large increase in the cross section at about 140 Mev. led them to believe that there was a resonance with one angular momentum state mainly contributing. The large $\cos^2\theta$ term in the angular distribution implied that this was due to the $P_{3/2}$ state. They therefore expected a large $P_{3/2}$ phase shift δ_{33}. Such a resonance in the $P_{3/2}$ amplitude is predicted by the static nucleon theory of Chew and Low (23). There is also a large $\cos\theta$ term which is negative. This they deduced could only be due to an S and P wave interference. They therefore expected a negative S-wave phase shift.

Analysis of the data completely justified these qualitative inferences. Fermi obtained one large and five small phase shifts. The phase shift δ_{33} was large and the phase shifts $\delta_{31}, \delta_{3}, \delta_{11}, \delta_{13}$ and δ_{1} were found to be small. Here we have used the notation of Fermi, the first index refers to the isospin, the second to the angular momentum. The phase shifts with only one index are S wave phase shifts. The phase shift δ_{33} is found to have an η^3 dependence on the meson momentum at low energies, but increases more rapidly near resonance. Certain general ambiguities were soon recognized in the phase shift solutions.

These ambiguities are a direct result of the symmetry in the phase shift dependence of the scattering amplitudes. To begin with it was soon recognized that a complete reversal in sign of all of the phase shifts would also give the same fit to any data which involved only pion-nucleon forces. This is because in the angular distributions, the sines of the phase shifts appear bilinearly. Coulomb interference experiments performed by Orear (25) indicated however that δ_{33} was positive.

When Fermi obtained his set of phase shifts, Yang (24) observed that a different type of solution would also fit the data. This solution was characterized by the following $(\delta_{31} - \delta_{33}) \approx (\delta'_{3} - \delta'_{31})$, $\delta_1 \approx \delta'_1$ and $(\delta_{33} + \delta_{31}) \approx (\delta'_{33} + \delta'_{31})$. The Yang phase shift solution was found however to be inconsistent with the dispersion relations. Further, this type of solution implies a large δ'_{31}. This is less reasonable from the theoretical point of view, since there is no known mechanism which would give this. There is a second type of ambiguity which occurs, which is not as directly related to the structure of the scattering amplitudes, but is rather a result of the non linear nature of the quantity which is being fitted. The result of this is that several phase shift solutions of the Fermi type differing mainly in the small phase shifts are possible which fit the same data. There are a number of ways of eliminating these incorrect solutions, or at least of making them seem unlikely. They should be consistent with the dispersion relations, they should be continuous with respect to the energy, and most important of all, they should be capable of fitting the data from new experiments which help determine the scattering matrix. Very important in this respect are the experiments designed to measure P_+ and P_- , the recoil proton polarization in the scattering of π^+ on p, and π^- on p respectively. In section III 2 we consider three solutions of the Fermi type obtained by Chiu and Lomon (26), we calculate the values of P_+ and P_- for these three solutions, in order to determine which are invalid. The other two tests have already been carried out by Chiu and Lomon (26). The importance of obtaining a unique set of phase shifts derives from the fact that such a set is essential, if one wishes to construct a model of the interaction.

SUMMARY

In recent years, polarization experiments and the calculations associated with them have become important in helping physicists to pin down the scattering matrix for the nucleon-nucleon and pion-nucleon interactions. Thus a large part of this thesis is devoted to a review of the formalism of scattering matrix calculations, followed by a few applications to current problems. It also includes a required extension of variational techniques. Section I is concerned with the theory of polarization experiments. The theory is developed from first principles, using a density matrix formalism for a description of states. Expressions are derived for the cross Section I_0 for the scattering of an unpolarized beam, and for the polarization P which then describes the azimuthal assymmetry observed in a double scattering. Expressions are also obtained for the triple scattering parameters R, A, R', and A', and for the correlation parameters $C_{nn}, C_{KP}, C^{p}_{nn}, C^{p}_{kp}, C^{p}_{kn}$ and C^{p}_{np}. These results are obtained quite generally for the scattering of a particle of spin S from a target of spin T. Triple scattering experiments are described, and the depolarization and rotation paramaters D and R are interpreted. Correlation experiments are discussed with respect to the measurement of C_{nn} and C_{KP}, and these parameters are also interpreted physically. Section II specializes the preceding to the scattering of a particle of spin 1/2 from a target of spin 1/2. The most general M-

matrix which satisfies the conditions of invariance under rotations, space reflection, time reversal and which is also charge independent is derived. Expressions are obtained for the elements of the M-matrix in terms of spherical harmonics, and the elements of the S-matrix, which are directly expressible in terms of phase shifts.

The M-matrix is modified to account for the identity of the two particles and for coulomb effects, in the case of the scattering of protons on protons. Two phase shift representations of the S-matrix are given, and the suitability of the "barred" representation for the treatment of coulomb effects is discussed. A physical interpretation of each representation is also given.

Calculations are made of C_{nn}, C_{KP} and R at 140 Mev. using phase shift fits of Harvard and Harwell data. These solutions, designated of type #6 are consistent with the boundary condition model of Feshbach and Lomon (10), and are

$^3P_0$ (?) characterized by a large negative $^3P_0$ phase shift. Calculations are also made of C_{nn} and C_{KP} at 310 Mev. using Stapp's (15) solution #6. Several arguments indicating the incorrectness of solution #6 are discussed, partly on the basis of our calculation of C_{KP}, and are shown to be invalid. Comparison of R with recent (27) experimental results however indicate that solution #6 does not fit the data well, particularly at small angles. This result is in agreement with that of Stabler and Lomon (28) at Cornell.

In Section III, the formalism for scattering matrix calculations in pion nucleon scattering, considering only S and P waves, is developed from first principles. Expressions are obtained for the differential cross section for elastic scattering of positive and negative pions on protons, and also for the charge exchange cross section. The expression for the recoil proton polarization is derived for both of these experiments. These expressions are then used to calculate P_+ and P_- for solutions A, B, and C of Chiu and Lomon (26) at 307 Mev. Our results indicate that the P_+ experiment does not distinguish between these three solutions, and the P_- experiment favours solution C. These conclusions are shown to be in agreement with those of Korenchenko, Polumordvinova and Zinovx (29).

In Section IV, variational principles for phase shifts are obtained, for the nucleon-nucleon interaction as described by Feshbach and Lomon (10), with the addition of an external potential of the form.

$$[V_c(r) + S_{12} V_T(r)]^{s\pi}$$

where $V_c(r)$ and $V_T(r)$ are central and tensor potentials respectively, s and π are the spin and parity labels, and S_{12} is the usual tensor operator. Methods for utilising these results are suggested. These variational techniques can be used in fitting the above interaction model to the data.

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SECTION I

THE THEORY OF POLARIZATION EXPERIMENTS

1. The Density Matrix (30)

We wish to describe the scattering of a particle of spin S by a target of spin T. The interaction will in general be spin dependent, and for a definite energy and momentum of the incident particle, there will be in general $(2S + 1)\ (2T + 1) = N$ independent states ψ_i with i running from 1.....N. Any state ψ^α which is a linear combination of these states is termed a pure state.

$$\psi^\alpha = \sum_i c_i^\alpha \psi_i \qquad 1.1$$

Where the c_i^α are arbitrary complex constants.
For such a state we can always define a complete experiment such that the result is predictable with absolute certainty for that state. Essentially this means that we can find a set of hermit~~e~~ian operators for which that state is an eigenstate. A complete experiment is then really a set of experiments determining the pertinent eigenvalues. Perhaps the most familiar example of a pure state would be 100% linearly polarized monochromatic light. Such light will always be completely transmitted by a suitably oriented Nic~~h~~ol Prism. For partially polarized light, there is no orientation for which we can predict with certainty that every photon will be transmitted. Such a state is called a mixed state, and is a state of less than maximum information, as opposed to a pure

state, which is a state of maximum information. A mixed state cannot be written in the form 1.1. The difficulty of representing a mixed state can be easily resolved, when we realize that a mixed state is merely an incoherent mixture of pure states. More precisely this means that every impure state can be written as an incoherent superposition of pure states ψ^α where ψ^α is some pure state. Thus the sum over α is actually a sum over all possible pure states, it is often referred to as an ensemble sum.

$$\Psi = \sum_\alpha e^{i\phi_\alpha}\psi^\alpha \qquad 1.4$$

The phase factors ϕ^α are real quantities which vary randomly with time.

Let us now evaluate the expectation value of an operator O in spin space.

$$\langle O \rangle_\Psi = \frac{\langle \Psi | O | \Psi \rangle}{\langle \Psi | \Psi \rangle} \qquad 1.5$$

$$= \frac{\sum_\alpha \langle \psi^\alpha | O | \psi^\alpha \rangle + \sum_{\alpha \neq \alpha'} e^{i(\phi_\alpha - \phi_{\alpha'})} \langle \psi^{\alpha'} | O | \psi^\alpha \rangle}{\sum_\alpha \langle \psi^\alpha | \psi^\alpha \rangle + \sum_{\alpha \neq \alpha'} e^{i(\phi_\alpha - \phi_{\alpha'})} \langle \psi^{\alpha'} | \psi^\alpha \rangle}$$

But since ϕ^α varies randomly with time, the quantities $e^{i(\phi_\alpha - \phi_{\alpha'})}$ average to zero in time. We thus get for the expectation value,

$$\langle O \rangle_\Psi = \frac{\sum_\alpha \langle \psi^\alpha | O | \psi^\alpha \rangle}{\langle \psi^\alpha | \psi^\alpha \rangle} \qquad 1.6$$

Writing this out explicitly, we get

$$\frac{\sum_{ij} O_{ij} \sum_\alpha c_i^{\alpha *} c_j^\alpha}{\sum_{ij} \sum_\alpha \delta_{ij} c_i^\alpha c_j^{\alpha *}} \qquad 1.7$$

If we define a matrix operator

$$\rho_{ij} = \sum_{\alpha} c_j^{\alpha} c_i^{\alpha *} \qquad 1.8$$

This gives

$$\langle O \rangle_{\Psi} = \frac{Tr(O\rho_{\Psi})}{Tr(\rho_{\Psi})} \qquad 1.9$$

The matrix ρ is known as the density matrix for the state. Evidently it completely describes the state, since 1.9 gives us a prescription for writing down the expectation value of any operator for a state whose density matrix is known.
The diagonal elements of the density matrix

$$\rho_{ii} = \sum_{\alpha} |c_i^{\alpha}|^2 \qquad 1.10$$

evidently gives the weighting of the state i in the ensemble.
$Tr\,\rho = \sum_i \rho_{ii}$ may be taken to be proportional to the intensity. It is essentially the normalization factor for the density matrix.

Let us now consider the space of all N × N matrices, call this M-space. This space is technically a vector space, that is it has all the properties of a vector space.

If A & B are elements of M, then

C = A + B is also an element of M (closure)

If A is an element of M and λ is a scalar then

B = λ A is also an element of M (scalar multiplication)

There is also a scalar product, which is invariant under unitary transformations.

$$A.B = Tr(A^{\dagger} B).$$

M-space is N^2 dimensional, it therefore can be represented by N^2 orthonormal base matrices $\underset{\sim}{S}$, which should also be hermitean, such that

$$T_r(\underset{\sim}{S}_\mu^+ S_\nu) = N\,\delta_{\mu\nu} \qquad 1.11$$

$\mu = 1.....N^2$.

Any N×N matrix operator, can then be expanded in terms of these base matrices.

$$O = N^{-1} \sum_{\mu=1}^{N^2} T_r(O\underset{\sim}{S}_\mu^+)\,\underset{\sim}{S}_\mu \qquad 1.12$$

In particular ρ can be expanded to give

$$\rho = N^{-1} \sum_{\mu=1}^{N^2} T_r(\rho\,\underset{\sim}{S}_\mu^+)\,\underset{\sim}{S}_\mu$$

which using 1.9

$$= N^{-1}(T_r\rho) \sum_{\mu=1}^{N^2} \underset{\sim}{S}_\mu < \underset{\sim}{S}_\mu > \qquad 1.13$$

We see therefore that to completely specify an impure state, we need to specify the value of N^2 operators. Then N^2 experiments are needed to determine the state. From equations 1.9 and 1.12, we find for the expectation value of an operator

$$<O>_I = N^{-1} \sum_{\mu=1}^{N^2} T_r(O\underset{\sim}{S}_\mu^+)\ <\underset{\sim}{S}_\mu> \qquad 1.14$$

2. Change of State in Scattering Problems, the M-matrix

When two particles originally in a state ρ_i scatter from each other, the final state is described by a transformed density matrix ρ_f. Each spin state ψ_i will be transformed by the interaction into a new state ψ_i'.

$$\psi_i \to \psi_i' = \sum_j M_{ji} \psi_j \qquad 2.1$$

$$\psi^\alpha \to \psi^{\alpha\prime} = \sum_i c_i^\alpha \psi_i'$$

$$= \sum_{ij} c_i^\alpha M_{ji} \psi_j = \sum_j c_j^{\alpha\prime} \psi_j$$

with $$c_j^{\alpha\prime} = \sum_i c_i^\alpha M_{ji} \qquad 2.2$$

Now $$(\rho_f)_{ji} = \sum_\alpha c_j^{\alpha\prime} c_i^{\alpha\prime *}$$

$$= \sum_\alpha \sum_{lk} M_{jl} c_l^\alpha M_{ik}^* c_k^{\alpha *}$$

$$= \sum_{lk} M_{jl} (\rho_i)_{lk} M_{ik}^*$$

that is 2.3

$$\rho_f = M \rho_i M^\dagger$$

Using 2.3 and 1.9 we get

$$\langle \not{S}_n \rangle_f = \frac{Tr(M \rho_i M^\dagger \not{S}_n)}{Tr\, \rho_f} \qquad 2.4$$

using 1.13

$$Tr\, \rho_f \langle \not{S}_n \rangle_f = N^{-1}\, Tr\, \rho_i \{ Tr\, M [\sum_\nu S_\nu \langle S_\nu \rangle_i] M^\dagger \not{S}_n \}$$

$$I_f \langle \not{S}_n \rangle_f = N^{-1} \sum \langle S_\nu \rangle_i\, Tr\,(M S_\nu M^\dagger \not{S}_n) \qquad 2.5$$

where $$I_f = \frac{Tr\, \rho_f\,(\theta\phi)}{Tr\, \rho_i} \qquad 2.6$$

is the differential cross section. This is evident from the interpretation of $Tr\,\rho$, see discussion following 1.10, and also from the fact that $M = M(\theta\phi)$ has no radial dependence.

3. Scattering of a Particle of Spin S from a Target of Spin T

The wave function which describes the scattering may be written, neglecting[1] for the moment any antisimmetry in the case of identical particles.

$$\psi_j^\alpha = \exp i \frac{\vec{p}\cdot\vec{r}}{\hbar} C_j^\alpha + r^{-1} \exp \frac{ipr}{\hbar} \sum_i M_{ji} C_i^\alpha \qquad 3.1$$

with ψ_j^α the j^{th} spin component of the state ψ^α and C_j^α the j^{th} spin component of the incident plane wave state. M_{ji} is the ji^{th} element of the M-matrix and $\vec{r}$ and $\vec{p}$ are the relative coordinate and momentum in the centre of mass. We use the following hermitean operators as our basis.

$1 1_t, 1\vec{\sigma}_t, 1_t\vec{\sigma}, \vec{\sigma}\vec{\sigma}_t$. The $\vec{\sigma}, \vec{\sigma}_t$ are essentially the spin operators for the incident and target particles respectively. For example, if both particles are spin 1/2 these will be the Pauli spin operators. From equation 2.5 we then obtain for the scattering of a polarised beam from an unpolarised target

$$\langle\vec{\sigma}\rangle_f I_f = N^{-1}\{\langle\vec{\sigma}\rangle_i \cdot \mathrm{Tr}(M_f\vec{\sigma} M_f^\dagger \vec{\sigma}) + \mathrm{Tr}(M_f M_f^\dagger \vec{\sigma})\} \qquad 3.2$$

$$I_f = I_{0f} + N^{-1}\langle\vec{\sigma}\rangle_i \cdot \mathrm{Tr}(M_f\vec{\sigma} M_f^\dagger) \qquad 3.3$$

Where we have written $\langle\vec{\sigma}\rangle_i$ for the initial polarization, I_f for the final cross section, and I_{0f} for the unpolarized final

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1 This effect is not pertinent to our discussion, it will be shown in a specific example that the incident and target particles may be treated as distinguishable if the M-matrix is suitably antisymmetrized. See Section II.

cross section.

We also have from 1.13 the relation

$$\rho = N^{-1}\, Tr\, \rho \{ \vec{\sigma} \cdot \langle \vec{\sigma} \rangle + 1 \} \qquad 3.4$$

For the density matrix for the scattering of a polarized beam from an unpolarized target.

4. Single and Double Scattering Experiments (14)

We have an initially unpolarized beam incident on an unpolarized target. The single scattering then gives

for the cross section

$$I_{01} = N^{-1}\, Tr\, (M_1 M_1^{\dagger}) \qquad 4.1$$

and for the polarization after scattering

$$I_{01} \langle \vec{\sigma} \rangle_1 = N^{-1}\, Tr\, (M_1 M_1^{\dagger} \vec{\sigma}) \qquad 4.2$$

A second scattering is then used to analyze the effect of the first, we obtain for the cross section using 3.3

$$I_2 = I_{02} + N^{-1} \langle \vec{\sigma} \rangle_1 \cdot Tr\, (M_2 \vec{\sigma} M_2^{\dagger}) \qquad 4.3$$

The quantities $\langle \vec{\sigma} \rangle_1$ and $Tr\, (M \vec{\sigma} M^{\dagger})$ are evidently measures of the polarizing and analyzing powers of the target respectively. We will show that both of these quantities can be described in terms of a single variable $P(\theta_1)$. To do this we observe that

$$\langle \vec{\sigma} \rangle_1 = (N I_{01})^{-1} [Tr\, (M_1 M_1^{\dagger} \vec{\sigma}) \cdot \vec{\mu}_1] \vec{\mu}_1 \qquad 4.4$$

where $\vec{\mu}_1$ is the direction of polarization, defined by the equation $\langle\vec{\sigma}\rangle_1 = |\langle\vec{\sigma}\rangle_1|\vec{\mu}_1$

we will now show that

$$Tr(M\vec{\sigma}M^{\dagger}) = Tr(MM^{\dagger}\vec{\sigma}) = NI_0P(\theta)\vec{n} \tag{4.5}$$

where $\vec{n}$ is the normal to the scattering plane. The proof of this statement involves certain general invariance properties of the M-matrix. (14) (31).

We begin by expanding M in terms of a complete set of operators in the spin space of the incident particle.

$$M = g1 + \vec{h}\cdot\vec{\sigma} \tag{4.6}$$

where g and $\vec{h}$ are operators in the spin space of the target particle.

Since M is a function of θ, ϕ angles which describe the relative orientation of $\vec{p}$ and $\vec{p}'$, the initial and final relative momentum of the two particles, it must therefore be independent of the euclidiean system used to describe the scattering process. It must therefore be invariant under rotations and space inversion. It must also be invariant under time reversal, i.e., the operation which changes $t \rightarrow -t$, and simultaneously $\vec{p} \rightarrow \vec{p}'$, since such an operation does not alter the process being described for a conservative system, i.e., a system for which direction in time is not significant. Thus g must be a scalar in coordinate space, and invariant under time reversal and $\vec{h}$ must be an axial vector, and change sign under time reversal. Evaluating the traces in equation 4.5, we obtain

$$Tr\,(M\vec{\sigma}M^{\dagger}) = (2s+1)\,Tr'\,(\vec{h}g^{\dagger}+g\vec{h}^{\dagger}-i\vec{h}\times\vec{h}^{\dagger})$$

$$Tr\,(MM^{\dagger}\vec{\sigma}) = (2s+1)\,Tr'\,(\vec{h}g^{\dagger}+g\vec{h}^{\dagger}-i\vec{h}^{\dagger}\times\vec{h})$$

where the primes indicate traces in target space. The two expressions are identical except for the final term in each trace. Now because of the properties of $\vec{h}$, these last terms must transform as axial vectors and be invariant under time reversal, however when the indicated traces are evaluated we are left with a function of $\vec{p}$ and $\vec{p}\,'$. The only function of $\vec{p}$ and $\vec{p}\,'$ which transforms like an axial vector is $\vec{p}\times\vec{p}\,'$. This function however changes sign under time reversal. We therefore conclude that

$$Tr'\,(\vec{h}\times\vec{h}^{\dagger}) = Tr'\,(\vec{h}^{\dagger}\times\vec{h}) = 0$$

so that $Tr\,(MM^{\dagger}\vec{\sigma}) = Tr\,(M\vec{\sigma}M^{\dagger})$

which is what we wished to prove. Further the remaining trace when evaluated must be proportional to $\vec{p}\times\vec{p}\,' = \vec{n}\,p^2\sin\theta$ so

$$Tr\,(MM^{\dagger}\vec{\sigma}) = Tr\,(M\vec{\sigma}M^{\dagger}) = \alpha\,p^2\sin\theta\,\vec{n} \qquad 4.7$$

Where α is an arbitrary function of the scattering angle. So from 4.2 and 4.7 we obtain

$$Tr\,(MM^{\dagger}\vec{\sigma}) = Tr\,(M\vec{\sigma}M^{\dagger}) = N I_0 P(\theta)\,\vec{n} \qquad 4.8$$

Where $P(\theta)\vec{n}$ is the polarization vector. Thus we note that in the scattering of an unpolarized beam from an unpolarized target, the final polarization is a function of θ, and further is normal to the scattering plane.

The cross section after the second scattering equation 4.3 may now be written

$$I_2 = I_{02} (1 + P_1 P_2 \cos \phi_{12}) \qquad 4.9$$

where $\vec{n}_1 \cdot \vec{n}_2 = \cos \phi_{12}$

The measured quantity is the asymmetry factor e defined by

$$e = \frac{I_2(\theta_2, \phi_{12}=0) - I_2(\theta_2, \phi_{12}=\pi)}{I_2(\theta_2, \phi_{12}=0) + I_2(\theta_2, \phi_{12}=\pi)} \qquad 4.10$$

$$= \frac{LL - LR}{LL + LR} = P_1(\theta_1) P_2(\theta_2)$$

here (LL) signifies a first scattering to the left and a second to the left. Similarly for (LR). The quantity e evidently gives the left right asymmetry in the scattering. If θ_1 is chosen equal to θ_2 and if both targets are identical, then neglecting the energy loss in the first scattering

$$e(\theta) = P^2(\theta)$$

$$e^{\frac{1}{2}}(\theta) = P(\theta)$$

which determines $P(\theta)$ since $e(\theta)$ is measurable. The geometry of this experiment is given in Fig. 2.

Further information about the M-matrix may be obtained from triple scattering experiments. Such experiments are designed to determine how the second scattering changes the magnitude or direction or both of the polarization of the scattered particles. Thus the first scatterer is a polarizer, the third is an analyzer.

5. Triple Scattering Experiments (14) (32)

After double scattering the polarization of the beam is given by

$$I_2 \langle \vec{\sigma} \rangle_2 = N^{-1} \{ Tr(M_2 M_2^{\dagger} \vec{\sigma}) + \langle \vec{\sigma} \rangle_1 \cdot Tr(M_2 \vec{\sigma} M_2^{\dagger} \vec{\sigma}) \} \qquad 5.1$$

The first term we immediately recognize from equation 4.8 as being

$$I_{o2} P_2 \vec{n}_2$$

The last we seek to evaluate. We write the first $\vec{\sigma}$ in the trace in the $\vec{n}_2, \vec{k}_2, \vec{s}_2$ representation, the second in the $\vec{n}_2, \vec{k}_2', \vec{s}_2'$ representation. (See Fig. 1.) This is more convenient since the first $\vec{\sigma}$ is roughly speaking linked with the first scattering, and the second is linked with the second scattering. We therefore get for the last term.

$$\begin{aligned} N^{-1} \{ & \langle \vec{\sigma} \rangle_1 \cdot \vec{n}_2 \, Tr(M_2 \sigma_{n_2} M_2^{\dagger} [\sigma_{n_2} \vec{n}_2 + \sigma_{k_2'} \vec{k}_2' + \sigma_{s_2'} \vec{s}_2']) \\ & + \langle \vec{\sigma} \rangle_1 \cdot \vec{k}_2 \, Tr(M_2 \sigma_{k_2} M_2^{\dagger} [\sigma_{n_2} \vec{n}_2 + \sigma_{k_2'} \vec{k}_2' + \sigma_{s_2'} \vec{s}_2']) \qquad 5.2 \\ & + \langle \vec{\sigma} \rangle_1 \cdot \vec{s}_2 \, Tr(M_2 \sigma_{s_2} M_2^{\dagger} [\sigma_{n_2} \vec{n}_2 + \sigma_{k_2'} \vec{k}_2' + \sigma_{s_2'} \vec{s}_2']) \end{aligned}$$

The various traces which appear in the above are observables and functions of $\theta_1, \theta_2, \phi_2$, they must therefore be independent of the euclidiean system which we use for the description of the scattering process. We therefore conclude that all pseudoscalar traces are zero. Let us look at the transformation properties of the quantities involved. $M_2, M_2^{\dagger}$ and σ_{n_2} both transform as scalars in coordinate space, while $\sigma_{k_2}, \sigma_{k_2'}, \sigma_{s_2}$ and $\sigma_{s_2'}$ transform like pseudoscalars.

Equation 5.1 therefore reads

$$N^{-1}\{<\vec{\sigma}>_1\cdot\vec{n}_2\,\mathrm{Tr}(M_2\sigma_{n_2}M_2^{\dagger}\sigma_{n_2})\,\vec{n}_2+<\vec{\sigma}>_1\cdot\vec{k}_2\,[\mathrm{Tr}(M_2\sigma_{k_2}M_2^{\dagger}\sigma_{k_2'})\,\vec{k}_2'+\mathrm{Tr}(M_2\sigma_{k_2}M_2^{\dagger}\sigma_{s_2'})\,\vec{s}_2']+<\vec{\sigma}>_1\cdot\vec{s}_2\,[\mathrm{Tr}(M_2\sigma_{s_2}M_2^{\dagger}\sigma_{k_2'})\,\vec{k}_2'+\mathrm{Tr}(M_2\sigma_{s_2}M_2^{\dagger}\sigma_{s_2'})\,\vec{s}_2']\} \qquad 5.3$$

Or rewriting it in a more useful form

$$I_2<\vec{\sigma}>_2 = I_{02}\{(P_2+D<\vec{\sigma}>_1\cdot\vec{n}_2)\,\vec{n}_2+(A<\vec{\sigma}>_1\cdot\vec{k}_2+R<\vec{\sigma}>_1\cdot\vec{s}_2)\,\vec{s}_2'+(A'<\vec{\sigma}>_1\cdot\vec{k}_2+R'<\vec{\sigma}>_1\cdot\vec{s}_2)\,\vec{k}_2'\} \qquad 5.4$$

Where

$$P = \mathrm{Tr}(MM^{\dagger}\sigma_n)/\mathrm{Tr}(MM^{\dagger})$$

$$D = \mathrm{Tr}(M\sigma_n M^{\dagger}\sigma_n)/\mathrm{Tr}(MM^{\dagger})$$

$$R = \mathrm{Tr}(M\sigma_s M^{\dagger}\sigma_{s'})/\mathrm{Tr}(MM^{\dagger}) \qquad 5.4a$$

$$A = \mathrm{Tr}(M\sigma_k M^{\dagger}\sigma_{s'})/\mathrm{Tr}(MM^{\dagger})$$

$$R' = \mathrm{Tr}(M\sigma_s M^{\dagger}\sigma_{k'})/\mathrm{Tr}(MM^{\dagger})$$

$$A' = \mathrm{Tr}(M\sigma_k M^{\dagger}\sigma_{k'})/\mathrm{Tr}(MM^{\dagger})$$

For simplicity we have dropped the subscript 2. throughout. Thus the triple scattering experiments will involve five new quantities. As will be shown later in Section II, all of these quantities are not independent. In fact for particles of equal mass we have (nonrelativisticly).

$$\frac{A + R'}{A' - R} = \tan\frac{\theta}{2} \qquad 5.5$$

As is evident from equations 3.3 and 4.5, polarization along the direction of motion cannot be detected in a single scattering, since the analyzing term which gives rise to the asymmetry in the analyzing portion of the scattering experiment is given by

$$P_3 I_{03} \langle\vec{\sigma}\rangle_2 \cdot \vec{n}_3 \qquad 5.6$$

Triple scattering experiments are therefore designed to measure either

$$I_2 \langle\vec{\sigma}\rangle_2 \cdot \vec{n}_2 = I_{02}\left(P_2 + D \langle\vec{\sigma}\rangle_1 \cdot \vec{n}_2\right)$$

or

$$I_2 \langle\vec{\sigma}\rangle_2 \cdot \vec{s}_2' = I_{02}\left(A \langle\vec{\sigma}\rangle_1 \cdot \vec{k}_2 + R \langle\vec{\sigma}\rangle_1 \cdot \vec{s}_2\right) \qquad 5.7$$

/ies The corresponding asymmetrys in the scattering are denoted by e_{3n} and $e_{3s'}$ respectively. The geometry of these experiments is given in Figs. 3, 4 and 5.

Thus in the e_{3n} experiment $\vec{n}_3 = \pm\vec{n}_2$ i.e., the second and third scatterings are in the same plane. We obtain for the cross section and left right asymmetry.

$$I_3 = I_{03}\left(1 + P_3 \langle\vec{\sigma}\rangle_2 \cdot \vec{n}_3\right) \qquad 5.8$$

$$e_{3n} = \frac{I_3(\vec{n}_3 = \vec{n}_2) - I_3(n_3 = -n_2)}{I_3(\vec{n}_3 = \vec{n}_2) + I_3(\vec{n}_3 = -\vec{n}_2)} \qquad 5.9$$

$$= \frac{LL - LR}{LL + LR}$$

Using 5.7 this gives

$$e_{3n} = \frac{2 I_{03} P_3 \langle \vec{\sigma} \rangle_2 \cdot \vec{n}_2}{2 I_{03}}$$

$$e_{3n} = \frac{P_3 (P_2 + D P_1 \cos\phi_{12})}{(1 + P_1 P_2 \cos\phi_{12})} \qquad 5.10$$

$\cos\phi_{12} = \vec{n}_1 \cdot \vec{n}_2$

Thus if we choose $\phi_{12} = 0$ or π i.e., all three planes parallel, we can determine D from 5.10 where we of course assume that the quantities $P_3 P_2$, $P_3 P_1$, $P_1 P_2$, have all been determined from double scattering experiments. Alternatively we can utilise both values of ϕ_{12} (see Fig. 4), assuming that the first scattering is to the left, one then has four scattering intensities LL, LR, RL and RR where RL for example is the intensity with the second and third scatterings to right and left respectively. Using 5.10 we obtain

$$D = \frac{(LL + RL - LR - RR)}{(LL + RL + LR + RR) P_3 P_1} \qquad 5.11$$

We notice that here only one double scattering paramater need be determined, i.e., $P_1 P_3$.

The paramater D is called the depolarization of the scattering. It is a measure of the extent to which the second scattering depolarizes the beam. To illustrate this let us take $P_1 = 1$ the beam completely polarized after the first scattering, and choose $\cos\phi_{12} = \pm 1$ then from 5.4 and 4.9

$$\langle \vec{\sigma} \rangle_2 = \vec{n}_2 (P_2 \pm D) / (1 \pm P_2) \qquad 5.12$$

$$\langle \vec{\sigma} \rangle_2 = \vec{n}_2 \left\{ \pm 1 + \frac{D - 1}{P_2 \pm 1} \right\} \qquad 5.13$$

We see that D is a measure of the amount to which the second scattering depolarizes the beam. If $D = 1$ there is no depolarization. If $D = P_2^2$ the final polarization is the same as if the initial beam were completely unpolarized, however D may be less than this so that it may not represent a depolarization but an actual reversal of spin. From 5.12 using the fact that $|\langle\vec{\sigma}\rangle_2| \leqq 1$ we obtain the limits on D.

$$-1 + 2|P_2| \leqq D \leqq 1 \qquad 5.14$$

In the second triple scattering experiment we seek to measure the $\vec{s}_2'$ component of the polarization after the second scattering. We therefore choose $\vec{n}_3 = \pm\vec{s}_2'$ i.e., the second and third scattering planes are normal to each other. We obtain for the cross section and left right asymmetry

$$I_3 = I_{03}(1 + P_3 \langle\vec{\sigma}\rangle_2 \cdot \vec{n}_3)$$

$$e_{3s'} = \frac{I_3(\vec{n}_3 = \vec{s}_2') - I_3(\vec{n}_3 = -\vec{s}_2')}{I_3(\vec{n}_3 = \vec{s}_2') + I_3(\vec{n}_3 = -\vec{s}_2')} \qquad 5.15$$

Now using 5.7 and 4.9

$$\langle\vec{\sigma}\rangle_2 \cdot \vec{s}_2' = \frac{I_{02}[A\langle\vec{\sigma}\rangle_1 \cdot \vec{k}_2 + R\langle\vec{\sigma}\rangle_1 \cdot \vec{s}_2]}{I_{02}[1 + P_2\langle\vec{\sigma}\rangle_1 \cdot \vec{n}_2]} \qquad 5.16$$

But

$$\langle\vec{\sigma}\rangle_1 \cdot \vec{k}_2 = 0$$

So

$$\langle\vec{\sigma}\rangle_2 \cdot \vec{s}_2' = \frac{R P_1 \vec{n}_1 \cdot \vec{s}_2}{1 + P_1 P_2 \vec{n}_1 \cdot \vec{n}_2} \qquad 5.18$$

$$= \frac{R P_1 \sin\phi_{12}}{1 + P_1 P_2 \cos\phi_{12}}$$

Then

$$e_{3s'} = \frac{R\,P_1 P_3 \sin\phi_{12}}{1 + P_1 P_2 \cos\phi_{12}} \qquad 5.19$$

We now choose $\phi_{12} = \pm\frac{\pi}{2}$ i.e., the first and second scattering planes are normal to each other. So that in the entire experiment, each of the three successive scattering planes are perpendicular. We then get

$$e_{3s'} = R\,P_1\,P_3 \qquad 5.20$$

Since P_1P_3 may be determined from double scattering experiments, R is obtained. In order to get some physical picture of what R is, again consider the case of $P_1 = 1$ then from 5.4 with $\phi_{12} = \frac{\pi}{2}$

$$\langle\vec{\sigma}\rangle_2 = P_2\vec{n}_2 + R\vec{s}_2{}' + R'\vec{k}_2{}' \qquad 5.21$$

Therefore we see that the polarization has been rotated out of the second scattering plane, giving it a component P_2 in the direction $\vec{n}_2$, R gives a measure of this rotation, it is therefore called the rotation paramater. From 5.21 we get

$$|R| \leq (1 - P_2^2)^{\frac{1}{2}} \qquad 5.22$$

We see therefore that triple scattering experiments can determine D or R, the first experiment performed with $\phi_{12} = 0$ or π and all three scattering planes parallel to each other. The second experiment is performed with $\phi_{12} = \pm\frac{\pi}{2}$ and the successive planes are normal to each other. If we wish to measure either A R$'$ or A$'$, we must utilise the fact that the particle which we are observing has a magnetic moment, so that it is possible to rotate the spin direction of the particle with the aid of a magnetic field. If we wish to measure A, it is necessary to rotate the spin after the first scattering in such a way that the polarization gets a component along the direction of motion $\vec{k}_1'$. In order to do this, a magnetic field normal to the $\vec{n}_1\ \vec{k}_1'$ plane, is placed between the first and second scatterers. In a similar manner a magnetic field placed between the second and third scatterers will rotate the $\vec{k}_2'$ component of the polarization, i.e., give it a component perpendicular to $\vec{k}_2'$. In this way, R$'$ may be obtained. In order to measure A$'$ we need magnetic fields both between the first and second scatterers and between the second and third scatterers. These experiments are very difficult to perform, and to date only the A experiment has been performed, see for example reference 15. For further details of these experiments, see reference 14, and the references contained therein.

6. The Theory of Correlation Experiments (14)

In these experiments, we look at the spin of the scattered particle in correlation with the spin of the recoiling target. The pertinent operator $\vec{\sigma}\,\vec{\sigma}_t$, has an expectation value given by

$$I_f \langle \vec{\sigma}\vec{\sigma}_t \rangle = N^{-1} \{ \mathrm{Tr}(MM^{\dagger}\vec{\sigma}\vec{\sigma}_t) + \langle \vec{\sigma} \rangle_i \cdot \mathrm{Tr}(M\vec{\sigma}M^{\dagger}\vec{\sigma}\vec{\sigma}_t) \} \qquad 6.1$$

Where the target particle is assumed to be initially unpolar-

ized. The subscript t refers to the target particle.
Now we can detect polarization of any particle in any direction normal to its direction of motion by a single scattering, i.e., in the directions $\vec{n}$ and $\vec{s}\,'$ for the scattered particle, and in the directions $\vec{n}$ and $\vec{s}_t\,'$ for the recoiling target.
Non-relativistic kinematics readily yields

$$\vec{k}\,' = \vec{s}_t\,' = \vec{p} \quad , \quad \vec{k}_t\,' = -\vec{s}\,' = -\vec{k}$$

See Fig. 1.

We therefore can measure the following components of the dyadic $\vec{\sigma}\vec{\sigma}_t$. $\vec{n}\vec{n}, \vec{k}\vec{n}, \vec{k}\vec{p}, \vec{n}\vec{p}$. Considering only these components, and restricting ourselves to components of $\langle\vec{\sigma}\rangle_i$ normal to the incident direction, we obtain

$$\begin{aligned} I_f \langle \vec{\sigma}\vec{\sigma}_t \rangle = N^{-1} \{ & Tr(MM^\dagger\sigma_n\sigma_{tn})\vec{n}\vec{n} + Tr(MM^\dagger\sigma_k\sigma_{tn})\vec{k}\vec{n} \\ & + Tr(MM^\dagger\sigma_k\sigma_{tp})\vec{k}\vec{p} + Tr(MM^\dagger\sigma_n\sigma_{tp})\vec{n}\vec{p}\,] + \\ & \langle\vec{\sigma}\rangle_i\cdot\vec{n}\,[Tr(M\sigma_n M^\dagger\sigma_n\sigma_{tn})\vec{n}\vec{n} + Tr(M\sigma_n M^\dagger\sigma_k\sigma_{tn})\vec{k}\vec{n} \\ & + Tr(M\sigma_n M^\dagger\sigma_k\sigma_{tp})\vec{k}\vec{p} + Tr(M\sigma_n M^\dagger\sigma_n\sigma_{tp})\vec{n}\vec{p}\,] \\ & + \langle\vec{\sigma}\rangle_i\cdot\vec{s}\,[Tr(M\sigma_s M^\dagger\sigma_n\sigma_{tn})\vec{n}\vec{n} + Tr(M\sigma_s M^\dagger\sigma_k\sigma_{tn})\vec{k}\vec{n} \\ & + Tr(M\sigma_s M^\dagger\sigma_k\sigma_{tp})\vec{k}\vec{p} + Tr(M\sigma_s M^\dagger\sigma_n\sigma_{tp})\vec{n}\vec{p}\,]\} \end{aligned} \tag{6.2}$$

A number of these traces are zero, by virtue of the space inversion arguments of Section I, 5. When these are eliminated we are left with

$$\begin{aligned} I_f \langle \vec{\sigma}\vec{\sigma}_t \rangle = N^{-1} \{ & [Tr(MM^\dagger\sigma_n\sigma_{tn}) + \langle\sigma_n\rangle_i\, Tr(M\sigma_n M^\dagger\sigma_n\sigma_{tn})] \\ & \vec{n}\vec{n} + [Tr(MM^\dagger\sigma_k\sigma_{tp}) + \langle\sigma_n\rangle_i\, Tr(M\sigma_n M^\dagger\sigma_k\sigma_{tp})]\vec{k}\vec{p} \\ & + \langle\sigma_s\rangle_i\, Tr(M\sigma_s M^\dagger\sigma_n\sigma_{tp})\vec{n}\vec{p} + \langle\sigma_s\rangle_i\, Tr(M\sigma_s M^\dagger\sigma_k\sigma_{tn}) \\ & \vec{k}\vec{n}\} \end{aligned} \tag{6.3}$$

Or in a more familiar form

$$I_f \langle \vec{\sigma}\,\vec{\sigma}_t \rangle = I_0 \{ [C_{nn} + \langle \sigma_n \rangle_i C^p_{nn}] \vec{n}\,\vec{n} + [C_{kp} + \langle \sigma_n \rangle_i C^p_{kp}] \vec{k}\,\vec{p} + \langle \sigma_s \rangle_i C^p_{kn} \vec{k}\,\vec{n} + \langle \sigma_s \rangle_i C^p_{np} \vec{n}\,\vec{p} \} \qquad 6.4$$

Where

$$\begin{aligned} C_{nn} &= Tr(MM^{\dagger}\sigma_n \sigma_{tn}) / Tr(MM^{\dagger}) \\ C_{kp} &= Tr(MM^{\dagger}\sigma_k \sigma_{tp}) / Tr(MM^{\dagger}) \\ C^p_{nn} &= Tr(M\sigma_n M^{\dagger}\sigma_n \sigma_{tn}) / Tr(MM^{\dagger}) \\ C^p_{kp} &= Tr(M\sigma_n M^{\dagger}\sigma_k \sigma_{tp}) / Tr(MM^{\dagger}) \\ C^p_{kn} &= Tr(M\sigma_s M^{\dagger}\sigma_k \sigma_{tn}) / Tr(MM^{\dagger}) \\ C^p_{np} &= Tr(M\sigma_s M^{\dagger}\sigma_n \sigma_{tp}) / Tr(MM^{\dagger}) \end{aligned} \qquad 6.4a$$

We thus have six kinds of correlation experiments, four with a polarized incident beam, and two with an unpolarized beam. The target of course is initially unpolarized. We will only be concerned with the latter two experiments. For these we have

$$\langle \vec{\sigma}\,\vec{\sigma}_t \rangle = C_{nn} \vec{n}\,\vec{n} + C_{kp} \vec{k}\,\vec{p} \qquad 6.5$$

It will be convenient to refer to the scattered particle as s, and the target particle as t. The density matrix for these two outgoing particles incident on two spin zero analyzers, is given using 1.13 by

$$\rho_i(st) = N^{-1} Tr\, \rho_i \{ 1 + \vec{\sigma}_s \cdot \vec{P}_s + \vec{\sigma}_t \cdot \vec{P}_t + C_{nn} (\vec{n}\,\vec{n}) \cdot (\vec{\sigma}_s\,\vec{\sigma}_t) + C_{kp} (\vec{k}\,\vec{p}) \cdot (\vec{\sigma}_s\,\vec{\sigma}_t) \} \qquad 6.6$$

Where we have used for our complete set of operators $1_s 1_t, 1_s \vec{\sigma}_t$ $\vec{\sigma}_s 1_t, \vec{\sigma}_s \vec{\sigma}_t$ all multiplied by $1_1 1_2$ where 1 and 2 refer to the two spin zero analyzers, with

$$\vec{P}_s = \mathrm{Tr}(MM^\dagger \vec{\sigma}_s) / \mathrm{Tr}(MM^\dagger)$$
$$\vec{P}_t = \mathrm{Tr}(MM^\dagger \vec{\sigma}_t) / \mathrm{Tr}(MM^\dagger)$$

the polarizations of the scattered particle and the recoiling target. In obtaining this we have used equation 3.4. The M-matrix for the scattering from two spin zero targets, is then given by

$$M^0(\theta_s \phi_s ; \theta_t \phi_t) = (f_1(\theta_s) + g_1(\theta_s)\vec{\sigma}_s \cdot \vec{N}_s) \cdot (f_2(\theta_t) + g_2(\theta_t)\vec{\sigma}_t \cdot \vec{N}_t) \qquad 6.7$$

This result will be derived in the next part of this section. $\vec{N}_s$ and $\vec{N}_t$ are the normals to the two scattering planes. The coincidence cross section is given by

$$I_f(\theta_s \phi_s ; \theta_t \phi_t) = \frac{\mathrm{Tr}\,\rho_f(\theta_s \phi_s ; \theta_t \phi_t)}{\mathrm{Tr}\,\rho_i} \qquad 6.8$$

Where $I_f(\theta_s \phi_s ; \theta_t \phi_t)\, d\Omega_s d\Omega_t$ is the probability that s scatters into $d\Omega_s$ about $\theta_s \phi_s$ while t scatters into $d\Omega_t$ about $\theta_t \phi_t$. Using 2.3 we obtain

$$I_f(\theta_s \phi_s ; \theta_t \phi_t) = I_0^s(\theta_s) I_0^t(\theta_t) \{ 1 + \vec{P}_s \cdot \vec{P}_s^0(\theta_s \phi_s) + \vec{P}_t \cdot \vec{P}_t^0(\theta_t \phi_t) + C_{nn}(\vec{P}_s^0 \vec{P}_t^0) \cdot (\vec{n}\vec{n}) + C_{kp}(\vec{P}_s^0 \vec{P}_t^0) \cdot (\vec{k}\vec{p}) \} \qquad 6.9$$

with

$$I_0^s(\theta_s) = |f_1(\theta_s)|^2 + |g_1(\theta_s)|^2$$
$$I_0^t(\theta_t) = |f_2(\theta_t)|^2 + |g_2(\theta_t)|^2$$
$$I_0^s(\theta_s)\vec{P}_s^0(\theta_s) = [f_1(\theta_s) g_1^*(\theta_s) + f_1^*(\theta_s) g_1(\theta_s)]\, \vec{N}_s(\theta_s \phi_s) \qquad 6.10$$
$$I_0^t(\theta_t)\vec{P}_t^0(\theta_t) = [f_2(\theta_t) g_2^*(\theta_t) + f_2^*(\theta_t) g_2(\theta_t)]\, \vec{N}_t(\theta_t \phi_t)$$

The quantities I_0^s and I_0^t are merely the cross sections for the scattering of unpolarized beams from spin zero targets, while $\vec{P}_0^s$ and $\vec{P}_0^t$ are the polarizations after scattering. These results will be derived in the next part. We had

$$I_f(\theta_s\phi_s;\theta_t\phi_t) = I_0^s I_0^t \{ 1 + P_s^0 \vec{P}_s\cdot\vec{N}_s + P_t^0 \vec{P}_t\cdot\vec{N}_t$$
$$+ P_s^0 P_t^0 C_{nn} \vec{N}_s\cdot\vec{n}\ \vec{N}_t\cdot\vec{n} + P_s^0 P_t^0 C_{KP} \vec{N}_s\cdot\vec{K}\ \vec{N}_t\cdot\vec{P} \}$$

$$\cos\phi_s = \vec{N}_s\cdot\vec{n}\ , \quad \cos\phi_t = \vec{N}_t\cdot\vec{n}$$

Write

$$I_f(\theta_s \tfrac{0}{\pi}, \theta_t 0) = \tfrac{LL}{RL}$$

$$I_f(\theta_s \tfrac{0}{\pi}, \theta_t \pi) = \tfrac{LR}{RR}$$

We will now define the correlation experiments with which we will be concerned.

Noting that $\vec{N}(\theta\ \phi=0) = -\vec{N}(\theta\ \phi=\pi)$

We get with $\vec{N}_s = \pm\vec{n}$ and $\vec{N}_t = \pm'\vec{n}$

$$C_{nn}(\theta) = \frac{LL+RR-RL-LR}{LL+RR+RL+LR}\cdot\frac{1}{P_s^0 P_t^0} \qquad 6.11$$

and with $\vec{N}_s = \pm\vec{K}$, $\vec{N}_t = \pm'\vec{P}$ we get

$$C_{KP}(\theta) = \frac{LL+RR-RL-LR}{LL+RR+RL+LR}\cdot\frac{1}{P_s^0 P_t^0} \qquad 6.12$$

This then defines the two experiments. They are shown in Figures 6 and 7.

The quantities C_{nn} and C_{KP} evidently have a simple physical interpretation.

C_{nn} gives a measure of the correlation of the spins of a scattered particle and the corresponding recoiling target in the direction $\vec{n}$. Thus a small value of C_{nn} would mean that if the spins of the scattered particle and the corresponding recoiling target were measured in coincidence, there would be very small chance of both having a large component of spin in the direction $\vec{n}$.

If however we have a large value of C_{nn}, this would indicate that there is a large probability for both particles to have large components of spin in the direction $\vec{n}$.

In a similar manner C_{KP} may be interpreted, where the direction $\vec{K}$ would refer to the scattered particle, and $\vec{P}$ to the recoiling target particle.

7. Scattering from a Spin 0 Target (14) (33)

From equation 4.6, the M-matrix is given by

$$g + \vec{h} \cdot \vec{\sigma}$$

where g is a scalar in coordinate space, and invariant under time reversal, and $\vec{h}$ is an axial vector and changes sign under time reversal. Evidently if we choose $\vec{p}$ and $\vec{p}\,'$ and $\vec{p} \times \vec{p}\,'$ as our coordinate axes, the M-matrix can be written

$$g(\theta) + h(\theta)\, \vec{\sigma} \cdot \vec{n} \qquad 7.1$$

Here g and h are arbitrary functions of the scattering angle and energy. The terms $\vec{\sigma} \cdot \vec{p}$ and $\vec{\sigma} \cdot \vec{p}\,'$ are ruled out since they change sign under space inversion.

From equation 3.4, and equation 2.3, we get for the cross section for scattering of an unpolarized beam.

$$I_0 = \mathrm{Tr}\,(MM^\dagger) = |g(\theta)|^2 + |h(\theta)|^2 \tag{7.2}$$

Also the polarization after the scattering is given by equation 2.5.

We obtain

$$\begin{aligned} I_0 \vec{P} &= N^{-1}\,\mathrm{Tr}\,(MM^\dagger \vec{\sigma}) \\ &= (h(\theta)\, g^*(\theta) + h^*(\theta)\, g(\theta))\,\vec{n} \end{aligned} \tag{7.3}$$

This proves the statements in the last section that I_0^s, I_0^t were cross sections for scattering of unpolarized beams, and $\vec{P}_0{}^s$, $\vec{P}_0{}^t$ were polarizations after scattering of unpolarized beams.

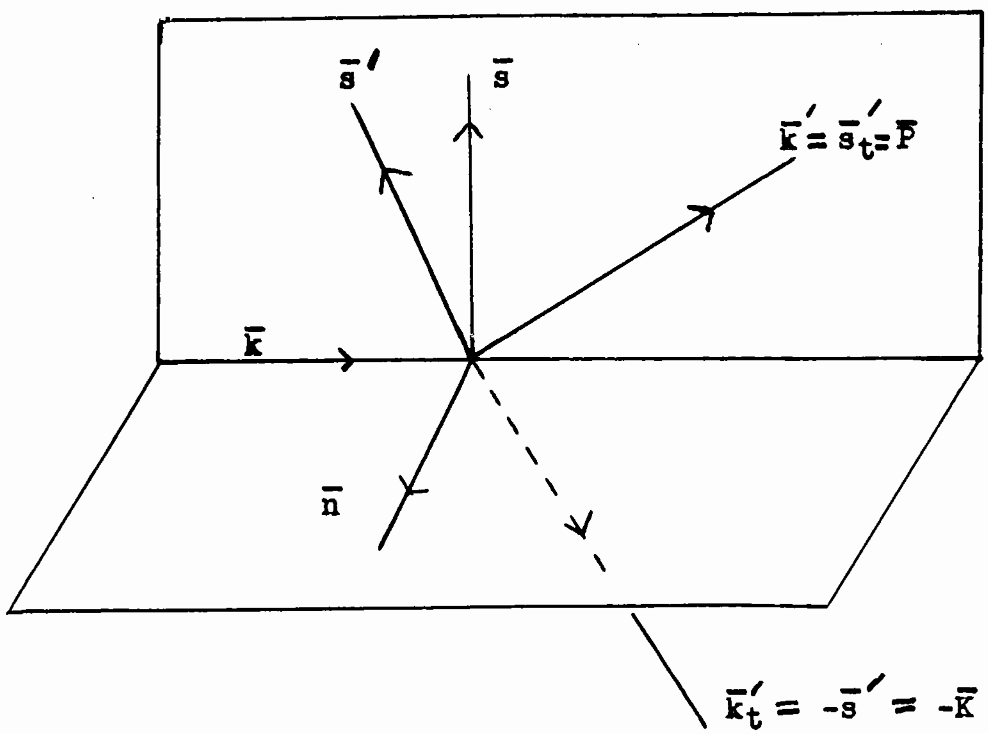

Fig. 1. Geometry used in the various types of double and triple scattering, and correlation experiments.

Here the various quantities are defined as follows.

$$\bar{P} = \frac{\bar{p} + \bar{p}'}{|\bar{p} + \bar{p}|}, \quad \bar{K} = \frac{\bar{p}' - \bar{p}}{|\bar{p}' - \bar{p}|}, \quad \bar{n} = \frac{\bar{k} \times \bar{k}'}{|\bar{k} \times \bar{k}'|}, \quad \bar{s} = \bar{n} \times \bar{k}, \quad \bar{s}' = \bar{n} \times \bar{k}'.$$

Where $\bar{p}'$ and $\bar{p}$ are the final and initial relative momenta, and $\bar{k}$ and $\bar{k}'$ are unit vectors in the direction of the initial and final lab momenta.

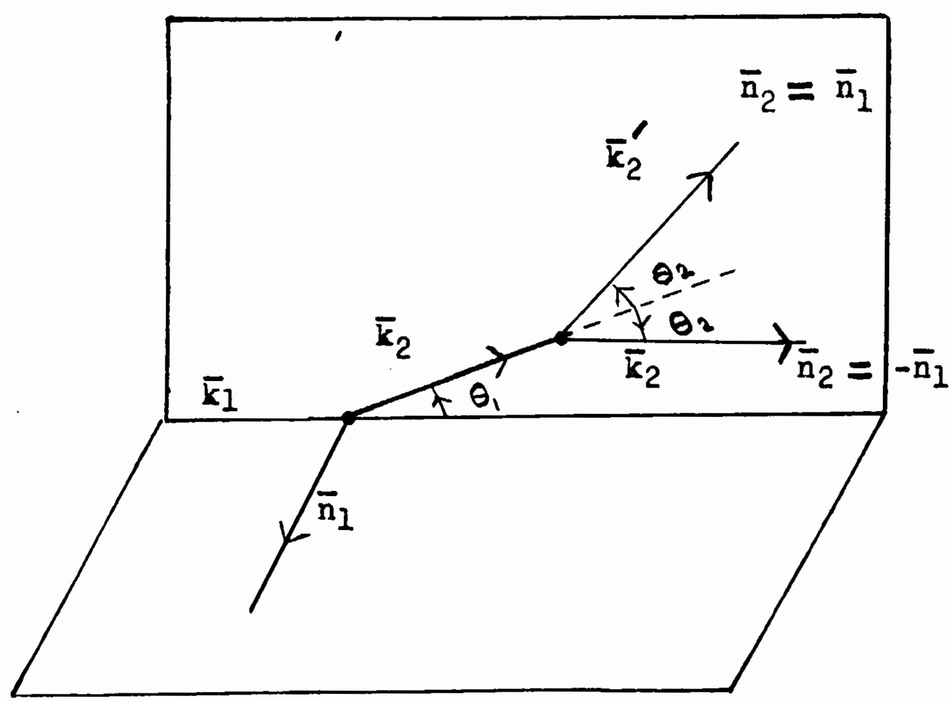

Fig. 2. Typical double scattering experiment for detecting the polarization produced in an unpolarized beam by a single scattering. The polarization is in the direction $\bar{n}_1$, and is detected by the assymetry in the second scattering, which is proportional to $P(\theta_1)\ P(\theta_2)$.

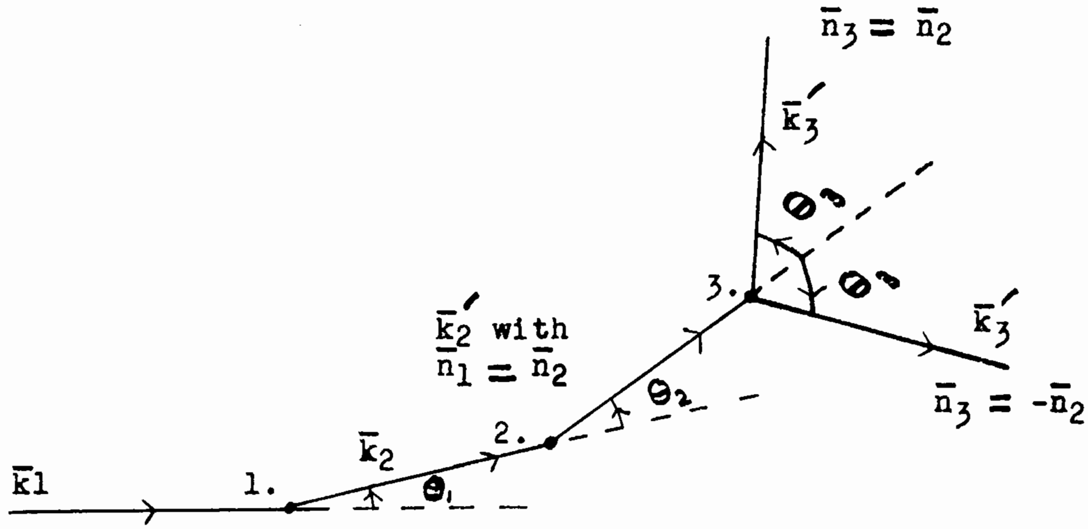

Fig. 3. The first type of experiment for the measurement of the depolarization $D(\theta)$. The beam incident on the first scatterer is unpolarized. The first scattering produces a componenent of polarization $P(\theta)$ normal to $\bar{k}_1$ and $\bar{k}_2$. The effect of the second scattering shown, is to alter this component by an amount proportional to $D(\theta_2)$. The third scattering possesses an azimuthal assymmetry, which enables us to obtain a value for $D(\theta_2)$.

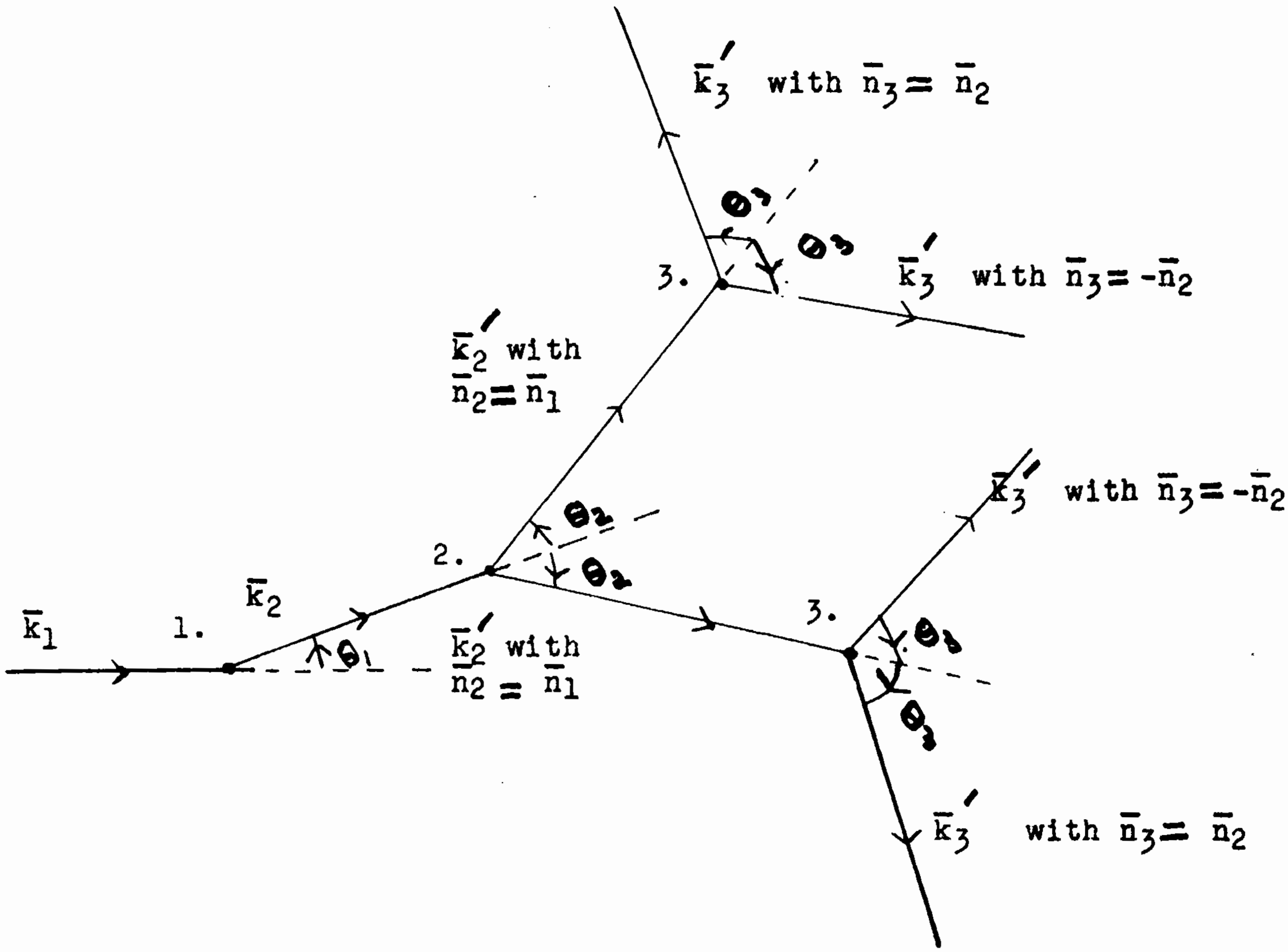

Fig. 4. A second type of experiment for the measurement of D(θ), which utilises both a left and a right scattering at the second scatterer. This type of experiment has the advantage that only one double scattering paramater is necessary, in order to compute D(θ) from the results of the triple scattering experiment. In the other experiment Fig. 3., three double scattering paramaters are needed.

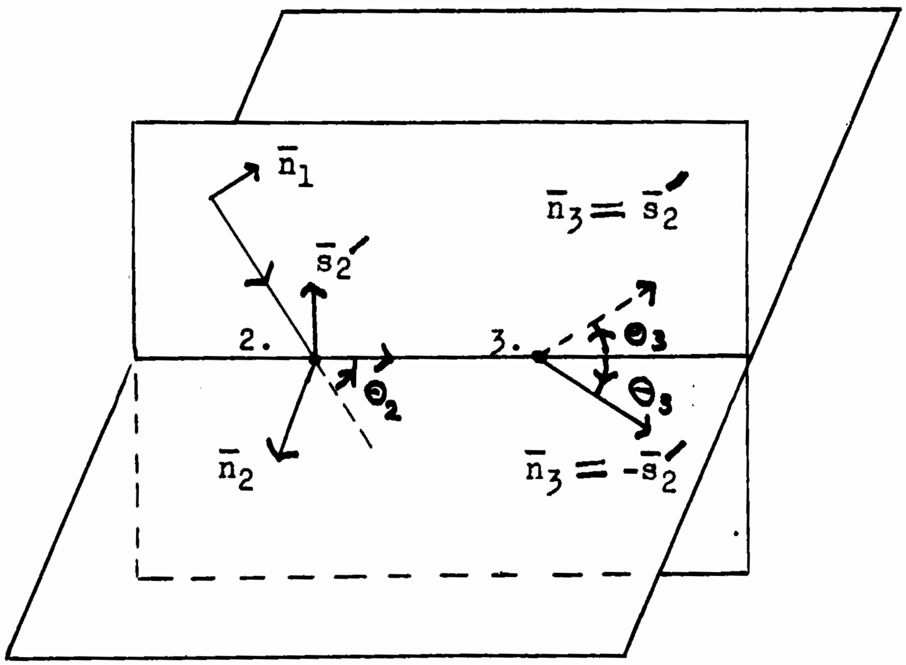

Fig. 5. An experiment for the measurement of the rotation paramater $R(\theta)$. The first scattering produces a polarization of the incoming unpolarized beam, in the direction $\bar{n}_1$. For simplicity, the incoming beam is not shown. The second scattering produces a component of polarization in the direction $\bar{s}_2$, which is proportional to $R(\theta_2)$. The third scattering, which is carried out in a plane normal to this direction, exhibits an azimuthal assymmetry from which $R(\theta_2)$ may be obtained.

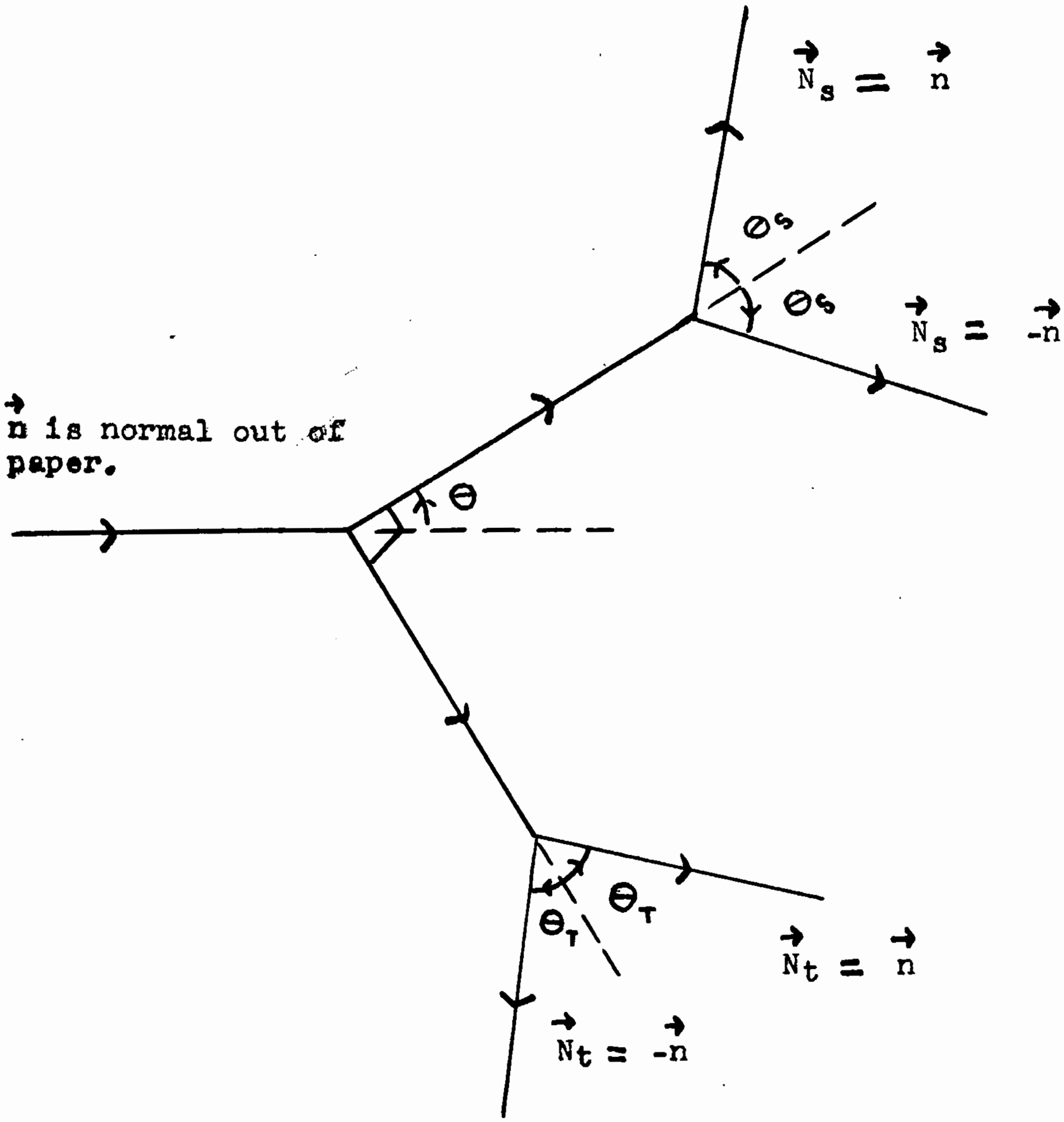

Fig. 6. An experiment for the measurement of the correlation paramater $C_{nn}(\theta)$. The incoming beam is completely unpolarized, as is the target. The scattering produces polarization both of the scattered particle and of the recoiling target. This experiment measures the $\bar{n}$ component of polarization of the scattered particle in correlation with the $\bar{n}$ component of polarization of the recoiling target.

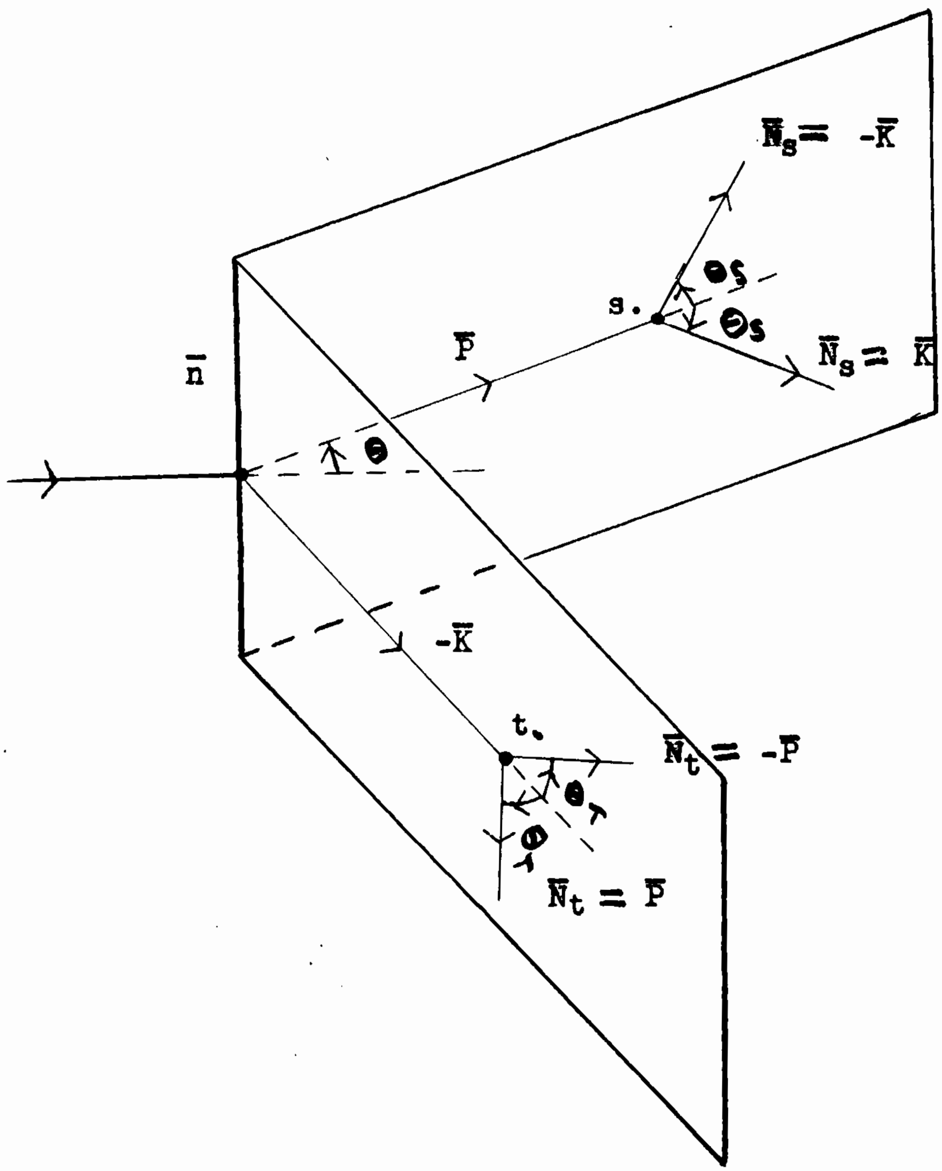

Fig. 7. An experiment for measuring the correlation paramater $C_{KP}(\theta)$. The beam incident on the first scatterer is unpolarized, as is the target, the scattering polarizes both and this experiment looks at the $\bar{K}$ component of polarization of the scattered particle in correlation with the P component of polarization of the recoiling target. This is done by scattering both particles off spin zero analyzers, with the scattering planes normal to $\bar{K}$ and $\bar{P}$ respectively.

SECTION II

THE FORMALISM FOR NUCLEON-NUCLEON M-MATRIX CALCULATIONS

1. The Nucleon-Nucleon M-Matrix (15) (34)

The base matrices which describe the spin space of two spin 1/2 particles are
$S_\nu = 1_1 1_2, 1_1 \vec{\sigma}_2, \vec{\sigma}_1 1_2, \vec{\sigma}_1 \vec{\sigma}_2$. Where 1 refers to the incident particle and 2 to the target.
As mentioned before, the M-matrix must be a scalar under rotations, space inversion and time reversal. It must be formed from contractions of the above operators with the vectors which describe the experiment, namely $\vec{n}, \vec{p}, \vec{k}$.
Let us list the properties of these operators and vectors under the three operations mentioned. Here a plus will denote no change under the operation, while a minus will denote change in sign under the operation.

Operator	Space Inversion	Time Reversal	Rotations
$1_1 1_2$	+	+	+
$1_1 \vec{\sigma}_2$	+	−	+
$\vec{\sigma}_1 1_2$	+	−	+
$\vec{\sigma}_1 \vec{\sigma}_2$	+	+	+
$\vec{n}$	+	−	+
$\vec{p}$	−	−	+
$\vec{k}$	−	+	+

The possible scalars are therefore.
$1_1 1_2, \vec{\sigma}_1 \cdot \vec{n}, \vec{\sigma}_2 \cdot \vec{n}, (\vec{\sigma}_1 \cdot \vec{p}\, \vec{\sigma}_2 \cdot \vec{p}), (\vec{\sigma}_1 \cdot \vec{k}\, \vec{\sigma}_2 \cdot \vec{k}),$
$(\vec{\sigma}_1 \cdot \vec{n}\, \vec{\sigma}_2 \cdot \vec{n})$.

We can then write the M-matrix in the form.

$$M = a(\theta) + c(\theta)[\sigma_{1n}+\sigma_{2n}] + b(\theta)[\sigma_{1n}-\sigma_{2n}] + m(\theta)\sigma_{1n}\sigma_{2n} + g(\theta)[\sigma_{1p}\sigma_{2p}+\sigma_{1k}\sigma_{2k}] + h(\theta)[\sigma_{1p}\sigma_{2p}-\sigma_{1k}\sigma_{2k}]. \quad 1.1$$

where $a(\theta) \cdots h(\theta)$ are arbitrary functions of θ and the energy. We will now show that for charge independent forces $b(\theta) = 0$. Consider

$$(\sigma_{1n}-\sigma_{2n})\ (\alpha_1\beta_2+\alpha_2\beta_1)$$

i.e., acting on a triplet state. Choose the $\vec{n}$ direction to be the direction of the z-axis, we then obtain.

$$(\sigma_{1n}-\sigma_{2n})\ (\alpha_1\beta_2+\alpha_2\beta_1) = (\alpha_1\beta_2-\beta_1\alpha_2)$$

thus a triplet state is transformed into a singlet state by this interaction. Thus since J the total angular momentum, and the parity are good quantum numbers, this means for example that a $^3P_0$ state would be transformed into a $^1P_0$ state. Since for nucleons the total (including isospin) wave function must be antysymmetric, this means that we must change the isospin function from singlet to triplet. But charge independence is a statement of conservation of isospin, this interaction is therefore not charge independent. We therefore deduce that $b(\theta)=0$ for charge independence. We therefore have as the final form of the M-matrix

$$M(\theta,\phi) = a(\theta) + c(\theta)[\sigma_{1n}+\sigma_{2n}] + m(\theta)\sigma_{1n}\sigma_{2n} + g(\theta)[\sigma_{1p}\sigma_{2p}+\sigma_{1k}\sigma_{2k}] + h(\theta)[\sigma_{1p}\sigma_{2p}-\sigma_{1k}\sigma_{2k}] \quad 1.2$$

It therefore appears that to determine the M matrix at a given value of the energy and of angle, we need to determine 10 real quantities. Actually, unitarity of the S-matrix imposes 5 additional conditions on the quantities $a(\theta) \cdots h(\theta)$. This means that we have to perform 5 experiments at all angles to completely determine the M-matrix at that energy. To show this, we follow the work of Smorodinski and Ryndin (35). We write the wave function for the scattering process.

$$\Psi_{\vec{k}}(r \to \infty) \cong \varphi_{\vec{k}}^{incid.} + \varphi_{\vec{k}}^{scat.} = e^{ikz}\chi + M(\vec{k}, \vec{k}')\frac{\chi}{r}e^{ikr} \qquad 1.3$$

Here the particle is incident in the direction $\vec{k}$, and is scattered into the direction $\vec{k}'$. In view of the unitarity of the S-matrix, the $\Psi_{\vec{k}}$ satisfy the same requirements of orthognality and normalization as the initial functions of the incident wave, and form when $r \to \infty$ a complete set of functions with respect to angular variables.

$$\langle \varphi_{\vec{k}'} | \varphi_{\vec{k}} \rangle = \langle \Psi_{\vec{k}'} | \Psi_{\vec{k}} \rangle \qquad 1.4$$

utilizing expression 1.3, the asymptotic form of the plane wave.

$$\frac{1}{2\pi}\varphi_{\vec{k}}^{incid.} \cong \frac{i}{k}\left\{\delta\left(1+\frac{\vec{k}\cdot\vec{r}}{kr}\right)\frac{e^{-ikr}}{r} - \delta\left(1-\frac{\vec{k}\cdot\vec{r}}{kr}\right)\frac{e^{ikr}}{r}\right\} \qquad 1.5$$

and the completeness of the spin functions.

We obtain

$$2\pi\left[M(\vec{k}, \vec{k}') - M^{\dagger}(\vec{k}', \vec{k})\right] = ik\int M^{\dagger}(\vec{k}', \vec{k}'')\, M(\vec{k}, \vec{k}'')\, d\omega_{\vec{k}''} \qquad 1.6$$

Writing the M-matrix in the form $\sum \alpha_\nu S_\nu$ where the S_ν are a complete set of spin operators, we then obtain the following integral conditions.

$$4\pi \, \mathrm{Im}\, \alpha_\nu \cdot N = k \int \mathrm{Tr}[S_\nu M(\vec{k},\vec{k}'') M^+(\vec{k}',\vec{k}'') \, d\omega_{\vec{k}''} \qquad 1.7$$

writing these out explicitly we obtain

$$4\pi \, \mathrm{Im}\, a(\theta) = \frac{k}{4} \int d\omega_{\vec{k}''} \, \mathrm{Tr}[M^+(\vec{k}',\vec{k}'') M(\vec{k},\vec{k}'')]$$
$$4\pi \, \mathrm{Im}\, c(\theta) = \frac{k}{4} \int d\omega_{\vec{k}''} \, \mathrm{Tr}[\sigma_{1n}\sigma_{2n} M^+(\vec{k}',\vec{k}'') M(\vec{k},\vec{k}'')]$$
$$4\pi \, \mathrm{Im}\, m(\theta) = \frac{ik}{8} \int d\omega_{\vec{k}''} \, \mathrm{Tr}[(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\vec{n} \, M^+(\vec{k}',\vec{k}'') M(\vec{k},\vec{k}'')]$$
$$4\pi \, \mathrm{Im}\, [g(\theta) - h(\theta)] = \frac{k}{4} \int d\omega_{\vec{k}''} \, \mathrm{Tr}[\sigma_{1K}\sigma_{2K} M^+(\vec{k}',\vec{k}'') M(\vec{k},\vec{k}'')]$$
$$4\pi \, \mathrm{Im}\, [g(\theta) + h(\theta)] = \frac{k}{4} \int d\omega_{\vec{k}''} \, \mathrm{Tr}[\sigma_{1P}\sigma_{2P} M^+(\vec{k}',\vec{k}'') M(\vec{k},\vec{k}'')] \qquad 1.8$$

These are the five integral relations which we referred to previously as limiting the arbitrariness of the functions $a(\theta) \ldots\ldots h(\theta)$.

We may obtain expressions for the functions $a(\theta) \ldots\ldots h(\theta)$ by multiplying both sides of equation 1.2 by S_ν and taking the trace of both sides. In this way we obtain

$$a = \frac{1}{4} \mathrm{Tr}\, M$$
$$c = \frac{1}{4} \mathrm{Tr}\, M\sigma_{1n} = \frac{1}{4} M \sigma_{2n}$$
$$m = \frac{1}{4} \mathrm{Tr}\, M \sigma_{1n}\sigma_{2n} \qquad 1.9$$
$$g = \frac{1}{4} \mathrm{Tr}\, M(\sigma_{1P}\sigma_{2P} + \sigma_{1K}\sigma_{2K})$$
$$h = \frac{1}{4} \mathrm{Tr}\, M(\sigma_{1P}\sigma_{2P} - \sigma_{1K}\sigma_{2K})$$

In order to compute the traces, a specific representation of the matrices must be introduced. The simplest representation to work with is the single particle representation. Here the basis vectors are

$$\begin{aligned} \alpha_1 \alpha_2 &= |11\rangle = \phi_{11} \\ \alpha_1 \beta_2 &= |1-1\rangle = \phi_{1-1} \\ \beta_1 \alpha_2 &= |-11\rangle = \phi_{-11} \\ \beta_1 \beta_2 &= |-1-1\rangle = \phi_{-1-1} \end{aligned} \qquad 1.10$$

where α_n, β_n are the usual spin up and down functions of the n^{th} particle. Our basis vectors are therefore denoted by a couple, while operators will have matrix elements denoted by a pair of couples, thus $O = O_1 O_2$ will have matrix elements

$$\langle ab|O_1 O_2|cd\rangle = \langle a|O_1|c\rangle\langle b|O_2|d\rangle \qquad 1.11$$

Then, taking the usual representation of the Pauli matrices, letting the 3-axis point along the direction of the incident beam, we get for the various operators which appear in equations 1.2 and 1.9

$$1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sigma_{1\eta} = \begin{pmatrix} 0 & 0 & -ie^{-i\phi} & 0 \\ 0 & 0 & 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 & 0 & 0 \\ 0 & ie^{i\phi} & 0 & 0 \end{pmatrix}$$

$$\sigma_{2\eta} = \begin{pmatrix} 0 & -ie^{-i\phi} & 0 & 0 \\ ie^{i\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & -ie^{-i\phi} \\ 0 & 0 & ie^{i\phi} & 0 \end{pmatrix}$$

$$\sigma_{1\eta}\sigma_{2\eta} = \begin{pmatrix} 0 & 0 & 0 & -e^{-2i\phi} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -e^{2i\phi} & 0 & 0 & 0 \end{pmatrix}$$

$$(\sigma_{1\rho}\sigma_{2\rho} + \sigma_{1\kappa}\sigma_{2\kappa}) = \begin{pmatrix} 1 & 0 & 0 & -e^{-2i\phi} \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ e^{2i\phi} & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \text{Cos}\ \theta & \text{Sin}\ \theta\ e^{-i\phi} & \text{Sin}\ \theta & -\text{Cos}\ \theta\ e^{-2i\phi} \\ \text{Sin}\ \theta & -\text{Cos}\ \theta & -\text{Cos}\ \theta & -\text{Sin}\ \theta\ e^{-i\phi} \\ \text{Sin}\ \theta\ e^{i\phi} & -\text{Cos}\ \theta & -\text{Cos}\ \theta & -\text{Sin}\ \theta\ e^{-i\phi} \\ -\text{Cos}\ \theta\ e^{2i\phi} & -\text{Sin}\ \theta\ e^{i\phi} & -\text{Sin}\ \theta\ e^{i\phi} & \text{Cos}\ \theta \end{pmatrix}$$

$$= (\sigma_{1\rho}\sigma_{2\rho} - \sigma_{1\kappa}\sigma_{2\kappa})$$

The M-matrix elements most easily expressible in terms of phase shifts are those in the singlet triplet representation. In this representation, the M-matrix may be written

$$\begin{pmatrix} M_{11} & M_{10} & M_{1-1} & 0 \\ M_{01} & M_{00} & M_{0-1} & 0 \\ M_{-11} & M_{-10} & M_{-1-1} & 0 \\ 0 & 0 & 0 & M_{ss} \end{pmatrix} \qquad 1.13$$

the subscripts $1, 0, -1, S$ refer to the basis

$$\begin{aligned} \chi_1 &= \alpha_1\alpha_2 \\ \chi_{-1} &= \beta_1\beta_2 \\ \chi_0 &= \frac{1}{\sqrt{2}}(\alpha_1\beta_2 + \beta_1\alpha_2) \\ \chi_S &= \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2) \end{aligned} \qquad 1.14$$

To obtain the traces needed in equation 1.9 we transform to the single particle representation by means of

$$\langle ab|M|cd\rangle = \langle ab|\mu\rangle\langle\mu|M|\nu\rangle\langle\nu|cd\rangle \qquad 1.15$$

using equation 1.14 to evaluate the Clebsh Gordon coefficients. We obtain

$$M(\theta\phi) = \begin{pmatrix} M_{11} & \frac{1}{\sqrt{2}}M_{10} & \frac{1}{\sqrt{2}}M_{10} & M_{1-1} \\ \frac{1}{\sqrt{2}}M_{01} & \frac{1}{2}(M_{00}+M_{ss}) & \frac{1}{2}(M_{00}-M_{ss}) & \frac{1}{\sqrt{2}}M_{0-1} \\ \frac{1}{\sqrt{2}}M_{01} & \frac{1}{2}(M_{00}-M_{ss}) & \frac{1}{2}(M_{00}+M_{ss}) & \frac{1}{\sqrt{2}}M_{0-1} \\ M_{-11} & \frac{1}{\sqrt{2}}M_{-10} & \frac{1}{\sqrt{2}}M_{-10} & M_{-1-1} \end{pmatrix} \qquad 1.16$$

Comparing this with the matrices equation 1.12 out of which it must be built, we obtain the following symmetrys.

$$M_{11}(\theta,-\phi) = M_{-1-1}(\theta,\phi) \quad , \quad M_{01}(\theta,\phi) = -M_{0-1}(\theta,-\phi)$$
$$M_{1-1}(\theta,-\phi) = M_{-11}(\theta,\phi) \quad , \quad M_{10}(\theta,\phi) = -M_{-10}(\theta,-\phi) \qquad 1.17$$

These four together with M_{00} and M_{ss} give six M-matrix elements. Using equation 1.9 we may obtain the five functions $a \ldots\ldots h$ in terms of the singlet triplet M-matrix elements. Inverting these equations and eliminating the functions $a \ldots\ldots h$ we obtain the following relationship between the $M_{\mu\nu}$

$$\frac{\sqrt{2}}{\sin\theta}(M_{10}+M_{01}) = \frac{1}{\cos\theta}(M_{11}-M_{1-1}-M_{00}) \quad \text{with } \phi = 0 \qquad 1.18$$

The functions $a \ldots\ldots h$ in terms of the $M_{\mu\nu}$ are given by

$$a = \frac{1}{4}(2M_{11}+M_{00}+M_{ss}) \quad , \quad h = \frac{1}{4}\left(\frac{1}{\cos\theta}\right)(M_{11}-M_{1-1}-M_{00})$$
$$c = \frac{1}{4}\frac{i}{\sqrt{2}}(M_{10}-M_{01}) \qquad\qquad = \frac{1}{4}\left(\frac{\sqrt{2}}{\sin\theta}\right)(M_{10}+M_{01})$$
$$m = \frac{1}{4}(-2M_{1-1}+M_{00}-M_{ss})$$
$$\text{with } \phi = 0 \qquad 1.18a$$
$$g = \frac{1}{4}(M_{11}+M_{1-1}-M_{ss})$$

From Section I equations 5.3, 5.4, 6.3, 6.4, noticing that $\vec{s}' = \vec{k}$ we obtain for the quantities of physical interest.

$$I_0 = Tr\, MM^{+}$$
$$I_0 P = \frac{1}{4} Tr(MM^{+}\sigma_{1n})$$
$$I_0 D = \frac{1}{4} Tr(M\sigma_{1n}M^{+}\sigma_{1n})$$
$$I_0 R = \frac{1}{4} Tr(M\sigma_{1s}M^{+}\sigma_{1k})$$
$$I_0 C_{kp} = \frac{1}{4} Tr(MM^{+}\sigma_{1k}\sigma_{2p}) \qquad 1.19$$
$$I_0 C_{nn} = \frac{1}{4} Tr(MM^{+}\sigma_{1n}\sigma_{2n})$$
$$I_0 A = \frac{1}{4} Tr(M\sigma_{1s}M^{+}\sigma_{1p})$$
$$I_0 R' = \frac{1}{4} Tr(M\sigma_{1k}M^{+}\sigma_{1p})$$
$$I_0 A' = \frac{1}{4} Tr(M\sigma_{1k}M^{+}\sigma_{1k})$$

Now it is easily seen that

$$\vec{\ell_2} = -\sin\frac{\theta}{2}\,\vec{K} + \cos\frac{\theta}{2}\,\vec{P}$$

and

$$\vec{S} = \sin\frac{\theta}{2}\,\vec{P} + \cos\frac{\theta}{2}\,\vec{K}$$

whence we obtain the following. With $\phi = 0$

$$I_0 = \frac{1}{2}\left(|M_{11}|^2 + |M_{10}|^2 + |M_{01}|^2 + |M_{1-1}|^2\right) + \frac{1}{4}\left(|M_{00}|^2 + |M_{ss}|^2\right)$$

$$I_0 P = \frac{\sqrt{2}}{4}\,\mathrm{Re}\, i\,(M_{10} - M_{01})\,(M_{11} - M_{1-1} + M_{00})^*$$

$$I_0 D = \frac{1}{2}\,\mathrm{Re}\left\{(M_{11} - M_{1-1})M_{00}^* + (M_{11} + M_{1-1})M_{ss}^* - 2M_{01}M_{10}^*\right\}$$

$$I_0 R = \frac{1}{2}\cos\frac{\theta}{2}\,\mathrm{Re}\left\{\left[M_{00} + (\cos\theta - 1)\sqrt{2}\,\frac{M_{10}}{\sin\theta}\right]\left[M_{11} + M_{1-1} + M_{ss}\right]^* + \frac{\sqrt{2}}{\sin\theta}(M_{10} + M_{01})M_{ss}^*\right\}$$

$$I_0 C_{KP} = \left(\frac{1}{2\sin\theta}\right)\left(|M_{01}|^2 - |M_{10}|^2\right) \qquad 1.20$$

$$I_0 C_{nn} = \frac{1}{2}\left(|M_{ss}|^2 + |M_{11} + M_{1-1}|^2\right)$$

also

$$I_0 = |a|^2 + |m|^2 + 2|c|^2 + 2|g|^2 + 2|h|^2$$

$$I_0 P = 2\,\mathrm{Re}\, c^*(a+m)$$

$$I_0(1-D) = 4\left(|g|^2 + |h|^2\right)$$

$$I_0 R = \left\{|a|^2 - |m|^2 - 4\,\mathrm{Re}\, hg^*\right\}\cos\frac{\theta}{2} + 2\,\mathrm{Re}\, i\,c\,(a-m)^*\sin\frac{\theta}{2}$$

$$I_0 C_{KP} = 4\,\mathrm{Re}\, i\,c\,h^*$$

$$I_0(1 - C_{nn}) = |a-m|^2 + 4|g|^2 \qquad 1.21$$

$$I_0 A = \left\{-\sin\frac{\theta}{2}\left(|a|^2 - |m|^2 - 4\,\mathrm{Re}\, g^*h\right) + \cos\frac{\theta}{2}\cdot 2\,\mathrm{Re}\, i\,c\,(a-m)^*\right\}$$

$$I_0 A' = \left\{\sin\frac{\theta}{2}\; 2\,\mathrm{Re}\, i\,c\,(a-m)^* + \cos\frac{\theta}{2}\left(|a|^2 - |m|^2 + 4\,\mathrm{Re}\, g^*h\right)\right\}$$

$$I_0 R' = \left\{-\sin\frac{\theta}{2}\left(|a|^2 - |m|^2 - 4\,\mathrm{Re}\, g^*h\right) + \cos\frac{\theta}{2}\; 2\,\mathrm{Re}\, i\,c\,(a-m)^*\right\}$$

It is easily seen that R A R$'$ and A$'$ are not independent, but are related to each other through

$$\tan\frac{\theta}{2} = -\frac{R' + A'}{R - A'} \qquad 1.22$$

Let us now obtain the singlet triplet M-matrix elements in terms of phase shifts.

2. The M-matrix Elements for the Singlet Triplet Representation in Terms of Phase Shifts

We will first treat the nucleons as distinguishable, and omit coulomb effects. Later we will show how to modify our results to account for these two effects. The M-matrix is defined through the relation.

$$\psi_i = e^{i\vec{k}\cdot\vec{r}} a_i + \frac{e^{ikr}}{r} \sum_j M_{ij} a_j \qquad 2.1$$

where we have used $\vec{k}\hbar$ rather than $\vec{p}$ for the centre of mass momentum.

So we have

$$f_i(\theta\phi) = \sum_j M_{ij} a_j \qquad 2.2$$

Where evidently a_j are the amplitudes of the spin state in the plane wave, see equation 3.1 in Section I. The $f_i(\theta\phi)$ are the amplitudes in the scattered wave. We wish to express the M-matrix elements in terms of the phase shifts. Now the phase shifts are related directly to the S-matrix. We therefore seek the relationship between the M-matrix and the S-matrix. The S-matrix is equal to the unit matrix plus the R-matrix. Where the R-matrix is defined through the equation

$$f'(l s m_l m_s) = \sum R(l s m_l m_s ; l' s' m_l' m_s') \, g(l' s' m_l' m_s') \qquad 2.3$$

where the amplitudes $g(l s m_l m_s)$ and $f'(l s m_l m_s)$ are related to the converging part of the incident plane wave and the scattered wave respectively, and are given by

$$\psi^{inc.}_{conv.} \approx -r^{-1} \sum \exp -i(kr - l\tfrac{\pi}{2}) g(l s m_l m_s) Y_l^{m_l} \chi_s^{m_s} \qquad 2.4$$

$$\psi^{sc.} \approx -r^{-1} \sum \exp i(kr - l\tfrac{\pi}{2}) f'(l s m_l m_s) Y_l^{m_l} \chi_s^{m_s}$$

where $\approx$ denotes equality asymptoticaly.

by looking at the expansion of a plane wave in the $l s m_l m_s$ representation, we see that

$$g(l s m_l m_s) = -\frac{\pi}{k}^{\frac{1}{2}} (2l+1)^{\frac{1}{2}} i^{l+1} a_s^{m_s} \delta^0_{m_l} \qquad 2.5$$

Evidently the $f'(l s m_l m_s)$ are related to the $f_\lambda(\theta\phi) = f_{s m_s}(\theta\phi)$ by

$$\psi_{sc} \approx r^{-1} e^{ikr} \sum_{s m_s} f_{s m_s}(\theta\phi) \chi_s^{m_s} \qquad 2.6$$

so that

$$f_{s m_s} = \sum e^{-il\frac{\pi}{2}} f'(l s m_l m_s) Y_l^{m_l} \qquad 2.7$$

so $\sum M_{sm_s;s'm_s'} a_{s'm_s'} =$

$$\sum e^{-il\frac{\pi}{2}} f'(lsm_lm_s) Y_l^{m_l}$$

$$= \sum e^{-il\frac{\pi}{2}} R(lsm_lm_s;l's'm_l'm_s') g(l's'm_l'm_s') Y_l^{m_l}$$

$$= \sum e^{-il\frac{\pi}{2}} R(lsm_lm_s;l's'm_l'm_s') [-\frac{\pi}{k}\delta_{m_l'0}(2l'+1)^{\frac{1}{2}} i^{l'+1} a_{s'}^{m_s'} \delta_{m_l0}].$$

So we get, noting that

$$e^{il'\frac{\pi}{2}} = i^{l'}$$

$$(ik)^{-1} \sum e^{-il\frac{\pi}{2}} R(lsm_lm_s;l's'0m_s') [\pi(2l'+1)]^{\frac{1}{2}} e^{il'\frac{\pi}{2}} \qquad 2.8$$

$$= \sum M(lm_lsm_s;s'm_s') Y_l^{m_l} = M_{sm_s;s'm_s'} \qquad 2.9$$

with

$$M(lsm_lm_s;s'm_s') = (ik)^{-1} e^{-il\frac{\pi}{2}} \sum R(lsm_lm_s;l's'0m_s') e^{il'\frac{\pi}{2}} [\pi(2l'+1)]^{\frac{1}{2}} \qquad 2.10$$

The most convenient phase shifts are those related to the R-matrix in the $lsjm_j$ representation. These matrix elements are related to those in the lsm_lm_s representation by means of the equation

$$\langle lsm_lm_s|R|l's'm_l'm_s'\rangle = \sum \langle lsm_lm_s|lsjm_j\rangle \langle lsjm_j|R|l's'j'm_j'\rangle \langle l's'j'm_j'|l's'm_l'm_s'\rangle \qquad 2.11$$

The quantities $\langle lsm_lm_s|lsjm_j\rangle$ are the Clebsh Gordon coefficients given in ref: 36 and there denoted by

$$C_{ls}(jm_j;lsjm_j)$$

So

$$R(lsm_lm_s;l's'm_l'm_s') = \sum{}' \langle lsm_lm_s|lsjm_j\rangle \langle l's'm_l'm_s'|l's'j'm_j'\rangle R(jm_jls;j'm_j'l's') \qquad 2.12$$

Where the prime indicates that there is no summation over

$l s l' s'$. The convenience of this representation derives from the fact that $j\, m_j$ are constants of the motion by virtue of the complete spherical symmetry of the Hamiltonian. S is a constant of the motion because of isospin conservation, parity conservation and the antysymmetry of the wave function (for nucleons).

Further the complete spherical symmetry of the problem, that is the fact that a complete rotation of the system (including spins) changes nothing in the problem, implies that the phase shifts are independent of m_j . The non-zero matrix elements may then be written

$$\begin{aligned} &R(l0l\,m_j;\,l0l\,m_j) = R_l \\ &R(l1j\,m_j;\,l1j\,m_j) = R_{lj} \qquad \text{with } l = \begin{matrix} j-1 \\ j+1 \\ j \end{matrix} \\ &R(j\pm 1\,1\,j\,m_j;\,j\mp 1\,1\,j\,m_j) = R^j_{\pm} = R^j \end{aligned} \qquad 2.13$$

The equality of R^j_+ and R^j_- is a result of the symmetry of the S-matrix, which fact is implied by time reversal invariance.

We then can write

$$\begin{aligned} R(l s m_l m_s;\, l' s' m_l' m_s') = {\sum}' &\langle l s m_l m_s | l s j m_j \rangle \\ \langle l' s' m_l' m_s' | l' s' j' m_j' \rangle &[R_l \delta_{s0} \delta_{ll'} \delta_{lj} + \\ R_{lj} \delta_{s1} \delta_{ll'} + R^j \delta_{s1} &\delta_{l\,j\pm 1} \delta_{l\,j\mp 1}] \end{aligned} \qquad 2.14$$

For the singlet case this gives

$$\begin{aligned} R(l0m_l0;\, l'0m_l'0) &= [\langle l0m_l0 | l0l\,m_l \rangle]^2 \\ R_l \delta_{m_l m_l'} &= R_l \delta_{ll'} \delta_{m_l m_l'}. \end{aligned} \qquad 2.15$$

And for the triplet case

$$R(l1m_l m_s; l'1m_l'm_s') = \sum{}' \langle l1m_l m_s|l1jm_j\rangle \langle l'1m_l'm_s'|l'1jm_j\rangle [R_{lj}\delta_{ll'} + R^j \delta_{lj\pm1}\delta_{l'j\mp1}] \quad 2.16$$

In the triplet state $l = j, j\pm1$, the second has parity odd with respect to the first.

So $l' = l = j, j\pm1$ gives the diagonal elements, and

$l' = 2j - l = l\pm2 = j\pm1$ gives the off diagonal elements.

There can be no elements between the states $l' = l\pm1$ since these states have opposite parity, and parity is assumed a good quantum number. We are further only interested in elements for which $m_l' = 0$ see equation 2.10. For these elements $m_l = m_s' - m_s$. We therefore get

For the Singlet Case

$$R(l000; l'000) R_l \delta_{ll'} \quad 2.17$$

and for the Triplet case

$$R(l1m_s'-m_s; l10m_s') = \sum \langle l1m_s'-m_s m_s|l1jm_s'\rangle \langle l10m_s'|l1jm_s'\rangle R_{lj} \quad 2.18$$

$$R(l1m_s'-m_s m_s; l'10m_s) = \sum \langle l1m_s'-m_s m_s|l1jm_s'\rangle \langle l'10m_s'|l'1jm_s'\rangle R^j \delta_{lj\pm1}\delta_{l'j\mp1}$$

We can now write down the non zero elements of the matrix

Singlet

$$M(l000; 00) = (ik)^{-1} \exp -i\frac{l\pi}{2} \sum R(l000; l'00) \exp il'\frac{\pi}{2}[\pi(2l'+1)]^{\frac{1}{2}} = (ik)^{-1} R_l [\pi(2l+1)]^{\frac{1}{2}} \quad 2.19$$

Triplet

$$M(l1\,m_s'-m_s\,m_s;1m_s') = (ik)^{-1}e^{-il\frac{\pi}{2}}\sum R(l1\,m_s'-m_s;l'10\,m_s')$$
$$e^{il'\frac{\pi}{2}}[\pi(2l'+1)]^{\frac{1}{2}}$$
$$= (ik)^{-1}[\pi(2l+1)]^{\frac{1}{2}}\sum_{j=l-1}^{l+1}\langle l1\,m_s'-m_s|l1j\,m_s'\rangle \qquad 2.20$$
$$\langle l10\,m_s'|l1j\,m_s'\rangle R_{lj} - \sum[\pi(2l'+1)]^{\frac{1}{2}}\langle l1\,m_s'-m_s\,m_s|l1j\,m_s'\rangle$$
$$\langle l10\,m_s'|l'1j\,m_s'\rangle R_j$$

with $l' = l\pm 2 = 2j-l = j\pm 1$

The above refers specifically to the case of distinguishable particles. If they are identical, as in the case of p-p scattering, we must antisymmetrize the wave function. The scattered wave for the case of distinguishable particles is given by equation 2.1.

$$\frac{e^{ikr}}{r} f_{sm_s} = \frac{e^{ikr}}{r}\sum M_{sm_s;s'm_s'}\,a_{s'}^{m_s'}\,\chi_{s'}^{m_s'}$$

We write the antisymmetrized scattered wave as follows.

$$= \frac{e^{ikr}}{r}\sum(1-TS)M_{sm_s;s'm_s'}\,a_{s'}^{m_s'}\,\chi_{s'}^{m_s'} \qquad 2.21$$

Here T and S are the spin and space exchange operators respectively. The above form takes into account both the antisymmetry of the wave function, and the fact that the particles are indistinguishable, i.e., the fact that we observe both the recoiling and scattered particles. It is evident from the above, that we may consider the particles as distinguishable, provided we suitably antisymmetrize the M-matrix. The antisymmetrized M-matrix is given by

$$M = (1-TS)M \qquad 2.22$$

From now on we will omit the superscript "a" for simplicity, it being understood.

An explicit form for the spin and space exchange operators is

$$S = -\frac{1}{4}\,(1+\vec{\sigma}_1\cdot\vec{\sigma}_2)\,(1+\vec{\tau}_1\cdot\vec{\tau}_2) \qquad 2.23$$
$$T = \frac{1}{2}\,(1+\vec{\sigma}_1\cdot\vec{\sigma}_2)$$

Here $\vec{\tau}_1$ and $\vec{\tau}_2$ are the isospin operators for the two particles. The S and T operators then have matrix elements given by

$$\langle s'l'|T|sl\rangle = (-1)^{s+1}\,\delta_{ss'}\,\delta_{ll'} \qquad 2.24$$
$$\langle s'l'|S|sl\rangle = (-1)^{l}\,\delta_{ss'}\,\delta_{ll'}$$

So only those parts of the M-matrix will be non-zero for which $l+s$ is even, i.e., triplet odd and singlet even.

3. <u>The S-matrix for Nucleon-Nucleon Scattering</u>

Following Blatt and Biedenharn (37) we write down the most general solution of the equations of motion for the scattering of two distinguishable nucleons, for a given $j m_j l s$. Here coulomb effects are omitted.

Asymptoticaly

$$\psi(j m_j l s) = \frac{1}{r}\,|j m_j l s\rangle \left\{ A(j m_j l s)\, e^{-i\left(kr - \frac{l\pi}{2}\right)} - B(j m_j l s)\, e^{i\left(kr - \frac{l\pi}{2}\right)} \right\} \qquad 3.1$$

The amplitudes of the diverging wave are related to those of the converging wave through.

$$B(j m_j l s) = \sum A(j' m_j' l' s')\, S(j m_j l s; j' m_j' l' s') \qquad 3.2$$

The operator whose elements are $S(j m_j l s; j' m_j' l' s')$ is called the scattering matrix, and it describes the effect of the interaction on the waves converging on the scattering centre.

There are many properties of the scattering matrix which are immediately evident.

(1) It cannot connect states of different $j\, m_j\, s$ or parity, since these are good quantum numbers, and the existence of such S-matrix elements would imply scattering between states with different good quantum numbers. Hence the matrix elements take the form.

$$S(j m_j l s ; j' m_j' l' s') = \delta_{jj'} \delta_{m_j m_j'} \delta_{ss'} S_{ll'} \qquad 3.3$$

where $l' = l \pm 2$

(2) The complete spherical symmetry of the S-matrix, i.e., with respect to the rotation operator $e^{\wedge \vec{J} \theta}$ implies that the S-matrix does not depend on m_j.

(3) The S-matrix is unitary, this is so because all particles are scattered elastically, that is there are no particles lost from this channel due to inelastic collisions of any kind.

(4) The S-matrix is symmetric. This is so because of time reversal invariance of the interaction. The direction of time does not enter into the problem.

With these restrictions, the S-matrix for a given j and any m_j may be written

$$\begin{pmatrix} S_{jj} & 0 & 0 & 0 \\ 0 & S_{j-1\,j} & S^{j} & 0 \\ 0 & S^{j} & S_{j+1\,j} & 0 \\ 0 & 0 & 0 & S_{j} \end{pmatrix} \qquad 3.4$$

Here $S_{jj} = e^{2i\delta_{jj}}$ and $S_j = e^{2i\delta_j}$, where δ_j, δ_{jj} are real. We are therefore left with the sub matrix.

$$\begin{pmatrix} S_{j-1\,j} & S_j \\ S_j & S_{j+1\,j} \end{pmatrix} \qquad 3.5$$

This is some general unitary symmetric matrix. The most general 2 X 2 unitary symmetric matrix is described by three real paramaters.

One way of writing it is as follows

$$S = u^{-1} \exp(2i\Delta)\, u \qquad 3.6$$

Where

$$\Delta = \begin{pmatrix} \delta_{j\alpha} & 0 \\ 0 & \delta_{j\beta} \end{pmatrix}; \quad u = \begin{pmatrix} \cos\epsilon_j & \sin\epsilon_j \\ -\sin\epsilon_j & \cos\epsilon_j \end{pmatrix} \qquad 3.7$$

The S-matrix is then given by

$$\begin{pmatrix} e^{2i\delta_{jj}} & 0 & 0 & 0 \\ 0 & \cos^2\epsilon_j\, e^{2i\delta_{j\alpha}} + \sin^2\epsilon_j\, e^{2i\delta_{j\beta}} & \frac{1}{2}\sin 2\epsilon_j\,(e^{2i\delta_{j\alpha}} - e^{2i\delta_{j\beta}}) & 0 \\ 0 & \frac{1}{2}\sin 2\epsilon_j\,(e^{2i\delta_{j\alpha}} - e^{2i\delta_{j\beta}}) & \sin^2\epsilon_j\, e^{2i\delta_{j\alpha}} + \cos^2\epsilon_j\, e^{2i\delta_{j\beta}} & 0 \\ 0 & 0 & 0 & e^{2i\delta_{j}} \end{pmatrix} \qquad 3.8$$

Let us write out the most general wave function for the coupled part of the S-matrix, i.e., the triplet, parity $(-1)^{j-1}$ part of the S-matrix. This is a 2-column vector, with elements $\psi(j+1)$ and $\psi(j-1)$. Suppressing the j m_j and s = 1 labels, we have the elements given by

$$\psi(j-1) \approx r^{-1}\,|j-1\rangle \left\{ A(j-1)\, e^{-i(kr - \overline{j-1}\frac{\pi}{2})} - B(j-1)\, e^{i(kr - \overline{j-1}\frac{\pi}{2})} \right\} \qquad 3.9$$

$$\psi(j+1) \approx r^{-1}\,|j+1\rangle \left\{ A(j+1)\, e^{-i(kr - \overline{j+1}\frac{\pi}{2})} - B(j+1)\, e^{i(kr - \overline{j+1}\frac{\pi}{2})} \right\}$$

with

$$B(j-1) = A(j-1)\, S_{j-1} + A(j+1)\, S^{j}$$
$$B(j+1) = A(j+1)\, S_{j+1} + A(j-1)\, S^{j} \qquad 3.10$$

In general, because of these equations

$$B(j \pm 1) \neq e^{2i\delta} A(j \pm 1) \qquad 3.11$$

This means that particles will be scattered between the channels $l = j+1$ and $l = j-1$. In general there will be a net loss in one channel and a corresponding gain in the other. If however we choose

$$\frac{A(j+1)}{A(j-1)} = \frac{B(j+1)}{B(j-1)} \qquad 3.12$$

there will be no net loss to either channel. In such a case the wave is an eigenwave of the scattering matrix. The effect of the scattering is merely to produce a change in phase of the outgoing wave with respect to the incoming wave. There are two such solutions denoted by α and β. The ratios are given by

$$\frac{A^{\alpha}(j+1)}{A^{\alpha}(j-1)} = \tan \epsilon_j \quad , \quad \frac{A^{\beta}(j+1)}{A^{\beta}(j-1)} = -\cot \epsilon_j \qquad 3.13$$

with

$$B^{\alpha}(j-1) = e^{2i\delta_{j\alpha}} A^{\alpha}(j-1)$$
$$B^{\beta}(j-1) = e^{2i\delta_{j\beta}} A^{\beta}(j-1) \qquad 3.14$$

The two eigenwaves are

$$\Psi^{\alpha} = \psi^{\alpha}(j+1) + \psi^{\alpha}(j-1)$$
$$\approx r^{-1} [\, |j+1\rangle \sin \epsilon_j\, e^{-i(kr - \frac{j+1\pi}{2})} + |j-1\rangle\, e^{-i(kr - \frac{j-1\pi}{2})}$$
$$- r^{-1} e^{2i\delta_{j\alpha}} [\, |j+1\rangle \sin \epsilon_j\, e^{i(kr - \frac{j+1\pi}{2})} + |j-1\rangle\, e^{-i(kr - \frac{j-1\pi}{2})}$$

and

$$\Psi^{\beta} = \psi^{\beta}(j+1) + \psi^{\beta}(j-1)$$

$$= u^{-1}\left[|j+1\rangle \cos\epsilon_j \, e^{-i(kr - \overline{j+1}\frac{\pi}{2})} - |j-1\rangle \sin\epsilon_j \, e^{-i(kr - \overline{j-1}\frac{\pi}{2})}\right]$$

$$- e^{2i\delta_{j\beta}} u^{-1}\left[|j+1\rangle \cos\epsilon_j \, e^{i(kr - \overline{j+1}\frac{\pi}{2})} - |j-1\rangle \sin\epsilon_j \, e^{i(kr - \overline{j-1}\frac{\pi}{2})}\right] \qquad 3.15$$

There are a few additional remarks that we should make concerning the S-matrix.

(1) We can add any multiple of π to any or all of $\delta_{j\alpha}, \delta_{j\beta}, \epsilon_j$ without altering the value of the S-matrix.

(2) No physical meaning has so far been attached to the labels α and β. We will now attempt to do so. We let the bombarding energy of the incoming particle go to 0. Near zero energy, the difference between the centrifugal potential barriers for $\ell = j-1$ and $\ell = j+1$ becomes so significant as to uncouple the two states. This means that they become separately eigenstates. From equation 3.13 this means that $\epsilon_j = 0$ or $\frac{\pi}{2}$. We define α and β so that as $E \to 0, \epsilon_j \to 0$ i.e. $\alpha \to j-1$ and $\beta \to j+1$. In future we therefore use $\alpha = j-1$ and $\beta = j+1$ as the labels for the $\delta_{j\alpha}$ and $\delta_{j\beta}$ phase shifts. We then can rewrite the S-matrix elements.

$$S_{jj} = e^{2i\delta_{jj}}$$

$$S_j = e^{2i\delta_j}$$

$$S_{j-1\,j} = \cos^2\epsilon_j \, e^{2i\delta_{j-1\,j}} + \sin^2\epsilon_j \, e^{2i\delta_{j+1\,j}} \qquad 3.11$$

$$S_{j+1\,j} = \sin^2\epsilon_j \, e^{2i\delta_{j-1\,j}} + \cos^2\epsilon_j \, e^{2i\delta_{j-1\,j}}$$

$$S^j = \frac{1}{2}\sin 2\epsilon_j \left(e^{2i\delta_{j-1\,j}} - e^{2i\delta_{j+1\,j}}\right)$$

There is another way of writing the S-matrix which makes it easier to interpret the three parameters. We define it in exactly the same way for the singlet case and also for the triplet parity $(-1)^j$ case, however for the triplet, parity $(-1)^{j+1}$ case, we define it through

$$S = \exp i\bar{\Delta} \exp 2i\bar{\epsilon}_j \exp i\bar{\Delta} \qquad 3.12$$

$$\text{with } \bar{\Delta} = \begin{pmatrix} \bar{\delta}_{j-1\,j} & 0 \\ 0 & \bar{\delta}_{j+1\,j} \end{pmatrix};\ \bar{\epsilon}_j = \begin{pmatrix} 0 & \bar{\epsilon}_j \\ \bar{\epsilon}_j & 0 \end{pmatrix} \qquad 3.13$$

This gives for the elements of the S-matrix

$$\begin{aligned} S_j &= e^{2i\delta_j} \\ S_{jj} &= e^{2i\delta_{jj}} \\ S_{j-1\,j} &= e^{2i\bar{\delta}_{j\,j-1}} \cos 2\bar{\epsilon}_j \\ S_{j+1\,j} &= e^{2i\bar{\delta}_{j\,j+1}} \cos 2\bar{\epsilon}_j \\ S^j &= i \sin 2\bar{\epsilon}_j\, e^{i(\bar{\delta}_{j\,j-1} + \bar{\delta}_{j\,j+1})} \end{aligned} \qquad 3.16$$

This representation of the S-matrix, is called the barred representation. As will be indicated in the section on coulomb effects, it is particularly useful where we have a mixture of nuclear and coulomb forces, as in p-p scattering. The equations connecting the two representation, are given below.

$$\begin{aligned} \delta_{j\,j+1} + \delta_{j\,j-1} &= \bar{\delta}_{j\,j+1} + \bar{\delta}_{j\,j-1} \\ \sin(\bar{\delta}_{j\,j-1} - \bar{\delta}_{j\,j+1}) &= \tan 2\bar{\epsilon}_j / \tan 2\epsilon_j \\ \sin(\delta_{j\,j-1} - \delta_{j\,j+1}) &= \sin 2\bar{\epsilon}_j / \sin 2\epsilon_j \\ \bar{\delta}_{jj} &= \delta_{jj} \\ \bar{\delta}_j &= \delta_j \end{aligned} \qquad 3.17$$

Let us write out the wave function equation 3.9 in the B.B. and barred representations for comparison. They are

B.B. Representation

$$\psi(j-1) = r^{-1} e^{-i(kr-(j-1)\frac{\pi}{2})} A(j-1)|j-1\rangle + r^{-1} e^{i(kr-(j-1)\frac{\pi}{2})}|j-1\rangle \cdot \{A(j-1)[\cos^2\epsilon_j e^{2i\delta_{jj-1}} + \sin^2\epsilon_j e^{2i\delta_{jj+1}}] + \tfrac{1}{2} A(j+1)\sin 2\epsilon_j \cdot [e^{2i\delta_{jj-1}} - e^{2i\delta_{jj+1}}]\} \qquad 3.18$$

with a similar expression for $\psi(j+1)$

Barred Representation

$$\psi(j-1) = r^{-1} e^{-i(kr-(j-1)\frac{\pi}{2})} A(j-1)|j-1\rangle - r^{-1} e^{i(kr-(j-1)\frac{\pi}{2})}|j-1\rangle \cdot \{e^{2i\bar{\delta}_{jj-1}}\cos 2\bar{\epsilon}_j A(j-1) + i\sin 2\bar{\epsilon}_j e^{i(\bar{\delta}_{jj-1}+\bar{\delta}_{jj+1})} A(j+1)\} \qquad 3.19$$

with a similar expression for $\psi(j+1)$

The B.B. phase shifts and mixing paramater may be interpreted in terms of the eigenwaves of the S-matrix. The mixing paramater in the quantity which determines the relative amounts of $j-1$ and $j+1$ wave necessary in order to have no particles scattered out of that channel. The B.B. phase shift, is then the shift in phase which occurs during such a scattering. The barred phase shifts on the other hand give the shift in phase of that part of the outgoing $j\pm1$ wave amplitudes which derive from the incoming $j\pm1$ channels, with respect to the incoming $j\pm1$ wave amplitudes. Also the mixing paramater $\bar{\epsilon}_j$ gives a measure of the extent to which ℓ is not conserved. A value of $\bar{\epsilon}_j = 0$ would mean that ℓ is conserved, and the further we get from $\bar{\epsilon}_j = 0$ the larger the degree of nonconservation of ℓ .

4. Coulomb Effects (38)

In the scattering of protons on protons, we must include in addition to nuclear effects, the contribution to the scattering of the coulomb repulsion between the protons. Consider the coulomb scattering of two particles of the same mass and unit charge each. The Schrodinger equation for the scattering is given by

$$[\nabla^2 + k^2(1 - \frac{n}{r})]\psi = 0 \qquad 4.1$$

with $n = \frac{e^2 \mu}{k \hbar^2} = \frac{e^2}{\hbar v}$

We try a solution of the form

$$u_c = e^{ikz} F(\vec{r}) \qquad 4.2$$

and get

$$[\nabla^2 + 2ik\frac{\partial}{\partial z} - \frac{nk}{r}]F = 0 \qquad 4.3$$

This has a solution $F(\zeta)$ with $\zeta = r - z$

The equation

$$[\zeta \frac{d^2}{d\zeta^2} + \frac{d}{d\zeta} - k(i\zeta\frac{d}{d\zeta} + \frac{n}{2})]F(\zeta) = 0 \qquad 4.4$$

There are two independent solutions of this equation, which we denote by

$$W_1(-in, 1, ik\zeta)$$

$$W_2(-in, 1, ik\zeta) \qquad 4.5$$

For large values of r these are given by

$$W_1 \approx \frac{(-ik\zeta)^{in}}{\Gamma(1+in)}\, g(-in,-in,-ik\zeta) \qquad 4.6$$

$$W_2 \approx \frac{(ik\zeta)^{-in-1}}{\Gamma(-in)}\, e^{ik\zeta}\, g(1+in,1+in,ik\zeta)$$

with

$$g(\alpha\beta z) \approx 1 + \frac{\alpha\beta}{z} + O\frac{1}{z^2} \qquad 4.7$$

thus

$$W_1 \approx \frac{e^{n\frac{\pi}{2}}}{\Gamma(1+in)}\left(1-\frac{n^2}{ik\zeta}\right)\exp(in\log k\zeta) \qquad 4.8$$

$$W_2 \approx -\frac{i e^{n\frac{\pi}{2}}}{\Gamma(-in)}\,\frac{e^{ik\zeta}}{k\zeta}\exp(-in\log k\zeta)$$

We wish the solution which is regular at the origin, this is given by $F = W_1 + W_2$, where F is the hyper- geometric function. The wave function is then given by

$$\psi(r\theta\phi) = c\, e^{ikz} F(-in; 1; ik\zeta) \qquad 4.9$$

Writing this out in the limit $r \sim \infty$ we get

$$\psi \approx \frac{c\, e^{n\frac{\pi}{2}}}{\Gamma(1+in)}\left[\mathcal{I} + S f(\theta)\right] \qquad 4.10$$

with

$$\mathcal{I} = e^{i\left[kz + n\ln k(r-z)\right]}\left[1 - \frac{n^2}{ik(r-z)}\right] \qquad 4.11$$

$$S = r^{-1} e^{i(kr - n\ln 2kr)}$$

$$f(\theta) = f_c(\theta)\, e^{2i\eta_0} = \frac{n}{2k\sin^2\theta/2}\, e^{\left(-in\ln\sin^2\frac{\theta}{2} + i\pi + 2i\eta_0\right)}$$

We wish a wave of unit amplitude, so we choose C

$$c = e^{-\frac{n\pi}{2}}\,\Gamma(1+in)$$

In 4.11 we have written $e^{2i\eta_0}$ for $\dfrac{\Gamma(1+in)}{\Gamma(1-in)}$

Partial Wave Treatment.

Write

$$\psi = \sum R_l Y_l^0 \cdot \frac{2\pi^{\frac{1}{2}}}{(2l+1)!} \qquad 4.12$$

Put $R_l = r^l e^{ikr} f_l(kr)$ to obtain the differential equation

$$\left\{ r\frac{d^2}{dr^2} + 2(ikr+l+1)\frac{d}{dr} + 2[ik(l+1) - nk] \right\} f_l(r) = 0 \qquad 4.13$$

with solution $f_l(r) = C_l F(l+1+in, 2l+2, -2ikr)$ which is regular at the origin. Asymptoticly this gives

$$R_l \approx \frac{C_l e^{(\frac{n\pi}{2}+i\eta_l)}}{(2k)^l \Gamma(l+1+in) kr} \sin(kr - \tfrac{l\pi}{2} - n\ln 2kr + \eta_l) \qquad 4.14$$

So we have

$$\psi = \frac{2\pi^{\frac{1}{2}}}{kr} \sum \frac{C_l e^{\frac{n\pi}{2}} (2l+1)!\, e^{i\eta_l}}{(2k)^l \Gamma(l+1+in)(2l+1)^{\frac{1}{2}}} \sin(kr - \qquad 4.15$$
$$\tfrac{l\pi}{2} - n\ln 2kr + \eta_l) Y_l^0 .$$

where $e^{2i\eta_l} = \dfrac{\Gamma(l+1+in)}{\Gamma(l+1-in)}$

with no coulomb forces, the asymptotic form of ψ is

$$\psi \approx \frac{2\pi^{\frac{1}{2}}}{ikr} \cdot \sum (2l+1)^{\frac{1}{2}} i^{l+1} \sin(kr - \tfrac{l\pi}{2}) Y_l^0$$

We therefore choose

$$C_l = e^{-\frac{n\pi}{2}} \frac{\Gamma(l+1+in)}{(2l)!} i^l (2k)^l \qquad 4.16$$

So we obtain

$$\psi \approx \frac{2\pi^{\frac{1}{2}}}{ikr} \sum (2l+1)^{\frac{1}{2}} i^{l+1} e^{i\eta_l} Y_l^0 \qquad 4.17$$
$$\sin(kr - \tfrac{l\pi}{2} - n\ln 2kr + \eta_l) .$$

$$\psi \approx \frac{\pi^{\frac{1}{2}}}{kr} \sum (2l+1)^{\frac{1}{2}} \left\{ e^{-i(kr - \frac{l\pi}{2} - n \ln 2kr)} - e^{2i\eta_l} e^{i(kr - \frac{l\pi}{2} - n \ln 2kr)} \right\} Y_l^0 \qquad 4.18$$

Hence the equations derived in Section II, 2, where we neglected coulomb effects, remain valid provided we replace kr by $kr - n \ln 2kr$. The S-matrix defined in this way will contain both nuclear and coulomb effects, and in the absence of nuclear forces, it reduces to the coulomb S-matrix $S_c = R_c + 1$. Because of the long range nature of the coulomb forces, it becomes convenient to write the R-matrix in the form

$$R = (S - S_c) + (S_c - 1) = \alpha + R_c \qquad 4.19$$

R_c is treated exactly, while since S differs from S_c only in nuclear effects which vanish for large l, α can be conveniently analyzed into partial waves.

Consider R_c. It must give rise to M-matrix elements given by

$$r^{-1} \exp i(kr - n \ln 2kr) f_c(\theta) e^{2i\eta_0} \chi_s^{m_s}$$

$$= r^{-1} \exp i(kr - n \ln 2kr) \chi_s^{m_s} \sum_{s'm_s'} M_{sm_s;s'm_s'} \delta_{ss'} \delta_{m_s m_s'}$$

therefore

$$M^{\text{Coul.}}_{sm_s;s'm_s'} = f_c(\theta) e^{2i\eta_0} \delta_{ss'} \delta_{m_s m_s'} \qquad 4.20$$

α on the other hand will evidently give rise to R-matrix elements given (using the B.B. representation by

$$\alpha_j = e^{2i\delta_j'} - e^{2i\eta_j}$$

$$\alpha_{lj} = e^{2i\delta_{lj}'} - e^{2i\eta_l} \quad l = j$$

$$\alpha_{j\pm1\,j} = \cos^2\epsilon_j\, e^{2i\delta_{j\pm1\,j}'} + \sin^2\epsilon_j\, e^{2i\delta_{j\mp1\,j}'} - e^{2i\eta_{j\pm1}} \qquad 4.21$$

$$\alpha^j = -\tfrac{1}{2}\sin 2\epsilon_j\,\left(e^{2i\delta_{j+1\,j}'} - e^{2i\delta_{j-1\,j}'}\right)$$

It proves convenient to multiply the M-matrix by a phase $e^{-2i\eta_0}$. This is of no physical significance. We then get

$$f(\theta) = f_c(\theta) = \frac{-n}{k(1-\cos\theta)} \exp - in \ln\tfrac{1}{2}(1-\cos\theta)$$

$$\alpha_j = e^{2i\delta_j} - e^{2i\phi_j}$$

$$\alpha_{lj} = e^{2i\delta_{lj}} - e^{2i\phi_l} \qquad 4.22$$

$$\alpha_{j\pm1\,j} = \cos^2\epsilon_j\, e^{2i\delta_{j\pm1\,j}} + \sin^2\epsilon_j\, e^{2i\delta_{j\mp1\,j}} - e^{2i\phi_{j\pm1}}$$

$$\alpha_j = -\tfrac{1}{2}\sin 2\epsilon_j\,\left(e^{2i\delta_{j+1\,j}} - e^{2i\delta_{j-1\,j}}\right)$$

where

$$\phi_l = \eta_l - \eta_0$$

$$e^{2i\eta_l} = \frac{\Gamma(l+1+in)}{\Gamma(l+1-in)} = \frac{(l+in)!}{(l-in)!} \qquad 4.23$$

$$e^{2i(\eta_l-\eta_0)} = \frac{(l+in)\cdots(1+in)}{(l-in)\cdots(1-in)}, \quad \eta_l-\eta_0 = \sum_{x=1}^{l}\arctan\frac{n}{x}$$

The complete M-matrix is now easily obtained. Using equations 4.19, 4.20, 2.9, 2.24.

We obtain for the complete M-matrix, antisymmetrized and with coulomb effects included.

$$M^a = (1-TS)M$$

$$M^a_{sm_s;s'm_s'} = [f_c(\theta) - f_c(\pi-\theta)(-1)^{1+s}]\delta_{ss'}\delta_{m_s m_s'} \qquad 4.24$$

$$+ 2\sum_{l\ \mathrm{even}} Y_s^{m_s'-m_s} M(l\,0\,m_s'-m_s\,0;0\,0) + 2\sum_{l\ \mathrm{odd}} Y_l^{m_s'-m_s}$$

$$M(l\,1\,m_s'-m_s\,m_s;1\,m_s')$$

The expressions for the $M(\ell s\, m_\ell m_s ; s' m_s')$ may be obtained from equations 2.19 and 2.20 with R replaced by α . We may now substitute in the expressions for from equation 4.22 in the B.B. representation, or alternately use the more convenient barred representation, equation 3.17. The coulomb corrections are put in as in the B.B. representation, only diagonal elements being affected. We obtain

$$
\begin{aligned}
\alpha_j &= e^{2i\bar{\delta}_j} - e^{2i\phi_j} \\
\alpha_{\ell j} &= e^{2i\bar{\delta}_{\ell j}} - e^{2i\phi_\ell} \qquad \ell = j \\
\alpha_{j\pm1\, j} &= \cos 2\bar{\epsilon}_j \, e^{2i\bar{\delta}_{j\pm1\, j}} - e^{2i\phi_{j\pm1}} \\
\alpha^j &= i \sin 2\bar{\epsilon}_j \, e^{i(\bar{\delta}_{j+1\, j} + \bar{\delta}_{j-1\, j})}
\end{aligned} \qquad 4.25
$$

and

$$
M_{ss} = f_c(\theta) + f_c(\pi-\theta) + 2(ik)^{-1} \sum_{\ell\ \mathrm{even}} P_\ell \left(\frac{2\ell+1}{2}\right) \alpha_\ell
$$

$$
\begin{aligned}
M_{11} = f_c(\theta) - f_c(\pi-\theta) + 2(ik)^{-1} \sum_{\ell\ \mathrm{odd}} P_\ell \Big\{ \\
\left(\frac{\ell+2}{4}\right)\alpha_{\ell\, \ell+1} + \left(\frac{2\ell+1}{4}\right)\alpha_{\ell\ell} + \left(\frac{\ell-1}{4}\right)\alpha_{\ell\, \ell-1} \\
- \frac{1}{4}\left[(\ell+1)(\ell+2)\right]^{\frac{1}{2}} \alpha^{\ell+1} - \frac{1}{4}\left[\ell(\ell-1)\right]^{\frac{1}{2}} \alpha^{\ell-1} \Big\}
\end{aligned} \qquad 4.26
$$

$$M_{00} = f_c(\theta) - f_c(\pi-\theta) + 2(ik)^{-1} \sum_{l\,\text{odd}} P_l \left\{ \left(\frac{l+1}{2}\right) \alpha_{l\,l+1} \right.$$

$$\left. + \left(\frac{l}{2}\right) \alpha_{l\,l-1} + \frac{1}{2}\left[(l+1)(l+2)\right]^{\frac{1}{2}} \alpha^{l+1} + \frac{1}{2}\left[l(l-1)\right]^{\frac{1}{2}} \alpha^{l-1} \right\}$$

$$M_{01} = 2(ik)^{-1} e^{i\phi} \sum P_l^1 \left\{ -\frac{\sqrt{2}}{4}\left(\frac{l+2}{l+1}\right)\alpha_{l\,l+1} + \right.$$

$$\frac{\sqrt{2}}{4}\left(\frac{2l+1}{l(l+1)}\right)\alpha_{l\,l} + \frac{\sqrt{2}}{4}\left(\frac{l-1}{l}\right)\alpha_{l\,l-1} + \frac{\sqrt{2}}{4}\left(\frac{l+2}{l+1}\right)^{\frac{1}{2}}\alpha^{l+1}$$

$$\left. - \frac{\sqrt{2}}{4}\left(\frac{l-1}{l}\right)^{\frac{1}{2}} \alpha^{l-1} \right\}$$

$$M_{10} = 2(ik)^{-1} e^{-i\phi} \sum_{l\,\text{odd}} P_l^1 \left\{ \frac{\sqrt{2}}{4}\alpha_{l\,l+1} - \right.$$

$$\left. \frac{\sqrt{2}}{4}\alpha_{l\,l-1} + \frac{\sqrt{2}}{4}\left(\frac{l+2}{l+1}\right)^{\frac{1}{2}}\alpha^{l+1} - \frac{\sqrt{2}}{4}\left(\frac{l-1}{l}\right)^{\frac{1}{2}}\alpha^{l-1} \right\}$$

$$M_{1\,-1} = e^{-2i\phi}\, 2(ik)^{-1} \sum_{l\,\text{odd}} P_l^2 \left\{ \left(\frac{1}{4(l+1)}\right)\alpha_{l\,l+1} \right.$$

$$- \left(\frac{2l+1}{4l(l+1)}\right)\alpha_{l\,l} + \left(\frac{1}{4l}\right)\alpha_{l\,l-1} - \frac{1}{4}\left[(l+1)(l+2)\right]^{-\frac{1}{2}}\alpha^{l+1}$$

$$\left. - \frac{1}{4}\left[l(l-1)\right]^{-\frac{1}{2}}\alpha^{l-1} \right\}$$

It was mentioned in Section II,3,that the barred phase representation was particularly useful when coulomb effects were present. The barred phase shifts $\bar{\delta}$ contain both coulomb and nuclear effects. If there were no nuclear effects, they would become merely the plain coulomb phase shifts ϕ . When both coulomb and nuclear effects are included, it is useful to remove the coulomb effects and obtain only the nuclear part. If, for example, the coulomb forces act only outside a given region and if the WKB approximation is valid in this outside region, then the barred phase shifts which would be obtained if coulomb forces were absent leaving only the nuclear forces are given by

$$S_N = e^{-i\phi} S e^{-i\phi}$$

$$\bar{\delta}^N_\ell = \bar{\delta}_\ell - \phi_\ell , \quad \bar{\delta}^N_{\ell j} = \delta_{\ell j} - \phi_\ell , \quad \bar{\epsilon}^N = \bar{\epsilon}$$

5. Scattering-Matrix Calculations in Proton-Proton Scattering

Polarization experiments of various kinds, have been performed at Berkley (15) at 310 Mev., and at Harvard and Harwell (39) at 140 Mev. These experimental results may be made to yield phase shift solutions of the scattering problem, by the following method. We write

$$M = \sum_n \left(\frac{y_n(\delta) - y_n}{\epsilon_n} \right)^2$$

here y_n is the observed value of the n^{th} observable, $y_n(\delta)$ is an expression for it in terms of phase shifts, obtained from equations 4.26 and 1.20, ϵ_n is the error associated with the measurement of that observable. Minimizing gives us a least squares fit to the data. A discussion of the methods used is contained in Ref. 13. With this procedure, one generally obtains a number of reasonable solutions. In order to obtain a unique solution, other experiments are necessary which distinguish between these solutions. It is the purpose of this section to obtain the values of some of these observables for particular phase shift solutions at 310 and 140 Mev., in order to compare them with more recently acquired experimental data. In this calculation, we are in fact interested in phase shift solutions which are consistent with the unmodified boundary condition model of Lomon and Feshbach (10). Since we will later have recourse to mentioning this model,see Section IV, a few words describing it may perhaps be in order. We represent the interaction in the following way, an external region in which the interaction is adequately described by a local potential of the form $V(\vec{r}\,\vec{\sigma}\,\vec{\tau})$. Here $\vec{r}$, $\vec{\sigma}$, and $\vec{\tau}$ are respectively the relative co-ordinate, spin and isospin of the two-nucleon system, and a core region of radius r_0, at the boundary of which the wave function satisfies an energy independent boundary condition of the form.

$$x_0 \left(\frac{d\Psi}{dx}\right)_{x_0} = F\,(\Psi)_{x_0} .$$

F is here an energy independent quantity. In the case of tensor coupled states of course, F is a 2×2 matrix and

Ψ a 2-column vector. The idea motivating this model is the following. Developments in the meson theory of nuclear forces, indicate that the description of the interaction by a local potential of the form $V(\vec{r},\vec{\sigma},\vec{\tau})$ is only valid for distances larger than ~ 0.7 fermis. For smaller distances, we enter a region in which several virtual mesons are exchanged, and a non local interaction is needed to describe the force. The fact that many meson exchanges occur means that the interaction is very strong. In this region the wave function is therefore quite insensitive to changes in the kinetic energy of the bombarding particle. The interaction within this region may then be approximately taken account of, by imposing an energy independent boundary condition on the logarithmic derivative of the wave function on the surface of the core region. With this model of the interaction, it is found that when the potential in the outside region is ignored, only one type of phase shift solution, that with a large negative 3Po phase shift, fits the data approximately.

The experiments at Berkley (15) were designed to measure I_0,P,R,D and A. We therefore use the solution #6 to obtain the values of the correlation paramaters C_{nn} and C_{KP} at $\Theta = 90^{\circ}$. The experiments at Harvard and Harwell (39) were first designed to measure I_0,P and D . We therefore calculate C_{nn},C_{KP} and the rotation paramater R. for the solution of type #6.

The three phase shift solutions in which we are interested are given below in the barred representation, in degrees.

Type ξ	Fit to Harvard Data at 140 Mev.	Fit to Harwell Data at 140 Mev.	Stapp #6 at 310 Mev.
$^1S_0$	12.8 ± 2.2	9.5 ± 4.5	-0.25 ± 2.3
$^1D_2$	5.3 ± 0.9	7.7 ± 1.8	13.8 ± 0.6
$^1G_4$	00.0	00.0	.27
$^3P_0$	-54.4 ± 0.8	-34.5 ± 2.2	-64.2 ± 1.9
$^3P_1$	4.2 ± 2.2	14.1 ± 1.8	-12.77 ± 0.9
$^3F_3$	1.0 ± 0.9	-0.7 ± 1.5	4.22 ± 1.1
$^3H_5$	00.0	00.0	-0.5
$^3H_6$	00.0	00.0	1.75
$^3P_2$	7.4 ± 0.3	12.0 ± 0.7	8.78 ± 0.5
$^3F_2$	2.6 ± 0.3	4.9 ± 0.7	-0.93 ± 0.7
ϵ_2	-0.1 ± 1.0	-0.6 ± 2.2	-0.2 ± 0.6
$^3F_4$	2.2 ± 0.2	2.8 ±0.4	4.42 ± 0.25
$^3H_4$	00.0	00.0	3.65
ϵ_4	00.0		1.3

Table 1. The three phase shift solutions. The first two i.e., the fits to the Harwell and Harvard data, were obtained from Stabler (21). The last is Stapp's solution #6 to the Berkley data. Using equations 4.25 and 4.26, we have used the above to obtain the M-matrix elements for Harvard and Harwell, and have calculated their value at 90° c.m. for Stapp's solution #6. The results are given below.

The M-Matrix for Harwell at 140 Mev.

$$M_{SS} = -\frac{.011380}{(1-x)} \exp(-i.014812 \ln.5(1-x)) - \frac{.011380}{(1+x)} \exp(-i.014812 \ln.5(1+x)) + \left\{ x^2(1.2756 + i.20122) + (-.17505 - i.025216) \right\}$$

$$M_{11} = -\frac{.011380}{(1-x)} \exp(-i.014812 \ln.5(1-x)) + \frac{.011380}{(1+x)} \exp(-i.014812 \ln.5(1+x)) + \left\{ x^3(-.041550 + i.061140) + (.98614 + i.20992)x \right\}$$

$$M_{00} = -\frac{.011380}{(1-x)} \exp(-i.014812 \ln.5(1-x)) + \frac{.011380}{(1+x)} \exp(-i.014812 \ln.5(1+x)) + \left\{ x^3(.92130 + i.047552) + x(-i.74634 + .57468) \right\}$$

$$M_{01} = e^{i\phi} (1-x^2)^{1/2} \left\{ x^2(-.030162 + i.055404) + (.048158 + i.0072866) \right\}$$

$$M_{10} = e^{-i\phi} (1-x^2)^{1/2} \left\{ x^2(-.23790 - i.002154) + (.76413 - i.30883) \right\}$$

$$M_{1-1} = e^{-2i\phi} \left\{ x^3(-.58376 - i.061714) + x(.58376 + i.061714) \right\}$$

The M-Matrix for Harvard at 140 Mev.

$$M_{SS} = -\frac{.011380}{(1-x)} \exp(-i.014812 \ln.5(1-x)) - \frac{.011380}{(1+x)} \exp(-i.014812 \ln.5(1+x)) + \left\{ x^2(.80528 + i.092660) + (.063552 + i.044540) \right\}$$

$$M_{11} = -\frac{.011380}{(1-x)} \exp(-i.014812 \ln.5(1-x) + \frac{.011380}{(1+x)} \exp(-i.014812 \ln.5(1+x)) + \left\{ x^3(.057692 + i.0093774) + x(.36284 + i.045100) \right\}$$

$$M_{00} = -\frac{.011380}{(1-x)} \exp(-i.014812 \ln.5(1-x)) + \frac{.011380}{(1+x)} \exp(-i.014812 \ln.5(1+x)) + \left\{ x^3(.36912 + i.0212180) + x(-.63082 + 1.0509) \right\}$$

$$M_{01} = e^{i\phi} (1-x^2)^{1/2} \left\{ x^2(-.050706 + i.00161444) + (-.081170 - i.019420) \right\}$$

$$M_{10} = e^{-i\phi} (1-x^2)^{1/2} \left\{ x^2(-.045764 - .00083708) + (-.66004 - i.70094) \right\}$$

$$M_{1-1} = e^{-2i\phi} \left\{ x^3(-.175001 - i.012940) + x(.17500 + i.0129400) \right\}$$

M-matrix for Stapp's Solution #6 at 310 Mev. and $\theta = 90^{\circ}$

$$M_{ss} = -.63500 \quad -i \quad .14807$$

$$M_{01} = -.37609 + i \quad .02800$$

$$M_{10} = -12751 \quad -i \quad .65561$$

$$M_{11} = M_{1-1} = M_{00} = 0.$$

With the aid of equation 1.20 we use the three calculated M-matrices to obtain the correlation paramaters C_{nn} and C_{KP} at $\theta = 90^{\circ}$ c.m., and further to calculate the rotation paramater for values of 0 in the range $\theta = 0^{\circ}$ to $\theta = 90^{\circ}$. The latter calculation is done only for the data at 140 Mev. The results are given in Table 2 and Fig. 8.

Quantity	Harvard	Harwell	Stapp's #6
C_{nn}	.9961	.9433	.4692
C_{KP}	-.9832	-.9649	-.3794

Table 2. The correlation paramaters C_{nn} and C_{KP} at $\theta = 90^{\circ}$, for the three solutions.

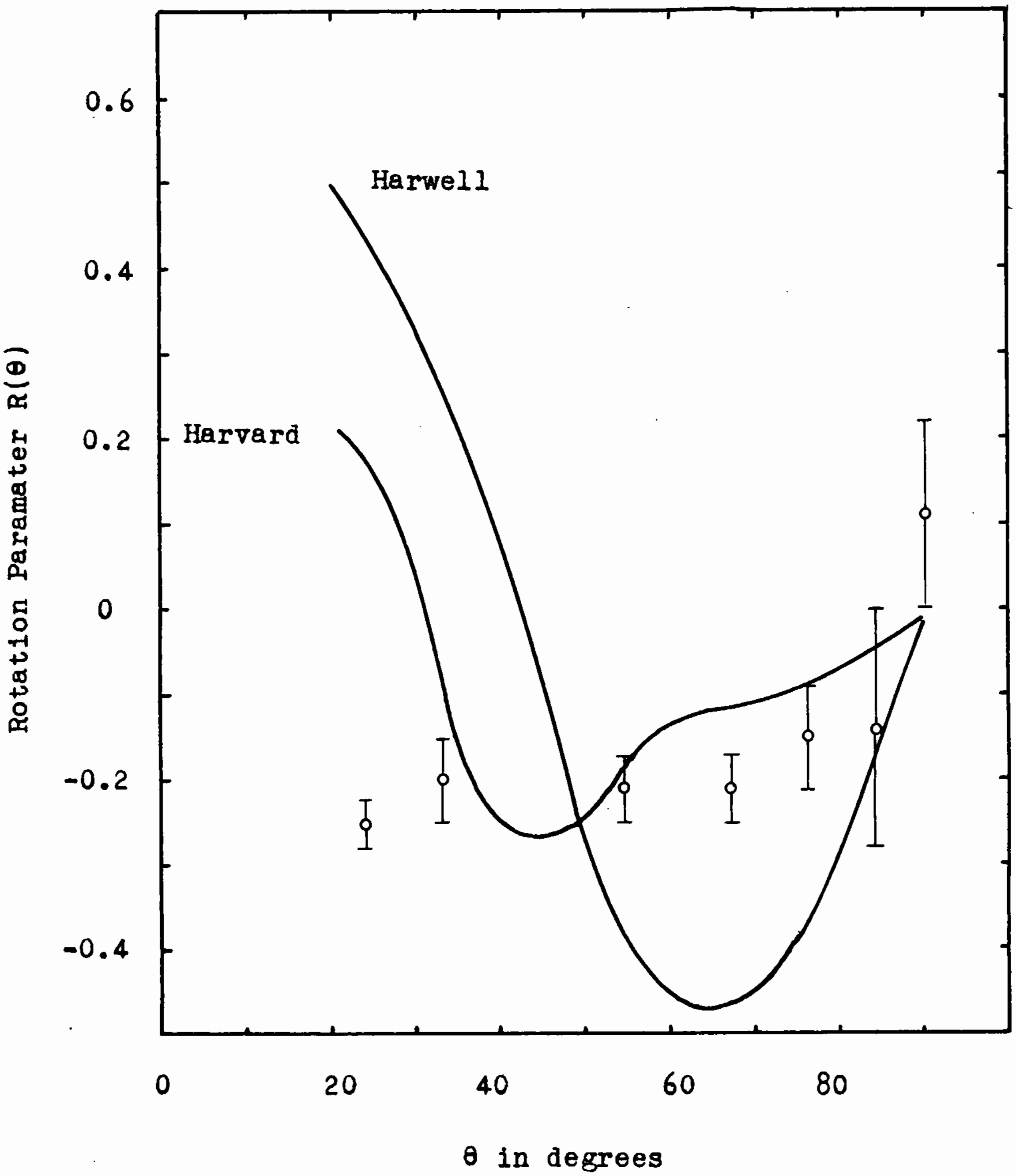

Fig. 8. Plots of the rotation paramater R (θ), for phase shift solutions of type #6 to Harvard and Harwell data. The experimental points are taken from reference 27.

6. Conclusions and Discussion.

In our work we have considered solution #6 of Stapp et al (15) at 310 Mev., and also two solutions obtained by Stabler (21) at Cornell in fitting data from Harvard and Harwell at 140 Mev. These solutions are also of the same type as #6. They are consistent with the boundary condition model of Lomon and Feshbach. At 310 Mev., a number of arguments are given by various authors suggesting that solution #6 is invalid. We will here reiterate these arguments and discuss them.

It is argued (see for example Gammel and Thaler ref. 40 ch. 9-4) that C_{KP} measured at 380 Mev. is positive, 0.6 ± 0.1, while in solution #6 C_{KP} at 310 Mev. is negative $\cong$ -0.38. This it is felt invalidates solution #6. However our calculations at 140 Mev. show that C_{KP} has a value $\cong$ -.98. It is therefore certainly not constant with energy and could conceivably change sign between 310 and 380 Mev.

The second argument is based on the work of Moravscik, MacGregor and Stapp (41). They use a modified method of analysing the data at 310 Mev. They assume that G H and all waves of higher angular momentum only see the one pion exchange part of the interaction at 310 Mev., due to the strong centrifugal barrier. They therefore elect to calculate these as functions of the pseudoscalar coupling constant g^2. In this way they decrease the number of degrees of freedom in the problem to 9 from the original 14, eliminating the four phase shifts ${}_1G_4$, ${}_3H_5$, ${}_3H_6$, ${}_3H_4$ and the mixing paramater ϵ_4. They then attempt to fit the data with the remaining 9 phase shifts, for several values of the coupling constant g^2. Doing this they observe that M plotted as a function of g^2 yield minima for solutions 1 and 2 at $g^2 \cong 12.0$ and 13.3 respectively, with the minima occurring at a value $M \cong 25$. Since the most probable value of $M \cong 27$, and the accepted value of $g^2 \cong 14.0$, these seem to be good solutions. For solution 6, they obtain a very shallow minimum, corresponding to a value of $g^2 \cong 20.$ and $M \cong 57$, with a negligible probability of M being this value. On these grounds they therefore rule out solution 6.

This argument is actually not physically complete, since it can be shown that the one pion exchange part of the interaction is not adequate to account for the G phase shifts, further the $^3H_4$ phase shift is coupled to the $^3F_4$ phase shift, and so is affected by the several pion exchange region of the interaction. If however waves of angular momentum larger than H only are treated in this manner, the most probable value is $\cong 22$ and for solution 6, $M \cong 35$ which is not too bad.

Stabler and Lomon (28) at Cornell have calculated P and D in the coulomb interference region for a solution of the same type as we used. They have found no agreement with the experimental results from Harvard. This is in agreement with what we found for R, since it seems to disagree with the new Harwell results for angles less than 30°, but is not too bad between 30° and 90°.

SECTION III

SCATTERING OF POSITIVE AND NEGATIVE MESONS BY NUCLEONS (42, 43)

1. The Recoil Proton Polarization and Elastic and Charge Exchange Cross Sections

It is convenient to use the isospin formalism. The pion has isospin 1 the nucleon has isospin 1/2. Hence the combined system has isospin 3/2 or 1/2. The eigenfunctions for these two cases are given below.

$$T/2 = \frac{3}{2} \qquad\qquad T/2 = \frac{1}{2}$$

$$\begin{aligned}
x_3 &= p^+ & x_1^1 &= \sqrt{\tfrac{1}{3}}\, p^0 - \sqrt{\tfrac{2}{3}}\, n^+ \\
x_3^1 &= \sqrt{\tfrac{2}{3}}\, p^0 + \sqrt{\tfrac{1}{3}}\, n^+ & & \\
x_3^{-1} &= \sqrt{\tfrac{2}{3}}\, n^0 + \sqrt{\tfrac{1}{3}}\, p^- & x_1^{-1} &= -\sqrt{\tfrac{1}{3}}\, n^0 + \sqrt{\tfrac{2}{3}}\, p^- \\
x_3^{-3} &= n^- & &
\end{aligned} \qquad 1.1$$

If we assume that the meson nucleon interaction is charge independent then the only dependence on isospin can be through $T/2$. We thus have two amplitudes that give the isospin dependence of the scattering.

$$\begin{aligned}
e^{ikz}\, x_1 &\rightarrow S_1 x_1 \\
e^{ikz}\, x_3 &\rightarrow S_3 x_3
\end{aligned} \qquad 1.2$$

Here the right hand side represents the scattered wave. For the case of π^+ on protons, the scattering is completely described by S_3 according to

$$e^{ikz} p^+ \rightarrow S_3 p^+ \qquad 1.3$$

For the case of   on protons, things are not quite so simple.

$$e^{ikz} p^- = \sqrt{\tfrac{1}{3}}\, e^{ikz} \chi_3 + \sqrt{\tfrac{2}{3}}\, e^{ikz} \chi_1$$

$$\rightarrow \sqrt{\tfrac{1}{3}}\, S_3 \chi_3 + \sqrt{\tfrac{2}{3}}\, S_1 \chi_1 = \sqrt{\tfrac{1}{3}} \left(\sqrt{\tfrac{2}{3}}\, n^0 + \sqrt{\tfrac{1}{3}}\, p^-\right) S_3 \qquad 1.4$$

$$+ \sqrt{\tfrac{2}{3}} \left(-\tfrac{1}{\sqrt{3}}\, n^0 + \sqrt{\tfrac{2}{3}}\, p^-\right) S_1 = n^0 \frac{\sqrt{2}}{3} (S_3 - S_1) + p^- \frac{1}{3} (S_3 + 2 S_1)$$

The first term describes the charge exchange scattering, i.e.,
      . The second term describes elastic scattering.
Hence we obtain a table of amplitudes for the various scattering processes.

Process	Amplitudes
$p^+ \rightarrow p^+$	S_3
$p^- \rightarrow p^-$	$\frac{1}{3}(S_3 + 2 S_1)$
$p^- \rightarrow n^0$	$\frac{\sqrt{2}}{3}(S_3 - S_1)$

1.5

It must also be of course recognized that the interaction is in general spin dependent. Hence each isospin amplitude, is really made up of four sub-amplitudes, given by

$$S^T_{\alpha\alpha},\ S^T_{\alpha\beta},\ S^T_{\beta\alpha},\ S^T_{\beta\beta}$$

describing what happens when the incoming wave is an α or β wave. Thus

$$e^{ikz} \chi_T \alpha \rightarrow (S^T_{\alpha\alpha}\, \alpha + S^T_{\alpha\beta}\, \beta)\, \chi_T$$

$$e^{ikz} \chi_T \beta \rightarrow (S^T_{\beta\alpha}\, \alpha + S^T_{\beta\beta}\, \beta)\, \chi_T \qquad 1.6$$

In carrying through a phase shift analysis, we will assume that only S and P   waves are scattered. This assumption is somewhat arbitrary, for energies $>$ 150 mev., but seems to work well up to 300 mev.

For an incident plane[1]wave $e^{ikz}\alpha$, the diverging part may be written

$$\approx \frac{e^{ikr}}{ikr} \pi^{\frac{1}{2}} \sum_{l=0}^{\infty} (2l+1)^{\frac{1}{2}} |l\,0\,\tfrac{1}{2}\rangle \qquad 1.7$$

While for an incident plane wave $e^{ikz}\beta$ the diverging part may be written

$$\approx \frac{e^{ikr}}{ikr} \pi^{\frac{1}{2}} \sum_{l=0}^{\infty} (2l+1)^{\frac{1}{2}} |l\,0\,-\tfrac{1}{2}\rangle \qquad 1.8$$

Where we have written this in the $l\,s\,m_l\,m_s$ representation, but have omitted the quantum number S = 1/2. Now these waves are not eigenwaves of the S-matrix, since m_l and m_s are not good quantum numbers. We therefore transform to the $j\,m_j\,l\,s$ representation. These four are good quantum numbers. We therefore obtain using the expansion formula

$$|l\,m_l\,m_s\rangle = |j\,m_j\,l\rangle\langle j\,m_j\,l\,|\,l\,m_l\,m_s\rangle \qquad 1.9$$

the following

$$\begin{aligned}
|0\,0\,\tfrac{1}{2}\rangle &= |\tfrac{1}{2}\,\tfrac{1}{2}\,0\rangle \\
|1\,0\,\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}\left(\sqrt{2}\,|\tfrac{3}{2}\,\tfrac{1}{2}\,1\rangle + |\tfrac{1}{2}\,\tfrac{1}{2}\,1\rangle\right) \\
|0\,0\,-\tfrac{1}{2}\rangle &= |\tfrac{1}{2}\,-\tfrac{1}{2}\,0\rangle \\
|1\,0\,-\tfrac{1}{2}\rangle &= \tfrac{1}{\sqrt{3}}\left(\sqrt{2}\,|\tfrac{3}{2}\,-\tfrac{1}{2}\,1\rangle - |\tfrac{1}{2}\,-\tfrac{1}{2}\,1\rangle\right)
\end{aligned} \qquad 1.10$$

---

1 We have here used $\vec{k}$ for the centre of mass momentum.

Here the Clebsch Gordon coefficients are easily worked out or may be found in ref. 36. We then obtain from 1.7 and 1.8

from 1.7

$$\approx \frac{e^{ikr}}{ikr}\pi^{\frac{1}{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}0\rangle + \sqrt{2}\,|\tfrac{3}{2}\tfrac{1}{2}1\rangle + |\tfrac{1}{2}\tfrac{1}{2}1\rangle\right) \qquad 1.11$$

from 1.8

$$\approx \frac{e^{ikr}}{ikr}\pi^{\frac{1}{2}}\left(|\tfrac{1}{2}-\tfrac{1}{2}0\rangle + \sqrt{2}\,|\tfrac{3}{2}-\tfrac{1}{2}1\rangle - |\tfrac{1}{2}-\tfrac{1}{2}1\rangle\right) \qquad 1.12$$

The phase shifts may be written $\delta(T\,j\,\ell)$. This follows from charge independence, the complete spherical symmetry of the interaction and parity conservation. We therefore make a summary of the pertinent phase shifts.

	S 1/2	P 1/2	P 3/2	
$\frac{T}{2} = 3/2$	δ_3	δ_{31}	δ_{33}	
$\frac{T}{2} = 1/2$	δ_1	δ_{11}	δ_{13}	1.13

We will call $e^{2i\delta} - 1 = \epsilon$. 1.14

Then the effect of the interaction is to alter the diverging part of the incident plane wave by $e^{2i\delta}$. The scattered wave is therefore given by

from 1.11

$$\approx \frac{e^{ikr}}{ikr}\pi^{\frac{1}{2}}\left\{\epsilon_T\,|\tfrac{1}{2}\tfrac{1}{2}0\rangle + \epsilon_{T3}\sqrt{2}\,|\tfrac{3}{2}\tfrac{1}{2}1\rangle + \epsilon_{T1}\,|\tfrac{1}{2}\tfrac{1}{2}1\rangle\right\} \qquad 1.15$$

and from 1.12

$$\approx \frac{e^{ikr}}{ikr}\pi^{\frac{1}{2}}\left\{\epsilon_T\,|\tfrac{1}{2}-\tfrac{1}{2}0\rangle + \sqrt{2}\,\epsilon_{T3}\,|\tfrac{3}{2}-\tfrac{1}{2}1\rangle - \epsilon_{T1}\,|\tfrac{1}{2}-\tfrac{1}{2}1\rangle\right\} \qquad 1.16$$

Writing these vectors in terms of the spherical harmonics and spin functions according to

$$|j m_j l\rangle = \sum_{l,\, m_l + m_s = m_j} Y_l^{m_l} \chi_s^{m_s} \langle l m_l l m_s | j m_j l\rangle \qquad 1.17$$

We obtain for the above

$$\frac{e^{ikr}}{ikr} \pi^{\frac{1}{2}} \left\{ \alpha \left[\epsilon_T Y_0^0 + \frac{1}{\sqrt{3}} (2\epsilon_{T3} + \epsilon_{T1}) Y_1^0 \right] + \beta \frac{1}{\sqrt{3}} (\epsilon_{T3} - \epsilon_{T1}) Y_1^1 \right\} \qquad 1.18$$

and

$$\frac{e^{ikr}}{ikr} \pi^{\frac{1}{2}} \left\{ \beta \left[\epsilon_T Y_0^0 + \frac{1}{\sqrt{3}} (2\epsilon_{T3} + \epsilon_{T1}) Y_1^0 \right] + \alpha \left[\frac{\sqrt{2}}{3} (\epsilon_{T3} - \epsilon_{T1}) Y_1^{-1} \right\} \qquad 1.19$$

The spherical harmonics are given by

$$Y_0^0 = \frac{1}{2\pi^{\frac{1}{2}}}, \qquad Y_1^0 = \frac{\sqrt{3}}{2\pi^{\frac{1}{2}}} \cos\theta$$

$$Y_1^{\pm 1} = \mp \frac{1}{2\pi^{\frac{1}{2}}} \sqrt{\frac{3}{2}} \sin\theta \, e^{\pm i\phi}$$

so we get

$$\frac{e^{ikr}}{2ikr} \left\{ \alpha \left[\epsilon_T + \cos\theta (2\epsilon_{T3} + \epsilon_{T1}) \right] - \beta \left[\sin\theta \, e^{i\phi} (\epsilon_{T3} - \epsilon_{T1}) \right] \right\} \qquad 1.20$$

$$\frac{e^{ikr}}{2ikr} \left\{ \beta \left[\epsilon_T + \cos\theta (2\epsilon_{T3} + \epsilon_{T1}) \right] + \alpha \left[\sin\theta \, e^{-i\phi} (\epsilon_{T3} - \epsilon_{T1}) \right] \right\} \qquad 1.21$$

by comparing with 1.6 we then obtain calling

$$S^T_{\alpha\alpha} = S^T_{\beta\beta} = f(k) \left[\epsilon_T + \cos\theta (2\epsilon_{T3} + \epsilon_{T1}) \right]$$

$$S^T_{\alpha\beta} = f(k) \left[\epsilon_{T1} - \epsilon_{T3} \right] \sin\theta \, e^{i\phi} \qquad 1.22$$

$$S^T_{\beta\alpha} = -f(k) \left[\epsilon_{T1} - \epsilon_{T3} \right] \sin\theta \, e^{-i\phi}$$

Let us now look at polarizations and differential cross sections for the scattering of π^+ on p and π^- on p. The pertinent formulae are given in Section I. They are

$$I = \frac{1}{2} \operatorname{Tr} (M M^\dagger)$$
$$I \vec{P} = \frac{1}{2} \operatorname{Tr} (M M^\dagger \vec{\sigma}) \qquad 1.16$$

The M-matrix will be given by

$$e^{ikz} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \rightarrow e^{ikz} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \frac{e^{ikr}}{r} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad 1.17$$

We now choose the z direction as the incident direction, and the x-z plane as the scattered plane, i.e., $\phi = 0$

We then have

$$S_{\alpha\alpha} = S_{\beta\beta} = \frac{e^{ikr}}{r} M_{\alpha\alpha}$$
$$S_{\alpha\beta} = -S_{\beta\alpha} = \frac{e^{ikr}}{r} M_{\alpha\beta} \qquad 1.18$$

So equation 1.17 can be written in the form

$$e^{ikz} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \rightarrow e^{ikz} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \frac{e^{ikr}}{r} \begin{pmatrix} M_{\alpha\alpha} & -M_{\alpha\beta} \\ M_{\alpha\beta} & M_{\alpha\alpha} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad 1.19$$

It is convenient to use a different representation for the spinors. We write M with respect to the basic spinors $\frac{1}{\sqrt{2}}(\alpha + i\beta) = \gamma$ and $\frac{1}{\sqrt{2}}(\alpha - i\beta) = \delta$. These are the spin functions for the positive y and negative y direction. In this representation equation 1.19 becomes

$$e^{ikz}\frac{1}{\sqrt{2}}\begin{pmatrix} a_1 - i a_2 \\ a_1 + i a_2 \end{pmatrix} \rightarrow e^{ikz}\frac{1}{\sqrt{2}}\begin{pmatrix} a_1 - i a_2 \\ a_1 + i a_2 \end{pmatrix} +$$

$$\frac{e^{ikr}}{r}\begin{pmatrix} M_{\alpha\alpha} - i M_{\alpha\beta} & 0 \\ 0 & M_{\alpha\alpha} + i M_{\alpha\beta} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} a_1 - i a_2 \\ a_1 + i a_2 \end{pmatrix} \qquad 1.20$$

Noticing that for an incident α wave, $a_2 = 0$ and for an incident β wave $a_1 = 0$. We see that $|M_{\alpha\alpha} - i M_{\alpha\beta}|^2$ and $|M_{\alpha\alpha} + i M_{\alpha\beta}|^2$ are porportional to the probabilities for scattering with the spin in the positive y or negative y directions. We further notice that the polarization in the xz plane is zero, which is what we expect.

For a given value of the isospin, the M-matrix is given by equation 1.20

$$M^T = \begin{pmatrix} M^T_{\alpha\alpha} - i M^T_{\alpha\beta} & 0 \\ 0 & M^T_{\alpha\alpha} + i M^T_{\alpha\beta} \end{pmatrix} \qquad 1.21$$

The scattering of π^+ on p, only involves $T = 3$. equation 1.3. The amplitude from 1.5 is S_3. The M-matrix may therefore be written

$$M^3 = \begin{pmatrix} M^3_{\alpha\alpha} - i M^3_{\alpha\beta} & 0 \\ 0 & M^3_{\alpha\alpha} + i M^3_{\alpha\beta} \end{pmatrix} \qquad 1.22$$

The polarization and differential cross sections are given by 1.16. They are

$$I_+ = \frac{1}{2}\left\{ |M^3_{\alpha\alpha} - i M^3_{\alpha\beta}|^2 + |M^3_{\alpha\alpha} + i M^3_{\alpha\beta}|^2 \right\}$$

$$I_+ P_+ = \frac{1}{2}\left\{ |M^3_{\alpha\alpha} - i M^3_{\alpha\beta}|^2 - |M^3_{\alpha\alpha} + i M^3_{\alpha\beta}|^2 \right\} \qquad 1.23$$

From 1.18 and 1.22 we have

$$M^3_{\alpha\alpha} - i M^3_{\alpha\beta} = \frac{1}{2ik}\left[\epsilon_3 + (2\epsilon_{33} + \epsilon_{31})\cos\theta - i(\epsilon_{31} - \epsilon_{33})\sin\theta \right]$$

$$1.24$$

$$M^3_{\alpha\alpha} + i M^3_{\alpha\beta} = \frac{1}{2ik}\left[\epsilon_3 + (2\epsilon_{33} + \epsilon_{31})\cos\theta + i(\epsilon_{31} - \epsilon_{33})\sin\theta \right]$$

We now introduce the notation

$$g(lT,2j\,|\,l'T',2j') = 4\sin\delta^l_{T,2j}\sin\delta^{l'}_{T',2j'}\cos(\delta^l_{T,2j} - \delta^{l'}_{T',2j'})$$

$$1.25$$

$$f(lT,2j\,|\,l'T',2j') = 4\sin\delta^l_{T,2j}\sin\delta^{l'}_{T',2j'}\sin(\delta^l_{T,2j} - \delta^{l'}_{T',2j'})$$

Putting 1.24 into 1.23 we obtain

$$\lambda\!\!\!{}^{-}{}^{-2} I_+ = A_+ + B_+\cos\theta + C_+\cos^2\theta \qquad 1.26$$

where

$$A^+ = \sin^2\delta_3 + \sin^2\delta_{31} + \sin^2\delta_{33} - \frac{1}{2}\,g(131\,|\,133)$$

$$B^+ = g(031\,|\,133) + \frac{1}{2}\,g(031\,|\,131) \qquad 1.27$$

$$C_+ = 3\sin^2\delta_{33} + \frac{3}{2}\,g(131\,|\,133)$$

and

$$2\lambda\!\!\!{}^{-}{}^{-2} I_+ P_+ = \sin\theta\left[f(031\,|\,133) - f(031\,|\,131) \right]$$

$$1.28$$

$$-\frac{3}{2}\sin 2\theta\, f(133\,|\,131)$$

In the case of π^- on protons, elastic scattering, the situation is not quite so simple. The amplitude is given equation 1.5 by 1/3 $(S_3 + 2\,S_1)$. The M-matrix may therefore be written $\frac{1}{3}\,(M^3 + 2M^1)$

$$= \frac{1}{3}\begin{pmatrix} [(M^3_{\alpha\alpha} - iM^3_{\alpha\beta}) + 2(M^1_{\alpha\alpha} - iM^1_{\alpha\beta})] & 0 \\ 0 & [(M^3_{\alpha\alpha} + iM^3_{\alpha\beta}) + 2(M^1_{\alpha\alpha} + iM^1_{\alpha\beta})] \end{pmatrix} \qquad 1.29$$

Using equation 1.16, the elastic scattering cross section, and the polarization of the recoiling proton, are given by

$$I_- = |x_3 + 2x_1|^2 + |y_3 + 2y_1|^2$$
$$I_-P_- = |x_3 + 2x_1|^2 - |y_3 + 2y_1|^2 \qquad 1.30$$

respectively, with

$$x_T = (M^T_{\alpha\alpha} - iM^T_{\alpha\beta})$$
$$y_T = (M^T_{\alpha\alpha} + iM^T_{\alpha\beta}) \qquad 1.31$$

Putting in the expressions for $M^T_{\alpha\alpha}$ and $M^T_{\alpha\beta}$ from equation 1.24 we obtain

$$\lambda\!\!\!^{-}{}^{-2} I_- = A_- + B_- \cos\theta + C_- \cos^2\theta \qquad 1.32$$

where

$$A_- = \frac{1}{9}\{(\sin^2\delta_3 + \sin^2\delta_{33} + \sin^2\delta_{31}) + 4(\sin^2\delta_1 + \sin^2\delta_{13} + \sin^2\delta_{11}) + g(131|111) + g(031|011) + g(133|113) - \tfrac{1}{2}g(133|131) - g(133|111) - 2g(113|111) - g(131|113)\}$$

$$B_- = \frac{1}{9}\{g(031|133) + \tfrac{1}{2}g(031|131) + 2g(031|113) + g(031|111) + 2g(011|133) + g(011|131) + 4g(011|113) + 2g(011|111)\} \qquad 1.33$$

$$C_- = \frac{1}{9}\{3\sin^2\delta_{33} + 12\sin^2\delta_{13} + \tfrac{3}{2}g(133|131) + 3g(133|113) + 3g(111|133) + 3g(131|113) + 6g(111|113)\}$$

and

$$18 \lambda^{-2} P_- I_- = 2\lambda^{-2} P_+ I_+ + 4 \sin\theta [f(111|011) - f(113|011)]$$
$$+ 6 \sin 2\theta \, f(111|113) + 2 \sin\theta [f(111|031) - f(113|031)$$
$$+ f(131|011) - f(\quad |011)] + 3 \sin 2\theta [f(131|113) \qquad 1.34$$
$$- f(133|111)]$$

For the charge exchange scattering, the amplitude is given equation 1.15 by $\frac{\sqrt{2}}{3}(S_3 - S_1)$. The M-matrix may therefore be written

$$= \frac{\sqrt{2}}{3} \begin{pmatrix} [(M^3_{\alpha\alpha} - i M^3_{\alpha\beta}) - (M^1_{\alpha\alpha} - i M^1_{\alpha\beta})] & 0 \\ 0 & [(M^3_{\alpha\alpha} + i M^3_{\alpha\beta}) - (M^1_{\alpha\alpha} - i M^1_{\alpha\beta})] \end{pmatrix} \qquad 1.35$$

Whence using equations 1.16 and 1.31 the cross section is given by

$$I_0 = \frac{1}{9} [|x_3 - x_1|^2 + |y_3 - y_1|^2] \qquad 1.36$$

Using equation 1.24 we then obtain

$$I_0 = A_0 + B_0 \cos\theta + C_0 \cos^2\theta \qquad 1.37$$

where

$$A_0 = \frac{2}{9} \{ (\sin^2\delta_3 + \sin^2\delta_1 + \sin^2\delta_{33} + \sin^2\delta_{31} + \sin^2\delta_{11}$$
$$+ \sin^2\delta_{13}) - \frac{1}{2} [g(011|031) + g(133|131) + g(133|113)$$
$$+ g(131|113) + g(131|111) + 2g(113|111)] \}$$
$$1.38$$
$$B_0 = \frac{1}{9} \{ [g(031|131) - g(031|111) - g(011|131)$$
$$+ g(011|111) + 2[g(031|133) - g(031|113) - g(011|133)$$
$$+ g(113|011)] \}$$
$$C_0 = \frac{1}{9} \{ 3[g(131|133) - g(131|113) - g(133|113)]$$
$$+ [g(113|111) - g(133|111) - 2g(131|111)] + 6\sin^2\delta_{33} + 6\sin^2\delta_3 \}$$

2. Scattering-Matrix Calculations in Pion-Nucleon Scattering at 307 Mev.

An analysis of the π -p scattering data at 307 Mev. was carried out by Chiu and Lomon, (26). A least squares fit of the type described in connection with the nucleon-nucleon calculation, was made utilising both $\pi^+ - p$ and $\pi^- - p$ elastic and charge exchange cross-sections. They obtained three phase shift solutions, designated A,B, and C. The solution A is characterized by small P-wave phase shifts, consistent with a k^3 extrapolation of the low energy data at 150 and 170 Mev. For this solution δ_{13} and δ_{31} differ in sign. A second solution B of similar nature appears for which δ_{13} has a k^5 or stronger dependence on the meson momentum. The third solution C corresponds to $\delta_{13} = \delta_{31}$ above resonance, this requires that both δ_{13} and δ_{11} change sign near resonance.

The purpose of this calculation is to evaluate the polarization of the recoiling proton in both $\pi^+ - p$ and $\pi^- - p$ experiments, and to compare the results with recently obtained experimental values, which will enable us to eliminate incorrect solutions. Expressions for the polarizations P_+ and P_- have been derived by Chiu, (44), these however omit all terms which do not contain $\sin \delta_{33}$, as these are quite small around resonance. These expressions are however not quite good enough at 307 Mev. We have therefore used the complete expressions equations 1.28 and 1.34.

The three solutions in which we are interested, are given in the paper by Chiu and Lomon, (44). They are reproduced below.

Phase Shift	Soln. A	Soln. B	Soln. C
δ_1	9.6 ± 8	20.2 ± 10	17.6 ± 10
δ_3	-24.1 ± 2	-24.7 ± 2.0	-24.7 ± 2.3
δ_{33}	132.8 ± 1.7	132.3 ± 1.5	132.4 ± 2.0
δ_{31}	-10.3 ± 3.0	-9.2 ± 3.0	-10.5 ± 3.0
δ_{13}	10.0 ± 4.0	3.4 ± 3.5	-5.9 ± 3.5
δ_{11}	-10.0 ± 5.9	0.9 ± 5.5	13.3 ± 5.7

Table 3. The phase shift solutions of Chiu and Lomon, (44), at 307 Mev.

Using equations 1.28 and 1.34 we then obtain the expressions for the recoil proton polarization at 307 Mev. for π^+ on p, and for π^- on p, as yielded by the above three solutions. They are given below

$$P_+(A) = \frac{.53985 \text{ Sin } \theta + .47262 \text{ Sin } 2\theta}{4.52 \text{ Cos}^2 \theta + 2.48 \text{ Cos } \theta + 1.06}$$

$$P_{+}(B) = \frac{.55447 \text{ Sin } \theta + .44168 \text{ Sin } 2\theta}{4.4 \text{ Cos}^2 \theta + 2.52 \text{ Cos } \theta + 1.12}$$

$$P_{+}(C) = \frac{.55502 \text{ Sin } \theta + .487.7 \text{ Sin } 2\theta}{4.56 \text{ Cos}^2 \theta + 2.56 \text{ Cos } \theta + 1.08}$$

and

$$P_{-}(A) = \frac{.41034 \text{ Sin } \theta + 1.7738 \text{ Sin } 2\theta}{4.212 \text{ Cos}^2 \theta + 1476 \text{ Cos } \theta + 2.34}$$

$$P_{-}(B) = \frac{-98505 \text{ Sin } \theta + .36093 \text{ Sin } 2\theta}{4.212 \text{ Cos}^2 \theta + 1.476 \text{ Cos } \theta + 2.34}$$

$$P_{-}(C) = \frac{-17261 \text{ Sin } \theta - 1.5031 \text{ Sin } 2\theta}{4.212 \text{ Cos}^2 \theta + 1.476 \text{ Cos } \theta + 2.34}$$

These results are plotted in Figures 9 and 10.

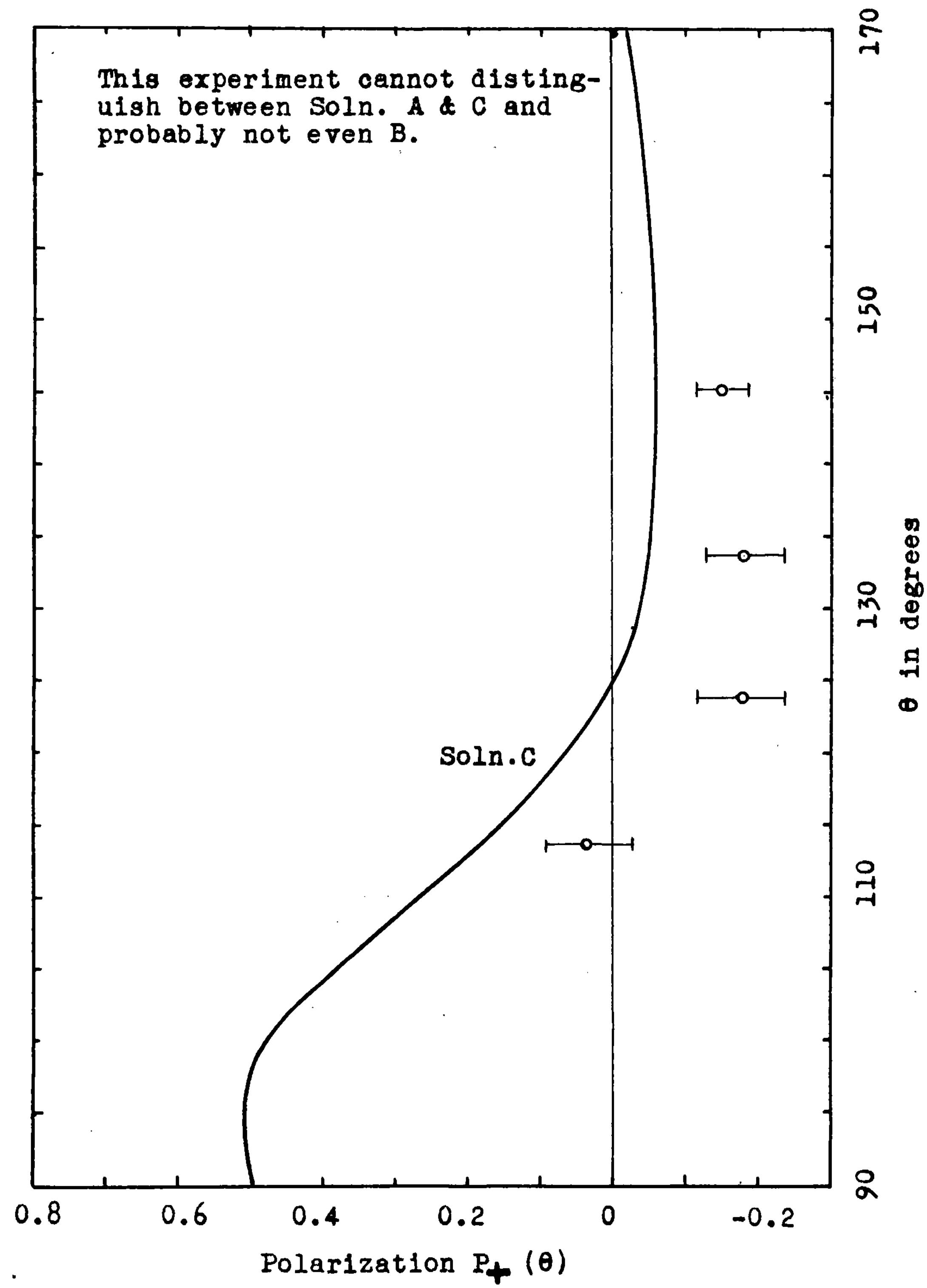

Fig. 9. A plot of the recoil proton polarization $P_+(\theta)$, in the scattering of protons on positive pions. The curve is obtained using the phase shift solution designated C of Chiu and Lomon (26). Solutions A and B, give essentially the same curve. The experimental points are taken from reference 45.

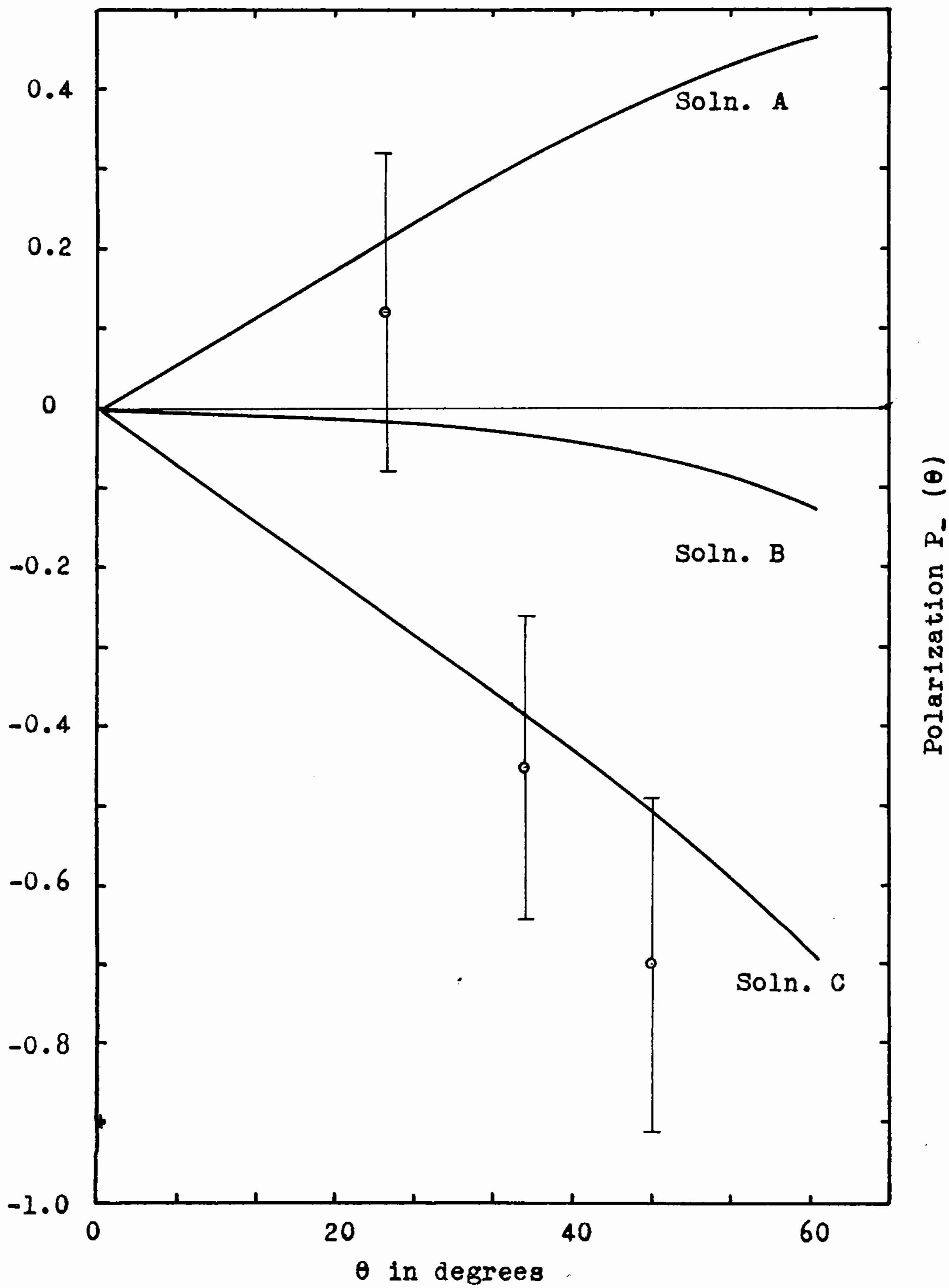

Fig. 10. A plot of the recoil proton polarization $P_-(\theta)$, in the scattering of protons on negative pions. The curves are obtained using the three solutions A, B, and C of Chiu and Lomon (26). The experimental points are obtained from reference 45.

3. Conclusions and Discussion

Before discussing the pertinence of our results, it would perhaps be in order to correct some unfortunate misstatements in the paper by Chiu and Lomon, (26), resulting from mislabelling of a graph. Their conclusions should be the following. At 220 and 307 Mev. three solutions designated A,B and C are found which fit the data. Of these solution A corresponds to a continuation of the low energy solutions, and fits the preliminary P_+ recoil proton polarization data at 220 Mev. Solution C is discontinuous in energy, with the solution below resonance, but also fits the preliminary polarization data. Solution B finally, does not fit the polarization data, and is further inconsistent with a form of dispersion relations sensitive to the small phase shifts with which A and C are consistent, in the region of resonance. These results therefore favour solutions A and C, solution B being definitely ruled out. From our calculations of P_+ and P_- at 307 Mev., we conclude that the P experiment does not really distinguish between the three phase shift sets. This is not surprising since $I_+ P_+$ only depends on the phase shifts through $\delta_3 \delta_{33}$ and δ_{31} which three are approximately the same for these three solutions, as are A_+ B_+ and C_+. The P_- experiment however does distinguish quite clearly between the three solutions, and from the preliminary results of Vasilevski and Vishniakov (45) at 300 Mev., we see that solution C is favoured over the other two.

Quite recently Korenchenko, Polumordvinova and Zinov (29) have performed an analysis of $\pi^+ \to \pi^+$, $\pi^- \to \pi^-$ and $\pi^- \to \pi^0$ data at several energies between 220 and 333 Mev. They obtained only two solution types designated a and b. At some energies they obtained other solutions but these they label as unphysical and related rather to the mathematical side of the problem, i.e., they are accidental. Of solutions a and b, a has an M value of $\cong$ 18, while b has an M value of $\cong$ 30, and in some cases as high as 71. Since the expected value of M is $\cong$ 19 (25 experimental points - 6 phase shifts), they conclude that solution a is a very likely one, while the probability of obtaining an average M value of $\cong$ 30 is $\cong$ 5%. They therefore conclude that solution a is the correct one. Further they attempt to fit the data starting with the Chiu and Lomon (26) solution A, at all energies (except 240 Mev.) they find that this solution led to their solution b. We reproduce their solution a below along with the Chiu and Lomon solution C at 307 Mev. We observe that they are essentially the same.

	Solution a	Solution C
δ_3	-23.9 ± 1.2	-24.7 ± 2.3
δ_{31}	-10.0 ± 2.0	-10.5 ± 3
δ_{33}	132.4 ± 0.9	132.4 ± 2
δ_1	17.1 ± 5.2	17.6 ± 10
δ_{11}	11.4 ± 3.3	13.3 ± 5.7
δ_{13}	-5.0 ± 1.2	-5.9 ± 3.5

Table 4. Solution a of Korenchenko et al and Solution C of Chiu and Lomon.

SECTION IV

VARIATIONAL PRINCIPLES FOR PHASE SHIFTS IN THE FESHBACH LOMON (10) BOUNDARY CONDITION MODEL

Perhaps the most convenient way of describing a short ranged interaction between two particles, is by means of a complete set of energy dependent phase shifts. It therefore follows that mathematical methods for obtaining the phase shifts, given the interaction, are quite important. One of the more fruitful approaches to this problem, is the variational approach. In brief, this method consists of obtaining for some function of the phase shifts, an integral expression over the interaction and the wave function, such that the stationary value of this expression with respect to arbitrary variations of the wave function, implies the correct equations of motion. Usually the equations of motion are written in integral form, and may be used in conjunction with the previous expression, to yield an iteration procedure for obtaining the phase shift. This method was used by F. Rohrlich and J. Eisenstein (46) for testing various exchange field theories of the nucleon-nucleon interaction, with respect to medium energy n-p cross sections, representing the interaction by rectangular and Yukawa well shapes, including tensor forces. In their work, variational principles were derived for both the case of uncoupled angular momentum states, and for the case of coupled states. The coupling of the angular momentum states, is of course due to the tensor term in the interaction. As pointed out by Lomon and Feshbach, the static potential method of representing the interaction is of limited validity, in particular it does not give a good description of the interaction at high or even moderate energies. In the region 100 to 380 Mev., that part of the interaction which can be represented by a local potential, is of less importance than the many pion exchange region. Lomon and Feshbach (see Section II) propose to represent the interaction by imposing an energy independent boundary condition on the logarithmic derivative of the wave function at the surface of a core region, and external to this core region, by a potential tail of the form.

$$[V_c(r) + V_T(r) S_{12}]^{S,\pi}$$

where $V_c(r)$ and $V_T(r)$ are central and tensor potentials, of arbitrary shape, and S_{12} is the usual tensor operator. S and π^+ are the spin and parity labels for the states being considered.

It is the purpose of this section, to obtain variational principles for the phase shifts for this model of the interaction. It is assumed that the phase shift solution of the pure boundary condition problem is known.

In Section IV, 2, we treat the case of uncoupled states, i.e., singlet, and triplet with parity $(-1)^{\ell}$. A Greens function is obtained which is reminiscent of that obtained by Rohrlich and Eisenstein, differing from theirs only in that the part containing the $u_<$ dependence contains a term $\tan\eta_0^j\, kx\, n_j(kx)$, η_0^j being the pertinent phase shift for the pure boundary condition problem. A variational principle for $\cot\eta$ is obtained, in which the R.H.S. is completely independent of η, except possibly through the variational wave functions.

In Section IV, 3, the case of coupled angular momentum states is treated, i.e., triplet, parity $(-1)^{j+1}$. A 2x2 matrix Greens function is obtained, which satisfies the differential equation

$$\begin{pmatrix} \frac{d^2}{dx^2} - \frac{j(j-1)}{x^2} + k^2 & \\ & \frac{d^2}{dx^2} - \frac{(j+1)(j+2)}{x^2} \end{pmatrix} G(xx') = -\delta(x-x')\,1$$

and the boundary condition

$$\frac{d}{dx} G(xx')\Big|_{x_0} = x_0^{-1} \begin{pmatrix} F_1+1 & f_c \\ f_c & F_2+1 \end{pmatrix} G(xx')\Big|_{x_0}$$

The 2-column wave vector satisfies the same boundary condition on the core surface x_0. In addition to this, reciprocity imposes the further symmetry condition

$$G(xx') = G^{\dagger}(x'x)$$

on the Greens function. The Greens function is also chosen so as to utilise the previously determined eigenphase shifts and mixing paramaters of the pure boundary condition problem. A variational principle is then obtained for the quantity.

$$\frac{\sin 2\epsilon_0 \tan 2\epsilon_0}{2k(\tan\eta - \tan\eta_0^{\alpha})(\tan\eta - \tan\eta_0^{\beta})}$$

Unlike the previous variational principles, this one contains quantities proportional to $(\tan\eta - \tan\eta_0^{\alpha})$ and $(\tan\eta - \tan\eta_0^{\beta})$, in the variational expression on the R.H.S. of the equation. A number of ways of utilising it are suggested.

1. The Case of Uncoupled States

We are concerned in this section with scattering in the states $j\, m_j\, 0\, (-1)^j$ and $j\, m_j\, 1\, (-1)^j$, where the first two quantum numbers refer to the total and z-component of angular momentum, and the last two refer to the spin and parity of the state respectively. In the first of these cases, the potential in the region external to the core may be written.

$${}_1V_C^{\pm} \qquad 1.1$$

and in the second case

$${}_3V_C^{\pm} + {}_3V_T^{\pm}\, S_{12} + {}_3V_{LS}^{\pm}\, \vec{L}\cdot\vec{S} \qquad 1.2$$

where $+$ refers to the even parity case, and $-$ refers to the odd parity case.

We have formally written in a spin orbit term in the external potential for the triplet case, but this actually will be a very short tail, since most of the spin orbit force dependence of the interaction will come from the core region. Further since the analysis for the first case is essentially a special case of the second case, (it is the second with s = 1 replaced by s = 0, and the tensor and spin orbit potentials put equal to zero), we will consider only the second case. The results for the first case may then readily be obtained from this.

The Schroedinger equation for this state is given by

$$\left[\nabla^2 + \frac{2\mu}{\hbar^2}(E-V) \right] \psi(j\, m_j\, 1\, (-1)^j) = 0$$

writing $\psi = \frac{u}{r} |j\, m_j\, 1\, (-1)^j\rangle$ with the labelling in the radial function suppressed, we obtain

$$\left[\frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}(E-V) - \frac{j(j+1)}{r^2} \right] u = 0 \qquad 1.3$$

where   is the reduced nucleon mass. Writing $x = \frac{r}{r_0}$ $k^2 = \frac{2\mu r_0^2}{\hbar^2}$ and $U = \frac{2\mu r_0^2}{\hbar^2} V$ where r_0 is some constant scale factor, say 10^{-13} cm., we obtain

$$\left[\frac{d^2}{dx^2} + k^2 - \frac{j(j+1)}{x^2} \right] u = \left[{}_3U_C^{\pi} - {}_3U_{LS}^{\pi} - 2\, {}_3U_T^{\pi} \right] u \qquad 1.4$$

Where we have used the following relations

$$S_{12} |j\, m_j\, 1\, (-1)^j\rangle = -2 |j\, m_j\, 1\, (-1)^j\rangle$$
$$\vec{L}\cdot\vec{S} |j\, m_j\, 1\, (-1)^j\rangle = - |j\, m_j\, 1\, (-1)^j\rangle \qquad 1.5$$

Equation 1.3 then becomes, writing the right hand side of

1.4 as $l(x)\ u(x)$.

$$\left[\frac{d^2}{dx^2} + k^2 - \frac{\ell(\ell+1)}{x^2} \right] u = l u \qquad 1.6$$

The Greens function for the problem satisfies

$$\left[\frac{d^2}{dx^2} + k^2 - \frac{\ell(\ell+1)}{x^2} \right] G(xx') = -\delta(x-x') \qquad 1.7$$

Multiplying 1.6 by $G(xx')$ and 1.7 by $u(x)$ subtracting and integrating over x from the core x_0 to ∞ , we obtain integrating by parts.

$$u(x') = -\int_{x_0}^{\infty} l\, u\, G(xx')\, dx + \left[G(xx')\, u'(x) - G'(xx')\, u(x) \right]_{x_0}^{\infty} \qquad 1.8$$

Where the derivatives are with respect to x .

The Feshbach-Lomon boundary condition on the wave function at x_0 is given by

$$u'(x_0) = x_0^{-1} \left(F'_{\ell\ell} + 1 \right) u(x_0) \qquad 1.9$$

where $F'_{\ell\ell}$ is the energy independent paramater for the state $\ell\, m_\ell\, 1\ (-1)^\ell$. From here on we will suppress the labels for simplicity. We choose our Greens functions to satisfy the same boundary condition on the core, with respect to the x variable. Interchanging x and x' , 1.8 then takes the form

$$u(x) = -\int_{x_0}^{\infty} G(x'x)\, l(x')\, u(x')\, dx' + \left[G(\infty x)\, u'(\infty) - G'(\infty x)\, u(\infty) \right] \qquad 1.10$$

For $x > x'$ we choose the solution

$$G^{>}(xx') = C\, kx\, n_\ell(kx) \qquad 1.11$$

For $x < x'$ we choose the solution

$$G^{<}(xx') = [A\, j_{j}(kx) + B\, n_{j}(kx)]\, kx \qquad 1.12$$

We have to satisfy the following conditions with these constants

$$G^{<\prime}(x_0 x') = (F+1)\, x_0^{-1}\, G^{<}(x_0 x') \qquad 1.13$$

$$G^{>}(x'x') = G^{<}(x'x') \qquad 1.14$$

$$G^{>\prime}(x'x') - G^{<\prime}(x'x') = -1 \qquad 1.15$$

Using these conditions to determine the unknown constants, we obtain

$$G(xx') = -k^{-1}\, kx_{>}\, n_{>}(kx)\,.\, kx_{<}\, [\, j(kx) - \tan\eta_0\, n(kx)]_{<} = G(x'x) \qquad 1.16$$

where we have for simplicity of notation omitted the subscript j on the Bessel functions which gives their order, and where the notation $>$ and $<$ means respectively the larger and the smaller of x and x'.

Here $$\tan\eta_0 = \frac{kx_0\, j'(kx_0) - F\, j(kx_0)}{kx_0\, n'(kx_0) - F\, n(kx_0)} \qquad 1.17$$

which is the phase shift for the pure boundary condition problem. See Ref. 10.

Writing

$$G(\infty x') \approx k^{-1}\,.\, kx\, (j - \tan\eta^0\, n)\, \cos(kx' - \tfrac{j\pi}{2}) \qquad 1.18$$

$$u(\infty) \approx \frac{A}{\cos\eta}\, \sin(kx' - \tfrac{j\pi}{2} + \eta)$$

We obtain for 1.10

$$u(x) = A\,kx[j - \tan\eta^{o} n] - \int_{x_0}^{\infty} l\,u\,G(x x')\,dx' \qquad 1.19$$

Looking at the asymptotic form of this we obtain

$$u(\infty) \approx \frac{A}{\cos\eta}\,\mathrm{Sin}\left(kx - \frac{l\pi}{2} + \eta\right) = A\,\mathrm{Sin}\left(kx - \frac{l\pi}{2}\right) +$$

$$\mathrm{Cos}\left(kx - \frac{l\pi}{2}\right)\left\{\tan\eta_0 A - \int_{x_0}^{\infty}(j - \tan\eta_0 n)\,x'\,dx'\right\}$$

whence

$$A = -(\tan\eta - \tan\eta_0)^{-1}\int_{x_0}^{\infty} l\,u\,(j - \tan\eta_0 n) \qquad 1.20$$

$$x'\,dx'$$

1.19 therefore becomes

$$u(x) = -\int_{x_0}^{\infty} l\,u\,G(x x')\,dx' - \{(\tan\eta - \tan\eta_0)^{-1} \qquad 1.21$$

$$\int_{x_0}^{\infty} l\,u\,(j - \tan\eta_0 n)\,x'\,dx'\}\,kx(j - \tan\eta_0 n)$$

which is the equation of motion.

Multiply both sides by $l(x)u(x)$ and integrate over x from x_0 to ∞, we obtain

$$-k(\tan\eta - \tan\eta_0)^{-1} = \frac{\int_{x_0}^{\infty} l\,u^2\,dx + \int_{x_0}^{\infty}\int_{x_0}^{\infty} l(x)u(x)G(xx')l(x')u(x')\,dx\,dx'}{\left[\int_{x_0}^{\infty} l\,u\,(j - \tan\eta_0 n)\right]^2} \qquad 1.22$$

This constitutes a variational principal, since when it is stationary with respect to arbitrary variations of u, δu, it implies the equation of motion. We will now show this. When it is stationary.

1.23

$$\frac{\delta\ \text{Numerator}}{\delta\ \text{Denominator}} = -k(\tan\eta - \tan\eta_0)^{-1}$$

Since the variation δu is arbitrary, we can then write, choosing δu to be proportional to a δ- function

$$\delta N = 2\delta u \left\{ \ell u + \ell \int_{x_0}^{\infty} G(x x') \ell(x') u(x') dx' \right\}$$

$$\delta D = 2\delta u \left\{ \ell(\partial - \tan\eta_0 n) x \int_{x_0}^{\infty} \ell u (\partial - \tan\eta_0 n) x' dx' \right\} \qquad 1.24$$

1.24 then gives

$$u = -\int_{x_0}^{\infty} G(x' x) \ell u x' dx' - kx(\partial - \tan\eta_0 n) \cdot (\tan\eta - \tan\eta_0)^{-1} \int_{x_0}^{\infty} \ell u (\partial - \tan\eta_0 n) x' dx'$$

which is the correct equation of motion, see 1.21. Therefore N/D stationary leads to the correct phase shift

2. The Case of Coupled States and Coupled Boundary Conditions

We are now interested in obtaining a variational principle for the eigenphase shifts for the coupled states $\partial m_\partial 1 \ (-1)^{\partial-1}$ i.e., $\ell = \partial - 1$ and $\ell = \partial + 1$. These states are coupled both by the tensor interaction in the tail and that implied by the coupling in the boundary condition. We have the following relations

$$S_{12} |\partial m_\partial 1 \partial - 1\rangle = -2\frac{\partial - 1}{2\partial + 1} |\partial m_\partial 1 \partial - 1\rangle + \frac{6[\partial(\partial+1)]^{\frac{1}{2}}}{2\partial+1} |\partial m_\partial 1 \partial + 1\rangle$$

$$S_{12} |\partial m_\partial 1 \partial + 1\rangle = -2\frac{\partial + 2}{2\partial + 1} |\partial m_\partial 1 \partial + 1\rangle + \frac{6[\partial(\partial+1)]^{\frac{1}{2}}}{2\partial+1} |\partial m_\partial 1 \partial - 1\rangle \qquad 2.1$$

$$\vec{L}\cdot\vec{S} \ |\partial m_\partial 1 \partial - 1\rangle = (\partial - 1) |\partial m_\partial 1 \partial - 1\rangle$$

$$\vec{L}\cdot\vec{S} \ |\partial m_\partial 1 \partial + 1\rangle = -(\partial + 2) |\partial m_\partial 1 \partial + 1\rangle$$

The first two relations may be obtained from Rohrlich and Eisenstein,(46), the last two by using the relation

$J^2 = L^2 + S^2 + \vec{L}\cdot\vec{S}$.

We will denote the radial dependence of the $J-1$ state by u_1 , and the radial dependence of the $J+$ state by u_2. The Schroedinger equation then reads

$$\left(\frac{d^2}{dx^2} + k^2 - \frac{J(J-1)}{x^2}\right) u_1 = \left({}_3V_C^{\pi} + (J-1)\,{}_3V_{LS}^{\pi} - \frac{2(J-1)}{2J+1}\,{}_3V_T^{\pi}\right) u_1 + 6\,\frac{[J(J+1)]^{\frac{1}{2}}}{2J+1}\,{}_3V_T^{\pi}\, u_2$$

$$\left(\frac{d^2}{dx^2} + k^2 - \frac{(J+1)(J+2)}{x^2}\right) u_2 = \left({}_3V_C^{\pi} - (J+2)\,{}_3V_{LS}^{\pi} - \frac{2(J+2)}{2J+1}\,{}_3V_T^{\pi}\right) u_2 + 6\,\frac{[J(J+1)]^{\frac{1}{2}}}{2J+1}\,{}_3V_T^{\pi}\, u_1 \qquad 2.2$$

We write these

$$\left(\frac{d^2}{dx^2} + k^2 - \frac{J(J-1)}{x^2}\right) u_1 = f u_1 + g u_2$$

$$\left(\frac{d^2}{dx^2} + k^2 - \frac{(J+1)(J+2)}{x^2}\right) u_2 = g u_1 + h u_2 \qquad 2.3$$

with

$$f = \left({}_3V_C^{\pi} + (J-1)\,{}_3V_{LS}^{\pi} - 2\,\frac{J-1}{2J+1}\,{}_3V_T^{\pi}\right).$$

$$h = \left({}_3V_C^{\pi} - (J+2)\,{}_3V_{LS}^{\pi} - 2\,\frac{J+2}{2J+1}\,{}_3V_T^{\pi}\right). \qquad 2.4$$

$$g = 6\,\frac{[J(J+1)]^{\frac{1}{2}}}{2J+1}\,{}_3V_T^{\pi}$$

The boundary condition at the surface of the core region now reads

$$\begin{pmatrix} u_1'(x_0) \\ u_2'(x_0) \end{pmatrix} = x_0^{-1} \begin{pmatrix} F_1+1 & f_c \\ f_c & F_2+1 \end{pmatrix} \begin{pmatrix} u_1(x_0) \\ u_2(x_0) \end{pmatrix} \qquad 2.5$$

Writing the equations of motion in matrix form, we obtain

$$\begin{pmatrix} \frac{d^2}{dx^2}+L_1^2 & 0 \\ 0 & \frac{d^2}{dx^2}+L_2^2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f & g \\ g & h \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad 2.6$$

The equation for the required Greens function is

$$\begin{pmatrix} \frac{d^2}{dx^2}+L_1^2 & 0 \\ 0 & \frac{d^2}{dx^2}+L_2^2 \end{pmatrix}\begin{pmatrix} G_1(xx') & G_{c_1}(xx') \\ G_{c_2}(xx') & G_2(xx') \end{pmatrix} = -\delta(x-x')\,1 \qquad 2.7$$

we now write 2.5, 2.6 and 2.7 in operator form

$$u'(x_0) = x_0^{-1}\,\phi\, u(x_0) \qquad 2.5'$$

$$\begin{pmatrix} D^2+L_1^2 & 0 \\ 0 & D^2+L_2^2 \end{pmatrix} u = F u \qquad 2.6'$$

$$\begin{pmatrix} D^2+L_1^2 & 0 \\ 0 & D^2+L_2^2 \end{pmatrix} G(xx') = -\delta(x-x')\,1 \qquad 2.7'$$

Multiply 2.6′ by $G^{\dagger}(xx')$ and 2.7′ by $u^{\dagger}(x)$ where the $\dagger$ interchanges rows and columns, i.e., $G^{\dagger}$ is the transpose of G. We then obtain

$$G^{\dagger}(xx')\begin{pmatrix} D^2+L_1^2 & 0 \\ 0 & D^2+L_2^2 \end{pmatrix} u = G^{\dagger}(xx')\, F u \qquad 2.8$$

$$u^{\dagger}\begin{pmatrix} D^2+L_1^2 & 0 \\ 0 & D^2+L_2^2 \end{pmatrix} G(xx') = -u^{\dagger}\,\delta(x-x')\,1 \qquad 2.9$$

Take the transpose of the second equation subtract it from the first and integrate from x_0 to ∞, we get

$$\int_{x_0}^{\infty}\left(G^{\dagger}(xx')\,D^2 u - [D^2 G(xx')]^{\dagger} u\right) dx = \qquad 2.10$$

$$\int_{x_0}^{\infty}[\,G^{\dagger}(xx')\,F\,u(x)\,]\,dx + u(x')$$

Integrating by parts, we then obtain

$$u(x') = -\int_{x_0}^{\infty} G^{+}(x x')\, F u\, dx + \left[G^{+}(x x')\, u'(x) - G^{+\prime}(x x')\, u(x) \right]_{x_0}^{\infty} \qquad 2.11$$

At the lower limit

$$u'(x_0) = x_0^{-1} \phi\, u(x_0)$$

We choose

$$G^{+}(x_0 x') = x_0^{-1} .\; G^{+}(x_0 x')\, \phi \qquad 2.12$$

Then since $\phi^{+} = \phi$, we have that.

$$G'(x_0 x') = x_0^{-1} \phi \,.\, G(x_0 x') \qquad 2.13$$

We then obtain for 2.11

$$u(x) = -\int_{x_0}^{\infty} G^{+}(x' x)\, F u\, dx' + \left[G^{+}(\infty x)\, u'(\infty) - G^{+\prime}(\infty x)\, u(\infty) \right] \qquad 2.14$$

From the boundary condition equation 2.13 satisfied by the Greens function, we obtain

$$G_1'(x_0 x') = x_0^{-1} (F_1 + 1)\, G_1(x_0 x') + x_0^{-1} f_c\, G_{c_2}(x_0 x') \qquad 2.15$$

$$G_{c_2}'(x_0 x') = x_0^{-1} (F_2 + 1)\, G_{c_2}(x_0 x') + x_0^{-1} f_c\, G_1(x_0 x') \qquad 2.16$$

$$G_2'(x_0 x') = x_0^{-1} (F_2 + 1)\, G_2(x_0 x') + x_0^{-1} f_c\, G_{c_1}(x_0 x') \qquad 2.17$$

$$G_{c_1}'(x_0 x') = x_0^{-1} (F_1 + 1)\, G_{c_1}(x_0 x') + x_0^{-1} f_c\, G_1(x_0 x') \qquad 2.18$$

In addition to these, the Greens function must satisfy the conditions

$$G^{>}(x'x') = G^{<}(x'x') \qquad 2.19$$

$$G^{>}(x'x') - G^{<}(x'x') = -1 \qquad 2.20$$

Where the $>$ and $<$ notation, is as before in the uncoupled case.

Writing these out in detail

$$G_1^{>}(x'x') = G_1^{<}(x'x') \qquad 2.21$$

$$G_1^{>\prime}(x'x') - G_1^{<\prime}(x'x') = -1 \qquad 2.22$$

$$G_2^{>}(x'x') = G_2^{<}(x'x') \qquad 2.23$$

$$G_2^{>}(x'x') - G_2^{<}(x'x') = -1 \qquad 2.24$$

We also demand that it satisfy the reciprocity condition.

$$G(xx') = G^{+}(x'x) \qquad 2.25$$

This gives

$$G_{c_1}(xx') = G_{c_2}(x'x) \qquad 2.26$$

$$G_{c_2}(xx') = G_{c_1}(x'x) \qquad 2.27$$

We use the Greens function

$$G(xx') = N\begin{pmatrix} -k^{-1}\cot\epsilon_0\,\partial^{\beta}_{1>}\partial^{\alpha}_{1<} & -k^{-1}\partial^{\beta}_{1}(x)\,\partial^{\alpha}_{2}(x') \\ -k^{-1}\partial^{\alpha}_{2}(x)\,\partial^{\beta}_{1}(x') & k^{-1}\cot\epsilon_0\,\partial^{\beta}_{2<}\partial^{\alpha}_{2>} \end{pmatrix} \qquad 2.28$$

Where $\partial_1^\alpha = (\partial_1 - \tan\eta_0^\alpha n_1) kx = J_1^\alpha y$, $y = kx$ etc.

The constants $N, \tan\eta_0^\alpha, \tan\eta_0^\beta, \tan\epsilon_0$ are to be determined by the boundary conditions 2.15 to 2.20 and 2.25. Let us now check equations 2.15 to 2.27. As one can see by inspection, equations 2.19, 2.25 are obviously satisfied by the proposed Greens function.

From equation 2.22 we obtain

$$N^{-1}\tan\epsilon_0 = y'^2 (J_1^{\beta\prime} J_1^\alpha - J_1^{\alpha\prime} J_1^\beta) \qquad 2.29$$

Working this out, we get

$$N^{-1}\tan\epsilon_0 = y'^2 (\tan\eta_0^\alpha - \tan\eta_0^\beta)(\partial_1 n_1' - n_1 \partial_1')$$

so that $N = \tan\epsilon_0 (\tan\eta_0^\alpha - \tan\eta_0^\beta)^{-1}$ 2.30

The equivalent equation for G_2 equation 2.24 is evidently consistent with this, since it may be obtained from this by interchanging α and β, and $\tan\epsilon_0$ by $-\tan\epsilon_0$, noting that the Wronskian is independent of the order of the Bessel functions.

From equations 2.15 and 2.16, we obtain

$$(J_1^{\alpha\prime} y_0 - F_1 J_1^\alpha) = f_c \tan\epsilon_0 J_2^\alpha \qquad 2.31$$

and

$$(y_0 J_2^{\alpha\prime} - F_2 J_2^\alpha) = f_c \cot\epsilon_0 J_1^\alpha \qquad 2.32$$

Multiplying 2.31 by 2.32 we obtain writing J_1^α, J_2^α explicitly

$$y_0 (\partial_1' - \tan\eta_0^\alpha n_1') = (\partial_1 - \tan\eta_0^\alpha n_1) \times$$

$$\left\{ F_1 + \frac{f_c^2}{\dfrac{y_0(\partial_2' - \tan\eta_0^\beta n_2')}{\partial_2 - \tan\eta_0^\alpha n_2} - F_2} \right\} = (\partial_1 - \tan\eta_0^\alpha n_1) F^\alpha \qquad 2.33$$

Which is just equation 18 of Lomon and Feshbach. This equation 2.33 yields $\tan\eta_0^\alpha$ and $\tan\eta_0^\beta$. Evidently it is symmetric in 1 and 2, as can be easily seen by looking at the product 2.31 × 2.32. If we had looked at the equations connecting G_2 and G_{c_1}, we would therefore have obtained the same result.

The mixing paramater $\tan\epsilon_0 = \tan\epsilon_0^\alpha$ is given by

$$\tan\epsilon_0 = \frac{J_1^\alpha}{J_2^\alpha f_c}\left(y_0 \frac{J_1^{\alpha\prime}}{J_1^\alpha} - F_1 \right) = \frac{(F^\alpha - F_1)}{f_c}\left[\frac{\partial_1 - \tan\eta_0^\alpha n_1}{\partial_2 - \tan\eta_0^\alpha n_2} \right] \qquad 2.34$$

which is identical with equation 24 of Lomon and Feshbach. To obtain this we have used equation 2.33.

So we see that our constants $\tan\eta_0^\alpha$, $\tan\eta_0^\beta$ and $\tan\epsilon_0$, are respectively the eigenphase shifts and mixing paramater for the pure boundary condition problem.

We now evaluate the integrated term of equation 2.14, it is

$$\begin{pmatrix} G_1(\omega x') u_1'(\omega) + G_{c_2}(\omega x') u_2'(\omega) - G_1'(\omega x') u_1(\omega) \\ - G_{c_2}'(\omega x') u_2(\omega) \\ G_2(\omega x') u_2'(\omega) + G_{c_1}(\omega x') u_1'(\omega) - G_2'(\omega x') u_2(\omega) \\ - G_{c_1}'(\omega x') u_1(\omega) \end{pmatrix} \qquad 2.35$$

We are interested in the eigen solutions to the scattering problem, so we choose

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} A(\mathcal{J}_1 - \tan\eta\, n_1) \\ B(\mathcal{J}_2 - \tan\eta\, n_2) \end{pmatrix} = A \begin{pmatrix} (\mathcal{J}_1 - \tan\eta\, n_1) \\ \tan\epsilon(\mathcal{J}_2 - \tan\eta\, n_2) \end{pmatrix} \quad 2.36$$

We have

$$u_1(\omega) \approx \frac{A}{\cos\eta} \sin\left(y - \mathcal{J} - 1\tfrac{\pi}{2} + \eta\right)$$

$$u_2(\omega) \approx \frac{A\tan\epsilon}{\cos\eta} \sin\left(y - \mathcal{J} + 1\tfrac{\pi}{2} + \eta\right)$$

$$G_1(\omega x') \approx -\frac{N k^{-1} \cot\epsilon_0}{\cos\eta_0^\beta}\, y' J_1^\alpha \sin\left(y - \mathcal{J} - 1\tfrac{\pi}{2} + \eta_0^\beta\right) \quad 2.37$$

$$G_2(\omega x') \approx \frac{N k^{-1} \cot\epsilon_0}{\cos\eta_0^\alpha}\, y' J_2^\beta \sin\left(y - \mathcal{J} + 1\tfrac{\pi}{2} + \eta_0^\alpha\right)$$

$$G_{c_2}(\omega x') \approx \frac{-N k^{-1}}{\cos\eta_0^\alpha}\, y' J_1^\beta \sin\left(y - \mathcal{J} + 1\tfrac{\pi}{2} + \eta_0^\alpha\right)$$

$$G_{c_1}(\omega x') \approx \frac{-N k^{-1}}{\cos\eta_0^\beta}\, y' J_2^\alpha \sin\left(y - \mathcal{J} - 1\tfrac{\pi}{2} + \eta_0^\beta\right)$$

Rewriting this more simply

$$u_1(\omega) \approx \frac{A}{\cos\eta} \sin(\phi_1 + \eta)$$

$$u_2(\omega) \approx \frac{A\tan\epsilon}{\cos\eta} \sin(\phi_2 + \eta)$$

$$G_2(\omega x') \approx \frac{N k^{-1}}{\cos\eta_0^\alpha} \cot\epsilon_0\, \mathcal{J}_2^\beta \sin(\phi_2 + \eta_0^\alpha) \quad 2.38$$

$$G_1(\omega x') \approx -\frac{N k^{-1}}{\cos\eta_0^\beta} \cot\epsilon_0\, \mathcal{J}_1^\alpha \sin(\phi_1 + \eta_0^\beta)$$

$$G_{c_1}(\omega x') \approx -\frac{N k^{-1}}{\cos\eta_0^\beta}\, \mathcal{J}_2^\alpha \sin(\phi_1 + \eta_0^\beta)$$

$$G_{c_2}(\omega x') \approx -\frac{N k^{-1}}{\cos\eta_0^\alpha}\, \mathcal{J}_1^\beta \sin(\phi_2 + \eta_0^\alpha)$$

The integrated term is therefore

1st element

$$-\frac{NA}{\cos\eta}\left\{\frac{\cot\epsilon_0}{\cos\eta_0^\beta}\,\partial_1^\alpha \sin(p_1+\eta_0^\beta)\cos(p_1+\eta)+\frac{\partial_1^\beta}{\cos\eta_0^\alpha}\tan\epsilon\cos(p_2+\eta)\right.$$
$$\sin(p_2+\eta_0^\alpha)-\frac{\cot\epsilon_0}{\cos\eta_0^\beta}\,\partial_1^\alpha\cos(p_1+\eta_0^\beta)\sin(p_1+\eta)-\frac{\partial_1^\beta}{\cos\eta_0^\alpha}\cos(p_2+\eta_0^\alpha)\cdot$$
$$\left.\tan\epsilon\sin(p_2+\eta)\right\} =$$
$$A(\tan\eta_0^\alpha-\tan\eta_0^\beta)^{-1}\left\{\partial_1^\alpha(\tan\eta-\tan\eta_0^\beta)+\partial_1^\beta\tan\epsilon\tan\epsilon_0\cdot\right.$$
$$\left.(\tan\eta-\tan\eta_0^\alpha)\right\} \qquad 2.39$$

2nd element

$$\frac{NA}{\cos\eta}\left\{\frac{\cot\epsilon_0}{\cos\eta_0^\alpha}\,\partial_2^\beta\sin(p_2+\eta_0^\alpha)\tan\epsilon\cos(p_2+\eta)-\frac{\partial_2^\alpha}{\cos\eta_0^\beta}\sin(p_1+\eta_0^\beta)\right.$$
$$\cos(p_1+\eta)-\frac{\cot\epsilon_0}{\cos\eta_0^\alpha}\,\partial_2^\beta\cos(p_2+\eta_0^\alpha)\tan\epsilon\sin(p_2+\eta)+\frac{\partial_2^\alpha}{\cos\eta_0^\beta}\cos(p_1+\eta_0^\beta)$$
$$\left.\sin(p_1+\eta)\right\}$$
$$=$$
$$A(\tan\eta_0^\alpha-\tan\eta_0^\beta)^{-1}\left\{-\partial_2^\beta\tan\epsilon(\tan\eta-\tan\eta_0^\alpha)+\partial_2^\alpha\tan\epsilon_0\cdot(\tan\eta-\tan\eta_0^\beta)\right\} \qquad 2.40$$

So rewriting the integrated term, it is

$$= A(\tan\eta_0^\alpha-\tan\eta_0^\beta)^{-1}\begin{pmatrix}\partial_1^\alpha(\tan\eta-\tan\eta_0^\beta)+\partial_1^\beta\tan\epsilon\tan\epsilon_0(\tan\eta-\tan\eta_0^\alpha)\\ \partial_2^\alpha(\tan\eta-\tan\eta_0^\beta)+\partial_2^\beta\tan\epsilon(\tan\eta-\tan\eta_0^\alpha)\end{pmatrix}$$

2.41

We wish to obtain $\tan\epsilon$ and A in terms of $\tan\eta$ and integrals over the potentials. To do this, we look at the asymptotic form of equation 2.14.

1st element

$$\frac{A}{\cos\eta}\sin(p_1+\eta) = -\int_{x_0}^{\infty}[G_1(x\omega)P_1(x)+G_{c_2}(x\omega)P_2(x)]\,dx$$
$$+\frac{A}{(\tan\eta_0^\alpha-\tan\eta_0^\beta)^{-1}}\left\{\frac{\sin(p_1+\eta_0^\alpha)}{\cos\eta_0^\alpha}(\tan\eta-\tan\eta_0^\beta)+\frac{\sin(p_1+\eta_0^\beta)}{\cos\eta_0^\beta}\tan\epsilon\,.\right.$$
$$\left.\tan\epsilon\,(\tan\eta-\tan\eta_0^\alpha)\right\}$$

with $P_1 = (f u_1 + g u_2)$, $P_2 = (g u_1 + h u_2)$

so $\dfrac{A}{\cos\eta}\sin(p_1+\eta) = k^{-1}\dfrac{\sin}{\cos\eta_0^\beta}(p_1+\eta_0^\beta)\int_{x_0}^{\infty}[\cot\epsilon_0\,\partial_1^\alpha P_1$
$+\partial_2^\alpha P_2]\,dx + A(\tan\eta_0^\alpha - \tan\eta_0^\beta)^{-1}\Big\{\dfrac{\sin}{\cos\eta_0^\alpha}(p_1+\eta_0^\alpha)\,.$
$(\tan\eta - \tan\eta_0^\beta) + \dfrac{\sin}{\cos\eta_0^\beta}(p_1+\eta_0^\beta)\tan\epsilon\tan\epsilon_0(\tan\eta - \tan\eta_0^\alpha)\Big\}$

Writing

$$I_1 = \int_{x_0}^{\infty}[\cot\epsilon_0\,\partial_1^\alpha P_1 + \partial_2^\alpha P_2]\,dx \qquad 2.42$$

We obtain

$$\sin p_1\,\{A(\tan\eta - \tan\eta_0^\alpha)(1+\tan\epsilon_0\tan\epsilon) + k^{-1}\tan\epsilon_0 I_1\}$$
$$+\cos p_1\,\tan\eta_0^\beta\{A(\tan\eta - \tan\eta_0^\alpha)(1+\tan\epsilon\tan\epsilon_0) + k^{-1}\tan\epsilon_0 I_1\} = 0$$

So we choose

$$A = -k^{-1}\tan\epsilon_0(\tan\eta - \tan\eta_0^\alpha)^{-1}(1+\tan\epsilon_0\tan\epsilon)I_1 \qquad 2.43$$

2nd element

$$\frac{A\tan\epsilon}{\cos\eta}\sin(p_2+\eta) = -\int_{x_0}^{\infty}[G_2(x\,\omega)P_2(x) + G_{c_1}(x\,\omega)P_1(x)]\,dx$$
$$+A(\tan\eta_0^\alpha - \tan\eta_0^\beta)^{-1} = \Big\{\frac{\sin}{\cos\eta_0^\alpha}(p_2+\eta_0^\alpha)\tan\epsilon_0(\tan\eta - \tan\eta_0^\beta)$$
$$-\frac{\sin}{\cos\eta_0^\beta}(p_2+\eta_0^\beta)\tan\epsilon(\tan\eta - \tan\eta_0^\alpha)\Big\}.$$

$$\frac{A\tan\epsilon}{\cos\eta}\sin(p_2+\eta) = k^{-1}\sin(p_2+\eta_0^\alpha)\tan\epsilon_0(\tan\eta_0^\alpha - \tan\eta_0^\beta)^{-1}$$
$$\int_{x_0}^{\infty}[-\cot\epsilon_0\,\partial_2^\beta P_2 + \partial_1^\beta P_1]\,dx + A(\tan\eta_0^\alpha - \tan\eta_0^\beta)^{-1}\Big\{\frac{\sin}{\cos\eta_0^\alpha}(p_2+\eta_0^\alpha).$$
$$\tan\eta_0^\alpha(\tan\eta - \tan\eta_0^\beta) - \frac{\sin}{\cos\eta_0^\beta}(p_2+\eta_0^\beta)\tan\epsilon(\tan\eta - \tan\eta_0^\alpha)\Big\}$$

Write $I_2 = \int_{x_0}^{\omega} (-\cot\epsilon_0 \partial_2^\beta P_2 + \partial_1^\beta P_1)\,dx$ 2.44

Then $\sin(p_2)\{-A(\tan\eta - \tan\eta_0^\beta)(\tan\epsilon - \tan\epsilon_0) + k^{-1}\tan\epsilon_0 I_2\} + \cos(p_2)\tan\eta_0^\alpha\{-A(\tan\eta - \tan\eta_0^\beta)\cdot(\tan\epsilon - \tan\epsilon_0) + k^{-1}\tan\epsilon_0 I_2\} = 0$

Hence we choose

$$A = k^{-1}\tan\epsilon_0 (\tan\eta - \tan\eta_0^\beta)^{-1}(\tan\epsilon - \tan\epsilon_0)^{-1} I_2 \qquad 2.45$$

Rewriting equations 2.43 and 2.45,

$$A = -\frac{k^{-1}\tan\epsilon_0 I_1}{(\tan\eta - \tan\eta_0^\alpha)(1 + \tan\epsilon_0\tan\epsilon)} \qquad 2.46$$

$$A = -\frac{k^{-1} I_2}{(\tan\eta - \tan\eta_0^\beta)(1 - \cot\epsilon_0\tan\epsilon)} \qquad 2.47$$

From equations 2.46 and 2.47 we have

$$\tan\epsilon = \frac{I_1\tan\epsilon_0 L_\beta - I_2 L_\alpha}{I_1 L_\beta + I_2\tan\epsilon_0 L_\alpha} \qquad 2.48$$

Where

$$L_\alpha = \tan\eta - \tan\eta_0^\alpha$$
$$L_\beta = \tan\eta - \tan\eta_0^\beta \qquad 2.49$$

From equations 2.46 to 2.48

$$A = -\frac{\sin 2\epsilon_0}{2k L_\alpha L_\beta}[I_1 L_\beta + I_2\tan\epsilon_0 L_\alpha] \qquad 2.50$$

$$A\tan\epsilon_0 = -\frac{\sin 2\epsilon_0}{2k L_\alpha L_\beta}[I_1\tan\epsilon_0 L_\beta - I_2 L_\alpha] \qquad 2.51$$

Whence we have for the equations of motion 2.14 using equations 2.50, 2.51 and 2.41

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\frac{\sin 2\epsilon_0}{2k L_\alpha L_\beta}\,(\tan\eta_0^\alpha - \tan\eta_0^\beta)^{-1}\cdot$$

$$\begin{pmatrix} \partial_1^\alpha L_\beta\,(I_1 L_\beta + I_2 \tan\epsilon_0 L_\alpha) + \partial_1^\beta \tan\epsilon_0\,(I_1 \tan\epsilon_0 L_\beta - I_2 L_\alpha)\,L_\alpha \\ \partial_2^\alpha \tan\epsilon_0 L_\beta (I_1 L_\beta + I_2 \tan\epsilon_0 L_\alpha) - \partial_2^\beta L_\alpha\,(I_1 \tan\epsilon_0 - I_2 L_\alpha) \end{pmatrix}$$

$$- \begin{pmatrix} \int_{x_0}^{\omega} (P_1 G_1(xx') + P_2 G_{c_2}(xx'))\,dx \\ \int_{x_0}^{\omega} (P_2 G_2(xx') + P_1 G_{c_1}(xx'))\,dx \end{pmatrix} \qquad 2.52$$

In the last term, we use the notation $\int P_1\, G(xx')\,dx$ for integration over x , and $\int G_1(xx')\, P_1\, dx'$ for integration over x' .

Also recall

$$I_1 = \int_{x_0}^{\omega} (\cot\epsilon_0\, \partial_1^\alpha P_1 + \partial_2^\alpha P_2)\,dx$$

$$I_2 = \int_{x_0}^{\omega} (-\cot\epsilon_0\, \partial_2^\beta P_2 + \partial_1^\beta P_1)\,dx$$

In order to obtain a variational principle, multiply both sides of equation 2.52 by

$$(u_1(x') \quad u_2(x'))\begin{pmatrix} f(x') & g(x') \\ g(x') & h(x') \end{pmatrix} = (P_1(x') \quad P_2(x'))$$

and integrate over x'. We obtain

$$\int_{x_0}^{\omega} (P_1 u_1 + P_2 u_2)\,dx' = -\frac{\sin 2\epsilon_0}{2k L_\alpha L_\beta}\,(\tan\eta_0^\alpha - \tan\eta_0^\beta)^{-1}\cdot$$

$$\Big\{ L_\beta\,(I_1 L_\beta + I_2 \tan\epsilon_0 L_\alpha) \int_{x_0}^{\omega} (\partial_1^\alpha P_1 + \partial_2^\alpha \tan\epsilon_0 P_2)\,dx'$$

$$+ L_\alpha\,(I_1 \tan\epsilon_0 L_\beta - I_2 L_\alpha) \int_{x_0}^{\omega} (\partial_1^\beta \tan\epsilon_0 P_1 - \qquad 2.53$$

$$\partial_2^\beta P_2)\,dx' \Big\} - \Big\{ \int_{x_0}^{\omega}\!\!\int_{x_0}^{\omega} (P_1 G_1 P_1 + P_2 G_2 P_2 + P_2 G_{c_2} P_1 + P_1 G_{c_1} P_2)\,dx\,dx'$$

This may be reduced to give

$$\int_{x_0}^{\infty}(P_1u_1+P_2u_2)\,dx' = -\frac{\sin 2\epsilon_0 \tan\epsilon_0}{2k L_\alpha L_\beta}(\tan\eta_0^\alpha - \tan\eta_0^\beta)^{-1}\{$$
$$I_1^2L_\beta^2 - I_2^2L_\alpha^2 + 2I_1I_2L_\alpha L_\beta\tan\epsilon_0\} - \{\int_{x_0}^{\infty}\int_{x_0}^{\infty}(P_1G_1P_1 +$$
$$P_2G_2P_2 + P_2G_{c_2}P_1 + P_1G_{c_1}P_2)\,dx\,dx'\}$$

So that

$$-\frac{\sin 2\epsilon_0\tan\epsilon_0}{2kL_\alpha L_\beta}(\tan\eta_0^\alpha - \tan\eta_0^\beta)^{-1} = [J] =$$
$$\frac{\int_{x_0}^{\infty}(P_1u_1+P_2u_2)\,dx' + \{\int_{x_0}^{\infty}\int_{x_0}^{\infty}(P_1G_1P_1 + P_2G_2P_2 + P_2G_{c_2}P_1 + P_1G_{c_1}P_2)\,dx\,dx'\}}{\{I_1^2L_\beta^2 - I_2^2L_\alpha^2 + 2I_1I_2L_\alpha L_\beta\tan\epsilon_0\}} \qquad 2.54$$

We must show that when $\delta[J]=0$ for arbitrary variations of u_1 and u_2, we get the correct equations of motion, equation 2.52. We will do this in detail for arbitrary variations of u_1. Evidently from the symmetry of equation 2.54, if when $[J]$ is stationary with respect to arbitrary variations of u_1, we get the correct equations of motion, then the same will be true with respect to arbitrary variations of u_2.

We have, since the variations are arbitrary.

$$\delta\int_{x_0}^{\infty}(P_1u_1+P_2u_2)\,dx' = 2\,(f u_1 + g u_2)$$

$$\delta\int_{x_0}^{\infty}\int_{x_0}^{\infty}(P_1G_1P_1 + P_2G_{c_2}P_1 + P_1G_{c_1}P_2 + P_2G_2P_2) =$$

$$f\{\int_{x_0}^{\infty}(2G_1P_1 + P_2G_{c_2} + G_{c_1}P_2)\,dx\}$$

$$g\{\int_{x_0}^{\infty}(G_{c_2}P_1 + P_1G_{c_1} + 2P_2G_2)\,dx\}$$

But

$$G_{c_2}(x'x) = G_{c_1}(xx')$$

So

$$\delta \int_{x_0}^{\omega}\int_{x_0}^{\omega} (P_2 G_{c_2} P_1 + P_1 G_{c_1} P_2 + P_1 G_1 P_1 + P_2 G_2 P_2)\, dx\, dx'$$

$$= 2f \int_{x_0}^{\omega} (P_1 G_1 + P_2 G_{c_2})\, dx + 2g \int_{x_0}^{\omega} (P_2 G_2 + P_1 G_{c_1})\, dx$$

$$\delta I_1^2 = 2I_1 \; (f \cot\epsilon_0\, \partial_1^\alpha + g\, \partial_2^\alpha)$$

$$\delta I_2^2 = 2I_2 \; (-g \cot\epsilon_0 \partial_2^\beta + f\, \partial_1^\beta)$$

$$2\,\delta I_1 I_2 = 2I_1 (-g \cot\epsilon_0\, \partial_2^\beta + f\, \partial_1^\beta) + 2I_2 (f \cot\epsilon_0\, \partial_1^\alpha + g\, \partial_2^\alpha)$$

Whence we have that

$\delta [J] = 0$ implies

$$f\left\{u_1 + \int_{x_0}^{\omega} (G_1 P_1 + P_2 G_{c_2})\, dx\right\} + g\left\{u_2 + \int_{x_0}^{\omega} (P_2 G_2 + P_1 G_{c_1})\, dx\right\}$$

$$= -\frac{\sin 2\epsilon_0}{2k L_\alpha L_\beta} (\tan\eta_0^\alpha - \tan\eta_0^\beta)^{-1} \Big\{ f [L_\beta^2 \cot\epsilon_0\, \partial_1^\alpha I_1 - L_\alpha^2\, \partial_1^\beta I_2 + \tan\epsilon_0\, I_1 L_\alpha L_\beta\, \partial_1^\beta + \tan\epsilon_0 \cot\epsilon_0\, L_\alpha L_\beta I_2 \partial_1^\alpha]$$

$$+ g [L_\beta^2 I_1 \partial_2^\alpha + L_\alpha^2 \cot\epsilon_0\, \partial_2^\beta I_2 - \tan\epsilon_0\, I_1 \cot\epsilon_0\, \partial_2^\beta L_\alpha L_\beta + \tan\epsilon_0\, \partial_2^\alpha I_2 L_\alpha L_\beta] \Big\}$$

i.e.,

$$f\Big\{u_1+\int_{x_0}^{\infty}(G_1P_1+P_2G_{c_2})\,dx+\frac{\sin 2\epsilon_0}{2kL_\alpha L_\beta}(\tan\eta_0^\alpha-\tan\eta_0^\beta)^{-1}\big[\partial_1^\alpha L_\beta(L_\beta I_1+\tan\epsilon_0 L_\alpha I_2)+\partial_1^\beta L_\alpha\tan\epsilon_0(L_\beta\tan\epsilon_0 I_1-L_\alpha I_2)\big]\Big\}+g\Big\{u_2+\int_{x_0}^{\infty}(P_2G_2+P_1G_{c_1})\,dx+\frac{\sin 2\epsilon_0}{2kL_\alpha L_\beta}(\tan\eta_0^\alpha-\tan\eta_0^\beta)^{-1}\big[L_\beta\tan\epsilon_0\,\partial_2^\alpha(L_\beta I_1+\tan\epsilon_0 L_\alpha I_2)-L_\alpha\partial_2^\beta(-L_\alpha I_2+\tan\epsilon_0 L_\beta I_1)\big]\Big\}=0$$

This vanishing then implies the equations of motion equation 2.52. Since each bracket is separately zero when u_1 and u_2 are the exact wave functions. We see then that the values of u_1 and u_2 which make $[J]$ stationary, are those which satisfy the correct equations of motion.

3. Discussion

The variational principles which we have derived, may be used in fitting the above interaction model to the data. It enables us to obtain the phase shifts produced by an interaction with a known boundary condition, for an arbitrary external potential tail.

The variational principle equation 1.23 may be utilised directly to yield the singlet and triplet parity $(-1)^j$ phase shifts. The variational principle equation 2.54 cannot be utilised quite so simply, since it contains the desired phase shift explicitly on both sides of the equation.

However if we are concerned with fairly high energies, where it is expected that the core region gives the more important contribution to the phase shift, then one can approximate the phase shift on the right hand side of the equation, by η_0^α or η_0^β, and utilise an iteration procedure, to obtain the true phase shift. As stated, this method will be most useful at high or moderate energies, where the first approximation to the phase shift is close to the correct value. At lower energies it may still work, but perhaps a better first guess may be necessary, or more iterations required.

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