SYMBOLIC METHODS IN COMBINATORIAL ANALYSIS

by

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Contents

Introduction

We consider n events $A_1, \ldots, A_n$ and let $p(A_{i_1} \ldots A_{i_k})$ denote the probability of the joint occurrence of $A_{i_1}, \ldots, A_{i_k}$. Then if $p(A_{i_1} \ldots A_{i_k})$ is either equal to zero or to a function ϕ_k of k alone, we call the n events $A_1, \ldots, A_n$ "quasi-symmetric events. Kaplansky [4] shows how algebraic symbolism can be applied to find the probabilities of these types of events, by making use of "the symbolic method". In order to explain this method, let A, B, C be events, p(A) the probability of A, p(AB) the joint probability of A and B, etc. Then the probability that none of A , B , C happen is:

$$(1-p(A))(1-p(B))(1-p(C))$$

provided we agree to write p(AB) for p(A)p(B), p(AC) for p(A)p(C), p(BC) for p(B)p(C) and p(ABC) for p(A)p(B)p(C). We call this "multiplication in the symbolic sense".

By making use of the usual indicator and expectation functions we will obtain a method which, although similar to the symbolic method, has more applications and is more convenient. We shall call this method "the indicator method". (For details on the indicator and expectation functions we refer to Loève [6]) For example, using this method, the usual formula for finding the probability that "exactly k out of n events occur", will be obtained. The method will mainly be used to enumerate certain types of restricted permutations, which can be considered as quasi-symmetric events.

In this connection we shall establish in Chapter 3 a formula (perhaps of independent interest) which gives an expression for the number of ways in which k objects can be chosen from n distinct objects arrayed in a row, so that no j consecutive objects are contained in each choice. (The case j = 3 has been submitted as a problem to the elementary problem section of The American Mathematical Monthly for one of the issues to appear this summer.) Also it is believed that for the first time, in Chapter 4, a readily applicable formula is given for the solution of a permutation problem set by Mendelsohn [9, p.238, example 5].

CHAPTER 1

Use of the Indicator and Expectation Functions.

Consider a finite number of sets $A_1, \ldots, A_n$ representing n events. Let A_i^c denote the complement of A_i, that is, the event "not A_i". We introduce the indicator function $I_{A_i}(x)$ in the usual way:

$$I_{A_i}(x) = \begin{cases} 1 \text{ if } x \in A_i \\ 0 \text{ if } x \notin A_i \quad (\text{i.e. if } x \in A_i^c), \end{cases}$$

If we write I_A instead of $I_A(x)$, then for all x,

if $A \subset B$ (1) $I_A \leq I_B$

if $A = B$ (2) $I_A = I_B$

(3) $I_{A^c} = 1 - I_A$

(4) $I_{A \cap B} = I_A \cdot I_B$

(5) $I_\emptyset = 0$ where $\emptyset$ is the empty set.

(6) $E(I_A) = 1 \cdot p(A) + 0 \cdot p(A^c) = p(A)$ where E is the expectation function and p(A) denotes the probability of the occurrence of an event A.

Let $A_1 \cap A_2 \cap \ldots \cap A_n$ be denoted by $A_1 A_2 \ldots A_n$. Then $p(A_1 A_2 \ldots A_n)$ denotes the probability of the joint occurrence of $A_1, A_2, \ldots, A_n$.

Suppose now, events A_1, A_2, A_3 are considered and the probability that none of these occur is required. We proceed as follows:

$$I_{A_1^c A_2^c A_3^c} = I_{A_1^c} I_{A_2^c} I_{A_3^c} \qquad \text{by (4)}$$

$$= (1-I_{A_1})(1-I_{A_2})(1-I_{A_3}) \qquad \text{by (3)}$$

$$= 1-I_{A_1}-I_{A_2}-I_{A_3}+I_{A_1A_2}+I_{A_1A_3}+I_{A_2A_3}-I_{A_1A_2A_3} \qquad \text{by}(4)$$

Taking the expectation on both sides of the equation we obtain:

$$E\,(I_{A_1^cA_2^cA_3^c}) = E\,(1-I_{A_1}-I_{A_2}-I_{A_3}+ \dots -I_{A_1A_2A_3}).$$

Then, $P(A_1^cA_2^cA_3^c) = 1-P(A_1)-P(A_2)-P(A_3)+P(A_1A_2)+P(A_1A_3)+P(A_2A_3)-P(A_1A_2A_3)$ by (6).

But $A_1^cA_2^cA_3^c$ is the event that none of the events A_1, A_2, A_3 occurs. Hence we have the required probability.

If the probability that A_1 occurs while A_2 and A_3 do not occur is required, we would proceed as follows:

$$\begin{aligned} I_{A_1A_2^cA_3^c} &= I_{A_1}\ I_{A_2^cA_3^c} \\ &= I_{A_1}\ (1-I_{A_2})(1-I_{A_3}) \\ &= I_{A_1} - I_{A_1A_2} - I_{A_1A_3} + I_{A_1A_2A_3} \end{aligned}$$

By taking the expectation on both sides of the equation, it follows that $P(A_1A_2^cA_3^c) = P(A_1) - P(A_1A_2) - P(A_1A_3) + P(A_1A_2A_3)$ where $A_1A_2^cA_3^c$ is the event considered.

The above example suggests the following rule:

<u>Rule of Replacement</u>

The probability that event $A_1A_2 \dots A_kA_{k+1}^c \dots A_n^c$ occurs can be obtained as follows.

a) Write down $I_{A_1}\ I_{A_2} \dots I_{A_k}\ (1-I_{A_{k+1}})(1-I_{A_{k+2}}) \dots (1-I_{A_n})$

b) Carry out the required multiplication putting all products of the form $I_{A_1} I_{A_2} \ldots I_{A_i}$ equal to $I_{A_1A_2 \ldots A_i}$.

c) Replace $I_{A_1A_2 \ldots A_i}$ by $p(A_1A_2 \ldots A_i)$

We then have the required probability.

We wish now to find the probability that exactly k events among the n events $A_1, A_2, \ldots, A_n$ occur simultaneously, regardless of which k events occur. Consider two cases a) and b):

a) $k > 0$

Let $i_1, i_2, \ldots, i_k$ be k distinct integers chosen from $1, 2, \ldots, n$, and $i_{k+1}, \ldots, i_n$ be the remaining integers.

Let $X = \sum I_{A_{i_1} A_{i_2} \ldots A_{i_k} A^c_{i_{k+1}} \ldots A^c_{i_n}}$ (summation over all possible selections $i_1, \ldots, i_k$)

$$= \sum I_{A_{i_1} A_{i_2} \ldots A_{i_k}} (1-I_{A_{i_{k+1}}})(1-I_{A_{i_{k+2}}}) \ldots (1-I_{A_{i_n}})$$

$$= \sum I_{A_{i_1} A_{i_2} \ldots A_{i_k}} - \sum (I_{A_{i_1} \ldots A_{i_k}} \sum_{a=1}^{n-k} I_{A_{i_{k+a}}})$$

$$+ \sum (I_{A_{i_1} \ldots A_{i_k}} \sum_{\substack{a,b=1 \\ a \neq b}}^{n-k} I_{A_{i_{k+a}} A_{i_{k+b}}})$$

$$- \sum (I_{A_{i_1} \ldots A_{i_k}} \sum_{\substack{a,b,c=1 \\ a \neq b \;\; b \neq c \;\; a \neq c}}^{n-k} I_{A_{i_{k+a}} A_{i_{k+b}} A_{i_{k+c}}})$$

$$+ \ldots \pm \sum I_{A_{i_1} \ldots A_{i_k}} (I_{A_{i_{k+1}} A_{i_{k+2}} \ldots A_{i_n}})$$

$$= J_k - \binom{k+1}{k} J_{k+1} + \binom{k+2}{k} J_{k+2} - \binom{k+3}{k} J_{k+3} + \dots \pm \binom{n}{k} J_n$$

where $J_v = \sum I_{A_{i_1} A_{i_2} \dots A_{i_v}}$.

Taking the expectation on both sides we obtain:

$$E(X) = S_k - \binom{k+1}{k} S_{k+1} + \binom{k+2}{k} S_{k+2} - \binom{k+3}{k} S_{k+3} + \dots \pm \binom{n}{k} S_n,$$

where $S_v = E(J_v) = \sum p(A_{i_1} A_{i_2} \dots A_{i_v})$. But E (X) gives us the probability that at least one of the events of the type $A_{i_1} A_{i_2} \dots A_{i_k} A^c_{i_{k+1}} \dots$ $\dots A^c_{i_n}$ occurs. That is, the probability that exactly k events among the n events $A_1, A_2, \dots, A_n$ occur. Denote this probability by $P_{[k]}$.

Hence,

$$P_{[k]} = S_k - \binom{k+1}{k} S_k + \binom{k+2}{k} S_{k+2} - \dots \pm \binom{n}{k} S_n \qquad \text{(A)}$$

b) $k = 0$

Here we consider simply, $(1-I_{A_1})(1-I_{A_2}) \dots (1-I_{A_n})$.

By the rule of replacement we obtain,

$$P_{[0]} = P_0 = 1 - S_1 + S_2 - S_3 + \dots \pm S_n \qquad \text{(B)}$$

where $P_{[0]}$ or P_0 denotes the probability that none of the events $A_1, A_2, \dots, A_n$ occurs.

Note that by putting $S_0 = 1$ and $k = 0$ in (A), the relation (B) follows. More simply we can write (A) as:

$$P_{[k]} = \sum_{r=k}^{n} (-1)^{r-k} \binom{r}{k} S_r,$$

which can be rewritten as:

$$P_k = \sum_{r=0}^{n} (-1)^k \binom{r}{k} (-1)^r S_r \qquad \text{(C)}$$

since $\binom{r}{k} = 0$ whenever $r < k$.

From this point onwards we will consider only events which have a special property, which Kaplansky [3] refers to as the "quasi-symmetric" property.

Definition

Given a set $A_1, A_2, \ldots, A_n$ of events. Then if for any subset $A_{i_1}, \ldots, A_{i_k}$, $p(A_{i_1} A_{i_2} \ldots A_{i_k})$ is either equal to zero or to a function ϕ_k of k alone, we say $A_1, A_2, \ldots, A_n$ are "quasi-symmetric" events.

The displacement operator E (no ambiguity will arise in its use with regard to the expectation function E) is defined by $E^k \phi_o = \phi_k$, $E^o \phi_o = 1$. For example, in the case of complete symmetry, that is, where each $p(A_{i_1}, \ldots, A_{i_k})$ is equal to ϕ_k, we can write $P_o = (1-E)^n \phi_o$ giving us the probability that none of the events $A_1, A_2, \ldots, A_n$ occurs as $P_o = 1 - n\,\phi_1 + \binom{n}{2} \phi_2 - \ldots$ (using (B))

We now give a rule in order to obtain probabilities in the case of quasi-symmetric events.

Rule of Replacement for Quasi-Symmetric Events

The probability that the event $A_1 A_2 \ldots A_k A_{k+1}^c \ldots A_n^c$ occurs can be obtained as follows:

(1) Write down $I_{A_1} I_{A_2} \ldots I_{A_k} (1-I_{A_{k+1}}) \ldots (1-I_{A_n})$

(2) Carry out the required multiplication putting all products of the form $I_{A_{i_1}} I_{A_{i_2}} \ldots I_{A_{i_v}}$ equal to $I_{A_{i_1} A_{i_2} \ldots A_{i_v}}$ and dropping any which are identically equal to zero.

(3) Replace each remaining term $I_{A_{i_1} A_{i_2} \ldots A_{i_v}}$ by E^v thus obtaining a polynomial f(E).

The required probability is then $f(E)\,\phi_o$. To obtain this rule we have only used the rule of replacement as given for the general case and observed that $p(A_{i_1} A_{i_2} \ldots A_{i_k}) = \phi_k$ when $A_{i_1} A_{i_2} \ldots A_{i_k}$ is not the empty set.

The advantage of the above method is that we may resort to all devices of formal algebra in computing f(E).

Our main application of the above rule of replacement will be to find the probability that none of the events $A_1, \ldots, A_n$ occurs. For, once we have found the required polynomial f(E), such that $P_o = f(E)\,\phi_o$, then, as pointed out by Fréchet [2], we can easily find $P_{[k]}$.

Multiplying each term S_v of (B) by $(-1)^k \binom{v}{k}$ we obtain formula (C).

Then,

$$P_{[k]} = f(E)\,\psi_o \qquad \text{(D)}$$

where $E^r \psi_o = \psi_r$, $\psi_r = (-1)^k \binom{r}{k} \phi_r$

and $P_o = f(E)\,\phi_o$ is the same as formula (B).

Further, the probability that at most v of the n events $A_1, A_2, \ldots A_n$ occur can easily be obtained.

Denote this probability by P_v.

$$P_v = \sum_{k=o}^{v} P_{[k]} = \sum_{k=o}^{v} \sum_{r=o}^{n} (-1)^k \binom{r}{k} (-1)^r S_r$$

$$= \sum_{r=o}^{n} \sum_{k=o}^{v} (-1)^k \binom{r}{k} (-1)^r S_r$$

Now $\sum_{k=o}^{v} (-1)^k \binom{r}{k} = (-1)^v \binom{r-1}{v}$. To verify, we first see it is true when $v = o$ and $v = 1$. To apply induction with respect to v assume that the formula is true for $v = u$.

Then,

$$\sum_{k=o}^{u+1} (-1)^k \binom{r}{k} = (-1)^u \binom{r-1}{u} + (-1)^{u+1} \binom{r}{u+1}$$

$$= (-1)^{u+1} \frac{(r-1)!}{(u+1)!\ (r-u-2)!}$$

$$= (-1)^{u+1} \binom{r-1}{u+1}$$

Thus the formula is proved.

(Note, when $r = o$, $(-1)^o \binom{r}{o} = (-1)^o \binom{r-1}{o} = 1$ as $\binom{m}{o}$ is taken as 1 for all values of m. Also $\binom{-1}{r} = (-1)^r$ where r is a positive integer and $\binom{n}{r} = o$ if $r > n$ or $r < o$, where n is a positive integer.)

Therefore $P_v = \sum_{r=o}^{n} (-1)^v \binom{r-1}{v} (-1)^r S_r$ which can be written as

$$P_v = f(E) \Lambda_o \qquad \text{(E)}$$

where $E^r \Lambda_o = \Lambda_r$, $\Lambda_r = (-1)^v \binom{r-1}{v} \phi_r$

and remembering that $P_o = f(E)\ \phi_o$.

Our main problem henceforth will be that of finding f(E) such that

$P_o = f(E)\,\phi_o$. Once this is done, $P_{[k]}$ and P_k are readily obtained as shown. We will use the word "evaluate" to mean the process of finding such a polynomial f(E) from the form $(1-I_{A_1})(1-I_{A_2})\ldots(1-I_{A_n})$ by using the rule of replacement for quasi-symmetric events. f(E) will also be referred to as "the required polynomial" for the given events "the corresponding polynomial", or as "the associated polynomial". However, it should be pointed out that f(E) will always be meant to be the polynomial such that $P_o = f(E)\,\phi_o$ unless stated otherwise, since we may obtain in the same manner, from the rule of replacement, a polynomial g(E) such that $g(E)\,\phi_o$ gives the probability that a certain number of given events occur while certain others do not.

As a simple example consider 8 horses entered in a race where any horse is just as likely to finish in any assigned position. Number the horses 1,2, ..., 8. The probability required is that:

	1	does	not	finish	first
and	2	"	"	"	second
"	3	"	"	"	first
"	4	"	"	"	second
"	5	"	"	"	first
"	6	"	"	"	second
"	7	"	"	"	first
"	8	"	"	"	second

Let (ij) denote the event that the i^{th} horse finishes in j^{th} position. The events have the "quasi-symmetry" property and $\phi_k = \frac{(8-k)!}{8!}$

Write I_{ij} instead of $I_{(ij)}$. Then we have to evaluate

$(1-I_{11})(1-I_{22})(1-I_{31})(1-I_{42}) \ldots (1-I_{82})$, that is, we have to find the associated polynomial f(E). By applying the rule of replacement, it is easily seen that $f(E) = (1 - 8E + 16E^2)$ as (ij)(mn) is equal to the empty set when j = n.

$$\text{Then } P_o = (1 - 8E + 16E^2)\, \Phi_o = \frac{8! - 8(8-1)! + 16(8-2)!}{8!} = \frac{2}{7}$$

Using formula (D) $P_{[1]} = E^o \Psi_o - 8(-1)^1\binom{1}{1}\frac{(8-1)!}{8!} + 16(-1)^1\binom{2}{1}\frac{(8-2)!}{8!}$

$$= 0 + 1 - \frac{4}{7} = \frac{3}{7}$$

$$P_{[2]} = E^o \Psi_o - 0 + 16(-1)^2 \binom{2}{2}\frac{(8-2)!}{8!} = \frac{2}{7}$$

Checking,

$$P_o = 1 - \left(P_{[1]} + P_{[2]}\right) \quad \text{as} \quad P_{[k]} = 0 \quad \text{for} \quad k > 2$$

$$\frac{2}{7} = 1 - \left(\frac{3}{7} + \frac{2}{7}\right)$$

Using formula (E) $\quad P_2 = f(E)\Lambda_o$

$$= (-1)^2 \binom{-1}{2}\Phi_o - 8(-1)^2\binom{0}{2}\Phi_2 + 16(-1)^2\binom{1}{2}\Phi_3$$

$$= 1 + 0 + 0 = 1$$

and we have $P_2 = P_{[o]} + P_{[1]} + P_{[2]}$

The above example is one of a general type of problem dealing with permutations. We are usually given that n distinguishable objects are arrayed in a straight line. The objects are numbered 1,2, ..., n according to their respective positions starting, say on the left. A permutation is then made of the n objects, and the probability is required that object number 1 is now in the i_1^{th} respective position starting from the left, object number 2 is now in the i_2^{th} respective position, and so on. Such restrictions need not apply to all of the n numbers. Our assumption here is that each object is equally likely to occupy any respective position as a result of the permutation. As in the previous example, (ij) denotes the event that the i^{th} object is placed into the j^{th} position or briefly, that "i is j". We have then $X = (i_1 j_1)(i_2 j_2) \ldots (i_k j_k)$ is equal to the empty set if $i_u = i_v \quad u \neq v$ or $j_u = j_v \quad u \neq v$. Otherwise $p(X) = \phi_k$ where $\phi_k = \frac{(n-k)!}{n!}$. Hence we are dealing here with quasi-symmetric events. Problems of this type are usually referred to in the literature as "restricted permutation" or "card-matching" problems. Note that $I_{(mn)} I_{(uv)} \equiv o$ implies that (mn)(uv) is equal to the empty set and hence the event (mn) and the event (uv) cannot occur simultaneously. Thus $p\left[(mn)(uv)\right] = o$.

CHAPTER 2

Generalization of the "Problème des Rencontres".

Perhaps the best known example dealing with restricted permutations is the game of "rencontre". The game is usually played as follows. Two equivalent decks of n different cards are put into random order and matched against each other. If a card occupies the same place in both decks, we say we have a "coincidence" or "rencontre". The probability that no "rencontre" occurs is simply the probability that none of the events (11), (22), ..., (nn) occur. Our problem is then to evaluate

$$(1 - I_{11})\,(1 - I_{22}) \ldots (1 - I_{nn}).$$

Since the intersection of any k of the considered events is not equal to the empty set we see that, $f(E) = (1 - E)^n$. Then,

$$P_o = \left(1 - \binom{n}{1} E + \binom{n}{2} E^2 - \ldots \pm \binom{n}{n} E^n\right) \phi_o$$

$$= 1 - \binom{n}{1} \frac{(n-1)!}{n!} + \binom{n}{2} \frac{(n-2)!}{n!} - \ldots \pm \binom{n}{n} \frac{(n-n)!}{n!}$$

$$= 1 - \frac{1}{1!} + \frac{1}{2!} - \ldots \pm \frac{1}{n!}$$

We note that the above formula gives the first n+1 terms of the expansion of e^{-1} and hence for n greater than, say, 5 P_o is "almost" independent of n. In fact $P_o \approx .36788$.

Using formula D of Chapter 1 we find

$$P_{[k]} = \frac{1}{k!}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^{n-k} \frac{1}{(n-k)!}\right)$$

and hence $P_{[k]} \approx \dfrac{e^{-1}}{k!}$

If $f_n(E)$ is the associated polynomial, in the general case, it follows that,

$$f_n(E) = (1-E)\, f_{n-1}(E) \text{ as } (1-E)^n = (1-E)(1-E)^{n-1}.$$

Extending the game of rencontre to 3 decks of n cards each, we let (i,j,k) denote the event that the j^{th} card of the second deck and the k^{th} card of the third deck occupy the i^{th} position. The probability of no coincidence, that is, for any $m = 1,2, \ldots,n$, the m^{th} card does not appear in the same position in all three decks, is found by first evaluating

$$(1 - I_{111})(1- I_{222}) \ldots (1 - I_{nnn}), \text{ where } I_{iii} = I_{(i,i,i)}$$

We obtain $f(E) = (1-E)^n$ and $P_o = (1-E)^n \phi_o$. However, here $\phi_k = \left[\frac{(n-k)!}{n!}\right]^2$.

In general, if s+1 decks of n cards each, are considered the probability that no rencontre occurs is $(1-E)^n\, \phi_o$ where $\phi_k = \left[\frac{(n-k)!}{n!}\right]^s$.

Instead of writing $(1-E)^n \phi_o$ we can write $(E-1)^n\ \frac{(0!)^s}{(n!)^s}$ where $E^K\,(0!)^s \equiv (K!)^s$ and by putting $E-1 = \Delta$, $P_o = \frac{\Delta^n\,(0!)^s}{(n!)^s}$

Let ${}_sH_n$ denote the number of possible permutations of the s+1 decks with no coincidences. Then we note that,

$${}_sH_n = P_o\,(n!)^s = \Delta^n\,(0!)^s,$$

in agreement with Riordan [12]. In particular for $s = 1$, ${}_1H_n = \Delta^n\, 0!$, the solution for the usual game with 2 decks.

The probability that none of a particular choice of s decks has any coincidence with the remaining deck is $\left[\frac{\Delta^n 0!}{n!}\right]^s$, although any two decks excluding the remaining deck may have coincidences.

Also noted by Riordan, are some interesting arithmetical properties of the numbers ${}_sH_n$. For their verification the operator E proves to be useful.

<u>Property a)</u> ${}_sH_{n+p} \equiv -\, {}_sH_n \pmod p$ for any prime p.

<u>proof:</u> $_{s}H_{n+p} = (E-1)^{n+p}\,(0!)^{s}$ where p is any prime

$= (E-1)^{n}\,(E-1)^{p}\,(0!)^{s}$

$\equiv (E-1)^{n}\,\left[E^{p} + (-1)^{p}\right]\,(0!)^{s} \pmod{p}$

$\equiv \left[E^{p}\,(E-1)^{n}+(-1)^{p}\,(E-1)^{n}\right]\,(0!)^{s} \pmod{p}$

$\equiv E^{p}\,(E-1)^{n}\,(0!)^{s} + (-1)^{p}\,(E-1)^{n}\,(0!)^{s} \pmod{p}$

$\equiv 0 + (-1)^{p}\;{}_{s}H_{n} \pmod{p}$

Note that $E^{p}\,(E-1)^{n}\,(0!)^{s} \equiv 0 \pmod{p}$ since every term of $E^{p}\,(E-1)^{n}$ is of the form KE^{p+m}, where K,m are integers; hence every term of $E^{p}\,(E-1)^{n}\,(0!)^{s}$ is of the form $K\,\left[(p+m)!\right]^{s}$, therefore every term contains the factor p.

Thus for every prime p, $p \neq 2$, ${}_{s}H_{n+p} \equiv -\,{}_{s}H_{n} \pmod{p}$. But for p = 2, $-1 \equiv 1 \pmod{2}$ and therefore property a) is true for all primes p.

For example, let n = 2, p = 3, s = 1. Then ${}_{1}H_{2+3} = 44$, ${}_{1}H_{2} = 1$ and $44 \equiv -1 \pmod{3}$.

<u>Property b)</u> ${}_{s+p-1}H_{n} \equiv {}_{s}H_{n} \pmod{p}$ for any prime p.

<u>proof:</u> ${}_{s+p-1}H_{n} = (E-1)^{n}\,(0!)^{s+p-1}$ where p is any prime.

Every term on the right side of the equation is of the form

$\binom{n}{k}\,(-1)^{n-k}\,(E)^{k}\,(0!)^{s+p-1}$, $= \binom{n}{k}(-1)^{n-k}(k!)^{s}(k!)^{p-1}$

$= t_{k}$. $(0 \leq k \leq n)$.

Now if p divides k!, then $t_{k} \equiv 0 \pmod{p}$. If p does not divide k!, then $(k!)^{p-1} \equiv 1 \pmod{p}$ which follows from Fermat's theorem on congruences. Thus $t_{k} \equiv \binom{n}{k}\,(-1)^{n-k}\,(k!)^{s}$

(mod p) when p does not divide k!

Let t_k^1 be the general term in $(E-1)^n\ (0!)^s$, that is,

$t_k^1 = \binom{n}{k}\ (-1)^{n-k}\ (k!)^s$. It is seen that $t_k \equiv t_k^1 \pmod{p}$.

Therefore ${}_{s+p-1}H_n \equiv {}_sH_n \pmod{p}$ for any prime p.

We now consider generalizations of the problem of rencontre as noted by Kaplansky [4, p.908].

The n integers 1,2, ...,n are divided into $\frac{n}{a}$ subsets of a integers each. What, then, is the probability that in a random permutation of the n integers, no integer appears in any of the a places originally occupied by the members of its subset? As an example, let a = 2 and assume n is even. Obviously we may "re-divide" the integers into the subsets, $\{1,2\}$, $\{3,4\}$, ..., $\{n-1,n\}$ as the required probability is the same. Hence we wish to evaluate,

$$\prod(1-I_{x_1y_1})\prod(1-I_{x_2y_2})\ \ldots\prod(1-I_{x_{\frac{n}{2}}y_{\frac{n}{2}}})$$

where $x_1 = 1,2,\ y_1 = 1,2;\ x_2 = 3,4,\ y_2 = 3,4;\ \ldots;\ x_{\frac{n}{2}} = n-1,n,\ y_{\frac{n}{2}} = n-1,n.$

Now $\prod(1-I_{x_iy_i})(1-I_{x_jy_j})\quad i \neq j\quad i,j = 1,2,\ \ldots,\frac{n}{2}$ doesnot have any products of the form $I_{x_iy_i}\ I_{x_jy_j} \equiv 0$. Further, the polynomial correspon-ding to $\prod(1-I_{x_iy_i})$ is equal to the polynomial corresponding to $\prod(1-I_{x_jy_j})$. Therefore let f(E) be the polynomial corresponding to $\prod(1-I_{x_iy_i})$ for any i. Then our required probability is

$$P_o = [f(E)]^{\frac{n}{2}}\ \Phi_o \qquad \text{where } \Phi_k = \frac{(n-k)!}{n!}$$

That is, the corresponding polynomial for the solution is $F_n(E) = [f(E)]^{\frac{n}{2}}$.
We find f(E).

$$\prod(1-I_{x_1y_1}) = (1-I_{11})\ (1-I_{12})\ (1-I_{21})\ (1-I_{22})$$

$$= 1-I_{(11)} - I_{(12)} - I_{(21)} - I_{(22)} + I_{(11)(22)} + I_{(12)(21)}$$

Hence $f(E) = 1-4E + 2E^2$ and finally,

$$F_n(E) = (1 - 4E + 2E^2)^{\frac{n}{2}}$$

Thus for example $F_2(E) = 1 - 4E + 2E^2$

$$F_4(E) = 1 - 8E + 20E^2 - 16E^3 + 4E^4$$

$$F_6(E) = 1 - 12E + 54E^2 - 112E^3 + 108E^4 + 8E^6$$

Suppose the n integers 1,2, ..., n are divided into subsets, not necessarily of the same size each, denoted by $a_1, a_2, \ldots, a_m$ containing $a_1, a_2, \ldots, a_m$ integers respectively. We introduce subsets $b_1, b_2, \ldots, b_m$ containing $b_1, b_2, \ldots, b_m$ integers respectively, all chosen from among 1,2, ...,n, with the condition that no integer occurs in two distinct b_i. What, then, is the probability that, given a random permutation of the n integers, none of the a_i members belonging to the subset a_i occupy any of the b_i places specified by the members belonging to the subset b_i, $(i = 1,2, \ldots, m)$? Obviously, we obtain the same answer if we consider the subsets $a_1, a_2, \ldots, a_m$ to be equal to the subsets

$$\{1,2, \ldots, a_1\}, \{a_1 + 1, a_1 + 2, \ldots, a_1 + a_2\}, \ldots$$

respectively. The problem then is to evaluate the polynomial

$$\Pi(1-I_{x_1y_1})\,\Pi(1-I_{x_2y_2})\,\ldots\,\Pi(1-I_{x_my_m})$$

where $x_1 = 1,2, \ldots, a_1$, $y_1 = 1,2, \ldots, b_1$; $x_2 = a_1 + 1, a_1 + 2, \ldots, a_1 + a_2$ $y_2 = b_1 + 1, b_1 + 2, \ldots, b_1 + b_2$ and so on. Now no product of the form $I_{x_iy_i} I_{x_jy_j} \equiv 0$ for $i \neq j$. Hence letting $f(a_ib_i)$ be the polynomial corresponding to the evaluation of $\Pi(1-I_{x_iy_i})$ $i = 1,2, \ldots,m$ it follows that:

$$P_o = [\Pi f(a_ib_i)]\,\phi_o \qquad i = 1,2, \ldots,m$$

where P_o is our required probability. That is, our associated polynomial $f(E)$ required to solve the initial problem is $f(E) = f(a_1b_1)\cdot f(a_2b_2) \ldots f(a_mb_m)$. An explicit expression for $f(a_ib_i)$ is therefore desired. We note that $f(a_ib_i)$ depends only on the number of elements in the subset a_i and the number in the subset b_i. Hence a general form $\Pi(1-I_{ij})$ $i = 1,2, \ldots,a$ $j = 1,2, \ldots,b$ is considered. Denote its associated polynomial by $f(a,b)$. Consider the event

$$(i_1j_1)(i_2j_2) \ldots (i_vj_v) = \mathcal{V}$$

where $i_1,i_2, \ldots,i_v$ are any v distinct integers chosen from among the integers $1,2, \ldots,a$. There are $\binom{a}{v}$ such choices. Also $j_1,j_2, \ldots,j_v$ are any v distinct integers chosen from among $1,2, \ldots,b$. There are $\binom{b}{v}$ such choices. We wish to obtain all events of the type $\mathcal{V}$, the occurrence of which is possible. For each particular selection of $i_1,i_2, \ldots,i_v$ and a particular selection $j_1,j_2, \ldots,j_v$ there are obviously $v!$ possible events of type V, the occurrence of which is possible. Hence there are $\binom{a}{v}\binom{b}{v}v!$ such possible events. It follows that,

$$f(a,b) = \sum_{v=0}^{a} \binom{a}{v}\binom{b}{v}v!\ (-E)^v$$

Note that it is not necessary to distinguish whether $a < b$ or $b < a$ since $\binom{a}{v} = 0$ if $v > a$ and $\binom{b}{v} = 0$ if $v > b$.

As an example, in the usual deck of 52 cards, consider each subset a_i to be precisely those 4 cards having the same denomination (for example aces, deuces, etc.) and $b_i = a_i$. Then the required polynomial $f(E) = [f(4,4)]^{13} = (1 - 16E + 72E^2 - 96E^3 + 24E^4)^{13}$.

Extending the above problem to the case of 3 random permutations of the n integers 1,2, ...,n, we have three sets of subsets, a_i, b_i, c_i, $i = 1,2, \ldots, m$. Here none of the members of b_i can at the same time be together with any member of c_i in any position designated by any member of a_i. Let $f(a_i, b_i, c_i)$ be the polynomial corresponding to the evaluation of $\prod(1-I_{x_i y_i z_i})$ where $x_i = 1,2, \ldots a_i$, $y_i = 1,2, \ldots b_i$, $z_i = 1,2, \ldots, c_i$.

Then as before, the required probability is:

$$P_o = \left[\prod f(a_i b_i c_i)\right] \phi_o \quad \text{where } i = 1,2, \ldots, m.$$

Note that here $\phi_k = \left[\frac{(n-k)!}{n!}\right]^2$. Consider $\prod(1-I_{ijk})$ where $i = 1,2, \ldots, a$; $j = 1,2, \ldots, b$; $k = 1,2, \ldots, c$. Let its corresponding polynomial be $f(a,b,c)$. As before, we wish to find all events of the type

$$(i_1 j_1 k_1)(i_2 j_2 k_2) \ldots (i_v j_v k_v) = V$$

whose occurrence is possible. Note that V is the empty set if $i_s = i_t$

for some $s \neq t$ or $j_s = j_t$ for some $s \neq t$ or $k_s = k_t$ for some $s \neq t$. There are then $\binom{a}{v}$ possible favourable choices of $i_1, i_2, \ldots, i_v$, $\binom{b}{v}$ possible favourable choices of $j_1, j_2, \ldots, j_v$ and $\binom{c}{v}$ favourable choices of $k_1, k_2, \ldots, k_v$. Now for a particular choice of $i_1, i_2, \ldots, i_v$, a particular choice of $j_1, j_2, \ldots, j_v$ and a particular choice of $k_1, k_2, \ldots, k_v$ there are $(v!)^2$ possible events of type V whose occurrence is possible. Hence there are, in total $\binom{a}{v}\binom{b}{v}\binom{c}{v}(v!)^2$ events of type V whose occurrence is possible. Finally we have,

$$f(a,b,c) = \sum_v \binom{a}{v}\binom{b}{v}\binom{c}{v}(v!)^2 \, (-E)^v/v!$$

or $f(a,b,c,) = \sum_v (a)_v \, (b)_v \, (c)_v \, (-E)^v/v!$ as in [4], where $(a)_v = a(a-1) \ldots (a-v+1)$.

The procedure considered above can be thought of as matching 3 decks of n cards each, having a_1, b_1, c_1 aces respectively, a_2, b_2, c_2 deuces respectively, and so on, with the cards being numbered so that $1, 2, \ldots, a_1$ are aces, $a_1 + 1$, $a_2 + 2$, ..., $a_1 + a_2$ are deuces, and so on. Similarly for the other decks.

In the case of $s = k + 1$, permutations of the n integers $1, 2, \ldots, n$, the subsets being $a_i, b_i, \ldots, S_i$, $i = 1, 2, \ldots, m$ we have

$$P_o = \left[\prod f(a_i, b_i, \ldots, S_i)\right] \phi_o, \; (i = 1, 2, \ldots, m), \; \phi_r = \left[\frac{(n-r)!}{n!}\right]^k$$

and

$$f(a, b, \ldots, S) = \sum_v \binom{a}{v}\binom{b}{v} \ldots \binom{S}{v}(v!)^k \, (-E)^v \qquad \text{(A)}$$

As an example, suppose that each of four decks has two suits of two cards each. Then we wish to find $[f(2,2,2,2)]^2$ which is the required polynomial f(E).

$$f(2,2,2,2) = \sum_{v=0}^{2} \binom{2}{v}^4 (v!)^3 (-E)^v = 1 - 16E + 8E^2$$

$$f(E) = (1 - 16E + 8E^2)^2 = 1 - 32E + 272E^2 - 256E^3 + 64E^4$$

$$P_o = f(E)\ \phi_o \text{ where } \phi_k = \left[\frac{(4-k)!}{4!}\right]^3 .$$

$$(4!)^3\ P_o = (4!)^3 - 32(3!)^3 + 272(2!)^3 - 256(1!)^3 + 64(0!)^3$$

$$= 13{,}824 - 6{,}912 + 2{,}176 - 256 + 64 = 8{,}896$$

$$P_o \approx .644$$

Further, $P_{[m]} = f(E)\ \psi_o$ where $\psi_k = (-1)^m \binom{k}{m} \phi_k$. Hence we have:

m	$(4!)^3\ P_{[m]}$	$P_{[m]}$ approx.
0	13,824 - 6,912 + 2176 - 256 + 64 = 8896	.644
1	6,912 - 2(2176) + 3(256) - 4(64) = 3072	.222
2	2176 - 3(256) + 6(64) = 1792	.130
3	256 - 4(64) = 0	0
4	64 = 64	.005

$\sum_{m=0}^{4} P_{[m]} = 1.001$, error due to approximations. The above also indicates a systematic way of obtaining $P_{[m]}$ once having obtained P_o.

The preceding example is one of an interesting class of examples. Suppose we have n balls of which a_1 are of one colour, a_2 of another colour, and so on. Suppose now that there are n holes arrayed in a straight line so that a ball fits into any hole. Then n balls are then placed into the n holes in such a manner that any ball is equally likely to be placed into any hole. This procedure is repeated S-1 times. What is the probability that no hole will have contained a ball of the same colour each of the S times? Using (A), our required polynomial f(E) is equal to

$$\left[\sum \binom{a_1}{v}^S (v!)^{S-1}(-E)^v\right] \left[\sum \binom{a_2}{v}^S (v!)^{S-1}(-E)^v\right] \cdots \left[\sum \binom{a_n}{v}^S (v!)^{S}(-E)^v\right]$$

where $a_1 + a_2 + \dots + a_n = n$. More simply we write:

$$f(E) = \prod_{i=1}^{n} \left[\sum \binom{a_i}{v}^S (v!)^{S-1}(-E)^v\right]$$

and when $a_1 = a_2 = \dots = a_n = a$,

$$f(E) = \left[\sum \binom{a}{v}^S (v!)^{S-1} (-E)^v\right]^{\frac{n}{a}} .$$

A more interesting and somewhat more intricate example of restricted permutations given by Kaplansky [4, p.909] will now be considered.

The integers 1,2, ...,n are divided into subsets of a integers each. What is the probability that, given a random permutation of the integers, no integer occupies any place designated by all the other members of its subset? That is, assuming the subset to be $\{1,2, \dots,a\}$,

$\{a + 1, a + 2, \ldots, 2a\}$, ..., any integer may occupy its own original position but not any position occupied originally by any other member of its subset. For $a = 1$, the required probability is obviously 1.

As an illustrative example, let $n = 4$ and $a = 2$. Then the two subsets are $\{1,2\}$ and $\{3,4\}$ and the following permutations satisfy the conditions,

1 2 3 4	3 4 1 2
1 3 2 4	3 4 2 1
1 4 3 2	4 3 1 2
3 2 1 4	4 3 2 1
4 2 3 1	

There are nine such permutations. Hence $P_o = \frac{9}{4!}$.

It is easily seen that in general, it is quite an involved process to calculate the required probability using ordinary combinatorial arguments. We wish then to find a polynomial $A(E)$, such that our required probability is equal to $A(E)\,\phi_o$ where as before $E^k\,\phi_o = \phi_k$ and $\phi_k = \frac{(n-k)!}{n!}$. Observe that $A(E)$ can not be obtained in a similar manner as in the previous examples. In order to find $A(E)$ we require two lemmas.

<u>Lemma 1</u> Given a finite number of events $\{B_i\}$ satisfying the condition of quasi-symmetry and the following events are considered,

$$\begin{matrix} A_1^1, & A_2^1, & \ldots, & A_{a_1}^1 \\ A_1^2, & A_2^2, & \ldots, & A_{a_2}^2 \\ \vdots & \vdots & & \vdots \\ A_1^r, & A_2^r, & \ldots, & A_{a_r}^r \end{matrix}$$

where each A^n_m is the intersection of N^n_m events taken from $\{B_i\}$ such that A^n_m is not the empty set, that is, $p(A^n_m)$ = a function of N^n_m alone. Further for any two events A^i_u, A^j_v, $i \neq j$, none of the N^i_u events of which A^i_u is the intersection is also one of the N^j_v events of which A^j_v is the intersection and $A^i_u A^j_v$ is not the empty set. Then,

$$\sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2} \cdots \sum_{s_r=1}^{a_r} p(A^1_{s_1} \; A^2_{s_2} \; \cdots A^r_{s_r}) = g(E)\, \phi_o$$

where $E^k \phi_o = \phi_k$, ϕ_k being some given function of k alone, for some polynomial g(E) which can be found.

<u>proof:</u> Let $A = (I_{A^1_1} + I_{A^1_2} + \cdots + I_{A^1_{a_1}})(I_{A^2_1} + I_{A^2_2} + \cdots + I_{A^2_{a_2}}) \cdots$

$$\cdots(I_{A^r_1} + I_{A^r_2} + \cdots + I_{A^r_{a_r}})$$

It is easily seen that, by taking the expectation of A, precisely the required sum of probabilities is obtained. It follows then, that if we substitute $E^{N^i_u}$ for $I_{A^i_u}$ in A we obtain the required polynomial g(E).

<u>Lemma 2</u> Given the same events and conditions as in Lemma 1 except that here each event A^n_m is the intersection of N^n_m events, each such event being taken from either $\{B_i\}$ or $\{B^c_i\}$. (Note that the set $\{B^c_i\}$ is the set of events obtained by taking for each B_i, its complement B^c_i.) Then we obtain a polynomial h(E) such that the sum of the probabilities

as in Lemma 1 is equal to $h(E)\,\phi_o$.

proof:

$$I_{A_u^i} = I_{B_1^i}\, {}_{B_2^i} \dots {}_{B_s^i}\ (1 - I_{B_{s+1}^i})(1 - I_{B_{s+1}^i}) \dots (1 - I_{B_{N_u^i}^i})$$

Then we substitute for $I_{A_u^i}$, $E^s f_{N_u^i}(E)$ where $f_{N_u^i}(E)$ is the polynomial corresponding to the evaluation of $(1 - I_{B_{s+1}^i}) \dots (1 - I_{B_{N_u^i}^i})$ and carry out the multiplication obtaining $h(E)$.

(Note that if events A_i^1, $i = 1,2, \dots, a_1$; A_i^2, $i = 1,2, \dots, a_2$ and so on are mutually exclusive, then $h(E)\,\phi_o$ is precisely the probability that at least one of the events of the type $A_{s_1}^1 A_{s_2}^2 \dots A_{s_r}^r$, $s_i = 1,2, \dots a_1$ occur.)

We now find $A(E)$. Consider the subset $1,2, \dots, a$ and any selection $i_1, i_2, \dots, i_r$ of r distinct integers chosen from among $1,2, \dots, a$. Then the event that these r integers occupy their own original positions while each of the remaining $a - r$ integers do not occupy any place originally occupied by any of $a - r$ integers is $(i_1 i_1)(i_2 i_2) \dots (i_r i_r) \cdot \prod (uv)^c$ where $u,v = i_{r+1}, \dots, i_a$. Denote this event by R. Using Lemma 2, let $\{A_j^1\}$ be the set of all possible events of the type R. There are in fact $\sum_{r=o}^{a} \binom{a}{r}$ such events. Considering any other subset, the same number of events of type R are obtained. Then using Lemma 2 and denoting by $\{A_j^i\}$ the set of all events of type R we consider all the subsets of $1,2, \dots, n$, that is, $i = 1,2, \dots, \frac{n}{a}$, and obtain,

$$\sum_{s_1} \sum_{s_2} \dots \sum_{s_r} p(A_{s_1}^1 A_{s_2}^2 \dots A_{s_m}^m) = \left[\sum_{r=o}^{a} \binom{a}{r} E^r f(a - r, a - r)\right]^{\frac{n}{a}} \phi_o$$

where $f(a - r, a - r)$ is the polynomial $f(E)$ corresponding to the evaluation of $\prod(1 - I_{ij})$, $i,j = 1,2, \ldots, a - r$ and $m = \frac{n}{a}$. We recall that $f(a,b) = \sum_{v} \binom{a}{v}\binom{b}{v} v! \, (-E)^v$. Further, we note that the event $A^1_{s_1} A^2_{s_2} \ldots A^m_{s_m}$ is the event that, in the first subset of a integers, s_1 of the integers are fixed, while each of the others do not occupy any of the $a - s_1$ places designated by themselves and similarly for the second subset, s_2 of the integers are fixed and so on for the rest of the subsets. Hence, using the note at the end of Lemma 2 we have:

$$A(E) = \left[\sum_{r=0}^{a} \binom{a}{r} E^r f(a - r, a - r)\right]^{\frac{n}{a}}$$

in agreement with Kaplansky [4, p.909] where the above is stated without proof, and the required probability is equal to $A(E)\,\phi_o$ where

$$E^k \phi_o = \phi_k, \quad \phi_k = \frac{(n-k)!}{n!}, \quad f(a - r, a - r) = \sum_{v=0}^{a-r} \binom{a-r}{v} v! \, (-E)^v.$$

Denote $A(E)\,\phi_o$ by P_a, then $P_1 = \left[\sum_{r=0}^{1} \binom{1}{r} E^r f(1-r,1-r)\right]^n \phi_o$,

$f(1,1) = 1 - E$, $f(0,0) = 1$ and therefore $P_1 = (1 - E + E)^n \phi_o = 1$.

Letting $n = 4$, $a = 2$, then $P_2 = \left[\sum_{r=0}^{2} \binom{2}{r} E^r f(2-r,2-r)\right]^2 \phi_o$,

$f(2,2) = 1 - 4E + 2E^2$. Therefore

$$\begin{aligned} P_2 &= \left[1 - 4E + 2E^2 + 2E(1-E) + E^2\right]^2 \phi_o \\ &= (1 - 2E + E^2)^2 \phi_o \\ &= (1 - 4E + 6E^2 - 4E^3 + E^4)\,\phi_o \end{aligned}$$

$$= \frac{4! - 4\cdot3! + 6\cdot2+ - 4! + 0!}{4!}$$

$$= \frac{9}{4!}$$

the nine favourable permutations being given at the outset of this particular topic.

Using similar restrictions in the case of s permutations of n objects a similar polynomial A(E) is obtained. For example, if the n integers were to be permuted twice then any element may occupy, in both permutations, its original position but may not occupy any position designated by any other members of its subset in both permutations. In this case,

$$A(E) = \left[\sum_{r=o}^{a} \binom{a}{r} E^r f(a-r,a-r,a-r)\right]^{\frac{n}{a}}$$

$$\text{and } P_A = A(E)\,\phi_o \text{ where } \phi_k = \left[\frac{(n-k)!}{n!}\right]^2$$

CHAPTER 3

Some Combinatorial Lemmas

In this chapter certain combinatorial lemmas will be given. These lemmas are quite helpful in obtaining the associated polynomials in order to solve certain types of restricted permutation problems; moreover they are in themselves of some interest. Lemmas 1 and 2 are given by Kaplansky [3]. However the proof to be given here for Lemma 1 is different. Lemmas 3, 4 and 5 do not seem to appear in the literature. Also to be noted is that, although no actual examples employing Lemmas 3 and 4 are to be given, the use of Lemmas 2 and 5 in Chapter 4 will give an idea of the type of example which may be encountered and require the use of Lemma 3 or 4.

The following lemmas deal with the number of ways of choosing k elements, independent of order, from among n distinct objects arrayed in a row subject to certain conditions. The number $\binom{n}{k}$ is the case where no restrictions are imposed. The proofs of the lemmas will all be based on the following observations. Suppose we are given n - k symbols "0" and k symbols "$\emptyset$". Then for every selection "k from n", there exists a corresponding arrayment of the symbols "0" and "$\emptyset$", in a straight line, in an obvious order. For example if $n = 5$, $k = 2$, and the n objects are numbered 1,2,3,4,5 then, corresponding to the selection "1,4" is the arrangement "$\emptyset$ 0 0 $\emptyset$ 0". Obviously there is a "one-one correspondence" between all possible selections of k and all possible arrangements of the symbols. Further, for given restrictions on the selection of k, there are corresponding restrictions on the arrangements of the symbols.

<u>Lemma 1</u> The number of ways of choosing k elements, from among n distinct elements $X_1, X_2, \ldots, X_n$ so that no two consecutive elements, that is, X_i, X_{i+1}, $i = 1,2, \ldots, n-1$ appear in each selection, is $\binom{n-k+1}{k}$.

<u>Proof:</u> Suppose the n - k symbols "0" are arrayed in a row. There are then n - k + 1 "spaces" between these symbols including the "space" before the first symbol and the "space" after the last symbol. The k symbols "$\emptyset$" may then be inserted in $\binom{n-k+1}{k}$ ways into the "spaces", that is with one "space" being occupied at most with one symbol "$\emptyset$". This in fact gives the total number of ways of arranging the total number of symbols along a straight line so that no two of the symbols "$\emptyset$" appear consecutively. Hence $\binom{n-k+1}{k}$ is the required total.

For example, the three possible selections X_1X_3, X_2X_4, X_1X_4 from X_1, X_2, X_3, X_4 for k = 2 correspond to $\emptyset\ 0\ \emptyset\ 0$, $0\ \emptyset\ 0\ \emptyset$, $\emptyset\ 0\ 0\ \emptyset$ respectively and $\binom{4-2+1}{2} = 3$. Note, for k = 1 we obtain simply $\binom{n}{k}$.

<u>Lemma 2</u> The number of ways of choosing k elements, from among n distinct elements $X_1, X_2, \ldots, X_n$ arrayed in a <u>circle</u> so that no two consecutive elements, that is X_i, X_{i+1} $i = 1,2, \ldots, n$, $X_{n+1} = X_1$, appear in each selection is $\binom{n-k}{k} \frac{n}{(n-k)}$.

<u>Proof:</u> Let R(n,k) denote the number of selections with the restrictions as in Lemma 1, that is when the given objects are arrayed in a row. Let C(n,k) be our required number. In order to obtain C(n,k) we must therefore exclude those number~~s~~ selections from R(n,k) which include X_1 and X_n together. There are in fact R(n-4,k-2) such selections each containing

both X_1 and X_n . (If X_1 and X_n are both in a selection, then certainly X_2 and X_{n-1} cannot also be therein.) Our required number is then,

$$C(n,k) = R(n,k) - R(n-4,k-2)$$

$$= \binom{n-k+1}{k} - \binom{n-k-1}{k-2}$$

$$= \frac{(n-k+1)!}{(n-2k+1)!k!} - \frac{(n-k-1)!}{(n-2k+1)!(k-2)!}$$

$$= \frac{(n-k-1)!}{(n-2k+1)!(k-2)!} \left[\frac{(n-k)(n-k+1)}{k(k-1)} - 1\right]$$

$$= \frac{(n-k-1)!}{(n-2k+1)!(k-2)!} \cdot \frac{n(n-2k+1)}{k(k-1)}$$

$$= \frac{(n-k-1)!n(n-k)}{(n-2k)!k!(n-k)}$$

$$= \frac{(n-k)!n}{(n-2k)!k!(n-k)}$$

$$= \binom{n-k}{k} \frac{n}{n-k}$$

<u>Lemma 3</u> As Lemma 1 except here the restriction is that no three consecutive elements appear. Denote the required number by $R_3(n,k)$. Then,

$$R_3(n,k) = \binom{n-k+1}{k} + \binom{k-1}{1}\binom{n-k+1}{k-1} + \binom{k-2}{2}\binom{n-k+1}{k-2} + \dots$$

<u>Proof:</u> Obviously $R_3(n,k) = R(n,k) + X$, where X is the number of selections wherein at least two elements appear consecutively, but not three. Again we consider the insertion of the k symbols "∅" among the

n-k+1 "spaces" of the n-k symbols "O" arrayed in a straight line. The number of ways of placing these symbols such that each arrangement has i pairs of consecutive symbols, each pair consisting of the two consecutive symbols $\emptyset\ \emptyset$ is $\binom{k-i}{i}\binom{n-k+1}{k-i}$, $i = 1,2, \ldots, \left[\frac{k}{2}\right]$, where $\left[\frac{k}{2}\right]$ is equal to the maximum integer $\leq \frac{k}{2}$. To see this, we find all possible distinct insertions of the k-i "elements"

$$\overset{1}{(\emptyset\ \emptyset)},\ \overset{2}{(\emptyset\ \emptyset)},\ \ldots,\ \overset{i}{(\emptyset\ \emptyset)},\ \overset{i+1}{\emptyset},\ \ldots,\ \overset{k-i}{\emptyset}$$

among the n-k+1 spaces, each being filled with at most one "element". There are $\binom{n-k+1}{k-i}$ different selections of "spaces" into which the "elements" may be placed. Further, the "elements" may themselves be arrayed in $\binom{k-i}{i}$ distinct ways. Hence the total number of possible distinct insertions is equal to $\binom{k-i}{i}\binom{n-k+i}{k-i}$. Hence $X = \sum_{i=1}^{\left[\frac{k}{2}\right]} \binom{k-i}{i}\binom{n-k+1}{k-i}$.

In fact we could have let $i = 0$ and obtain also $R(n,k)$. Letting $k = 1$ we obtain $R_3(n,1) = \binom{n}{1}$.

As an example, let $n = 5$, $k = 3$, then $R_3(5,3) = 7$. Numbering the 5 distinct elements, the favourable selections are,

1, 3, 5	1, 4, 5
1, 2, 4	2, 3, 5
1, 2, 5	2, 4, 5
1, 3, 4	

In order to simplify the writing of the following lemmas, we introduce the notations:

$$\begin{pmatrix} a_m \\ \vdots \\ a_2 \\ a_1 \end{pmatrix} \equiv \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} \begin{pmatrix} a_3 \\ a_2 \end{pmatrix} \cdots \begin{pmatrix} a_m \\ a_{m-1} \end{pmatrix}$$

and hence $\begin{pmatrix} a_m \\ \vdots \\ a_2 \\ a_1 \end{pmatrix} = \dfrac{a_m!}{a_1!\,(a_2-a_1)!\,\ldots\,(a_m-a_{m-1})!}$

Thus by Lemma 3 we have $R_3(n,k) = \sum_{i=0}^{[\frac{k}{2}]} \begin{pmatrix} n-k+1 \\ k-i \\ i \end{pmatrix}$

<u>Lemma 4</u> As Lemma 1, except here the restriction is that no four consecutive elements appear. Denote the required number by $R_4(n,k)$. Then,

$$R_4(n,k) = R_3(n,k) + \sum_{i=1} \begin{pmatrix} n-k+1 \\ k-1-i \\ i \\ 1 \end{pmatrix} + \sum_{i=2} \begin{pmatrix} n-k+1 \\ k-2-i \\ i \\ 2 \end{pmatrix} + \sum_{i=3} \begin{pmatrix} n-k+1 \\ k-3-i \\ i \\ 3 \end{pmatrix} + \ldots$$

<u>Proof:</u> $R_4(n,k) = R_3(n,k) + X$ where X is the total number of distinct insertions of all possible elements, each of the form

$$\overset{1}{(\emptyset\,\emptyset\,\emptyset)},\ \overset{2}{(\emptyset\,\emptyset\,\emptyset)},\ \ldots,\ \overset{j}{(\emptyset\,\emptyset\,\emptyset)},\ \overset{j+1}{(\emptyset\,\emptyset)},\ \ldots,\ \overset{i}{(\emptyset\,\emptyset)},\ \overset{i+1}{\emptyset},\ \ldots,\ \overset{k-j-i}{\emptyset}$$

into the n-k+1 "spaces", with $j > 1$. For each fixed j and i there are

$\begin{pmatrix} n-k+1 \\ k-j-i \\ i \\ j \end{pmatrix}$ possible distinct insertions, hence,

$$R_4(n,k) = R_3(n,k) + \sum_{j=1} \sum_{i=1} \begin{pmatrix} n-k+1 \\ k-j-i \\ i \\ j \end{pmatrix}$$

The reader will no doubt suspect a general formula to include Lemmas 1,3, and 4. In fact by using the same ideas therein, the following general formula is obtained and stated without proof.

Let $R_j(n,k)$ denote the number of ways of selecting k objects from n, arrayed in a straight line so that no j are consecutive. Then,

$$R_j(n,k) = \sum_{i_1} \sum_{i_2} \cdots \sum_{i_{j-1}} \begin{pmatrix} n-k+1 \\ i_j \\ \vdots \\ i_2 \\ i_1 \end{pmatrix}$$

where $i_j = k-i_1 - i_2 - \ldots - i_{j-1}$. For $i_1 = 0$ fixed, R_{j-1} is obtained.

<u>Lemma 5</u> The number of ways of choosing k elements from among the n distinctive elements $X_1, X_2, \ldots, X_n$ so that no two elements X_i, X_{i+2}, $i = 1,2, \ldots, n-2$ are contained in each choice is,

$$\bar{R}\,(n,k) = \sum_{i=o} \begin{pmatrix} n-2k+2+i \\ k-i \\ i \end{pmatrix}$$

<u>Proof:</u> The required number is precisely the number of possible ways of placing the n-k symbols "0" and the k symbols "Ø" along a straight line so that three consecutive symbols of the form Ø Ø Ø or Ø 0 Ø are never encountered. Again here we insert the k symbols "Ø" into the n-k+1 "spaces" accordingly, as in the previous Lemmas. Here k-i elements each of the form

$$\overset{1}{(\emptyset\ \emptyset)},\ \overset{2}{(\emptyset\ \emptyset)} \ldots \overset{i}{(\emptyset\ \emptyset)},\ \overset{i+1}{\emptyset},\ \ldots,\ \overset{k-i}{\emptyset} \qquad i = 0,1, \ldots, \left[\frac{k}{2}\right]$$

are considered. Each such set of k-i elements is to be inserted among the "spaces". First we find all possible selections of k-i "spaces". Because of our restrictions, two consecutive "spaces" can not be included in our selection, for if so there would appear "Ø 0 Ø", somewhere in the final arrangement. Hence, using Lemma 1, there are $\binom{n-2k+2+i}{k-i}$ possible selections of "spaces" for each selection of the k-i "elements". Further, the "elements" themselves may be arrayed in $\binom{k-i}{i}$ distinct ways in a row. Therefore, $\binom{k-i}{i}\binom{n-2k+2+i}{k-i}$ gives the total number of favourable arrangements for the total number of symbols, using the set of k-i "elements". Finally,

$$\bar{R}\,(n,k) = \sum_{i=o} \left(\begin{matrix} n-2k+2+i \\ k-i \\ i \end{matrix}\right) \quad . \text{ For } k = 1,\ \bar{R}\,(n,1) = n.$$

CHAPTER 4

The "Problème des Ménages" and Others.

E.Lucas in his "Théorie des Nombres" published in Paris in 1891, states the following question.

"Des femmes, en nombre n, sont rangées autour d'une table, dans un ordre déterminé; on demande quel est le nombre des manières de placer leurs maris respectifs, de telle sorte qu'un homme soit placé entre deux femmes, sans se trouver à côté de la sienne?"

The above is the so-called "reduced problem des ménages". The well known "problème des ménages" itself asks for the number of ways of seating at a circular table n married couples, husbands and wives alternating, so that no husband is next to his own wife. Obviously, once the "reduced" problem is solved, then so is the "general" one. We now proceed to solve the problem in terms of probability.

Suppose the n women are seated such that there is an empty seat between each two women. Then we number the women 1,2, ...,n following a clockwise path. We number the seat immediately to the left of wife number i by the same integer i. The husbands are also numbered 1,2, ...,n each having the same number as his wife. Then the problem is to find the probability that, given a random permutation of the n integers, 1,2, ...,n ,

$$
\begin{gathered}
1 \text{ is not } 1\\
1 \text{ is not } 2\\
2 \text{ is not } 2\\
2 \text{ is not } 3\\
\vdots\\
n \text{ is not } n
\end{gathered}
$$

n is not 1

It is required then to find the polynomial $f_n(E)$ corresponding to the evaluation of

$$X = (1-I_{11})\,(1-I_{12})\,(1-I_{22})\,(1-I_{23})\,\ldots\,(1-I_{nn})\,(1-I_{n1})$$

again writing $I_{(ij)} = I_{ij}$ where (ij) is the event that "i is j". Observe that two events cannot possibly occur simultaneously if and only if they appear in consecutive factors of X. Also any k events can occur simultaneously if and only if they do not contain any two events appearing in consecutive factors of X. Hence, using Lemma 2 of the previous chapter, it follows that,

$$f_n(E) = \sum_{k=0}^{n} \binom{2n-k}{k} \frac{2n}{2n-k} (-E)^k$$

and $_nP_o = f_n(E)\,\phi_o$, where $\phi_k = \frac{(n-k)!}{n!}$, [For n = 1 set $f_1(E) = 1-E$] gives the probability of the "reduced" problem.

$$_nP_o = \sum_{k=0}^{n} (-1)^k \left(\frac{2n}{2n-k}\right) \binom{2n-k}{k} \frac{(n-k)!}{n!}$$

Using formula (D) of Chapter 1, the probability that exactly r conditions will be violated is

$$_nP_{[r]} = \sum_{k=r}^{n} (-1)^{r+k} \left(\frac{2n}{2n-k}\right)\binom{2n-k}{k}\binom{k}{r} \frac{(n-k)!}{n!}$$

This simple method is due to Kaplansky [3];

Following Kaplansky and Riordan [5, pp.117-19] a recurrence formula for $f_n(E)$ will now be obtained by an algebraic method.

Let $Y_m = (1-I_{11})(1-I_{12})\,\ldots\,(1-I_{m-1m-1})(1-I_{m-1m})(1-I_{mm})$

$m = 1,2, \ldots, n$

Let $Y^1_m = Y_m\,(1-I_{m\,m+1})$ $\quad m = 1,2,\ \ldots,n-1$

Let $X_m = Y_m\,(1-I_{m1})$

Denote the corresponding polynomial of Y_m by $\overline{Y}_m$, that of Y^1_m by $\overline{Y^1}_m$ and that of X_m by $\overline{X}_m$

$$\begin{aligned} Y_m &= Y^1_{m-1}\,(1-I_{mm}) \\ &= Y^1_{m-1} - Y^1_{m-1}\,I_{mm} \\ &= Y^1_{m-1} - Y_{m-1}\,(1-I_{m-1m})\,I_{mm} \\ &= Y^1_{m-1} - I_{mm}\,Y_{m-1} \end{aligned}$$

Hence $\overline{Y}_m = \overline{Y^1}_{m-1} - E\,\overline{Y}_{m-1}$ $\quad$ (a)

Similarly $\overline{Y^1}_{m-1} = \overline{Y}_{m-1} - E\,\overline{Y^1}_{m-2}$ $\quad$ (b)

$$\begin{aligned} X_m &= Y_m\,(1-I_{m1}) \\ &= Y_m - I_{m1}\,(1-I_{11})(1-I_{mm})\,Y^{11}_m \end{aligned}$$

where $Y^{11}_m = (1-I_{12})(1-I_{22})\ \ldots\ (1-I_{m-1m-1})(1-I_{m-1m})$

$$X_m = Y_m - I_{m1}\,Y^{11}_m$$

Hence $\overline{X}_m = \overline{Y}_m - E\,\overline{Y^{11}}_m$

But $\overline{Y^{11}}_m = \overline{Y}_{m-1}$

Therefore $\overline{X}_m = \overline{Y}_m - E\,\overline{Y}_{m-1}$ (c)

$$\overline{X}_n = \overline{Y^1}_{n-1} - 2E\,\overline{Y}_{n-1} \quad \text{using (a) and (c)}$$

$$= (1-2E)\,\overline{Y}_{n-1} - E\,\overline{Y^1}_{n-2} \quad \text{using (b)}$$

$$= (1-2E)(\overline{Y}_{n-1} - E\,\overline{Y}_{n-2}) - 2E^2\,\overline{Y}_{n-2} + E^2\,Y^1_{n-3}$$

by subtracting and adding $(1-2E)\,E\,\overline{Y}_{n-2}$ and using (b).

Now $-2E^2\,\overline{Y}_{n-2} + E^2\,\overline{Y^1}_{n-3} = -E^2\,(\overline{Y}_{n-2} - E\,\overline{Y}_{n-3})$ by (a)

Hence $\overline{X}_n = (1-2E)\,\overline{X}_{n-1} - E^2\,\overline{X}_{n-2}$

Therefore we have the recurrence formula:

$$f_n(E) = (1-2E)\,f_{n-1}(E) - E^2\,f_{n-2}(E)$$

$f_n(E)$ is usually referred to as the ménage polynomial.

Setting $u_n = n!\;{}_nP_o$ we have the so-called "ménage number" giving us the number of ways of seating the men, once the women are seated. It is interesting to note that the above formula was first given, without proof by Touchard [15] in 1934. MacMahon [8] observed the fact that u_n is the coefficient of $X_1 \cdot X_2 \cdots X_n$ in $(y-x_1-x_2)(y-x_2-x_3)\ldots(y-x_n-x_1)$ where $y = x_1 + x_2 + \ldots + x_n$. Lucas [7, p.495] gives the recurrence formula:

$$(n-2)\,u_n = (n^2-2n)\,u_{n-1} + nu_{n-2} + 4(-1)^{n-1}$$

and using this, tabulates the values of u_n up to $n = 20$. M.Wyman and L.Moser [10] give a table of values of u_n up to $n = 65$, computed by

F.L.Muksa. Also included are some new explicit solutions for u_n, via an exponential generating function which leads to some new results concerning the ménage numbers.

As noted in [5, p.118], it is perhaps somewhat more elegant to have E operate directly on $\frac{o!}{n!}$ in lieu of ϕ_o. To achieve this we form a new polynomial,

$$g_n(E) = \sum_{k=o}^{n} (-1)^k \binom{2n-k}{k} \left(\frac{2n}{2n-k}\right) E^{n-k}$$

and consider $g_n(E) \frac{o!}{n!}$ where $E^{n-k}\frac{o!}{n!} = \frac{(n-k)!}{n!}$. As $E^k \phi_o = \frac{(n-k)!}{n!}$ it follows that each term of $f_n(E) \phi_o$ is equal to its "corresponding" term in $g_n(E) \frac{o!}{n!}$ and hence ${}_nP_o = f_n(E) \phi_o = g_n(E) \frac{o!}{n!}$. Further, $u_n = n!P_o = g_n(E)o!$ where $E^r o! = r!$. Touchard in [14, p.111] shows that $g_n(E)$ may be written as a Tchebycheff polynomial,

$$g_n(E) = 2 \cos \left[2n \cos^{-1} \left(\frac{E^{\frac{1}{2}}}{2}\right) \right]$$

That is, $g_n(E) = 2\, T_{2n} \left(\frac{E^{\frac{1}{2}}}{2}\right)$ with T_m a Tchebycheff polynomial. Following Chrystal [1, p.278] we let $x = \cos \theta + i \sin \theta$. Then $\frac{1}{x} = \cos \theta - i \sin \theta$ and $\cos m\theta = \frac{1}{2}(x^m + \frac{1}{x^m})$.

Therefore $2^{2n} \cos^{2n} \theta = (x+\frac{1}{x})^{2n}$

$$= \sum_{r=o}^{n-1} \binom{2n}{r} x^{2n-r} \left(\frac{1}{x}\right)^r + \sum_{r=n+1}^{2n} \binom{2n}{r} x^{2n-r} \left(\frac{1}{x}\right)^r$$

$$+ \binom{2n}{n} x^n \left(\frac{1}{x}\right)^n$$

$$= \sum_{r=o}^{n-1} \binom{2n}{r} x^{2n-r} \left(\frac{1}{x}\right)^r + \sum_{v=o}^{n-1} \binom{2n}{v} x^v \left(\frac{1}{x}\right)^{2n-v} + \binom{2n}{n}, \quad v = 2n-r$$

$$= \sum_{r=o}^{n-1} \binom{2n}{r} \left(x^{2n-2r} + \frac{1}{x^{2n-2r}}\right) + \binom{2n}{n}$$

$$= 2 \sum_{r=o}^{n-1} \binom{2n}{r} \cos (2n-2r)\, \theta + \binom{2n}{n}$$

As $g_{n-r}(E) = 2 \cos \left[(2n-2r) \cos^{-1} \left(\frac{E^{\frac{1}{2}}}{2}\right)\right]$ it follows that,

$$2^{2n} \left(\frac{E^{\frac{1}{2}}}{2}\right)^{2n} = \sum_{r=o}^{n-1} \binom{2n}{r} g_{n-r}(E) + \binom{2n}{n}$$

$$E^n = \sum_{r=o}^{n-1} \binom{2n}{r} g_{n-r}(E) + \binom{2n}{n}$$

We can write $E^n = \sum_{r=o}^{n} \binom{2n}{r} g_{n-r}(E)$ with $g_o(E) = 1$. (In the explicit expression given above for $g_n(E)$, $g_o(E)$ is undefined.) Hence, by letting both sides of the equation operate on o! we obtain the inverse relation

$$n! = \sum_{r=o}^{n} \binom{2n}{r} u_n$$

as developed by Riordan [13] and used for the investigation of the residues of the numbers $u_{n,m} \left(= n!\, {}_nP_{[m]}\right)$ to a prime modulus.

For further literature on the game of ménages we refer to Kaplansky and Riordan [8] which together with an already mentioned article by Moser and Wyman [10] contain an extensive bibliography.

Omitting the restriction that "n is 1" in the game of ménages we obtain the "Problème des ménages non-circulaires". That is, here the married couples are seated at a straight table and hence the omission of "n is 1." Using Lemma 1 of the previous chapter, the requisite formulae are:

$$f_n(E) = \sum_{k=o}^{n} \binom{2n-k}{k} (-E)^k$$

$$_nP_o = \sum_{k=o}^{n} (-1)^k \binom{2n-k}{k} \frac{(n-k)!}{n!}$$

$$_nP_{[r]} = \sum_{k=r}^{n} (-1)^{r+k} \binom{2n-k}{k} \binom{k}{r} \frac{(n-k)!}{n!}$$

Using the same notation as was used to derive the recurrence formula for the ménage polynomial,

$$Y_m = (1-I_{11}) (1-I_{12}) \ldots (1-I_{m-1m}) (1-I_{mm})$$

$$Y'_m = Y_m (1-I_{m\ m+1}) \text{ , } I_{n\ n+1} \text{ not defined}$$

$\overline{Y}_m$, $\overline{Y'}_m$ are the corresponding polynomials of Y_m, Y'_m respectively and using the derived relations,

$$\overline{Y}_m = \overline{Y'}_{m-1} - E\,\overline{Y}_{m-1} \qquad \text{(a)}$$

$$\overline{Y'}_{m-1} = \overline{Y}_{m-1} - E\,\overline{Y'}_{m-2} \qquad \text{(b)}$$

we obtain, $\overline{Y}_n = (1-E)\,\overline{Y}_{n-1} - E\,\overline{Y'}_{n-2}$

$$= (1-2E)\,Y'_{n-2} - (1-E)\,E\,\overline{Y}_{n-2}$$

$$= (1-2E)\,(\overline{Y'}_{n-2} - E\,\overline{Y}_{n-2}) - E^2\,\overline{Y}_{n-2}$$

Hence we obtain the recurrence formula

$$f_n(E) = (1-2E)\,f_{n-1}(E) - E^2\,f_{n-2}(E)$$

That is, the recurrence formula for the ménage polynomial involving a round table in the game is the same for the ménage polynomial obtained using a straight table.

A general type of restricted permutation is now discussed. This is the question of "discordant permutations". Two permutations are said to be discordant with each other when no element is in the same position in both. The case when dealing with permutations discordant with a given one is in fact the usual "problème des rencontres", which has been considered in Chapter 2. The case of permutations discordant with two permutations, one permutation being (1,2, ...,n), the other (n,1,2, ...,n-1) is the "problème des ménages". We consider now Lucas' problem [7, p.491] of permutations discordant with <u>any</u> two given ones.

First, it is easily seen that for any two given permutations we can in fact solve the problem by considering two other permutations chosen accordingly such that one is of the type (1,2, ...,m). For example suppose the given permutations are $A^1 = (5,3,2,4,1)$ and $B^1 = (3,4,2,1,5)$. Then we write instead of A^1, $A = (1,2,3,4,5)$ and instead of B^1, $B = (2,4,3,5,1)$. The problem of discordant permutations will now be solved using the five integers 1,2, ...,5 and A and B as the given permutations. The general solution for n integers can then be easily seen. Consider B to be a permutation of A. That is, write down

$$B = \binom{1,2,3,4,5}{5,1,3,2,4} = (2,4,3,5,1)$$

where $\binom{1,2,3,4,5}{5,1,3,2,4}$ is a permutation of (1,2,3,4,5) placing 1 in 5^{th} place, 2 in 1^{st} place, 3 in 3^{rd} place, 4 in 2^{nd} place, and 5 in 4^{th} place, obtaining a new permutation. Hence we can write B as B = $\binom{1,5,4,2,3}{5,4,2,1,3}$ or more simply as B = (1,5,4,2)(3), Now the problem is to find that in a random permutation of 1,2,3,4,5,

1 is not 1 or 5
2 " " 2 or 1
3 " " 3 or 3
4 " " 4 or 2
5 " " 5 or 4

But these are precisely the "ménages conditions" imposed on the 4 integers 1,5,4,2 only and on the integer 3 alone. It follows that our required polynomial $D_5(E) = f_4(E)\ f_1(E)$, $f_n(E)$ being the ménage polynomial and the required probability ${}_5P_o = D_5(E)\ \phi_o$ where $\phi_k = \frac{(n-k)!}{n!}$ and note that $f_1(E)$ was said to be equal to 1 - E. Then,

$$f_4(E)\ f_1(E) = \left[\sum_{k=o}^{4} (-1)^k \binom{8-k}{k} \left(\frac{8}{8-k}\right) E^k\right] (1-E)$$

$$= 1 - 9E + 28E^2 - 36E^3 + 18E^4 - 2E^5$$

and ${}_5P_o = f_4(E)\ f_1(E)\ \phi_o = \frac{16}{5!}$. The following are the 16 permutations discordant with (1,2,3,4,5) and (2,4,3,5,1):

3 1 5 2 4	4 1 5 2 3	4 5 1 3 2	5 1 4 3 2
3 5 1 2 4	4 1 5 3 2	4 5 2 1 3	5 3 1 2 4
3 5 2 1 4	4 3 5 1 2	5 1 2 3 4	5 3 2 1 4
3 5 4 1 2	4 5 1 2 3	5 1 4 2 3	5 3 4 1 2

No doubt the general solution is now "almost" apparent. However before writing such a solution, another example is considered. Here $n = 6$, $A = (1,2,3,4,5,6)$ and $B = (4,5,1,3,2,6)$. Then $B = (3,4,1)(2,5)(6)$. Hence $D_6(E) = f_3(E)\ f_2(E)\ f_1(E)$, and ${}_6P_o = D_6(E)\ \phi_o$.

In general, suppose two permutations of the integers 1,2, ...,n are given. Then the problem is that of finding the probability that a random permutation is discordant with the permutations $A = (1,2, \ldots,n)$ and $B = (i_1,i_2, \ldots,i_{r_1})(j_1,j_2, \ldots,j_{r_2}) \ldots (k_1,k_2, \ldots,k_{r_m})$.

Then the required probability is,

$${}_nP_o = f_{r_1} f_{r_2} \cdots f_{r_m}\ \phi_o$$

where $f_{r_i} \equiv f_{r_i}(E)$, and $f_{r_i}(E)$ is the ménage polynomial.

An interesting application of the question of discordant permutations is now discussed. The problem here is to find the probability that a term chosen at random from those of an n by n determinant contains no "element" from either of the main diagonals. By an "element" of the term $a_1\ a_2\ \ldots\ a_n$ is simply meant any factor a_i. This question was first treated by Netto [11, p.80 et seq.] in a rather complicated manner.

We examine now an n by n determinant having the following form.

$$\begin{vmatrix} a_1^1 & a_1^2 & \cdot & \cdot & \cdot & a_1^{n-1} & a_1^n \\ a_2^1 & a_2^2 & \cdot & \cdot & \cdot & a_2^{n-1} & a_2^n \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & \cdot \\ a_{n-1}^1 & a_{n-1}^2 & \cdot & \cdot & \cdot & a_{n-1}^{n-1} & a_{n-1}^n \\ a_n^1 & a_n^2 & \cdot & \cdot & \cdot & a_n^{n-1} & a_n^n \end{vmatrix}$$

We recall that the value of the determinant shown above is found by summing all possible terms, each of the form $\pm\, a_{i_1}^1 a_{i_2}^2 \ldots a_{i_n}^n$, $i_1, i_2, \ldots, i_n$ being any permutation of $1, 2, \ldots, n$. The question as to which sign, + or - is affixed to each term, is irrelevant to our main problem. As can easily be seen from the above determinant, the question is then to find the probability that given any term of the determinant at random,

1 is not i_1

1 " " i_n

2 " " i_2

2 " " i_{n-1}

$\vdots$ $\vdots$ $\vdots$ $\vdots$

n is not i_n

and n is not i_1

More simply, we wish to find the probability that, given a random permutation of 1,2, ...,n none of the events "1 is 1", "1 is n", "2 is 2", "2 is n-1", ..., "n is n","n is 1" occur. This is the same as finding the probability for a permutation of 1,2, ...,n to be discordant with the given permutations A = (1,2, ...,n) and B = (n,n-1, ...,1). Further,

$$B = (1,n)\ (2,n-1)\ (3,n-2)\ \ldots\ (\tfrac{n}{2}, \tfrac{n}{2} + 1)\ , \quad n \text{ even}$$

$$B = (1,n)\ (2,n-1)\ (3,n-2)\ \ldots\ (\tfrac{n-1}{2}, \tfrac{n+1}{2} + 1)\ (\tfrac{n+1}{2})\ , \quad n \text{ odd}$$

Therefore our required probability is,

$$ {}_nP_o = \left[f_2(E)\right]^{\frac{n}{2}} \Phi_o, \quad n \text{ even}$$

$$ = \left[f_2(E)\right]^{\frac{n-1}{2}} f_1(E)\ \Phi_o\ , \quad n \text{ odd}$$

where $f_i(E)$ is the ménage polynomial. In fact, $f_1(E) = 1-E$ and $f_2(E) = 1 - 4E + 2E^2$. A list of the probabilities for n = 1,2, ...,8 is now given.

n	required polynomial f(E)	${}_nP_o$
1	$1 - E$	0
2	$1 - 4E + 2E^2$	0
3	$1 - 5E + 6E^2 - 2E^3$	0
4	$1 - 8E + 20E^2 - 16E^3 + 4E^4$	$\frac{4}{4!} = \frac{1}{6}$
5	$1 - 9E + 28E^2 - 36E^3 + 20E^4 - 4E^5$	$\frac{16}{5!} = \frac{2}{15}$
6	$1 - 12E + 54E^2 - 112E^3 + 108E^4 - 48E^5 + 8E^6$	$\frac{80}{6!} = \frac{1}{9}$
7	$1 - 13E + 66E^2 - 166E^3 + 220E^4 - 156E^5 + 56E^6 - 8E^7$	$\frac{672}{7!} = \frac{2}{15}$

$$8 \quad 1 - 16E + 104E^2 - 352E^3 + 664E^4 - 704E^5 + 416E^6 - 128E^7 + 16E^8 \quad \frac{4757}{8!} = \frac{33}{280}$$

The above values are in agreement with Touchard [14, pp.117-8] who, by means of generating functions, produces relations which make for easier computation of values for large n.

We now consider a problem treated for the first time in the literature in 1956 by Mendelsohn [9, p.238]. Here the restrictions imposed on a random permutation of n integers 1,2, ...,n is: "1 is 2nd", "n is (n-1)th" and for i = 2,3,4, ...,n-1, "i is (i-1)th" and "i is (i+1)th". Using our method it is required then to evaluate

$$X_n = (1-I_{12})(1-I_{21})(1-I_{32})(1-I_{23})(1-I_{34})(1-I_{43}) \ldots (1-I_{n,n-1})$$

Note the last factor of X_n is $(1-I_{n,n-1})$ which is so for n even. For n odd the last factor is $(1-I_{n-1,n})$. However the evaluation of X_n is obtained for general n and X_n is used only for illustration. First it is observed that there are 2(n-1) factors of X_n. Denote the factors by x_i, $(i = 1,2, \ldots, 2(n-1))$, as they appear in their natural ordering from left to right. Then it is seen that $I_{ij}I_{mn} \equiv 0$ if and only if I_{ij} is contained in a factor x_i and I_{mn} in a factor x_{i+2} $(i = 1,2, \ldots, 2n-4)$. Hence, using Lemma 5 of the previous chapter, the associated polynomial is,

$$f_n(E) = \sum_{k=0}^{n} \sum_{i=0}^{\left[\frac{k}{2}\right]} \binom{2n-2k+i}{k-i}\binom{k-i}{i} (-E)^k$$

where $\left[\frac{k}{2}\right]$ is the maximum integer $\leq \frac{k}{2}$. Then, for example, $f_2(E) = 1-2E+E^2$,

$f_3(E) = 1 - 4E + 4E^2$ and $f_4(E) = 1 - 6E + 11E^2 - 6E^3 + E^4$.

We now proceed to obtain a recurrence formula. Write down, $X_n = (1-I_{12})(1-I_{21})\, X_{n-1}$, X_n being given above.

Then $\quad X_n = X_{n-1} - (I_{12} + I_{21})\, X_{n-1} + I_{12}\, I_{21}\, X_{n-1}$

Now $\quad I_{12}\, I_{21}\, X_{n-1} = I_{12}\, I_{21}\,(1-I_{32})(1-I_{23})\, X_{n-2}$

$$= I_{12}\, I_{21}\, X_{n-2}$$

By the rule of replacement the associated polynomial of $I_{12}\, I_{21}\, X_{n-2} = E^2 f_{n-2}(E)$, where $f_n(E)$ was given above.

Further, $\quad I_{12}\, X_{n-1} + I_{21}\, X_{n-1} = (I_{12} + I_{21})(1-I_{32})(1-I_{23})\, X_{n-2}$

$$= I_{12}\, X_{n-2} - I_{12}\, I_{32}\, X_{n-2} - I_{12}\, I_{23}\, X_{n-2}$$

$$+ I_{12}\, I_{32}\, I_{23}\, X_{n-2} + I_{21}\, X_{n-2} - I_{21} I_{32} X_{n-2}$$

Now, $\quad I_{12}\, I_{32}\, I_{23}\, X_{n-2} = I_{12}\, I_{32}\, I_{23}\,(1-I_{34})(1-I_{43})\, X_{n-3}$

$$= I_{12}\, I_{32}\, I_{23}\, X_{n-3}$$

We now evaluate $I_{12}\, X_{n-1}$. Suppose E is written instead of I_{12} <u>before</u> actual multiplication. Then the associated polynomial is

$$Ef_{n-1}(E) - E^3 f_{n-3}(E) + E\, g(E)$$

where $g(E)$ corresponds to the evaluation of $I_{32}\, X_{n-2}$. Moreover, the associated polynomial of $I_{21}\, X_{n-1}$ is

$$Ef_{n-2}(E) - E\, g(E).$$

Hence the associated polynomial of $I_{12}\ X_{n-1} + I_{21}\ X_{n-1}$ is simply,

$$E\ f_{n-1}(E) + E\ f_{n-2}(E) - E^3\ f_{n-3}(E)$$

and finally we obtain the recurrence relation

$$f_n(E) = (1-E)\ f_{n-1}(E) + (-E+E^2)\ f_{n-2}(E) + E^3\ f_{n-3}(E)$$

It is interesting to note, that although Mendelsohn [9, p.238] gives the recurrence formula, he does not give an exact formula for the required probabilities.

CHAPTER 5

A Second Type of Restricted Permutation.

In all of the preceding examples, (ij) represented the event "i is j". We introduce now a second type of restricted permutation. Here (ij) denotes the event "i immediately precedes j". Here also the fundamental requirement of quasi-symmetry is fulfilled and in fact, as before, the probability of k events occurring simultaneously is either equal to 0, or to $\frac{(n-k)!}{n!}$. The occurrence of an event $(i_1j_1)(i_2j_2) \ldots (i_kj_k)$, is impossible if $i_m = i_n$, $m \neq n$, or $j_m = j_n$, $m \neq n$. Further, any event of the form $(i_1j_1)(j_1j_2)(j_2j_3) \ldots (j_{k-1}i_1)$ cannot occur, that is, has probability zero. For example, in a random permutation of 1,2, ...,n , the probability that i never immediately precedes i+1, $(i = 1,2, \ldots, n-1)$ is required. The required polynomial is found by evaluating $(1-I_{12})(1-I_{23}) \ldots (1-I_{n-1,n})$, where $I_{(ij)} = I_{ij}$. As no products in the multiplication vanish, the required polynomial is $f(E) = (1-E)^{n-1}$. The required probability is then $(1-E)^{n-1}\phi_o$, where $E^k\phi_o = \phi_k$ and $\phi_k = \frac{(n-k)!}{n!}$

Kaplansky [4, pp.911-13] treats two interesting problems dealing with permutations of this second type.

The first is the so called "n - kings problem". We consider the n by n chess-board. What is the probability that if n kings be placed on the board one on each row and one on each column, no two attack each other? Kaplansky refers to this problem as "a first cousin, albeit a shabby relative of the famous 'n-queens problem". In order to realize the conditions imposed in the general case, consider the case when $n = 5$ and the following diagram.

	1	2	3	4	5
1				X	
2		X			
3					X
4	X				
5			X		

The numbers at the top of the columns represent the five kings. King "number 1" is placed in row 4, "king number 2" in row 2 and so on, giving us a "4 2 5 1 3" permissible arrangement. Note that once the digit 4 is the first digit of this arrangement "4 2 5 1 3", the second digit cannot be 3 or 5. Once the second digit is 2, the third cannot be 1 or 3. Once the third is 5, the fourth cannot be 4 and once the fourth is 1, the fifth cannot be 2. In general then, if king "number i" is in the j^{th} row, then king number $i+1$ cannot be placed in the $(j-1)$th or $(j+1)$th row. If $j = 1$, then the restriction of the $(j-1)$th row is ignored and similarly for the $(j+1)$th row for $j = n$.

Hence given a permutation of the n integers $1,2, \ldots,n$ the restrictions are, "1 immediately precedes 2", "n immediately precedes n-1" and for $i = 2,3,4, \ldots,n-1$, "i immediately precedes (i-1)" and "i immediately precedes (i+1)". Therefore our problem is to evaluate the product,

$$A_n = (1-I_{12})(1-I_{21})(1-I_{23})(1-I_{32}) \ldots (1-I_{n-1,n})(1-I_{n,n-1})$$

However no method has been found in order to give explicitly, as in the previous examples, the associated polynomial. Following Kaplansky [4, pp.912-13] we proceed as follows:

Let B_n be defined by the equation $A_n = B_n(1-I_{n,n-1})$.

Let $\bar{A}_n$, $\bar{B}_n$ be the associated polynomials of A_n, B_n respectively.

Then the following statements are true.

$$\bar{A}_n = \bar{B}_n - E\,\bar{B}_{n-1} \qquad \text{(a)}$$

$$\bar{B}_n = (1-E)\,\bar{B}_{n-1} - E\,\bar{B}_{n-2} \qquad \text{(b)}$$

<u>Proof of (a).</u> $\quad A_n = B_n(1-I_{n,n-1})$

$$= B_n - B_n I_{n,n-1}$$

$$= B_n - B_{n-1}(1-I_{n-1,n-2})(1-I_{n-1,n})\, I_{n,n-1}$$

$$= B_n - B_{n-1}(1-I_{n-1,n-2})\, I_{n,n-1}$$

Now $B_{n-1}(1-I_{n-1,n-2})\, I_{n,n-1} = C_{n-1}(1-I_{n-2,n-1})(1-I_{n-1,n-2})\, I_{n,n-1}$

$$= C_{n-1}(1-I_{n-1,n-2})\, I_{n,n-1}$$

Consider $B_{n-1}(1-I_{n-1,n-2})\, I_{n-1,n} = C_{n-1}(1-I_{n-2,n-1})\, I_{n-1,n}$

Also, it can be easily seen that the associated polynomial of $C_{n-1}(1-I_{n-1,n-2})$ is the same as that of $C_{n-1}(1-I_{n-2,n-1})$.

Further, $B_{n-1}(1-I_{n-1,n-2})\, I_{n-1,n} = B_{n-1}\, I_{n-1,n}$

It follows that $\bar{A}_n = \bar{B}_n - E\,\bar{B}_{n-1}$

<u>Proof of (b).</u> $$B_n = A_{n-1}(1-I_{n-1,n})$$

$$= A_{n-1} - B_{n-1}(1-I_{n-1,n-2})\ I_{n-1,n}$$

$$= A_{n-1} - B_{n-1}\ I_{n-1,n}$$

Hence $\overline{B}_n = \overline{A}_{n-1} - E\ \overline{B}_{n-1}$

Using (a) and (b) we obtain

$$\overline{B}_n = (1-E)\ \overline{B}_{n-1} - E\ \overline{B}_{n-2} \qquad \text{(c)}$$

Solving (c) now as in the usual case of difference equations, with initial values. $B_1 = 1$ and $B_2 = 1-E$ we consider, $u^n = (1-E)\ u^{n-1} - E\ u^{n-2}$. Solving $u^{n-2}\ u^2 = u^{n-2}\ [(1-E)\ u-E]$ we obtain two solutions,

$$s,t = \frac{(1-E) \pm (1-6E+E^2)^{\frac{1}{2}}}{2} \qquad \text{(d)}$$

The general solution is then of the form $\overline{B}_n = Y\ s^n + Z\ t^n$. As $\overline{B}_1 = 1$, $\overline{B}_2 = 1-E$ we can define $\overline{B}_o = 0$. Therefore solving $Y + Z = 0$ and $Y\ s + Z\ t = 1$ for Y and Z it follows that,

$$\overline{B}_n = \frac{s^n - t^n}{s - t} \qquad \text{(e)}$$

Thus using (a) and (e) $\overline{A}_n$ is found.

For example for $n = 4$,

$$\overline{B}_4 = \frac{s^4 - t^4}{s - t} = s^3 + s^2t + st^2 + t^3 \qquad \text{using (e)}$$

$$= \frac{(1-E)\left[(1-E)^2 + 1 - 6E + E^2\right]}{2} \qquad \text{using (d)}$$

$$= 1 - 5E + 5E^2 - E^3$$

$$\bar{B}_3 = \frac{s^3 - t^3}{s - t} = s^2 + st + t^2$$

$$= \frac{1}{4}\left[3(1-E)^2 + 1 - 6E + E^2\right]$$

$$= 1 - 3E + E^2$$

and $\bar{A}_4 = \bar{B}_4 - E\,\bar{B}_3 = 1 - 6E + 8E^2 - 2E^3$

The required probability is,

$$\bar{A}_4\,\Phi_o = (4! - 6\cdot 3! + 8\cdot 2! - 2\cdot 1!)\;/\;4!$$

$$= \frac{2}{4!}$$

The two corresponding favourable permutations are 3 1 4 2 and 2 4 1 3.

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