GENERALISING THE STRUCTURE-SEMANTICS ADJUNCTION: OPERATIONAL CATEGORIES

by

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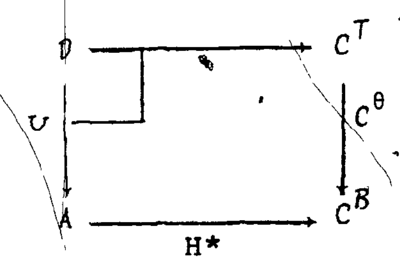
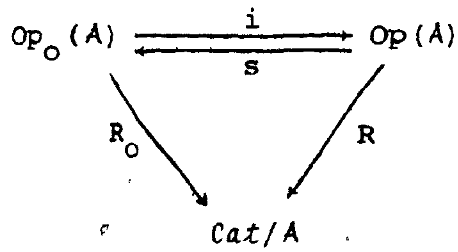
ABSTRACT

The idea of an operational category over $\mathcal{A}$ generalizes the notions of tripleable and equational category over $\mathcal{A}$, and also the dual notions of cotripleable and coequational category. An operational category, $U: \mathcal{D} \to \mathcal{A}$ is given by a presentation (θ, H)



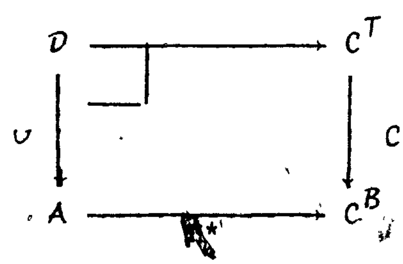
where θ is a functor bijective on objects and $\mathcal{D}$ is a specified pullback. $R: Op(\mathcal{A}) \to \mathcal{C}at/\mathcal{A}$ is defined as the category of operational categories (and functors) with given presentations. Another category, $Op_o(\mathcal{A})$ over $\mathcal{C}at/\mathcal{A}$ of operational categories with standard presentations is also defined. There is a fixed theory θ_o, employed in every standard presentation. $Op_o(\mathcal{A})$ is a retract of $Op(\mathcal{A})$ over $\mathcal{C}at/\mathcal{A}$:



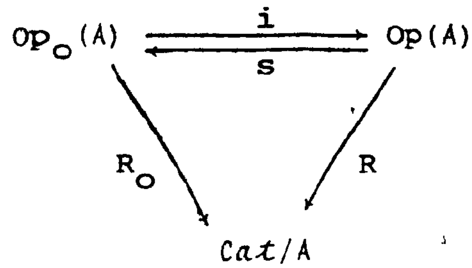
i.e. every operational category (and functor) has a standard presentation (but not $s \dashv i$!). Also R_O has a left adjoint L_O and $Op_O(A)$ is complete. Finally, there is a category of algebras, $\underline{S}_*$-Alg over Cat/A such that $Op_O(A) \approx S_*$-Alg over Cat/A. Thus, the operational categories can be determined by their internal structure, without reference to any presentation. Some properties of operational categories and some special cases are also examined.

RESUME

Le concept de catégorie opérationnelle sur A généralise les notions de catégorie triplable et équationnelle sur A et aussi les notions duales de catégorie cotriplable et coéquationnelle. Une catégorie opérationnelle, $U: \mathcal{D} \to A$ est donnée par une présentation (θ, H)



où θ est un foncteur bijectif sur les objets et $\mathcal{D}$ est un produit fibré spécifié. $R: \mathrm{Op}(A) \to \mathcal{C}at/A$ est défini comme étant la catégorie des catégories opérationnelles (et foncteurs) dont les présentations sont spécifiées.
Nous définissons aussi une autre catégorie, $\mathrm{Op}_o(A)$ sur $\mathcal{C}at/A$ de catégories opérationnelles avec présentations standards. Il y a une théorie fixe θ_0, employée dans toutes les présentations standards. $\mathrm{Op}_o(A)$ est un rétracte de $\mathrm{Op}(A)$ sur $\mathcal{C}at/A$:



c'est-à-dire toute catégorie opérationnelle possède une présentation standard (mais pas $s \dashv i$!). De plus R_O possède une adjonction à gauche L_O et $Op_O(A)$ est complète. Enfin, il existe une catégorie d'algèbres, $\underline{S}_*$-Alg sur Cat/A telle que $Op_O(A) \simeq S_*$-Alg over Cat/A. Ainsi, les catégories opérationnelles peuvent être déterminées par leur structure interne sans faire référence à quelque présentation que ce soit. Nous examinons aussi quelques propriétés et cas particuliers des catégories opérationnelles.

ACKNOWLEDGEMENTS

I would like to thank Professor J. Lambek for posing the problem of characterising operational categories and encouraging me to find its solution.

I would also like to thank the Association of Universities and Colleges of Canada for the Commonwealth Scholarship and the Department of Mathematics and Statistics at McGill University for its financial support.

TABLE OF CONTENTS

INTRODUCTION

This paper considers operational categories, a generalization of the algebraic categories of Lawvere [5] and the equational categories of Linton [6,7]. As well as all the tripleable categories, the cotripleable categories are operational too.

Lawvere's idea was to represent an algebraic theory, say that of groups, by a product preserving functor which is bijective on objects (b.o.)

$$\theta : \underline{N}^{op} \to T$$

where $\underline{N}^{op}$ is the free category with finite products on one generator. Then, relative to the base functor

$$H: \mathit{Sets} \times \underline{N}^{op} \longrightarrow \mathit{Sets}$$
$$(X, n) \longrightarrow X^n$$

the category of models ($\mathcal{D}$) for the theory, with forgetful functor (algebraic) to *Sets*, is constructed by the pullback

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & Sets^{\mathcal{T}} \\ {\scriptstyle U}\downarrow & \lrcorner & \downarrow{\scriptstyle Sets^{\theta}} \\ Sets & \xrightarrow[H^*]{} & Sets^{\underline{N}^{op}} \end{array}$$

where H^* is the exponential transpose of H. (Here $\mathcal{D}$ is the category of product preserving functors from $\mathcal{T}$ to *Sets* with forgetful functor U being evaluation at 1). For *Grp*, $\mathcal{T}$ is generated by $\underline{N}^{op}$, a multiplication map $m:2\to 1$, an identity $e:0\to 1$ and an inverse map $i:1\to 1$, closed up with respect to finite products, composition and some equations e.g. $m(id,i) = et$, or $x.x^{-1} = e$ (t is the terminal morphism $1\to 0$).

Linton generalized this idea by letting the base functor be the homfunctor

$$\text{Hom}: Sets \times Sets^{op} \to Sets$$

or

$$\text{Hom}: A \times A^{op} \to Sets$$

(where *Sets* is the category in which the homsets of A live). An equational theory is a product preserving, b.o. functor

$$\theta: A^{op} \to T$$

and the equational functor is created by pulling back as before:

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathit{Sets}^{\mathcal{T}} \\ {\scriptstyle U}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathit{Sets}^{\theta}} \\ \mathcal{A} & \xrightarrow[\text{Yoneda}]{} & \mathit{Sets}^{\mathcal{A}^{op}} \end{array}$$

Note that $\mathcal{T}$ may be large with respect to *Sets*. It is assumed, where necessary, that there is a Grothendieck universe V containing a universe U. *Sets* is the category of small sets with respect to U; *Ens* is the category of small sets with respect to V. With $\mathcal{T}$ V-small, the pullback is constructed in the category of (*Ens*-) small categories. Linton showed that all tripleable categories are equational. Also equational are non-tripleable categories such as Complete Boolean Algebras ($\mathcal{CBA}$) over *Sets* (see [4]), and Burroni's [2] categories of graphical algebras, which are tripleable over *Gph*, the category of (directed multi-) graphs and graph homomorphisms.

The notion of operational category was introduced by J. Lambek at a meeting of the Midwestern Category Theory Seminar at Waterloo University in 1968. He freed the base functor to be any

$$H:\mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$$

and a theory to be any b.o. functor

$$\theta:\mathcal{B} \longrightarrow \mathcal{T}$$

An operational category $\mathcal{D}$ (with presentation (θ, H)) is given by the pullback

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & C^{T} \\ \big\downarrow & \lrcorner & \big\downarrow C^{\theta} \\ A & \xrightarrow[H^*]{} & C^{B} \end{array}$$

(H^* is the transpose of H.) Thus all algebraic and equational categories are operational. In particular, all tripleable categories are operational. By a duality argument, the cotripleable categories are too.

A category of operational categories (with presentations) and operational functors (with presentations) over A, $Op(A)$, is constructed with a forgetful functor $R: Op(A) \to Cat/A$. However, much of the focus of this work will be on operational categories and functors without regard to any particular presentation. There are several reasons for this. Firstly, a given operational category will usually have many presentations. Two different methods of standardizing the presentation will be given. Secondly, $Op(A)$ is not a very attractive category. There is no left adjoint to the forgetful functor and there is no easy construction of limits in $Op(A)$. Finally, one of the motivations for this work was to characterise the operational categories in Cat/A, where no presentation is given. Unfortunately, the operational categories and functors (without presentations) don't form a category, since two operational functors acting with respect to incompatible presentations may not compose to form an operational functor (c.f. Proposition III.3.12). Hence, the category of operational categories most gainfully employed is a category of operational categories with standard presentations, $Op_0(A)$. Its forgetful functor $R_o: Op_0(A) \to Cat/A$ has a left adjoint and $Op_0(A)$ has all limits. Also it is equivalent to a category of algebras which enables the operational categories and functors to be characterized in terms of their internal structure.

Of course, this construction of models from a theory (relative to H) is a generalised semantics functor, $Sem: Th(B)^{op} \to Cat/A$ with a left adjoint $Str: Cat/A \to Th(B)^{op}$ sending $U: \mathcal{D} \to A$ to the full image of $B \to C^A \to C^{\mathcal{D}}$ (c.f. [6]).

Other, generalisations of tripleable and equational categories and their duals have been explored in [3], [10] and [11]. In [3], Davis considers 'equational systems of functors'. The constructions employ a base functor

$A \times A^A \to A$ which exploits the ability to compose endofunctors. The only examples that he gives which aren't operational are categories of machines. Thiébaud, in his unpublished thesis [10], constructs a generalised Structure-Semantics adjunction, based on the theory of bimodules. As it happens, this adjunction is related to that for $R_0 : Op_0(A) \to Cat/A$, though no reference is made to operationality in the sense of pullbacks. Once he has created the adjunction, most of his work is devoted to studying the algebras for the resulting triple, $\underline{S}$. Here it is the operational categories themselves which are considered. Wyler, in [11], studies categories of sets with relations and mappings which preserve the structure. Aside from categories operational over *Sets*, his examples include fields (with field extensions) and small categories (with functors). None of these authors characterise their objects of study through internal properties or provide standard presentations. Here, the name 'operational' implies the exclusion of situations in which relations are inherent, such as Wyler's examples of fields and categories.

In chapter I operational categories and functors are defined, the standard presentations are given and the left adjoint for R_0 is created. Aside from Beck's Tripleability Theorem, §1 is due to J. Lambek. Chapter 2 is devoted to shuffle retracts, the algebraic material which culminates in the construction of the $\underline{S}_*$-algebras. In Chapter III the triple induced by $L_o \dashv R_o$ is related to shuffle retracts and $Op_o(A)$ is shown to be equivalent to $\underline{S}_*$-Alg over Cat/A. Examples of operational categories are given, as well as counter-examples to some appealing hypotheses. Finally, mild conditions are given for $L_oR_o \approx 1$, for operationality to be equivalent to Beck's Tripleability Conditions (given below), and for limits and colimits to exist in an operational category.

Except where stated above to the contrary, all work presented is due to the author.

CHAPTER I

OPERATIONAL CATEGORIES

§1. Operational Categories

Definition 1.1: Let $\theta : \mathcal{B} \to \mathcal{T}$ be a functor bijective on objects and let $H : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be any functor.

Construct the pullback $U : \mathcal{D} \to \mathcal{A}$

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\quad \P' \quad} & \mathcal{C}^{\mathcal{T}} \\ {\scriptstyle U}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathcal{C}^{\theta}} \\ \mathcal{A} & \xrightarrow[\quad H^* \quad]{} & \mathcal{C}^{\mathcal{B}} \end{array}$$

Then $(\mathcal{D}, U)$ is an operational category with presentation (θ, H). By convention, objects $(\mathcal{D}, U)$ of $\mathcal{C}at/\mathcal{A}$ may be denoted $\mathcal{D}$ or $\mathcal{D}$ over $\mathcal{A}$, with the U suppressed. An object of $\mathcal{D}$ is an algebra i.e. a pair $(X, \Phi) \in \mathcal{A} \times \mathcal{C}^{\mathcal{T}}$ such that

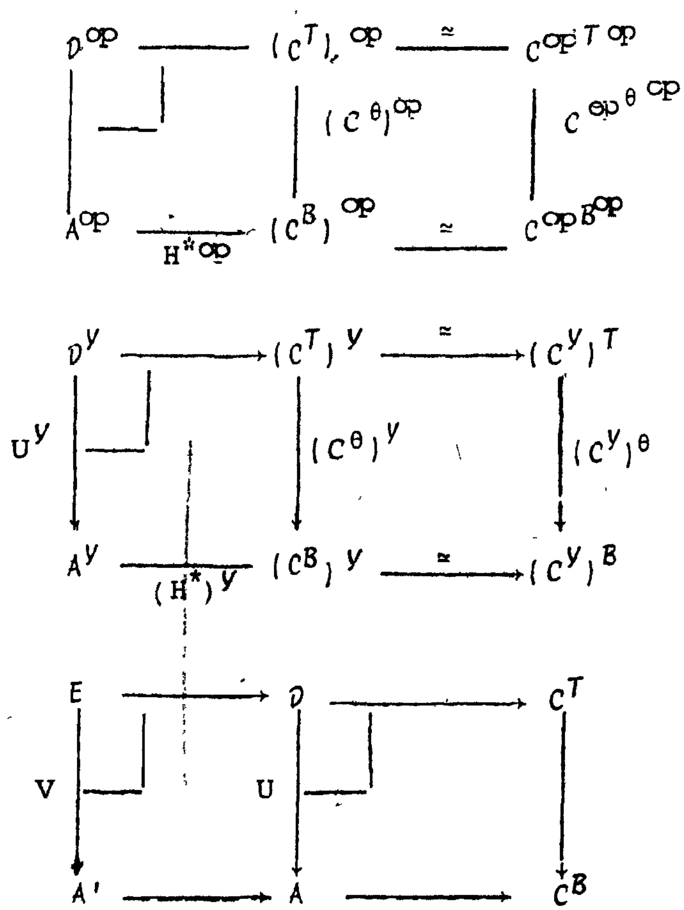
$$\Phi\theta = H^*X$$

A morphism of $\mathcal{D}$ is a homomorphism i.e. a pair (f, t) in $\mathcal{A} \times \mathcal{C}^{\mathcal{T}}$ such that

$$t\theta = H^*f$$

Proposition 1.2: Let $U:\mathcal{D}\to\mathcal{A}$ be operational with presentation (θ,H). Then $U^{op}:\mathcal{D}^{op}\to\mathcal{A}^{op}$ is operational. Further, for any category $\mathcal{Y}$, $U^{\mathcal{Y}}:\mathcal{D}^{\mathcal{Y}}\to\mathcal{A}^{\mathcal{Y}}$ is operational. Finally, given $F:\mathcal{A}'\to\mathcal{A}$ then $V:\mathcal{E}\to\mathcal{A}'$, the pullback of U along F, is operational.

Proof: The conclusions follow immediately from the following three pullbacks:



Lemma 1.3: Every operational category is operational with respect to an evaluation functor $ev:\mathcal{A}\times C^{\mathcal{A}}\to C$ i.e. the exponential transpose of the identity on $C^{\mathcal{A}}$.

<u>Proof</u>: Given $U: \mathcal{D} \to \mathcal{A}$ operational with presentation (θ, H), construct the pushout

$$\begin{array}{ccc} T & \longrightarrow & T' \\ {\scriptstyle\theta}\uparrow & & \uparrow{\scriptstyle\theta'} \\ B & \xrightarrow[H^{**}]{} & C^A \end{array}$$

where H** is the other transpose of H. Since the object functor, $|\ |: \mathcal{Cat} \to \mathcal{Set}$ preserves pushouts (it has a right adjoint), θ bijective on objects implies θ' bijective on objects. Applying $\mathcal{C}^{(-)}$ to the pushout yields the pullback

$$\begin{array}{ccccc} \mathcal{D} & \longrightarrow & C^{T'} & \longrightarrow & C^T \\ \downarrow & & \downarrow{\scriptstyle C^{\theta'}} & & \downarrow{\scriptstyle C^{\theta}} \\ \mathcal{A} & \xrightarrow[\text{sub}]{} & C^{C^A} & \xrightarrow[C^{H^{**}}]{} & C^B \end{array}$$

where sub (substitution) is the transpose of evaluation. For $\alpha: P \to Q$ in C^A we have

$$(\text{sub}X)(P) = PX$$

$$(\text{sub}X)(\alpha) = \alpha_X$$

Now

$$\begin{aligned} (C^{H^{**}}\text{sub})(X)B &= \text{sub}X(H^{**}B) \\ &= (H^{**}B)X \\ &= (H^{*}X)B \end{aligned}$$

Morphisms of $\mathcal{B}$ are dealt with similarly. So

$C^{H^{**}}\mathrm{sub} = H^*$ and the category constructed with respect to the evaluation is $\mathcal{D}$.

With respect to an evaluation, an algebra is a pair (X,Φ) where

$$\Phi\theta = \mathrm{sub}X$$

That is, given $\alpha: P \to Q$ in $C^{\mathcal{A}}$, we have

$$\begin{array}{ccc} \Phi\theta P & & PX \\ {\scriptstyle \Phi\theta\alpha}\downarrow & = & \downarrow{\scriptstyle \alpha_X} \\ \Phi\theta P & & QX \end{array}$$

Let $(f,t): (X,\Phi) \to (X',\Phi')$ be a homomorphism. Then for each $\omega \in \mathcal{T}(\theta P, \theta Q)$

$$\begin{array}{ccc} \Phi\theta P & \xrightarrow{t_{\theta P}} & \Phi'\theta P \\ {\scriptstyle \Phi\omega}\downarrow & & \downarrow{\scriptstyle \Phi'\omega} \\ \Phi\theta Q & \xrightarrow[t_{\theta Q}]{} & \Phi'\theta Q \end{array}$$

commutes by the naturality of t. As before, $\Phi\theta P = PX$ etc. but also

$$t_{\theta P} = (\mathrm{sub}f)(P) = Pf \qquad 1.1$$

$$\begin{array}{ccc} PX & \xrightarrow{Pf} & PX' \\ {\scriptstyle \Phi\omega}\downarrow & & \downarrow{\scriptstyle \Phi'\omega} \\ QX & \xrightarrow[Qf]{} & QX' \end{array} \qquad 1.2$$

Conversely, let (X,Φ) and (X',Φ') be two algebras and assume that $f:X\to X'$ in A satisfies (1.2) for each ω in T. Then there is a natural transformation $t:\Phi\to\Phi'$ defined by (1.1). This defines all components of t since θ is bijective on objects. Obviously, $(f,t):(X,\Phi)\to(X',\Phi')$ is a a homomorphism. Thus homorphisms correspond to morphisms of A satisfying (1.2). Consequently, we have

<u>Lemma 1.4</u>:

$U:\mathcal{D}\to A$ operational implies U faithful.

Often, a morphism, f, of $\mathcal{D}$ and its underlying morphism, Uf, in A will be given the same name.

Recall Beck's Tripleability Theorem[8]. Given $U:\mathcal{D}\to A$, a U-split coequalizer is a diagram in A of the form

$$UC \underset{Ug}{\overset{Uf}{\rightrightarrows}} UD \underset{x}{\overset{y}{\rightleftarrows}} X \qquad t: UD \to UC$$

such that

$$y.Uf = y.Ug$$
$$yx = id \qquad 1.3$$
$$Uf.t = id$$
$$Ug.t = xy$$

These equations force y to be the coequalizer of Uf and Ug in $\mathcal{A}$. Moreover, since equations are preserved by any functor, this is an absolute coequalizer (see Paré [9]). U creates coequalizers of U-split coequalizers if, whenever such a system as (1.3) occurs, then y = Uy' where y' is a coequalizer of f and g. The theorem may then be stated as follows

Theorem 1.5 (Beck): Let $U: \mathcal{D} \to \mathcal{A}$ have a left adjoint. Then U is tripleable iff the following conditions hold:

i) U reflects isomorphisms
ii) U creates coequalizers of U-split coequalizers.

Here i) and ii) will be called Beck's Tripleability Conditions (B.T.C.). Trivially, this theorem and the conditions can be dualized to yield a theorem about cotriples.

Proposition 1.6: $U: \mathcal{D} \to \mathcal{A}$ operational implies U satisfies

B.T.C. and their duals.

Corollary 1.7: $U:\mathcal{D}\to\mathcal{A}$ operational with a left (respectively right) adjoint implies U is tripleable (respectively cotripleable).

Proof of 1.7: For triples just combine Beck's Tripleability Theorem with Proposition 1.6. Then dualize to obtain the result for cotriples.

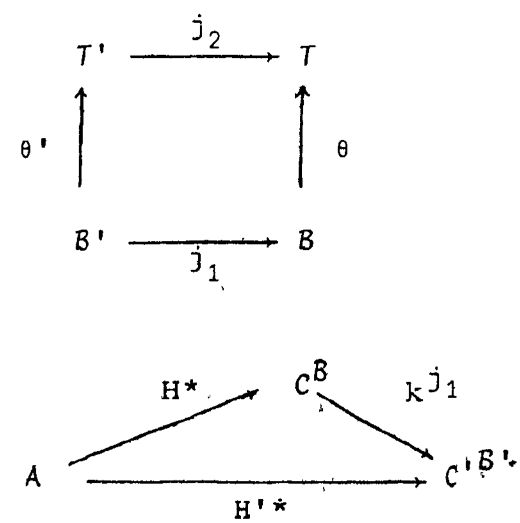
Proof 1.6: Let $U:\mathcal{D}\to\mathcal{A}$ be operational with a presentation (θ,ev). Inspection shows that U reflects isomorphisms. For the second condition of B.T.C., consider a U-split coequalizer as in (1.3) with $C = (UC,\Phi_C)$ and $D = (UD,\Phi_D)$. Then, given any $\omega \in \mathcal{T}(\theta P,\theta Q)$ we have

$$
\begin{array}{ccccc}
PUC & \rightrightarrows & PUD & \xrightarrow{Py} & PX \\
{\scriptstyle \Phi_C\omega}\downarrow & & \downarrow{\scriptstyle \Phi_D\omega} & & \downarrow{\scriptstyle \Phi\omega} \\
QUC & \rightrightarrows & QUD & \xrightarrow[Qy]{} & QX
\end{array}
$$

with Py and Qy coequalizers (since y is an absolute coequalizer and the left-hand squares commute). Hence there is a unique map $\Phi\omega:PX\to QX$ making the diagram commute. By the uniqueness condition for coequalizers, Φ is a functor $\mathcal{T}\to\mathcal{C}$ and for $\alpha:P\to Q$ in C^A, $\Phi\theta\alpha = \alpha_X$. Also by uniqueness, y yields a homomorphism. The dual result holds by (1.2).

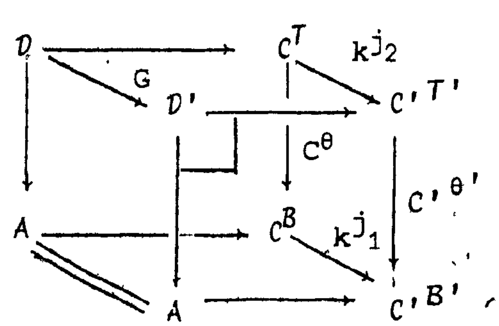
§2: Operational Functors

Definitions 2.1: Let $U:\mathcal{D}\to\mathcal{A}$ and $U':\mathcal{D}'\to\mathcal{A}$ be two operational categories with presentations (θ,H) and (θ',H') respectively. A <u>morphism of presentations</u>, $(\theta,H) \to (\theta',H')$, is a pair of functors $(j_1,j_2) = j$ and a functor $k:\mathcal{C}\to\mathcal{C}'$ such that the following diagrams commute



2.1

Let $G:\mathcal{D} \to \mathcal{D}'$ be a functor over A. G is an <u>operational functor with presentation (j,k)</u> if $(j,k): (\theta,H)\to(\theta',H')$ is a morphism of presentations such that G is the induced functor into the pullback in



2.2

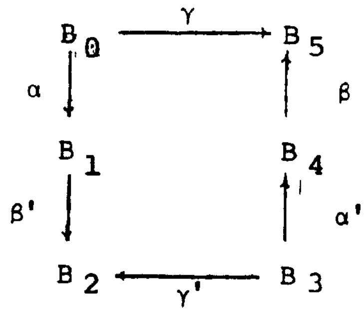
Hence there is a category, <u>Op(A)</u> with objects $(\mathcal{D},\theta,H)$ where (θ,H) is a presentation of $\mathcal{D}$ and morphisms the triplets $(G,j,k):(\mathcal{D},\theta,H) \to (\mathcal{D}',\theta',H')$ where G is the operational functor induced by $(j,k):(\theta,H) \to (\theta',H')$. Composition and identities are given by those of $\mathcal{C}at/A$. Hence there is a forgetful functor $R:\mathrm{Op}(A) \to \mathcal{C}at/A$,

sending $(\mathcal{D}, \theta, H)$ to $\mathcal{D}$ and (G,j,k) to G.

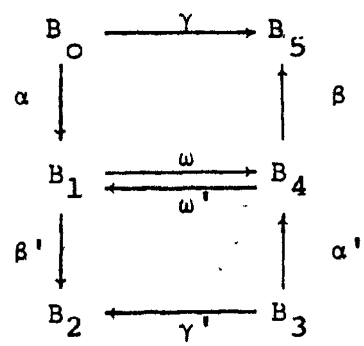
Note that since operational categories and functors can have more than one presentation, the composite of two operational functors in which the presentations are incompatible may not be operational (c.f. Example III.3.12). Hence they do not form a category over $\mathcal{C}at/A$.

Now a functor $L: \mathcal{C}at/A \longrightarrow \mathrm{Op}(A)$ will be built. The construction of $L\mathcal{D}$ employs a standard theory, θ_o independent of $\mathcal{D}$.

Definition 2.2: The standard theory $\theta_o:\mathcal{B}_o \to \mathcal{T}_o$ is given by a subcategory inclusion where $\mathcal{B}_o$ is generated by the graph



and $\bar{T}_o$ is generated by the graph



subject to the equations:

$$\begin{aligned} \beta\omega\alpha &= \gamma \\ \beta'\omega'\alpha' &= \gamma' \\ \omega'\omega &= id \\ \omega\omega' &= id \end{aligned} \qquad 2.3$$

Note that both B_o and T_o are finite categories.

When considering functors on T_o it will be the choice of ω (and ω') which will determine the algebra. The other peripheral morphisms are there to prevent unwanted choices of ω, in some sense to guard the ω's.*

The base functor $H\mathcal{D}: A\times B_0 \to C(\mathcal{D})$ for the construction of $L\mathcal{D}$ is contained in the following definition of C.

<u>Definition 2.3</u>: C: $\mathcal{Cat}/A \to \mathcal{Cat}$ is defined as follows:

$C\mathcal{D}$ is given by the pushout in $\mathcal{Cat}$

$$\begin{array}{ccc} \mathcal{D}\times B_0 & \xrightarrow{\mathcal{D}\times\theta_0} & \mathcal{D}\times T_0 \\ {\scriptstyle U\times B_0}\big\downarrow & & \big\downarrow{\scriptstyle P_{\mathcal{D}}} \\ A\times B_0 & \xrightarrow[H\mathcal{D}]{} & C\mathcal{D} \end{array} \qquad 2.4$$

For $G:\mathcal{D}\to\mathcal{D}'$ over A, $CG: C\mathcal{D}\to C\mathcal{D}'$ is given by the universal functor from the pushout

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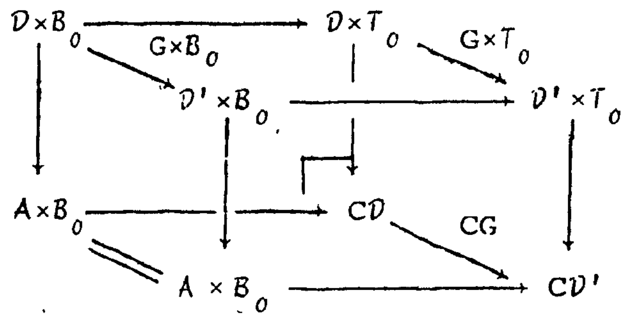
* A naïve approach to the problem of choosing a standard theory would be to consider the theory

$$B \xrightarrow[\theta]{} T \circlearrowleft \omega \qquad \omega^2 = \omega$$

and to set up a base functor so that for each object of $\mathcal{D}$ there is a morphism, ω_D, with $\omega_D{}^2 = \omega_D$. The problem is that for each object of A, its identity is also idempotent, so that there would be an extra copy of A in our resulting operational category. Also $\Phi\omega = \omega_D{}^n$ would generate unwanted algebras. By the choice of θ_0, these nuisances are excluded.

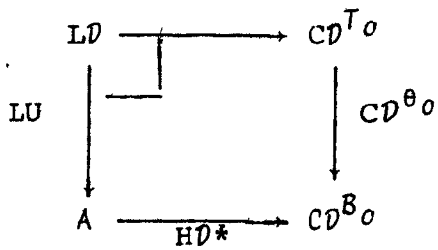


2.5

The uniqueness property of pushouts guarantees the functoriality of C.

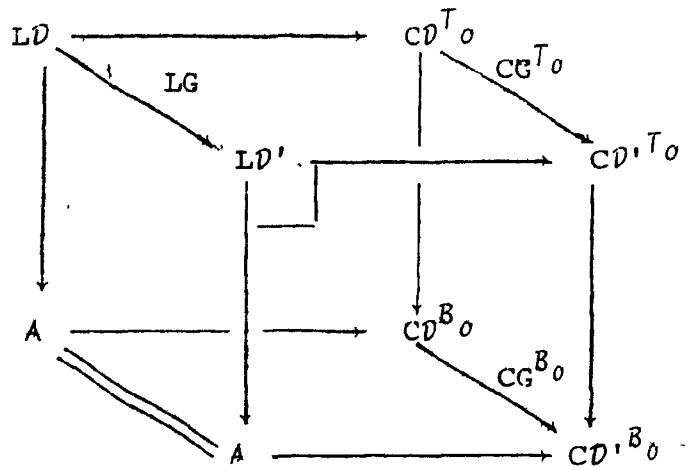
Definition 2.4: Let $U:\mathcal{D} \to A$ and $U':\mathcal{D}' \to A$ be in $\mathcal{C}at/A$ and let $G:\mathcal{D} \to \mathcal{D}'$ be a functor over A.

Then $LU:L\mathcal{D} \to A$ is the operational category given by the pullback



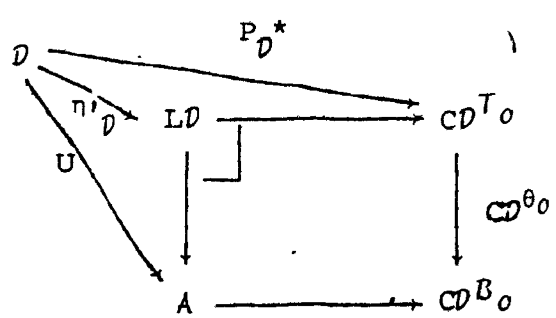
2.6

$LG:L\mathcal{D} \to L\mathcal{D}'$ over A is the operational functor induced by CG



2.7

(The base of the cube commutes by (2.5). By the universal property of pullbacks, L is a functor. There is no conflict between the two uses of L applied to U, $LU:L\mathcal{D}\to A$ and $LU:L\mathcal{D}\to LA$, since $LA \cong A$. Since (2.4) commutes, there is a $\eta_{\mathcal{D}}':\mathcal{D}\to RL\mathcal{D}$ over A



which yields a natural transformation $\eta':1\to RL:Cat/A\to Cat/A$.

<u>Lemma 2.5</u>: $C\mathcal{D}$ is generated by the following graphs and

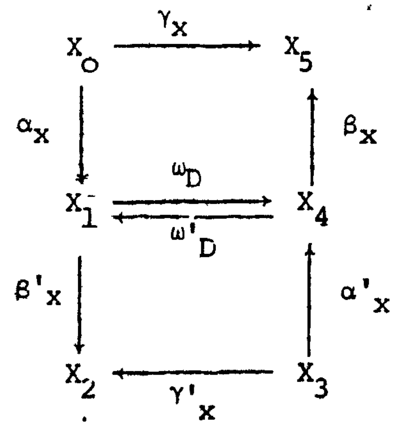
equations (for a discussion of this type of construction see Barr and Wells [1])

i) the coproduct of six copies of A; $A_0+\ldots+A_5$ (regarded as a graph with equations.

ii) for each $X\in|A|$, $\alpha_X, \beta_X, \gamma_X, \alpha'_X, \beta'_X, \gamma'_X$ (subscripts normally omitted).

iii) for each $D\in|\mathcal{D}|$, ω_D, ω'_D,

with domains and codomains given by



with the ω_D's occuring only if $X=UD$. These morphisms are subject to the equations

iv) for each $f\in A(X,X')$:

$$f_1\alpha_X = \alpha_{X'}f_o$$
$$f_5\beta_X = \beta_{X'}f_4$$
$$f_5\gamma_X = \gamma_{X'}f_o$$
$$f_4\alpha'_X = \alpha'_{X'}f_3$$
$$f_2\beta'_X = \beta'_{X'}f_1$$
$$f_2\gamma'_X = \gamma'_{X'}f_3$$

(subscripts on f's often omitted).

v) For $X = UD$

$$\beta_X\omega_D\alpha_X = \gamma_X$$
$$\beta'_X\omega'_D\alpha'_X = \gamma'_X$$

vi) for each $f \varepsilon \mathcal{D}(D,C)$

$$f_4\omega_D = \omega_C f_1$$
$$f_1\omega'_D = \omega'_C f_4$$

vii) for each $D \varepsilon |\mathcal{D}|$.

$$\omega'_D\omega_D = id$$
$$\omega_D\omega'_D = id$$

The inclusions $H\mathcal{D}: A \times B_o \to C\mathcal{D}$ and $P_{\mathcal{D}}: \mathcal{D} \times T_o \to C\mathcal{D}$ are given by:

$$H\mathcal{D}(X,B_k) = X_k \qquad P_{\mathcal{D}}(D,B_k) = (UD)_k$$

$$H\mathcal{D}(X,\alpha) = \alpha_X \qquad P_{\mathcal{D}}(D,\alpha) = \alpha_{UD}$$

$$\vdots$$

$$H\mathcal{D}(X,\gamma') = \gamma'_X \qquad P_{\mathcal{D}}(D,\gamma') = \gamma'_{UD}$$

$$H\mathcal{D}(f,B_k) = f_k \qquad P_{\mathcal{D}}(D,\omega) = \omega'_D$$

$$P_{\mathcal{D}}(D,\omega') = \omega'_D$$

$$k = 0,\ldots,5 \qquad P_{\mathcal{D}}(f,B_k) = f_k$$

Proof: Consider the category C and the functors $H\mathcal{D}$ and $p_{\mathcal{D}}$ constructed in the lemma. A quick inspection shows that $H(U\times B_0)=p(\mathcal{D}\times\theta_0)$. Now consider a pair of functors $F:A\times B_0\to X$ and $F':\mathcal{D}\times T_0\to X$ such that $F(U\times B_0)=F'(\mathcal{D}\times\theta_0)$. Define $F'':C\to X$ by its action of the generators of C

$$F''X_k = F(X,B_k)$$

$$F''f_k = F(f,B_k)$$

$$F''\alpha_X = F(X,\alpha)$$

$$\vdots$$

$$F''\gamma'_X = F(X,\gamma)$$

$$F''\omega_D = F'(D,\omega)$$

$$F''\omega'_D = F'(D,\omega')$$

It is trivial to check that F'' preserves the equations in the definition of C and hence is a functor. Also $F''H\mathcal{D} = F$ and $F''p_{\mathcal{D}} = F'$. Thus C is a pushout and may be identified with $C\mathcal{D}$.

§3: Standard Presentations

The idea of operational retracts (defined below) leads to a standard presentation of operational categories and functors similar to the presentation of L.

Definition 3.1: Let $U: \mathcal{D} \to A$ be operational with presentation (θ, H). Let $D = (UD, \Phi_D)$ be an algebra (Φ_D will always be the functor part of D with respect to the given presentation) and X a retract of UD in A

$$X \underset{y}{\overset{x}{\rightleftarrows}} UD \qquad\qquad yx = id$$

satisfying the following condition: for any composable pair of morphisms ω_1, ω_2 in T,

$$\theta B_1 \xrightarrow{\omega_1} \theta B_2 \xrightarrow{\omega_2} \theta B_3 \qquad 3.1$$

the following equation holds

$$(H^*y)_{\theta B_3} \Phi_D \omega_2 (H^*xy)_{\theta B_2} \Phi_D \omega_1 (H^*x)_{\theta B_1} = (H^*y)_{\theta B_3} \Phi_D \omega_2 \omega_1 (H^*x)_{\theta B_1} \qquad 3.2$$

Then {y,D,x} is called an operational retract.

Given the presentation by an evaluation, we have

$$\theta P \xrightarrow{\omega_1} \theta Q \xrightarrow{\omega_2} \theta R \qquad 3.1a$$

$$(Ry)\Phi_D \omega_2 (Qxy) \Phi_D \omega_1 (Px) = (Ry) \Phi_D \omega_2 \omega_1 (Px) \qquad 3.2a$$

The value of operational retracts is that if $\{y,D,x\}$ is one, then X underlies an algebra (X,Φ) with Φ given by $\Phi\theta P = PX$ and for $\alpha:\theta P\to\theta Q$ in T

$$\Phi\omega = Qy\Phi_D\omega Px$$

$$\begin{array}{ccc} PX & \xrightarrow{Px} & PUD \\ \Phi\omega \downarrow & & \downarrow \Phi_D\omega \\ QX & \xleftarrow[Qy]{} & QUD \end{array} \qquad 3.3$$

(3.2) guarantees that Φ is a functor i.e. for ω_1 and ω_2 as in (3.1a) we have:

$$\begin{aligned} \Phi\omega_2\Phi\omega_1 &= [(Ry)\Phi_D\omega_2(Qx)][(Qy\Phi_D\omega_1(Px)] \\ &= (Ry)\Phi_D\omega_2\omega_1(Px) \\ &= \Phi\omega_2\omega_1 \end{aligned}$$

While if $\omega_1 = \theta\alpha$ then:

$$\begin{aligned} \Phi\theta\alpha &= (Qy)\Phi_D\theta\alpha(Px) \\ &= Qy\,\alpha_{UD}\,Px \qquad\qquad \Phi_D\theta = subUD \\ &= \alpha_X PyPx \\ &= (subX)\alpha \end{aligned}$$

For Φ defined by (3.3) we write:

$$\begin{aligned} \Phi &= H^*y\Phi_D H^*x \\ &= y\Phi_D x \end{aligned}$$

There are many situations under which (3.2) may hold e.g. if xy is a homomorphism or, y and x derive from a U-split coequalizer (c.f. Lemma III.3.1) In more complicated situations, the commutativity of homomorphisms with terms such as $\Phi\omega$ may be invoked many times, in a back-and-forth process, to establish (3.2). Also, the equations $\Phi = y\Phi_D x$, for $\{y,D,x\}$ an operational retract, may be used (c.f. Example III.3.13).

Lemma 3.2 Let $G:\mathcal{D}\to\mathcal{D}'$ be an operational functor as in (2.2). Then if $\{y,D,x\}$ is an operational retract for $\mathcal{D}$ with $y\Phi_D x = \Phi_{D'}$, then $\{y,GD,x\}$ is an operational retract and $y\Phi_{GD}x = \Phi_{GD'}$.

Proof:

$$
\begin{aligned}
\Phi_{GD'} &= \P' GD' \\
&= k^{j}2\P' D' \\
&= k^{j}2(H^*y\Phi_D H^*x) \\
&= (H'^*y)(k^{j}2\P' D)(H'^*x) \\
&= (H'^*y)(\P' GD)(H'^*x) \\
&= (H'^*y)\Phi_{GD}(H'^*x) \\
&= y\Phi_{GD}x
\end{aligned}
$$

Definition 3.3: Define $C':Op(A)\to Cat$ as follows

$C'(\mathcal{D}, \theta, H)$ is generated by:

ia) the underlying graph and equations of $C(\mathcal{D})$ and

viii) if $\{y,D,x\}$ is an operational retract for $(\mathcal{D},\theta,H)$ with $y\Phi_D x = \Phi_{D'}$ then $y\omega_D x = \omega_{D'}$ and $y\omega'_D x = \omega'_{D'}$.

The quotient functor $C(\mathcal{D}) \to C'(\mathcal{D},\theta,H)$ is called $q_{\mathcal{D}}$. Now let $(G,j,k): (\mathcal{D},\theta,H) \to (\mathcal{D}',\theta',H')$ be in $Op(A)$. Since, by Lemma 3.2, CG preserves the equations viii) it induces $C'(G,j,k): C'(\mathcal{D},\theta,H) \to C'(\mathcal{D}',\theta',H')$. The functoriality of C guarantees that of C', and the $q_{\mathcal{D}}$'s form the components of a natural transformation $q: C \to C': Op(A) \to \mathcal{C}at/A$. There is also the base functor $H'(\mathcal{D},\theta,H) = q_{\mathcal{D}}H\mathcal{D}: A\times B \to C\mathcal{D} \to C'(\mathcal{D},\theta,H)$.

Lemma 3.4:

i) If

$$y\omega_D x = y_n\omega_{D_n}y_{n-1}\omega_{D_{n-1}}y_{n-2}\cdots\cdots y_1\omega_{D_1}y_o \qquad 3.4$$

in $C'(\mathcal{D},\theta,H)$, then there is an i $(1\leq i\leq n)$ such that

$$y\Phi_D x = y_n y_{n-1}\cdots\cdots y_i\Phi_{D_i}y_{i-1}\cdots\cdots y_1 y_o \qquad 3.5$$

ii) If

$$y\omega'_C z\omega_D x = y_n\omega'_{D_n}y_{n-1}\omega_{D_{n-1}}y_{n-2}\cdots\cdots y_1\omega_{D_1}y_o$$

in $C'(\mathcal{D},\theta,H)$, then there are $i<j$ such that

$$y\Phi_C z\Phi_D x \quad = \quad y_n y_{n-1}\cdots\cdots y_j \Phi_{D_j} y_{j-1}\cdot\cdots\cdots y_2 \Phi_{D_2} y_{2-1}\cdot\cdots\cdots y_o$$

and

$$yz\Phi_D x \quad = \quad y_n y_{n-1}\cdot\cdots\cdots y_i \Phi_{D_i} y_{i-1}\cdot\cdots\cdots y_o$$

$$y\Phi_C zx \quad = \quad y_n y_{n-1}\cdot\cdots\cdots y_j \Phi_{D_j} y_{j-1}\cdot\cdots\cdots y_o$$

iii) Hence, if $y'x' = id$, $yx = id$ and

$$y'\omega_{D'}' , x'y\omega_D \quad = \quad id \qquad 3.6$$

then $y'\Phi_{D'}x' = y\Phi_D x$ and $\{y,D,X\}$ is an operational retract.

<u>Proof</u>:

i) The proof is by induction on the length of the proof of (3.4) where the length of the proof is measured by the number of applications of the generating equations of $C'(\mathcal{D}, \theta, H)$. The case for $n = 0$ is trivial. Assume that (3.4) is proved in n steps and (3.5) holds for some i. Now consider all proofs of length n+1 which can be obtained from (3.4). Trivially, applications of i) cause no problems while iv) and v) are inapplicable. The only type of application of vi) which is of interest is one of the form $f\omega_{D_i} = \omega_{D_i'} f$ (for $f: D \to D'$) or $\omega_{D_i} f = f\omega_{D_i'}$. Without loss of generality, consider only the first case. Then

$f\Phi_{D_i} = \Phi_{D_i'} f$ and so:

$$y\omega_D x = y_n \omega_{D_n} y_{n-1} \cdot\cdots\cdots y_1 \omega_{D_1} y_o$$

$$= y_n \omega_{D_n} y_{n-1} \cdot\cdots\cdots y_i' f \omega_{D_i} y_{i-1} \cdot\cdots\cdots y_o$$

$$= y_n \omega_{D_n} y_{n-1} \cdot\cdots\cdots y_i' \omega_{D'_i} f y_{i-1} \cdot\cdots\cdots y_o$$

Also

$$y\Phi_D x = y_n y_{n-1} \cdot\cdots\cdots y_i \Phi_{D_i} y_{i-1} \cdot\cdots\cdots y_o$$

$$= y_n y_{n-1} \cdot\cdots\cdots y_i' f \Phi_{D_i} y_{i-1} \cdot\cdots\cdots y_o$$

$$= y_n y_{n-1} \cdot\cdots\cdots y_i' \Phi_{D'_i} f y_{i-1} \cdot\cdots\cdots y_o$$

Trivially, after any applications of vii) the result still holds and, a priori, applications of viii) preserve (3.5).

ii) This proof follows exactly the same lines as i).

iii) For (3.6) to hold in $C'(\mathcal{V}, \theta, H)$ the ω 's must be eliminated, which in this case can only be done by an application of vii). Hence,

$$y'\omega'_{D'}x'y\omega_D x = z'\omega'_C id\omega_C z$$

for some z, z' and C and so by ii) of the lemma:

$$y'\Phi_{D'}x'y\Phi_D x = z'\Phi_C\Phi_C z$$

and

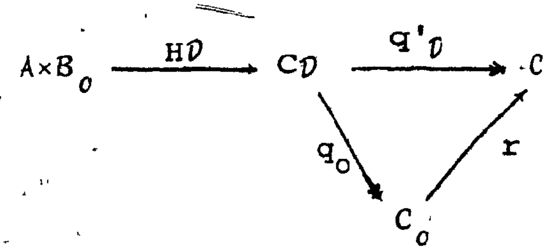
$$\begin{aligned} y'\Phi_{D'}x' &= y'\Phi_{D'}x'yx \\ &= z'\Phi_C z \\ &= y'x'y\Phi_D x \\ &= y\Phi_D x \end{aligned}$$

Hence, in the notation of (I.3.2)

$$\begin{aligned} (H^*y)_{\theta B_3}\Phi_D\omega_2(H^*xy)_{\theta B_2}\Phi_D\omega_1(H^*x)_{\theta B_1} &= (H^*y')_{\theta B_2}\Phi_{D'}\omega_2(H^*xy)_{\theta B_2}\Phi_D\omega_1(H^*x)_{\theta B_1} \\ &= (H^*z')_{\theta B_3}\Phi_C\omega_2\Phi_C\omega_1(H^*z)_{\theta B_1} \\ &= (H^*y)_{\theta B_3}\Phi_D\omega_2\omega_1(H^*x)_{\theta B_1} \end{aligned}$$

Hence $\{y,D,x\}$ is an operational retract.

<u>Definition 3.5</u>: $Op_0(A)$ is the subcategory of $Op(A)$ (with inclusion i) with objects those $(\mathcal{D}, \theta, H)$ such that $\theta = \theta_0$ and $H = q'_{\mathcal{D}}H\mathcal{D}: A\times B_0 \to C\mathcal{D} \to C$ where $q'_{\mathcal{D}}$ is a 'minimal epimorphism' in the following sense: if $q'_{\mathcal{D}} = rq_0$ for some epimorphism $q_0: C\mathcal{D} \to C_0$ and $\mathcal{D}$ is presented by $(\theta_0, q_0H\mathcal{D})$, then r is an isomorphism.



The morphisms of $Op_0(A)$ are those $(G,j,k):(\mathcal{D},\theta_0,H) \to (\mathcal{D}_1,\theta_0,H_1)$ such that $j = (id,id)$ and k is induced by CG:

$$\begin{array}{ccc} C\mathcal{D} & \xrightarrow{q'_{\mathcal{D}}} & C \\ {\scriptstyle CG}\downarrow & & \downarrow{\scriptstyle k} \\ C\mathcal{D}' & \xrightarrow[q'_{\mathcal{D}'}]{} & C' \end{array}$$

Hence the $q'_{\mathcal{D}}$'s are the components of a natural transformation $C \to \mathcal{C}_0 : Op_0(A) \to \mathcal{C}at$ where $C_0(\mathcal{D},\theta_0,H) = C$ and $C_0(G,\theta_0,k) = k$. The objects and morphisms of $Op_0(A)$ are said to have <u>standard presentations</u>. The forgetful functor, $Op_0(A) \to \mathcal{C}at/A$ is called R_0.

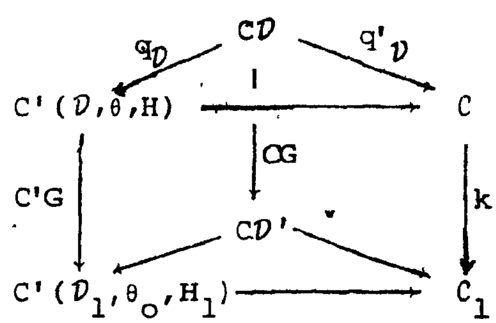
<u>Lemma 3.6:</u>

Let $(G,\theta_0,k):(\mathcal{D},\theta_0,H) \to (\mathcal{D}_1,\theta_0,H_1)$ be in $Op_0(A)$. Then $q'_{\mathcal{D}}:C\mathcal{D} \to \mathcal{C}$ factorises through $q_{\mathcal{D}}:C\mathcal{D} \to C'(\mathcal{D},\theta_0,H)$ i.e. $q'_{\mathcal{D}} = rq_{\mathcal{D}}$ for some r. Further, we have the commuting diagram

$$\begin{array}{ccc} C'(\mathcal{D},\theta_0,H) & \xrightarrow{r} & C \\ {\scriptstyle C'(G,\theta_0,k)}\downarrow & & \downarrow{\scriptstyle k} \\ C'(\mathcal{D}_1,\theta_0,H_1) & \xrightarrow[r_1]{} & C_1 \end{array}$$

Hence any presentation of the form $(\theta_{\mathcal{D}}, H'(\mathcal{D}, \theta_0, H))$ is standard.

Proof: Let $y\omega_D x = \omega_{D'}$ be a generating equation of $C'(\mathcal{D}, \theta_o, H)$. Then $y\phi_D x = \phi_{D'}$ for $(\mathcal{D}, \theta_o, H)$. Hence $y\omega_D x = \omega_{D'}$ in $\mathcal{C}$. Hence, there is a functor $r\colon C'(\mathcal{D}, \theta_o, H) \to \mathcal{C}$ induced by the identity on $C\mathcal{D}$. Now since both k and C'G are induced by CG and $q_{\mathcal{D}}$ is an epimorphism, we have the desired commuting square.



The rest follows.

Theorem 3.7: There is a functor $s\colon Op(A) \to Op_0(A)$ over Cat/A $(R_o s = R)$ such that $si \simeq id$. In other words, every operational category (resp. functor) has a standard presentation.

Proof: Let $(\mathcal{D}, \theta, H)$ be in $Op(A)$. Let $\mathcal{P}$ be the operational category given by the standard presentation $(\theta_0, H(\mathcal{D}, \theta, H))$.

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & C'(\mathcal{D},\theta,H)^{T_0} \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & C'(\mathcal{D},\theta,H)^{B_0} \end{array}$$

There is an $F: \mathcal{D} \to \mathcal{P}$ given by the definition of C'. F is faithful since $U: \mathcal{D} \to \mathcal{A}$ is. To see that F is a monomorphism, assume FD = FD'. Then $\omega_D = \omega_{D'}$ in $C'(\mathcal{D}, \theta, H)$. Now apply Lemma 3.4.

Conversely, let (X,Φ) be an algebra in $\mathcal{P}$. Then, from the equations (2.3) for T_0

$$\beta_X \Phi \omega \alpha_X = \gamma_X \qquad 3.7$$

For source-target reasons, $\Phi\omega$ may not include α 's, β 's and γ 's (or their primed versions). Hence it is a composite of morphisms in $\mathcal{A}$, ω 's and ω' 's. In fact, $\Phi\omega$ can always be written without ω' 's. To see this, assume $\Phi\omega$ cannot be written without using an ω'. Then, since no equation in Lemma 2.5 allows the interaction of ω' 's and α 's or β 's (e.g. no commutivity), equations of type v) cannot be invoked. Thus, α 's and β 's cannot be eliminated and (3.7) cannot be established. Contradiction. Hence, $\Phi\omega$ can be written using only morphisms in A and ω 's. Also, for source-target reasons, only one ω occurs. That is

$$\begin{aligned}\Phi\omega &= y_4 \omega_D x_1 \\ &= y \omega_D x\end{aligned}$$

with x,y morphisms in A. Then by the construction of $C'(\mathcal{D}, \theta, H)$,

$$\begin{aligned}\gamma &= \beta(y\omega_D x)\alpha \\ &= y(\beta\omega_D\alpha)x \qquad 3.8 \\ &= y\gamma x\end{aligned}$$

Now recall diagram (2.4). Then

$$\begin{aligned}\P(U\times\mathcal{B}_0) &= U \\ U\,\P(\mathcal{D}\times\theta_0) &= U\end{aligned}$$

where all $\P$'s are projections to the first factor. Since C is the pushout, we can define $F = \langle U, U\P \rangle : C'(\mathcal{D}, \theta, H) \to A$. In the notation of Lemma 2.5

$$
\begin{aligned}
FX_k &= X \\
Ff_k &= f \\
F\alpha_x &= F\beta_x \\
&= F\gamma_x \\
&\vdots \\
&= F\omega_{D'} \\
&= id_x \qquad UD = X
\end{aligned}
$$

Hence F respects equations viii) from the definition of $C'(\mathcal{D}, \theta, H)$ and so $F = F_1 q'_{\mathcal{D}}$ for some F_1. Thus, applying F_1 to (3.8) yields

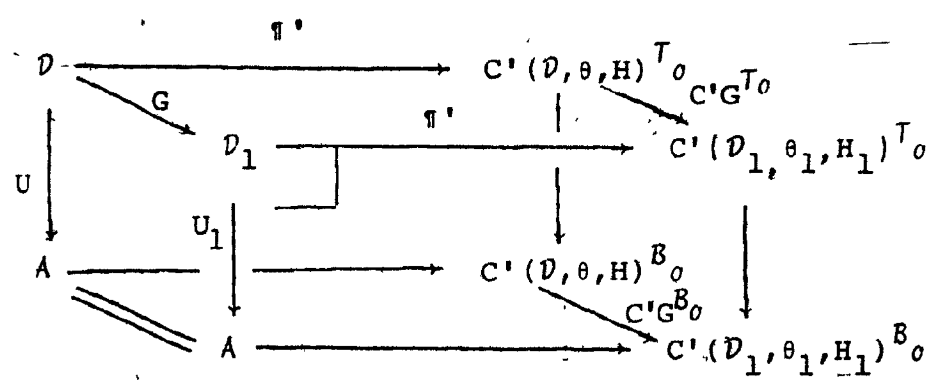
$$id = yx$$

Applying the same arguments to another equation in the definition of T_0 (c.f. (2.3)), $\Phi\omega' = y'\omega'_{D'}x'$ with y', x' in A and $y'x' = id$. Now since $\omega'\omega = id$,

$$
\begin{aligned}
id &= \Phi\omega'\Phi\omega \\
&= y'\omega_{D'}x'y\omega_D x
\end{aligned}
$$

Hence, by Lemma 3.4, $\Phi\omega' = y\omega'_D x$ with $\{y, D, x\}$ an operational retract. Hence, $y\Phi_D x = \Phi_{D'}$ for some D' and so $\Phi\omega = y\omega_D x = \omega_{D'}$ i.e. $(X, \Phi) = FD'$. Now if $f: X \to X'$ is a homorphism $FD \to FD'$ then $f\omega_D = \omega_{D'}f$. Hence by Lemma 3.4, $f\Phi_D = \Phi_{D'}f$ and so f is a homomorphism of $\mathcal{D}$. Thus F is also surjective and so is an isomorphism. Since P was defined up to isomorphism we identify $\mathcal{P}$ with $\mathcal{D}$.

Now let (G,j,k): $(\mathcal{D}, \theta, H) \to (\mathcal{D}_1, \theta_1, H_1)$ be an operational functor. Then consider the operational functor $(G_0, \theta_0, C'G)$: $(\mathcal{D}, \theta_0, H(\mathcal{D}, \theta, H)) \to (\mathcal{D}_1, \theta_0, H(\mathcal{D}_1, \theta_1, H_1))$



By definition, $U_1G = U$. But also

$$\begin{aligned}
(\P' GD) &= \Phi_{GD}\omega \\
&= \omega_{GD} \\
&= C'G\omega_D \\
&= (C'G^{T_0}\Phi_D)\omega \\
&= (C'G^{T_0}\P' D)\omega \\
&= (\P' G_0 D)\omega
\end{aligned}$$

So $\P' GD = \P' G_0 D$ for each D in $\mathcal{D}$. Now, by the faithfulness of U_1, $Gf = f = G_0 f$ for each morphism f of $\mathcal{D}$. Thus $G_0 = G$ i.e. G has a standard presentation.

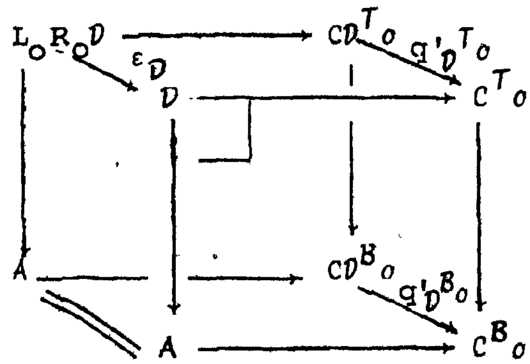
Define $s(\mathcal{D}, \theta, H) = (\mathcal{D}, \theta_0, H(\mathcal{D}, \theta, H))$ and $s(G,j,k) = (G, \theta_0, C'G)$. s is a functor and $R_0 s = R$.

Now for $(G,\theta_0,k): (\mathcal{D}\ \theta_0, H)^{\circ} \to (\mathcal{D}_1, \theta_0, H_1)$ standard as in Definition 3.5, $(\theta_0, H(\mathcal{D}, \theta_0, H))$ is a presentation of $\mathcal{D}$ and by Lemma 3.6 $q'_{\mathcal{D}}: C\mathcal{D} \to C$ factors through $C'(\mathcal{D}, \theta_0, H)$. So, by the 'minimality' of q' (see Definition 3.5) $C'(\mathcal{D}, \theta_0, H) = C$. Similarly for $C'(\mathcal{D}_1, \theta_1, H)$. Thus, since both k and C'G are induced by CG, $C'G \approx k$. Hence $si \simeq id$.

From its construction, it is clear that $L: \mathcal{C}at/A \to Op(A)$ factors through i i.e. $L = iL_0$, for $L_0: \mathcal{C}at/A \to Op_0(A)$. Because of the problem of non-matching presentations, L is not an adjoint for R (c.f. Proposition III.3.14). However, we have

<u>Theorem 3.8</u>: L_0 is left adjoint to R_0.

<u>Proof</u>: The unit η, is given by the η' found in Definition 2.4 (since $R_0 L_0 \approx RL$). The counit ε, is given by the operational functors presented by the quotient functors $q'_{\mathcal{D}}: C\mathcal{D} \to C$.



The naturality of $q'_{\mathcal{D}}$ guarantees that the counit morphisms define a natural transformation $L_0R_0 \to 1$. The identities for the adjunction are easily checked.

CHAPTER II

SHUFFLE RETRACTS

§ 1: Shuffle retracts

In order to characterise the operational categories and functors, the idea of being closed under operational retracts must be translated into a property which can be searched for without reference to presentations. The notion of shuffle retract will be substituted for that of operational retract. In fact, n-shuffle retracts will be defined for n a positive integer. However, the essence of the idea occurs in (1-)shuffle retracts.

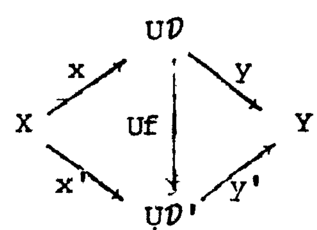
Definition 1.1: Given $U: \mathcal{D} \to \mathcal{A}$, construct a graph $Sh(\mathcal{D})$ $(= Sh(\mathcal{D},1))$ the graph of shuffles, with objects triplets (y,D,x) where

$$X \xrightarrow{x} UD \xrightarrow{y} Y$$

lies in $\mathcal{A}$ (by convention, X will always be the domain of x and Y the codomain of y etc.). Let (y',D',x') be given by

$$X' \xrightarrow{x'} UD' \xrightarrow{y'} Y'$$

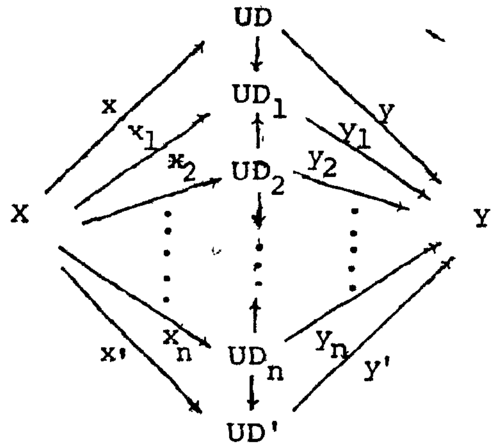
There are arrows between (y,D,x) and (y',D,x') only if X=X' and Y=Y'. Then arrows f: (y,D,x,)→(y',D',x') are given by those f:D → D' in $\mathcal{D}$ such that both triangles of



commute. Composition is that of $\mathcal{D}$. The arrows of this category are called <u>right shuffles</u>. We write

$$\begin{aligned}(y,D,x) &= (y'(Uf),D,x)\\ &\to (y',D',(Uf)x)\\ &= (y',D',x')\end{aligned}$$

Arrows in $Sh(\mathcal{D})^{op}$ are called <u>left shuffles</u>. Define (y,D,x) and (y',D',x') to be <u>shuffle equivalent ((y,D,x) ≡ (y',D',x'))</u> if they lie in the same component of $Sh(\mathcal{D})$ i.e. in A we have the commuting diagram



In other words, the equivalence is obtained by making a sequence of right and left shuffles. The equivalence class of (y,D,x) is denoted $[y,D,x]$.

Definitions 1.2:

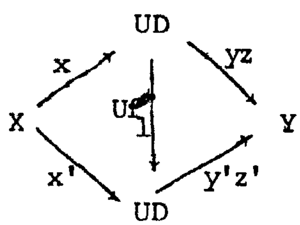
To cope fully with (I.3.2) a larger graph $Sh'(\mathcal{D},1)$ $(= Sh'(\mathcal{D}))$ is required. Its objects are of the form (y,C,z,D,x) where

$$X \xrightarrow{\;x\;} UD \xrightarrow{\;z\;} UC \xrightarrow{\;y\;} Y$$

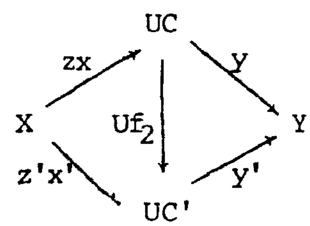
lies in $\mathcal{A}$. Note that z is a morphism in $\mathcal{A}$, not, in general, in the image of $\mathcal{D}$. Let $(y',C';z',D',x')$ be another object of $Sh'(\mathcal{D})$ given by the diagram

$$X' \xrightarrow{x'} UD' \xrightarrow{z'} UC' \xrightarrow{y'} Y'$$

There are arrows between them only if X = X' and Y = Y'. Then they are either arrows f_1: (yz,D,x) → (y'z',D',x') or arrows f_2: (y,C,zx) → (y',C',z'x') in *Sh*(𝒟) i.e. either



or



commutes. Once again, the arrows of *Sh'*(𝒟) (resp. *Sh'*(𝒟)^{op}) are called <u>right (resp. left) shuffles</u>, and objects lying in the same component of *Sh'*(𝒟) are <u>shuffle equivalent (≡')</u> with equivalence classes denoted by [y,C,z,D,x].

<u>Definition 1.3:</u>

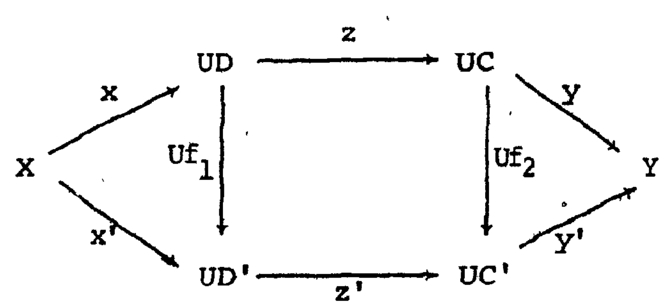
The idea of shuffle equivalence allows us to define the following central notions. (y,C,z,D,x) <u>shuffles out</u> if (y,C,z,D,x) ≡' (y',D',id,D',x') for some y',D', and x'. A <u>shuffle retract</u> is an equivalence class [y,D,x] of *Sh*(𝒟) such that X=Y, yx=id and (y,D,xy,D,x) shuffles out. Note that it doesn't follow that y or x is a homomorphism (c.f. Example III.3.7). A <u>shuffle homomorphism</u> between shuffle retracts $[y,D,x]$ and $[y',D',x']$ is a morphism $f \in A(X,X')$ such that

$$(y',D'x'f) \equiv (fy,D,x) \qquad 1.1$$

Shuffle homomorphisms are well-defined since if $(y,D,x) \equiv (t,C,z)$ then $(t,C,zf) \equiv (y,D,xf)$ etcetera. From (1.1) it follows that the shuffle retracts and shuffle homomorphisms form a category called $S(\mathcal{D})$ with an underlying functor $S(U)$ to A

$$S(U) \; : \; S(\mathcal{D}) \longrightarrow A$$
$$(y,D,x) \longmapsto X$$
$$f \longmapsto f$$

$Sh(\mathcal{D})$ and $Sh'(\mathcal{D})$ can adopt the structure of categories by using the composition rules of $\mathcal{D}$. An arbitrary morphism of $Sh'(\mathcal{D})$ is then a pair (f_2,f_1) of morphisms of $\mathcal{D}$ such that all paths commute in



Lemma 1.4:

There is an embedding $Sh(\mathcal{D}) \to Sh'(\mathcal{D})$ sending (y,D,x) to (y,D,id,D,x) and $f:(y,D,x) \to (y',D',x')$ to (f,f). There are also two 'contractions' $Sh'(\mathcal{D}) \to Sh(\mathcal{D})$ sending (y,C,z,D,x) to (yz,D,x) (resp. (y,C,zx)) and (f_2,f_1) to f_1 (resp. f_2)). They preserve shuffle equivalence. Thus, if $[y,D,x]$ is a shuffle retract with

$$(y,D,xy,D,x) \equiv' (z',C,id,C,z)$$

Then

$$\begin{aligned}[y,D,x] &= [yxy,D,x] \\ &= [z',C,z]\end{aligned}$$

and so

$$\begin{aligned}(y,D,xy,D,x) &\equiv' (z',C,id,C,z) \\ &\equiv' (y,D,id,D,x)\end{aligned}$$

Lemma 1.5: Let $U:\mathcal{D} \to A$ be operational with presentation (θ,H).

i) If $(y,D,x) \equiv (y',D',x')$ in $Sh(\mathcal{D})$ then $y\phi_D x = y'\phi_{D'}x'$.

ii) Similarly, if $(y,C,z,D,x) \equiv' (y',C',z',D',x')$, then

$$y\Phi_C z\Phi_D x = y'\Phi_{C'} z'\Phi_{D'} x'$$

iii) Hence, all shuffle retracts (resp. homomorphisms) are operational retracts (resp. homomorphisms).

Proof: The method of the proofs of i) and ii) are simplified versions of that of Lemma I.3.4 i.e. it is sufficient to check the result for right shuffles, which is easy. For iii) just apply i), ii) and Lemma 1.4.

Lemma 1.6: Let $G: \mathcal{D} \to \mathcal{D}'$ be a functor over $\mathcal{A}$. If $(y,D,x) \equiv (y_1,D_1,x_1)$ then $(y,GD,x) \equiv (y_1,GD_1,x_1)$. Similarly, if $(y,C,z,D,x) \equiv' (y_1,C_1,z_1,D_1,x_1)$ then $(y,GC,z,GD,x) \equiv' (y_1,GC_1,z_1,GD_1,x_1)$.

Proof: As before, it is sufficient to check the hypothesis for the generating equivalences, in fact just for the right shuffles, and then apply induction. The second statement is proved exactly like the first.

Consider a right shuffle

$$\begin{aligned} (y,D,x) &= (tUf,D,x) \qquad f: D \to C \text{ in } \mathcal{D} \\ &\to (t,C,(Uf)x) \\ &= (t,C,z) \end{aligned}$$

Then

$$
\begin{aligned}
(y,GD,x) &= (tU'(Gf),D,x) \qquad U'G = U \\
&\to (t,GC,U'(Gf)x) \\
&= (t,GC,z)
\end{aligned}
$$

S can be extended to be a functor $S: \mathcal{C}at/A \to \mathcal{C}at/A$. Let $G: \mathcal{D} \to \mathcal{D}'$ be a functor in $\mathcal{C}at/A$. Then we have

$$
\begin{aligned}
S(G):\quad S(\mathcal{D}) &\to S(\mathcal{D}') \\
[y,D,x] &\mapsto [y,GD,x] \\
f &\mapsto f
\end{aligned}
$$

By the lemma, SG is well-defined i.e. its image lies in $S\mathcal{D}'$ and its definition is independent of the representative of the equivalence class chosen.

Lemma 1.7:

i) If $(y',[y,D,x],x') \equiv (t',[t,C,z],z')$ in $Sh(S\mathcal{D})$ then $(y'y,D,xx') \equiv (t't,C,zz')$ in $Sh(\mathcal{D})$.

ii) Similarly, if

$(y_3',[y_1,D_1,x_1],y_2',[y,D,x],y_1') \equiv' (t_3',[t_1,C_1,z_1],t_2',[t,C,z],t_1')$ then $(y_3'y_1,D_1,x_1y_2'y,D,xy_1') \equiv' (t_3't_1,C_1,z_1t_2't,C,zt_1')$.

iii) Thus if

$[y',[y,D,x],x']$ is a shuffle retract with respect to $S\mathcal{D}$

then $[y'y'D, xx']$ is one with respect to $\mathcal{D}$. Similarly, shuffle homomorphisms remain shuffle homomorphisms.

Proof:

i) We need only check for right (and left) shuffles and then apply induction.

If

$$(y', [y,D,x], x') = (t'SUf, [y,D,x], x')$$
$$\rightarrow (t', [t,C,z], (SUf)x') \quad f: [y,D,x] \rightarrow [t,C,z]$$
$$= (t', |t,C,z|, z')$$

Then

$$((SUf)y, D, x) \equiv (t, C, zSUf)$$

Thus

$$(y'y, D, xx') = (t'(SUf)y, D, xx')$$
$$\equiv (t't, C, (SUf)x')$$
$$= (t't, C, zz')$$

ii) This is proved as in i).

iii) Let $[y', [y,D,x], x']$ be a shuffle retract for $S\mathcal{D}$.

Then

$$[y'y, D, xx'y'y, D, xx'] \equiv' [y'y, D, xy, D, xx']$$
$$\equiv' [y'y, D, id, D, xx']$$

<u>Proposition 1.8:</u> There is a triple on Cat/A, the shuffle triple, $\underline{S} = (S,\eta,\mu)$ with S given as above, and for $\mathcal{D}$ over A, the unit $\eta_{\mathcal{D}}$ is given by

$$\eta_{\mathcal{D}} D = [id, D, id]$$
$$\eta_{\mathcal{D}} f = f$$

and multiplication $\mu_{\mathcal{D}}$ given by

$$\mu_{\mathcal{D}}[y', [y,D,x], x'] = [y'y, D, x'x]$$
$$\mu_{\mathcal{D}} f = f$$

<u>Proof:</u> Clearly η is a natural transformation. $\mu_{\mathcal{D}}$ is well-defined by Lemma 1.7. Now let $G:\mathcal{D} \to \mathcal{D}'$ over A. Then

$$\begin{aligned}\mu_{\mathcal{D}'} S^2 G[y', [y,D,x], x'] &= [y', [y,GD,x], x'] \\ &= [y'y, GD, xx'] \\ &= SG[y'y, D, xx'] \\ &= SG\mu_{\mathcal{D}}[y', [y,D,x], x']\end{aligned}$$

Thus μ is a natural transformation. The proofs that $\mu S\eta = id = \mu\eta S$ and $\mu S\mu = \mu\mu S$ are left as easy exercises.

The category of algebras for $\underline{S}$ will be denoted $\underline{S}$-Alg with the corresponding adjunction being $F^{\underline{S}} \dashv U^{\underline{S}}$.

§2: n-Shuffle Retracts

Operational categories have a more detailed structure than arbitrary $\underline{S}$-algebras e.g. if $\mathcal{D}$ is operational with respect to (θ,H) and $[y,D,x]$ is a shuffle retract, and so an operational retract, with $y\Phi_D x = \Phi_{D'}$, then this equation can be employed to obtain new operational retracts which are not shuffle retracts c.f. Example III.3.13. To capture these properties of operational categories, a countable sequence of triples $\underline{S}_n = (S_n, \eta_n, \mu_n)$ must be constructed with $\underline{S}_1 = \underline{S}$ and $S_n : \underline{S}_{n-1}\text{-Alg} \to \underline{S}_{n-1}\text{-Alg}$ ($\underline{S}_0\text{-Alg} = Cat/A$). Of necessity, the triples must be constructed inductively. The information required for the induction is contained in the following hypothesis.

Hypothesis 2.n: For each $m \leq n$ there is a triple $\underline{S}_m = (S_m, \eta_m, \mu_m)$ on $\underline{S}_{m-1}$-Alg such that for $(\mathcal{D}, d_1, d_2, \ldots d_{m-1})$ an $\underline{S}_{m-1}$-algebra (d_k is the structure morphism for $\underline{S}_k$), $S_m(\mathcal{D}, d_1, d_2, \ldots d_{m-1})$ has as objects some triplets (y,D,x) such that

$$X \xrightarrow{x} UD \xrightarrow{y} Y$$

lies in A, and has as morphisms $(y,D,x) \to (y',D',x')$ some $f: X \to X'$. Let the structure morphisms for $S_m(\mathcal{D}, d_1, d_2, \ldots d_{m-1})$ be $(S_m d)_k$. For each $\underline{S}_{m-1}$-algebra, there is also a graph, $Sh(\mathcal{D}, m)$. They are defined so that the following statements hold: for $\mathcal{D}$ in Cat/A, $Sh(\mathcal{D},1) = Sh(\mathcal{D})$ and given an $\underline{S}_{m-1}$-algebra $(\mathcal{D}, d_1, d_2, \ldots d_{m-1})$, $Sh(\mathcal{D},m)$ is generated by

i) the underlying graph of $Sh(\mathcal{D},m-1)$ and

ii) if $D = d_{m-1}[y',D',x']$ then $(y,D,x) \to (yy',D',x'x)$

ii) is called an <u>m-expansion (with respect to D)</u>. Note that for $k < m$, k-expansions are also m-expansions. An m-expansion as an arrow of $Sh(\mathcal{D},m)$ is called an <u>m-contraction</u>. Define an equivalence relation on the objects of $Sh(\mathcal{D},m)$ by $(y,D,x) \equiv_m (y',D',x')$ iff they lie in the same component of $Sh(\mathcal{D},m)$. Equivalence classes are denoted $[y,D,x]$. Now $Sh'(\mathcal{D},m)$ (and its equivalence relation $\equiv'_m$) are constructed relative to $Sh(\mathcal{D},m)$ just as $Sh'(\mathcal{D})$ (and its equivalence relation $\equiv'$) were generated relative to $Sh(\mathcal{D})$. Then (y,C,z,D,x) m-shuffles out if $(y,C,z,D,x) \equiv_m' (t',D',id,D',t)$ for some t,t' and D'. Finally, any (y,D,x) such that $yx = id$ and (y,D,xy,D,x) m-shuffles out is an m-shuffle retract. An m-shuffle homomorphism $[y,D,x] \to [y',D',x']$ is an $f: X \to X'$ in A such that $(fy,D,x) \equiv_m (y',D',x'f)$. Then, $S_m(\mathcal{D},d_1,d_2,\ldots d_{m-1})$ is the category of m-shuffle retracts and homorphisms for $\mathcal{D}$ over A with an underlying functor $\underline{S}_m U: \underline{S}_m \mathcal{D} \to A$.

The unit, η_m, of $\underline{S}_m$ applied to $\mathcal{D}$ over A is given by

$$\begin{aligned} D &\longmapsto [id,D,id] \\ f &\longmapsto f \end{aligned}$$

and the multiplication, μ_m, at $\mathcal{D}$ over A is given by

$$\begin{aligned} [y',[y,D,x],x'] &\longmapsto [y'y,D,xx'] \\ f &\longmapsto f \end{aligned}$$

Definition 2.1: Assuming (2.n), define $Sh(\mathcal{D},n+1)$ etc. and $\overline{S}_{n+1}: \underline{S}_n\text{-Alg} \to \underline{S}_n\text{-Alg}$ just as in (2.n+1).

<u>Lemma 2.2:</u> Assume (2.n). Let $(\mathcal{D}, d_1, \ldots d_n)$ be an $\underline{S}_n$-algebra. Given that $S_{n+1}\mathcal{D}$ is an $\underline{S}_k$-algebra for some $k \geq 0$ with $(S_{n+1}d)_k [y', [y, D, x], x'] = [y'y, D, xx']$ (if $k > 1$) and $(S_{n+1}d)_k f = f$, then

i) if $[y', [y, D, x], x'] \equiv_{k+1} [t', [t, C, z], z']$ in $Sh(S_{n+1}\mathcal{D}, k+1)$ then $[y'y, D, xx'] \equiv_{n+1} [t't, C, zz']$

ii) Further, if

$[y_3', [y_1, D_1, x_1], y_2', [y, D, x], y_1'] \equiv_{n+1} [t_3', [t_1, C_1, z_1] t_2' z [t, C, z], t_1']$ then $[y_3' y_1, D_1, x y_2' y, D, x y_1'] \equiv_{n+1} [t_3' t_1, C_1, z t_2' t' C, z t_1']$.

<u>Proof:</u>

i) As before, it is sufficient to check the equivalences for generating shuffle equivalences i.e. for right (and left) shuffles and for k+1-expansions (and contractions). Right shuffles are dealt with as in Lemma 1.7. If there is the k+1-expansion

$$[y, D, x] = (S_{n+1}d)_k [t', [t, C, z], z'] = [t't, C, zz']$$

then $t't = y$, $D = C$ and $zz' = x$. Thus

$$[y'y, D, xx'] = [y't't, C, zz'x']$$

ii) This is proved as in i).

<u>Lemma 2.3:</u> Assume (2.n) and let $(\mathcal{D},d_1,\ldots,d_n)$ be an $\underline{S_n}$-algebra. Then $S_{n+1}\mathcal{D}$ is an $\underline{S_k}$-algebra for every k.

<u>Proof:</u> Since $S_{n+1}\mathcal{D}$ is an $\underline{S_0}$-algebra, induction based on Lemma 2.2 shows that $(S_{n+1}d)_k$ is well-defined for all k. Clearly, it is a structure morphism for $S_{n+1}\mathcal{D}$ i.e. $S_{n+1}\mathcal{D}$ is an $\underline{S_k}$-algebra for every k.

<u>Lemma 2.4:</u> Assume (2.n). Let $G:\mathcal{D}\to\mathcal{D}'$ over A be a homomorphism of $\underline{S_n}$-Alg.

i) If $(y,D,x) \equiv_{k+1} (y_1,D_1,x_1)$ for some k, $0<k<n$, then $(y,GD,x) \equiv_{k+1} (y_1,GD_1,x_1)$.

ii) Similarly, if $(y,C,z,D,x) \equiv'_{k+1} (y_1,C_1,z_1,D_1,x_1)$ then $(y,GC,z,GD,x) \equiv'_{k+1} (y_1,GC_1,z_1,GD_1,x_1)$.

<u>Proof:</u>

i) As this lemma is an extension of lemma 1.6, it is sufficient for the inductive proof to consider k+1-expansions (and contractions). Let $D' = d_k[y,D,x]$. Then $(y',D',x') \equiv_{k+1} (y'y,D,xx')$. But

$$\begin{aligned} GD' &= Gd_k[y,D,x] \\ &= d'_k SG[y,D,x] \\ &= d'_k[y,GD,x] \end{aligned}$$

Thus $[y',GD',x'] \equiv_{k+1} [y'y,GD,xx'] = [y_1,GD_1,x_1]$ as required.

ii) This is proved as in i).

Thus, assuming (2.n), S_{n+1} can be extended to a functor $\underline{S}_n$-Alg $\to \underline{S}_n$-Alg as follows. Let $G:(\mathcal{D},d_1,d_2,\ldots d_n) \to (\mathcal{D}',d'_1,d'_2,\ldots,d'_n)$ be an $\underline{S}_n$-algebra homomorphism. Then $S_{n+1}G$ is defined by

$$[y,D,x] \longmapsto [y,GD,x]$$
$$f \longmapsto f$$

By the lemma, $S_{n+1}G$ is well-defined. Also

$$\begin{aligned}(S_{n+1}d')_k S_k(S_{n+1}G)[y',[y,D,x],x'] &= (S_{n+1}d')_k[y',[y,GD,x],x'] \\ &= [y'y,GD,xx'] \\ &= S_{n+1}G[y'y,D,xx'] \\ &= (S_{n+1}G)(S_{n+1}d)_k[y',[y,D,x],x']\end{aligned}$$

Hence $S_{n+1}G$ is an $\underline{S}_k$-homomorphism for $1 \leq k \leq n$ i.e. an $\underline{S}_n$-homomorphism.

<u>Theorem 2.5</u>: For each n, (2.n) holds. In particular, $\underline{S}_n = (S_n,\eta_n,\mu_n)$ is a triple.

<u>Proof</u>: Proposition 1.8 is just (2.1). Given (2.n), define everything as in (2.n+1). Trivially, η_{n+1} is a natural transformation. μ_{n+1} is well-defined and a natural

transformation, by Lemma 2.3. Thus (2.n+1) holds.

Proposition 2.6: $\eta_{n+1}S_n : S_n \to S_{n+1}S_n$ is an isomorphism which has, for each $(\mathcal{D}, d_1, d_2, \ldots d_{n-1})$, the inverse $(S_n d)_{n+1}$ as in Lemma 2.3.

Proof: Let $\eta_{n+1}S_n\mathcal{D} = \eta$ and $(S_n d)_{n+1} = d_{n+1}$. Then $d_{n+1}\eta = id$ since d_{n+1} is a structure morphism. Now let $[y', [y,D,x], x'] \in |S_{n+1}S_n\mathcal{D}|$. Then by Lemma 2.2 $[y'y, D, xx'] \in S_n\mathcal{D}$. Hence $[y'y, [id,D,id], xx'] \in S_n^2\mathcal{D}$ and $(S_n d)_n[y'y, [id,D,id], xx'] = [yy', D, xx']$. Also, $(S_n d)_n[y, [id,D,id], x] = [y,D,x]$. So

$$\begin{aligned} \eta d_{n+1}[y', [y,D,x], x'] &= [id, [y'y, D, xx'], id] \\ &\equiv_{n+1} [y'y, [id,D,id], xx'] \\ &\equiv_{n+1} [y', [y,D,x], x'] \end{aligned}$$

Trivially $\eta d_{n+1} f = f$.

Thus $\eta_{n+1}S_n\mathcal{D}$ is an isomorphism for each $\mathcal{D}$ and so $\eta_{n+1}S_n$ is an isomorphism.

§3: $\underline{S_*}$-algebras

Definition 3.1: Let $(\mathcal{D}, \{d_n\})$ be such that, for each n, $(\mathcal{D}, d_1, d_2, \ldots d_n)$ is an $\underline{S_n}$-algebra. Then $(\mathcal{D}, \{d_n\})$ is an $\underline{S_*}$-algebra. Similarly, if $(\mathcal{D}', \{d'_n\})$ is another $\underline{S_*}$-algebra and $G: \mathcal{D} \to \mathcal{D}'$

is an $\underline{S}_n$-homomorphism for each n, then G is an $\underline{S}_*$-homomorphism. These categories and functors form a category called $\underline{S}_*$-Alg with a forgetful functor $U_*: \underline{S}_*\text{-Alg} \to \mathcal{C}at/A$.

Proposition 3.2: U_* has a left adjoint F_*.

Proof: Let $\mathcal{D}$ be a category over A. Since $S_{n+1}S_n \simeq S_n$ for all n by Proposition 2.6, $S_nS_1 \simeq S_1$ for all $n > 1$. Call the isomorphism $\nu_n: S_nS\mathcal{D} \simeq S\mathcal{D}$. Define $\nu_1 = \mu$. Then $F_*\mathcal{D} = (\mathcal{D}, \{\nu_n\})$ is an $\underline{S}_*$-algebra. Now let $G: \mathcal{D} \to \mathcal{D}'$ be a functor over A. Then $F_*G = SG$ is, by Theorem 1.8, an $\underline{S}$-homomorphism and so, since $\eta_n S$ is a natural transformation for each n, SG is an $\underline{S}_*$-homomorphism. Hence, F_* is a functor $\mathcal{C}at/A \to \underline{S}_*$-Alg.

Now the unit for the adjunction is $\eta: 1 \to S = U_*F_*$. The counit at $(\mathcal{D}, \{d_n\})$ is $\varepsilon = d_1: F_*\mathcal{D} \to (\mathcal{D}, \{d_n\})$. ε is an $\underline{S}_*$-homomorphism since

$$\begin{aligned} d_1 S_1 \varepsilon &= d_1 S_1 d_1 \\ &= d_1 \mu \\ &= \varepsilon \nu_1 \end{aligned}$$

and for $n > 1$

$$d_n S_n \varepsilon [y', [y,D,x], x'] = d_n [y', d_1 [y,D,x], x']$$
$$= d_n [y'y, D, xx']$$
$$= d_1 [y'y, D, xx']$$
$$= \varepsilon \nu_n [y', [y,D,x], x']$$

Now the equations for the adjunction are

$$U_* \varepsilon_{(\mathcal{D}, \{d_n\})} \eta_{\mathcal{D}} = d_1 \eta_{\mathcal{D}}$$
$$= id$$

and

$$\varepsilon_{F_* \mathcal{D}} F_* \eta_{\mathcal{D}} = \nu_1 S \eta_{\mathcal{D}}$$
$$= \mu_{\mathcal{D}} S \eta_{\mathcal{D}}$$
$$= id$$

Thus $F_* \dashv U_*$.

CHAPTER III

A CHARACTERISATION OF OPERATIONAL CATEGORIES

In this chapter the operational categories are characterised by their internal structure. Specifically, the operational categories are exactly the $\underline{S}_*$-algebras. We approach this result by first showing that the triple associated to the adjunction $L_o \dashv R_o$ is just S.

§1: $R_o L_o \cong S$

The following lemma links the constructions of $L\mathcal{D}$ and $S\mathcal{D}$.

Lemma 1.1:

i) if

$$y\omega_D x = y_n \omega_{D_n} y_{n-1} \omega'_{D_n} y_{n-2} \cdot \cdots \cdot y_1 \omega_{D_1} y_o$$

in $C\mathcal{D}$, then there is an i $(1 \leq i \leq n)$ such that

$$(y, D, x) \equiv (y_n y_{n-1} \cdot \cdots \cdot y_i, D_i, y_{i-1} \cdot \cdots \cdot y_o)$$

ii) If

$$y\omega_C' z\omega_D x = y_n\omega_{D_n}' y_{n-1}\cdots\cdots y_1\omega_{D_1} y_o$$

then there are i<j such that

$$(y,C,z,D,x) \equiv' (y_n y_{n-1}\cdots y_j, D_j, y_{j-1}\cdots y_i, D_i, y_{i-1}\cdots y_o)$$

iii) Hence, if $y'x' = id$, $yx = id$ and $y'\omega_{D}' x' y\omega_D x = id$ then $y'\omega_{D}' x' = y\omega_D' x$ and $[y'D,x]$ is a shuffle retract.

Proof: The proofs all run parallel to parts of that of I.3.4. Lemma II.1.4 may also be used.

Theorem 1.2: $R_0 L_0 \cong S: \mathcal{Cat}/A \to \mathcal{Cat}/A$. Hence, the comparison functor $K: Op_0(A) \to \underline{S}$-Alg provides each operational category (resp. functor) with a standard presentation, with the structure of an $\underline{S}$-algebra (resp. $\underline{S}$-homomorphism). For $\mathcal{D}$ operational with a standard presentation, the structure morphism d is given by

$$\begin{aligned} d:\ S\mathcal{D} &\longrightarrow \mathcal{D} \\ [y,D,x] &\longmapsto (X, y\phi_D x) \\ f &\longmapsto f \end{aligned}$$

Proof: Let (X,ϕ) be an algebra in $L\mathcal{D}$. Then, by duplicating the arguments of Theorem I.3.7,

$$\Phi\omega = y\omega_D x \qquad yx = id$$

$$\Phi\omega' = y'\omega'_{D'}x' \qquad y'x' = id$$

$$id = y'\omega'_{D'}x'y\omega_D x$$

Hence, by Lemma 1.1, $y'\omega'_{D'}x' = y\omega_D' x$ and $[y,D,x]$ is a shuffle retract. Similarly, morphisms in $L_0\mathcal{D}$ are shuffle homomorphisms. Thus, there is a functor $\phi_{\mathcal{D}}:R_0L_0\mathcal{D}\to S.\mathcal{D}$.

Conversely, define $\phi_{\mathcal{D}}':S\mathcal{D}\to R_0L_0\mathcal{D}$

by

$$[y,D,x] \mapsto (X, y\Phi_D x)$$

$$f \mapsto f$$

with respect to the given presentation of $L\mathcal{D}$. $\phi'_{\mathcal{D}}$ is well-defined by Lemma II.1.5. Clearly, $\phi'_{\mathcal{D}}\phi_{\mathcal{D}} =$ id. By Lemma 1.1, $\phi_{\mathcal{D}}\phi'_{\mathcal{D}} =$ id, too.

To see that $R_0L_0G \cong S(G)$ for a functor $G:\mathcal{D}\to\mathcal{D}'$ over A, we check that $S(G)$ satisfies the definition of LG (c.f. (I.2.7)). Trivially, $(LU')S(G) = LU$. For the other arm of the pullback note that

$$\Phi_{\eta GD} = \Phi_{LG\eta D}$$

$$= \P' LG\eta_D$$

$$= CG^T o \P'\eta D$$

$$= CG^T o \Phi_{\eta D}$$

Hence

$$
\begin{aligned}
\P' S(G)[y,D,x] &= \P'[y,GD,x] \\
&= (H\mathcal{D}')^{*}y\Phi_{\eta G\mathcal{D}}(H\mathcal{D}')^{*}x \\
&= (CG.H\mathcal{D})^{*}y(CG^{T}o\Phi_{\eta\mathcal{D}})(CG.H\mathcal{D})^{*}x \\
&= CG^{T}o(H\mathcal{D})^{*}y\Phi_{\eta\mathcal{D}}(H\mathcal{D})^{*}x \\
&= CG^{T}o\P'[y,D,x] \\
&= \P' R_oL_oG[y,D,x]
\end{aligned}
$$

Also $\P' SGf = H^{*}Uf = R_oL_of$. Thus $SG \approx R_oL_oG$. Hence $S \approx R_oL_o$.

§2: The Characterisation Theorem

Lemma 2.1: Let $U:\mathcal{D} \to A$ be operational with standard presentation (θ_0,H). Assume that n-shuffle retracts for $\mathcal{D}$ are operational retracts and $\mathcal{D}$ is an $\underline{S_n}$-algebra for some $n \geq 1$ with $d_k:S_k\mathcal{D} \to \mathcal{D}$ given by

$$
\begin{aligned}
[y,D,x] &\mapsto (X, y\Phi_D x) \\
f &\mapsto f
\end{aligned}
$$

for $1 \leq k \leq n$. Then if, for some $k, 1 \leq k \leq n+1$

$$(y,D,x) \equiv_k (y',D',x') \qquad 2.1$$

then $y\Phi_D x = y'\Phi_{D'}x'$. Further, if

$\int(y,C,z,D,x) \equiv'_k (y',C',z',D',x')$ then $y\phi_C z\phi_D x = y'\phi_{C'} z'\phi_{D'} x'$. Hence, if $[y,D,x]$ is an n+1-shuffle retract then it is an operational retract and $\mathcal{D}$ is an $\underline{S}_{n+1}$-algebra.

Proof: The hypothesis for n = 0 is just Lemma II.1.5. The proof there can be expanded to cover the general case simply by checking equivalence by shuffle expansions (and contractions). Assume that the result holds for n and that (2.1) holds. Let $D = d_n[t,D',z]$. Then

$$\begin{aligned}(y,D,x) &\equiv_{n+1} (yt,D',zx)\\ &= (y',D',x')\end{aligned}$$

Where yt = t' and zx = x'. Then

$$\phi_D = t\phi_{D'}z$$

and

$$\begin{aligned}y\phi_D x &= yt\phi_{D'}zx\\ &= y'\phi_{D'}x'\end{aligned}$$

Theorem 2.2: There is an equivalence $K':Op_0(A) \to \underline{S}_*$-Alg over $\mathcal{C}at/A$.

Proof: Define $K':Op_0(A) \to \underline{S}_*$-Alg by $K'(\mathcal{D},\theta_0,H) = (\mathcal{D},\{d_n\})$, and $K'(G,id,k) = G$ where d_n is given by

$$\begin{aligned}[y,D,x] &\mapsto (X, y\phi_D x)\\ f &\mapsto f\end{aligned}$$

By Theorem 1.2, $\mathcal{D}$ is an $\underline{S}$-algebra with structure morphism d_1. Assume now that $\mathcal{D}$ is an $\underline{S}_n$-algebra with structure morphisms d_n. Then Lemma 2.1 shows that $\mathcal{D}$ is an $\underline{S}_{n+1}$-algebra with structure morphism d_{n+1}. Hence $\mathcal{D}$ is an $\underline{S}_*$-algebra by induction. Now let $(G,id,k):(\mathcal{D},\theta_0,H) \to (\mathcal{D}_1,\theta_0,H_1)$ be an operational functor with a standard presentation. By Theorem 1.2, G is an $\underline{S}$-homomorphism. Assume that G is an $\underline{S}_n$-homomorphism. Then

$$\begin{aligned} Gd_{n+1}[y,D,x] &= G(X,y\phi_D x) \\ &= (X,y\phi_{GD}x) \\ &= d'_{n+1}[y,GD,x] \\ &= d'_{n+1}S_{n+1}G[y,D,x] \end{aligned}$$

where the second line holds since, by Lemma 2.1, $[y,D,x]$ is an operational retract and Lemma I.3.2 gives operational functors this property. Also, $Gd_{n+1}f = f = d'_{n+1}S_{n+1}Gf$. So G is an $\underline{S}_{n+1}$-homomorphism. Thus, by induction, G is an $\underline{S}_*$-homomorphism.

Clearly, $U_0K' = R_0$. Also, by the definition of F_* and Theorem 1.2, $K'L_0 \cong F_*$.

Now the $\underline{S}_*$-algebras $(\mathcal{D},\{d_n\})$ (respectively

$\underline{S}_*$-homomorphisms G) will be shown to be operational with respect to a standard presentation, $(\theta_o, H''\mathcal{D})$ (resp. (id, C"G)). First $H''(\mathcal{D},\{d_n\})$ must be constructed and some of its properties established.

Definition 2.3: Define $C'':\underline{S}_*\text{-Alg} \to \text{Cat}$ as follows; for $(\mathcal{D},\{d_n\})$ an $\underline{S}_*$-algebra, $C''(\mathcal{D},\{d_n\})$ is generated by

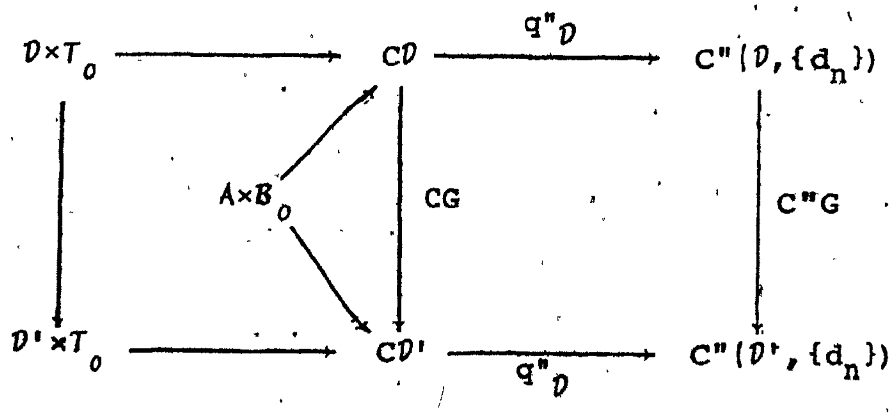
ia) the underlying graph and equations of $C\mathcal{D}$

viii) if there is an n such that $d_n[y,D,x] = D'$ in $S_n\mathcal{D}$, then

$$y\omega_D x = \omega_{D'}$$

$$y\omega'_D x = \omega'_{D'}$$

If $G:\mathcal{D}\to\mathcal{D}'$ is an $\underline{S}_*$-homomorphism then CG preserves the equations viii) and so induces $C''G:C''(\mathcal{D},\{d_n\})\to C''(\mathcal{D}',\{d'_n\})$. The quotient functors $q''_{\mathcal{D}}:C\mathcal{D}\to C''(\mathcal{D},\{d_n\})$ yield a natural transformation $q'':C\to C'':\underline{S}_*\text{-Alg}\to\text{Cat}$. The base functor is $H''(\mathcal{D},\{d_n\}) = q''H\mathcal{D}:A\times B_o\to C''(\mathcal{D},\{d_n\})$. There is also the commuting diagram, from the definition of $C\mathcal{D}$:


$\mathcal{D}\times T_0$
$C\mathcal{D}$
$q''_{\mathcal{D}}$
$C''(\mathcal{D}, \{d_n\})$
$A\times B_0$
CG
$C''G$
$\mathcal{D}'\times T_0$
$C\mathcal{D}'$
$q''_{\mathcal{D}}$
$C''(\mathcal{D}', \{d_n\})$

2.2

Lemma 2.4:

i) if

$$y\omega_D x = y_m \omega_{D_m} y_{m-1} \cdots\cdots y_1 \omega_{D_1} y_o$$

in $C''(\mathfrak{D}; \{d_n\})$ then there is an i such that $1 \leq i \leq m$ and an n such that $(y,D,x) \equiv_n (y_m y_{m-1} \cdots y_i, D_i, y_{i-1} \cdots y_o)$.

ii) if

$$y\omega'_C z\omega_D x = y_m \omega'_{D_m} y_{m-1} \cdots\cdots y_1 \omega_{D_1} y_o$$

then there are $i<j$ and an n such that

$$(y,C,z,D,x) \equiv'_n (y_m y_{m-1} \cdots\cdots y_j, D_j, y_{j-1} \cdots\cdots y_i, D_i, y_{i-1} \cdots\cdots y_o)$$

iii) Further, if $y'x' = id$, $yx = id$ and $y'\omega'_D x' y\omega_D x = id$ then $y'\omega'_D x' = y\omega'_D x$ and there is an n such that $[y,D,x]$ is an n-shuffle retract.

Proof:

i) This lemma is an extension of Lemma 1.1. Hence it is sufficient to check the hypothesis for expansions (and contractions). If equivalence is by, say, an n-expansion of type viii) then we have, $D = d_n[y_1, D', x_1]$ and $\omega_D = y_1 \omega_{D'} x_1$. So $y\omega_D x = yy_1\omega_{D'}x_1 x = y'\omega_{D'}x'$. where $y' = yy_1$ and $x_1 x = x'$. Thus

$$(y,D,x) \equiv_{n+1} (yy_1, D', x_1 x)$$
$$= (y', D', x')$$

Shuffle contractions are dealt with similarly. Since only finitely many expansions and contractions can be used in many proof, let n be the highest level of equivalence used.

ii) and iii) follow as in 1.1.

<u>Proof of Theorem 2.2 (cont'd):</u>

Let $(\mathcal{D},\{d_n\})$ be an $\underline{S}_*$-algebra. Construct the operational category $\mathcal{P}$ $(\doteq K''(\mathcal{D},\{d_n\}))$ using the presentation $(\theta_0, H''(\mathcal{D},\{d_n\}))$.

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & C''(\mathcal{D},\{d_n\})^{T_0} \\ \big\downarrow & \lrcorner & \big\downarrow \\ A & \longrightarrow & C''(\mathcal{D},\{d_n\})^{B_0} \end{array}$$

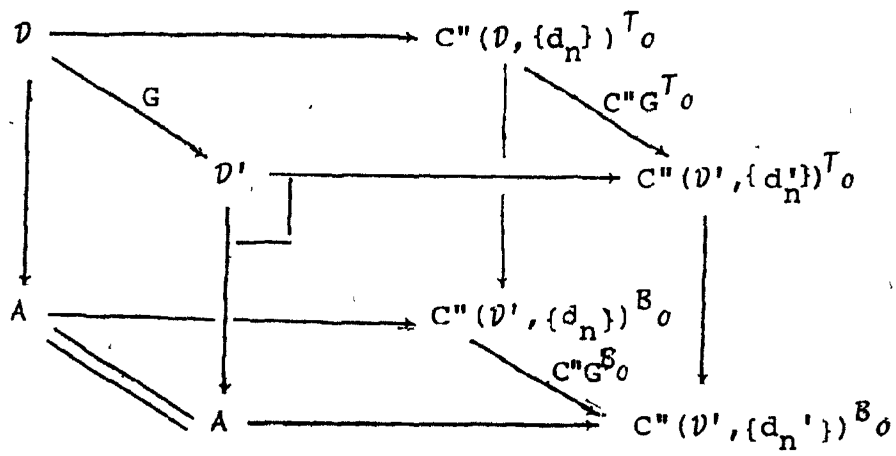
Since (I.2.4) is a commuting diagram, there is a functor $F: \mathcal{D} \to \mathcal{P}$ over A, sending D to (UD, Φ_D) and f to f, where $\Phi_D: T_0 \to C'(\mathcal{D},\{d_n\})$ maps ω to ω_D. F is faithful since U is. To see F is a monomorphism note that if $\omega_D = \omega_{D'}$ in $C''(\mathcal{D},\{d_n\})$ then by Lemma 2.4, $\Phi_D = \Phi_{D'}$ and so $D = D'$.

Now let (X,Φ) be an algebra in $\mathcal{P}$. By repeating the arguments of Theorems I.3.7 or III.1.2 we show that $\Phi\omega = y\omega_D x$, $\Phi\omega' = \bar{y}'\omega'_{D'}x$ and $id = y'\omega_D x' x\omega_D x$. Hence, by Lemma 2.4, $\Phi\omega' = y\omega'_D x$ and, for some n, $[y,D,x]$ is an n-shuffle retract. Let $D' = d_n[y,D,x]$. Then

$$\Phi\omega = \omega_{D'}$$
$$\Phi\omega' = \omega'_{D'}$$

Thus $(X,\Phi) = FD'$. Now if $f:FD \to FD'$ is a homomorphism in $\mathcal{P}$ then $f\omega_D = \omega_{D'}f$. Hence, by Lemma 2.4, $(f,D,id) \equiv_n (id,D',f)$ for some n and f is a morphism of $S_n\mathcal{D}$. Hence $f = d_n f$ is a morphism $D \to D'$ of $\mathcal{D}$. Thus, $\mathcal{P} \simeq \mathcal{D}$ over A, i.e., that every $\underline{S}_*$-algebra has a standard presentation as an operational category.

Now consider an $\underline{S}_*$-homomorphism $G:\mathcal{D} \to \mathcal{D}'$. Then the commutativity of (2.2) shows that G is the induced operational functor (K"G) in the diagram



Thus K" is inverse to K' as far as operational categories and functors are concerned. Now it must be checked that presentations are respected too. The presentation for $K''K'(\mathcal{D}, \theta_0, H)$ is specified by the generating equations in $C''(\mathcal{D},\{d_n\})$; $y\omega_D x = \omega_{D'}$ if $[y,D,x]$ is an n-shuffle retract with $\partial_n[y,D,x] = D'$. But then $[y,D,x]$ is an operational retract with $y\Phi_D x = \Phi_{D'}$ and so, in $\mathcal{C}$, $y\omega_D x = \omega_{D'}$. Hence, the identity on $\mathcal{C}\mathcal{D}$ induces a functor $C''(K'(\mathcal{D}, \theta_0, H)) \to \mathcal{C}$. Thus, by the 'minimality' of q , $C''(\mathcal{D},\{d_n\}) \cong \mathcal{C}$. Also, if (G, θ_0, k) is a morphism of $Op_0(A)$, then, since both k and C'G are induced by CG, $k \cong C'G$. Thus $K''K' \simeq id$.

Consider $K'K''(\mathcal{D},\{d_n\}) = (\mathcal{D},\{d'_n\})$. Assume that $d_n = d'_n$ for $0 < k \le n$. Let $[y,D,x]$ be an n+1-shuffle retract with $d'_{n+1}[y,D,x] = D'$. Then $y\omega_D x = \omega_{D'}$ in $C''(\mathcal{D},\{d_m\})$.

Hence, by Lemma 2.4, $[y,D,x]$ is an m-shuffle retract for some m, with $d_m[y,D,x] = D'$. Without loss of generality, $m>n+1$. So

$$\begin{aligned} D' &= d_m[y,D,x] \\ &= d_m[id, d_{n+1}[y,D,x], id] \\ &= d_{n+1}[y,D,x] \end{aligned}$$

Thus $d'_{n+1} = d_{n+1}$. So $d'_n = d_n$ for all n and hence $K'K''(\mathcal{D}, \{d_n\}) \cong (\mathcal{D}, \{d_n\})$. Trivially, $K'K''G \cong G$ for any $\underline{S}_*$-homomorphism. Thus K' is an equivalence.

<u>Corollary 2.5</u>: $Op_0(A)$ is complete with limits preserved by R_o.

<u>Proof</u>: Cat/A is complete. So $\underline{S}$-Alg is complete and U preserves limits. By induction, $\underline{S}_n$-Alg is complete and $U^{\underline{S}_n}$ preserves limits. Hence, given any diagram in $\underline{S}_*$-Alg, its underlying diagram in Cat/A has a limit which exists in $\underline{S}_n$-Alg for each n, and so in $\underline{S}_*$-Alg. Hence, by Theorem 2.2, $Op_o(A)$, being equivalent to $\underline{S}_*$-Alg, is complete with limits preserved by R_o.

§3: Examples

Under this heading are collected a diverse assortment of examples. Some are quite general propositions, such as the demonstration that all slice categories are operational. Others illustrate the manipulation of shuffles, with a view towards providing counter-examples to some reasonable (but false) hypotheses.

Lemma 3.1: Let $U:\mathcal{D} \to \mathcal{A}$ be a functor. Then for any U-split coequalizer as in (I.1.3), $[y,D,x]$ is a shuffle retract.

Proof:

$$\begin{aligned}(y,D,xy,D,x) &= (y,D,(Ug)t,D,x)\\ &\to (yUg,C,t,D,x)\\ &= (y\,Uf,C,t,D,x)\\ &\to (y,D,(Uf)t,D,x)\\ &= (y,D,id,D,x)\end{aligned}$$

Proposition 3.2: Let $U:\mathcal{D} \to \mathcal{A}$ have a left adjoint F. Then $S\mathcal{D} \simeq \mathcal{A}^{\underline{T}}$ where $\underline{T}$ is the triple associated with the adjunction. In particular, if $\mathcal{D}$ is tripleable, then $S\mathcal{D} \simeq \mathcal{D}$. Dually, if $U:\mathcal{D} \to \mathcal{A}$ has a right adjoint then

$S\mathcal{D} \simeq A_{\underline{G}}$ where $\underline{G}$ is the associated cotriple, and if $\mathcal{D}$ is cotripleable then $S\mathcal{D} \simeq \mathcal{D}$. Thus tripleable and cotripleable categories are operational.

Proof: Let $[y,D,x]$ be a shuffle retract. Then there is a $\underline{T}$-algebra $(X, ydTx)$ (where $d = U\varepsilon_D : TUD \to UD$) since

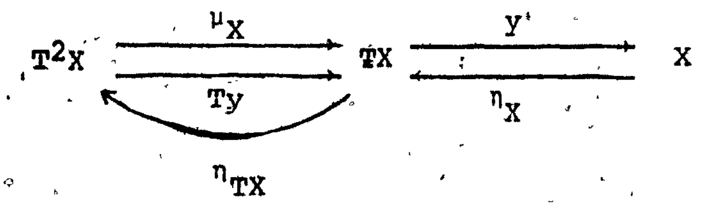
$$\begin{aligned} ydTx\eta_X &= yd\eta_{UD}x \\ &= yx \qquad\qquad U\varepsilon\eta_U = id \\ &= id \end{aligned}$$

and

$$\begin{aligned} (ydTx)T(ydTx) &= ydT(xy)TdT^2x \\ &= ydTdT^2x \qquad \varepsilon \text{ is a natural transformation} \\ &= yd\mu_{UD}T^2x \qquad \mu = U\varepsilon F \\ &= ydTx\mu_X \end{aligned}$$

If f is a shuffle homomorphism $[y,D,x] \to [y',D',x']$ then $(y',D',x'f) \equiv (fy,D,x)$ and so $(y'd'Tx')Tf = (ydTx)$. Hence f is a $\underline{T}$-homomorphism. These constructions respect the equivalence relation and so define a functor $L\mathcal{D} \to A^{\underline{T}}$.

Now, let (X,y) be a $\underline{T}$-algebra. Every algebra for a triple induces a U-split coequalizer with respect to any adjunction defining the triple.



Thus by Lemma 3.1, $[y, FX, \eta_X]$ is shuffle retract. Clearly, $\underline{T}$-homomorphisms yield shuffle homomorphisms. Hence there is a functor $A^{\underline{T}} \to \mathcal{SD}$.

It is easy to check that these functors are inverse. The dual results follow from Proposition I.1.2.

Examples 3.3: By Proposition I.1.2, the pullback of any operational category is operational. Thus, for example, the category of finite groups over the category of finite sets ($\mathcal{G}rp_f \to \mathcal{S}ets_f$) is operational.

Example 3.4: Any full subcategory closed under retracts is operational. Thus, given a group G, U:Sub(G) $\to \mathcal{G}rp$ (where Sub(G) is the full subcategory of subgroups of G) is operational. Also $\mathcal{G}rp_f \to \mathcal{S}ets$ is operational.

Example 3.5: If $U:\mathcal{D} \to A$ is a fibration then all shuffle retracts are trivial.

Proof: Recall that, given $U:\mathcal{D} \to A$, $f:D \to D'$ is a cartesian morphism in $\mathcal{D}$ if, for any $f_1:D_1 \to D'$ such that $Uf_1 = Uf$, then there is a $\xi:D_1 \to D$ such that $U\xi = id_{UD}$ and $f_1 = f\xi$. $U:\mathcal{D} \to A$ is a fibration if, given $x:X \to Y$ in A with $UD = Y$, then there is a cartesian morphism $f:D' \to D$ with $Uf = x$. Also cartesian morphisms must be closed under composition.

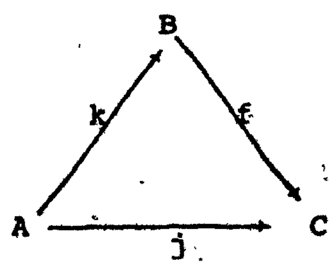
Now let $[y,D,x]$ be a shuffle retract and let $x':X' \to D$ be a cartesian morphism over x. Then $[y,D,x] = [id,X',id]$. However, if there is an $x'':X'' \to D$ such that $Ux'' = x$ then $[id,X,id] = [id,X',id]$. Hence, many interesting fibrations aren't operational.

Proposition 3.6: Slice categories are operational.

Proof: Let $A/X \to A$ be a slice category (the objects of A/X are pairs (A,a), $a:A \to X$). A quick induction shows that if $(y,(A,a),x) \equiv (y',(A',a'),x')$ then $ax = a'x'$. Also $(y,(A,a),x) \equiv (id,(x,ax),id)$. It easily follows that $S(A/X) \simeq A/X$.

Proposition 3.7: Satisfying B.T.C. and being faithful does not imply that a functor $U:\mathcal{D} \to A$ is operational.

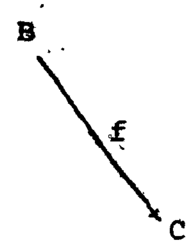
Proof: Generate $\mathcal{A}$ by the graph



and the equations

$$jfk = id$$
$$fkjf = f$$

Let $U:\mathcal{D} \to \mathcal{A}$ be given by the subcategory inclusion



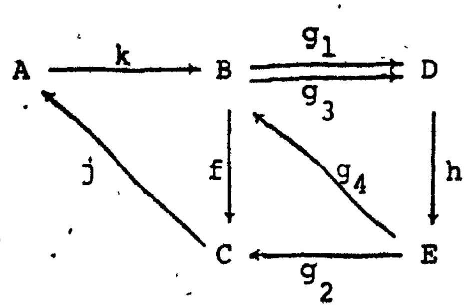
Trivially, U satisfies B.T.C.. Now

$$\begin{aligned}(jf,B,kjf,B,k) &\equiv' (j,C,fkjf,B,k)\\ &= (j,C,f,B,k)\\ &\equiv' (jf,B,id,B,k)\end{aligned}$$

So (jf,B,k) is a shuffle retract over A, yet no object of $\mathcal{D}$ lies over A. Hence, there can be no S-algebra structure for $\mathcal{D}$ i.e. $\mathcal{D}$ is not operational.

<u>Example 3.8</u>: Here is constructed a category for which $S^{n+1}\mathcal{D} \cong S^n\mathcal{D} + 1$ (where 1 is the terminal category). This category will be used below to create counter-examples to various attractive hypotheses.

Let A be generated by the graph



and the equations

i) $jfk = id$

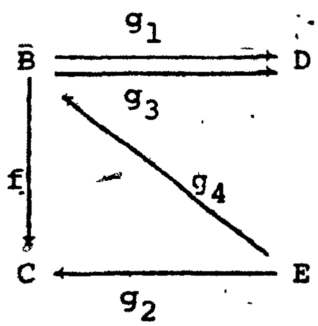
ii) $fkjf = g_2hg_1$

iii) $jg_2 = jfg_4$

iv) $g_1k = g_3k$

v) $g_4hg_3 = id$

Let $U:\mathcal{D} \to \mathcal{A}$ be the inclusion generated by



Claim 1: All non-trivial identities (i.e. composites of generators equalling an identity morphism of $\mathcal{A}$) are contained in the following list (note that not all the morphisms in the list are identities!).

a) g_2hg_1, g_2hg_3, g_4hg_1, g_4hg_3 and all defined composites of these

b) jfk, all defined morphisms of the forms 3.1
$jf\lambda k$ and $j\lambda k$ where λ is in a), and all composites of these.

Proof: All the generating identities in the definition of $\mathcal{A}$ are in the list and it is closed under application of generating equations.

Now all candidates for shuffle retract for $\mathcal{D}$ can be obtained by 'splitting' an identity in the list into a pair of maps. When this is done all but one of any composition of identities must vanish. Hence, after taking equivalence into account, there is only one possible, non-trivial shuffle retract, namely, $(jf,B,k) \equiv (j,C,fk)$. Now

$$\begin{aligned}(jf,B,kjf,B,k) &\to (j,C,fkjf,B,k)\\ &= (j,C,g_2hg_1,B,k)\\ &\equiv (jg_2,E,h,D,g_1k)\\ &= (jfg_4,E,h,D,g_3k)\\ &\to (jf,B,g_4hg_3,B,k)\\ &= (jf,B,id,B,k)\end{aligned}$$

So, $[jf,B,k]$ is a non-trivial shuffle retract.

Now assume that there is a non-trivial shuffle homomorphism $f_o\colon[id,D_o,id] \to [id,D_o',id]$. Then

$$(f_o,D_o,id) \equiv (id,D_o',f_o). \qquad 3.2$$

Assume further that the shortest proof of equivalence (the length of a proof is the number of shuffles employed) is begun by 'splitting' the identity in the left-hand term of the equivalence (3.2). Then $id = f_2f_1$ where $f_2 = \eta_{\mathcal{D}}f_2$ i.e. f is a morphism of $\mathcal{D}$. By examining the list (3.1) it is apparent that $f_2f_1 = g_4hg_3$

$$(f_o, D_o, id) = (f_o, B, g_4hg_3)$$
$$(f_og_4, E, hg_3)$$

The next step in the proof must be a right shuffle. The only equation involving something of the form f_og_4 is $jfg_4 = jg_2$. Hence $f_o = f_1jf$ for some f_1. Thus

$$(f_og_4, E', hg_3) = (f_1jfg_4, D, hg_3)$$
$$= (f_1jg_2, D, hg_3)$$

and the 'j' cannot be eliminated except by reversing the equivalences already used, contradicting minimality. Thus, in any minimal proof of an equivalence (3.2), the first step must be a right shuffle.

Lemma 3.9: Given $U: \mathcal{D} \to A$, assume that no minimal proof of equivalence

$$(f, D, id) \equiv (id, D', f) \qquad 3.3$$

in $Sh(\mathcal{D})$ can begin with a left shuffle (from the left-hand term). Then every non-trivial shuffle homomorphism f (i.e not in the image of η) has a minimal proof of (3.3) which begins by

$$(f,D,id) = (f_3f_3',D,id)$$
$$\rightarrow (f_3,D_1,f_3')$$
$$= (f_3,D_1,f_2f_1)$$
$$\rightarrow (f_3f_2,D_2,f_1)$$

where f_1 doesn't underlie any $f_1':D \rightarrow D_2$.

Proof: The proof is by induction on the minimum length of proof of (3.3). If the proof is in one step then $f \in Im(\eta)$. Assume the hypothesis for the cases with the minimum length of proof being n. Let f satisfy (3.3) with a minimum proof length of n+1. By assumption, the proof must begin with a right shuffle

$$(f,D,id) \rightarrow (f_3,D_1,f_3')$$

By minimality, the next shuffle must be to the left

$$(f_3,D_1,f_3') \rightarrow (f_3f_2,D_2f_1)$$

Now if $f_1 = Uf_1'$ for $f_1': D \rightarrow D_2$,

$$(f,D,id) \rightarrow (f_3f_2,D_2,f_1)$$

and so the proof of (3.3) may be given in n steps. Contradiction. Hence f_1 is as required.

Returning to the particular $\mathcal{D}$ at hand, the search for morphisms $f_3' \varepsilon Im(\eta)$ such that $f_3' = f_2 f_1$ in a non-trivial way with f_2 and f_1 as above shows that they are all of the form $f_3' = (f' g_4)(hg_3)$. Hence the first two steps of the minimal proof look like

$$(f_o, D_o, id) \rightarrow (f_3, D_1, f_3')$$
$$\rightarrow (f_3 f_3' g_4, E, hg_3)$$

But this equivalence can be obtained by

$$(f_o, D_o, id) = (f_o, B, g_4 hg_3)$$
$$(f_o g_4, E, hg_3)$$

in one step. Hence, no such f_3' as considered here can be employed in a minimal proof of (3.3). So the only shuffle homomorphisms between trivial shuffle objects are from $\mathcal{D}$ i.e. η is full.

To complete the characterization of $L\mathcal{D}$ the shuffle homomorphisms into and out of $[jf,B,k]$ must be analysed. The morphisms out of A in A are all of the form $f_o k$. Assume $(f_o kjf, A, k) \equiv (id, X, f_o k)$ for some f_o and X. Then by the usual arguments, in any minimal proof we have $f_o = f_1 f$ and

$$(f_o kjf, B, k) = (f_1 fkjf, B, k)$$
$$= (f_1 g_2 hg_1, B, k)$$

Now to eliminate the 'g_2' a 'j' must be introduced. So $f_1 = f_2 j$ and

$$f_o k = f_2 jfk$$
$$= f_2$$

Hence, the proof above is not minimal. Contradiction. So there are no such shuffle homomorphisms.

In the same way it is shown that there are no shuffle homomorphisms into $[jf, A, k]$ from any shuffle retract.

Thus, $S\mathcal{D} \simeq \mathcal{D} + 1$. Since the isolated $[jf, A, k]$ can have no influence on the construction of shuffle retracts all of the above work generalises to show that $S^{n+1}\mathcal{D} \simeq S^n\mathcal{D} + 1$.

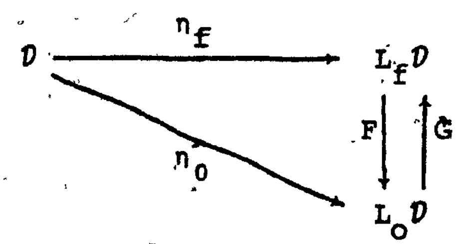
<u>Proposition 3.10</u>: For an arbitrary A, $S^2 \neq S$. Hence, $L_o R_o \neq 1$ and so the image of Op(A) is not a full subcategory of *Cat*/A. (Note that Op(A) isn't a subcategory of *Cat*/A since presentations aren't unique.)

<u>Proof</u>: With $\mathcal{D}$ as in Example 3.8, $S^2\mathcal{D} \simeq S\mathcal{D} + 1$.

Now $S^2 = R_oL_oR_oL_o$. Hence $L_oR_oL_o \neq L_o$ and so $L_oR_o \neq 1$.

<u>Proposition 3.11</u>: Let $Op_f(A)$ be the full subcategory of $\mathcal{C}at/A$ of operational categories with forgetful functor R_f. Then, there is an A such that R_f has no left adjoint.

<u>Proof</u>: Assume R_f has a left adjoint L_f. Then, for any category $\mathcal{D}$ over A, we have



where F and G are the universal functors (for notational convenience, the forgetful functors will often be ignored). So GF = id and F is a monomorphism. Consider the category $\mathcal{D}$ of Example 3.8. By Lemma (II.1.6), $[jf, \eta B, k]$ is a shuffle retract for $L_f\mathcal{D}$. Hence there is an object over A in $L_f\mathcal{D}$. Thus, $L_f\mathcal{D} \simeq L_o\mathcal{D}$ and $\eta_f \simeq \eta_o$.

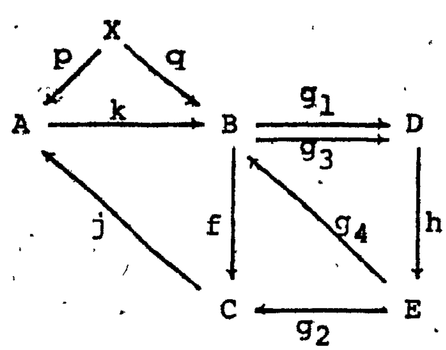
Now, $\eta_{\mathcal{D}o}$ equalizes $\eta_{oL\mathcal{D}}$ and $L\eta_{o\mathcal{D}}: L_o\mathcal{D} \to L_o^2\mathcal{D}$ and so $L_f\mathcal{D}$ fails to have the required universal property. Contradiction. Thus, R_f has no left adjoint.

<u>Proposition 3.12</u>: The operational categories and functors don't form a subcategory of $\mathcal{C}at/\mathcal{A}$.

<u>Proof</u>: A pair of operational functors

$$\mathcal{D}_1 \xrightarrow{G_1} \mathcal{D}_2 \xrightarrow{G_2} \mathcal{D}_3$$

are constructed whose composite G_2G_1 isn't operational. This is possible because G_1 and G_2 are operational with respect to two different presentations of $\mathcal{D}_2$. The example is based on a modification of Example 3.8. Let $\mathcal{A}$ be generated by the graph



and equations:

i) $jfk = id$

ii) $fkjf = g_2hg_1$

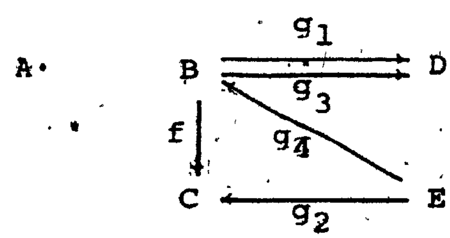
iii) $g_1k = g_3k$

iv) $jg_2 = jfg_4$

v) $g_4hg_3 = id$

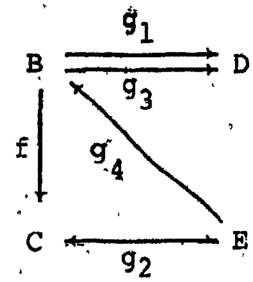
vi) $kp = q$

Let $\mathcal{D}_1$ be the subcategory generated by

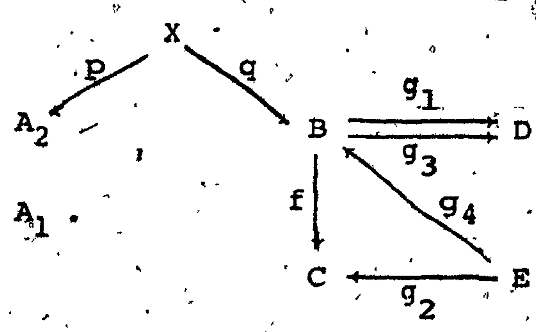


$S\mathcal{D}_1 = \mathcal{D}_1 + 1$ (where the 'new' object is $A_2 = [jf,B,k]$) by the same reasoning as in Example 3.8.

Let $\mathcal{D}_2 = S\mathcal{D}_1$. $G_1 = \eta_{\mathcal{D}_1}: \mathcal{D}_1 \longrightarrow S\mathcal{D}_1$ is operational since $\mathcal{D}_1 = S\mathcal{D}_o$, where $\mathcal{D}_o$ is the subcategory generated by



Let $\mathcal{D}_3$ be generated by



with $U_3A_1 = A = U_3A_2$.

$G_2 : \mathcal{D}_2 \longrightarrow \mathcal{D}_3$ is given by

$$G_2(id, A, id) = A_1$$

$$G_2(jf, B, k) = A_2$$

Now $\mathcal{D}_3$ and G_2 must be presented operationally. Let $\mathcal{C}$ be generated by

i) the graph and equations of $C\mathcal{D}_3$ and

ii) the equations

$$jf\omega_B k = \omega_{A_2}$$

$$jf\omega'_B k = \omega'_{A_2}$$

H is given by $A\times B_0 \to Q\mathcal{D}_3 \to C$. As before, the only non-trivial shuffle retract is $[jf,B,k]$. Clearly, the only shuffle homomorphism is $p:[id,X,id] \to [jf,B,k]$. Thus $\mathcal{D}_3$ is presented by (θ_0,H).

In $\mathcal{D}_2$, call $[id,A,id] = A_1$ and $[jf,B,k] = A_2$. Then, $G_1 = \eta_{\mathcal{D}}$ is operational through the presentation for $\mathcal{D}_2$ whose $\underline{S}$-algebra map sends $[jf,B,k]$ to A_1. By symmetry, $\mathcal{D}_2$ is also operational with respect to a presentation mapping $[jf,B,k]$ to A_2.

Now, since $CG_2:C\mathcal{D}_2 \to C\mathcal{D}_3$ preserves the equations for the 'second' presentation at $\mathcal{D}_2$, namely $jf\omega_B k = \omega_{A_2}$ etc. it induces a presentation for G_2 i.e. G_2 is operational.

G_2G_1 can never be an $\underline{S}$-homomorphism. This is because any structure map d for $\mathcal{D}_3$ must map $p:[id,X,id] \to [jf,B,k]$ to $p;X \to A_2$. Hence

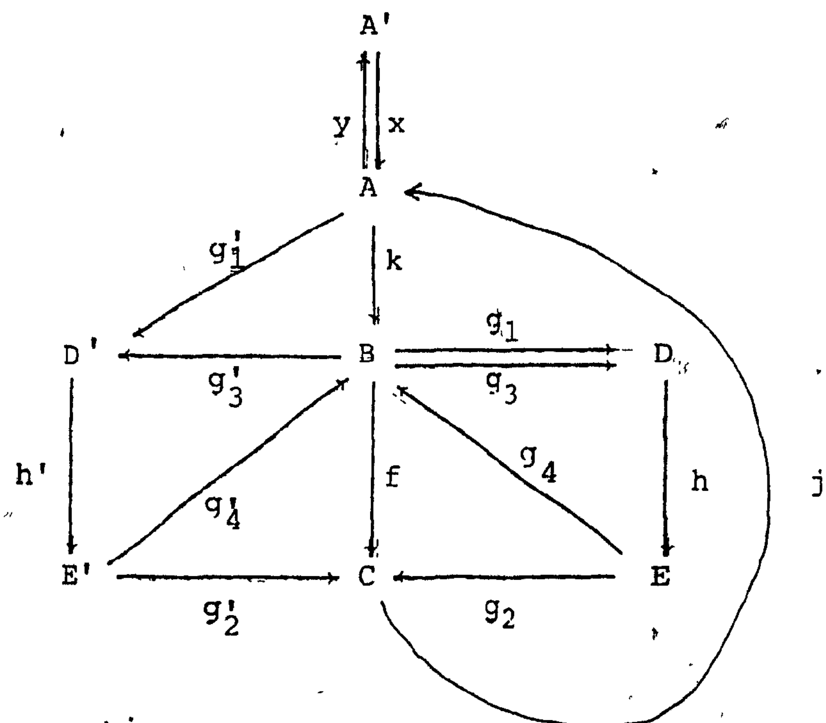
$$\begin{aligned} dS(G_2G_1)[jf,B,k] &= d[jf,B,k] \\ &= A_2 \end{aligned}$$

But

$$\begin{aligned} G_2G_1\, d[jf,B,k] &= G_2G_1A \\ &= A_1 \end{aligned}$$

<u>Proposition 3.13</u>: In general, $\underline{S}_2 \neq id$. Here, Example 3.8 will be extended to demonstrate this.

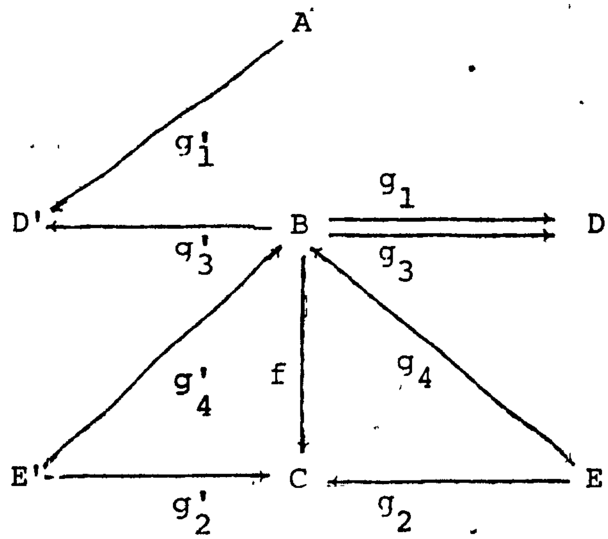
Let A be generated by the graph



and the equations:

i)	jfk	=	id	vi)	yx	=	id
ii)	$fkjf$	=	g_2hg_1	vii)	$fkxy$	=	$g_2'h'g_1'$
iii)	jg_2	=	jfg_4	viii)	jg_2'	=	jfg_4'
iv)	g_1k	=	g_3k	ix)	$g_1'x$	=	$g_3'kx$
v)	g_4hg_3	=	id	x)	$g_4'h'g_3'$	=	id

Let $U:\mathcal{D}\to\mathcal{A}$ be the inclusion generated by



Claim 1: All non-trivial identities are contained in the following list:

a) g_2hg_1, g_2hg_3, g_4hg_1, g_4hg_3, $g_2'h'g_1'$, $g_2'h'g_3'$, $g_4'h'g_1'$, $g_4'h'g_3'$, and all defined composites of these

b) jfk, all defined morphisms of the forms $jf\lambda k$ and $j\lambda k$ where λ is in a), and all composites of these. 3.1

c) yx, $y\lambda x$ where λ is in b), and all composites of these.

Proof: This is proved as in Example 3.8.

Hence, by splitting identities, the only possible shuffle retracts are (jf,B,k), (y,A,x), (yjf,B,kx).

<u>Claim 2</u>: If $(f_2,A,f_1) \equiv (f_2',X,f_1')$ then either $f_2 = f_2'$, $f_1 = f_1'$ and $X = A'$, <u>or</u> $f_2 = f_3 g_1'$.

<u>Proof</u>: The only morphism of $\mathcal{D}$ into or out of A is g_1'.

Hence, as before, [jf,B,k] is a non-trivial shuffle retract, since [id,A,id] can only be equivalent to itself in $Sh(\mathcal{D})$.

(y,A,xy,A,x) shuffles out iff the central 'x' is eliminated. This can only be done by applying vi) or vii). Now vi) cannot be employed since no 'y' can be introduced before (to the right of) this 'x'. vii) can only be used by introducing a jfk = id

$$(y,A,xy,A,x) = (y,A,jfkxy,A,x)$$
$$= (y,A,jg_2'h'g_1',A,x)$$

The central 'j' can only be eliminated by i) or ii). i) and ii) are innapplicable since no 'f' can be introduced before this 'j' except by vii), which is futile. So (y,A',x) isn't a shuffle retract. Also, consider (yjf,A,kxyjf,A,kx). The central 'j' can only be eliminated by applying i) or ii). To employ ii), i) must be used to provide the required 'fk'. But any application of i) introduces another unwanted 'j'. So (yjf,B,kx) isn't a shuffle retract either. Thus the only non-trivial shuffle retract in $S\mathcal{D}$ is $[jf,B,k]$.

By an argument parallel to that in Example 3.8, there are no non-trivial shuffle homomorphisms in $S\mathcal{D}$. Hence, $\mathcal{D}$ is an $\underline{S}$-algebra. However,

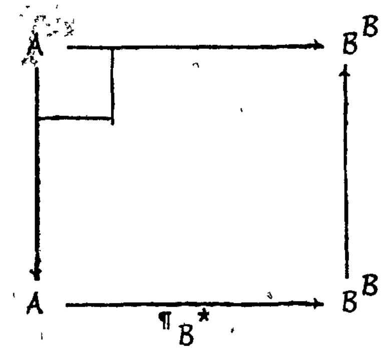
$$\begin{aligned}(y,A',xy,A',x) &\equiv_2' (yjf,B,kxy,A',x)\\ &\equiv' (yj,C,fkxy,A',x)\\ &= (yj,C,g_2'h'g_1',A',x)\\ &\equiv (yjg_2',D',h',C',g_1'x)\\ &= (yjfg_4',D',h',C',g_3'kx)\\ &\equiv (yjf,B,g_4'h'g_3',B,kx)\\ &= (yjf,B,id,B,kx)\end{aligned}$$

So $[y,A,x]$ is a 2-shuffle retract lying over X. But $\mathcal{D}$ has no objects lying over A', so $\mathcal{D}$ is not an $\underline{S}_2$-algebra and $\underline{S}_2$ is a non-identity triple. This example can be further extended to show that $\underline{S}_n \not\cong$ id.

Proposition 3.14: $R:Op(A) \to Cat/A$ never has a left adjoint.

Proof: Assume that R has a left adjoint L'. Given

$U:\mathcal{D} \to \mathcal{A}$, let $L\mathcal{D} = (L\mathcal{D}, \theta_{\mathcal{D}}, H_{\mathcal{D}})$ with $\theta_{\mathcal{D}}: \mathcal{B}_{\mathcal{D}} \to \mathcal{T}_{\mathcal{D}}$. Consider any (small) category $\mathcal{B}$. Then there is the operational category $(\mathcal{A}, 1_{\mathcal{B}}, \P_{\mathcal{B}})$



and the functor $U:\mathcal{D} \to \mathcal{A}$. Hence there is an operational functor $(G,j,k):(L\mathcal{D}, \theta_{\mathcal{D}}, H_{\mathcal{D}}) \to (\mathcal{A}, 1_{\mathcal{B}}, \P_{\mathcal{B}})$ such that (I.2.1) holds. Now, for each A in $\mathcal{A}$, $\P_{\mathcal{B}}^{*}A = 1_{\mathcal{B}}$ which is a monomorphism. Hence $j_1:\mathcal{B} \to \mathcal{B}_{\mathcal{D}}$ is a monomorphism. Since $\mathcal{B}$ was chosen arbitrarily, $\mathcal{B}_{\mathcal{D}}$ must be larger than any small category i.e. $\mathcal{B}_{\mathcal{D}}$ is large. Contradiction. Thus, R has no left adjoint.

For those familiar with Thiébaud's thesis, we present the following result.

<u>Proposition 3.15</u>: Consider Thiébaud's Structure-Semantics adjunction, Comonoids$(\mathcal{A}) \rightleftarrows \mathcal{C}at/\mathcal{A}$. This adjunction factors through that for $Op_0(\mathcal{A})$ with $K_1K_2 \simeq$ id.

$$\begin{array}{ccc} \text{Comonoids}(A) & \overset{K_1}{\underset{K_2}{\rightleftarrows}} & Op_o(A) \\ \text{Str} \dashv \text{Sem} & & L_o \dashv R_o \\ & Cat/A & \end{array}$$

<u>Proof</u>: Only a sketch of the proof will be given here; the functors between Comonoids(A) and $Op_o(A)$ will be defined, while the proofs of the details will be left to the reader.

A (bimodule) comonoid (or cotriple) $\underline{G}$ on A is a triplet (G,ε,δ) where G is a bimodule on A i.e. G is a functor $A^{op}\times A \to \mathcal{S}ets$ and $\varepsilon : G \to A = \text{Hom}_A$ and $\delta : G \to G\otimes G$ are natural transformations satisfying the usual kinds of cotriple equations.

The semantics functor sends $\underline{G}$ to its co-algebras. For exactly the same reasons that the co-algebras for a cotriple are operational with a given, standard presentation, those for $\underline{G}$ are too. Hence, the Semantics functor factors through R_o.

Conversely, given an operational category with standard presentation, $(\mathcal{D}, \theta_0, H)$ define an equivalence relation on the triplets (y, D, x)

$$X \xrightarrow{x} UD \xrightarrow{y} \underline{Y}$$

in A by $(y,D,x) \equiv_* (y',D',x')$ iff $y\Phi_D x = y'\Phi_{D'}x'$ with respect to the standard presentation. Equivalence classes are denoted $\{y,D,x\}$. Now define a comonoid $\underline{G}$ on A by

$$G(X,Y) = \{\{y,D,x\} \mid \text{dom } x = X,\ \text{cod } y = \underline{Y}\}$$

$$\varepsilon: G \longrightarrow A$$
$$[y,D,x] \longmapsto yx$$

$$\delta: G \longrightarrow G \otimes G$$
$$[y,D,x] \longmapsto [y,D,id_{UD}] \otimes [id_{UD},D,x]$$

Thus, there is a functor $Op_O(A) \to \text{Comonoids}(A)$ over Cat/A.

§4: Limits, Colimits and Equational Categories

Although Proposition 3.7 showed that B.T.C. doesn't imply operationality, under mild conditions the two concepts are equivalent.

Lemma 4.1:

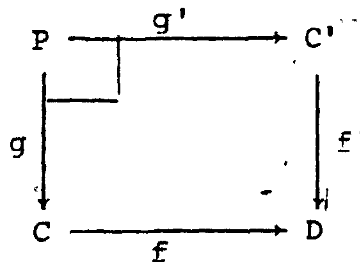
Let $U: \mathcal{D} \to A$ be faithful with $\mathcal{D}$ having and U preserving pullbacks. Then, if $(y,D,x) \equiv (y',D',x')$, it is so via a left-right shuffle equivalence i.e. a left shuffle followed by a right shuffle. Dually, if $\mathcal{D}$ has and U preserves pushouts, then all shuffle equivalences are achieved through right-left shuffles.

Proof:

Consider a right-left shuffle

$$\begin{aligned}(yUf,C,x) &\to (y,D,(Uf)x)\\ &= (y,D,(Uf')x')\\ &\to (yUf',C',x')\end{aligned}$$

Now consider the pullback in $\mathcal{D}$



Since U preserves pullbacks and $(Uf)x=(Uf')x'$, there is a z in $\mathcal{D}$ such that

$$x = (Ug)z$$
$$x' = (Ug')z$$

Hence

$$(yUf, C, x) = (yUf, C, (Ug)z)$$
$$\to (yU(fg), P, z)$$
$$= (yU(f'g'), P, z)$$
$$\to (yUf', C', (Ug')z)$$

Thus, any right-left shuffle can be converted to a left-right shuffle. Since right shuffles compose (as do left shuffles), this commutivity allows any sequence of left and right shuffles to be reduced to a left-right pair. The proof for the dual is straight forward.

Proposition 4.2:

Consider $U:\mathcal{D} \to A$ with $\mathcal{D}$ having, and U preserving pullbacks and assume U satisfies B.T.C.. Then $S\mathcal{D} \simeq \mathcal{D}$; in particular, $\mathcal{D}$ is operational. Dually, if $\mathcal{D}$ has and U preserves pushouts, and U satisfies the duals of B.T.C. then $S\mathcal{D} \simeq \mathcal{D}$.

Thus, under the conditions on U above we see, using Proposition I.1.6, that $\mathcal{D}$ is operational iff $S\mathcal{D} \simeq \mathcal{D}$ iff U satisfies B.T.C..

Proof: Given a shuffle system [y,D,x] we have (y,D,xy,D,x) shuffles out. By Lemma 4.1, this can be done by first making two shuffles to the left, over the right and left D's and then two to the right. Without loss of generality, the first shuffle on (xy,D,x) is unnecessary, since if it derives from x = (Uf)t then [y,D,x] ≡ [yUf,D',t] where f:D → D'. Thus the pattern of shuffles is as follows

$$
\begin{aligned}
(y,D,xy,D,x) &= (y,D,(Ug)t,D,x) \\
&\to (yUg,C,t,D,x) \\
&= (y'Uf,C,t,D,x) \\
&\to (y',D',(Uf)t,D,x) \\
&= (y',D',Uh,D,x) \\
&\to (y',D',id,D',(Uh)x)
\end{aligned}
$$

But Uh could also, at the last move, shuffle to the left! Applying Lemma 4.1, the set-up can be modified so that Uf.t=id. Thus

$$
\begin{aligned}
y &= yxy \\
&= y(Ug)t \\
&= y'(Uf)t \\
&= y'
\end{aligned}
$$

and the actual shuffle process is

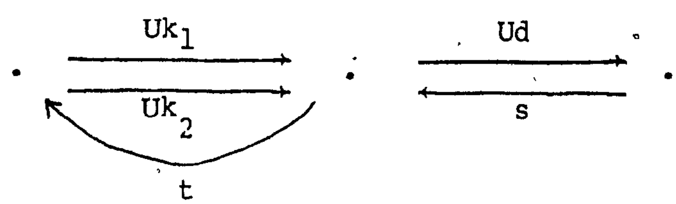
$$
\begin{aligned}
(y,D,xy,D,x) &= (y,D,(Ug)t,D,x) \\
&\to (yUg,C,t,D,x) \\
&= (yUf,C,t,D,x) \\
&\to (y,D,id,D,x)
\end{aligned}
$$

This system yields a U-split coequalizer (see (I.1.3)). Thus y=Uy' for some y' and [y,D,x] is a trivial shuffle retract. A shuffle homomorphism is an f; D → D' such that (f,D',id) ≡ (id,D,f). Applying Lemma 4.1 shows that we must have

$$
\begin{aligned}
(f,D',id) &= (f,D',(Ud)s) \\
&\to (fUd,D'',s) \\
&= (Ul,D'',s) \\
&\to (id,D,f)
\end{aligned}
$$

for some s, d, and l.

Now construct the kernel pair (k_1,k_2) of d in $\mathcal{D}$. U preserves this kernel pair by hypothesis and d equalizes the pair (id,sd). Thus, there is a t in $\mathcal{A}$ such that $k_1t=id$ and $k_2t=sd$. Putting all this together yields a U-split coequalizer



Thus, d is the coequalizer of its kernel pair. Now, 1 coequalizes k_1 and k_2. Thus there is an f' in $\mathcal{D}$ such that $f'd=1$. Hence $Uf'Ud=U1$. But $fUd=U1$ and d is an epimorphism. Thus $Uf'=f$ i.e. every shuffle homomorphism is a morphism of $\mathcal{D}$ and $S\mathcal{D} \approx \mathcal{D}$.

<u>Proposition 4.3</u>: If $\mathcal{D}$ is operational with respect to the presentation (θ,H) where the base functor $H:\mathcal{A}\times\mathcal{B}\longrightarrow\mathcal{C}$ is such that $\mathcal{A}$ and $\mathcal{C}$ have and H preserves a given class of limits e.g. finite products, pullbacks, finite limits, all limits, then U creates these limits. The dual results about colimits also hold.

<u>Proof</u>: The proof for finite products will be shown. The other proofs follow exactly the same pattern and the dual results follow from I.1.1. Let (X,Φ) and (X',Φ') be two algebras in $(\mathcal{D},\theta,H)$. Then $\Phi\times\Phi'$ is well-defined since $\mathcal{C}$ has

products. Thus $(X \times X', \Phi \times \Phi')$ is an algebra since

$$\begin{aligned}(\Phi\times\Phi')\theta &= \Phi\theta\times\Phi'\theta \\ &= H^*X\times H^*X' \\ &= H^*(X\times X')\end{aligned}$$

<u>Theorem 4.4</u>: Let $U: \mathcal{D} \to \mathcal{A}$ be equational. Then if $\mathcal{A}$ has some class of limits then U creates limits of that class. Further, if $\mathcal{A}$ has pullbacks then $S\mathcal{D} \simeq \mathcal{D}$. From this follows the well-known result that the equational categories over *Sets* (as a full subcategory of Op(*Sets*) lie over a full subcategory of *Cat*/*Sets* *
i.e. every functor over $\mathcal{A}$ between equational categories is operational.

<u>Proof</u>: Since *Sets* has all small limits and the Yoneda functor preserves any limits which exist in $\mathcal{A}$, the technique of Proposition 4.3 yields the first result. For the second, just apply Proposition 4.2.

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* Here *Sets* is a (small) category of sets inside a larger universe. *Cat* is the category of small categories in this universe.

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