A VARIATIONAL APPROACH
TO THE
EQUATIONS OF STELLAR STRUCTURE

by

Ivan Maurice Aron

Submitted in partial fulfillment of the requirements for the degree of Master of Science at McGill University

1954

CHAPTER I

THE EQUATIONS OF STELLAR STRUCTURE

The Observational Material

The fact that a star can only be studied through its light may seem, at first, to impose severe limitations on the knowledge to be acquired concerning the nature of these bright pinpoints on the night sky. When Mother Goose said

"Twinkle, twinkle little star,
How I wonder what you are",

she expressed not only the astronomer's interest in his subject, but also his continual battle with the atmosphere that introduces such uncertainty into his observations.

In spite of the difficulties observational astronomy is able to provide information on several quantities with a reasonable degree of accuracy. Three quantities which will concern us are the mass, the radius and the luminosity. The total mass of a star may be determined if it is a member of a well-observed binary system. For other stars masses are determined largely through inference, from relations to other observables as determined for binaries. The radius is less well defined than mass, since stars do not have an abrupt border. However, as density of stellar matter decreases, the properties change with respect to absorbtion and emission of light. The effect is to produce a reasonably sharp 'edge' with the uncertainty in the precise placing of this point only a negligible part of the radius itself. The sun appears to have a sharp cleavage between the chromosphere and the corona. Eclipsing variables generally show a very sharp drop in the light curve at the instant of totality. Again much of our knowledge of radii depends on binaries. Luminosity is defined as the total radiant energy in all wave-lengths emitted from the surface of the star per unit time.

Another observational feature of interest is the spectral class. The apparently bewildering differences in appearance of stellar spectra can be arranged in an orderly array of gradual changes. The system in its early (and still largely valid) form is the work of Miss A. J. Cannon and her co-workers and is described in detail in the preface to the Henry Draper catalogue (1). Further improvements and revisions are given by Morgan, Keenan and Kellman (2) and elsewhere(3).

The spectral sequence was explained by Saha (4) as an effect of surface temperature. If, from the spectral class or otherwise, we can assign a surface temperature to a star, then we have fixed the rate at which a square centimeter of surface radiates. If we know the luminosity of the star and the contribution to that luminosity of unit area, then the radius, can be determined. The luminosity itself affects the spectrum so that if the distance to the star is unknown, so that the apparent magnitude cannot be directly converted into absolute magnitude, the spectral effects still permit a reasonable estimate. The discussion by Kuiper (5) in 1938 fixed a temperature scale which has been revised and modified by Morgan and Keenan (6) on the basis of recent photoelectric colours.

Systematic studies of the luminosities and spectral classes were made by Hertzsprung in a series of papers starting in 1905 (7) and by Russell (8). The culmination of their work is the Hertzsprung-Russell diagram. The H-R diagram is a plot of observations of individual stars, the co-ordinates being spectral class and luminosity, or absolute magnitude. It is found that a large majority of stars fall in a well-defined band on the H-R diagram. This band, the Main Sequence, runs from bright stars of early spectral type to faint late ones in a rough diagonal. Midway up in brightness a branch springs from the Main Sequence, but separated, and runs roughly horizontal towards late types. (Fig. 1). This is called the Giant Branch. If we examine the stars in a globular cluster, we find that the H-R diagram is very different in appearance. (Fig. 2.)

Baade finds (9) that stars fall into two classes which he calls populations. These correspond to the two types of H-R diagrams. Various suggestions as to the reason for the differences have been made, principally differences in chemical constitution and in age (10).

The spectral classes correspond to differences in energy production per unit surface area. They are related to differences in luminosity or total energy production. Hence implicit in the H-R diagram is an empirical relation between Luminosity and Radius. If a study is made of stars whose masses are sufficiently well known, it is found that there is also a relation between Luminosity and Mass.

Any theory of stellar structure has as its goal to satisfy the empirical Luminosity-Mass-Radius relation.

The Differential Equations of Stellar Structure

A stellar model may be defined as the specification of the values of physical variables throughout a star. More particularly, we wish to know at any point in the star the temperature, pressure and density of matter, and the rate of variation of these quantities. We want to know certain properties of the matter, notably which form of Gas Law or Equation of State is valid. Is energy being generated at the point in question, and, if so, by what mechanism and at what rate? How is energy transported- by convection, conduction or radiation, or some combination? What is the nature and spectral distribution of the radiation? How is the radiation affected on passing through matter at this point? What is the chemical composition?

We answer some of these questions by making what appear to be reasonable assumptions. For example, we may decide that the energy generation is a certain function of temperature and density. These are the basic postulates of the model. With these postulates we solve the differential equations of stellar structure, either exactly or by a suitable approximation method. Our final goal is to determine the mass, radius and luminosity of the model for comparison with some specific star, or with the Hertzsprung-Russell diagram.

Let the variables be defined as follows:

r is the radius to point inside the star.

ρ is the density at the point.

$P(r)$ is the total pressure at the point.

$L(r)$ is the net rate of flow of radiant energy through the sphere of radius r.

$M(r)$ is the mass interior to radius r.

k is Boltzmann's constant.

H is the mass of the proton.

G is the gravitational constant.

c is the velocity of light.

μ is the mean molecular weight per particle in units of the proton mass.

Also let L, M, R be the surface values of $L(r), M(r), r$.

As for units, besides the standard c.g.s. units, it is often useful to express r, $L(r)$, $M(r)$ as fractions of their surface values; $P(r)$, $\rho(r)$ as fractions of their central values, and L, M, R in terms of the values for the Sun $L_\odot, M_\odot, R_\odot$.

We are to concern ourselves here with stars showing spherical symmetry and in a state of equilibrium. Everything happens along a radius vector. The star does not change while we examine it, and, indeed, is assumed not to have changed for a fairly long period preceeding our study. These assumptions characterize by far the greater portion of stellar studies to date.

Consider a star in hydrostatic equilibrium under its own gravitation. This means the pressure exerted outwards by the elastic properties of a small mass of gas exactly balances the force due to the weight of all the gas above it. If g is the value of gravity at r, then the change in pressure between the two sides of a thin slab of matter dr thick, of mass dm is given by

1.1 $$dP = -g\,dm = -g\rho\,dr$$

But by Newton's law

1.2 $$g = \frac{GM(r)}{r^2}$$

Therefore

1.3 $$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho$$

The mass of a spherical shell dr in thickness at radius r is the volume of the shell multiplied by the density.

1.4 $$\frac{dM(r)}{dr} = 4\pi r^2 \rho$$

1.3 and 1.4 are the first two equations of stellar structure. We shall find it useful to combine them. We differentiate 1.3 with respect to r and substitute 1.4

1.5 $$\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -4\pi r^2 \rho$$

The total pressure P is composed of two parts, the gas pressure P_g and the radiation pressure p_r.

1.6 $$P = p_g + p_r$$

The gas pressure depends on the equation of state. It can be shown (11 p. 600 ff) that for practically all cases of interest the correct equation of state is the Perfect Gas Law.

1.7 $$p_g = \frac{k}{\mu H}\rho T$$

The radiation pressure is given by the Stefan-Boltzmann Law.

1.8 $$p_r = \frac{1}{3}aT^4$$

where a is the radiation constant.

For stars of mass up to two or three times that of the Sun we can neglect p_r compared to p_g in 1.6 (11 p.603 ff.)

When radiation passes through a gas which absorbs part of it, the momentum of the radiation is decreased. It is this momentum transferred to the matter which is the radiation pressure. The fraction of the radiation absorbed by a mass $dm = \rho dr$ is proportional to dm and the constant of proportionality K is called the opacity coefficient. The momentum transferred is equal to the energy absorbed divided by the velocity of light. The pressure is the momentum per unit area, and the energy passing through unit area is the total energy through the sphere of radius r which is $L(r)$ divided by the surface area of the sphere.

The above is expressed as

1.9 $$-dp_r = \frac{L(r)}{4\pi r^2} \frac{K\rho\, dr}{c}$$

Radiation pressure decreases outwards.

If 1.8 and 1.9 are combined we may write

1.10 $$\frac{dT(r)}{dr} = \frac{-3K\, L(r)\, \rho}{16\pi a c r^2 T^3}$$

The luminosity will increase as we move outward through a thin layer if the layer contains sources of energy. If the energy generated per unit mass is ϵ then

1.11 $$dL(r) = 4\pi r^2 \epsilon \rho\, dr$$

1.10 and 1.11 are the remaining two equations of stellar structure. We must give some suitable specification to K and ϵ.

A very useful form in which to put the opacity is known as Kramers' Law or rule. The opacity coefficient varies with wave-length. Rosseland defined the proper mean opacity coefficient which will be referred to as the opacity κ (11 p. 611)

1.12 $$\kappa = \kappa_0 \frac{\rho}{T^{3.5}}$$

This is an approximate form obtained by use of the correspondence principle of quantum mechanics. Corrections to 1.11 are often expressed by multiplying the right hand side by a quantity τ , known as the guillotine factor, because its principal purpose is to correct for the sharp cut-off at absorbtion edges. It is obtained by comparison with an opacity table constructed with better approximations to atomic absorbtion mechanisms. The guillotine factor often includes another correction called the Gaunt Factor.

Study of opacity tables reveals that the guillotine factor for a particular model can be well represented by

1.13 $$\tau \propto \frac{T^{\eta}}{\rho^{\sigma}}$$

This depends partly on the manner in which ρ and T vary together in a star. Entering a table of opacities which is arranged by density and temperature, we would take a sort of diagonal path across the table to find the entries of interest at different levels. This series of values is well represented by the guillotine factor 1.13.

Combining 1.13 with 1.12

1.14 $$K = K_0 \frac{\rho^{1-\sigma}}{T^{3.5-\eta}}$$

A useful expression is derived if we insert 1.14 into 1.10, take the derivative of the resulting expression with respect to r and finally substitute from 1.11

1.15 $$\frac{d}{dr}\left[\frac{r^2}{\rho^{2-\sigma}} \frac{dT^{7.5-\eta}}{dr}\right] = \frac{-3(7.5-\eta) K_0 r^2 \rho \epsilon}{4ac}$$

The generation of energy is by atomic fusion with the compound nucleus formed being less massive than the total of the original masses. This mass defect appears as energy. There are currently two well-studied processes known and others are being investigated. The general form of the energy generation equation is

1.16 $$\epsilon = \epsilon_0 \rho^{\xi} T^n$$

where ϵ_0, ξ and n depend on the particular process.

CHAPTER II

CHANGING VIEWS OF THE SUN

The history of the study of stellar structure up to the end of 1938 forms almost a complete unit, as much as any subject constantly under study can be said to divide itself into stages. Chandrasekhar's Monograph (12) published early in 1939 was an impressive summing up. A broad picture was presented, leaving the reader with the feeling of a field not completely explored, but with the major outlines drawn. It is perhaps only to be expected that Chandrasekhar would apply himself dilligently to rendering the book obsolete as quickly as possible. Others have, independently, assisted him.

In 1951 Chandrasekhar contributed the chapter on Stellar Structure to a volume commemorating the fiftieth anniversary of the Yerkes Observatory. A comparison of the two treatments is very instructive, particularly with regard to how a field of science may change in just over a decade. The basic equations are the same. Certain arguments are repeated almost verbatim. Gone completely, however, are the detailed examinations of the behaviour of Polytrope solutions. Instead there is everywhere the mark of new knowledge of the atom and its properties. Most notable in its effect is the still-growing understanding of the mechanisms of energy generation.

The proposal, by Bethe and Critchfield (13), of a mechanism whereby four protons could fuse to become one alpha particle with release of energy, was made before the war, in 1939. It was not until after the war that information was available on which reasonably precise calculations could be based. Further changes may be expected, partly from new investigations and partly when the "lead curtain", which hinders radiation of information at the time of writing, is lifted. The work of Salpeter (14) may be noted for its effect on the results of Epstein and Motz.

Cowling (15) had studied the problem of convective stability. Consider a small element of matter in the interior of a star which suddenly increases slightly in temperature due to some natural fluctuation. The heating causes it to expand, and, being now less in density than the surrounding matter, will tend to rise. As it rises, the pressure of the surrounding gases decreases, and there is further expansion. This expansion cools the element. If the process takes place with negligible transfer of heat to or from the surroundings, we have very close to an adiabatic expansion and cooling. If the temperature gradient of the surrounding gases is small enough, the element in time reaches the same temperature and pressure as the layer it is in, and the eddy dies out.

If however, the expanding element cannot cool itself fast enough compared with the surrounding matter, the eddy persists, and will increase. This instability leads to the existence of a convective zone. The radiative gradient predominates provided it is smaller than the adiabatic gradient. If the radiative gradient is greater than adiabatic, a convective zone results. We express the condition for stability as

$$\left(\frac{dT}{T}\right)_{ad.} > \left(\frac{dT}{T}\right)_{rad.} \tag{2.1}$$

Cowling assumed that energy generation took place at a rate depending on some, as yet unknown, power of the temperature. He found that if this exponent was greater than a " number between 6 and 7", the star must have a central region in convective equilibrium.

Naturally when Bethe and Critchfield proposed their Carbon-Nitrogen cycle, dependent on the seventeenth or eighteenth power of temperature, convective cores were assumed for all models. It was true that an alternative reaction existed, the proton-proton mechanism, dependent on the fourth power of temperature. It was clear, however, that in a reasonably hot stellar interior, the Carbon-Nitrogen cycle would provide all but a negligible portion of the energy. Only the faintest of stars, the red dwarfs, were expected to generate their energy by the proton-proton reaction.

Epstein (16a) sought to refine solar models by taking into account both energy sources to see if the sun did have any appreciable source of luminosity from the proton-proton reaction. The results were unexpected. The sun, Epstein found, gets almost all its energy generation in this fashion. The carbon-nitrogen cycle plays a minor role. Using Salpeter's energy generation formula, Epstein and Motz recalculated the model (16b). The results are similar, but the composition is almost pure hydrogen which is not reasonable for a star which has been burning hydrogen into helium for 4,000,000,000 years or more.

Epstein and Motz believed the difficulty with the second model was in the opacity figures. These depend on the composition, which is usually determined only after the model is completed. Accordingly they calculated a third model. This was treated as a double eigen value problem. A model is calculated with assumed composition and opacity. The composition is then determined for the resulting model by comparing it with the sun for mass, luminosity and radius. The new composition is used for a new model. An iterative process is set up which leads to a final model whose initially assumed and finally determined composition and opacity agree. In brief, this is a self-consistent model.

This model (16c) contains 93.1% hydrogen, 6.7% helium, and 0.2% heavier elements. This is a reasonable composition. The proton-proton reaction is found to provide all by a negligible fraction of the energy. Consequently, the convective core would be small if it exists at all. Epstein and Motz find the core occupys about 8% of the solar radius and includes 4% of the mass. Only 29% of the energy is generated inside the core.

The temperature and density distributions given by Epstein and Motz will be of importance further on in this paper. They are plotted in Figure 3.

Opacity calculations are also affected by the newer knowledge. Estimates of the probable composition of stellar gas mixtures have changed considerably, and better wave-functions are known. The opacity tables published before and during the war are badly outdated. Keller and Meyerott have undertaken the computation of an extensive series of opacity tables. They have published their preliminary investigations into the factors which will affect the results, such as electron screening and occupation numbers of the bound levels in ions. (17). Preliminary tables are being readied for publication which will omit some of the refinements to be ultimately included.

CHAPTER III

REMARKS ON THE CALCULUS OF VARIATIONS

The Original Problem

The Calculus of Variations came into being as a method of handling a certain class of problems. Here are a few typical examples.

a. The Isoperimetric Problem: Find a closed curve of given perimeter and greatest area.

b. The Brachistochrone: Two points, one higher than the other are to be connected by a curve along which a frictionless mass point moves, under the acceleration of gravity, in the shortest possible time. This is the problem which gave rise to the calculus of variations. It was treated by Jakob Bernouilli in 1696.

c. Least Surface of Revolution: The curve $y = y(x) \geq 0$ is revolved about the x axis. The resulting surface has as its ends two circles of fixed radius in the planes $x = x_0$ and $x = x_1$. What is the curve if the surface of revolution is to be the smallest possible. This least surface is the one formed if two wire circles held the proper distance apart coaxially are dipped in a soap solution and withdrawn with a soap film joining them. The film over one of the circles must be pierced so that air pressure is the same on both sides.

Each of these problems can be stated in the form of a definite integral to be minimized subject to some sort of restraining condition. The third problem, for example is equivalent to finding the minimum value of the surface area

3.1 $$A = 2\pi \int_{x_0}^{x_1} y\left(1+\left\{\frac{dy}{dx}\right\}^2\right)^{1/2} dx$$

subject to

3.2 $$y(x_0) = a_0 \qquad y(x_1) = a_2$$

The Variation

The problem, then, is to minimize the integral

3.3 $$J[y] = \int_{x_0}^{x_1} F(x, y, y')\,dx$$

where the values x_0, x_1, $y(x_0)$, $y(x_1)$ are given. F is twice continuously differentiable with respect to its three arguments. The second derivative of y is also assumed continuous. Let $y = f(x)$ be the desired function giving the minimum. That is, in a sufficiently small neighborhood of the function f(x), the integral J(y) is smallest when $y = f(x)$.

Let $\bar{y} = y + \epsilon\eta(x)$ where $\eta(x)$ is a function with a continuous second derivative defined in the neighborhood of y(x). $\eta(x)$ Vanishes at $x = x_0$, $x = x_1$ but is otherwise arbitrary. ϵ is a small parameter. The quantity $\delta y = \epsilon\eta(x)$ is known as the variation of the function y.

The integral $J[\bar{y}] = J[y + \epsilon\eta]$ may be regarded as a function $\Phi(\epsilon)$ which has a minimum at $\epsilon = 0$ relative to all values of Φ in a sufficiently small neighborhood of $\epsilon = 0$, and therefore $\Phi'(0) = 0$. It is permissible to differentiate the integral

$$3.4 \qquad \Phi(\epsilon) \int_{x_0}^{x_1} F(x, y + \epsilon\eta, y' + \epsilon\eta')\, dx$$

under the integral sign. Then a necessary condition for a minimum is

$$3.5 \qquad \Phi'(0) = \int_{x_0}^{x_1} \left(\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'}\right) dx = 0$$

which holds for all functions $\eta(x)$ satisfying the requirements.

We transform 3.5 by performing a partial integration on the second part of the integral, noting that $\eta(x_0) = \eta(x_1) = 0$

3.6 $$\int_{x_0}^{x_1} \eta \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] dx = 0$$

which is valid of every one of our functions η . Since the functions η are arbitrary, except for the end points, 3.6 can hold only if

3.7 $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Naturally the vanishing of a first derivative does not necessarily signify a minimum, but rather an extremum. Most of the problems of the variational calculus are minimum problems, however.

We shall introduce the notation $\delta J = \epsilon \Phi(0)$, the first variation of J. Then it may be shown that δ is an operator with the following properties.

δ operates through an integral sign.

δ operates on functions exactly as the differential operator d does

δ may be restricted to operate on only one of the variables in an integrand leaving the rest unchanged, and does not operate on the variable of integration.

If variation is taken with respect to the variable z, then $\delta z = 0$ at both the limits of integration. δ commutes with d.

If an integral I is set equal to a constant, then $\delta I = 0$.

The general aim of techniques as described above is to derive a differential equation such as 3.7 by minimizing an integral of the form 3.3 where additional variables and derivatives of variables may be contained in the argument of F. The technique is to set the variation with respect to a particular variable z of the integral equal to zero, and then, by partial integration, transform the integrand into a differential expression multiplied by the variation of z. Then since δz is arbitrary, the differential expression must vanish, which is the desired differential equation.

Subsidiary Conditions

The variational calculus goes beyond the problem of minimizing integrals between fixed end points. There often exist one or more restraining conditions.

Consider for example the problem of the shape of a uniform string hanging under the influence of gravity between two fixed points. The integral to be minimized is that which gives the height of the centre of gravity of the string above some reference plane. But if the string is not elastic, then its total length, expressible as an integral in arc-lengths, is constant. A subsidiary condition of this form, expressible as the fact that a certain integral remains constant is called a Holonomic Constraint. Other constraints not expressible as integrals are called Nonholonomic. We are not here concerned with these.

We may write the integral to be minimized and the holonomic condition, respectively, as

3.8 $$J = \int_{x_0}^{x_1} F(x, y, y')\,dx \qquad K = \int_{x_0}^{x_1} G(x, y, y')\,dx = C$$

The condition for a minimum is $\delta J = 0$ Since K is a constant we have $\delta K = 0$ always. Then if λ is some arbitrary constant we have

3.9 $$\delta J + \lambda \delta K = 0$$

It can be shown that this condition is equivalent to writing 3.7 with $F^* = F + \lambda K$. The value of λ and of the constants of integration determined from the boundary conditions and from $K = C$. λ is known as a Lagrange Multiplier.

Many fine detailed treatments of the methods of the calculus of variations exist. That of Courant and Hilbert (18) quite extensive.

The Inverse Problem

The manner in which the calculus of variations leads from a problem in an integral to be minimized to a differential equation raises the inverse problem: Can a given differential equation be profitably treated as an integral to be minimized subject to a constraint? Unfortunately no general answer can be given. Most theorems are conditions of necessity, and sufficiency conditions are rare. While some techniques exist which will aid in attacking a particular problem, success is largely a matter of good fortune and ingenuity. This aspect of the calculus of variations may be called "more of an Art than a Science."

One sufficiency condition which will be of importance to us in the next chapter, is quoted by Courant and Hilbert (p.257) from Bolza. For a given ordinary second order differential equation $y'' = f(x, y, y')$ one can always find functions F and G and a Lagrange multiplier λ such that $F^* = F + \lambda G$ will make 3.7 when solved for y'' identical with the differential equation.

Let us assume, however, that we have been successful in obtaining the explicit form of the two integrals in 3.8. We now proceed to find an approximate solution of the differential equation by the method of trial functions. A trial function is a function which is thought to be a close approximation of the actual solution of the equation and which contains a number of undetermined parameters. A polynomial expansion

3.10 $$y = a_0 + a_1 x + a_2 x^2 + \cdots\cdots a_n x^n$$

is one example.

The trial function is chosen so that when it and its first derivatives are substituted in the integrands 3.8, the integration may be carried out in closed form. We reason that if the differential equation is derived by minimizing J with K constant, then the values of the parameters in the trial function which will make the trial function most closely fit the exact solution are to be obtained by the same process.

We write J', K' to indicate that the trial function has been substituted and the integration carried out.

3.11 $$J' = J(a_0, a_1, \cdots\cdots a_n) \qquad K' = K(a_0, a_1, \cdots\cdots a_n)$$

3.12 $$\delta J' + \lambda \delta K' = 0$$

The Lagrange multiplier λ is known. Recalling that δ has the properties of the differential operator we have

3.13 $$\frac{\partial J'}{\partial a_0}\delta a_0 + \frac{\partial J'}{\partial a_1}\delta a_1 + \cdots\cdots \frac{\partial J'}{\partial a_n}\delta a_n + \lambda\left(\frac{\partial K'}{\partial a_0}\delta a_0 + \frac{\partial K'}{\partial a_1}\delta a_1 + \cdots\cdots \frac{\partial K'}{\partial a_n}\delta a_n\right) = 0$$

or

3.14 $$\left(\frac{\partial J'}{\partial a_0} + \lambda \frac{\partial K'}{\partial a_0}\right)\delta a_0 + \left(\frac{\partial J'}{\partial a_1} + \lambda \frac{\partial K'}{\partial a_1}\right)\delta a_1 + \cdots$$
$$\cdots + \left(\frac{\partial J'}{\partial a_n} + \lambda \frac{\partial K'}{\partial a_n}\right)\delta a_n = 0$$

The variations $\delta a_0, \delta a_1, \cdots\cdots \delta a_n$ are completely independent of each other. Therefore

3.15
$$\frac{\partial J'}{\partial a_0} + \lambda \frac{\partial K'}{\partial a_0} = 0$$
$$\cdots\cdots\cdots\cdots$$
$$\frac{\partial J'}{\partial a_n} + \lambda \frac{\partial K'}{\partial a_n} = 0$$

3.15 is a set of n equations in the n parameters. When they are solved, the trial function with these parameters is the variational approximation to the solution of the differential equation.

Clearly much depends on the choice of a trial function which is capable of close approximation to the actual solution. The published numerical integrations of stellar models are very useful attacking the particular problem of this investigation.

CHAPTER IV

THE VARIATIONAL FORM OF THE EQUATIONS

Introductory

At the beginning of the present investigation it was hoped that a single variational integral might be found which would generate all four of the stellar structure differential equations (1.2, 1.4, 1.10 and 1.11), or the two equivalent second order differential equations (1.5 and 1.15). Considerable labour was expended in the attempt without success. The lack of sufficiency conditions makes a definite proof of impossibility a problem which must be solved almost from first principles, an effort which did not seem worthwhile.

Let us turn our attention to the two second order equations 1.5 and 1.15, namely.

4.1 $$\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -4\pi G r^2 \rho$$

4.2 $$\frac{d}{dr}\left[\frac{r^2}{\rho^{2-\sigma}}\frac{dT^{7.5-\eta}}{dr}\right] = \frac{-3(7.5-\eta)K_0\epsilon_0 r^2\rho^2 T^n}{4ac}$$

These are both of the form $y'' = f(r, y', \rho)$

In each case ρ is related to y, so essentially $y'' = f(r, y, y')$. Thus we can find an integral (possibly with a restraining condition) which will, on variation, lead to the required differential equation.

For convenience we shall differentiate between the two equations and their respective families of integrals, derivatives, consequent algebraic equations etc., by referring to those arising from 4.1 as the Hydrostatic set and those from 4.2 as the Energy set.

The Hydrostatic Set

Consider the integrals

4.3 $$H = \frac{1}{2}\int_a^b \frac{r^2}{\rho}\left[\frac{dP}{dr}\right]^2 dr \qquad \text{and} \qquad I = \int_a^b \rho P r^2 dr = 1$$

Let P be the variable. The variational operator is not to operate on ρ.

4.4 $$I = 1 \qquad \therefore \delta I = 0$$

If C_1 is a Lagrange multiplier, then

4.5 $$\delta H + c_1 \delta I = 0$$

$$\delta\left[\frac{1}{2}\int_a^b \frac{r^2}{\rho}\left(\frac{dP}{dr}\right)^2 dr\right] + c_1\,\delta\left[\int_a^b \rho P r^2 dr\right] = 0$$

4.6 $$\int_a^b \frac{r^2}{\rho}\frac{dP}{dr}\frac{d}{dr}(\delta P)\,dr + c_1\int_a^b \rho(\delta P) r^2 dr = 0$$

The first term of 4.6 is integrated by parts to give

4.7 $$\frac{r^2}{\rho}\frac{dP}{dr}\,\delta P\Big|_a^b - \int_a^b \frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right]\delta P\,dr$$

The integrated part vanishes since $\delta P = 0$ at both limits. Hence 4.6 becomes

4.8 $$-\int_a^b \frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] \delta P\, dr + c_1 \int_a^b \rho r^2\, \delta P\, dr = 0$$

or

4.9 $$\int_a^b \left\{\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] - c_1 \rho r^2\right\} \delta P\, dr = 0$$

Since δP is arbitrary, the bracketed part of the integrand must vanish. But this is identical with 4.1 provided

4.10 $$c_1 = -4\pi G$$

Multiply 4.1 by P and integrate between a and b

4.11 $$\int_a^b P \frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] dr = -4\pi G \int_a^b \rho P r^2\, dr$$

The right hand integral is $I = 1$ from 4.3. Partial integration of the left hand side gives

4.12 $$\frac{Pr^2}{\rho}\frac{dP}{dr}\Big|_a^b - \int_a^b \frac{r^2}{\rho}\left[\frac{dP}{dr}\right]^2 dr = \frac{Pr^2}{\rho}\frac{dP}{dr}\Big|_a^b - 2H$$

In the right hand side of 4.12 the integrated part will vanish in many cases. If the upper limit is the boundary of the star, then densities and pressures are low enough for the perfect gas law to operate and $P/\rho \propto T$. Hence the term vanishes at an upper limit corresponding to the boundary. If the lower limit is the centre of the star the term vanishes there. Otherwise the term could hardly be expected to vanish at the lower limit in any real star. It is unlikely that the term would vanish for non-vanishing upper and lower limits.

The relation resulting from 4.11 and 4.12 when the integrated part vanishes (e.g. for a star in radiative equilibrium throughout) is

4.13 $$H = 2\pi G$$

Care must be taken regarding the use of 4.13 in evaluating constants after the process of minimizing is complete. In both the examples to follow it will be noted that the normalizing condition on I together with 4.10 and 4.13 are equivalent to one of the minimizing equations.

If the star is not in radiative equilibrium throughout, we have

4.14 $$H = 2\pi G - \frac{1}{2}\frac{P_r r^2}{\rho}\frac{dP}{dr}\Big|_a^b$$

For a radiative envelope fitted on to a convective core, the last term in 4.14 may be evaluated and the expression becomes a useful fitting condition.

We are to evaluate H and I after inserting our chosen trial functions $\rho(r)$ and $P(r)$. Minimizing the results subject to the condition that $\rho(r)$ is not to be varied, which is implicit in the derivation above, we obtain certain relations between our parameters. Then 4.13 will prove useful in disentangling combinations of variables in these relations. Finally, with our trial functions now containing some of the parameters in numerical form we may evaluate those remaining from the boundary conditions, namely, by integrating 1. 4 and 1. 11 to obtain L and M.

Examples are given following the derivation of the Energy Set.

The Energy Set

We shall find it convenient to introduce new variables into 4.2

4.15 $$y = T^{7.5-\eta} \qquad \text{so} \qquad T^{\eta} = y^{\frac{2n}{15-\eta}} = y^{\lambda-1}$$

4.16 $$c' = \frac{-3(7.5-\eta)\,K_0\,\epsilon_0}{4ac} \qquad \text{and} \qquad \lambda = \frac{2n}{15-\eta} + 1$$

Then we have 4.2 in the form

4.17 $$\frac{d}{dr}\left[\frac{r^2}{\rho^{2-\sigma}}\frac{dy}{dr}\right] = c' r^2 \rho^2 y^{\lambda-1}$$

Consider the integrals

4.18 $$J = \frac{1}{2}\int_a^b \frac{r^2}{\rho^{2-\sigma}}\left[\frac{dy}{dr}\right]^2 dr \qquad \text{and} \qquad K = \int_a^b r^2 \rho^2 y^{\lambda} dr = 1$$

The variable is now y and, again, δ does not operate on ρ.

4.19 $$K = 1 \,, \quad \therefore \quad \delta K = 0$$

Let C_2 be the Lagrange multiplier.

4.20 $$\delta\left\{\frac{1}{2}\int_a^b \frac{r^2}{\rho^{2-\sigma}}\left[\frac{dy}{dr}\right]^2 dr\right\} + C_2\,\delta\left\{\int_a^b r^2\rho^2 y^2\,dr\right\} = 0$$

4.21 $$\int_a^b \frac{r^2}{\rho^{2-\sigma}}\frac{dy}{dr}\frac{d}{dr}(\delta y)\,dr + C_2\lambda\int_a^b r^2\rho^2 y^{\lambda-1}\,\delta y\,dr = 0$$

Integrating the first term of 4.21 by parts gives

4.22 $$\frac{r^2}{\rho^{2-\sigma}}\frac{dy}{dr}\,\delta y\Big|_a^b - \int_a^b \frac{d}{dr}\left[\frac{r^2}{\rho^{2-\sigma}}\frac{dy}{dr}\right]\delta y\,dr$$

As before the integrated part vanishes at both limits where $\delta y = 0$

4.23 $$\int_a^b \left\{\frac{d}{dr}\left[\frac{r^2}{\rho^{2-\sigma}}\frac{dy}{dr}\right] - \lambda C_2 r^2\rho^2 y^{\lambda-1}\right\}\delta y\,dr = 0$$

The bracketed part of the integrand must vanish. This is identical with 4.17 provided

4.24 $$C' = \lambda C_2$$

Multiply 4.17 by y and integrate between a and b.

4.25 $$\int_a^b y\frac{d}{dr}\left[\frac{r^2}{\rho^{2-\sigma}}\frac{dy}{dr}\right]dr = C'\int_a^b r^2\rho^2 y^\lambda\,dr$$

The right hand integral is $K=1$ from 4.18.
Partial integration of the left hand side gives

$$4.26 \qquad \frac{r^2 y}{\rho^{2-\sigma}} \frac{dy}{dr}\Big|_a^b - \int_a^b \frac{r^2}{\rho^{2-\sigma}} \left[\frac{dy}{dr}\right]^2 dr = \frac{r^2 y}{\rho^{2-\sigma}} \frac{dy}{dr}\Big|_a^b - 2J$$

The integrated part can be transformed with the aid of 1.10 , 1.14, and 4.15.

$$4.27 \qquad \frac{r^2 y}{\rho^{2-\sigma}} \frac{y\,dy}{dr} = \frac{-3(7.5-n)K_0 L(r)}{4\pi ac} y$$

The right hand side vanishes at the boundary, the left hand side will insure vanishing at the centre of a star even if a point source model should make $L(0) \neq 0$.

The discussion following 4.12 now applies here with obvious modifications. The analogues of 4.13 and 4.14 are

$$4.28 \qquad 2J = \frac{r^2 y}{\rho^{2-\sigma}} \frac{dy}{dr}\Big|_a^b - c'$$

CHAPTER V

A SOLAR MODEL: FIRST APPROXIMATION

The Choice of a Trial Function

An examination of graphs of the temperature and density distributions of solar models, and of stars of less than solar mass presents a striking resemblance to the "bell-shaped" curve of the Gauss error function. (11.p.629) The values given by Epstein and Motz (16c) were plotted and the impression was confirmed. This particular model was chosen because it seemed to give a truer picture than earlier models based solely on the carbon cycle, and the conclusion that convective core is small makes it unimportant in a first approximation variational treatment.

The plots made were of log r vs log log $\frac{\rho_c}{\rho}$ and log r vs log log $\frac{T_c}{T}$ (Fig. 3). The density curve is almost a straight line curving slightly at the upper end. A straight line in these coordinates indicates exponential variation with a power of r given by the slope. The temperature line is more curved throughout, and the slope is generally less than for the density.

Measuring the slopes of the lines, it is seen that

5.1 $$\rho = \rho_c e^{-\alpha r^2}$$

is a very good approximation to the density distribution. The function

$$5.2 \qquad T = T_c e^{-\beta r^2}$$

is not so good a fit, but will serve as a first approximation.

Accordingly 5.1 and 5.2 are chosen as trial functions with $\alpha, \beta, \rho_c, T_c$ as parameters to be determined. The equation of state is the perfect gas law, hence from 5.1 and 5.2 we obtain

$$5.3 \qquad P = P_c e^{-(\alpha+\beta)r^2} = P_c e^{-\gamma r^2}$$

The substitution $\gamma = \alpha + \beta$ is made so that the parameters in P may easily be kept distinct from those in ρ. The former are to be varied, the latter kept constant.

The opacity is treated as Kramer's law without guillotine factor as in 1.12 which is

$$5.4 \qquad K = K_0 \frac{\rho}{T^{3.5}}$$

The energy is generated by the proton-proton reaction.

$$5.5 \qquad \epsilon = \epsilon_0 \rho T^4$$

Then in the defining equations for variables (4.15 and 4.16), for J and K (4.18) and in the differential equation (4.17), we have

5.6 $\eta = \sigma = 0 \qquad n = 4 \qquad \lambda = \frac{23}{15}$

We shall also use

5.7 $\nu = n + 7.5 = \frac{15}{2}\lambda = \frac{23}{2}$

Concerning Infinite Stars

A star is generally regarded as a physical entity with definite bounds. Therefore solutions in which pressure, density and temperature vanish only at infinity are regarded with suspicion. They are 'limiting cases' considered of interest only as indicating the limit of a family of finite-radius solutions. Chandrasekhar's chapter on polytropes (12) contains an example of this view. If, however, the physical variables fall off together and vanish assymtotically, the model may not be objectionable.

At some point the density is so low that interatomic spacing is very large, and the equations of stellar structure no longer hold for such rarified material. Assume, next, that by the time the density has become this small, the temperature and pressure are also negligible compared with their values deep in the interior. Also assume that energy generation does not occur in the outer layers; a reasonable assumption since the temperature is low. Then, if the density decreases sufficiently more rapidly than the radius increases, the mass of the star will be finite, even if the extent is infinite.

Under the above conditions there should be no objection to an infinite-radius stellar model regarded as an idealization. Every stellar model is idealized to some extent, frequently to a very great degree.

The functions 5.1, 5.2 and 5.3 meet the requirements. The mass is finite, and the difference in mass between the infinite star and one of this form, cut short at the sun's radius, is negligible. The Epstein-Motz results show that the convective core is small, and the difference between their convective core solution and the Gauss error curve at small radii is also negligible.

To the above arguments, let us add the well-known beautiful behavior of error-curve integrals when the limits are 0 and ∞ . These limits are accordingly adopted.

<u>The Hydrostatic Set</u>

5.8 $$P = P_c e^{-\gamma r^2} \qquad \rho = \rho_c e^{-\alpha r^2} \qquad \frac{dP}{dr} = -2\gamma r P_c e^{-\gamma r^2}$$

5.9 $$H = \frac{1}{2}\int_0^\infty \frac{r^2}{\rho}\left[\frac{dP}{dr}\right]^2 dr = \frac{2P_c^2\gamma^2}{\rho_c}\int_0^\infty r^4 e^{-(2\gamma-\alpha)r^2} dr$$
$$= \frac{3\sqrt{\pi}}{4}\frac{P_c^2}{\rho_c}\frac{\gamma^2}{(2\gamma-\alpha)^{5/2}}$$

It is necessary that

5.10 $$2\gamma \geqslant \alpha \geqslant 0$$

for this integral to exist.

$$I = \int_0^\infty \rho P r^2 dr = \rho_c P_c \int_0^\infty r^2 e^{-(\gamma-\alpha)r^2} dr$$

5.11
$$= \frac{\rho_c P_c \sqrt{\pi}}{4(\gamma+\alpha)^{3/2}}$$

Constants and combinations of variable may now be conveniently grouped. The definitions below are chosen to agree with those in the second approximation.

5.12
$$A = \frac{3\sqrt{\pi}}{16} \frac{P_c^2}{\rho_c} \qquad \theta = 2\alpha$$

$$B = \frac{P_c \rho_c \sqrt{\pi}}{8}$$

Then

5.13
$$H = \frac{2^{9/2} A \gamma^2}{(4\gamma-\theta)^{5/2}} \quad \text{and} \quad I = \frac{2^{5/2} B}{(2\gamma+\theta)^{3/2}}$$

Taking derivatives

5.14
$$\frac{\partial H}{\partial \gamma} = \frac{-2^{11/2} A \gamma(\gamma+\theta)}{(4\gamma-\theta)^{7/2}}$$

5.15
$$\frac{\partial I}{\partial \gamma} = \frac{-2^{5/2} \cdot 3B}{(2\gamma+\theta)^{5/2}}$$

5.16
$$\frac{\partial H}{\partial P_c} = \frac{2H}{P_c} \qquad \frac{\partial I}{\partial P_c} = \frac{I}{P_c}$$

5.16 follows from the definition of A and B in 5.12.

$$\delta H + c_1 \delta I = 0$$

5.17 $$\left(\frac{\partial H}{\partial \gamma} \delta\gamma + \frac{\partial H}{\partial P_c} \delta P_c \right) + c_1 \left(\frac{\partial H}{\partial \gamma} \delta\gamma + \frac{\partial I}{\partial P_c} \delta P_c \right) = 0$$

$$\left(\frac{\partial H}{\partial \gamma} + c_1 \frac{\partial I}{\partial \gamma} \right) \delta\gamma + \left(\frac{\partial H}{\partial P_c} + c_1 \frac{\partial I}{\partial P_c} \right) \delta P_c = 0$$

Since $\delta\gamma$ and δP_c are independent

5.18 $$\frac{\partial H}{\partial \gamma} + c_1 \frac{\partial I}{\partial \gamma} = 0 \quad \text{and} \quad \frac{\partial H}{\partial P_c} + c_1 \frac{\partial I}{\partial P_c} = 0$$

Inserting 5.14, 5.15 and 5.16 into the two parts of 5.18, we obtain, respectively, after transposition:

5.19 $$\frac{8A}{c_1 B} \frac{(2\gamma+\theta)^{5/2}}{(4\gamma-\theta)^{7/2}} \gamma(\gamma+\theta) + 3 = 0$$

5.20 $$\frac{8A}{c_1 B} \frac{(2\gamma+\theta)^{3/2}}{(4\gamma-\theta)^{5/2}} \gamma^2 + 1 = 0$$

These are two simultaneous equations to be solved. Since A and B both contain P_c and ρ_c, there are too many variables and we can only obtain certain of them in terms of others. The boundary and normalization conditions will later enable us to solve for each variable explicity. We reduce the number of variables to two by writing

5.21 $$t = \frac{\gamma}{\theta} \qquad \text{and} \qquad Q = \frac{8A}{C_1 B} \frac{(2t+1)^{5/2}}{(4t-1)^{1/2}} \theta$$

5.19 and 5.20 become

5.22 $$Qt(t+1) + 3 = 0$$

5.23 $$Qt^2(4t-1) + (2t+1) = 0$$

Eliminating Q:

5.24 $$10t^2 - 6t - 1 = 0$$

Expressed in terms of t, the condition (5.10) that the integral 5.9 exist is $t \geqslant 1/4$. The desired root of 5.24 is

5.25 $$t = 0.7359$$

Instead of evaluating Q directly, we shall take the expression for Q (5.23) and compare it with the definition 5.21, in which we substitute the definitions for A, B (5.21) and C_1 (4.10). We obtain

5.26 $$\frac{3t^2(2t+1)^{3/2}}{\pi G(4t-1)^{5/2}} \left[\frac{P_c \theta}{\rho_c^2}\right] = 1$$

From the normalizing condition $I = 1$ and by expressing 5.11 in terms of t and θ we obtain

5.27 $$\frac{\sqrt{\pi}}{2^{1/2}(2t+1)^{3/2}} \left[\frac{P_c \rho_c}{\theta^{3/2}}\right] = 1$$

Here we have two equations in three unknowns when the value of t is inserted. Recall 4.13 $(H = 2\pi G)$, a form which applies to this case. This expression is of no value here, however, which may be verified by expressing 5.9 in terms of t. Then 4.13 is the product of 5.26 and 5.27. The reason for this is that 5.26 and 5.27 depend on the definition of C_1 and the normalization, and these two conditions together with 4.13 are equivalent to the second of 5.18.

We shall leave further discussion of the solution until after the evaluation of the energy expressions.

The Energy Set

In this approximation

5.28 $$y = T^{15/2} = T_c^{15/2} e^{-15/2 \beta r^2}$$

$$y^2 = T^{\nu} \qquad \nu = 23/2$$

Let $\phi = 15\beta$ and $\theta = 2\alpha$

5.29 $$J = \frac{1}{2}\int_0^{\infty} \frac{r^2}{\rho^2}\left[\frac{dy}{dr}\right]^2 dr = \frac{(15)^2}{2} \frac{T_c^{15}}{\rho^2} \beta^2 \int_0^{\infty} r^4 e^{-(\phi-\theta)r^2} dr$$

$$= \frac{3\sqrt{\pi}}{16} \frac{T_c^{15}}{\rho^2} \frac{\phi^2}{(\phi-\theta)^{5/2}}$$

It is necessary that

5.30 $$\phi \geqslant \theta \geqslant 0$$

5.31 $$K = \int_0^\infty r^2\rho^2 y^\lambda dr = T_c^{\nu}\rho_c^2 \int_0^\infty r^2 e^{-(\frac{\nu}{15}\phi+\theta)r^2} dr$$

$$= \frac{T_c^{\nu}\rho_c^2\sqrt{\pi}}{4\left(\frac{\nu}{15}\phi+\theta\right)^{3/2}} = \frac{T_c^{\nu}\rho_c^2\sqrt{\pi}\,(15)^{3/2}}{4(\nu\phi+15\theta)^{3/2}}$$

Keeping definitions consistent with the second approximation, let

5.32 $$E = \frac{3\sqrt{\pi}\;T_c^{15}}{16^2\rho_c^2} \qquad F = \frac{T_c^{\nu}\rho_c^2\sqrt{\pi}\,(15)^{3/2}}{8}$$

Then

5.32a $$J = \frac{16E\,\phi^2}{(\phi-\theta)^{5/2}} \qquad K = \frac{2F}{(\nu\phi+15\theta)^{3/2}}$$

Taking derivatives

5.33 $$\frac{\partial J}{\partial\phi} = \frac{-8E\phi(\phi+4\theta)}{(\phi-\theta)^{7/2}}$$

5.34 $$\frac{\partial K}{\partial\phi} = \frac{-3\nu F}{(\nu\phi+15\theta)^{5/2}}$$

Also from the definition of E and F.

5.35 $$\frac{\partial J}{\partial T_c} = \frac{15J}{T_c} \qquad \frac{\partial K}{\partial T_c} = \frac{\nu K}{T_c}$$

From

5.36 $$\frac{dJ}{d\phi} + C_2\frac{dK}{d\phi} = 0 \quad \text{and} \quad \frac{dJ}{dT_c} + C_1\frac{dK}{dT_c} = 0$$

we obtain, respectively

5.37 $$\frac{8E}{3\nu C_2 F}\frac{(\nu\phi+15\theta)^{5/2}}{(\phi+\theta)^{7/2}}\phi(\phi+4\theta)+1=0$$

5.38 $$\frac{120E}{\nu C_2 F}\frac{(\nu\phi+15\theta)^{3/2}}{(\phi-\theta)^{5/2}}\phi^2+1=0$$

As before, we must reduce the number of variables.

Let

5.39 $$v=\frac{\phi}{\theta} \qquad N=\frac{2^{3/2}E(23v+30)^{5/2}\theta}{69\,C_2F(v-1)^{7/2}}$$

Since $\nu = 23/2$ Then the equations are

5.40 $$Nv(v+4)+1=0$$

5.41 $$90\,Nv^2(v-1)+(23v+30)=0$$

Eliminating N

5.42 $$67v^2-212v-120=0$$

5.43 $$v = 3.6543$$

From 5.39 and 5.41 we have

5.44 $$\frac{3}{2^{5/2}15^{1/2}23C_2}\frac{(23v+30)^{3/2}}{(v-1)^{5/2}}\left[\frac{T_c^{1/2}\theta}{\rho_c^4}\right]=-1$$

From the normalization

5.45 $$\frac{\sqrt{\pi}(15)^{3/2}}{2^{1/2}(23v+30)^{3/2}}\left[\frac{T_c^{23/2}\rho_c^2}{\theta^{3/2}}\right]=1$$

Parameters and Units

By the choice of normalization we define a relation in our units. The product $C_1 K$ is unaffected by a change in the value of K as long as C_1 is also changed accordingly. We may fix our units by noting

5.46 $$K = \int_0^R r^2 \rho P dr = \frac{1}{4\pi}\int_0^M P dM(r) = M\bar{P} = 1$$

Let these define the relation between our units of mass and our units of pressure numerically. Then inserting values of t into 5.26, and 5.27 (with 5.46)

5.47 $$\frac{P_c \Theta^2}{\rho_c^2} = 1$$

5.48 $$\frac{P_c^2 \rho_c^2}{\theta^3} =$$

Integrating $dM(r)$

5.49 $$M^2 = \frac{8\pi^3 \rho_c^2}{\theta^3}$$

From 5.48 and 5.49 $P_c = 3.2334\bar{P}$

From 5.48 and 5.47

5.50 $$\frac{M^2}{\bar{P}} = \frac{143.8}{\theta}$$

5.51 whence $\theta = 143.8$ (units of length)$^{-2}$

Putting this into 5.47

5.52 $\rho_c = 109.5 \times 10^9$ (units of density)

Inserting the known value of the solar mass we can convert to c.g.s. units.

Solution of the energy set involves a knowledge of chemical composition to obtain opacity and energy generation coefficients. Usually we require the model to give us this information. A discussion of how this may be done is found in Chapter VII. For such an unsophisticated model as this example only the methods merit discussion. The results themselves would be of no value.

Epstein and Motz give, at the centre of their small convective core

$$\log P_c = 17.30$$
$$\log \rho_c = 1.99$$
$$\log T_c = 7.11$$
$$\kappa_0 = 1.506 \times 10^{23}$$
$$\epsilon_0 = 6.550 \times 10^{-30} \qquad \text{all c.g.s. units}$$

CHAPTER VI

A SOLAR MODEL: SECOND APPROXIMATION

By examining the graph (Fig. 3) and measuring the slopes of the two curves we see that

6.1 $$\rho = \rho_c e^{-\alpha r^2}$$

is a good approximation. The curve is very nearly a straight line, which indicates exponential variation with some power of r. The "slope", whose value indicates the power of r, is nearly equal to 2.

The temperature curve displays more curvature, and the slope of the tangent is less than 2 in general. The discrepancy is not very severe, and it may be possible to improve the model considerably with a small correction term. We retain the previous density trial function 6.1, and adopt a temperature function.

6.2 $$T = T_c (1 + b r^2) e^{-(\alpha+\beta) r^2} = P_c (1 + b r^2) e^{-\gamma r^2}$$

where it is assumed that b is small compared to α and β.

Then

6.3 $$P = P_c (1 + b r^2) e^{-(\alpha+\beta) r^2} = P_c (1 + b r^2) e^{-\gamma r^2}$$

We note that the differential equation 1.10 will have its right hand side proportional to r near the origin where temperate and density (and hence energy generation per unit volume) are sensibly constant. Thus $\frac{dT}{dr} = 0$ at the origin and 6.2 has no first power term in r. The second derivative does not vanish. We would also expect the physical variables to be even functions of r and hence only even powers would appear in an expansion.

With few exceptions the variables are identical with those used in Chapter V. The procedure is an obvious extension of that previously used: hence comments are made only when differences appear.

<u>The Hydrostatic Set</u>

6.4 $$\frac{dP}{dr} = 2P_c r\left[(b-\gamma) - b\gamma r^2\right]$$

6.5 $$H = \frac{1}{2}\int_0^\infty \frac{r^2}{\rho}\left[\frac{dP}{dr}\right]^2 dr =$$

$$= \frac{2P_c^2}{\rho_c}\int_0^\infty \left\{(b-\gamma)^2 r^4 - 2b\gamma(b-\gamma)r^6 + b^2\gamma^2 r^8\right\} e^{-(2\gamma-\alpha)r^2}$$

$$= \frac{2P_c^2}{\rho_c}\left[\frac{3\sqrt{\pi}(b-\gamma)^2}{8(2\gamma-\alpha)^{5/2}} - \frac{2\gamma b(b-\gamma)\cdot 3\cdot 5\sqrt{\pi}}{16(2\gamma-\alpha)^{7/2}} + \frac{3\cdot 5\cdot 7\, b^2\gamma^2\sqrt{\pi}}{32(2\gamma-\alpha)^{9/2}}\right]$$

As before

6.6 $$A = \frac{3\sqrt{\pi}\, P_c^2}{16\rho_c} \qquad \text{and} \qquad \theta = 2\alpha$$

Then

6.7 $$H = \frac{A}{(2\gamma-\alpha)^{7/2}}\left\{ b^2(11\gamma^2 + 2\gamma\theta + \theta^2) + 2b\gamma(4\gamma-\theta)(\gamma+\theta) + \gamma^2(4\gamma-\theta)^2 \right\}$$

The integral does not exist unless

6.8 $$4\gamma \geqslant \theta \geqslant 0$$

6.9 $$I = \int_0^\infty \rho P r^2 dr = P_c \rho_c \int_0^\infty r^2(1+br^2)e^{-(\gamma+\alpha)r^2} dr$$

$$= P_c \rho_c \left[\frac{\sqrt{\pi}}{4(\gamma+\alpha)^{3/2}} + \frac{3b\sqrt{\pi}}{8(\gamma+\alpha)^{5/2}} \right]$$

Put

6.10 $$B = \frac{P_c \rho_c \sqrt{\pi}}{8}$$

Then

6.11 $$I = \frac{2^{5/2} B(2\gamma + \theta + 3b)}{(2\gamma+\theta)^{5/2}}$$

Take derivatives with respect to the parameters of P.

6.12 $$\frac{dH}{db} = \frac{2^{11/2} A}{(4\gamma-\theta)^{9/2}} \left\{ b(11\gamma^2 + 2\gamma\theta + \theta^2) + \gamma(4\gamma-\theta)(\gamma+\theta) \right.$$

6.13 $$\frac{dH}{d\gamma} = \frac{-2^{11/2} A}{(4\gamma-\theta)^{11/2}} \left\{ 5b^2(11\gamma^2 + 5\gamma\theta + 2\theta^2) + b(4\gamma-\theta) \cdot \right. \\ \left. \cdot (6\gamma^2 + 12\gamma\theta + \theta^2) + \gamma(4\gamma-\theta)^2(\gamma+\theta) \right\}$$

6.14 $$\frac{dH}{dP} = \frac{2H}{P_c}$$

6.15 $$\frac{dI}{db} = \frac{2^{5/2} \cdot 3B}{(2\gamma+\theta)^{5/2}}$$

6.16 $$\frac{dI}{d\gamma} = \frac{-2^{5/2} \cdot 3B[2\gamma + \theta + 5b]}{(2\gamma+\theta)^{7/2}}$$

6.17 $$\frac{dI}{dP_c} = \frac{I}{P_c}$$

We will have to reduce the number variables to equal the number of equations by expressing certain of them in terms of others. There is a slight difference between C and Q of Chapter V.

6.18 $$C = \frac{8A}{C_1 B} \frac{(2\gamma+\theta)^{5/2}}{(4\gamma-\theta)^{9/2}} \theta \qquad s = \frac{b}{\theta} \qquad t = \frac{\gamma}{\theta} \qquad t > 1/4$$

Because the variations of the parameters are independent we can write

6.19 $$\frac{dH}{db} + C_1 \frac{dI}{db} = 0$$

6.20 $$\frac{dH}{d\gamma} + C_1 \frac{dI}{d\gamma} = 0$$

6.21 $$\frac{dH}{dP_c} + C_1 \frac{dI}{dP_c} = 0$$

These lead, respectively, to

6.22 $$C\left\{s(11t^2+2t+1) + t(4t-1)(t+1)\right\} + 3 = 0$$

6.23 $$C(2t+1)\left\{5s^2(11t^2+5t+2) + s(4t-1)(6t^2+12t+1) + t(4t-1)^2 (t+1)\right\} + 3(4t-1)\left\{(2t+1)+5s\right\}$$

6.24 $$C\left\{s^2(11t+2t+1) + 2st(4t-1)(t+1) + t^2(4t-1)^2\right\} + (2t+1) + 3s = 0$$

These three equations in the unknowns C, s and t may require considerable labour to solve. We desire, if possible to take advantage of the recurrence of certain terms. The most rigourous methods of attack lead to impossibly complicated expressions. We take refuge in a result of the Theory of Equations: any method of eliminating a variable between two equations which does not violate the fundamental laws of algebra will not "destroy" a root. It will at most introduce extraneous roots which can be tested afterwards for validity.

Eliminat C by solving 6.22 and introducing the result into 6.23 and 6.24. Because of recurrence of terms in the equations, there is cancellation leading to unexpected simplification. We obtain, respectively

6.25 $$S = \frac{-3t(4t-1)(2t-1)}{22t^3-24t^2-7t-3}$$

6.26 $$S = \frac{t(4t-1)(10t^2-6t-1)}{10t^3+6t^2+7t+1}$$

6.25 and 6.26 may be regarded as simultaneous linear equations in S whose coefficients are polynomials in t. We discard the trivial roots $S=0$, $t=0$ and $S=0$, $t=1/4$. Then the determinant condition for simultaneous roots is

$$70t^4 - 84t^3 + 16t^2 + 6t + 1 = 0 \tag{6.27}$$

Recall that we must have $t \geqslant 1/4$. This fourth degree polynomial is easily treated by Horner's or Newton's approximation methods.

6.27 has no real solution.

We turn back to equations 6.22, 6.23 and 6.24. In these let s = 0. Then 6.22 and 6.23 become identical. If we express C in terms of Q of Chapter V, we have 6.23 and 6.24 identical with 5.22 and 5.23. This is to be expected since s = 0 implies b = 0 and the trial functions of Chapter V are precisely this special case of our present trial functions.

As far as the hydrostatic equations are concerned, our small correction term is vanishingly small.

The Energy Equations

In the energy integrals we have the temperature raised to the $\frac{15}{2}$ and $\frac{23}{2}$ powers. Since the correction term b is assumed small, we employ the bionomial expansion to two terms.

$$y = T^{15/2} \simeq T_c^{15/2}(1 + 15/2\, br^2)e^{-15/2\,\beta r^2} \tag{6.28}$$

$$T^{\nu} = T_c(1 + \nu br^2)e^{-\nu\beta r^2} \qquad \nu = 23/2$$

As before we take $\phi = 15\beta$ Then

6.29
$$J = \frac{1}{2}\int_0^\infty \frac{r^2}{\rho^2}\left(\frac{dy}{dr}\right)^2 dr$$

$$= \frac{15^2}{2}\,\frac{T_c^{15}}{\rho_c^2}\int_0^\infty r^4\left[(b-\beta)^2 - 15\beta b(b-\beta)r^2 + \frac{b^2}{4}15\beta^2 r^4\right] e^{-(\phi-2\alpha)r^2}\,dr$$

$$\frac{3\sqrt{\pi}\,T_c^{15}}{16^2\rho_c^2(\phi-2\alpha)^{9/2}}\Big\{225 b^2(11\phi^2+16\phi\alpha+64\alpha^2)+120\phi(\phi-2\alpha)(\phi-8\alpha)b$$
$$+16\phi^2(\phi-2\alpha)^2$$

The integral exists only if 6.30 $\quad \phi \geqslant 2\alpha \geqslant 0$

6.31
$$K = \int_0^\infty r^2\rho^2 T^\nu dr = T_c^\nu \rho_c^2\int_0^\infty r^2(1+\nu b r^2)e^{-\left(\frac{\nu\phi}{15}-2\alpha\right)r^2} dr$$

$$= \frac{T_c^\nu \rho_c^2\sqrt{\pi}}{8}\,(15)^{3/2}\,\frac{\left(2(\nu\phi+30\alpha)+45\,\nu b\right)}{(\nu\phi+30\alpha)^{5/2}}$$

We define E and F as in Chapter V

6.32
$$E = \frac{3\sqrt{\pi}\,T_c^{15}}{16^2\rho_c^2}$$

$$F = \frac{T_c^\nu\rho_c\sqrt{\pi}\,(15)^{3/2}}{8}$$

Taking derivatives

6.33
$$\frac{dJ}{db} = \frac{30E}{(\phi-2\alpha)^{9/2}}\Big\{15b(11\phi^2+16\phi\alpha+64\alpha^2)+4\phi(\phi-2\alpha)(\phi+8\alpha)\Big\}$$

6.34 $$\frac{dJ}{d\phi} = \frac{-E}{2(\phi-2\alpha)^{1/2}}\left\{1125\,b^2(11\phi^2+40\phi\alpha+128\alpha^2) + 120\,b(\phi-2\alpha)(3\phi^2+48\phi\alpha+32\alpha^2) + 16\phi(\phi-2\alpha)^2(\phi+8\alpha)\right\}$$

6.35 $$\frac{dJ}{dT_c} = \frac{15J}{T_c}$$

6.36 $$\frac{dK}{d\phi} = \frac{45\,\nu D}{(\gamma\phi+30\alpha)^{5/2}}$$

6.37 $$\frac{dK}{d\phi} = \frac{-3\,\gamma D\left(2(\gamma\phi+30\alpha)+75\,\gamma b\right)}{3\,(\gamma\phi+30\alpha)^{7/2}}$$

6.38 $$\frac{dK}{dT_c} = \frac{\nu K}{T_c}$$

We have

6.39 $$\frac{dJ}{db} + C_2\frac{dK}{db} = 0$$

6.40 $$\frac{dK}{d\phi} + C_2\frac{dK}{d\phi} = 0$$

6.41 $$\frac{dJ}{dT_c} + C_2 \frac{dK}{dT_c}$$

Similar to the definitions of Chapter V

6.42 $$\frac{b}{2\alpha} = u \qquad \frac{\theta E}{2\,\nu C_2 D}\,\underline{(\gamma v + 15)^{5/2}}$$

$$\frac{\phi}{2\alpha} = v \qquad\qquad v \geqq 1$$

$$\nu = 23/2$$

Then from 6.39, 6.40 and 6.41 respectively

6.43 $$2F\left\{15u(11v^2 + 8v + 16) + 4v(v-1)(v+4)\right\} + 1 = 0$$

6.44 $$F(23v + 30)\left\{1125u^2(11v^2 + 20v + 32) + 120u(v-1)(3v^2 + 24v + 8)\right.$$
$$\left. + 16v(v-1)^2(v+4)\right\} + (v-1)\left\{2(23v + 30) + 1725u\right\} = 0$$

6.45 $$90F\left\{225u^2(11v^2 + 8v + 16) + 120uv(v-1)(v+4) + 16v^2(v-1)^2\right\}$$
$$+ 2(23v + 30) + 1035u = 0$$

Solving 6.43 for D and substituting 6.44 and 6.45

6.46 $$u = \frac{-4(v-1)(69v^2 - 16v - 424)}{15(253v^3 - 928v^2 - 968v - 1696)}$$

$$6.47 \quad 2700(11v^2+8v+16)w^2+15(211v^3+388v^2+776v+480)w - \\ -4v(v-1)(67v^2-212v-120) = 0$$

To solve these equations, we change variables to eliminate the trivial roots $u=0$, $v=0$ and $w=0, v=1.$ Write

$$6.48 \qquad w = \frac{-4}{15} v(v-1)\mathbf{w}$$

Then 6.46 and 6.47 become

$$6.49 \qquad \mathbf{w} = \frac{+(69v^2-16v-424)}{(253v^3-928v^2-968-1696)}$$

$$6.50 \qquad 48v(v-1)(11v^2+8v+16)\mathbf{w}^2-(211v^3+388v^2+776v+480)\mathbf{w} \\ -(67v^2-212v-120)=0$$

The numerical solution of 6.49 and 6.50 is carried out by setting the left hand side equal to X. A value is chosen for v and $\mathbf{w}$ is found from 6.49. Then this pair of values v and $\mathbf{w}$ is inserted in the left hand side of 6.50 and X is found. A graph is plotted to find where X vanishes.

The only admissable root is

6.51 $$v = 6.680 \qquad w = 0.0986$$

Then

6.52 $$\mu = 0.998$$

whence

6.53 $$b = 2.24\,\beta = 1.99\,\alpha$$

The use of the binomial expansion to two terms was made on the assumption that b would be small compared with the other parameters. Since this is not the case, this root has no validity. We are no better off than we were in the hydrostatic set with no root at all.

As in the hydrostatic set, the equations 6.43, 6.44, 6.45 reduce to the two equations of the first approximation if $\mu = 0$.

We are forced to conclude that we must turn to some other type of correction term or some other type of trial function if we wish to elaborate and improve the results of Chapter V.

CHAPTER VII

ADDITIONAL REMARKS

Improving the Trial Functions

The failure to obtain any improvement by adding the small correction term to the temperature distribution of Chapter V raises the question of what other means may be used to improve our results. We can broaden the question by asking, "What is to be done when our simple density trial function is no longer valid?" We may further inquire, "Are we restricted to the study of stars in a range where previous numerical integration provides a guide?" An answer to the last question will be found to deal as well with the first two.

Suppose we turn from our previous models of infinite radius and consider finite stars. The first obvious advantage is that we now have a definite radius to speak of. We may now think of the r of our integrals as being defined as a fraction of the total radius. Let us write our trial functions as series expansions in periodic functions of r . We must, of course, be careful that we have proper behavior of the trial functions and their first and second derivatives at the origin and boundary. We will probably prefer, however, to choose our functions in such a way that a convective core can be fitted with the aid of the fitting conditions arising out of 4.14 and 4.28.

An examination of 4.3 and 4.18 shows that it will be very much to our advantage to have the orthogonal property in the first derivative. A cosine series should be satisfactory if taken to enough terms. The very considerable labour of evaluating the derivatives in Chapter VI, and of solving the algebraic equations will be much diminished.

The Disposable Constants

We cannot expand both our temperature and density functions to an unlimited number of terms. How many parameters are at our disposal?

In the temperature expansion the limit is set only by the number of simultaneous equations we are willing to solve. Every parameter a_i in temperature provides an equation of the form

$$\frac{\partial J}{\partial a_i} + c_2 \frac{\partial K}{\partial a_i} = 0 \tag{7.1}$$

We have, say, n parameters of temperature and m of density, and a total of n equations arising from 7.1. We have a normalization condition K = 1. We have an empirical relation between mass, luminosity and radius. This can be broken down into two relations mass in terms of radius and luminosity in terms of radius. Thus, including the normalization, there are three conditions available to fix additional parameters. The radius itself is not an independent condition.

If we assume a composition and a particular form of opacity law (i.e. fix μ, σ and η) and if we treat the hydrostatic and energy sets independently, then we have three disposable density parameters. We can compromise between the two sets of solutions by taking some form of mean, such as the least-squares mean of the density distributions. The closeness of agreement of the distributions from the two sets is a measure of how closely we approach the exact solution when we take a limited number of terms in the expansion.

We are more likely to want our model to determine for us the composition. If we assume a certain percentage of heavy elements and the proportions of these elements then we can fix the ratio of hydrogen to helium, and two density parameters. We may want more available constants. In this case we treat the hydrostatic and energy equations as related. This is preferable in any case as the separation is somewhat artificial. Both second order differential equations are necessary for complete specification of the interior. Now there are two normalization conditions, an extra parameter can be fixed.

Here we encounter the difficulty that our energy equations 7.1 and the corresponding set in the hydrostatic case

$$7.2 \qquad \frac{\partial H}{\partial a_i} + C_1 \frac{\partial K}{\partial a_i} = 0 \qquad\qquad i = 1 \cdots\cdots n$$

will not lead to the same n relations between the temperature and density parameters. Examining the two variational principles involved, we see that the hydrostatic set is fixed the moment we set up trial functions and equation of state. The energy set depends on the composition assumed, directly as far as ϵ_0 and K_0 are concerned, indirectly in choosing σ and η to fit the guillotine factor. Composition enters into μ when we relate the two sets by the equation of state, since the hydrostatic set is expressed in ρ_c and P_c, and the energy set in ρ_c and T_c

We can write ϵ_0, K_0 and μ as numerical constants multiplied by simple functions of the hydrogen and helium abundances. Then let 7.1 be solved in the form

$$7.3 \qquad a_i = f_i(a_j, X, Y, \sigma, \eta) \qquad\qquad \begin{matrix} i = 1 \cdots\cdots n \\ j = 1 \cdots\cdots m \end{matrix}$$

Where a_j are the density parameters, X the fractional abundance of hydrogen and Y that of helium. 7.2 has solutions

$$7.4 \qquad a_i' = f_i'(a_j, X, Y, \sigma, \eta)$$

Then X, Y, σ and η are determined from the condition that $a_i = a_i'$ for all i. Then we have two normalizations and the mass and luminosity conditions to be satisfied. If $n = 4$, then $a_i = a_i'$ are exactly sufficient to fix X, Y σ and η, and we have four disposable density parameters. If $n > 4$, we have more equations than needed and will need to take a least squares solution. We still have four density parameters. If $n < 4$ we must use one or more of the remaining conditions and sacrifice one or more density parameters. The number of density and temperature parameters are now equal.

The test for consistency of the model is by comparison of the predicted march of opacity as temperature and density vary with tabulated opacities.

In Conclusion

The model described in Chapter V was simple to develop. The integrations, taking of derivatives and algebraic solution were rapidly done. The addition of the correction term in Chapter VI multiplied the labour manyfold. Such complications are capable of destroying the advantage to be expected over numerical integration methods, especially with the increased use of high speed computers in the field of stellar models. If, however, the trial functions are carefully chosen, then the work of evaluating the parameters may be greatly reduced.

Certain advantages now become apparent. The trial function may be expected to be valid over a reasonable range of radii. By decreasing the amount of hydrogen and increasing correspondingly the amount of helium, the evolutionary track of the star may be followed with very little additional work. It should also be noted that it will not be necessary to recompute the entire model each time a new opacity table appears. Only a small portion of the job need be repeated. The variational method may have considerable utility in making a preliminary exploration of regions later to be more fully studied by other means. Only further investigation can show in detail its capabilities and limitations.

Hertzsprung-Russell Diagrams

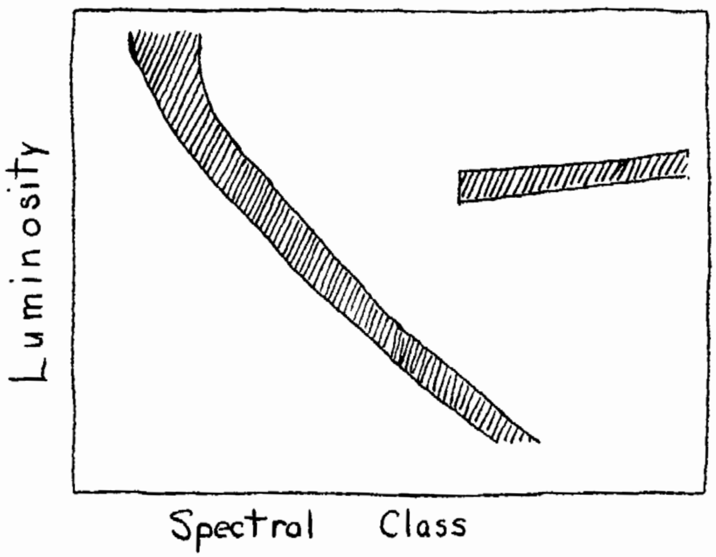

Figure 1
Population I
The Sun's Neighbors

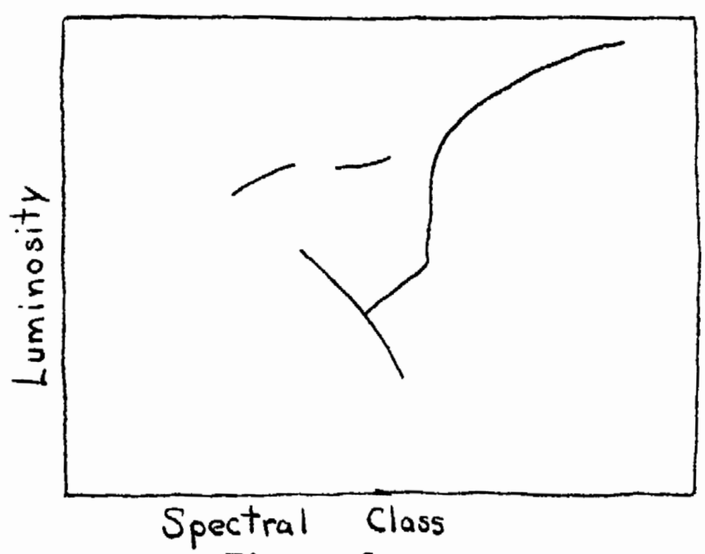

Figure 2
Population II
A Globular Cluster

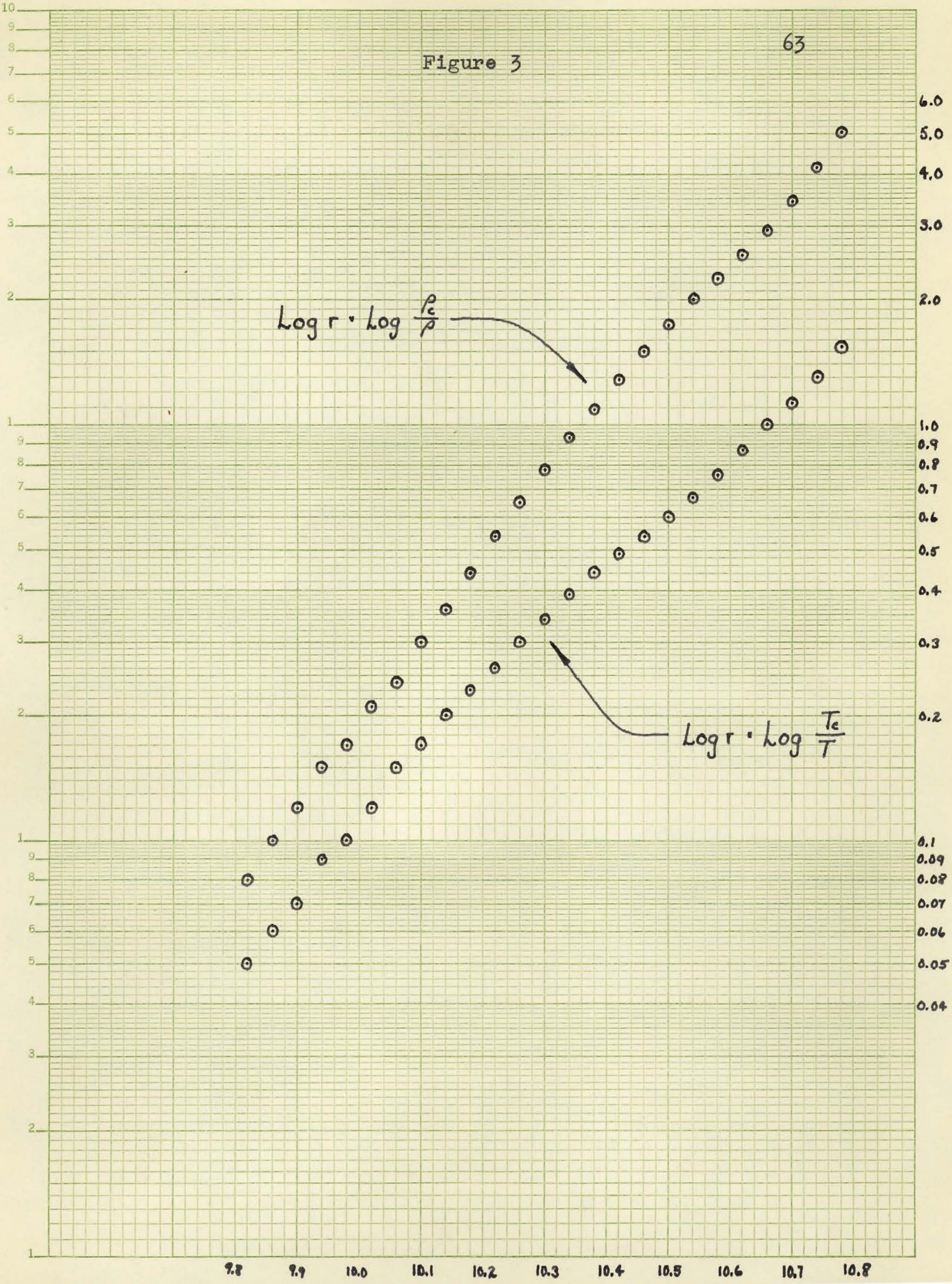
Log r · Log ρc/ρ
Log r · Log Tc/T
6.0
5.0
4.0
3.0
2.0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.09
0.08
0.07
0.06
0.05
0.04
9.8
9.9
10.0
10.1
10.2
10.3
10.4
10.5
10.6
10.7
10.8

Figure 3

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The following abbreviations are used.

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Astr. J.	Astronomical Journal
A. N.	Astronomische Nachrichten
Ap. J.	Astrophysical Journal
MN	Monthly Notices of the Royal Astronomical Society
Phys.Rev.	Physical Review

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