Topics in Cosmological Fluctuations: Linear Order and Beyond

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Abstract

The object of this thesis is to present various applications of the theory of cosmological perturbations. Within are contained a number of manuscripts, each concerned with a separate aspect of the theory. The thesis itself begins with a general overview of cosmological perturbation theory designed to be accessible to the non-specialist. Both the classical and quantum first order theory are considered. Back-reaction, via the formalism of the Effective Energy Momentum Tensor (EEMT) is reviewed. Subsequent chapters are more specialized dealing with various applications of the theory. At first order, topics discussed include the classicalization of cosmological perturbations (chapter 2), and the effects of including the dilaton and its fluctuations on a novel mechanism for the production of inhomogeneities in string gas cosmology (chapter 3). At second order, an original solution to the Dark Energy problem is proposed (chapter 4), and the effects of back-reaction on the power spectrum, including the spectral index and the gaussianity, are examined (chapter 5).

Résumé

Cette thèse vise à présenter diverses applications de la théorie des perturbations cosmologiques. En ce but, plusieurs manuscripts, traitant de différents aspects de cette theéorie, seront introduits. La thèse elle-même débute par un survol de la théorie des perturbations cosmologiques, se voulant accessible aux non-initiés. Les formalismes classique et quantique sont tous deux abordés, ainsi que la méthode du Effective Energy Momentum Tensor (EEMT). Les chapitres suivants se concentrent sur diverses applications de la théorie. En premier lieu, les sujets discutés comprennent une classification des perturbations cosmologiques (deuxième chapitre), ainsi que les effets de l'ajout du dilaton à un nouveau mécanisme de production d'inhomogénéités dans la cosmologie des cordes (troisième chapitre). Ensuite, le quatrième chapitre présente une solution originale au problème de l'énergie manquante, et les effets de la "backreaction" sur le spectre de puissance, y compris l'index spectral et la gaussianité, sont examinés au cinquième chapitre.

Contribution of Authors/ Acknowledgements

The following thesis contains published works completed in collaboration with a number of authors. Throughout my years at McGill, I have been fortunate to collaborate with a number of talented individuals. These include: R.H. Brandenberger, C.P. Burgess, J. Cline, D.A. Easson, H. Firouzjahi, S. Kanno, J. Khoury, A. Nayeri, S.P. Patil, F. Quevedo, R. Rabadan, G. Rajesh, J. Soda, G. Tasinato, I. Zavala C., and R.J. Zhang. I have profitted greatly through my interactions with all of them and this text contains a sampling of the results

For the work presented in this thesis, the division of labour varied from article to article. Chapter 2 consists of a single author paper and the current author is only person willing to claim credit for the result, although he is deeply indebted to both R.H. Brandenberger and C.P. Burgess who provided him with invaluable advice. The third chapter consists of the author's notes which lead to his contribution to the article "More on the Spectrum of Perturbations in String Gas Cosmology", contained in appendix 1. For the articles which make up chapters 4 and 5, the authors contribution was consistent with him being the lead author on both articles - nonetheless, the contribution of his co-author, R.H. Brandenberger, cannot be overstated.

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Chapter 1

Introduction

Modern cosmology is a field dominated by a relatively small number of extraordinary challenges. These problems run the gamut of cosmological epochs ranging from the very early universe (What was the mechanism for inflation? What set the initial conditions?) to current times (What is the source of Dark Energy? What sets the scale of Dark Energy?). One of the most exciting things about cosmology is that it serves as a testing ground for fundamental theories, many of which would be hard to connect to reality except through their impact on the universe on a large scale. Many attempts to resolve the above-mentioned problems draw on physical theories which are radical departures from conventional cosmology. By this, we have in mind string theory in its various guises, ranging from String Gas Cosmology [1] to Brane World models [2], as well as slightly less exotic models such as Loop Quantum Cosmology [3], and Causal Set Quantum Gravity [4]. Despite the author's own acquaintance with string-motivated model-building (see, for example, [5],[6],[7],[8]), the aim of this thesis is not (for the most part) to introduce new theories on the origins of the universe or to posit entirely new theoretical frameworks but rather to examine the effects of more conventional physics and to explore their ramifications more fully. The unifying theme of this thesis is the theory of primordial fluctuations, and the aim is to show that the theory is rich and can be used to explain a variety of phenomena.

The aim of the current chapter is to provide the reader with a comprehensive overview of the theory of cosmological perturbations [1]. The approach mimics that of [9] and particularly that of [10] which we've shamelessly pilfered for stylistic touches. The approach is conventional and follows the standard treatment: we begin by motivating and developing the Newtonian theory, we extend the results to the relativistic theory, and then review the quantum formulation. Due to the importance of back-reaction to this thesis, the classical theory is considered up to second-order and the formalism of the Effective Energy-Momentum Tensor (EEMT) [11],[12] is presented. It is hoped that this provides an accessible introduction to the basic tenets of the theory of cosmological fluctuations but it should by no means be seen as a complete overview: numerous important topics are only briefly mentioned (for example, issues of gauge) while some are omitted entirely (for example, the theory of CMB anisotropies). Nonetheless, the topics presented in the introduction should provide a basis to understand the salient features of the overall thesis and it is hoped that the reader will find the presentation clear. This author has no intention of rivaling the excellent presentations already extant in the literature - we have in mind [9],[10],[13],[14], for example - and the reader who finds our explanation of a certain point lacking is encouraged to refer to those sources.

1.1 Motivation

Before embarking on our exposition of the theory, some motivation for its existence would be fitting.

The field of experimental cosmology concerns itself with observing a number of phenomena: many of these, such as the anisotropies in the cosmic microwave background (CMB) or the statistical properties of clusters of galaxies would, at first glance, appear to be completely unrelated. However, the theory of cosmological perturbations unifies these effects and shows them to be a consequence of a single physical principle operating over an enormously large range of scales. Using a minimal set of parameters, the theory of cosmological perturbations proves able to predict and explain many of the results

[1]Throughout, the terms cosmological perturbations and primordial fluctuations are used interchangeably. By both, we mean small amplitude perturbations of both the background metric and matter.

experimentalists obtain from their observations of the cosmos.

Out of the basic 11 cosmological parameters [15], 3 are directly pertaining to cosmological fluctuations [2], while attempts have been made to explain others in terms of the fluctuations themselves (see, for example, chapter 4). Clearly, the ability to predict and reduce the number of cosmological parameters is a great boon for the theory - not to mention a benefit to cosmology in general.

In addition to this, the theory can be used, indirectly, to obtain accurate values of many of the remaining cosmological parameters [3]. An important result of experimental cosmology is the observed spectrum of anisotropies in the cosmic microwave background. The anisotropies are themselves predicted to exist and their subsequent behaviour falls within the realm of the theory. Matching theory and experiment leads to strict bounds on the value of cosmological parameters.

Another use of the theory is in distinguishing between different models of the early universe. The standard interpretation of cosmology - where a period of inflation is assumed to have taken place - leads to the requirement that cosmological fluctuations were created in the epoch that preceded reheating. Again, requiring agreement between theory and observation limits the theories pertaining to the very early universe to that class which predicts a nearly scale-invariant ($n \approx 1$) spectrum of perturbations. In this way, not only do fluctuations limit the parameter space of cosmology as explained above, they also help constrain the "theory space" available to researchers.

Having briefly motivated the importance of the role played by fluctuations, we now turn to their study.

[2] $\Delta^2_{\mathcal{R}}(k_\star)$, the density perturbation amplitude, n, the density perturbation spectral index, and r, the tensor to scalar ratio

[3] By remaining, we mean those parameters that are neither directly related to the fluctuations themselves nor those can be explained in terms of them, such as H, ρ_M, η, etc.

1.2 The Classical Theory

1.2.1 The Basics - Newtonian Gravity

Despite Newtonian gravity being of limited practicality in cosmology [4], it is useful in helping to develop an intuition for the properties and behaviours of gravitational perturbations. Since General Relativity reduces to the Newtonian theory, we expect the results obtained here to be valid in the context of cosmology proper, (at least in the Newtonian limit of small masses and large distances).

Consider a flat, non-expanding background permeated by homogeneous matter, the latter characterized by the single parameter ρ, representing the energy density of the matter. In the limit of pure homogeneity, no non-trivial dynamics are possible and the theory is hardly more interesting than the case of empty space.

Introducing inhomogeneities, $\delta\rho$, - specifically, a local excess of energy density - results in the creation of attractive gravitational forces distributed in concordance with the inhomogeneities throughout the background. In accord with Newtonian dynamics, the magnitude of the attractive force is proportional to the local matter excess, which leads to the following form for the equation of motion:

$$\delta\ddot{\rho} \sim G\delta\rho, \tag{1.1}$$

where we've introduced G, the gravitational constant, and the overdot denotes a time derivative. As this is an attractive force, we see that the introduction of fluctuations in a static background leads directly to runaway, exponential instabilities. The severity of the stability can be attenuated by considering a dynamical background, as we will see later. Without specifying the properties of the matter, there is little else that can be said about this model aside from emphasizing its drastic instability. For this reason, we turn to the next simplest scenario, namely the case where our matter is a perfect fluid.

[4]By this, we mean that the theory can't adequately take into account the case of an expanding universe. In other words, it can only be used in the study of sub-Hubble scale phenomenoa. For example, Newtonian gravity does provide a satisfactory explanation of non-relativistic phenomena i.e. structure formation

1.2.2 One Step Up - The Perfect Fluid

We now focus our attention on a more sophisticated type of matter, the perfect fluid [5].

In this case, the dynamics of the fluid are governed by the equation of state as well as the basic hydrodynamical equations,

$$\begin{aligned} \dot{\rho} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\ \dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{\rho}\nabla p + \nabla \varphi &= 0 \\ \nabla^2 \varphi &= 4\pi G \rho \\ \dot{S} + (\mathbf{v} \cdot \nabla) S &= 0 \\ p &= p(\rho, S)\,. \end{aligned} \tag{1.2}$$

The variables are identified as follows: p is the pressure, ρ the energy density, S the entropy density, $\mathbf{v}$ the fluid velocity, and φ is the Newtonian potential. The derivative with respect to time is indicated by the over-dot.

As before, the case of a homogeneous and isotropic (with respect to $\mathbf{v}$) distribution is of no physical interest. In order to make use of this model, we introduce perturbations in our fluid variables of the form

$$\begin{aligned} \rho &= \rho_0 + \delta\rho \\ \mathbf{v} &= \delta\mathbf{v} \\ p &= p_0 + \delta p \\ \varphi &= \varphi_0 + \delta\varphi \\ S &= S_0 + \delta S, \end{aligned} \tag{1.3}$$

where it is understood the that background quantities (denoted by the subscript $_0$) comply with the requirement of strict homogeneity and isotropy.

Introducing these new variables in (1.3) and linearizing immediately yields,

$$\delta\dot{\rho} + \nabla \cdot (\delta\rho \mathbf{v} + \rho \delta\mathbf{v}) = 0$$

[5]The reader is reminded that a perfect fluid is completely characterized by its energy density ρ, and pressure p as measured in its rest frame. Furthermore, perfect fluids experience no shear stresses, viscosity or heat conduction. Clearly, this has the advantage of minimizing the amount of phenomenological parameters while still maintaining a certain level of realism.

$$
\begin{aligned}
\delta\dot{\mathbf{v}} + (\delta\mathbf{v}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\delta\mathbf{v} + \frac{1}{\rho}\nabla\delta\mathbf{p} - \frac{\delta\rho}{\rho}\nabla\mathbf{p} + \nabla\delta\varphi &= 0 \\
\nabla^2\delta\varphi &= 4\pi G\delta\rho \qquad (1.4) \\
\delta\dot{S} + (\delta\mathbf{v}\cdot\nabla)S + (\mathbf{v}\cdot\nabla)\delta S &= 0 \\
\delta p &= \delta p(\rho, S, \delta\rho, \delta S),
\end{aligned}
$$

which, when combined, lead to the dynamical equation

$$
\delta\ddot{\rho} - c_s^2\nabla^2\delta\rho - 4\pi G\rho_0\delta\rho = \sigma\nabla^2\delta S, \qquad (1.5)
$$

and the constraint equation

$$
\delta\dot{S} = 0. \qquad (1.6)
$$

In the above, we've introduced the speed of sound, c_s, defined by

$$
c_s^2 = \frac{dp}{d\rho}\,|_S, \qquad (1.7)
$$

and σ is a quantity which appears in the equation of state and can be defined through the relation

$$
\sigma\delta S = \delta p - c_s^2\delta\rho. \qquad (1.8)
$$

(1.5) is our first non-trivial result. It tells us that there exist two distinct types of fluctuations: entropy fluctuations, δS, and density fluctuations, $\delta\rho$. Furthermore, we see that entropy fluctuations act as a source for density fluctuations. As we've identified the dynamics of the system, we are in position to make some general statements about adiabatic and entropy fluctuations. The first, and least interesting, is that the δS are necessarily static. Secondly, if follows from the above that adiabatic perturbations are generally time dependent. Finally, although the presence of δS implies the presence of $\delta\rho$, it is important to realize that the reverse does not necessarily follow.

Having pointed out the existence of two distinct types of fluctuations, we now introduce some jargon. In the case of a vanishing δS (as occurs when one considers a fluid with only a single component), the fluctuations are referred to as adiabatic. Otherwise, the non-vanishing δS is, to no great surprise, termed an entropy fluctuation.

We've discussed entropy perturbations for the sake of completeness but the results of this thesis are concerned exclusively with the effects of adiabatic fluctuations. As a consequence, we ignore the effects of entropy perturbations in the discussion that follows.

An intuitive physical interpretation of (1.5) is readily available. Ignoring entropy perturbations, our dynamics are identical to those in the previous section save for the presence of fluid pressure (denoted by the gradient term). We find that the dynamics of adiabatic perturbations are governed by the competing effects of fluid pressure, which act to smooth out the perturbation, and gravitational attraction, which acts to amplify it.

An important point to mention is that (1.5) implies the existence of an intrinsic scale λ_J, denoted the Jeans length, given by

$$\lambda_J = c_s(\frac{\pi}{G\rho_0})^{1/2}. \tag{1.9}$$

The Jeans length is important in that it serves to distinguish between two distinct qualitative behaviours of the solutions to (1.5). For modes with wavelengths far longer than the Jeans length, the equation of motion reduces to

$$\delta\ddot{\rho} - 4\pi G\rho_0\delta\rho \approx 0, \tag{1.10}$$

whose solutions grow exponentially, while those with wavelength less than λ_J obey the equation

$$\delta\ddot{\rho} - c_s^2\nabla^2\delta\rho \approx 0 \tag{1.11}$$

with oscillatory solution.

In this model, the presence of the Jeans length tells us that there are two different forms of solutions. For those solutions of large spatial extent, we see that the pressure is incapable of keeping the gravitational attraction in check with the result that the amplitude of these modes face unrestrained growth. Meanwhile, small-scale perturbations experience an interminable cycle of alternating pressure domination followed by gravity domination which serves to limit the strength of these fluctuations.

As the theory increases in complexity, we will find that the presence of a homolog to the Jeans scale persists, although the particular behaviour experienced by modes on both sides of the scale are markedly different than in this specific case.

1.2.3 The Effects of a Dynamical Spacetime

We are now able to consider the effects of embedding our simple hydrodynamical model within an expanding spacetime. In order to accommodate this new level of complexity, we introduce physical, x, and co-moving, q, coordinates related through

$$x(t) \;=\; a(t)q(t), \tag{1.12}$$

with $a(t)$ playing the role of the scale factor.

Following the same procedure as in the previous section, our new dynamical equation takes the form

$$\delta\ddot{\rho} + 2(\frac{\dot{a}}{a})\delta\dot{\rho} - \frac{c_s^2}{a^2}\nabla_q^2\delta\rho - 4\pi G\rho_0\delta\rho \;=\; \frac{\sigma\nabla^2}{a^2\rho_0}\delta S, \tag{1.13}$$

and

$$\delta\dot{S} \;=\; 0. \tag{1.14}$$

The novelty of a dynamical background has led to the introduction of a new term within the equation of motion (proportional to the expansion rate, $\dot{a}$) which acts to dampen the perturbation - clearly, an expanding spacetime acts to diffuse the amplitude of the fluctuations.

As before, the presence of the Jeans scale separates our solution space into two distinct regions. An approximate solution valid on small scales has the form

$$\delta_k(t) \;\sim\; \frac{1}{\sqrt{a(t)}}\exp{(\pm ic_s k\int^t \frac{dt'}{a(t')})}, \tag{1.15}$$

while, on larger scales, and if $p = 0$, our solutions take on the form of power laws such as

$$\delta_k(t) \;\sim\; c_1 t^{2/3} + c_2 t^{-1}. \tag{1.16}$$

As we can see, the inclusion of cosmological expansion has significantly changed the qualitative behaviour of the fluctuations. It has, to a certain extent, tamed the instabilities present on large scales as well as ensured that perturbations on small scales are not free to persist indefinitely; rather, we see that they must eventually dampen out.

Having built up a certain level of intuition about the behaviour of our solutions, we now turn an eye to the general relativistic theory, which we will make exclusive use of in this thesis. We will see that the results on small scales will correspond to those of the present section, but that large scale effects will be different yet again.

1.2.4 The General Relativistic Theory

So far, the theory presented is perfectly adequate for describing perturbations on scales much smaller than the Hubble length. However, for large scale fluctuations - in a cosmological context, the separation between large and small is naturally effected by the Hubble scale - we'd expect general relativistic corrections to be important. General arguments in inflationary cosmology suggest (see, [11]) that the phase space of fluctuations is dominated by the infra-red sector. As well, promoting the metric to a dynamical degree of freedom allows for the possibility of metric fluctuations in addition to matter perturbations. This detail is crucial in determining the effects of super-Hubble modes on the CMB (see, for example, [16]). For these reasons, we now consider the fully relativistic theory.

Our approach is to consider the spacetime given by

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}, \tag{1.17}$$

where $g^{(0)}_{\mu\nu}$ denotes a FLRW metric - homogeneous and isotropic so that

$$g^{(0)}_{\mu\nu} = g^{(0)}_{\mu\nu}(t), \tag{1.18}$$

while $\delta g_{\mu\nu}$ remains a completely general (albeit, small by assumption) perturbation

$$\delta g_{\mu\nu} = \delta g_{\mu\nu}(\mathbf{x}, \mathbf{t}). \tag{1.19}$$

In general, the above expression will be decomposed in terms of scalars, vectors, and tensors. Below, we examine the contributions to $\delta g_{\mu\nu}$ arising from each type of fluctuation. If we decompose the components of $\delta g_{\mu\nu}$ entirely in terms of scalar quantities, we find that we can write [6]

[6]About notation: commas denote partial derivatives, semicolons indicate covariant derivatives, and δ_{ij} is the Kronecker symbol

$$\delta g_{\mu\nu} = a^2 \begin{pmatrix} 2\phi & -B_{,i} \\ -B_{,i} & 2(\psi\delta_{ij} - E_{,ij}) \end{pmatrix}, \tag{1.20}$$

where we've identified our four separate degrees of freedom - one for each spacetime dimension. We will see below that constraints arising from the need for gauge invariance will reduce the number of degrees of freedom. The presence of a^2 is conventional and serves to reduce the complexity of the resulting equations of motion. Its presence is perfectly natural as the background can be written (in conformal coordinates)

$$g_{\mu\nu} = a^2\eta_{\mu\nu}, \tag{1.21}$$

where $\eta_{\mu\nu}$ represents Minkowski space.

The vector contributions to the perturbation take the form

$$\delta g_{\mu\nu} = a^2 \begin{pmatrix} 0 & -S_i \\ -S_i & F_{i,j} + F_{j,i} \end{pmatrix}, \tag{1.22}$$

where S_i and F_i are divergenceless vectors. The requirement that they be divergenceless arises from the following reason: if S_i or F_i weren't divergenceless, these vector perturbations would contain a hidden, scalar degree of freedom of the form $\Phi = \delta^i S_i$.

Finally, the contribution to the fluctuation in terms of tensors gives

$$\delta g_{\mu\nu} = -a^2 \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}, \tag{1.23}$$

where h_{ij} is trace-free (again, to avoid the presence of a scalar perturbation) and divergenceless (to eliminate any vectors). It can be shown, (see, for example [17]) that the number of independent degrees of freedom in the tensor case is 2 - in other words, gravitational waves have two distinct polarization states.

In summary, the most general expression for a perturbed spacetime is

$$\delta g_{\mu\nu} = \delta g_{\mu\nu}^{(scalar)} + \delta g_{\mu\nu}^{(vector)} + \delta g_{\mu\nu}^{(tensor)}. \tag{1.24}$$

This potentially imposing expression gets simplified in two important ways: requiring gauge invariance substantially reduces the number of independent degrees of freedom, and some very general properties allows us to ignore other variables.

It can be shown by examining the equations of motion that the amplitude of the vector perturbations decay rapidly in time. Furthermore, their presence at recombination would lead to an important B-mode component in the CMB polarization, which has not been observed. As a result, we can safely ignore any vector contributions to $\delta g_{\mu\nu}$. Furthermore, tensor contributions to $\delta g_{\mu\nu}$ can also be disregarded on the basis that gravitational waves don't couple to matter (at least, scalar matter) at first order due to the impossibility of writing a fully covariant term in the stress-energy tensor linear in h_{ij}. The result is that $\delta g_{\mu\nu}$ is well approximated by its expansion in terms of scalars alone. Therefore, we are left to consider a perturbed line of element of the form

$$ds^2 = a^2][(1+2\phi)dt^2 - B_{,i}dx^i dt + (\delta_{ij} + 2(\psi\delta_{ij} - E_{,ij}))dx^i dx^j], \quad (1.25)$$

which acts as an excellent approximation to our physical spacetime [7]

We must now turn to issues of gauge transformations and determine the corresponding effects on $\delta g_{\mu\nu}$.

1.2.5 Gauge Transformations

In line with the philosophy of General Relativity, the spacetime labels $(\mathbf{x}, \mathbf{t})$ carry no intrinsic physical meaning. As our theory must apply in all coordinate systems, it follows that our scalar variables ϕ, ψ, B and E, being components of a tensor, should not be expected to represent physical (observable) degrees of freedom. The observables we seek must be true scalars i.e. they must be invariant under coordinate transformations. Although we can't anticipate ϕ, ψ, B and E as having this property, we can expect (or, at least, hope) that certain linear combinations of these variables will be gauge invariant. Our present goal is to show precisely what these combinations are and to express $\delta g_{\mu\nu}$ in terms of new, gauge independent variables.

In a four dimensional spacetime, we have four independent gauge degrees of freedom: these correspond to the four infinitesimal coordinate transformations

$$x^\mu \longrightarrow \tilde{x}^\mu = x^\mu + \xi^\mu . \quad (1.26)$$

[7] The fact that this is a successful approximation is confirmed by the accuracy with which the observed CMB anisotropies can be predicted by the theory using this form of the line element.

We impose a space and time separation on our gauge transformations so that they have the form

$$\xi^0 = \xi^0, \tag{1.27}$$

$$\xi^i = \xi^i_{tr} + g^{ij}\beta_{,}j. \tag{1.28}$$

ξ^i_{tr} is transverse and g^{ij} represents the spatial components of the background metric. Note that by this separation, we've introduced two independent scalar functions, ξ^0 and β. The three components of ξ^i_{tr} reduce to two independent components due to the constraint imposed by transverseness. Thus, overall, we find that our four independent gauge transformations are parameterized by 2 scalars (ξ^0,β) and 2 vector components (the two independent components of ξ^i_{tr}). Had we included tensor modes, we would have found that the combined requirements of symmetry, tracelessness, and transversality lead immediately to gauge covariance - our two tensor degrees of freedom would correspond directly to two physical, tensor degrees of freedom. Again, we have two linearly independent tensor degrees of freedom. For reasons noted above, we disregard all but the scalars and determine the transformation properties of our set of (ϕ,ψ,E,B) in terms of (ξ^0,β).

It can be shown that [9]

$$\begin{aligned}
\tilde{\phi} &= \phi - \frac{a'}{a}\xi^0 - (\xi^0)' \\
\tilde{B} &= B + \xi^0 - \beta' \\
\tilde{E} &= E - \beta \\
\tilde{\psi} &= \psi + \frac{a'}{a}\xi^0 ,
\end{aligned} \tag{1.29}$$

where we've used conformal time, η, and $' = \partial_\eta$.

The above completely characterizes the transformations properties of our metric variables in terms of our gauge parameters. As we only have two independent scalar gauge parameters, we must have only two independent scalar observables. At this point, we can pick one of two ways to resolve the gauge ambiguity [8]. The cleanest approach is to form a pair of gauge-invariant metric variables. By inspection, we see that

$$\Phi = \phi + \frac{1}{a}[(B - E')a]' \tag{1.30}$$

[8] By gauge ambiguity, we mean the existence of four metric variables but only two gauge variables.

$$\Psi = \psi - \frac{a'}{a}(B - E'). \tag{1.31}$$

satisfy this requirement. Therefore, any effect calculated in terms of Φ and Ψ must be observable in virtue of their gauge invariance.

The other option open to us is to fix the gauge by hand. This has the drawback (or, in certain cases, the advantage) or presenting us with a large number of equivalent choices. The transformation equations do not suggest a preferred gauge but a few gauge choices prevail in the literature. Synchronous gauge, defined by $\delta g_{0i} = 0$, or,

$$ds^2_{synch} = a^2(\eta)[-(1 + 2\phi)d\eta^2 + (\delta_{ij} + 2(\psi\delta_{ij} - E_{,ij}))dx^i dx^j]. \tag{1.32}$$

is a popular choice with an important drawback - it is not, properly speaking, a gauge choice as it leaves three degrees of freedom [9], while another popular choice is longitudinal gauge $(B = E = 0)$.

Longitudinal gauge has the interesting property that the gauge-fixed variables ψ, ϕ coincide exactly with the gauge-invariant variables Ψ, Φ, as can be seen from 1.31.

In this thesis, we will exclusively make use of longitudinal gauge (or, equivalently we only employ gauge-invariant variables).

1.2.6 The Fully Relativistic Equations of Motion

Our starting point is, naturally, the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.33}$$

A note on the stress-energy tensor, $T_{\mu\nu}$: in order to maintain generality but still benefit from simplicity, we model all matter in terms of a general scalar field, φ. As discussed above, in addition to our matter fluctuations given by

$$\varphi(\vec{x}, \eta) = \varphi_0(\eta) + \delta\varphi(\vec{x}, \eta), \tag{1.34}$$

(in the above, φ_0 is the matter field which leads to the background configuration $g^{(0)}_{\mu\nu}$) we must consider the presence of a perturbation about a homogeneous and isotropic metric which we write as

[9] Corresponding to four degrees of freedom with only one constraint.

$$g_{\mu\nu}(\vec{x},\eta) = g^{(0)}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\vec{x},\eta). \tag{1.35}$$

In order to determine the dynamics of the fluctuations, we expand the Einstein equations in terms of the amplitude of the fluctuations (which, by assumption, is small) and linearize.

A complication arises when we realize that the components of δG^μ_ν and δT^μ_ν are not gauge-invariant: this difficulty can be circumvented by employing the gauge-invariant quantities defined above.

Our gauge-invariant metric ansatz is

$$ds^2 = a^2(\eta)[(1+2\phi)d\eta^2 - (1-2\psi)\delta_{ij}dx^i dx^j], \tag{1.36}$$

and the perturbed Einstein equations take the form

$$\begin{aligned} -3\mathcal{H}(\mathcal{H}\phi + \psi') + \nabla^2\psi &= 4\pi G a^2 \delta T^0_0 \\ (\mathcal{H}\phi + \psi')_{,i} &= 4\pi G a^2 \delta T^0_i, \end{aligned} \tag{1.37}$$

and

$$\begin{aligned} [(2\mathcal{H}' + \mathcal{H}^2)\phi + \mathcal{H}\phi' + \psi'' + 2\mathcal{H}\psi']\delta^i_j &+ \\ \frac{1}{2}\nabla^2(\phi - \psi)\delta^i_j - \frac{1}{2}g^{ik}(\phi-\psi)_{,kj} &= -4\pi G a^2 \delta T^i_j, \end{aligned} \tag{1.38}$$

where

$$\mathcal{H} = \frac{a'}{a}. \tag{1.39}$$

Our first useful result is immediate: in the absence of anisotropic stress (i.e. $\delta T^i_j = 0$), (1.38) implies that $\phi = \psi$.

We can simplify our equations further by making an ansatz for the form of the matter Lagrangian. If we take

$$S = \int d^4x \sqrt{-g}[\frac{1}{2}\varphi^{,\alpha}\varphi_{,\alpha} - V(\varphi)], \tag{1.40}$$

we obtain

$$\begin{aligned} \nabla^2\phi - 3\mathcal{H}\phi' - (\mathcal{H}' + 2\mathcal{H}^2)\phi &= 4\pi G(\varphi_0'\delta\varphi' + V'a^2\delta\varphi) \\ \phi' + \mathcal{H}\phi &= 4\pi G\varphi_0'\delta\varphi \\ \phi'' + 3\mathcal{H}\phi' + (\mathcal{H}' + 2\mathcal{H}^2)\phi &= 4\pi G(\varphi_0'\delta\varphi' - V'a^2\delta\varphi), \end{aligned} \tag{1.41}$$

Our system of two degrees of freedom with one constraint can be reduced even further to

$$\phi'' + 2\left(\mathcal{H} - \frac{\varphi_0''}{\varphi_0'}\right)\phi' - \nabla^2\phi + 2\left(\mathcal{H}' - \mathcal{H}\frac{\varphi_0''}{\varphi_0'}\right)\phi = 0\,. \tag{1.42}$$

As we could have expected, our equation of motion shows many features in common with the Newtonian EOM. As before, the dynamics are essentially those of a damped harmonic oscillator. The cosmological expansion again plays a role in the damping with an additional contribution arising from the matter field.

On small scales ($\ll H^{-1}$), our solutions are again represented by damped oscillations - as they must be as this corresponds to the Newtonian limit. We again note the presence of an intrinsic scale in the EOM. In this case, the Jeans length is replaced by the Hubble scale ($H = a(\eta)\mathcal{H}^{-1}$). For solutions on scales greater than H^{-1}, we find that the amplitude "freezes out" - in other words, the amplitude becomes static. This result is crucial in understanding the observable effects of the fluctuations and serves to greatly simplify calculations.

Properties of the Solutions

Having discussed the behaviour of the individual modes of 1.2.6, we now consider the characteristics of the spectrum as a whole.

The standard lore is that fluctuations originated during an early period of inflation. These were spontaneously produced from quantum vacuum perturbations. Rapid cosmological expansion stretched the vast majority of modes out to scales far larger than the Hubble scale As was discussed above, modes on scales larger than H freeze out. .

Rapid expansion had the effect of obliterating any features present in the spectrum and any indication of a preferred scale. The result is a scale-invariant (or Harrison-Z'eldovich) power spectrum $P_\varphi(k)$ defined by

$$P_\varphi(k) = k^3|\delta\varphi(k)|^2, \tag{1.43}$$

such that

$$P_\varphi(k) \sim k^{n-1}, \tag{1.44}$$

with $n = 1$, where n is the spectral index.

A more intuitive quantity is the density profile, $\delta_\epsilon = \frac{\delta\rho}{\rho}$ and this can be related to the above quantity via the Poisson equation to yield:

$$|\delta_\epsilon|^2 \sim k^4 |\delta\varphi(k)|^2. \tag{1.45}$$

The position space quantity of relevance is the mass fluctuation contained within a sphere of radius r, which we take to be of the form

$$(\frac{\delta M}{M}(r, t_H(r))^2 \sim r^{1-n}. \tag{1.46}$$

Having assumed a reasonably long period of inflation - which is substantiated by a number of observations i.e. the measured flatness of the universe, the paucity of topological defects, etc. - has led to a theoretical prediction for the spectral index, n. The experimentally determined value of n is $0.951^{+0.015}_{-0.019}$ [15], which lends weight to our naive explanation of the form of the power spectrum of fluctuations.

1.3 Quantum Theory

In the previous section, we made allusion to the mechanism of production of primordial fluctuations and we pointed out that the characteristic scale of production corresponds to the scale of inflation. The magnitude of H during inflation is generally considered to be a few orders of magnitude below the Planck scale, M_{Pl}. Observational constraints from gravitational wave spectrum suggest that the scale of inflation would have been $< 10^{17} GeV$ [157] while the usual value is taken to be $H \sim 10^{13} GeV$ which follows from COBE. Any processes occurring on such minute length scales are expected to be quantum mechanical in nature - clearly, our classical theory should not be used to describe the behaviour of perturbations during the earliest cosmological epoch. With this in mind, we now turn to the quantum theory of cosmological perturbations.

In the absence of a quantum theory of gravity, we find ourselves in a bit of a quandary: how are we to reconcile quantum mechanics with general relativity in order to produce a sensible, renormalizable theory? Luckily, the theory is saved by an experimental fact - the amplitudes of fluctuations on scales of interest are much smaller than unity. This immediately suggests that a perturbative, semi-classical approach might bear fruit.

Our starting point is the Einstein-Hilbert action coupled to scalar field matter:

$$S = \int d^4x\sqrt{-g}[\frac{-R}{16\pi G} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)]. \tag{1.47}$$

Our approach will consist of introducing fluctuations in our physical degrees of freedom and expanding our action about the background up to second order in the fluctuation amplitude - correction terms will be negligible since the amplitude of the fluctuations is measured to be of order 10^{-5}. So far, the treatment seems identical to our approach in the classical case. The difference arises when, after having expanded, we quantize our action [10] which leaves us with a semi-classical theory - a quantum field propagating in a classical spacetime.

We take as our ansatz an FLRW background in conformal coordinates and a spatially homogeneous background scalar field, $\varphi_0(\eta)$:

$$ds^2 = a^2(\eta)[(1+2\phi)d\eta^2 - (1-2\psi)d\vec{x}^2], \tag{1.48}$$

$$\varphi(\eta,\vec{x}) = \varphi_0(\eta) + \delta\varphi(\eta,\vec{x}). \tag{1.49}$$

Note that we've dispensed with issues of gauge by choosing gauge-invariant quantities. If we assume an anisotropic background, the Einstein equations impose the constraint $\phi = \psi$.

The next step is trivial but considerably long - expressing our action in terms of the relevant degrees of freedom. We spare the reader the tedious algebra and simply quote the result from [9]:

$$\begin{aligned} S &= \frac{1}{16\pi G}\int d^4x(a^2[-6(\psi')^2 - 2\psi_{,i}(2\phi_{,i} - \psi_{,i}) + 8\pi G(\delta\varphi'^2 - \delta\varphi_{,i}^2 - V_{,\varphi\varphi}a^2\delta\varphi^2) \\ &+ 16\pi G(\varphi_0'(\phi+3\psi)'\delta\varphi - 2V_\varphi a^2\phi\delta\varphi) + -(16\pi G a^2\varphi_0'\delta\varphi'(\phi+3\psi))' \end{aligned} \tag{1.50}$$

We are now forced to introduce a change of variable in order to reduce the above action to canonical form. For this purpose, we introduce the so-called Mukhanov variable,

$$v = a[\delta\varphi + \frac{\varphi'}{\mathcal{H}}\phi]. \tag{1.51}$$

In terms of v, our action becomes

[10] The action is renormalizable since no terms are of order great than 2

$$S = \frac{1}{2}\int d^4x[v'^2 - v_{,i}v^{,i} + \frac{z''}{z}v^2], \tag{1.52}$$

with

$$z = a\frac{\varphi_0'}{\mathcal{H}}, \tag{1.53}$$

$$\mathcal{H} = \frac{a'}{a}, \tag{1.54}$$

and

$$' = \frac{d}{d\eta}. \tag{1.55}$$

Quite generally, direct calculation shows that $z(\eta) \sim a(\eta)$ in slow-roll and power law inflation. This follows immediately from that fact that $H \approx$ *constant* and thus $\mathcal{H} \sim \varphi_0 \sim 1/a$

We recognize (1.52) as the action for a harmonic oscillator (not surprising as we've expanded to second order) with a negative, time-dependent mass. In fact,

$$\frac{z''}{z} \sim a^2(\eta). \tag{1.56}$$

This result is quite clear. By placing the field v in our dynamical background, the mass of each mode varies as the scale factor. We can easily interpret this result using a particle interpretation - as the spacetime inflates, the number of particles in each mode grows in step with the scale factor.

Quantizing (1.52) is straightforward and can be accomplished in a number of ways [20] as it describes a field with time-dependent mass propagating in a flat, static spacetime.

The initial conditions for our field are taken to be vacuum - we don't trouble ourselves with the lack of an unique vacuum in de Sitter space - usually taken to be the Bunch-Davies vacuum.

1.3.1 The Quantum Hamiltonian and Squeezed States

The Hamiltonian corresponding to the above action can be written down in second quantized form:

$$H = \int d^3\vec{k}[\,k\,(a^{\dagger}_{\vec{k}}\,a_{\vec{k}} + a^{\dagger}_{-\vec{k}}\,a_{-\vec{k}} + 1) - i\frac{z'}{z}\,(a_{\vec{k}}\,a_{-\vec{k}} - h.c.)]. \tag{1.57}$$

Here, $a^{\dagger}_{\vec{k}}$ and $a_{\vec{k}}$ correspond to creation and annihilation operators, respectively.

The first term in the brackets represents back-to-back harmonic oscillators, in phase such that the system has no net momentum. The second term leads to the "squeezing" of the oscillators on scales larger than the Hubble radius $H^{-1}(t)$ (on these scales the second term in (1.57) dominates over the spatial gradient terms coming from the first term in the equation of motion for v). On these scales, the squeezing results in an increase in the mode amplitude

$$v_k(\eta) \sim z(\eta) \sim a(\eta)\,, \tag{1.58}$$

where the second proportionality holds if the equation of state of the background geometry does not change in time. We take this to be the case in our analysis.

This Hamiltonian leads immediately to a particular set of states known as squeezed quantum states as we will see below.

There exists an extensive literature on squeezed states. We refer the reader to [99] and [100] for the mathematical properties of squeezed states. For their physical interest, we direct the reader to [101].

The evolution of a state of a system governed by the Hamiltonian (1.57) can be described by the following evolution operator:

$$U = S(r_k, \varphi_k)\, R(\theta_k), \tag{1.59}$$

where

$$S_{\vec{k}}(\eta) = \exp[\frac{r_k(\eta)}{2}(e^{-2i\varphi_{\vec{k}}(\eta)}a_{-\vec{k}}a_{\vec{k}} - h.c.)], \tag{1.60}$$

and

$$R(\theta_k) = \exp[-i\theta_k(a^{\dagger}_k a_k + a^{\dagger}_{-k}a_{-k})], \tag{1.61}$$

where $S(r_k, \varphi_k)$ is the two-mode squeeze operator, $R(\theta_k)$ is the rotation operator, the real number r_k is known as the squeeze factor, φ_k is the squeezing phase, and θ_k is the rotation angle.. The rotation operator and the phase θ_k play no important role in what follows hence we ignore them from now on.

The action of the squeezing operator on the vacuum results in squeezed vacuum states

$$S_{\vec{k}}(\eta)|0\rangle \equiv |k\rangle = \sum_{n=0}^{\infty} \frac{1}{\cosh(r_k(\eta))}(-e^{2i\varphi_k(\eta)} \tanh(r_k(\eta)))^n \, |n,k;n,-k> . \quad (1.62)$$

The behaviour of the squeezing parameter r_k is completely determined by the background geometry. The evolution of the squeezing parameters is typically very complicated, but an exact solution is known in the case of a de Sitter background [86]:

$$r_k = \sinh^{-1}(\frac{1}{2k\eta}), \quad (1.63)$$

$$\varphi_k = -\frac{\pi}{4} - \frac{1}{2}\arctan(\frac{1}{2k\eta}), \quad (1.64)$$

$$\theta_k = k\eta + \tan^{-1}(\frac{1}{2k\eta}), \quad (1.65)$$

where the vacuum state being operated upon corresponds, again, to the Bunch-Davies vacuum.

The squeezing operator has the property of being a unitary operator acting on a normalizable vacuum state, so that

$$\langle k|k\rangle = 1. \quad (1.66)$$

Although squeezed states do not provide a basis (as they are overcomplete), they do form an orthogonal set of states:

$$\langle l|k\rangle = \delta^l_k. \quad (1.67)$$

This follows from the properties of many particle states.

An important property of squeezed states of which we will make use is the fact that the number of particles in such a state can be expressed entirely in terms of the squeezing parameter via

$$\langle k|N_k|k\rangle = \sinh^2(r_k), \quad (1.68)$$

where N_k is the number operator for the k-mode. Physically, squeezed states represent states which have minimal uncertainty in one variable (high squeezing) of a pair of canonically conjugate variables - the uncertainty in the other is fixed by the requirement that the state saturates the Heisenberg uncertainty bound. For those states of cosmological interest, we take the squeezing to be in momentum.

For our applications, the squeezing parameter will be quite large. As shown in [102],

$$r_k \approx \ln(\frac{a(t_2)}{a(t_1)}), \tag{1.69}$$

where $a(t_1)$ $(a(t_2))$ is the scale factor at first (second) Hubble crossing. For current cosmological scales, $r_k \approx 10^2$.

1.4 Considering the Next Order - Backreaction

In all previous sections, we limited our analysis to the linear theory. It can be expected that the inclusion of higher order terms - interactions between different perturbations modes - will lead to interesting effects.

1.4.1 The Effective Energy Momentum Tensor

The approach we follow is that of the effective Energy Momentum Tensor (EEMT) [11]. Generally speaking, the EEMT can be used to determine the effects of backreaction on the background quantities.

We proceed as follows. Consider the Einstein equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{1.70}$$

We expand both sides in powers of the amplitude of the perturbations

$$G^{(0)}_{\mu\nu} + G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} + ... = 8\pi(T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu} + ...). \tag{1.71}$$

We now *define* the linear fluctuations to be those quantities that satisfy the first order equations, $G^{(1)}_{\mu\nu} = 8\pi T^{(1)}_{\mu\nu}$, so that the above reduces to (we tacitly assume that backreaction has no effect on the linear fluctuations):

$$G^{(0)}_{\mu\nu} + G^{(2)}_{\mu\nu} = 8\pi(T^{(0)}_{\mu\nu} + T^{(2)}_{\mu\nu}). \tag{1.72}$$

The next step consists of rewriting the above equation in the suggestive form

$$G^{(0)}_{\mu\nu} = 8\pi T^{(0)}_{\mu\nu} + 8\pi T^{(2)}_{\mu\nu} - G^{(2)}_{\mu\nu}. \tag{1.73}$$

We now have an equation with which we can determine the effects of the perturbations on the background through the second order terms.The above equation provides us with the means of determining the effect of backreaction on the homogeneous background. As such, solving (1.73) provides us with a metric tensor - different from the original metric tensor which solved the zeroth order part of (1.71) - which incorporates the effects of the fluctuations on the background.

An important point that needs to be addressed is that (1.73) isn't valid in the form presented. The background metric is a strictly spatially homogeneous function - as a result, it cannot be sourced by local functions. We circumvent this problem by averaging over large scales (large compared to H^{-1}) which results in

$$G^{(0)}_{\mu\nu} = 8\pi T^{(0)}_{\mu\nu} + \langle 8\pi T^{(2)}_{\mu\nu} - G^{(2)}_{\mu\nu} > . \tag{1.74}$$

We identify the new term with an effective contribution to the stress-energy tensor:

$$\tau_{\mu\nu} =< T^{(2)}_{\mu\nu} - \frac{G^{(2)}_{\mu\nu}}{8\pi} >, \tag{1.75}$$

so that

$$G_{\mu\nu} = 8\pi[T_{\mu\nu} + \tau_{\mu\nu}] \tag{1.76}$$

We don't consider issues related to gauge transformation here (for that, the reader is referred to [12] and [11]) - suffice it to say, it has been shown that expressing $\tau_{\mu\nu}$ in terms of our first-order gauge invariant variables leads to a first order gauge-invariant quantity.

In order to have a concrete expression for the EEMT, we adopt our usual metric ansatz, namely

$$ds^2 = (1+2\phi)dt^2 - a^2(t)(1-2\phi)\delta_{ij}dx^i dx^j, \tag{1.77}$$

and assume that the spacetime is filled with scalar field matter,

$$\mathcal{L}_{matter} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi). \tag{1.78}$$

This leads to

$$\begin{aligned}
\tau_{00} &= \frac{1}{8\pi G}\left[+12H\langle\phi\dot{\phi}\rangle - 3\langle(\dot{\phi})^2\rangle + 9a^{-2}\langle(\nabla\phi)^2\rangle\right] \\
&+ \langle(\delta\dot{\varphi})^2\rangle + a^{-2}\langle(\nabla\delta\varphi)^2\rangle \\
&+ \frac{1}{2}V''(\varphi_0)\langle\delta\varphi^2\rangle + 2V'(\varphi_0)\langle\phi\delta\varphi\rangle \quad ,
\end{aligned} \tag{1.79}$$

and

$$\begin{aligned}
\tau_{ij} &= a^2\delta_{ij}\left\{\frac{1}{8\pi G}\left[(24H^2 + 16\dot{H})\langle\phi^2\rangle + 24H\langle\dot{\phi}\phi\rangle\right.\right. \\
&+ \left.\langle(\dot{\phi})^2\rangle + 4\langle\phi\ddot{\phi}\rangle - \frac{4}{3}a^{-2}\langle(\nabla\phi)^2\rangle\right] + 4\dot{\varphi_0}^2\langle\phi^2\rangle \\
&+ \langle(\delta\dot{\varphi})^2\rangle - a^{-2}\langle(\nabla\delta\varphi)^2\rangle - 4\dot{\varphi}_0\langle\delta\dot{\varphi}\phi\rangle \\
&- \left.\frac{1}{2}V''(\varphi_0)\langle\delta\varphi^2\rangle + 2V'(\varphi_0)\langle\phi\delta\varphi\rangle\right\} \quad .
\end{aligned} \tag{1.80}$$

This expression is quite general and we will make good use of it chapter 4.

1.4.2 The Infrared Sector

Expanding in the amplitude of the fluctuations would appear to provide us with a rapidly converging perturbative series - this follows from the fact that subsequent terms in the series are at least five orders of magnitude smaller than those at leading order. Although we can claim that cosmology is experiencing its golden age and that the accuracy with which observations are made has vastly improved compared to just a few years ago, we do not yet have the ability to accurately detect effects to five significant figures. In light of this, we could question the utility of developing the theory to second order - at least, at this point in time.

At first glance, $\tau_{\mu\nu}$ would be expected to provide a negligible contribution to the overall energy momentum tensor. However, a crucial observation needs to be made. Having averaged over small scales, we are left to consider the effects of the infrared sector of the theory.

A typical term in $\tau_{\mu\nu}$ has the form

$$\langle\phi^2\rangle_{IR} = \int_{IR}\frac{dk}{k}|\delta_k\phi|^2. \tag{1.81}$$

At all times, the ultraviolet sector is bounded above by the scale of inflation and below by the Hubble scale. On the other hand, the infrared is bounded above by the Hubble scale but it is unbounded from below. Furthermore, during a period of inflation, the number of modes populating the IR phase space grows exponentially due to cosmological redshifting. If we postulate an extended period of inflation it should be expected that the effects $\tau_{\mu\nu}$ has on the background, despite initially being negligible, will rapidly grow in importance. In [12], it was found that during an epoch of inflation, $\tau_{\mu\nu}$ acts as a growing, negative contribution to the cosmological constant. Because of this, backreaction was initially postulated as a mechanism which could serve as a graceful exit mechanism for inflation.

Subsequent to a period of inflation, the UV phase space grows at the expense of the IR - this follows from the fact that, as inflation ends, the Hubble scale gets redshifted and the separation between the IR and UV starts to migrate into the IR. As a result, the importance of backreaction diminishes.

In light of this behaviour, and the fact that $\tau_{\mu\nu}$ acts as a negative contribution to the cosmological constant, it was postulated [12] that spacetime could undergo an endless cycle of periods of exponential inflation punctuated by periods of power law expansion (when the contribution from the IR sector was substantial) - this was to contingent on having a large, positive, bare cosmological constant.

The phenomenology of this model was investigated and the results are included in chapter 4.

1.5 Preface to the rest of the Thesis

This introduction should serve as sufficient background to understand the thesis in its entirety.

The chapters that follow are, effectively, reprints of articles or, as in the case of chapter 3, collections of notes which have led to an article (in this case, the article is reprinted as an appendix).

The chapters are organized as follows: results pertaining to the theory at linear order are presented first (chapters 2 and 3). Backreaction (2nd order) results follow (chapters 4 and 5). Despite this, all of the following chapters can be read independently of each other. Each chapter is self-contained and relies in no way on any other chapter except for the current introductory chapter.

Chapter two is concerned with an important theoretical point that has received relatively little attention in the literature. The quantum and classical theories are both independently understood: what hasn't been as well studied is how to make contact between the two - a necessary consequence of the theory of inflation. This chapter contains a comprehensive review of classicalization and decoherence, along with an original model detailing the quantum-to-classical transition that would have occurred concomitantly with inflation. This work originally appeared as a preprint as astro-ph/0601134.

The aim of the next chapter is to map out the effects of the dilaton and its fluctuations on an innovative mechanism to produce a scale-invariant spectrum of fluctuations in the context of string gas cosmology. This is the only contact this thesis makes with string theory. Specifically, the viability of the mechanism is examined when taking the dilaton into consideration.

Chapter four, the first of the two chapters which discuss backreaction, looks at the late-time phenomenology of the EEMT. What is found is that backreaction can provide a natural solution to the Dark Energy problem without the need to introduce new physics. In fact, it is found that Dark Energy is a natural consequence of the universe having undergone a period of inflation. These results were first presented in astro-ph/0510523.

Chapter five again examines the effects of backreaction. However, instead of looking at the effects on the background, the effects of the backreaction on the perturbation themselves is determined.

The appendix contains a reproduction for an article entitled *More on the spectrum of perturbations in string gas cosmology.* The author's contribution to this publication came about as a result of the work first presented in chapter 3. The reference for this work is JCAP 0611:009,2006.

Chapter 2

First Order - The Quantum to Classical Transition of Linear Perturbations

2.1 Foreword

The introductory chapter presented both the classical and quantum theory of perturbations. Modern cosmology requires that fluctuations originated in the quantum realm and cosmological evolution necessitated a progression to the classical stage. A quantum-to-classical transition is a natural consequence of inflation.

The current chapter presents the results of the author's investigations on the quantum-to-classical transition of cosmological perturbations begun under the supervision of C.P. Burgess and R.H. Brandenberger. This work first appeared as astro-ph/0601134.

2.2 Introduction

As is well known, temperature fluctuations in the CMB and the inhomogeneities that seed structure formation in the universe share a common origin. Both are a result of the scalar metric perturbations produced during inflation. However, these perturbations are of a purely quantum mechanical nature while no cosmological systems of interest (CMB anisotropies, clusters etc.) display any quantal signatures. Presumably, for this to be the case, the primordial density perturbations underwent a quantum-to-classical transition some time between generation during inflation and recombination, when structure first became apparent.

Decoherence is a much studied process (see [82] for a comprehensive review). Although not all conceptual issues have been resolved, it is understood that it can occur whenever a quantum system interacts with an "environment". In other words, this effect can be said to pervade open systems due to the difficulty of creating a truly closed, macroscopic quantum system. Along with its ubiquity, it is also known to be a practically irreversible process, since the loss of quantum correlations in the system is accompanied by an increase in entropy.

Early studies of the classicalization of primordial perturbations focussed on intrinsic properties of the system (see, for example [83],[84]). This was made possible by the application of ideas of quantum optics to the theory of cosmological perturbations. Primordial density fluctuations (the scalars as well as the tensors) evolve into a peculiar quantum state - a *squeezed* vacuum state [85],[86]. By studying the large squeezing limit of these states, it was found that quantum perturbations become indistinguishable from a classical stochastic process. In other words, quantum expectation values in a highly squeezed state are identical to classical averages calculated from a stochastic distribution, up to corrections which vanish in the limit of infinite squeezing. The authors of [84] refer to this as "decoherence without decoherence" while [87] endows the phenomenon with the more technical epithet "quantum non-demolition measurement". We emphasize that these works focussed on the classical properties of the states and not on the coherence properties of the system.

As is well understood, in order to study true classicalization, one must consider two distinct aspects of a system. First the quantum states must evolve, in some limit, into a set of states analogous to classical configurations. The second is that these resultant states interfere with each other in a negligible fashion. This last property constitutes decoherence and is equivalent to the vanishing of the off-diagonal elements of the density matrix.

A truly closed gravitational system is a practical impossibility (unless one considers the totality of the universe to constitute the system as in, for example, quantum cosmology). Since the gravitational interaction has infinite range and couples to all sources of energy, interactions with some sort of environment are an inevitability. As such, environmentally induced decoherence must also be present and would play an important role in the classicalization of primordial density fluctuations.

The purpose of the present chapter is to determine precisely the effects by the "inflationary environment" (we will elucidate this notion below) on cosmological perturbations and to study the resultant decoherence. Other authors have also examined this problem (see, for example [87],[88],[89],[90],[91]) - however, we are the first to present an exact analytic expression for the density matrix with a realistic environment-system interaction.

The chapter is organized as follows: in the next section, we review some basic properties of decoherence of which we will make use. After reviewing the quantum theory of cosmological perturbations in section III, we make clear our concept of the environment and motivate some realistic interactions in section IV. Subsequently, we develop necessary formalism which, in section VI, we make use of to demonstrate the classical nature of the system and calculate the decoherence time scale.

2.3 Decoherence

In the present section, we intend to present an extremely (but, hopefully, not exceedingly) terse account of the theory of decoherence. The physics of classicalization is elegant and subtle and a thorough exposition of its finer points would bring us too far afield from the purpose of this chapter. We confine our attention solely to the cardinal features and disregard any peripheral aspects. The reader unsatisfied by our presentation is encouraged to consult any of a number of excellent reviews of which we mention but a few [92],[93],[82].

From an operational perspective, the process of decoherence usually refers to the disappearance of off-diagonal elements of the density matrix. These elements (phase relations) represent the interference of states inherent in any quantum system. Evidently, their disappearance is an integral part of a quantum-to-classical transition.

Having mathematically defined decoherence, we now turn to the physical processes responsible for it. At the heart lies the concept of the open system and the near impossibility of forming a macroscopic closed state. Virtually all realistic systems must interact with an environment of some sort where, by environment, we refer to degrees of freedom which interact with degrees of freedom in our system but which are not witnessed by some observer intent only on the evolution of the system. This leads to the first important characteristic of decoherence - its *ubiquity.*

Next, we come upon the concept of entangled states. Initially, if we disregard all correlations between system and environment, our composite wave function (system + environment) can be expressed as the outer product of the system and environment states (more generally, it will be the outer product of ensembles of states, as is the case when one makes use of density matrices). Though initially factorizable, interactions between the environment-system pair rapidly change this: the total state evolves from the form

$$|\Psi\rangle = (\sum_i \alpha_i |\phi_i^{system}\rangle) \otimes (\sum_j \beta_j |\Phi_j^{environment}\rangle), \tag{2.1}$$

to

$$|\Psi\rangle = \sum_{i,j} \gamma_{ij} |\phi_i^{system}\rangle |\Phi_j^{environment}\rangle, \tag{2.2}$$

where (2.2) represents an entangled state and, as such, is non-factorizable in this basis. Entanglement is key to the whole process for the following reason - an entangled state produces a density matrix which is non-factorizable. The operational equivalent of an observer ignoring the environmental degrees of freedom is to trace out (partial trace) these degrees of freedom. Due to the orthogonality of the environment states, the observer is left with a density matrix which diagonalizes as the states entangle (the fact that the decoherence rate is related to the rate of entanglement has been used to estimate decoherence times. See, for example, [87],[94]). An interesting property of classicalization follows from this - the interference terms are still present, but are unobservable by a "local" observer (local in the sense that he only observes the system).

These "hidden" interference terms lead us to our next point. By tracing out the environmental degrees of freedom, an observer throws away all the correlation terms, leading to a decrease in the amount of information available in the system - hence, this leads to an increase in the entropy from which we can conclude that decoherence is a practically irreversible process.

The system being decohered, it can only be found in a much smaller subset of the states that were previously allowed - this is what prevents us, in part, from seeing "Schroedinger's Cat" states at a macroscopic level. The states that diagonalize the density matrix of the system are referred to as pointer states [95], and these states remain in the subset of physical states after decoherence. If the evolution of the system is dominated by the self-Hamiltonian of the system, the pointer basis is composed of the eigenstates of the self-Hamiltonian while, if the interaction dominates, the eigenstates of the interaction form the basis [96]. Pointer states are also those states for which the production of entropy during decoherence is minimized (predictability sieve)[97].

Finally, we conclude with a heuristic view of decoherence. Neglecting certain interacting degrees of freedom in a theory will generally lead to an apparent loss of unitarity. Thus, one should expect a flow of probability out of the system which, in turn, manifests itself as a vanishing of certain elements of the density matrix.

2.4 Quantum Perturbations in an Inflationary Universe

2.4.1 The Action for Quantum Perturbations

We provide in this section an overview of the quantum theory of cosmological perturbations in an inflationary background. For a more in-depth treatment, the reader is referred to [9] or [10].

The classical action for an inflationary model is given by (in this and in what follows, we set $G = \hbar = 1$)

$$S = \int d^4x \sqrt{-g}(\frac{1}{16\pi} R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)). \tag{2.3}$$

If the potential $V(\phi)$ for the matter scalar field ϕ is sufficiently flat and if, in addition, initial conditions are chosen for which the kinetic and spatial gradient terms in the energy density are negligible, this action leads to a period of inflation during which the space-time background is close to de Sitter

$$ds^2 = (\frac{\alpha}{\eta})^2(-d\eta^2 + (dx^i)^2), \tag{2.4}$$

where η is conformal time.

During the course of inflation, any pre-existing classical fluctuations are diluted exponentially. However, quantum fluctuations are present at all times in the vacuum state of the matter and metric fluctuations about the classical background space-time. Their wavelengths are stretched exponentially, become larger than the Hubble radius $H^{-1}(t)$ and re-enter the Hubble radius after inflation ends. These fluctuations are hypothesized to be the source of the currently observed density inhomogeneities and microwave background anisotropies. In order for this hypothesis to be correct, the fluctuations must decohere.

The quantum theory of linear fluctuations about a classical background space-time is a well-established subject (see e.g. the reviews [9] or [10]). If the matter has no anisotropic stress (which is the case if matter is described by a collection of scalar fields), then a gauge (coordinate system) can be chosen in which the metric including its (scalar metric) fluctuations [1] (ψ) can be written as

$$ds^2 = (\frac{\alpha}{\eta})^2(-(1+2\psi(x,\eta))\,d\eta^2 + (1-2\psi(x,\eta))\,(dx^i)^2), \tag{2.5}$$

and the matter including its perturbation ($\delta\phi$) is

$$\phi \longrightarrow \phi + \delta\phi(x,\eta). \tag{2.6}$$

The quantum theory of cosmological perturbations is based on the canonical quantization of the metric and matter fluctuations about the classical background given by $a(\eta)$ and $\phi(\eta)$. Since the metric and matter fluctuations are coupled via the Einstein constraint equations, the scalar metric fluctuations contain only one independent degree of freedom. To identify this degree of freedom, we expand the action (2.3) to second order in $\delta\phi$ and ψ, and combine the terms by making use of the so-called Mukhanov variable [98, 24]

$$v = a(\eta)\,[\delta\phi + \frac{\phi'}{\mathcal{H}}\psi], \tag{2.7}$$

in terms of which the perturbed action S_2 takes on a canonical form (the kinetic term is canonical) and the perturbations can hence readily be quantized:

$$S_2 = \frac{1}{2}\int d^4x\,[v'^2 - v_{,i}v_{,i} + \frac{z''}{z}v^2], \tag{2.8}$$

where $z = \frac{a\phi'}{\mathcal{H}}$, and a prime indicates a derivative with respect to η. This action contains no interaction terms: it represents the evolution of a free

[1] We are not considering the vector and tensor metric fluctuations. In an expanding background, the vector perturbations decay, and the tensor fluctuations are less important than the scalar metric modes.

scalar field with a time-dependent square mass

$$m^2 = -\frac{z''}{z}, \tag{2.9}$$

propagating in a flat, static spacetime. This action leads directly to a well-defined quantum theory via the canonical commutation relations.

The Hamiltonian corresponding to the above action S_2 can be written down in second quantized form:

$$H = \int d^3\vec{k}[\, k\,(a^\dagger_{\vec{k}}\, a_{\vec{k}} + a^\dagger_{-\vec{k}}\, a_{-\vec{k}} + 1) - i\frac{z'}{z}\,(a_{\vec{k}}\, a_{-\vec{k}} - h.c.)]. \tag{2.10}$$

The first term in the brackets represents back-to-back harmonic oscillators, in phase such that the system has no net momentum. The second term leads to the "squeezing" of the oscillators on scales larger than the Hubble radius $H^{-1}(t)$ (on these scales the second term in (2.4.1) dominates over the spatial gradient terms coming from the first term in the equation of motion for v). On these scales, the squeezing leads to an increase in the mode amplitude

$$v_k(\eta) \sim z(\eta) \sim a(\eta)\,, \tag{2.11}$$

where the second proportionality holds if the equation of state of the background geometry does not change in time. We take this to be the case in our subsequent analysis.

2.4.2 Properties of Squeezed States

There exists an extensive literature on squeezed states. We refer the reader to [99] and [100] for the mathematical properties of squeezed states. For their physical interest, we direct the reader to [101].

The evolution of a state of a system governed by the Hamiltonian (2.4.1) can be described by the following evolution operator:

$$U = S(r_k, \varphi_k)\, R(\theta_k), \tag{2.12}$$

where

$$S_{\vec{k}}(\eta) = \exp[\frac{r_k(\eta)}{2}(e^{-2i\varphi_{\vec{k}}(\eta)}a_{-\vec{k}}a_{\vec{k}} - h.c.)], \tag{2.13}$$

and

$$R(\theta_k) = \exp[-i\theta_k(a^\dagger_k a_k + a^\dagger_{-k}a_{-k})], \tag{2.14}$$

where $S(r_k, \varphi_k)$ is the two-mode squeeze operator, $R(\theta_k)$ is the rotation operator, and the real number r is known as the squeeze factor. The rotation operator and the phase θ_k play no important role in what follows hence we ignore them from now on.

The action of the squeezing operator on the vacuum results in squeezed vacuum states

$$S_{\vec{k}}(\eta)|0\rangle \equiv |k\rangle = \tag{2.15}$$
$$\sum_{n=0}^{\infty} \frac{1}{\cosh(r_k(\eta))}(-e^{2i\varphi_k(\eta)} \tanh(r_k(\eta)))^n \, |n, k; n, -k> .$$

The behaviour of the squeezing parameter r_k is completely determined by the background geometry. The evolution of the squeezing parameters is typically very complicated, but an exact solution is known in the case of a de Sitter background [86]:

$$r_k = \sinh^{-1}(\frac{1}{2k\eta}), \tag{2.16}$$

$$\varphi_k = -\frac{\pi}{4} - \frac{1}{2} \arctan(\frac{1}{2k\eta}), \tag{2.17}$$

$$\theta_k = k\eta + \tan^{-1}(\frac{1}{2k\eta}), \tag{2.18}$$

where the vacuum state being operated upon corresponds to the Bunch-Davies vacuum.

The squeezing operator has the property of being unitary so that

$$\langle k|k\rangle = 1. \tag{2.19}$$

Although squeezed states do not provide a basis (as they are overcomplete), they do form an orthogonal set of states:

$$\langle l|k\rangle = \delta^l_k. \tag{2.20}$$

This follows from the properties of many particle states.

An important property of squeezed states of which we will make use is the fact that the number of particles in such a state can be expressed entirely in terms of the squeezing parameter via

$$\langle k|N_k|k\rangle = \sinh^2(r_k), \tag{2.21}$$

where N_k is the number operator for the k-mode. Physically, squeezed states represent states which have minimal uncertainty in one variable (high

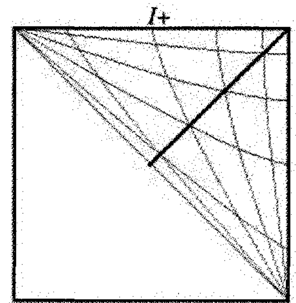

Figure 2.1: The Penrose diagram for de Sitter space in planar coordinates. Note that these coordinates only cover half the spacetime. Blue lines indicate lines of constant t, red lines constant r, and the solid black line represents the horizon.

squeezing) of a pair of canonically conjugate variables - the uncertainty in the other is fixed by the requirement that the state saturates the Heisenberg uncertainty bound. For those states of cosmological interest, we take the squeezing to be in momentum.

For our application, the squeezing parameter will be quite large. As shown in [102],

$$r_k \approx \ln(\frac{a(t_2)}{a(t_1)}), \tag{2.22}$$

where $a(t_1)$ $(a(t_2))$ is the scale factor at first (second) Hubble crossing. For current cosmological scales, $r_k \approx 10^2$.

2.4.3 The Hidden Sector

An essential ingredient in the theory of decoherence is the presence of unobserved, "hidden" degrees of freedom: their interaction with our system degrees of freedom causes the delocalization of the phase relations. In this section, we show that de Sitter space naturally provides us with a hidden sector and that the borderline between the visible and invisible in our theory is naturally given by the Hubble scale.

Although de Sitter space is geodesically complete, a geodesic observer will be subject to the effects from both a particle horizon and an event horizon [103],[104]. That the latter constitutes a true event horizon can best be seen by examining the behaviour of null geodesics in Painleve-de Sitter coordinates (see, for example [105]), which remain finite across the horizon, in contrast to static coordinates. Specifically, we have

$$ds^2 = -(1-\frac{r^2}{l^2})\, dt^2 - 2\frac{r}{l}\, dt\, dr + dr^2 + r^2\, d\Omega^2. \tag{2.23}$$

Here, l($= 1/H$) denotes the de Sitter radius. Clearly, setting $r = l$ causes our timelike coordinate to become spacelike (the characteristic feature of an event horizon). Timelike observers that cross from $r - |\epsilon|$ to $r + |\epsilon|$ find themselves incapable of getting back, trapped outside of a sphere of radius l.

Now, if one transforms to the coordinates typically used when discussing inflation (the so-called planar coordinates) and examines the behaviour of timelike geodesics, one finds that all timelike observers originating within the horizon must eventually cross.

The zero-point fluctuations induced by the horizon [18] can be thought of as the seeds for metric perturbations [106], [107]. Heuristically, the horizon can be thought of as a source of thermal radiation with a temperature $H/2\pi$ (in complete analogy with the black hole case). This radiation then produces gravitational metric perturbations, with the same spectrum, which are stretched out by subsequent cosmological evolution and ultimately lead to the formation of structure in the post-inflationary universe.

Note, however, that this naive picture is not quite correct - the equation of state of the produced radiation is not thermal [19], and including the effects of gravitational back-reaction leads to corrections to the thermal spectrum (this is also true in the black hole case [108]). However, our ensuing discussion in no way relies on strict thermality.

We consider our observer to be to the left of the horizon in fig.1. In accord with our discussion above, we take our radiation to be produced at the horizon with a continuous distribution such that a non-vanishing subset of our modes have wavelengths less than l (or H^{-1}). It follows that our observer in planar coordinates, due to the event horizon, will be prevented from observing certain radiation modes. We conclude that those modes which are unobservable are those associated with physical wavelengths less than the horizon scale. Of course, gravitational redshifting will cause these modes to stretch and eventually cross the horizon. The point is that particle production is a continuous process and we expect that, at all times, a certain set of modes will be unobservable, and these modes will be associated with physical wavelengths less than H^{-1}. As a result of this, decoherence is an inevitability and we define our environment to be a set of modes whose physical momenta are greater than the Hubble scale.

Having identified the modes of the theory which we must trace out, we ask what happens if we trace out additional modes. For example, if an observer was only interested in very low energy modes ($k \ll H$) he could ignore (or trace out) modes with ($k < H$, but not $k \ll H$) - surely this would provide an additional source of decoherence as it increases the environment. However, compare this to the case of an observer who is interested in all super-Hubble modes. The second observer would see less decoherence than the first. Decoherence is, after all, an observer dependent effect - an observer who could monitor every degree of freedom in the universe wouldn't expect to see any decoherence. However, our goal is to determine a lower bound on the amount of decoherence as measured by any observer in the "out" region of our Penrose diagram. In this case, we trace out only those modes which we must (i.e. all modes on sub-horizon scales) and take our system to be composed of the rest.

2.5 Interactions with the Environment

Key to our investigation of decoherence is the notion of the environment. Such an environment can take on many different guises. As was stated above, we define ours in the following fashion: expanding the background fields (gravity and the inflaton) in terms of fluctuations, we identify our environment with the fluctuations whose wavelengths are less than some cut-off, while our system consists of those wavelengths greater than this cutoff. As explained above, since we are operating in a de Sitter background, the natural scale to pick for the cutoff is the Hubble scale.

In order to determine the precise form of interactions inherent to a system of cosmological perturbations, we expand (2.8) to the next order (recall that expanding to second order is what led to a free field theory) in the fluctuations, and express the result in terms of $v(x, \eta)$. Interactions can either be purely gravitational in nature (backreaction), or they can arise in the matter sector through $V(\phi)$, the inflaton potential.

2.5.1 Gravitational Backreaction

To focus on the interactions due to gravitational backreaction, we must expand the gravitational action to third order in the amplitude of the perturbations and write down the potential in terms of the Mukhanov variable v. Expanding to higher order simply introduces more complicated interactions. For our purposes, we restrict our attention to the simplest terms that arise.

In the case where the metric, including its fluctuation field ψ, is given by

$$ds^2 = a^2(\eta)[-(1+2\psi)d\eta^2 + (1-2\psi)(dx^i)^2], \tag{2.24}$$

we can expand the Ricci scalar in powers of ψ to obtain

$$R = \frac{6\,a(\eta)''}{a^3(\eta)(1+2\psi)} = 6\frac{a(\eta)''}{a^3(\eta)}(1-2\psi+4\psi^2-8\psi^3...) \tag{2.25}$$

(where terms with derivatives either temporal or spatial of the ψ have been ignored as they are sub-dominant) from which we can extract our term of interest, $R^{(3)}$,

$$R^{(3)} \equiv -48\frac{a(\eta)''}{a^3(\eta)}\psi^3, \tag{2.26}$$

which is the leading order gravitational self-interaction term. Recalling the definition (2.7) of the Mukhanov variable in a slow-roll inflationary background, our potential, expressed in terms of v, becomes (neglecting $\delta\phi$ when substituting v for ψ and we use the fact that, for our inflationary background, $a(\eta) = 1/(H\eta)$)

$$V = \frac{1}{16\pi M_{Pl}^2}\int d^3x\,\sqrt{-g}R^{(3)} \tag{2.27}$$

$$= \frac{1}{M_{Pl}^2}\int d^3x\,a^4(\eta)\,\frac{4}{\pi}\,\frac{a''(\eta)}{a^3(\eta)}(\frac{\mathcal{H}v}{(\phi)'a})^3$$

$$= \frac{3}{\sqrt{2}\pi}\int d^3x\frac{H^2}{M_{Pl}}\frac{a(\eta)}{(2\epsilon)^{3/2}}v^3, \tag{2.28}$$

so that

$$V \equiv \int d^3x\lambda v^3, \tag{2.29}$$

with

$$\lambda = \frac{3}{\sqrt{2}\pi}\frac{H^2}{M_{Pl}}\frac{1}{(2\epsilon)^{3/2}}a(\eta) = a(\eta)\lambda_0, \tag{2.30}$$

and where we've used the slow roll conditions

$$H^2 = V(\phi)/(3M_{Pl}^2), \quad 3H\dot{\phi} = -V', \tag{2.31}$$

and

$$\epsilon \equiv \frac{M_{Pl}^2}{2}(\frac{V'}{V})^2, \tag{2.32}$$

is one of the slow-roll parameters. Our dimensionful coupling is explicitly time-dependent - this is to be expected since it is associated with a fixed physical scale and our theory (2.8) is written entirely in terms of co-moving quantities.

2.5.2 Inflaton Interactions

In addition to the gravitational backreaction terms, there are also interactions due to non-linearities in the matter evolution equation. Consider a model of chaotic inflation with a potential of the form

$$V = \int d^3x\,\sqrt{-g}\mu\phi^4\,, \tag{2.33}$$

where μ is a dimensionless coupling constant. The perturbations produced during inflation are joint matter and metric fluctuations. The matter part of the fluctuation, denoted by $\delta\phi$, give rise to a cubic term in the interaction potential of the form

$$V \sim \int d^3x\,4\sqrt{-g}\mu\phi(\delta\phi)^3, \tag{2.34}$$

where, in the case of slow-roll inflation, we can treat ϕ as a constant. Now, writing the potential in terms of the Mukhanov variable (and this time neglecting ψ in the process of substitution), we have

$$V \sim \int d^3x 4a^4(\eta)\mu\phi(\frac{v}{a})^3 = \int d^3x\, a(\eta)\,4\mu\phi v^3, \tag{2.35}$$

so that

$$\lambda = 4\mu\phi\, a(\eta). \tag{2.36}$$

How do the coupling strengths of the two potentials compare? Taking the ratio of the two, we find

$$\begin{aligned}\frac{\lambda_{inf}}{\lambda_{grav}} &= \frac{4\mu\phi a(\eta)}{\frac{3\pi}{\sqrt{2}}\frac{H^2}{M_{Pl}}\frac{1}{(2\epsilon)^{3/2}}a(\eta)} \\ &= \frac{4\sqrt{2}}{3\pi}(2\epsilon)^{3/2}\frac{\mu\phi}{H^2}M_{Pl} \\ &= \frac{4\sqrt{2}}{\pi}(2\epsilon)^{3/2}\frac{M_{Pl}^3}{\phi^3}. \end{aligned} \tag{2.37}$$

(2.38)

Since the observationally allowed value for ϕ at times when fluctuations relevant to current observations are generated is of the order $10^{-3}M_{pl}$, we find that the gravitational coupling could conceivably dominate depending on the value of ϵ. Since we are only interested in obtaining a lower bound on the decoherence rate, and due to the fact that the exact form of the inflaton potential (along with the initial conditions that determine ϵ) is model dependent, we consider gravitational backreaction to be the main source of decoherence in what follows. Nonetheless, the above demonstrates that inflaton interactions have the potential to be important.

We couple our system to the environment by writing

$$V = \int d^3x \lambda v^3 \equiv \int d^3x \lambda v^2 \varphi, \tag{2.39}$$

where v now refers only to the expansion of the Mukhanov variable in momenta greater than some cutoff and φ is the same field but expanded in terms of the environment modes.

2.6 The Density Matrix

Having determined a candidate interaction between our system and the environment, we now face the task of deriving an appropriate master equation in order to determine the time dependence of our density matrix. Several approaches exist (for example, [109],[110]) which have been used by a number of authors - rather, we follow the method of [111] which we now review.

We assume that our system of interest is weakly interacting with some environment. The Von Neumann equation for the full density matrix (ρ) reads (note that we make use of conformal time. This is due to the fact that our action (2.8) is expressed in terms of conformal time)

$$\frac{d\rho}{d\eta} = -i[H,\rho], \tag{2.40}$$

where H is the total Hamiltonian of the system and can be written as

$$H = H_0 + V, \tag{2.41}$$

where H_0 is the self-Hamiltonian and V couples the system to the environment. Note that (ρ) denotes the full density matrix for the system *and* the environment.

Switching to the interaction representation (2.40) takes on the form

$$\frac{d\overline{\rho}}{d\eta} = -i[\overline{V},\overline{\rho}], \tag{2.42}$$

where

$$\overline{\rho} = \exp(iH_0\eta)\,\rho\,\exp(-iH_0\eta), \tag{2.43}$$

with a similar expression for $\overline{V}$.

A perturbative solution of (2.42) is found to be given by the following:

$$\begin{aligned} \overline{\rho} &= \rho_0 - i\int_0^{\eta} d\tau\,[\overline{V}(\tau),\rho_0] \\ (-i)^2 & \int_0^{\eta} d\tau_2 \int_0^{\tau_2} d\tau_1 [\overline{V}(\tau_2),[\overline{V}(\tau_1),\rho_0]] + ... \end{aligned} \tag{2.44}$$

Our ultimate goal is to derive an equation of motion for the reduced density matrix ($\rho_A = Tr_B\rho$, where A denotes the system quantities while B refers to the environment. We use this notation throughout the rest of the chapter). To this end, we trace out the environmental degree of freedoms to obtain

$$\overline{\rho_A} = \rho_0^A - \int_0^{\eta} d\tau_2 \int_0^{\tau_2} d\tau_1\, Tr_B[\overline{V}(\tau_2),[\overline{V}(\tau_1),\rho_0]] + ... \tag{2.45}$$

Note that the first order term has vanished - this is due to the specific form of our system-environment coupling. Had we used a potential in which an even power of the environment field had appeared, we would have obtained a non-vanishing contribution at this order. Had this been the case, the first order term could have been neglected on the grounds that it would lead to unitary evolution of the system - since our goal is to study the decoherence of the system (a non-unitary process), we can safely ignore such terms.

We find that (see the appendix)

$$Tr_B\,(\overline{V}(x_1,\eta_1)\,\overline{V}(x_2,\eta_2)\rho) \;=\; \frac{8\pi^2\,a^2(\eta)}{V\mathcal{H}^5}\,\delta(\eta_1-\eta_2)\delta(x_1-x_2). \tag{2.46}$$

In light of the fact that this equation was derived in the limit of small time intervals, we can approximate the integral in (2.45) by the product of the integrand with the time interval, η. Bringing (ρ_0) to the left-hand side and dividing both sides by time allows us to write

$$\frac{\overline{\rho}-\rho_0}{\eta} = \frac{d\overline{\rho}}{d\eta}, \tag{2.47}$$

in the limit of small η.

As the initial time $(\eta = 0)$ is arbitrary, we conclude that our equation for the reduced density matrix may be written as (in terms of physical time)

$$\frac{d\overline{\rho}(t)}{dt} \simeq -a(t)\frac{8\pi^2\lambda^2(t)}{V\mathcal{H}^5}\int d^3x[v^2,[v^2,\overline{\rho}(t)]], \tag{2.48}$$

where the details have been relegated to the appendix. V is a normalization volume, $\mathcal{H} = a(t)H$, with H the physical Hubble scale, and we note that in order to obtain the condition (2.46), it was necessary to eliminate non-local terms by coarse-graining over scales of order $\mathcal{H}$ in both time and space.

The differential equation (2.48) is the master equation for our system. In order to proceed, we obtain a matrix representation in the basis of squeezed states. Again, we point out that these do not form a true basis for the Hilbert space (note, however that the use of an overcomplete basis poses no difficulties when it comes to obtaining representations of the density matrix [92]). However, in the limit of large squeezing, squeezed states become orthogonal to other states in the system. Since squeezed states are the "natural" states of the system, we view all other states as being spurious and truncate our Hilbert space so that it contains only the former. Furthermore, as our interactions are small compared to (1.57), we identify the squeezed states as our pointer basis [112].

Finding a matrix representation of eq.(2.48) is a relatively simple affair - due to the nature of the squeezed states, the expectation value of the operator v^{2n}, with n an integer, must be diagonal in this (discrete) basis of states. This, along with the identities [99]:

$$\begin{aligned} S(r_k,\varphi_k)\,a_{\pm k}\,S^{\dagger}(r_k,\varphi_k) \;&=\; a_{\pm k}\,\cosh(r_k) \\ &+\; a^{\dagger}_{\mp}\,e^{2i\varphi_k}\,\sinh(r_k), \end{aligned} \tag{2.49}$$

and

$$S^{-1}(r_k,\varphi_k) = S^\dagger(r_k,\varphi_k) = S(-r_k,\varphi_k), \tag{2.50}$$

renders the calculation relatively straightforward. Note that $\langle k|N_k|k\rangle = \sinh^2(r_k)$, where N_k is the number operator [99]. With this in mind, we find that (2.48) reduces to

$$\begin{aligned} \frac{d\rho_{ij}}{dt} &\simeq -a(t)\frac{128\pi^2}{V^2}\frac{\lambda^2(t)}{\mathcal{H}^5}(\frac{\sinh^4(r_i)}{k_i^2}+\frac{\sinh^4(r_j)}{k_j^2} \\ &- 2\frac{\sinh^2(r_i)\sinh^2(r_j)}{k_ik_j})\rho_{ij}, \end{aligned} \tag{2.51}$$

where, for simplicity, we've replace the $\cosh^2(r)$ terms with $\sinh^2(r)$ since we are interested in the limit of large r.

The combination $\frac{\sinh^2(r_i)}{V} \equiv n_i(t) = a^2(t)n_i(0)$ is to be interpreted as the particle density, a quantity which is finite in the thermodynamic limit. Clearly, the decoherence rate increases as the difference between the two momenta increases. For this reason, we take our states of interest to have approximately the same momenta, and the above reduces to

$$\frac{d\rho_{ij}}{dt} \simeq -128\pi^2a^2(t)\frac{\lambda_0^2n_i^2(0)}{H^5}\frac{(k_i-k_j)^2}{k_i^2k_j^2}\rho_{ij}, \tag{2.52}$$

in terms of physical time and co-moving momenta and volume.

A few things are immediately obvious:

1) The diagonal elements suffer no loss of coherence. This actually could have been surmised much earlier from eq.(2.45) by noticing that the trace over the system degrees of freedom must vanish.

2) The rate of decoherence grows extremely rapidly. In fact, in order to decohere the system within 60 e-foldings (approximately the minimal time permissible for the duration of inflation), the initial particle density (n_0) can be as low as 10^{-25} particles per Hubble volume. [2]

3) The particular time $t = 0$ for a pair of modes should be taken to correspond to the the time that the shortest of the pair (the higher energy mode) crosses the horizon.

So far, we've argued that a certain sector of the theory is unobservable (thus justifying a minimal amount of tracing), determined an interaction

[2]In arriving at this estimate, we have considered the case where $k_i \approx k_j$, $k_j \approx H$, $H \approx 10^{-3}M_{Pl}$, $\epsilon \approx 10^{-2}$.

between our visible and invisible sectors, and obtained a lower bound on the parameters of the theory such that decoherence takes place within 60 e-foldings of inflation. The question remains: in a realistic cosmological model, are the parameters of the theory such that decoherence can take place during the inflationary period, and be caused by the leading order gravitational back-reaction term? In other words, is the bound we found satisfied in conventional models?

In order to answer that question, we must obtain the number density of particles in a typical super-Hubble mode at first Hubble crossing.

Consider the square of the substitution we used to obtain our potential in terms of the Mukhanov variable:

$$v^2 = a^2(\eta)(\frac{\phi'}{\mathcal{H}})^2\psi^2. \tag{2.53}$$

To determine the number of particles of the v field in terms of physically meaningful quantities, we must first quantize the Mukhanov field. However, once the theory is quantized, the expression (2.53) is meaningless - the left-hand side is an operator, while the right is a classical field. In light of this, we follow the usual route [20] in semi-classical gravity and replace v with it's vacuum expectation value:

$$\langle v^2\rangle = a^2(\eta)(\frac{\phi'}{\mathcal{H}})^2\psi^2. \tag{2.54}$$

In the limit of large squeezing, we have that

$$\langle v^2\rangle = \frac{1}{2\pi^3}\int\frac{d^3k}{k}N_k(t), \tag{2.55}$$

where $N_k(t)$ is the number of particles in the k-mode at time t, which scales in time as

$$N_k(t) \propto a^4(t), \tag{2.56}$$

where we now consider only physical (as opposed to co-moving as in the previous discussions) quantities. The extra factors of $a(t)$ in the particle number appear because we are now considering the red-shifting of the momenta (see (2.16)). We expect the spectrum to be exponentially suppressed at high (sub-Hubble) momenta: therefore, to a good approximation, the integral in (2.55) can be taken to be over the infrared sector only. Furthermore, rather than performing the integral over the modes, we reparameterize and integrate over the times which these particular modes first crossed the horizon. In other words, we let

$$k = \frac{H}{a(t)}, \tag{2.57}$$

and

$$N_k(t) = a^4(t)\, N_H(0), \tag{2.58}$$

where, as above, $t = 0$ denotes first Hubble crossing for a particular mode. We now have,

$$\langle v^2 \rangle \simeq \frac{2}{\pi^2} N_H(0) H^3 \int_0^{t_r} dt a^2(t) = \frac{H^2}{\pi^2} N_H(0) a^2(t_r), \tag{2.59}$$

with t_r denoting the time of reheating and where we've ignored the time-dependence of the Hubble scale.

During reheating, the inflaton will undergo periods when it's total energy is dominated by it's kinetic term. So, during reheating, we can make the substitution $\dot{\phi}^2 \simeq \rho_r$ to obtain

$$\frac{N_H(0)H^2}{\pi^2} a^2(t_r) \simeq a^2(t_r) \frac{\rho_r \psi^2}{H^2}. \tag{2.60}$$

We identify $N_H H^3 = n_H(0)$ with the number of particles of momentum H per Hubble volume and taking the reheating temperature as H so that $\rho_r \simeq H^4$. We can now make use of the fact that, observationally, $\psi^2 \approx 10^{-9}$, to deduce that $n_H(0) \approx 10^{-8}$ particles/Hubble volume. This is well above the lower bound we found. In this case, we find that the modes will decohere approximately 20 e-foldings after crossing the horizon.

2.7 Conclusion

In this chapter, we have studied the decoherence of cosmological fluctuations during a period of cosmological inflation, taking the effects of squeezing into account. We have determined realistic interactions for our system of perturbations and have found that, at the same order, gravitational interactions and matter (inflaton) interactions are comparable, depending on the scale of inflation and the slow-roll parameter ϵ. Furthermore, we have justified the use of Hubble scale as a cutoff.

Having considered the leading order gravitational correction to the action of quantized cosmological perturbations, we find that super-Hubble modes decohere long before the end of inflation. Of course, we have only obtained a lower bound on the decoherence rate - interactions more complicated than the ones considered here will generally lead to much faster decoherence times [94].

2.8 Appendix: Tracing out the Environment

In this appendix, we explicitly calculate the partial trace of eq.(2.45).

The expansion of the Mukhanov variable in a spatially flat background takes the form

$$v = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{k} \frac{1}{\sqrt{2|k|}} (a_k e^{-ikx} + a_k^\dagger e^{ikx}), \tag{2.61}$$

and we restrict our attention to modes within a sphere of radius H in momentum space. Since our calculation will be performed in terms of comoving quantities and we take our cutoff to correspond to a fixed physical scale, our cutoff acquires a time dependence of the form

$$\mathcal{H} = a(\eta)H. \tag{2.62}$$

Our normalization conventions are as follows:

$$|k\rangle = \sqrt{2E_k} a_k^\dagger |0\rangle, \ \ \langle k|k'\rangle = (2\pi)^3 2E_k \delta^{(3)}(k-k'), \tag{2.63}$$

$$[a_k, a_{k'}^\dagger] = (2\pi)^3 \, \delta^{(3)}(k-k'). \tag{2.64}$$

The identity operator has the form

$$1 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k}. \tag{2.65}$$

For simplicity, we ignore the effects of squeezing until the very last. As our initial conditions, we do not take the environment to be in the vacuum - this would be contrary to the basic idea of the generation of inhomogeneities. We take our states to be 2 particle zero-momentum states. Were we to explicitly include the effects of squeezing, we would find that our scattering amplitude $\langle i|\phi^n|j\rangle$ would scales as $a^n(\eta)$. Our approach is as follows: we calculate the scattering amplitude for a fixed particle number (2) and, at the last step, include the additional factors of $a(\eta)$ in order to embody the effects of particle production (squeezing). Note that we must take into account squeezing since (2.16) tells us that all modes in de Sitter space get squeezed.

We take these states to be populated according to a distribution which falls off exponentially in the UV, with temperature parameter $T = \beta^{-1} = \frac{\mathcal{H}}{2\pi}$. In other words

$$\rho_{env} = C\exp(-\beta H), \tag{2.66}$$

where this H refers to the Hamiltonian. The precise form of the distribution is immaterial - after tracing, the only information that the systems retains about the environment is it's "size" (the cutoff scale). As another simplification, we take the energy of the state to be dominated by it's momentum. Due to the nature of squeezing and in view of our comments about the distribution, this is a perfectly justifiable assumption. C is a normalization constant which we determine by the condition that the trace of the left hand side of the equation be ρ_{sys} i.e. $Tr_{env}\rho = \rho_{sys}$.

$$\begin{aligned} Tr_{env}\,\rho &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\frac{\langle k,-k|\rho|k,-k\rangle}{\langle k,-k|k,-k\rangle} \\ &= \frac{C}{2\pi^2}\rho_{sys}\int_{\mathcal{H}}^{\infty} dk\frac{E_k}{2}\exp\left(-2\beta E_k\right) \\ &= \frac{C}{2\pi^2}\rho_{sys}\left(\frac{1}{32\pi^2}\mathcal{H}^2 e^{-4\pi}(1+4\pi)\right) \equiv \rho_{sys}. \end{aligned} \tag{2.67}$$

Therefore, we set $C \approx 16\pi^3 e^{4\pi}/\mathcal{H}^2$.

The terms on the right hand side will all have the basic form (aside from the trace of ρ_0, which is the same as the above):

$$\begin{aligned} RHS &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\frac{\langle k,-k|v(x)v(x')\rho|k,-k\rangle}{\langle k,-k|k,-k\rangle} \\ &= \frac{8\pi}{\delta(0)\mathcal{H}^4}\int_{\mathcal{H}}^{\infty} dk\frac{\sin[k(x-x')]}{k(x-x')}(e^{-i\omega_k(\eta-\eta')})e^{-2\beta E_k}, \end{aligned} \tag{2.68}$$

where the delta function arises from the normalization of the states. Since we are only interested in physics on scales much greater than $\mathcal{H}$, we coarse-grain over time and use the relation

$$\langle e^{i\omega_k(\eta-\eta')}\rangle \approx \frac{\delta(\eta-\eta')}{\mathcal{H}}. \tag{2.69}$$

Thus, we find that

$$RHS = \frac{8\pi}{\delta(0)\mathcal{H}^5}\delta(\eta-\eta')\int_{\mathcal{H}}^{\infty} dk\frac{\sin[k(x-x')]}{k(x-x')}e^{-2\beta E_k}. \tag{2.70}$$

Again, as our interest lies in scales such that $\mathcal{H}(x-x\prime) \gg 1$, we perform the substitution

$$\langle\frac{\sin[k(x-x')]}{k(x-x')}\rangle = \pi\delta(\mathcal{H}(x-x')). \tag{2.71}$$

Finally, we obtain

$$RHS \simeq \frac{8\pi^2}{\delta(0)}\frac{\delta(\eta-\eta')\,\delta(x-x')}{\mathcal{H}^5}a^2(\eta). \tag{2.72}$$

Note that we identify $\delta(0)$ with the volume of space and we've included the additional factors of $a(\eta)$ as dicussed above.

Chapter 3

First Order - Including the Effects of the Dilaton in the NBV Mechanism

3.1 Foreword

String Gas Cosmology began with the seminal paper by Brandenberger and Vaffa [1] which served to usher in an era where string-based cosmological models have become commonplace. This early work, pre-dating the appearance of branes, dispensed with the complexities of string theory, and treated the system as a gas of strings embedded in a universe where the extra dimensions were toroidally compactified. Despite the model's appearance in 1989, it wasn't until 2005 [133] that a mechanism was proposed which could account for the presence of density fluctuations.

The mechanism as presented made no reference to the dilaton other than to state that it was fixed. In this chapter, the author's work on the role played by the dilaton is discussed.

3.2 Introduction

The purpose of the following it to explore the robustness of the mechanism to produce a scale-invariant spectrum in string gas cosmology as presented in [133].

Assuming that a spectrum with the correct power can be produced during the Hagedorn phase, scale-invariance can be broken in at least two indepedent ways:

1. The presence of fluctuations in the dilaton could lead to non-trivial dynamics for the metric perturbation causing deviations from an $n = 1$ spectrum in either the Hagedorn phase or the subsequent radiation epoch.

2. During the Hagedorn phase, a dilaton evolving non-trivially would lead to modes exiting the Hubble radius with amplitudes that differ from the predictions of scale-invariance - this follows immediately from the result that the amplitude of the fluctuation modes is set by the dilaton as can be seen below.

We begin by examining the effects of introducing fluctuations in the dilaton.

3.3 Basic setup

We take the basic action of dilaton cosmology to be given by [28]

$$S = \frac{1}{4\pi\alpha'}\int d^Dx\sqrt{-g}e^{-2\varphi}\Big(R+4(\nabla\varphi)^2-\frac{1}{12}H^2\Big)+\int d^Dx\sqrt{-g}\,\mathcal{L}_{matter}, \quad (3.1)$$

where $\mathcal{L}_{matter}$ is the Lagrangian for a string gas which we treat as a perfect fluid. In what follows, we make a number of simplifying assumptions. We turn off all fluxes $(H = 0)$, and consider the case $D = 4$, without concerning ourselves with the effects of the scalars that arise from the compactification. We trust that the simplifying assumptions retain the essence of the scenario while dispensing with extraneous complications.

Defining our full, higher-dimensional background as [29]

$$ds^2 = e^{2\lambda(\eta)}\Big((1+2\phi)d\eta^2-(1-2\psi)\delta_{ij}dx^idx^j\Big)-e^{2\nu(\eta)}(1-2\xi)\delta_{mn}dx^mdx^n, \quad (3.2)$$

the perturbed equations of motion read (here, $T \equiv T^\mu_\mu$ is the trace and χ is the dilaton fluctuation, $\frac{\delta\phi}{\phi}$. Here and below, repeated latin indices imply summation over space only)

$$\vec{\nabla}^2\psi + 3\vec{\nabla}^2\xi - 9\mathcal{H}\xi' - 3\mathcal{H}\psi' - 3\mathcal{H}^2\phi = \frac{1}{2}e^{2\varphi+2\lambda}\Big(2\chi T^0_0 + \delta T^0_0\Big) - 6\mathcal{H}\phi\phi'$$
$$-3\Psi'\phi' - 6\xi'\phi' - \vec{\nabla}^2\chi + 3\mathcal{H}\psi' + 2\phi\varphi'^2 - 2\chi'\phi', \quad (3.3)$$

$$\partial_i\psi' + 3\partial_i\xi' + \mathcal{H}\partial_i\phi - 3\mathcal{H}\partial_i\xi = \frac{1}{2}e^{2\varphi+2\lambda}\delta T^i_0 + \partial_i\phi\phi' - \partial_i\chi' + \mathcal{H}\partial_i\chi, \quad (3.4)$$

$$\partial_i\partial_j\Big(\phi - \psi - 6\xi - 2\chi\Big) = 0 \qquad i \neq j, \quad (3.5)$$

$$\Big(\partial_i^2 - \vec{\nabla}^2\Big)\Big(\phi - \psi - 6\xi\Big) - 2\psi'' - 6\xi'' - 4\mathcal{H}\psi' - 6\mathcal{H}\xi' - 2\mathcal{H}^2\phi - 4\mathcal{H}'\phi - 2\Phi'\mathcal{H}$$
$$= e^{2\varphi+2\lambda}\Big(2\chi T^i_i + \delta T^i_i\Big) + 2\partial_i^2\chi - 4\phi\phi'' - 2\Phi'\phi' - 4\mathcal{H}\phi\phi' - 4\Psi'\Phi' - 12\xi'\phi'$$
$$+2\chi'' - 2\vec{\nabla}^2\chi + 2\mathcal{H}\chi' + 4\phi\varphi'^2 - 4\chi'\phi' \quad (3.6)$$

$$-\vec{\nabla}^2\phi + 5\vec{\nabla}^2\xi - 5\xi'' + 2\vec{\nabla}^2\psi - 3\psi'' - 10\mathcal{H}\xi' - 3\Phi'\mathcal{H} - 9\mathcal{H}\psi' - 6\mathcal{H}^2\phi - 6\mathcal{H}'\phi$$
$$= e^{2\varphi+2\lambda}\Big(2\chi T^m_m + \delta T^m_m\Big) - 4\phi\phi'' - 2\Phi'\phi' - 8\mathcal{H}\phi\phi' - 6\psi'\phi' - 10\xi'\phi' + 2\chi''$$
$$-2\vec{\nabla}^2\chi + 4\mathcal{H}\chi' + 4\phi\varphi'^2 - 4\chi'\Phi' \quad (3.7)$$

$$-2\phi\varphi'^2 + 2\phi'\chi' + \phi\phi'' - 6\psi\mathcal{H}\phi' + \frac{1}{2}\Phi'\phi' - 2\mathcal{H}\phi\phi' - \frac{3}{2}\psi'\phi' - 3\xi'\phi' - \frac{1}{2}\chi''$$
$$+\frac{1}{2}\vec{\nabla}^2\chi - \mathcal{H}\chi' = \frac{1}{4}e^{2\varphi+2\lambda}\Big(2\chi T + \delta T\Big) \quad (3.8)$$

As stated above, we are interested in the 4 dimensional scenario with fixed dilaton. In this case, the above equations simplify considerably to (taking $\psi = \phi$ and the Newtonian limit)

$$\vec{\nabla}^2\psi = \frac{1}{2}e^{2\varphi}\Big(2\chi T^0_0 + \delta T^0_0\Big) - \vec{\nabla}^2\chi, \quad (3.9)$$

$$\partial_i\psi' = \frac{1}{2}e^{2\varphi}\delta T^i_0 - \partial_i\chi', \quad (3.10)$$

$$\partial_i\partial_j\chi = 0 \qquad i \neq j, \quad (3.11)$$

$$-2\psi'' = e^{2\varphi}\Big(2\chi T^i_i + \delta T^i_i\Big) + 2\chi'',$$

$$-\frac{1}{2}\chi'' + \frac{1}{2}\vec{\nabla}^2\chi = \frac{1}{4}e^{2\varphi}\Big(2\chi T + \delta T\Big), \quad (3.12)$$

As in the standard case, our physical degree of freedom is a linear combination of the metric and scalar perturbations - something not taken into consideration in [133]. A gauge invariant quantity would be

$$\gamma \equiv \psi + \chi. \quad (3.13)$$

In order to determine the power spectrum, we make use of (3.9) and write it as

$$\vec{\nabla}^2\gamma = \frac{1}{2}e^{2\varphi}\big(2\chi T^0_0 + \delta T^0_0\big). \tag{3.14}$$

There is an important difference with the results of [133] in that the overall amplitude of the fluctuations is set by the dilaton - this result will be important later. This result could not have been obtained by the authors of [133] as they disregarded the effects of the dilaton throughout - it was assumed that it was fixed and ignored its dynamics.

3.4 Effects of χ

In order to quantify the effects of the dilaton (and it's fluctuation, χ) on the setup, we need to solve the equations presented in the previous section. The system of equations we're concerned with is the following:

$$\begin{aligned}
\nabla^2(\psi+\chi) &= \frac{e^{2\varphi}}{2}(2\chi T^0_0 + \delta T^0_0),\\
\partial_t^2(\psi+\chi) &= \frac{e^{2\varphi}}{2}(2\chi T^i_i + \delta T^i_i),\\
(-\partial_t^2 + \nabla^2)\chi &= \frac{e^{2\varphi}}{2}(2\chi T + \delta T).
\end{aligned} \tag{3.15}$$

In the case of a radiation equation of state, the dilaton is fixed, and we conclude that

$$T = 0, \delta T = 0, \tag{3.16}$$

and (in 3+1 dimensions)

$$T^0_0 = 3T^i_i, \delta T^0_0 = 3\delta T^i_i. \tag{3.17}$$

Combining the equations leads to the following:

$$(\nabla^2 - \partial_t^2)\psi = \frac{e^{2\varphi}}{3}\delta T^0_0, \tag{3.18}$$

during radiation domination while, in the case of matter domination, the above reduces to

$$(\nabla^2 - \partial_t^2)\psi = 0. \tag{3.19}$$

We've made an important simplifying assumption in order to obtain the above - we've thrown away all terms propotional to the anisotropic stress tensor. This is justified since we expect terms proportional to χ to be subdominant due to supression by factors of the Planck mass. As a result of this assumption, we find that χ can play no role in the dynamics of ψ during the radiation era since $p = 0$ during the Hagedorn phase. In other words, we find that the mechanism is impervious to fluctuations in the dilaton.

3.5 Effects of φ

We now turn to the case where φ is allowed to run. The dilaton freezes in the case of radiation domination so we need only concern ourselves with its dynamics during the Hagedorn phase. We begin with the Tseytlin-Vafa equations [28]:

$$-N(\frac{d\lambda}{dt})^2 + (\frac{d\varphi}{dt})^2 - e^{\varphi}E = 0, \tag{3.20}$$

$$\frac{d^2\lambda}{dt^2} - \frac{d\varphi}{dt}\frac{d\lambda}{dt} - e^{\varphi}P = 0, \tag{3.21}$$

$$2(\frac{d^2\varphi}{dt^2} - N(\frac{d\varphi}{dt})^2) - e^{\varphi}E = 0. \tag{3.22}$$

In the case of interest, $P = 0$, $N = 4$, and the equations reduce to a coupled, first-order, non-linear system:

$$-4\Lambda^2 + 9\Phi^2 - 2\frac{d\Phi}{dt} = 0, \tag{3.23}$$

$$\frac{d\Lambda}{dt} - \Phi\Lambda = 0, \tag{3.24}$$

via the change of variables

$$\Lambda = \frac{d\lambda}{dt}, \quad \Phi = \frac{d\varphi}{dt}, \tag{3.25}$$

in anticipation of the fact that the relevant quantity is not the amplitude of the dilaton but its rate of change - in order to obtain scale-invariant spectrum, the rolling of the dilaton (Φ) would have to be slow compared to the expansion rate of the background (Λ). Deviations from scale-invariance due to the rolling of the dilaton can be quantified in terms of the parameter

$$\alpha = \frac{\Phi}{\Lambda} \tag{3.26}$$

which we require to be $\ll 1$.

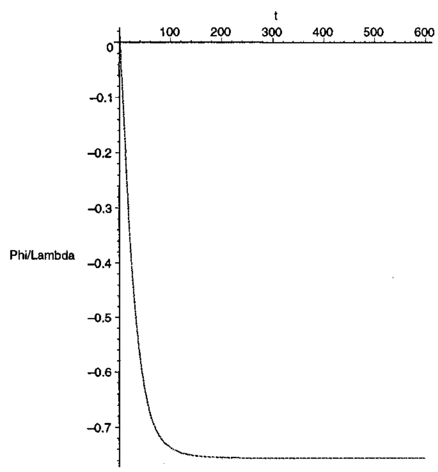

Figure 3.1: Evolution of the quantity $\frac{\Phi}{\Lambda}$. Initial conditions are such that $\frac{\Phi}{\Lambda}$=0.01.

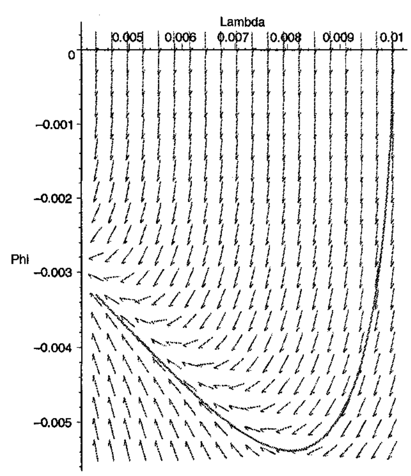

Figure 3.2: Initial conditions are such that Φ/Λ=0.01

The reason we require $\alpha \ll 1$ is as follows: in order to obtain perfect scale-invariance, we require that φ be constant. Short of that, we can expect approximate scale-invariance if the rate at which the amplitude evolves is much slower than the rate at which modes cross the horizon. In other words, we require $\frac{\frac{d\varphi}{dt}}{\frac{d\lambda}{dt}} \equiv \alpha \ll 1$.

The results of solving the TV equations numerically are presented in figure 3.1 and the phase portait is figure 3.2.

According to fig.3.1, for an initially small α $(= 0.01)$, the system evolves to a configuration with $\alpha \simeq 1$ with the phase portrait showing us that this is a stable solution in the Hagedorn phase.

At the transition between the Hagedorn phase and the radiation era, the equation of state varies in a continuous way - we model it as the following: with T being the characteristic time scale associated with the transition.

3.6 Conclusion

In conclusion, we find the following

1. The overall amplitude of the perturbations are set by the value of the dilaton.

2. The mechanism is robust to fluctuations in the dilaton. Such perturbations do not seem to affect the scale-invariance of the produced spectrum.

3. As long as the dilaton remains fixed, the spectrum is scale-invariant. However, when φ is allowed to run, corrections to the spectrum rapidly become important - this is analogous to the case of inflation, when the inflaton begins to roll before reheating.

Chapter 4

Backreaction - Effects on the Background

4.1 Foreword

We begin our examination of backreaction by making use of the EEMT. Our goal in initiating this study was to determine the late-time behaviour of the effective energy momentum tensor. What was discovered was that it provides a natural explanation to the Dark Energy problem.

4.2 Introduction

The nature and origin of dark energy stand out as two of the great unsolved mysteries of cosmology. Two of the more popular explanations are either a cosmological constant Λ, or a new, slowly rolling scalar field (a quintessence field). If the solution of the dark energy problem proved to be a cosmological constant, one would have to explain why it is not 120 orders of magnitude larger (as would be expected in a non-supersymmetric field theory), nor exactly zero (as it would be if some hidden symmetry were responsible for the solution of the cosmological constant problem), and why it has become dominant only recently in the history of the universe. These are the "old" and "new" cosmological constant problems in the parlance of [57]. To date, this has not been accomplished satisfactorily, despite intensive efforts. If, instead of Λ, the solution rested on quintessence, one would need to justify the existence of the new scalar fields with the finely tuned properties required of a quintessence field (e.g. a tiny mass of about 10^{-33}eV if the field is a standard scalar field). Clearly, both of the above approaches to explaining dark energy lead directly to serious, new cosmological problems. In this chapter, we will explore an approach to explaining dark energy which does not require us to postulate any new matter fields.

There exist tight constraints on Λ from various sources - Big Bang Nucleosynthesis (BBN) [58], cosmic microwave background (CMB) anisotropies [59], cosmological structure formation [60] - which rule out models where the vacuum energy density is comparable to the matter/radiation energy density at the relevant cosmological times in the past. However, it could still be hoped that a variable Λ model might be compatible with observation since the value of ρ_Λ is constrained only for certain redshifts. In fact, the above constraints taken together with the results from recent supernovae observations [61],[62] leads one to posit that the vacuum energy density might be evolving in time.

This leads directly to the proposal of tracking quintessence [63]. However, some of the drawbacks of quintessence were mentioned above. A preferable solution would combine the better features of both quintessence and a cosmological constant: a tracking cosmological "constant".

In this chapter, we discuss the possibility that the energy-momentum tensor of long wavelength cosmological perturbations might provide an explanation of dark energy. The role of such perturbations in terminating inflation and relaxing the bare cosmological constant was investigated some time ago in [11, 12] (see also [64]). However, this mechanism can only set in if the number of e-foldings of inflation is many orders of magnitude larger than the number required in order to solve the horizon and flatness problems of Standard Big Bang cosmology. Here, we are interested in inflationary models with a more modest number of e-foldings. We discover that, in this context, the EMT of long wavelength cosmological perturbations results in a tracking cosmological "constant" of purely gravitational origin and can be used to solve the "new" cosmological constant problem.

We begin by reviewing the formalism of the effective EMT of cosmological perturbations in Section 2. We recall how, in the context of slow-roll inflation, it could solve the graceful exit problem of certain inflationary models. We then extend these results beyond the context of slow-roll inflation in Section 3. In Section 4, we investigate the behaviour of the EMT during the radiation era and show that the associated energy density is sub-dominant and tracks the cosmic fluid. We examine the case of the matter era and show how the

EMT can solve the dark energy problem in section 5. In Section 6 we consider the effects of back-reaction on the scalar field dynamics. We then summarize our results and comment on other attempts to use the gravitational back-reaction of long wavelength fluctuations to explain dark energy.

4.3 The EMT

The study of effective energy-momentum tensors for gravitational perturbations is not new [65, 66]. The interests of these early authors revolved around the effects of high-frequency gravitational waves. More recently, these methods were applied [11, 12] to the study of the effects of long-wavelength scalar metric perturbations and its application to inflationary cosmology.

The starting point was the Einstein equations in a background defined by

$$\begin{aligned} ds^2 &= a^2(\eta)((1+2\Phi(x,\eta))d\eta^2 \\ &- (1-2\Phi(x,\eta))(\delta_{ij}dx^i dx^j)) \end{aligned} \tag{4.1}$$

where η is conformal time, $a(\eta)$ is the cosmological scale factor, and $\Phi(x,\eta)$ represents the scalar perturbations (in a model without anisotropic stress). We are using longitudinal gauge (see e.g. [9] for a review of the theory of cosmological fluctuations, and [10] for a pedagogical overview). Matter is, for simplicity, treated as a scalar field φ.

The modus operandi of [11] consisted of expanding both the Einstein and energy-momentum tensor in metric (Φ) and matter ($\delta\varphi$) perturbations up to second order. The linear equations were assumed to be satisfied, and the remnants were spatially averaged, providing the equation for a new background metric which takes into account the back-reaction effect of linear fluctuations computed up to quadratic order

$$G_{\mu\nu} = 8\pi G\,[T_{\mu\nu} + \tau_{\mu\nu}], \tag{4.2}$$

where $\tau_{\mu\nu}$ (consisting of terms quadratic in metric and matter fluctuations) is called the effective EMT.

The effective energy momentum tensor, $\tau_{\mu\nu}$, was found to be

$$\begin{aligned} \tau_{00} &= \frac{1}{8\pi G}\left[+12H\langle\phi\dot{\phi}\rangle - 3\langle(\dot{\phi})^2\rangle + 9a^{-2}\langle(\nabla\phi)^2\rangle\right] \\ &+ \langle(\delta\dot{\varphi})^2\rangle + a^{-2}\langle(\nabla\delta\varphi)^2\rangle \\ &+ \frac{1}{2}V''(\varphi_0)\langle\delta\varphi^2\rangle + 2V'(\varphi_0)\langle\phi\delta\varphi\rangle \quad , \end{aligned} \tag{4.3}$$

and

$$
\begin{aligned}
\tau_{ij} &= a^2\delta_{ij}\left\{\frac{1}{8\pi G}\left[(24H^2+16\dot{H})\langle\phi^2\rangle+24H\langle\dot{\phi}\phi\rangle\right.\right. \\
&+ \left.\langle(\dot{\phi})^2\rangle+4\langle\phi\ddot{\phi}\rangle-\frac{4}{3}a^{-2}\langle(\nabla\phi)^2\rangle\right]+4\dot{\varphi_0}^2\langle\phi^2\rangle \\
&+ \langle(\delta\dot{\varphi})^2\rangle-a^{-2}\langle(\nabla\delta\varphi)^2\rangle-4\dot{\varphi}_0\langle\delta\dot{\varphi}\phi\rangle \\
&- \left.\frac{1}{2}V''(\varphi_0)\langle\delta\varphi^2\rangle+2V'(\varphi_0)\langle\phi\delta\varphi\rangle\right\} \quad , \qquad (4.4)
\end{aligned}
$$

where H is the Hubble expansion rate and the $\langle\rangle$ denote spatial averaging.

Specializing to the case of slow-roll inflation (with φ as the inflaton) and focusing on the effects of long wavelength or IR modes (modes with wavelength larger than the Hubble radius), the EMT simplifies to

$$
\tau_0^0 \cong \left(2\frac{V''V^2}{V'^2}-4V\right) < \phi^2 > \cong \frac{1}{3}\tau_i^i, \qquad (4.5)
$$

and

$$
p \equiv -\frac{1}{3}\tau_i^i \cong -\tau_0^0 . \qquad (4.6)
$$

so that $\rho_{eff} < 0$ with the equation of state $\rho = -p$.

The factor $\langle\phi^2\rangle$ is proportional to the IR phase space so that, given a sufficiently long period of inflation (in which the phase space of super-Hubble modes grows continuously), τ_0^0 can become important and act to cancel any positive energy density (i.e. as associated with the inflaton, or a cosmological constant) and bring inflation to an end - a natural graceful exit, applicable to any model in which inflation proceeds for a sufficiently long time.

Due to this behaviour during inflation, it was speculated [67] that this could also be used as a mechanism to relax the cosmological constant, post-reheating - a potential solution to the old cosmological constant problem. However, this mechanism works (if at all - see this discussion in the concluding section) only if inflation lasts for a very long time (if the potential of φ is quadratic, the condition is that the initial value of φ is larger than $m^{-1/3}$ in Planck units).

4.4 Beyond Slow-Roll

Here, we will ask the question what role back-reaction of IR modes can play in those models of inflation in which inflation ends naturally (through the reheating dynamics of φ) before the phase space of long wavelength modes has time to build up to a dominant value. In order to answer this question, we require an expression for $\tau_{\mu\nu}$ unfettered by the slow-roll approximation. Doing this provides us with an expression for the EMT which is valid during preheating and, more importantly, throughout the remaining course of cosmological evolution.

In the long wavelength limit, we have [1],

$$\tau_{00} = \frac{1}{2}V''(\varphi_0)\langle\delta\varphi^2\rangle + 2V'(\varphi_0)\langle\phi\delta\varphi\rangle, \tag{4.7}$$

and

$$\begin{aligned}\tau_{ij} &= a^2\delta_{ij}\left\{\frac{1}{8\pi G}\left[(24H^2 + 16\dot{H})\langle\phi^2\rangle\right] + 4\dot{\varphi_0}^2\langle\phi^2\rangle\right\} \\ &- \left.\frac{1}{2}V''(\varphi_0)\langle\delta\varphi^2\rangle + 2V'(\varphi_0)\langle\phi\delta\varphi\rangle\right\}.\end{aligned} \tag{4.8}$$

As in the case of slow-roll, we can simplify these expressions by making use of the constraint equations which relate metric and matter fluctuations [9], namely

$$-(\dot{H} + 3H^2)\phi \simeq 4\pi G V'\delta\varphi\,. \tag{4.9}$$

Then, (4.7) and (4.8) read

$$\tau_{00} = (2\kappa^2\frac{V''}{(V')^2}(\dot{H} + 3H^2)^2 - 4\kappa(\dot{H} + 3H^2))\langle\phi^2\rangle, \tag{4.10}$$

$$\begin{aligned}\tau_{ij} &= a^2\delta_{ij}(12\kappa(\dot{H} + H^2) + 4\dot{\varphi_0(t)}^2 \\ &- 2\kappa^2\frac{V''}{(V')^2}(\dot{H} + 3H^2)^2)\langle\phi^2\rangle,\end{aligned} \tag{4.11}$$

with $\kappa = \frac{M_{Pl}^2}{8\pi}$.

The above results are valid for all cosmological eras. With this in mind, we now turn an eye to the post-inflation universe and see what the above implies about its subsequent evolution.

In what follows, we take the scalar field potential to be $\lambda\varphi^4$. As was shown in [68], the equation of state of the inflaton after reheating is that of radiation, which implies $\varphi(t) \sim 1/a(t)$.

[1] We've ignored terms proportional to $\dot{\phi}$ on the basis that such terms are only important during times when the equation of state changes. Such changes could lead to large transient effects during reheating but would be negligible during the subsequent history of the universe.

4.5 The Radiation Epoch

The radiation epoch followed on the heels of inflation. The EMT in this case reads

$$\tau_{00} = (\frac{1}{16}\kappa^2\frac{V''}{(V')^2}\frac{1}{t^4} - \frac{\kappa}{t^2})\langle\phi^2\rangle, \tag{4.12}$$

$$\tau_{ij} = a^2(t)\delta_{ij}(-3\frac{\kappa^2}{t^2} + 4(\dot{\varphi})^2 - \frac{1}{16}\kappa^2\frac{V''}{(V')^2}\frac{1}{t^4})\langle\phi^2\rangle. \tag{4.13}$$

The first thing we notice is that, if the time dependence of $\langle\phi^2\rangle$ is negligible, the EMT acts as a tracker with every term scaling as $1/a^4(t)$ (except for the $\dot{\varphi}$ which scales faster and which we ignore from now on).

We now determine the time dependence of $\langle\phi^2\rangle$, where

$$\langle\phi^2\rangle = \frac{\psi^2}{V}\int d^3\vec{x}\; d^3\vec{k_1}\, d^3\vec{k_2}\, f(\vec{k_1})f(\vec{k_2})e^{i(\vec{k_1}+\vec{k_2})\cdot\vec{x}}, \tag{4.14}$$

with

$$f(\vec{k}) = \sqrt{V}(\frac{k}{k_n})^{-3/2-\xi}k_n^{-3/2}e^{i\alpha(\vec{k})}. \tag{4.15}$$

Here, ψ represents the amplitude of the perturbations (which is constant in time), ξ represents the deviation from a Harrison-Zel'dovich spectrum, $\alpha(\vec{k})$ is a random variable, and k_n is a normalization scale.

Taking $\frac{\Lambda}{a(t)}$ as a time-dependent, infra-red cutoff and the Hubble scale as our ultra-violet cutoff, and focusing on the case of a nearly scale-invariant spectrum, the above simplifies to

$$\langle\phi^2\rangle = 4\pi\psi^2 k_n^{-2\xi}\int_{\frac{\Lambda}{a(t)}}^{H} dk_1\frac{1}{k_1^{1-2\xi}} \tag{4.16}$$

$$\tag{4.17}$$

In the limit of small ξ, the above reduces to

$$\langle\phi^2\rangle \cong 4\pi\psi^2\ln(\frac{a(t)H}{\Lambda}). \tag{4.18}$$

The time variation of the above quantity is only logarithmic in time and hence not important for our purposes. As well, given the small amplitude of the perturbations, $\langle\phi^2\rangle \ll 1$. Note that this condition is opposite to what needs to happen in the scenario when gravitational back-reaction ends inflation.

Now that we have established that the EMT acts as a tracker in this epoch, we still have to determine the magnitude of τ_{00} and the corresponding equation of state. In order to do this, as in [68], we assume that the preheating temperature is $T = 10^{12}$GeV, the quartic coupling $\lambda = 10^{-12}$, and the inflaton amplitude following preheating is $\varphi_0 = 10^{-4}M_{Pl}$. Making use of

$$a(t) = (\frac{32\pi\rho_0}{3M_{Pl}^2})^{1/4}t^{1/2}, \tag{4.19}$$

where ρ_0 is the initial energy density of radiation, we find

$$\tau_{00} = -\kappa(\frac{32\pi\rho_0}{3M_{Pl}^2})\frac{1}{a^4(t)}[1-\frac{1}{8}]\langle\phi^2\rangle \cong -\frac{4}{3}\frac{\rho_0}{a^4(t)}\langle\phi^2\rangle, \tag{4.20}$$

$$\tau_{ij} = -a^2(t)\delta_{ij}\kappa(\frac{32\pi\rho_0}{3M_{Pl}^2})\frac{1}{a^4(t)}[3+\frac{1}{8}]\langle\phi^2\rangle \cong -4\frac{\rho_0}{a^4(t)}\langle\phi^2\rangle. \tag{4.21}$$

We find that, as in the case of an inflationary background, the energy density is negative. However, unlike during inflation, the equation of state is no longer that of a cosmological constant. Rather, $w \cong 3$. Clearly, due to the presence of $\langle\phi^2\rangle$, this energy density is sub-dominant. Using the value of ψ in (4.16) determined by the normalization of the power spectrum of linear fluctuations from CMB experiments [69], we can estimate the magnitude to be approximately four orders of magnitude below that of the cosmic fluid. Any observational constraints that could arise during the radiation era (e.g. from primordial nucleosynthesis, or the CMB) will hence be satisfied.

4.6 Matter Domination

During the period of matter domination, we find that the EMT reduces to

$$\tau_{00} = (\frac{2}{3}\frac{\kappa^2}{\lambda}\frac{a^4(t)}{\varphi^4}\frac{1}{t^4} - \frac{8}{3}\frac{1}{t^2})\langle\phi^2\rangle. \tag{4.22}$$

$$\tau_{ij} = (-\frac{2}{3}\frac{\kappa^2}{\lambda}\frac{a^4(t)}{\varphi^4}\frac{1}{t^4} - \frac{8}{3}\frac{1}{t^2})\langle\phi^2\rangle. \tag{4.23}$$

In arriving at these equations, we are assuming that the matter fluctuations are carried by the same field φ (possibly the inflaton) as in the radiation epoch, a field which scales in time as $a^{-1}(t)$ [2]. This result is quite different from what was obtained in the radiation era for the following reason: previously, we found that both terms in τ_{00} scaled in time the same way. Now,

[2]Even if we were to add a second scalar field to represent the dominant matter and add a corresponding second matter term in the constraint equation (4.9), it can be seen that the extra terms in the equations for the effective EMT decrease in time faster than the dominant term discussed here.

we find (schematically)

$$\tau_{00} \propto \frac{\kappa^2}{a^2(t)} - \frac{\kappa}{a^3(t)}. \tag{4.24}$$

The consequences of this are clear: the first term will rapidly come to dominate over the second, which is of approximately the same magnitude at matter-radiation equality. This will engender a change of sign for the energy density and cause it to eventually overtake that of the cosmic fluid. The same scaling behaviour is present in τ_{ij} and so the equation of state of the EMT will rapidly converge to that of a cosmological constant, but this time one corresponding to a positive energy density.

Matter-radiation equality occurred at a redshift of about $z \approx 10^4$ and we find that

$$\tau_{00}(z=0) \simeq \rho_m(z=0), \quad w \simeq -1, \tag{4.25}$$

and thus we are naturally led to a resolution of the both aspects of the dark energy problem. We have an explanation for the presence of a source of late-time acceleration, and a natural solution of the "coincidence" problem: the fact that dark energy is rearing its head at the present time is directly tied to the observationally determined normalization of the spectrum of cosmological perturbations.

4.7 Dark Energy Domination and Inflaton Back-reaction

Does this model predict that, after an initial stage of matter domination, the universe becomes perpetually dominated by dark energy? To answer this question, one needs to examine the effects of back-reaction on the late time scalar field dynamics.

The EMT predicts an effective potential for φ that differs from the simple form we have been considering so far. During slow-roll, we have that

$$V_{eff} = V + \tau_0^0. \tag{4.26}$$

One might expect that this would lead to a change in the spectral index of the power spectrum or the amplitude of the fluctuations. To show that this is not the case, we can explicitly calculate the form of V_{eff} for the case of an arbitrary polynomial potential and see that, neglecting any φ dependence of $\langle\phi^2\rangle$, (4.26) implies an (a priori small) renormalization of the scalar field coupling. We find that the inclusion of back-reaction does not lead to any change in the spectral index (in agreement with [70]) or to any significant change in the amplitude of the perturbations.

During radiation domination, we find that the ratio of $\frac{\tau^0_0}{V}$ is fixed and small, so that scalar field back-reaction does not play a significant role in this epoch. In fact, back-reaction on the scalar field does not become important until back-reaction begins to dominate the cosmic energy budget. In that case,

$$V_{eff} \sim \frac{1}{\varphi^4}, \tag{4.27}$$

causing the φ to "roll up" it's potential. Once φ comes to dominate, the form of the effective potential changes to

$$V_{eff} \sim \varphi^4, \tag{4.28}$$

and φ immediately rolls down it's potential, without the benefit of a large damping term (given by the Hubble scale).

Thus, this model predicts alternating periods of dark energy/matter domination, which recalls the ideas put forth in [67].

From the point of view of perturbation theory, we see that in the regime where the higher-order terms begin to dominate and the series would be expected to diverge, these corrections are then suppressed and become subdominant again.

4.8 Discussion and Conclusions

To recap, we find that, in the context of inflationary cosmology, the EMT of long wavelength cosmological perturbations can provide a candidate for dark energy which resolves the "new cosmological constant" (or "coincidence" problem in a natural way. Key to the success of the mechanism is the fact that the EMT acts as a tracker during the period of radiation domination, but redshifts less rapidly than matter in the matter era. The fact that our dark energy candidate is beginning to dominate today, at a redshift 10^4 later than at the time of equal matter and radiation is related to the observed amplitude of the spectrum of cosmological perturbations.

We wish to conclude by putting our work in the context of other recent work on the gravitational back-reaction of cosmological perturbations. We are making use of non-gradient terms in the EMT (as was done in [11, 12]). As was first realized by Unruh [32] and then confirmed in more detail in [33, 71], in the absence of entropy fluctuations, the effects of these terms are not locally measurable (they can be undone by a local time reparametrization). It is important to calculate the effects of back-reaction on local observables measuring the expansion history. It was then shown [31] (see also [72]) that in the presence of entropy fluctuations, back-reaction of the non-gradient terms is physically measurable, in contrast to the statements recently made in [73] [3]. In our case, we are making use of fluctuations of the scalar field φ at late times. As long as this fluctuation is associated with an isocurvature mode, the effects computed in this chapter using the EMT approach should also be seen by local observers.

Our approach of explaining dark energy in terms of back-reaction is different from the proposal of [75]. In that approach, use is made of the leading gradient terms in the EMT. However, it has subsequently been shown [76] that these terms act as spatial curvature and that hence their magnitude is tightly constrained by observations. Other criticism was raised in [51] where it was claimed that, in the absence of a bare cosmological constant, it is not possible to obtain a cosmology which changes from deceleration to acceleration by means of back-reaction. This criticism is also relevant for our work. However, as pointed out in [78], there are subtleties when dealing with spatially averaged quantities, even if the spatial averaging is over a limited domain, and that the conclusions of [51] may not apply to the quantities we are interested in.

There have also been attempts to obtain dark energy from the back-reaction of short wavelength modes [42, 79, 80]. In these approaches, however, nonlinear effects are invoked to provide the required magnitude of the back-reaction effects.

We now consider some general objections which have been raised regarding the issue of whether super-Hubble-scale fluctuations can induce locally measurable back-reaction effects. The first, and easiest to refute, is the issue of causality. Our formalism is based entirely on the equations of general relativity, which are generally covariant and thus have causality built into them. We are studying the effects of super-Hubble but sub-horizon fluctuations [4].

[3]There are a number of problems present in the arguments of [73], in addition to this point. We are currently preparing a response that addresses the criticisms of these authors. See [74].

[4]We remind the reader that it is exactly because inflation exponentially expands the

Another issue is locality. As shown in [81], back-reaction effects such as those discussed here can be viewed in terms of completely local cosmological equations. For a more extensive discussion, the reader is referred to [74].

In conclusion, we have presented a model which can solve the dark energy problem without resorting to new scalar fields, making use only of conventional gravitational physics. The effect of the back-reaction of the super-Hubble modes is summarized in the form of an effective energy-momentum tensor which displays distinct behaviour during different cosmological epochs.

horizon compared to the Hubble radius that the inflationary paradigm can create a causal mechanism for the origin of structure in the universe. In our back-reaction work, we are using modes which, like those which we now observe in the CMB, were created inside the Hubble radius during the early stages of inflation, but have not yet re-entered the Hubble radius in the post-inflationary period.

Chapter 5

Backreaction - Effects on the Perturbations

5.1 Foreword

Forsaking the EEMT for a mores straightforward perturbative approach, we now take a look at the extent of the effects of backreaction on the fluctuations. Specifically, we examine the consequences of including higher-order terms on the form of the power spectrum and make an estimate as to the extent of the non-gaussianity induced by backreaction.

5.2 Introduction

The study of cosmological fluctuations is one of the cornerstones of modern cosmology. In order for a cosmological model to be considered successful, it must be able to reproduce, among other things, the power spectrum of the perturbations. These perturbations leave their mark as anisotropies in the cosmic microwave background (CMB) and go on to act as seeds for structure formation. The theory of cosmological perturbations establishes the bridge between observations (namely observations of fluctuations in the CMB and in the distribution of structure in the universe) and the physics of the very early universe which is responsible for providing the generation mechanism for the fluctuations.

At the present time, however, inflationary cosmology does not quite have the status of a *theory*. It is best thought of as a successful scenario that resolves many of the problems that plague Big Bang cosmology. There are a large number of different *models* that result in accelerated (i.e. inflationary) expansion. However, many models share a common feature in that they involve a scalar field (the "inflaton") which undergoes a period in which it rolls slowly down it's potential - leading to what is known as "slow-roll inflation". Our analysis will be in the context of a general slow roll inflation model.

The theory of cosmological perturbations (see e.g. [9] for a comprehensive review, and [10] for a recent abbreviated overview), a formalism crucial to the understanding of CMB anisotropies, is usually studied within the framework of linearized gravity. One writes down an ansatz for the form of the perturbed metric about a homogeneous and isotropic background space-time, linearizes the Einstein equations in the amplitude of the perturbations, and solves the resulting equations. In this scheme, all Fourier modes of the fluctuations evolve independently, as must be the case in a linear approximation. The Einstein equations which govern the evolution of space-time and matter are, however, nonlinear. Thus, retaining terms quadratic and higher in the perturbation amplitude leads to interactions between different perturbation modes. These interactions determine the "gravitational back-reaction", the difference between the full evolution of the space-time and what would be obtained in linear theory, and will lead to potentially important modifications of the results obtained at linear order. In particular, they may effect the key qualitative predictions of inflation, namely the scale invariance of the spectrum and its Gaussianity.

The period of inflation in most scalar field-driven inflationary models is very long (measured in units of the Hubble time during inflation). Thus, the red-shifting of scales leads to the population of a large phase space of long wavelength modes (modes with a wavelength larger than the Hubble radius). The back-reaction of such long wavelength modes on the background space-time was first studied in [11, 12] (making use e of the concept of an effective energy-momentum tensor of fluctuations first used in studies of short wavelength gravitational waves [30]). It was found that this effective energy-momentum tensor acts like a negative cosmological constant with a magnitude which increases in time as the phase space of long wavelength modes grows. A physical explanation of this effect in the quasi-homogeneous approximation to the evolution equations was recently provided in [81]. The effect can become non-perturbatively large [12] if the period of inflation is sufficiently long and leads to a change in the Hubble expansion rate. This change is physically measurable in models with at least two matter fields [31]. In models with only one matter field, however, the leading infrared back-reaction terms are not physically measurable by a local observer [32, 33, 34] (see also [35], and see [36] for a review of previous work on gravitational back-reaction in inflationary cosmology [1]).

Given that the back-reaction of first order cosmological perturbations on the background cosmology can become large, it is important to determine whether this gravitational back-reaction can also lead to large effects on the fluctuations themselves, potentially changing their key characteristics like almost scale-invariance and Gaussianity. It is the intent of this chapter to solve the perturbed Einstein equations to quadratic order, and determine the modifications to the results of the linear analysis, focusing on the effects on long-wavelength fluctuations. There has been a significant body of previous work devoted to second order cosmological perturbations, see e.g. [37, 38, 39, 40, 41, 43] and papers quoted therein. The effect of long-wavelength modes has also been studied in the "separate universe" approach [44, 45, 35], in which the effect of long-wavelength perturbations in encoded as a change in the background geometry. Our approach is similar. In particular, we will neglect spatial gradients in the equations of motion, and thus are focusing on the leading infrared contributions to back-reaction. What differentiates our analysis from previous work is the emphasis on the fact that, in an inflationary universe, modes continuously move into the infrared sector (wavelengths greater than the Hubble radius), and that thus the infrared phase space grows. This leads to the concern that back-reaction effects grow without limits, a concern which is the main motivation for our work.

The main result of our study, however, is that the leading infrared contributions of back-reaction to the power spectrum of cosmological fluctuations is very small. Assuming that the linear fluctuations have random phases, as they do in the simplest inflationary models, the relative contribution of our back-reaction terms to the power spectrum is suppressed by the product of a dimensionless inflationary slow-roll parameter and the amplitude of the linear density fluctuations.

We free ourselves of constraints imposed by any specific inflationary model

[1]There has been a lot of recent interest in the possibility that the leading gradient terms of long-wavelength modes might back-react on local observables in a way that mimics dark energy [47] (see also [48]). However, objections to this possibility have been raised [49, 50, 77, 42, 52].

by abstaining from picking a specific form of the inflaton potential. Rather, we assume that the inflationary slow-roll conditions are satisfied and we employ a very general form of the potential, one that allows us to easily interpret our results within the context of an explicit slow-roll realization. Thus, with regards to cosmological models, our approach is extremely general, while still retaining its ability to be specific.

Section 2 of this chapter is concerned with the general setup of the calculation. Here, the background equations are solved to second order in the slow-roll parameter (ϵ), neglecting spatial gradients, and the corrections to both the background scalar and the Hubble constant are determined, again, to second order in ϵ. Section 3 begins with a brief review of classical, relativistic, cosmological perturbation theory. The linear perturbation equations are solved using the background obtained in the previous section. The non-linear equations make their appearance in Section 4, where the next-to-leading order terms (those due to back-reaction) are solved for. Section 5 interprets these results as modifications to the linear terms, and their effect on the spectrum of perturbations is determined. Finally, the effects of back-reaction on the Gaussianity of the perturbations is investigated in Section 6.

5.3 Background

Before solving for the perturbations, we must determine the appropriate background for our model. We confine our attention to a model consisting of pure gravity (with vanishing cosmological constant) and a single, homogeneous, scalar field ϕ, which we presumes satisfies the slow-roll conditions. We write

$$\phi(x,t) \;=\; \phi_0 + \epsilon f_1(t) + \epsilon^2 f_2(t)\,, \tag{5.1}$$

where ϕ_0 is a constant and ϵ is a dimensionless slow-roll parameter whose value depends on the specific model in question. In consequence of (5.1), our background space-time will be approximately de Sitter space,

$$a(t) = e^{H(t)t}\,. \tag{5.2}$$

Here, the Hubble rate $H(t)$ is slowly time-dependent and can be expanded as

$$H(t) = H_0 + \epsilon h_1(t) + \epsilon^2 h_2(t)\,, \tag{5.3}$$

where H_0 a constant.

The dynamics of the scalar field is determined by its potential $V(\phi)$. In order not to limit our results to any specific model, but, rather, to show them to be a generic feature of slow-roll inflation, we use the following expansion (related to a power series expansion of the potential about the field value ϕ_0):

$$V(\phi) = \mu\phi_0 + \epsilon\lambda\phi(t) + \epsilon^2\Lambda\phi^2(t) \equiv V^{(0)} + V^{(1)} + V^{(2)}, \tag{5.4}$$

where superscripts indicate the order in ϵ and

$$\mu\phi_0 = V(\phi_0) - \phi_0 V'(\phi_0) + \frac{1}{2}\phi_0^2 V''(\phi_0), \tag{5.5}$$

$$\epsilon\lambda = V'(\phi_0) - \phi_0 V''(\phi_0) \tag{5.6}$$

$$\epsilon^2\Lambda = \frac{1}{2}V''(\phi_0), \tag{5.7}$$

the primes denoting a derivative with respect to the field. Thus, we see that μ,λ, and Λ are dimensionful parameters depending entirely on the form of the potential in the neighborhood of ϕ_0. In essence, this represents the series expansion (in ϵ) of any potential that can be used to generate slow-roll inflation.

In light of this, we can use the Klein-Gordon

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{\partial V}{\partial \phi} = 0, \tag{5.8}$$

and the Friedmann equations

$$3\dot{a}^2 + 4\pi a^2\dot{\phi}^2 + 8\pi a^2 V = 0, \tag{5.9}$$

$$\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} + 4\pi\dot{\phi}^2 - 8\pi V = 0 \tag{5.10}$$

(solved order by order in ϵ) to solve for the background, accurate to second order in the slow roll parameter.

We find that

$$H_0^2 = 8\pi\mu\phi_0, \tag{5.11}$$

$$f_1(t) = \frac{\lambda}{3H_0}(\frac{1}{3H_0} - t - \frac{e^{-3H_0 t}}{3H_0}), \tag{5.12}$$

$$f_2(t) = \frac{\lambda^2}{18H_0^2\mu}(-2 + 3H_0 t + e^{-3H_0 t}(3H_0 t + 2)), \tag{5.13}$$

$$h_1(t) = \lambda\sqrt{\frac{2\phi_0}{3\mu\pi}}, \tag{5.14}$$

and

$$h_2(t) = -\frac{1}{1296\pi\mu^2 H_0^4 t}(288H_0^2\lambda^2 t^2\pi^2\mu^2 + 162H_0^5\lambda^2 t\pi - 243\Lambda H_0^7 t \\ +16\lambda^2\pi^2\mu^2 e^{-6H_0 t} - 128\lambda^2\pi^2\mu^2 e^{-3H_0 t} - 288\lambda^2\pi^2\mu^2 H_0 t + 112\lambda^2\pi^2\mu^2)\,. \quad (5.15)$$

Note that the corrections to the scalar field have the property that they and their first temporal derivatives vanish at $t = 0$, which coincides with the onset of slow-rolling. Despite the overall factor of t^{-1} preceding $h_2(t)$, this term is not singular at the origin, as can be seen by expanding in a power series about $t = 0$.

5.4 Linearized Theory

The theory of cosmological perturbations is usually restricted to an analysis of the linearized (in the amplitude of the perturbations) Einstein equations. In this section, we review the salient results (see e.g. [9, 10] for comprehensive reviews).

The most general perturbed line element can be written as

$$ds^2 = a^2(\eta)[(1+2\Phi)d\eta^2 - 2(B_{;i}+S_i)dx^i d\eta - [(1-2\Psi)\gamma_{ij} + 2E_{;ij} + (F_{i;j}+F_{j;i}) + h_{ij}]dx^i dx^j] \quad (5.16)$$

where Φ, Ψ, B, E represent scalar, S_i, F_i vector, and h_{ij} tensor metric perturbations, respectively. These are distinguished by their transformation properties under three dimensional rotations. For inflationary cosmology and in linear perturbation theory, we can discard vector modes since they do not grow, and tensor modes because they grow at a slower rate than scalar metric fluctuations. Hence, we focus on the scalars. It is possible to choose a gauge in which $E = B = 0$, the so-called 'longitudinal' or 'conformal Newtonian' gauge. This gauge is convenient for calculational purposes. For matter without anisotropic stresses to linear order in the matter field perturbations, it follows from the off-diagonal space-space Einstein equations that $\Phi = \Psi$. Hence, for such matter the metric in longitudinal gauge takes the form

$$ds^2 = (1 + 2\kappa\Phi(\vec{x}, t))dt^2 - a^2(t)(1 - 2\kappa\Phi(\vec{x}, t))(dx^2 + dy^2 + dz^2)\,. \quad (5.17)$$

In the above, κ is a dimensionless parameter which indicates the order of the term in gravitational perturbation theory.

At second order, the perturbed metric is in general much more complicated. In particular, there is mixing between scalar, vector and tensor modes, and an anisotropic stress in generated, leading to $\Phi \neq \Psi$. However, it can be seen explicitly that all of the complicating terms contain spatial gradients and can hence be neglected in our study of the back-reaction effects of long wavelength modes. Thus, if we work to leading order in the infrared terms, we can neglect vector, tensor modes and anisotropic stress. Thus, we can apply longitudinal gauge also at second order. We will use this gauge in the following.

To incorporate effects due to slow-rolling, we expand Φ in powers of ϵ to obtain

$$\Phi_1(\vec{x},t) \;=\; \Psi_1(\vec{x},t) + \epsilon\alpha_1(\vec{x},t) + \epsilon^2\beta_1(\vec{x},t)\,. \tag{5.18}$$

The subscript $_1$ denotes the fact that the effects are linear in κ.

In order to source our first order metric perturbations, matter perturbations $\delta_1(\vec{x},t)$ must be present, i.e.

$$\phi(\vec{x},t) \longrightarrow \phi(t) + \kappa\delta_1(\vec{x},t)\,, \tag{5.19}$$

where $\phi(t)$ is the background matter solution given by (5.1)

At the linear level, perturbations decouple in Fourier space. It is thus convenient to track the evolution of each Fourier mode individually. At higher orders in perturbation theory, there will be mixing between the modes. Typically, cosmological perturbations are classified into two distinct sectors: sub-Hubble (UV) and super-Hubble (IR) modes. During inflation, the phase space associated with super-Hubble modes grows exponentially, while that of the sub-Hubble modes remains constant. In addition, the modes which are important from the point of view of structure formation and CMB anisotropies are super-Hubble during the last 50 e-foldings of inflation. It is for these reasons that we focus our attention on these modes and ignore their UV counterparts. This choice justifies our dropping of spatial derivatives in the equations of motion.

Using the standard result for the energy-momentum tensor of a scalar field

$$T_{\mu\nu} \;=\; \phi_{;\mu}\phi_{;\nu} - g_{\mu\nu}(\frac{\phi^{;\alpha}\phi_{;\alpha}}{2} - V(\phi))\,, \tag{5.20}$$

expanding the Einstein equations in a power series in κ, and truncating after first order leads to the equations of motion for scalar cosmological perturbations, which read (see e.g. [9])

$$(ii): \qquad -\frac{\dot{a}}{a}\dot{\Phi}_1 + 8\pi\dot{\phi}\dot{\delta}_1 - 2\ddot{\Phi}_1 - 8\frac{\ddot{a}}{a}\Phi_1 - 4\Phi_1\frac{\dot{a}^2}{a^2} - 16\pi\Phi_1(\dot{\phi})^2 \tag{5.21}$$

$$-8\pi V^{(1)}(\phi) + 16\pi V^{(0)}(\phi)\Phi_1 = 0,$$

$$(00): \quad 6\dot{\Phi}_1 \frac{\dot{a}}{a} + 8\pi\dot{\phi}\dot{\delta}_1 + 8\pi V^{(1)}(\phi) + 16\pi V^{(0)}(\phi)\Phi_1 = 0\,, \tag{5.22}$$

where the overdot denotes a derivative with respect to cosmic time, and the (ii) and (00) indicate the tensor indices of the Einstein equations in question.

These equations can be solved to yield

$$\Psi_1(\vec{x},t) = 0\,, \tag{5.23}$$

$$\alpha_1(\vec{x},t) = \psi_1 \int d^3\vec{k} f(\vec{k}) e^{i\vec{k}\cdot\vec{x}} e^{i\alpha_{\vec{k}}} V^{1/2} \tag{5.24}$$

$$\beta_1(\vec{x},t) = -\psi_1 \frac{2\Lambda\mu\phi_0 - \lambda^2}{\lambda\mu}(-1 + e^{-H_0 t}) \int d^3\vec{k} f(\vec{k}) e^{i\vec{k}\cdot\vec{x}} e^{i\alpha_{\vec{k}}} V^{1/2} \tag{5.25}$$

$$\delta_1(\vec{x},t) = -2\frac{\mu\phi_0\psi_1}{\lambda} \int d^3\vec{k} f(\vec{k}) e^{i\vec{k}\cdot\vec{x}} e^{i\alpha_{\vec{k}}} V^{1/2}\,, \tag{5.26}$$

where ψ_1 a constant representing the amplitude of the spectrum of linear fluctuations, $f(\vec{k})$ is the function describing the shape of the spectrum, $\alpha_{\vec{k}}$ are the phases (assumed later on to be random), and V is the cutoff volume used in the definition of the Fourier transform. Note that $\vec{k}$ denotes the co-moving momentum. We have introduced the cutoff volume in the definition of the Fourier transform in order that the dimension of the Fourier transform $\tilde{\Phi}(\vec{k})$ of $\Phi(\vec{x})$ is $k^{3/2}$, which in turn ensures that there is no volume arising in the relation between the power spectrum $P_\Phi(k)$ and $\tilde{\Phi}(\vec{k})$ (see Section 5).

Let us comment briefly on the interpretation of this solution. Note first that the fact that Ψ_1 vanishes is a consistency check on our analysis. In the pure de Sitter limit (no rolling of the scalar field), there are no scalar metric fluctuations to first order in perturbation theory (in κ). Since the equation of motion is second order, there are two fundamental solutions for each mode. In the long wavelength (super-Hubble) limit, the dominant solution is constant in time, and the sub-dominant solution is a decaying mode. We see that α_1 is a the constant perturbation sourced by δ_1, while β_1 is generated by the rolling of the scalar field and is a combination of the constant and decaying mode.

5.5 Back-Reaction

To second order in gravitational perturbation theory (expansion in κ), there are interactions between the Fourier modes of the fluctuation variables caused by the non-linearities of the Einstein equations. In order to determine the effects of non-linearities, we make the substitutions for the metric perturbation variable

$$\Phi(\vec{x},t) \longrightarrow \Phi_1(\vec{x},t) + \kappa\Phi_2(\vec{x},t)\,, \tag{5.27}$$

where Φ_2 can be in turn expanded in terms of the slow-roll parameter ϵ as

$$\Phi_2(\vec{x},t) \;=\; \Psi_2(\vec{x},t) + \epsilon\alpha_2(\vec{x},t) + \epsilon^2\beta_2(\vec{x},t)\,, \tag{5.28}$$

and for the matter field

$$\phi(\vec{x},t) \longrightarrow \phi(\vec{x},t) + \kappa^2\delta_2(\vec{x},t)\,. \tag{5.29}$$

Note that Φ_2 and δ_2 represent the effects of gravitational back-reaction.

In the following we will neglect spatial gradient terms in the equations of motion, since we are interested in the infrared modes and in the coupling between different infrared modes [2]. To order κ^2 in the perturbative expansion, the Einstein equations then become

$$\begin{aligned}
(ii): \quad & 2\ddot{\Phi}_2 + 8\dot{\Phi}_2\frac{\dot{a}}{a} + 8\Phi_2(\frac{\ddot{a}}{a} + 2\pi\dot{\phi}^2 + \frac{\dot{a}^2}{2a^2} - 2\pi V^{(2)}(\phi)) - 8\pi\dot{\phi}\dot{\delta}_2 \quad (5.30)\\
= \quad & 4\Phi_1(\ddot{\Phi}_1 + 2\Phi_1\frac{\dot{a}^2}{a^2} + 6\dot{\Phi}_1\frac{\dot{a}}{a} + 4\Phi_1\frac{\ddot{a}}{a} + 8\pi\Phi_1\dot{\phi}\dot{\delta}_1 + 4\pi V^{(1)}(\phi)\Phi_1,\\
(00): \quad & 6\dot{\Phi}_2\frac{\dot{a}}{a} + 8\pi\dot{\phi}\dot{\delta}_2 - 16\pi V^{(0)}(\phi)\Phi_2 \quad (5.31)\\
= \quad & 3\dot{\Phi}_1{}^2 - 4\pi\dot{\delta}_1{}^2 - 12\Phi_1\dot{\Phi}_1\frac{\dot{a}}{a} - 16\pi V^{(1)}(\phi)\Phi_1 - 8\pi V^{(2)}(\phi).
\end{aligned}$$

These equations can be solved to yield

$$\begin{aligned}
\Psi_2(\vec{x},t) &= 0\,, && (5.32)\\
\alpha_2(\vec{x},t) &= 0\,, && (5.33)\\
\beta_2(\vec{x},t) &= \psi_1^2\frac{(\Lambda\mu\phi_0 - \lambda^2)}{\lambda^2}(-1 + e^{-H_0t})g^2(\vec{x})\,, && (5.34)\\
\delta_2(\vec{x},t) &= 0\,, && (5.35)
\end{aligned}$$

with

$$g(\vec{x}) \;=\; \int d^3\vec{k}f(\vec{k})e^{i\vec{k}\cdot\vec{x}}e^{i\alpha_{\vec{k}}}V^{1/2}\,. \tag{5.36}$$

Let us briefly comment on the physical interpretation of these results. First, the vanishing of Ψ_2 is a consistency check since there are no scalar metric fluctuations in pure de Sitter space. Since the linear fluctuations are first order in ϵ, they will only contribute to the second order perturbations to

[2]It is also in this approximation that we can justify writing the perturbed metric in the form (5.17) - see e.g. [35] for a discussion of this point.

quadratic order, and hence the vanishing of α_2 is another consistency check on the algebra. From the expression for Φ_2 it is manifest that the second order perturbations are generated by the linear inhomogeneities Φ_1 and δ_1 at quadratic order. The vanishing of δ_2 is an interesting and unexpected result. It says that, to this order, there are no back-reaction effects on the evolution of the background scalar field.

5.6 Effects of Back-Reaction on the Power Spectrum

Having now determined the form of the back-reaction terms, it is important to estimate their amplitude. From observations of CMB anisotropies, we know that the linear perturbations are of order $\kappa\psi_1\epsilon \sim 10^{-5}$. The back-reaction terms should be expected to be of order $(\kappa\psi_1\epsilon)^2$. However, the second order correction to a fixed Fourier mode receives contributions from all linear Fourier modes to this order. Hence, one could expect the back-reaction effect to be amplified by a phase space factor which measures the phase space of Fourier modes which contribute. Since in inflationary cosmology, the phase space of infrared modes is growing, and the linear fluctuations do not decrease in amplitude on scales larger than the Hubble radius, the effects of back-reaction could be expected to grow in time and become non-perturbatively large. In this section we show that, provided that the linear fluctuations have random phases, the leading infrared quadratic back-reaction effects of linear fluctuations on the power spectrum of Φ do not show any large phase-space enhancement.

The total power spectrum $\mathcal{P}_{total}(\vec{k})$ of Φ, including the leading infrared terms of second order in κ, can be written as

$$\begin{aligned} \mathcal{P}_{total}(\vec{k}) &= k^3|\tilde{\Phi}_{\vec{k}}|^2 = \mathcal{P}_1(\vec{k}) + \mathcal{P}_2(\vec{k}), \\ &= |\tilde{\Phi}_1(\vec{k}) + \tilde{\Phi}_2(\vec{k})|^2 k^3, \end{aligned} \tag{5.37}$$

where $\tilde{\Phi}_i(\vec{k})$ are the Fourier transforms (using the definition of Fourier transform including the cutoff volume as in Equation (5.24)) of $\Phi_i(\vec{x})$. Making use of the results of Sections 3 and 4 we have

$$\begin{aligned} \tilde{\Phi}_1(\vec{k}) &= \epsilon\psi_1 f(k) e^{i\alpha_{\vec{k}}}[1 - \epsilon E(t)] \\ \tilde{\Phi}_2(\vec{k}) &= \epsilon^2\psi_1^2 E(t) h(k)\,, \end{aligned} \tag{5.38}$$

where $h(k)$ is the Fourier transform of $g^2(x)$ and where we have introduced the symbol $E(t)$ for the function

$$E(t) \equiv \frac{2\Lambda\mu\phi_0 - \lambda^2}{\lambda^2}(-1 + e^{-H_0 t}) . \tag{5.39}$$

The leading back-reaction correction to the power spectrum, denoted by the function $\mathcal{P}_{BR}(\vec{k})$, comes from the cross term in (5.38) and is thus linear in $\tilde{\Phi}_2(\vec{k})$:

$$\mathcal{P}_{BR}(\vec{k}) = 2\kappa|\tilde{\Phi}_1(\vec{k})\tilde{\Phi}_2(\vec{k})|k^3 . \tag{5.40}$$

Thus, the fractional correction to the power spectrum due to the leading back-reaction contributions is

$$\begin{aligned} \frac{\mathcal{P}_{BR}(k)}{\mathcal{P}_1(k)} &= 2\kappa\frac{\epsilon^2\psi_1^2 E(t)h(k)}{\epsilon\psi_1 f(k) + \mathcal{O}(\epsilon^{\epsilon})} \\ &\simeq 2\kappa\epsilon\psi_1\frac{h(k)}{f(k)} , \end{aligned} \tag{5.41}$$

from which it follows that, modulo the ratio of $h(k)$ over $f(k)$, the back-reaction terms are suppressed, as expected, by $\kappa\epsilon\psi_1$. The ratio of $h(k)$ over $f(k)$ is the possible large phase space enhancement factor.

Before continuing, we specify the linear power spectrum. We choose a normalization wavenumber k_n and choose ψ_1^2 to be the amplitude of the power spectrum at $k = k_n$. The function $f(k)$ describes the spectral shape. We choose a power law with a tilt ζ away from scale-invariance, i.e. we write

$$f(k) = \left(\frac{k}{k_n}\right)^{-3/2-\zeta} k_n^{-3/2} . \tag{5.42}$$

It can easily be checked that $\mathcal{P}_1(k_n) = \psi_1^2$.

We now evaluate the magnitude of $h(k)$, assuming that the phases $\alpha_{\vec{k}}$ are random:

$$\begin{aligned} h(\vec{k}) &= \frac{1}{(2\pi)^3}V^{-1/2}\int d^3x g^2(x)e^{-i\vec{k}\vec{x}} \\ &= \int d^3k_1 f(\vec{k}_1)f(\vec{k}-\vec{k}_1)e^{i(\alpha_{\vec{k}_1}+\alpha_{\vec{k}-\vec{k}_1})}V^{1/2} . \end{aligned} \tag{5.43}$$

Given that we are considering the effects of long-wavelength fluctuations, we must restrict the above integral over $\vec{k}_1$ to run only over super-Hubble modes, i.e.

$$|\vec{k}_1| \leq H . \tag{5.44}$$

To estimate the magnitude of $h(\vec{k})$, we insert the spectrum (5.42) into (5.43). If we consider the effects of back-reaction on modes $\vec{k}$ which are sub-Hubble now, we can apply the approximation

$$\left(\frac{k - k_1}{k_n}\right) \simeq \frac{k}{k_n}, \tag{5.45}$$

in which case the integral simplifies.

Assuming constant phases for the moment, the integral (5.43) can be easily estimated

$$h(k) \sim k^{-3/2-\zeta} k_n^{2\zeta} H^{3/2-\zeta} V^{1/2}. \tag{5.46}$$

Note in particular from (5.46) that the k-dependence of $\mathcal{P}_{BR}(k)$ is the same as that of the linear power spectrum. The leading effect of back-reaction thus does not change the power index of the spectrum. However, for wavelengths close to the Hubble radius, the approximation (5.45) is no longer good, and the correction terms will yield changes to the index of the power spectrum. The second fact to notice about the result (5.46) is the cutoff volume divergence. This stems from the fact that as V increases, more and more infrared modes are contributing to the back-reaction. For constant phases, the effect is additive. The volume divergence thus represents the phase space enhancement which is the focus of this investigation.

Let us now consider the more realistic situation - realized in typical inflationary models - in which the phases are random. A simple way to estimate the effects of the random phases in (5.43) is to add up the amplitudes of the back-reaction contributions of all infrared modes $\vec{k}_1$ as a random walk. This means dividing the amplitude obtained previously by $N(V)^{1/2}$, where $N(V)$ is the number of modes. Since for a finite volume V the wavenumbers are quantized in units of $\Delta k \sim V^{-1/3}$, the number $N(V)$ becomes

$$N(V) \sim (\frac{H}{\Delta k})^3 \tag{5.47}$$

in which case the result (5.43) becomes

$$h(k) \sim k^{-3/2-\zeta} k_n^{2\zeta} H^{-\zeta}. \tag{5.48}$$

Inserting this into (5.41), we obtain our final result

$$\frac{\mathcal{P}_{BR}(k)}{\mathcal{P}_1(k)} \sim 2\kappa\epsilon\psi_1 \left(\frac{k_n}{H}\right)^{\zeta}. \tag{5.49}$$

The main conclusion we draw from (5.49) is that there is no phase space enhancement of the back-reaction of long wavelength modes on the spectrum of cosmological perturbations, in contrast to the positive enhancement found for the back-reaction on the background metric. Given the absence of such a phase space enhancement, we find - as expected - that the back-reaction terms in the power spectrum are suppressed compared to the terms coming from the linear perturbations by $\kappa\epsilon\psi_1$. Thus, they are completely negligible in the case of a COBE-normalized spectrum of almost scale-invariant linear fluctuations. In addition, we find that the leading back-reaction terms do not change the spectral index.

5.7 Non-Gaussianity of the Spectrum Due to Back-Reaction

Having established that the back-reaction of infrared modes cannot substantially modify the amplitude and spectral tilt of the power spectrum of perturbations, we make some comments regarding the effects of back-reaction on the Gaussianity of the spectrum.

The inclusion of higher-order terms implies correlations between different modes, thus breaking strict Gaussianity. However, the question remains: how badly broken is it? To estimate this, we turn our attention to the bispectrum (three-point function).

In the case of purely Gaussian distribution, all odd moments are identically zero. Therefore, the non-vanishing of the bispectrum indicates that the distribution cannot be Gaussian.

We take it for granted that the bispectrum does not vanish (for examples of the three-point function see [46, 43]. In the context of higher order perturbation theory (see e.g. [38]), however, its amplitude is exceedingly small. We estimate it to be no larger than of order $\kappa^4\epsilon$, thus making it quite unlikely to be detected experimentally. Thus, we conclude that, although back-reaction modifies the distribution, Gaussianity remains an excellent approximation.

5.8 Conclusions

In this chapter, we have studied the back-reaction of long wavelength linear fluctuations on the power spectrum, produced by the mode mixing which occurs as a consequence of the non-linearity of the Einstein equations. We find that, assuming that the phases of the linear fluctuations are random, there is no phase space enhancement of the back-reaction effect. The leading infrared back-reaction contributions are suppressed by $\kappa\epsilon\psi_1$ compared to the contribution of the linear fluctuations, where $\kappa\psi_1$ is a measure of the amplitude of the linearized metric fluctuations, and ϵ is an inflationary slow-roll parameter. These leading back-reaction terms do not modify the tilt of the power spectrum on scales substantially smaller than the Hubble radius. Note that in the case of correlated phases of the linear fluctuations, a much larger back-reaction effect is possible.

We have also seen that the modifications to the Gaussianity of the leading order perturbations are negligible. The small size of these modifications goes a long way towards justifying the linear approximation to cosmological perturbation theory. However, our results do not exclude the possibility that large amplitude local fluctuations can effect the measured fluctuations, as very recently suggested in [53] (based on the second order formalism developed in [54]) [3].

Our work differs from previous work on second order fluctuations in that it emphasizes the fact that, in an accelerating universe, the phase space of super-Hubble modes is increasing in time. In contrast to what occurs in the case of the back-reaction on the homogeneous mode, the back-reaction on the fluctuating modes themselves does not increase without limits as a function of time. Compared to previous analyses, our work also gives an easier way to derive the leading-order effects of long-wavelength cosmological fluctuations. Our results have been derived in the context of an arbitrary slow-roll inflationary model and are thus valid for a wide range of cosmological scenarios.

[3]The possibility that local fluctuations can have a measurable effect on background quantities such as the deceleration parameter has recently been suggested in [55, 56].

Chapter 6

Summary

The previous four chapters represent the bulk of the author's contribution to the theory of cosmological perturbation. This has included results at both the linear and second order, on the background and on the perturbations themselves.

Problems addressed have ranged from the purely theoretical - for example, the classicalization process described in chapter two, whose exact details have virtually no impact on the late time universe, to the potentially crucial phenomenological implications of the effects of backreaction - here, we have in mind the results discussed in chapters four and five.

In conclusion, the author hopes he's conveyed something of the importance of cosmological perturbations as well as the potential utility of gravitational backreaction.

Chapter 7

Appendix: More on the spectrum of perturbations in string gas cosmology

7.1 Foreword

This appendix contains a reproduction of the article *More on the spectrum of perturbations in string gas cosmology.* The work presented in chapter 3 forms the basis of the author's contribution to this report.

7.2 Introduction

String gas cosmology is a model of superstring cosmology which is based on coupling to a classical dilaton gravity background a gas of classical strings with a mass spectrum corresponding to one of the consistent perturbative superstring theories [1, 113] (see also [114] for early work, and [115, 36, 116] for reviews). String gas cosmology has been developed in some detail in recent years [117, 118, 119]. In particular, it was shown [120, 121, 122, 123, 124, 125, 126] that string modes which become massless at enhanced symmetry points lead to a stabilization of the volume and shape moduli of the six extra spatial dimensions (see [1, 127] for arguments in the context of string gas cosmology on how to naturally obtain the separation between three large and six string-scale dimensions [1]) .

[1]See, however, [128, 129] for some caveats.

String gas cosmology is usually formulated in the string frame, the frame in which stringy matter couples canonically to the background dilaton space-time. The existence of a maximal temperature [130] of a gas of weakly interacting strings in thermal equilibrium has crucial consequences for string cosmology. As discussed in [1], as we follow our universe back in time through the radiation phase of standard cosmology, then when the temperature approaches its limiting value, the energy shifts from the radiative modes to the string oscillatory and winding modes. Thus, the pressure approaches zero. In the string frame, and for zero pressure, as follows from the equations derived in [113, 131], the universe is quasi-static. There is an attractive fixed point of the dynamics in which the scale factor of our large three dimensions is constant, but the dilaton is dynamical (we consider the branch of solutions in which the dilaton is a decreasing function of time). We call this phase the quasi-static Hagedorn phase. Since the string frame Hubble radius is extremely large in the Hagedorn phase (infinite in the limiting case that the scale factor is exactly constant), but decreases dramatically during the transition to the radiation phase of standard cosmology, all comoving scales of interest in cosmology today are sub-Hubble initially, propagate on super-Hubble scales for a long time after the transition to the radiation phase before re-entering the Hubble radius at late times. Thus, it appears in principle possible to imagine a structure formation mechanism driven by local physics.

As was recently suggested [133], string thermodynamic fluctuations in the Hagedorn phase may lead to a nearly scale-invariant spectrum of cosmological perturbations. There would be a slight red tilt for the spectrum of scalar metric perturbations. A key signature of this scenario would be a slight blue tilt for the spectrum of gravitational waves [134] (see also [135, 136] for more detailed treatments). Since this result is surprising from the point of view of particle cosmology, and since the analyses of [133, 134] contained approximations, it is an interesting challenge to analyze the cosmology from the point of view of the Einstein frame, the frame in which cosmologists have a better physical intuition.

In this chapter, we analyze both the background dynamics of string gas cosmology and the generation and evolution of cosmological perturbations in the Einstein frame. A naive extrapolation of the background solutions of [113] into the past would yield a cosmological singularity. Such an extrapolation is clearly not justified once the dilaton reaches values for which we enter the strong coupling regime of string theory. Rather, at early times one must have a new phase of the theory in which the dynamics is consistent with the qualitative picture which emerges from string thermodynamics. This phase is meta-stable and will decay into a phase with rolling dilaton. The modified background evolution can solve the horizon and singularity problems in the context of string gas cosmology.

An improved analysis of generation of fluctuations in the string frame shows that the conclusions of [133, 134] concerning the spectra of cosmological perturbations and gravitational waves (we are considering the case where our three large dimensions are toroidal) are only obtained if the dilaton velocity can be neglected.

7.3 Background Dynamics in the Einstein Frame

String gas cosmology is based on T-duality symmetry and on string thermodynamics. String thermodynamics yields the existence of a maximal temperature of a gas of strings in thermal equilibrium, the Hagedorn temperature [130]. If we consider [1] adiabatic evolution of a gas of strings in thermal equilibrium as a function of the radius of space R, T-duality [137] yields a temperature-radius curve (see Fig. 1) which is symmetric about the self-dual radius. Being at the self-dual radius must be a fixed point of the dynamics. The higher the entropy of the string gas is at a fixed radius, the larger is the flat region of the curve, the region where the temperature remains close to the Hagedorn temperature. This implies that the duration of the Hagedorn phase will increase the larger the energy density is.

In a regime in which it is justified to consider the dynamics in terms of a gas of strings coupled to background dilaton gravity, the string frame action for the background fields, the metric and the dilaton (we will set the antisymmetric tensor field to zero) is

$$S = -\int d^{1+N}x\sqrt{-g}e^{-2\phi}[R + 4g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi]\,, \tag{7.1}$$

where R is the string frame Ricci scalar, g is the determinant of the string frame metric, N is the number of spatial dimensions, and ϕ is the dilaton field. Note that we are working in units in which the dimensionful pre-factor appearing in front of the action is set to 1.

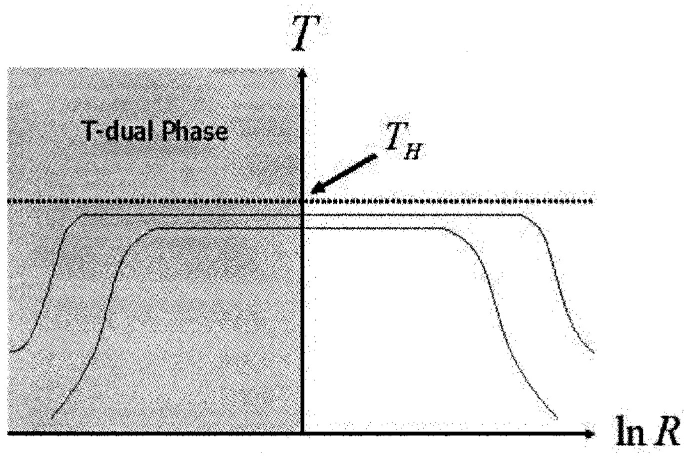

Figure 7.1: Sketch (based on the analysis of [1] of the evolution of temperature T as a function of the radius R of space of a gas of strings in thermal equilibrium. The top curve is characterized by an entropy higher than the bottom curve, and leads to a longer region of Hagedorn behaviour.

The background is sourced by a thermal gas of strings. Its action S_m is given by the string gas free energy density f (which depends on the string frame metric) via

$$S_m = \int d^{1+N}x\sqrt{-g}f\,. \tag{7.2}$$

The total action is the sum of S and S_m. Note that the factor $e^{-2\phi}$ gives the value of Newton's gravitational constant.

In the case of a spatially flat, homogeneous and isotropic background given by

$$ds^2 = dt^2 - a(t)^2 d\mathbf{x}^2\,, \tag{7.3}$$

the three resulting equations of motion of dilaton-gravity (the generalization of the two Friedmann equations plus the equation for the dilaton) in the string frame are [113] (see also [131])

$$-N\dot{\lambda}^2 + \dot{\varphi}^2 = e^{\varphi}E \tag{7.4}$$

$$\ddot{\lambda} - \dot{\varphi}\dot{\lambda} = \frac{1}{2}e^{\varphi}P \tag{7.5}$$

$$\ddot{\varphi} - N\dot{\lambda}^2 = \frac{1}{2}e^{\varphi}E\,, \tag{7.6}$$

where E and P denote the total energy and pressure, respectively, and we have introduced the logarithm of the scale factor

$$\lambda(t) = \log(a(t)) \tag{7.7}$$

and the rescaled dilaton

$$\varphi = 2\phi - N\lambda . \tag{7.8}$$

where N is the number of spatial dimensions (in the following, the case of $N = 3$ will be considered).

The Hagedorn phase is characterized by vanishing P and, therefore, constant total energy E. Thus, combining (7.4) and (7.6) to eliminate the dependence on $\dot{\lambda}$, yields a second order differential equation for φ with the solution

$$e^{-\varphi(t)} = \frac{E_0}{4}t^2 - \dot{\varphi}_0 e^{-\varphi_0} t + e^{-\varphi_0} \tag{7.9}$$

subject to the initial condition constraint

$$\dot{\varphi_0}^2 = e^{\varphi_0} E_0 + N\dot{\lambda_0}^2 \tag{7.10}$$

which follows immediately from (7.4). In the above, the subscripts stand for the initial values at the time $t = 0$.

In the Hagedorn phase, the second order differential equation (7.5) for λ can easily be solved. If the initial conditions require non-vanishing $\dot{\lambda}_0$, the solution is

$$\lambda(t) = \lambda_0 + \frac{1}{\sqrt{N}} \ln[\frac{\sqrt{N}\dot{\lambda}_0 - \mathcal{G}}{\sqrt{N}\dot{\lambda}_0 + \mathcal{G}}] , \tag{7.11}$$

where $\mathcal{G}$ is an abbreviation which stands for

$$\mathcal{G} = \frac{Et}{2} e^{\varphi_0} - \dot{\varphi}_0 . \tag{7.12}$$

For vanishing initial value of the derivative of the scale factor, the solution is simply

$$\lambda(t) = \lambda_0 , \tag{7.13}$$

i.e. a static metric. In the static case, the result (7.9) simplifies to

$$e^{-\varphi(t)} = e^{-\varphi_0} (\frac{\dot{\varphi}_0 t}{2} - 1)^2 . \tag{7.14}$$

These solutions are slight generalizations of the solutions given in the appendix of [113]. These solutions have also very recently been discussed in [132]. We are interested in the branch of solutions with $\dot{\phi} < 0$.

One important lesson which follows from the above solution is that, although the metric is static in the Hagedorn phase, a dilaton singularity develops at a time t_s given by

$$t_s = -\frac{2}{|\dot{\varphi}_0|} . \tag{7.15}$$

In fact, already at a slightly larger time t_c, the dilaton has reached the critical value $\phi = 0$, beyond which string perturbation theory breaks down. The times $|t_s|$ and $|t_c|$ are typically of string scale. Thus, unless the current value of the dilaton is extremely small, the duration of the phase in which the above solution is applicable will be short.

Note, however, that the solution (7.14) is not consistent with the qualitative picture which emerges from string thermodynamics [1] (see Fig. 1) according to which the evolution of all fields close to the Hagedorn temperature should be almost static. We know that the dilaton gravity action ceases to be justified in the region in which the theory is strongly coupled. This leads to the conclusion that the phase during which (7.14) is applicable must be preceded by another phase of Hagedorn density, a phase in which the dynamics reflects the qualitative picture which emerges from Fig. 1, and corresponds to fixed scale factor and fixed dilaton. We call this phase the *strong coupling Hagedorn phase* [2] Note that the Einstein action is not invariant under T-duality. Hence, we expect that intuition based on Einstein gravity will give very misleading conclusions when applied to the strong coupling Hagedorn phase. In particular, constant energy density should *not* lead to a tendency to expansion. The strong coupling phase is long-lived but meta-stable and decays into a solution in which the dilaton is free to roll, a phase described by the equations (7.4 - 7.6).

If the equation of state is that of radiation, namely $P = 1/NE$, then a solution with static dilaton is an attractor. For static dilaton, the equations (7.5) and (7.6) then reduce to the usual Friedmann-Robertson-Walker-Lemaitre equations.

Figure 2 shows a space-time sketch from the perspective of string frame coordinates. The Hagedorn phase lasts until the time t_R (the time interval from t_c to close to t_R being describable by the equations of motion of dilaton gravity) when a smooth transition to the radiation phase of standard cosmology takes place. This transition is governed by the annihilation of string winding modes into oscillatory modes and is described by Boltzmann-type

[2] Another argument supporting the assumption that in the strong coupling Hagedorn phase the dilaton is fixed can be given making use of S-duality. Under S-duality, the dilaton ϕ is mapped to $-\phi$. It is reasonable to assume that close to the maximal temperature, the system is in a configuration which is self S-dual, and in which the dilaton is hence fixed - we thank C. Vafa for stressing this point. Note that we are assuming that the existence of the maximal temperature remains true at strong coupling.

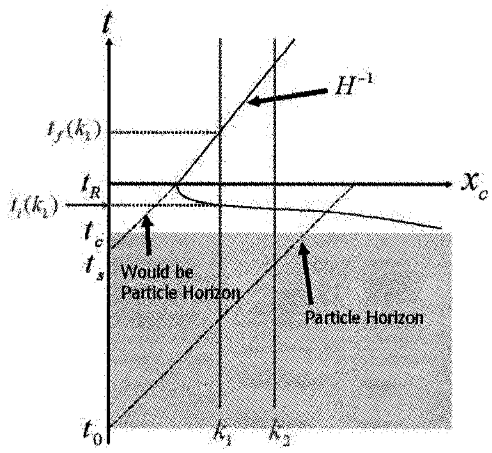

Figure 7.2: Space-time diagram (sketch) showing the evolution of fixed comoving scales in string gas cosmology. The vertical axis is string frame time, the horizontal axis is comoving distance. The Hagedorn phase ends at the time t_R and is followed by the radiation-dominated phase of standard cosmology. The solid curve represents the Hubble radius H^{-1} which is cosmological during the quasi-static Hagedorn phase, shrinks abruptly to a micro-physical scale at t_R and then increases linearly in time for $t > t_R$. Fixed comoving scales (the dotted lines labeled by k_1 and k_2) which are currently probed in cosmological observations have wavelengths which are smaller than the Hubble radius during the Hagedorn phase. They exit the Hubble radius at times $t_i(k)$ just prior to t_R, and propagate with a wavelength larger than the Hubble radius until they reenter the Hubble radius at times $t_f(k)$. Blindly extrapolating the solutions (7.13, 7.14) into the past would yield a dilaton singularity at a finite string time distance in the past of t_R, at a time denoted t_s. However, before this time is reached (namely at time t_c) a transition to a strong coupling Hagedorn phase with static dilaton is reached. Taking the initial time in the Hagedorn phase to be t_0, the forward light cone from that time on is shown as a dashed line. The shaded region corresponds to the strong coupling Hagedorn phase.

equations [118] (with corrections pointed out in [128, 129]). These Boltzmann equations are analogs of the equations used in the cosmic string literature (see [138] for reviews) to describe the transfer of energy between "long" (i.e. super-Hubble) strings and string loops. Note that the decay of string winding modes into radiation is the process that "reheats" the universe.

Close to the time t_s, we reach the strong coupling Hagedorn phase, the phase responsible for generating a large horizon.

In order that the cosmological background of Figure 2 match with our present cosmological background, the radius of space at the end of the Hagedorn phase needs to be of the order of 1 mm, the size that expands into our currently observed universe making use of standard cosmology evolution beginning at a temperature of about 10^{15} GeV. This is many orders of magnitude larger than the string size. Thus, without further assumptions, there is a cosmological "horizon" and "entropy" problem, similar to the one present in Standard Big Bang (SBB) cosmology. Provided that the strong coupling Hagedorn phase is long-lived (and, based on Fig. 2 this is more likely the higher the initial energy density is chosen), string gas cosmology will be able to resolve these problems. In particular, there will be enough time to establish thermal equilibrium over the entire spatial section, a necessary condition for the structure formation scenario outlined in [133] to work.

The above issues become more manifest when the cosmological background is rewritten in the Einstein frame. It is to this subject to which we now turn.

The conformal transformation of the metric to the Einstein frame is given by

$$\tilde{g}_{\mu\nu} = e^{-4\phi/(N-1)} g_{\mu\nu} = e^{-2\phi} g_{\mu\nu} \tag{7.16}$$

where quantities with a tilde refer to those in the Einstein frame, and in the final expression we have set $N = 3$. The dilaton transforms as (for $N = 3$)

$$\tilde{\phi} = 2\phi . \tag{7.17}$$

Under this transformation, the action (7.1) becomes

$$S_E = -\int d^4x \sqrt{-\tilde{g}} (\tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\phi} \tilde{\nabla}_\nu \tilde{\phi}) . \tag{7.18}$$

The matter action in the Einstein frame becomes

$$S_m = \int d^4x e^{2\phi} f[\tilde{g}, \phi] \sqrt{\tilde{g}} , \tag{7.19}$$

from which we see that the factor $e^{2\phi}$ plays the role of the gravitational constant.

We apply the conformal transformation for the metric (7.3) of a homogeneous and isotropic universe. To put the resulting Einstein frame metric into the FRWL form, we have to re-scale the time coordinate, defining a new Einstein frame time $\tilde{t}$ via

$$d\tilde{t} = e^{-\phi} dt . \tag{7.20}$$

The resulting scale factor $\tilde{a}$ in the Einstein frame then is given by

$$\tilde{a} = e^{-\phi} a . \tag{7.21}$$

The comoving spatial coordinates are unchanged.

The Hubble parameters $\tilde{H}$ and H in the Einstein and string frames, respectively, are related via

$$\tilde{H} = e^{\phi}(H - \dot{\phi}) . \tag{7.22}$$

It is important to note that they denote very different lengths.

Let us now calculate the evolution of the Einstein frame scale factor. Making use of the solution for the string frame dilaton (7.14), it follows from (7.21) that

$$\tilde{a} = e^{\lambda_0} e^{-\phi_0}(1 - \dot{\phi}_0 t) . \tag{7.23}$$

By integrating (7.20) we find that the physical time in the Einstein frame is given by

$$\tilde{t} = e^{-\phi_0} t (1 - \frac{1}{2}\dot{\phi}_0 t) . \tag{7.24}$$

It is important to keep in mind that $\dot{\phi} < 0$.

It follows that, in the Einstein frame, the evolution looks like that of a universe dominated by radiation. This is easy to verify for large times, when the factors of -1 within the parentheses in (7.23) and (7.24) are negligible, and it thus it follows that

$$\tilde{a}(\tilde{t}) \sim \tilde{t}^{1/2} . \tag{7.25}$$

The same conclusion can also be reached for times close to the singularity, by explicitly inverting (7.24) and inserting into (7.23).

We thus see that the expansion is non-accelerated and the Hubble radius is expanding linearly. The dilaton singularity in the string frame is translated into a curvature singularity in the Einstein frame, a singularity which occurs at the Einstein frame time

$$\tilde{t}_s = -\frac{exp(-\phi_0)}{2} |\dot{\phi}_0|^{-1} . \tag{7.26}$$

From the constraint equation (7.10) it follows that this of the order of the string scale.

As stressed earlier, these solutions cannot be applied when the value of the dilaton is larger than 0 (when the string theory enters the strong coupling regime). Instead, we will have a strong coupling Hagedorn phase

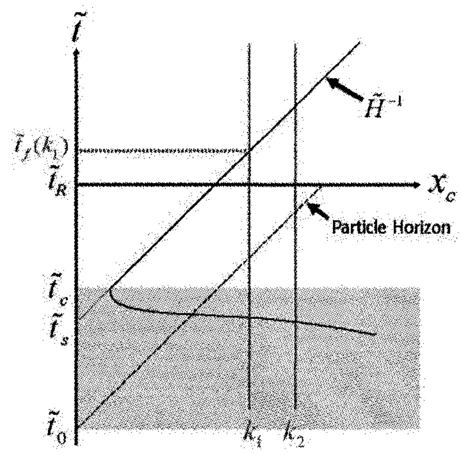

Figure 7.3: Space-time diagram (sketch) showing the evolution of fixed comoving scales in string gas cosmology. The vertical axis is Einstein frame time, the horizontal axis is comoving distance. The solid curve represents the Einstein frame Hubble radius $\tilde{H}^{-1}$ which is linearly increasing after $\tilde{t}_c$. Fixed comoving scales (the dotted lines labeled by k_1 and k_2) which are currently probed in cosmological observations have wavelengths which are larger than the Einstein frame Hubble radius during the part of the Hagedorn phase in which the dilaton is rolling. However, due to the presence of the initial strong coupling Hagedorn phase, the horizon becomes much larger than the Hubble radius. The shaded region corresponds to the strong coupling Hagedorn phase.

characterized by constant dilaton, and hence also almost constant Einstein frame scale factor.

The space-time sketch of our cosmology in the Einstein frame is sketched in Figure 3. In the absence of the strong coupling Hagedorn phase, the horizon (forward light cone beginning at $\tilde{t}_s$) would follow the Einstein frame Hubble radius (up to an irrelevant factor of order unity), thus yielding a horizon problem. However, during the strong coupling Hagedorn phase and, in particular, during the transition between the strong coupling Hagedorn phase and the phase described by the rolling dilaton, the horizon expands to lengths far greater than the Einstein frame Hubble radius. This phase can, in particular, establish thermal equilibrium on scales which are super-Hubble in the rolling dilaton phase. In the last section of this chapter we will come back to a discussion of how to model the strong coupling Hagedorn phase.

We will close this section with some general comments about the relationship between the string and the Einstein frames. Obviously, since the causal structure is unchanged by the conformal transformation, the comoving horizon is frame-independent. The Hubble radius, on the other hand, is a concept which depends on the frame. The string frame and the Einstein frame Hubble radii are two very different length scales. Both can be calculated in any frame, but they have different meanings.

7.4 Cosmological Perturbations in the String Frame

From the point of view of the string frame, the scale of cosmological fluctuations is sub-Hubble during the Hagedorn phase. In [133] it was assumed that thermal equilibrium in the Hagedorn phase exists on scales of the order of 1 mm. It was proposed to follow the string thermodynamical matter fluctuations on a scale k until that scale exits the string frame Hubble radius at the end of the Hagedorn phase, to determine the induced metric fluctuations at that time, and to follow the latter until the present.

To study cosmological perturbations, we make use of a particular gauge choice, longitudinal gauge (see [9] for an in depth review article on the theory of cosmological perturbations and [10] for a pedagogical introduction), in which the metric takes the form

$$ds^2 = e^{2\lambda(\eta)}\Big((1+2\Phi)d\eta^2 - (1-2\Psi)\delta_{ij}dx^i dx^j\Big)\,, \tag{7.27}$$

where η is conformal time, and where Ψ and Φ are the fluctuation variables which depend on space and time. In Einstein gravity, and for matter without anisotropic stress, the two potentials Φ and Ψ coincide, and Φ is the relativistic generalization of the Newtonian gravitational potential. In the case of dilaton gravity, the two potentials are related via the fluctuation of the dilaton field.

According to the proposal of [133], the cosmological perturbations are sourced in the Hagedorn phase by string thermodynamical fluctuations, in analogy to how in inflationary cosmology the quantum matter fluctuations source metric inhomogeneities [3]. On sub-Hubble scales, the matter fluctuations dominate. Hence it was suggested in [133, 135] to track the matter

[3] In spite of this analogy, there is a key difference, a difference which is in fact more important: In inflationary cosmology, the fluctuations are quantum vacuum perturbations, whereas in our scenario they are classical thermal fluctuations.

fluctuations on a fixed comoving scale k until the wavelength exits the Hubble radius at time $t_i(k)$ (see Fig. 2). At that time, the induced metric fluctuations are calculated making use of the Poisson equation

$$\nabla^2\Psi = 4\pi G a^2 \delta T^0_0 . \tag{7.28}$$

The key feature of string thermodynamics used in [133] (and discussed in more detail in [136, 135]) is the fact that the specific heat C_V scales as R^2, where R is the size of the region in which we are calculating the fluctuations. This result was derived in [139] and holds in the case of three large dimensions with topology of a torus. The specific heat, in turn, determines the fluctuations in the energy density. The scaling

$$C_V \sim R^2 \tag{7.29}$$

leads to a Poisson spectrum

$$P_{\delta\rho}(k) \sim k^4 \tag{7.30}$$

for the energy density fluctuations, and a similar spectrum for the pressure fluctuations [134, 135]. Making use of the Poisson equation (7.28), this leads to a scale-invariant spectrum for the gravitational potential Φ [4].

However, in the context of a relativistic theory of gravity, what should be used to relate the matter and metric fluctuations is the time-time component of the perturbed Einstein equation, or - more specifically - its generalization to dilaton gravity, and not simply the Poisson equation. There are correction terms compared to (7.28) coming from the expansion of the cosmological background and from the dynamics of the dilaton. The analysis of [133, 134] was done in the string frame. In this frame, during the Hagedorn phase the correction terms coming from the expansion of space are not present, but the dilaton velocity is important. The important concern is whether the resulting correction terms will change the conclusions of the previous work.

The perturbation equations in the string frame were discussed in [140]. Denoting the fluctuation of the dilaton field by χ, the equations read

$$\begin{aligned} \vec{\nabla}^2\Psi &- 3\mathcal{H}\Psi' - 3\mathcal{H}^2\Phi \\ &= \frac{1}{2}e^{2\phi+2\lambda}\left(2\chi T^0_0 + \delta T^0_0\right) - 6\mathcal{H}\Phi\phi' \\ &\quad -3\Psi'\phi' - \vec{\nabla}^2\chi + 3\mathcal{H}\chi' + 2\Phi\phi'^2 - 2\chi'\phi', \end{aligned} \tag{7.31}$$

[4]Note that thermal particle fluctuations would not give rise to a scale-invariant spectrum - see [155] for an interesting study of the role of thermal particle fluctuations in cosmological structure formation.

$$\partial_i \Psi' + \mathcal{H}\partial_i \Phi = \frac{1}{2}e^{2\phi+2\lambda}\delta T^i_0 + \partial_i \Phi \phi' - \partial_i \chi' + \mathcal{H}\partial_i \chi \tag{7.32}$$

$$\partial_i \partial_j \Big(\Phi - \Psi - 2\chi\Big) = 0 \qquad i \neq j, \tag{7.33}$$

$$-2\Psi'' - 4\mathcal{H}\Psi' - 2\mathcal{H}^2\Phi - 4\mathcal{H}'\Phi - 2\Phi'\mathcal{H} \tag{7.34}$$
$$= e^{2\phi+2\lambda}\Big(2\chi T^i_i + \delta T^i_i\Big) - 4\Phi\phi'' - 2\Phi'\phi' - 4\mathcal{H}\Phi\phi' - 4\Psi'\phi' + 2\chi'' + 2\mathcal{H}\chi' + 4\Phi\phi'^2 - 4\chi'\phi',$$

and

$$\begin{aligned} -2\Phi\phi'^2 &+ 2\phi'\chi' + \Phi\phi'' + \frac{1}{2}\Phi'\phi' + 2\mathcal{H}\Phi\phi' \\ &+ \frac{3}{2}\Psi'\phi' - \frac{1}{2}\chi'' + \frac{1}{2}\vec{\nabla}^2\chi - \mathcal{H}\chi' \\ &= \frac{1}{4}e^{2\phi+2\lambda}\Big(2\chi T + \delta T\Big), \end{aligned} \tag{7.35}$$

where $T \equiv T^\mu_\mu$ is the trace. The first equation is the time-time equation, the second the space-time equation, the next two the off-diagonal and diagonal space-space equations, respectively, and the last one is the matter equation.

The times $t_i(k)$ when the metric perturbations were computed are in the transition period between the Hagedorn phase and the radiation phase of standard cosmology. Space is beginning to expand. In this case, neither the terms containing the Hubble expansion rate nor those containing the dilaton velocity vanish.

We will first consider the case when the dilaton velocity is negligible (we will come back to a discussion of when this is a reasonable approximation), and then the case when the dilaton velocity is important.

If the dilaton velocity is negligible, then the equations simplify dramatically. We first note that at the time $t_i(k)$, the comoving Hubble constant $\mathcal{H}$ is of the same order of magnitude as k. Consider now, specifically, the time-time equation of motion (7.31). The terms containing $\mathcal{H}$ and its derivative on the left-hand side of this equation are of the same order of magnitude as the first term. We will, therefore, neglect all terms containing $\mathcal{H}$. Hence, the equation simplifies to

$$\vec{\nabla}^2\Psi = \frac{1}{2}e^{2\phi+2\lambda}\Big(2\chi T^0_0 + \delta T^0_0\Big) - \vec{\nabla}^2\chi. \tag{7.36}$$

Similarly, the perturbed dilaton equation simplifies to

$$-\frac{1}{2}\chi'' + \frac{1}{2}\vec{\nabla}^2\chi = \frac{1}{4}e^{2\phi+2\lambda}\Big(2\chi T + \delta T\Big), \tag{7.37}$$

From the latter equation, it follows that the Poisson spectrum of δT induces a Poisson spectrum of the dilaton fluctuation χ. Subtracting (7.37) from (7.36) and keeping in mind that the background pressure vanishes in the Hagedorn phase yields

$$\vec{\nabla}^2\Psi = \frac{1}{2}e^{2\phi+2\lambda}\left(\delta T^0_0 - \delta T\right) - \chi'' \tag{7.38}$$

from which it follows that the induced spectrum of Ψ will be scale-invariant

$$P_\Psi \sim k^0 \,. \tag{7.39}$$

It is easy to check that the other equations of motion are consistent with this scaling.

If the Hagedorn phase is modeled by the equations (7.4 - 7.6), then the dilaton velocity is not negligible, since it is related to the energy density via the constraint equation (7.4):

$$\dot{\phi}^2 = \frac{1}{4}e^{\varphi}E \,. \tag{7.40}$$

Thus, the terms containing the dilaton velocity are as important as the other terms in the Hagedorn phase. Now, the prescription of [133, 135] was to use the constraint equation at the time $t_i(k)$ when the scale k exits the Hubble radius. This time is towards the end of the Hagedorn phase. However, in the context of our action, Eq. (7.4) always holds. The right-hand side of this equation must be large since it gives the (square of the) Hubble expansion rate at the beginning of the radiation phase. At the time $t_i(k)$, then, for scales which are large compared to the Hubble radius at the beginning of the radiation phase, the value of the terms containing $\dot{\lambda}$ in (7.4) are negligible, and hence the dilaton velocity term is non-negligible.

Let us now compute the induced metric fluctuations in the Hagedorn phase taking into account the terms depending on the dilaton velocity. The time-dependence of the dilaton introduces a critical length scale into the problem, namely the inverse time scale of the variation of the dilaton. Translated to the Einstein frame, this length is the Einstein frame Hubble radius (this radius is, up to a numerical constant, identical to the forward light cone computed beginning at the time of the dilaton singularity). On smaller scales, the dilaton-dependent terms in the time-time Einstein constraint equation (7.31) are negligible, Eq. (7.31) reduces to the Poisson equation (7.28) and we conclude that the Poisson spectrum of the stringy matter induces a flat spectrum for the metric potential Ψ.

On larger scales, however, it is the dilaton-dependent terms in (7.31) which dominate. If we insist on the view that it is the string gas matter fluctuations which seed all metric fluctuations, then we must take all terms independent of the string sources to the left-hand side of the equations of motion. If we do this and keep the terms on the left-hand side of the equations which dominate in a gradient expansion, then the time-time equation becomes

$$3\Psi'\phi' - 2\Phi\phi'^2 + 2\chi'\phi' = \frac{1}{2}e^{2\phi+2\lambda}\left(2\chi T^0_0 + \delta T^0_0\right), \tag{7.41}$$

and the analogous approximation scheme applied to the dilaton equation yields

$$\begin{aligned} -2\Phi\phi'^2 &+ 2\phi'\chi' + \Phi\phi'' + \frac{1}{2}\Phi'\phi' + \frac{3}{2}\Psi'\phi' - \frac{1}{2}\chi'' \\ &= \frac{1}{4}e^{2\phi+2\lambda}\left(2\chi T + \delta T\right). \end{aligned} \tag{7.42}$$

Inspection of (7.42) shows that a Poisson spectrum of δT will induce a Poisson spectrum of χ. Subtracting two times (7.42) from (7.41) shows that, given a Poisson spectrum of χ, the resulting equation is no longer consistent with a scale-invariant spectrum for Ψ and Φ, since terms which have a scale-invariant spectrum would remain on the left-hand side of the equation. Hence, we conclude that the induced spectrum of Φ and Ψ will also be Poisson:

$$P_\Psi \sim k^4 . \tag{7.43}$$

The above conclusion is consistent with the Traschen Integral constraints [141] which state that in the absence of initial curvature fluctuations, motion of matter cannot produce perturbations with a spectrum which is less red than Poisson on scales larger than the horizon. This view is consistent with the fact that on small scales, the spectrum is scale-invariant: on small scales it is possible to move around matter by thermal fluctuations to produce new curvature perturbations.

As we have stressed earlier, however, the equations (7.4 - 7.6) are definitely not applicable early in the Hagedorn phase, namely in the strong coupling Hagedorn phase. In that phase, the dilaton is fixed, and thus the arguments of [133] imply the presence of scale-invariant metric fluctuations seeded by the string gas perturbations [5]. These fluctuations will persist in

[5]However, in this phase one must reconsider the computation of the string thermodynamic fluctuations, since our analysis implicitly assumes weak string coupling.

the phase in which the dilaton is rolling (the fluctuations cannot suddenly decrease in magnitude). Hence, we believe that the conclusions of [133] are robust.

7.5 Cosmological Perturbations in the Einstein Frame

From the point of view of the Einstein frame, the scales are super-Hubble during the phase of dilaton rolling. How is this consistent with their sub-Hubble nature from the point of view of the string frame? The answer is that, whereas the causal structure of space-time (and thus concepts like horizons) are frame-independent, the Hubble radius depends on the frame. The physical meaning of the Hubble radius is that it separates scales on which matter oscillates (sub-Hubble) from scales where the matter oscillations are frozen in (super-Hubble). Matter which is coupled minimally to gravity in the string frame feels the string frame Hubble radius [6], matter which is minimally coupled to gravity in the Einstein frame feels the Einstein frame Hubble radius. Strings couple minimally to gravity in the string frame and hence feel the string frame Hubble radius.

If we take into account the presence of the strong coupling Hagedorn phase, then it becomes possible, also in the Einstein frame, to study the generation of fluctuations. During the strong coupling Hagedorn phase, the dilaton is fixed and hence the fluctuation equations are those of Einstein gravity. Since scales of cosmological interest today are sub-Hubble during this phase, a scale-invariant spectrum of metric fluctuations is induced by the string gas fluctuations, as discussed in the previous section. If the strong coupling Hagedorn phase is long in duration, then a scale-invariant spectrum can be induced consistent with the Traschen integral constraints. Note that the a long duration of the strong coupling phase is required in order to justify the assumption of thermal equilibrium on the scales we are interested in.

We can also obtain the Einstein frame initial conditions by conformally transforming the initial conditions obtained in the string frame. The transformation of the perturbation variables in straightforward:

$$\tilde{\Psi} = (\Psi + \chi) \tag{7.44}$$

[6]Note, however, that the dilaton coupling to stringy matter can produce friction effects which are similar to Hubble friction.

$$\tilde{\Phi} = (\Phi - \chi) \quad (7.45)$$

$$\tilde{\chi} = 2\chi\,, \quad (7.46)$$

where, as before, tilde signs indicate quantities in the Einstein frame.

Note, in passing, that the string frame off-diagonal spatial equation of motion (7.33) immediately implies that

$$\tilde{\Psi} = \tilde{\Phi}\,, \quad (7.47)$$

which is the well-known result for Einstein frame fluctuations in the absence of matter with anisotropic stress.

Given the transformation properties (7.44 - 7.46) of the fluctuation variables, it is obvious that the conclusions about the initial power spectra of the fluctuations variables are the same as in the string frame: if the dilaton velocity can be neglected, the spectrum of Φ and Ψ is scale-invariant, if the dilaton velocity is important, the spectra of these variables are Poisson.

Whereas setting the initial conditions for the fluctuations may look more ad hoc in the Einstein frame, the evolution of the perturbations is easier to analyze since we can use all of the intuition and results developed in the context of fluctuations in general relativity. In particular, we can use the Deruelle-Mukhanov [142] matching conditions to determine the fluctuations in the post-Hagedorn radiation phase of standard cosmology from those at the end of the Hagedorn phase. The Deruelle-Mukhanov conditions are generalization to space-like hyper-surfaces of the Israel matching conditions [143] which state that the induced metric and the extrinsic curvature need to be the same on both sides of the matching surface. Applied to the case of cosmological perturbations, the result [142] is that (in terms of longitudinal gauge variables) both $\tilde{\Phi}$ and ζ need to be continuous [7] where ζ is defined as [147, 148, 149]

$$\zeta \equiv \tilde{\Phi} + \frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'}(\tilde{\Phi}' + \mathcal{H}\tilde{\Phi})\,, \quad (7.48)$$

where here $\mathcal{H}$ the Einstein frame Hubble expansion rate with respect to conformal time.

In the Einstein frame, the universe is radiation-dominated both before and after the transition. Hence, in both phases the dominant mode of the

[7] Note that the application of Israel matching conditions is, in our case, well justified. The concerns raised in [144] regarding the application of the matching conditions in the base of the Pre-Big-Bang [145] and Ekpyrotic/Cyclic [146] scenarios do not apply since in our case the matching conditions are satisfied at the level of the background solution.

equation of motion for $\tilde{\Phi}$ is a constant. The constant mode in the phase $\tilde{t} < \tilde{t}_R$ couples dominantly to the constant mode in the phase $\tilde{t} > \tilde{t}_R$. The initial value of the spectrum of $\tilde{\Phi}$ will seed both the constant and the decaying mode of $\tilde{\Phi}$ with comparable strengths and with the same spectrum. Hence, the late-time value of $\tilde{\Phi}$ is given, up to a factor of order unity, by the initial value of $\tilde{\Phi}$ at the time $\tilde{t}_i(k)$

$$P_{\tilde{\Phi}}(k,t) \simeq P_{\tilde{\Phi}}(k,\tilde{t}_i(k)) \sim k^0 \quad \tilde{t} >> \tilde{t}_R\,. \tag{7.49}$$

7.6 Discussion and Conclusions

In this chapter, we have recast string gas cosmology in the Einstein frame rather than in the string frame in which the analysis usually takes place. Our analysis sheds new light on several important cosmological issues.

At the level of the background evolution, it becomes clear that solutions of the dilaton gravity equations (7.4 - 7.6) with decreasing dilaton contain an initial singularity. From the point of view of the Einstein frame, there is an initial curvature singularity which follows from the presence of a singularity in the dilaton field in the string frame. Obviously, however, these solutions are not applicable at very early times since they correspond to times when the string theory is strongly coupled. Hence, there must be, prior to the phase of rolling dilaton, a strong coupling Hagedorn phase in which both the size of space and the dilaton must be quasi-static. Provided that this phase lasts sufficiently long, thermal equilibrium over all scales relevant to current observations can be established.

Note that during the period when the solutions of (7.4 - 7.6) have a rolling dilaton, then in the Einstein frame the expansion of space never accelerates. The evolution corresponds to that of a radiation-dominated universe. The Einstein frame Hubble radius increases linearly in time throughout. However, the presence of the strong coupling Hagedorn phase can solve the horizon problem in the sense of making the comoving horizon larger than the comoving scale corresponding to our currently observed universe.

How to model the strong coupling Hagedorn phase now becomes a crucial question for string gas cosmology [8]. Since the singularity of string gas cosmology is (from the point of view of the string frame) associated with the dilaton becoming large, and thus with string theory entering a strongly

[8]See also [156] for an interesting discussion of these issues.

coupled phase, it is interesting to conjecture that a process like tachyon condensation [151] will occur and resolve the singularity (like in the work of [152]). If this phase lasts for a long time, it will produce a large space in thermal equilibrium.

There are other possible scenarios in which there is a precursor phase of the rolling dilaton period which establishes thermal equilibrium on large scales. One such possibility was discussed in [150] and makes use of a pre-Hagedorn phase in which the extra spatial dimensions initially expand, driven by a gas of bulk branes. The resulting increase in the energy stored in the branes leads to the increase in size and entropy which solves both the horizon and entropy problems. Once the extra spatial dimensions have contracted again to the string scale, the size of our three spatial dimensions can be macroscopic while the temperature of matter is of string scale.

Another possibility to obtain a solution to the horizon problem and to justify thermal equilibrium over large scales is to invoke a bouncing cosmology such as obtained in the context of higher derivative gravity models in [153] (see also [154] for an earlier construction). The phase of contraction could produce the high densities required to form a string gas with the necessary requirements.

We have also studied the mechanism for the generation of fluctuations proposed in [133, 135] in more detail, both from the point of view of the string frame and the Einstein frame. We have shown that a small value of the dilaton velocity in the Hagedorn phase is required in order that the string thermodynamic fluctuations are able to generate a scale-invariant spectrum of cosmological fluctuations. If dilaton velocity terms are important, then a Poisson spectrum is produced. If the dilaton velocity is negligible, and if thermal equilibrium on the scales of interest can be justified, then a scale-invariant spectrum of metric fluctuations is induced.

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