BIRATIONAL ENDOMORPHISMS OF THE AFFINE PLANE

by

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A thesis submitted to the Faculty of Graduate Studies and Research, in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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March 1987

ISBN 0-315-38132-9

ABSTRACT

Birational morphisms $f : X \to Y$ of nonsingular surfaces are studied first. Properties of the surfaces X and Y are shown to be related to certain numerical data extracted from the configuration of "missing curves" of f, that is, the curves in Y whose generic point is not in $f(X)$. These results are then applied to the problem of decomposing birational endomorphisms of the plane into a succession of irreducible ones.

A graph-theoretic machinery is developed to keep track of the desingularisation of the divisors at infinity of the plane. That machinery is then used to investigate the problem of classifying all birational endomorphisms of the plane, and a complete classification is given in the case of two fundamental points.

RESUME

On étudie d'abord les morphismes birationnels $f : X \to Y$ de surfaces non singulières. On montre que les propriétés de X et Y sont liées à certains nombres et matrices extraits de la configuration des "courbes manquantes" de f, i.e., les courbes sur Y dont le point générique n'est pas dans $f(X)$. Ces résultats sont ensuite appliqués au problème de décomposer les endomorphismes birationnels du plan en successions d'endomorphismes irréductibles.

Une théorie des graphes est développée pour contrôler la désingularisation des diviseurs "à l'infini" du plan affine. Ces outils sont alors exploités pour étudier le problème de la classification des endomorphismes birationnels du plan et une classification complète est donnée dans le cas où il n'y a que deux points fondamentaux.

ACKNOWLEDGEMENTS

I wish to express my gratitude to all those who have contributed to this thesis, especially:

R. Ganong, who kindly explained to me some computational methods, in connection with questions treated here, and who examined early versions of this thesis and made many comments about them;

A. Lascu, for his encouragement, and for his help with some problems related to this research;

H. Kurke, for his interest in my work;

to those who made it possible for me to use the computer equipments, at the department of Mathematics, for typing my thesis. In this regard, many thanks go to Professor M. Barr.

I owe very special thanks to my professor, K. P. Russell, for the help and encouragement he provided, both personally and professionally, during the several years it took to complete this project. I also want to thank him for having introduced me to the very nice problems treated in this thesis, and for his enthusiasm to discuss them and to share his valuable ideas.

Finally, I want to thank my family and especially my wife, Anne-Marie, without whom it wouldn't have been possible to make it happen.

CONTENTS

INTRODUCTION

The questions at the origin of this thesis arose in the early seventies in Abhyankar's seminar at Purdue University. The participants were interested in several problems related to the geometry of the affine plane. One of those problems was the following:

Let X, Y be algebraically independent over a field $\mathbf{k}$. What are all *field generators* $f \in \mathbf{k}[X,Y]$, i.e., polynomials f such that there exists a rational function $g \in \mathbf{k}(X,Y)$ with $\mathbf{k}(f,g) = \mathbf{k}(X,Y)$? Of particular interest are the *good* field generators, that is, those for which the complementary function g can be chosen to be a polynomial in $\mathbf{k}[X,Y]$. (It turned out, though, that not all field generators are good [7], [16].)

Clearly, the study of good field generators is more or less equivalent to the study of birational endomorphisms of $\mathbf{A}^2$, since these are just given by homomorphisms $\phi : \mathbf{k}[X,Y] \to \mathbf{k}[X,Y]$ such that $\mathbf{k}(\phi(X), \phi(Y)) = \mathbf{k}(X,Y)$, i.e., ϕ can be viewed as a pair of elements f, g of $\mathbf{k}[X,Y]$ such that $\mathbf{k}(f,g) = \mathbf{k}(X,Y)$. A well known "non-trivial" (not automorphic) birational endomorphism of $\mathbf{A}^2$ is the "standard affine contraction in $\mathbf{A}^2$" (see III.2.1) given by

$$\begin{aligned} \phi : \mathbf{k}[X,Y] &\longrightarrow \mathbf{k}[X,Y] \\ X &\longmapsto X \\ Y &\longmapsto XY. \end{aligned}$$

ϕ gives a quadratic transformation of $\mathbf{P}^2$, and in view of the Noether-Castelnuovo factorisation theorem for birational transformations of $\mathbf{P}^2$ [11, theorem 6], [12], it was natural to ask whether every birational endomorphism of $\mathbf{A}^2$ is a composite of standard affine contractions and, of course, automorphisms.

There was some surprise when Russell, in conversations with A. Lascu, constructed a counterexample (III.2.6), namely a birational endomorphism $\psi : \mathbf{A}^2 \to \mathbf{A}^2$ such that ψ has three fundamental points and is irreducible. (The degrees of f and g in this example are 7, which may well be the minimum possible, or very close to it, for an irreducible birational endomorphism that is not a standard affine contraction.)

The methods by which the example was constructed, and its irreducibility proved, are as interesting as the example itself. They consist in a detailed analysis of the configuration of "missing curves" (I.1.3f) of ψ, that is, the curves in the target $\mathbf{A}^2$ whose generic point is not in $\psi(\mathbf{A}^2)$. These methods underlie large parts of this thesis. Russell soon exhibited a whole zoo of irreducible endomorphisms, some of them having infinitely near fundamental points. Its diversity shows that to give a reasonably complete *classification* of all birational endomorphisms of $\mathbf{A}^2$ is likely to be interesting and difficult. The aim of this thesis is to make some contributions to this problem.

In the first part of this thesis we study birational morphisms of nonsingular surfaces. This is partly because we find it interesting to see what is the contribution of various properties of $\mathbf{A}^2$ taken alone. (We consider properties such as affineness, factoriality, the property of having trivial units, and others.) But we also believe that it was psychologically necessary to adopt that general point of view, i.e., that our excursion helped us to prove things that we could not have understood by staying in the plane. The influence of Russell's methods is most visible in sections 2 and 3. The material in these sections consists essentially in generalizations of facts that Russell knew in the special case of ordinary fundamental points.

In many cases, however, even the correct statement of the generalised result was not obvious. Section 4 is devoted to the theory of weighted graphs. References are given at the beginning of that section. Apart from the last result (5.7), section 5 consists of simple observations and of facts that the author learned from his professor. The lemma (5.7), which is due to the author, is not used within this thesis; however, we believe that it may become useful in future investigations.

Part II is due to the author. It contains the graph-theoretic machinery that is used in part III, namely, the theory of local trees. Without doubt, this is the technical heart of the thesis. Of particular importance are the results numbered (3.8), (3.27), (3.28) and (3.32). In the author's opinion, the methods developed here are very appropriate for studying the divisors at infinity of $\mathbf{A}^2$.

Part III contains the material to which the title of this thesis refers. Section 1 (the preliminaries) contains several known facts, including the characterisation of $\mathbf{A}^2$ proved by Fujita [2] and Miyanishi and Sugie [8]. (1.4), (1.11) and (1.12) are due to the author. We don't know, however, if (1.4) has been noticed by other people. (1.11) will be used several times in part III, namely, whenever we prove that some curve is a coordinate line (1.9). Its corollary (1.12) is a characterisation of the coordinate lines in terms of the multiplicity sequence at infinity. Section 2 describes the (rather poor) state of our knowledge on the general problem of classifying the birational endomorphisms of $\mathbf{A}^2$. From the beginning to (2.5), we gather the pieces of information obtained from part I. (2.6) and (2.7) are examples that Russell found several years ago. It might be a good idea if the reader looks at (2.6) before reading anything else in this thesis. Whatever comes after (2.7) has been found by the author. (2.8) settles the case "n=1" of one fundamental point (including infinitely near ones); we don't know if that fact was known before we proved it. (2.11) is the author's contribution to the "zoo" of examples constructed by Russell. It consists in a family of irreducible endomorphisms exhibiting a particularly "nasty" behaviour with respect to infinitely near fundamental points. (In these examples we have "$j > 0$" and, in most cases, "$\delta > 0$"; see (I.1.3) for definitions. Before we found (2.11), all known irreducible examples had "$j = 0$" and a fortiori "$\delta = 0$".) In the last section, which is due to the author, we give a complete classification in the case "$n = 2$" of two fundamental points (including *i.n.* ones). That classification is given by theorems (3.1) and (3.2). The proofs make extensive use of the theory of local trees developed in part II. Note that some parts of the proofs generalise to the case "$n \geq 2$". We hope to eventually use these methods to get insight into the general theory.

Prerequisites and language. For the language and theory of basic algebraic geometry, we refer to [5], and in particular to sections V.1, V.3 and V.5. Our ground field is a fixed algebraically closed field $\mathbf{k}$, of arbitrary characteristic. An algebraic variety is an integral separated scheme of finite type over $\mathbf{k}$ (but only quasi-projective varieties will be considered). A surface (resp. a curve) is a variety of dimension two (resp. one); in particular, **curves and surfaces are irreducible and reduced.** All surfaces encountered in this thesis are nonsingular. All varieties have dimension ≤ 2, except at one or two places where the dimension is arbitrary and where the extra generality is irrelevant to us anyway. The words "complete" and "projective" are used interchangeably, and so are "blowing-up" and "monoidal transformation" (every blowing-up considered here is a blowing-up of a surface at a point). If X is a surface and Y a subset of X, a point Q is said to be *infinitely near* Y if it belongs to $\pi^{-1}(Y)$, for some composition $\pi : X' \to X$ of monoidal transformations. "Infinitely near" is abbreviated "*i.n.*".

For a variety X, the statement "X is factorial" means that it is the spectrum of a U.F.D.

supp (D) denotes the support of a divisor D. When D is effective and reduced, we will sometimes write "D" for the support of D. Div (X) is the group of divisors of a variety X and Cl (X) is the divisor class group, i.e., the group of divisors modulo linear equivalence.

$\mathcal{O}_X$ denotes the structure sheaf of a variety X and $\Gamma(X, \mathcal{O}_X)$ the ring of global sections. For a ring R (commutative, with 1), the group of units is R^*. We say X *has trivial units* if $\Gamma(X, \mathcal{O}_X)^* = \mathbf{k}^*$.

$\mathbf{A}^n$ denotes the affine n-space, $\mathbf{P}^n$ the projective n-space.

The domain and codomain of a morphism f are sometimes denoted dom (f) and codom (f) respectively.

$\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}$ denote respectively the set of positive integers, the ring of integers, the field of rational numbers, the field of real numbers and the field of complex numbers.

The cardinality of a set S is denoted by $|S|$. The g.c.d. of two integers a and b is denoted (a, b). The symbol $(C, C')_P$ stands for the local intersection number of curves C and C' at a point P.

The most important of all these comments is the following. Whenever possible, when we consider monoidal transformations, the same notation is used for a divisor D of a surface and for the *strict* transforms of D.

I. BIRATIONAL MORPHISMS OF NONSINGULAR SURFACES

Throughout this thesis, we are going to consider birational morphisms $f : X \to Y$ of nonsingular (algebraic) surfaces, over some fixed algebraically closed ground field $\mathbf{k}$, the characteristic of $\mathbf{k}$ being arbitrary. In this first part we will develop, to some extent, a general theory of such morphisms. However, since our ultimate goal is to understand the birational endomorphisms of $\mathbf{A}^2$, we will be primarily interested in those facts which are relevant to that special case. In particular, we will often consider the problem of describing the relations between the *structure* of f and the *properties* of the surfaces X and Y (properties such as affineness, factoriality, etc., i.e., properties that $\mathbf{A}^2$ has). By structure of f, we mean certain configurations of curves and points determined by f, or certain numerical data which can be extracted from these configurations; these notions will be defined in the first two sections.

1. Basic Concepts.

1.1. DEFINITIONS. Let X, Y be nonsingular surfaces. A morphism $f : X \to Y$ is called a *birational morphism* if it is dominant and if the induced inclusion of function fields is an isomorphism. Equivalently, there are open subsets X', Y' of X, Y respectively such that f restricts to an isomorphism $f' : X' \to Y'$.

From now on, the domain and codomain of any birational morphism under consideration will be tacitly assumed to be nonsingular surfaces.

Two birational morphisms $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ are *equivalent* if there are isomorphisms $x : X_1 \to X_2$, $y : Y_1 \to Y_2$ such that $f_1 = y^{-1} \circ f_2 \circ x$; we denote that relation by $f_1 \sim f_2$. Notice that it is not compatible with composition of morphisms, i.e., if $f_i : X_i \to Y_i$, $g_i : Y_i \to Z_i$, $(i = 1, 2)$ and $f_1 \sim f_2$, $g_1 \sim g_2$, it may happen that $g_1 \circ f_1 \not\sim g_2 \circ f_2$.

It is very well known (see for instance [5, chap. V, §5]) that, if X, Y are complete nonsingular surfaces, any birational morphism $X \to Y$ is a composition of monoidal transformations and any birational transformation $f : X \dashrightarrow Y$ can be written as $f = h \circ g^{-1}$ where g and h are birational morphisms of complete nonsingular surfaces. The following is an elementary consequence of these facts.

1.2. LEMMA. *Let $f : X \to Y$ be a birational morphism. Then there is a commutative diagram*

$$\begin{tikzcd} & Y_n \arrow[d, "\pi_n"] \\ & \vdots \arrow[d, "\pi_1"] \\ X \arrow[uur, hook] \arrow[r, "f"'] & Y = Y_0 \end{tikzcd} \qquad (n \geq 0)$$

such that $\pi_i : Y_i \to Y_{i-1}$ is the blowing-up of Y_{i-1} at some closed point $(1 \leq i \leq n)$, *and $X \hookrightarrow Y_n$ is an open immersion.*

1.3. DEFINITIONS AND REMARKS. Let $f : X \to Y$ be a birational morphism.

(a) The smallest $n \geq 0$ such that there exists a diagram as in (1.2) is denoted by $n(f)$. Clearly, $f \sim g \Rightarrow n(f) = n(g)$.

(b) A *fundamental point* of f is a point P of Y such that $f^{-1}(P)$ contains more than one point. By (1.2), there are at most $n(f)$ fundamental points.

Given a diagram as in (1.2) and $i > 0$, a fundamental point of $X \hookrightarrow Y_n \to \cdots \to Y_i$ which belongs to a curve that is contracted by $\pi_1 \circ \cdots \circ \pi_i$ is sometimes called an *infinitely near fundamental point of* f; such a point is not a fundamental point of f, according to definition (a). If f has no infinitely near (abbreviated *i.n.*) fundamental points we say f has ordinary fundamental points; that is the case iff f has $n(f)$ distinct fundamental points in its codomain.

(c) A *contracting curve* of f is a curve E in X such that $f(E)$ is a (fundamental) point. By (1.2), each such curve is isomorphic to an open subset of the projective line. The number of contracting curves is denoted by $c(f)$; it is an invariant of $\sim$ and we have $c(f) \leq n(f)$. Notice that if P is a fundamental point then $f^{-1}(P)$ is a union of contracting curves by (1.2); so f has at most $c(f)$ fundamental points.

(d) f is an open immersion iff f is injective, iff $c(f) = 0$, iff $n(f) = 0$. f will be said to be *trivial* if $n(f) = 0$, *nontrivial* if $n(f) > 0$.

(e) Consider a diagram as in (1.2), where n is not necessarily $n(f)$. For $i = 1, \ldots, n$, let P_i be the center of π_i and for $i = 0, \ldots, n$ let $f_i : X \to Y_i$ be the composite

$$X \hookrightarrow Y_n \to \cdots \to Y_i.$$

Then $n = n(f)$ iff P_i is a fundamental point of f_{i-1}, $1 \leq i \leq n$.

(f) The one dimensional irreducible components of the closure (in Y) of $Y \setminus f(X)$ are called the *missing curves* of f. The number of missing curves is denoted by $q(f)$; clearly, $f \sim g \Rightarrow q(f) = q(g)$. Given a curve C in Y, the following are equivalent:

- C is a missing curve
- $C \cap f(X)$ is contained in the set of fundamental points
- for some diagram as in (1.2) (equivalently for every such diagram) the strict transform of C in Y_n is disjoint from X.

(g) Let $q_0(f)$ denote the number of missing curves disjoint from $f(X)$. Clearly $q_0(f)$ is an invariant of $\sim$ and $q_0(f) \leq q(f)$. We will see later that $q_0(f) = 0$ whenever X has trivial units and Y has trivial divisor class group.

(h) A *minimal decomposition* of f is a diagram as in (1.2), with $n = n(f)$, together with an ordering of the set of missing curves (i.e., the missing curves are labelled $C_1, \ldots, C_q$ where $q = q(f) \geq 0$). Minimal decompositions will be denoted by $\mathcal{D}$, $\mathcal{D}'$, etc. Each time we choose a minimal decomposition $\mathcal{D}$, the following notations will be used:

- For the diagram, the notation is as in (1.2).
- The center of π_i is the point P_i of Y_{i-1} and the corresponding exceptional curve is E_i ($1 \leq i \leq n$).
- The missing curves are $C_1, \ldots, C_q$ where $q = q(f)$.
- **Whenever possible, the same notation will be used for a curve in some Y_i and for its strict transform in Y_j ($j > i$).**
- $\mathcal{D}$ determines a subset $J = J_{\mathcal{D}}$ of $\{1, \ldots, n\}$, defined by

$$J = \{\, i \mid E_i \cap X = \emptyset \text{ in } Y_n \,\}.$$

Thus the curves of Y_n which are disjoint from X are precisely $C_1, \dots, C_q$ and the E_i with $i \in J$. On the other hand, the contracting curves of f are the $E_i \cap X$ such that $i \in \{1, \dots, n\} \setminus J$. We see that $|J| + c(f) = n(f)$, so $|J|$ is an invariant of $\sim$. That number will be denoted by $j(f)$. Hence

$$c(f) + j(f) = n(f).$$

- D determines a subset $\Delta = \Delta_D$ of $\{1, \dots, n\}$, defined by

$$\Delta = \{ i \mid P_i \notin C_1 \cup \dots \cup C_q \text{ in } Y_{i-1} \}.$$

  One sees that the cardinality of Δ depends only on f, i.e., is independent of the choice of a minimal decomposition. That number is denoted by $\delta(f)$ and is, in fact, an invariant of $\sim$.

(i) If $i \in J$ then there is a j such that $i < j \le n$ and $P_j \in E_i$ (in Y_{j-1}).
(Indeed, if there is no such j then the inverse image of P_i in Y_n is E_i, which is disjoint from X. This means that $P_i \notin f_{i-1}(X)$, which is a contradiction (see part (e)) with the fact that P_i is a fundamental point of f_{i-1}.)

(j) At this level of generality, i.e., when no further conditions are imposed on X and Y, the philosophy is that there exists an f having a given property whenever there is no obvious reason that prevents it from existing. The following (trivial) fact is an illustration of this principle:

Given nonnegative integers n, c, q, q_0, j, δ, there exists an f with $n(f) = n$, $c(f) = c$, $q(f) = q$, $q_0(f) = q_0$, $j(f) = j$ and $\delta(f) = \delta$ iff

$$q_0 \le q, \quad \delta \le n, \quad j + c = n \quad \text{and} \quad \begin{cases} c = 0 \Rightarrow n = 0 \\ n = 0 \Rightarrow q_0 = q \\ q_0 = q \Leftrightarrow \delta = n. \end{cases}$$

(k) If $g : Y \to Z$ is a birational morphism, we denote by $\Delta c(f, g)$ the number of missing curves of f which are contracted by g.

1.4. LEMMA. *Let $f : X \to Y$ and $g : Y \to Z$ be birational morphisms.*
(a) $c(g \circ f) = c(f) + c(g) - \Delta c(f, g)$ and $q(g \circ f) = q(f) + q(g) - \Delta c(f, g)$.
(b) $n(g \circ f) \le n(f) + n(g)$ and $j(g \circ f) \le j(f) + j(g) + \Delta c(f, g)$.
(c) If $q_0(f) = 0$ then $n(g \circ f) = n(f) + n(g)$ and $j(g \circ f) = j(f) + j(g) + \Delta c(f, g)$.

PROOF: Let $h = g \circ f : X \to Z$ and let $\Gamma_1, \dots, \Gamma_{c(h)}$ be the contracting curves of h, labelled in such a way that $\Gamma_1, \dots, \Gamma_{c(f)}$ are the contracting curves of f. If Γ is a contracting curve of g, Γ is not a missing curve of f iff Γ is the closure in Y of some $f(\Gamma_i)$ with $c(f) < i \le c(h)$. Hence the equation $c(h) - c(f) = c(g) - \Delta c(f, g)$ is clear; an equally straightforward argument proves the second equation of (a), i.e., (a) is proved.

Choose a minimal decomposition of f and one of g, and consider the corresponding commutative diagram:

$$
\begin{tikzcd}
 & & & & Z_n \arrow[d] \\
 & & & & \vdots \arrow[d] \\
 & & Y_{n(f)} \arrow[d] \arrow[uurr, hook] & & Z_{n(g)} \arrow[d] \\
 & & \vdots \arrow[d] & & \vdots \arrow[d] \\
X \arrow[r, "f"'] \arrow[uurr, hook] & Y \arrow[r, phantom, "="] & Y_0 \arrow[r, "g"'] \arrow[uurr, hook] & Z \arrow[r, phantom, "="] & Z_0
\end{tikzcd}
$$

where $\hookrightarrow$ means open immersion and $n = n(f) + n(g)$. By definition, $n(g \circ f) \leq n$. The second inequality of (b) follows from this and (a), so (b) is clear. To prove (c), denote the center of $Z_i \to Z_{i-1}$ by P_i $(1 \leq i \leq n)$ and let h_i be the composite $X \hookrightarrow Y_{n(f)} \hookrightarrow Z_n \to \cdots \to Z_i$ $(0 \leq i \leq n)$. By (1.3e), it's enough to check that P_i is a fundamental point of h_{i-1} $(1 \leq i \leq n)$. If $n(g) < i \leq n$ then that condition holds, by (1.3e) applied to the minimal decomposition of f. If $1 \leq i \leq n(g)$ then by (1.3e) P_i is a fundamental point of $Y_0 \hookrightarrow Z_{n(g)} \to \cdots \to Z_i$, so there is a curve Γ in Y which contracts to P_i. If $q_0(f) = 0$ then $f^{-1}(\Gamma)$ contains a curve, so P_i is a fundamental point of h_{i-1}. Hence $n(g \circ f) = n(f) + n(g)$, and the second equation follows from that and (a).

REMARK. From the proof of (1.4), we see that if $f : X \to Y$ and $g : Y \to Z$ are birational morphisms and $q_0(f) = 0$ then each pair $(\mathcal{D}_f, \mathcal{D}_g)$ of minimal decompositions (of f, g respectively) determines a minimal decomposition $\mathcal{D}$ of $g \circ f$. More precisely, the commutative diagram is as in the proof and the missing curves are labelled as follows: If $C_1, \ldots, C_{q(g)}$ are the missing curves of g, and if $\Gamma_{i_1}, \ldots, \Gamma_{i_k}$ are (the images in Z of) those missing curves of f which are not contracted, where $i_1 < \cdots < i_k$, then the missing curves of $g \circ f$ are $C_1, \ldots, C_{q(g)}, \Gamma_{i_1}, \ldots, \Gamma_{i_k}$, in that order.

1.5 COROLLARY. *Let $f : X \to Y$ and $g : Y \to Z$ be birational morphisms. Then*

$$c(g \circ f) - q(g \circ f) = \big(c(f) - q(f)\big) + \big(c(g) - q(g)\big),$$

i.e., the number $c - q$ is "additive".

1.6. For a given $f : X \to Y$, minimal decompositions can be obtained one from another by relabelling the missing curves and by changing the order of the blowings-up (hence by relabelling the points $P_1, \ldots, P_n$). More precisely, if $\mathcal{D}$ and $\mathcal{D}'$ are minimal decompositions of f (where the notation of (1.3h) is used for $\mathcal{D}$, and P_i', E_i', C_{i_k}' etc. for $\mathcal{D}'$), then there is a unique pair $(\sigma, \tau) = (\sigma^{\mathcal{D},\mathcal{D}'}, \tau^{\mathcal{D},\mathcal{D}'})$ of permutations of $\{1, \ldots, q\}$ and $\{1, \ldots, n\}$ respectively, such that

(a) $C_i = C_{\sigma i}'$ for $1 \leq i \leq q$

(b1) $\mu(P_i, \Gamma) = \mu(P_{\tau i}', \Gamma)$ for $1 \leq i \leq n$ and for all curves Γ in Y

(b2) $\mu(P_i, E_j) = \mu(P_{\tau i}', E_{\tau j}')$ for all $i, j \in \{1, \ldots, n\}$

where $\mu(P_i, \Gamma)$ and $\mu(P_i, E_j)$ are defined as in (2.4), below, and similarly for $\mu(P_i', \Gamma)$ and $\mu(P_i', E_j')$. From (b2), we deduce that

(b3) P_i is i.n. P_j iff $P_{\tau i}'$ is i.n. $P_{\tau j}'$

(b4) $\tau_i > \tau_j$ whenever P_i is i.n. P_j and $i \neq j$

where (b4) follows from (b3). A permutation of $\{1,\dots,n\}$ which satisfies (b4) is called a $\mathcal{D}$-allowable permutation. Clearly, if τ is $\mathcal{D}$-allowable and σ is any permutation of $\{1,\dots,q\}$ then $(\sigma,\tau)=(\sigma^{\mathcal{D},\mathcal{D}'},\tau^{\mathcal{D},\mathcal{D}'})$ for some $\mathcal{D}'$. Moreover, we have the rules $\sigma^{\mathcal{D},\mathcal{D}}=id$ and $\sigma^{\mathcal{D}',\mathcal{D}''}\circ\sigma^{\mathcal{D},\mathcal{D}'}=\sigma^{\mathcal{D},\mathcal{D}''}$, and similarly for τ.

We will often find ourselves in the situation where, given $\mathcal{D}$ and $A\subseteq\{1,\dots,n\}$, we want the blowings-up at $\{P_i\,|\,i\in A\}$ to be performed first, i.e., we want to find $\mathcal{D}'$ such that if $\tau=\tau^{\mathcal{D},\mathcal{D}'}$ then $\tau(A)=\{1,\dots,s\}$, where $s=|A|$. For which A is that possible? Say that A is $\mathcal{D}$-closed if for all $i,j\in\{1,\dots,n\}$, $i\in A$ and P_i *i.n.* P_j imply $j\in A$. Notice that a topology on $\{1,\dots,n\}$ is obtained, and that if $\tau=\tau^{\mathcal{D},\mathcal{D}'}$, A is $\mathcal{D}$-closed iff $\tau(A)$ is $\mathcal{D}'$-closed. It is also clear that the existence of a $\mathcal{D}'$ such that $\tau^{\mathcal{D},\mathcal{D}'}(A)=\{1,\dots,s\}$ is equivalent to the $\mathcal{D}$-closedness of A.

For instance, the set $\Delta_{\mathcal{D}}$ (see (1.3h)) is $\mathcal{D}$-open, so we can always find a minimal decomposition satisfying $\Delta=\{n-\delta+1,\dots,n\}$.

2. Affineness, Factoriality and Trivial Units.

We will now study the relations between the structure of a birational morphism $f:X\to Y$ and certain properties of X and Y. The first such property is affineness.

2.1 PROPOSITION. *Let $f:X\to Y$ be a birational morphism, with missing curves $C_1,\dots,C_q$ $(q\geq 0)$. Consider the following conditions:*

(a) Y is affine, X is connected at infinity and no contracting curve of f is complete;

(b) X is affine;

(c) all fundamental points of f are in $C_1\cup\dots\cup C_q$ and the interior of $f(X)$ ($\operatorname{int} f(X)$) is $Y\setminus(C_1\cup\dots\cup C_q)$ and is affine.

Then (a)$\Rightarrow$(b)$\Rightarrow$(c).

The main ingredients of the proof are the following two facts. For the first one, see [3, theorem 2, p. 168] or [4, theorem 4.2, p. 69]; for the second, see [5, chap. V, theorem 1.10].

2.1.1 THEOREM. *Let U be an open subset of a complete nonsingular surface S. Then U is affine iff $S\setminus U$ is the support of an effective ample divisor of S.*

2.1.2 THEOREM (NAKAI-MOISHEZON CRITERION). *A divisor D on a complete nonsingular surface S is ample iff $D^2>0$ and $D.C>0$ for all curves C in S. (In particular, if D is effective then it is ample iff $D.C>0$ for all curves C in S.)*

Before we prove the proposition, we find it convenient to define some terminologies and symbols, and to prove some lemmas about them. The definitions are local to this discussion and to the proof of (2.1). These considerations are elementary and probably exist, in one form or another, in the literature.

2.1.3. DEFINITIONS. Let S be a complete nonsingular surface. For $D\in\operatorname{Div}(S)$, let the symbol $D\gg 0$ mean that D is effective, $D\neq 0$ and every irreducible component C of D satisfies $C.D>0$. Then the set $P(S)$ of divisors D such that $D\gg 0$ is a nonempty additive semigroup. Say that a subset Z of S is *positive* (in S) if $Z=\operatorname{supp}(D)$ for some $D\gg 0$. Then the set of positive subsets of S is stable under finite unions.

2.1.4. Lemma. *Let S be a complete nonsingular surface and Z a subset of S. Then the following are equivalent:*

(a) Z is positive;

(b) Z is closed, $Z \neq \emptyset$ and every connected component of Z is positive;

(c) Z is closed, $\emptyset \neq Z \neq S$ and every connected component of Z contains a positive set.

Indeed, (a)$\Leftrightarrow$(b)$\Rightarrow$(c) is trivial and to prove (c)$\Rightarrow$(b) amounts to proving that if Z is a finite union of curves which is connected and which contains a positive set Z', then Z is positive. Now a straightforward argument shows that if C is an irreducible component of Z such that $\emptyset \neq C \cap Z' \neq C$ then $C \cup Z'$ is positive; hence we are done.

2.1.5. Lemma. *Let $\pi : \widetilde{S} \to S$ be the blowing-up of a nonsingular complete surface S at a closed point P. Then:*

(a) If Z is a positive subset of $\widetilde{S}$ then $\pi(Z)$ is a positive subset of S;

(b) If $Z \subseteq S$, then Z is positive in S iff $\pi^{-1}(Z)$ is positive in $\widetilde{S}$.

Proof: (a) Let Z be positive in $\widetilde{S}$; then Z is not the exceptional curve E, so $Z = \operatorname{supp}(D)$, $D \in P(\widetilde{S})$ and $D = \widetilde{D}_0 + nE$ (where "$\widetilde{\ \ }$" means strict transform) for some nonnegative integer n and some effective $D_0 \in \operatorname{Div}(S)$, $D_0 \neq 0$. If C is an irreducible component of D_0 and $\widetilde{C}$ its strict transform, then

$$0 < \widetilde{C}.D = \widetilde{C}.\widetilde{D}_0 + n\widetilde{C}.E = C.D_0 - \mu(P,C)\cdot[\mu(P,D_0) - n]$$

and $0 \leq E.D = \mu(P,D_0) - n$, i.e., $C.D_0 > 0$. Hence $\pi(Z) = \operatorname{supp}(D_0)$ is positive in S.

(b) Let Z be positive in S. If $P \notin Z$ then the assertion is trivial. Assume $P \in Z$ and let $\mathcal{C}$ be the set of irreducible components C of Z such that $P \in C$. Let $D_0 \in P(S)$, such that $Z = \operatorname{supp}(D_0)$. For each $C \in \mathcal{C} \neq \emptyset$ we have

$$\mu(P,D_0) > \mu(P,D_0) - \frac{C.D_0}{\mu(P,C)}.$$

Hence we may choose positive integers a, b such that

$$\mu(P,D_0) > \frac{b}{a} > \mu(P,D_0) - \frac{C.D_0}{\mu(P,C)}, \qquad \text{all } C \in \mathcal{C}.$$

One checks that $a\widetilde{D}_0 + bE \in P(\widetilde{S})$, i.e., $\pi^{-1}(Z)$ is positive.

For the converse, observe that $Z = \pi(\pi^{-1}(Z))$ and use (a).

Proof of (2.1): Assume that (a) or (b) holds. Choose a minimal decomposition of f, with notation as in (1.3h), imbed Y_0 in a complete nonsingular surface $\overline{Y}_0$ and "complete the diagram":

$$\begin{tikzcd}
 & & Y_n \arrow[r, hook] \arrow[d, "\pi_n"] & \overline{Y}_n \arrow[d, "\overline{\pi}_n"] \\
 & & \vdots \arrow[d, "\pi_1"] & \vdots \arrow[d, "\overline{\pi}_1"] \\
X \arrow[r, "f"'] \arrow[urruu, hook, dashed] & Y = & Y_0 \arrow[r, hook] & \overline{Y}_0
\end{tikzcd}$$

where $\overline{\pi}_i$ is the blowing-up of $\overline{Y}_{i-1}$ at P_i ($1 \leq i \leq n$). Then $\overline{Y}_n \setminus X$ is connected and contains a curve, hence is a nonempty union of curves. So $Y_n \setminus X$ is a (possibly empty) union of curves, i.e.,

$$Y_n \setminus X = C_1 \cup \cdots \cup C_q \cup \bigcup_{j \in J} E_j$$
$$\overline{Y}_n \setminus X = C_1 \cup \cdots \cup C_q \cup \bigcup_{j \in J} E_j \cup L_1 \cup \cdots \cup L_p$$

where, in the second equation, C_i stands for the closure in $\overline{Y}_n$ of the missing curve C_i and $L_1, \ldots, L_p$ are curves in $\overline{Y}_n$, distinct from $C_1, \ldots, C_q$ and from the E_j with $j \in J$. From now on, in fact, C_i will be the closure of the missing curve C_i in any $\overline{Y}_j$ under consideration. Let $\pi = \pi_1 \circ \cdots \circ \pi_n$ and $\overline{\pi} = \overline{\pi}_1 \circ \cdots \circ \overline{\pi}_n$ then $\overline{\pi}$ does not contract L_i, so $L_i \subseteq \overline{Y}_n$ is the strict transform of a curve $L_i \subseteq \overline{Y}_0$ ($1 \leq i \leq p$). We see that $L_1, \ldots, L_p$ in $\overline{Y}_0$ (resp. in $\overline{Y}_n$) are the one-dimensional irreducible components of $\overline{Y}_0 \setminus Y_0$ (resp. $\overline{Y}_n \setminus Y_n$). Let

$$\Lambda_0 = L_1 \cup \cdots \cup L_p \quad \text{in } \overline{Y}_0$$
$$\Lambda_n = L_1 \cup \cdots \cup L_p \quad \text{in } \overline{Y}_n$$
$$\Gamma_0 = C_1 \cup \cdots \cup C_q \quad \text{in } \overline{Y}_0$$
$$\Gamma_n = C_1 \cup \cdots \cup C_q \quad \text{in } \overline{Y}_n$$
$$Z_0 = \Gamma_0 \cup \Lambda_0 \quad \text{in } \overline{Y}_0$$
$$Z_n = \Gamma_n \cup \bigcup_{j \in J} E_j \cup \Lambda_n \quad \text{in } \overline{Y}_n$$

and denote by F the set of fundamental points of f.

Claim.
(1) $\overline{Y}_n \setminus X = Z_n$ is connected and $\overline{Y}_0 \setminus Y_0 = \Lambda_0 \cup$ points
(2) $F \subset \Gamma_0$ and $\overline{\pi}^{-1}(F) = E_1 \cup \cdots \cup E_n$
(3) $\overline{\pi}(Z_n) = Z_0$
(4) $\overline{\pi}^{-1}(\Lambda_0) = \Lambda_n$
(5) $\operatorname{int} f(X) = Y_0 \setminus \Gamma_0 = \overline{Y}_0 \setminus Z_0$
(6) $\overline{Y}_0 \setminus \operatorname{int} f(X) = Z_0$.

In fact, (1) is trivial. If $a \in F$ then $\pi^{-1}(a)$ can't contain Z_n (indeed, suppose $Z_n \subseteq \pi^{-1}(a)$ then $Z_n = \bigcup_{j \in J} E_j$ and $p = q = 0$; in particular $\overline{Y}_0 \setminus Y_0$ contains no curve, so Y_0 is not affine; since (a) or (b) holds by assumption, X must be affine, so Z_n is positive by (2.1.1) and (2.1.2), so is $\pi(Z_n) = \{a\}$ by (2.1.5) and this is absurd) and $Z_n \cap \pi^{-1}(a) \neq \emptyset$ because no contracting curve of f is complete. Thus there is an irreducible component C of Z_n such that $\emptyset \neq C \cap \pi^{-1}(a) \neq C$, by connectedness of Z_n. Clearly, $C \subseteq \Gamma_n$, so $a \in \Gamma_0$ and $F \subset \Gamma_0$. From (1.3e) we see that $\pi^{-1}(F) = E_1 \cup \cdots \cup E_n$, so (2) holds. Now (3) follows immediately, (4) is trivial, (6) is an immediate consequence of (5) and (5) is proved by observing that

$$\operatorname{int} f(X) \subseteq Y_0 \setminus \Gamma_0 \subseteq Y_0 \setminus Z_0 \subseteq \overline{Y}_0 \setminus Z_0 \subseteq f(X),$$

where the only non obvious fact (if any) is the last inclusion. Now let $y \in \overline{Y}_0 \setminus Z_0$; since $F \subseteq \Gamma_0$ by (2), we have $\overline{\pi}^{-1}(Z_0) \supseteq Z_n$, so

$$\overline{\pi}^{-1}(y) \subseteq \overline{Y}_n \setminus \overline{\pi}^{-1}(Z_0) \subseteq \overline{Y}_n \setminus Z_n = X \quad \text{by (1)},$$

hence $y \in f(X)$ and the claim is proved.

Proof of (a)$\Rightarrow$(b). If (a) holds then $\overline{Y}_0 \setminus Y_0 = \Lambda_0$ and Λ_0 is positive, by (1), (2.1.1) and (2.1.2). Hence Λ_n is positive, by (4) and (2.1.5b), and Z_n is positive by connectedness of Z_n and (2.1.4). Let $D \in P(\overline{Y}_n)$ be such that $Z_n = \operatorname{supp}(D)$; since a straightforward argument shows that Z_n meets every curve in $\overline{Y}_n$, D is ample by (2.1.2) and X is affine by (1) and (2.1.1). Hence (b) holds.

Proof of (b)$\Rightarrow$(c). Statements (2) and (5) show that f restricts to an isomorphism

$$f^{-1}(\operatorname{int} f(X)) \longrightarrow \operatorname{int} f(X),$$

and that $f^{-1}(\operatorname{int} f(X)) = X \setminus \overline{\pi}^{-1}(Z_0)$, which is just the open set obtained by removing the contracting curves from X. But if (b) holds then X is affine, thus so is X minus the contracting curves, since removing a curve from an affine nonsingular surface yields an affine surface (by, say, a straightforward argument using (2.1.1) and (2.1.2)). Hence we are done.

2.2. COROLLARY. *Let $f : X \to Y$ be a birational morphism and suppose that Y is affine. Then the following are equivalent:*

(a) X is affine,

(b) X is connected at infinity and no contracting curve of f is complete.

The next properties (for a surface S) that will interest us are (1) the property of having a trivial divisor class group, i.e., $\mathrm{Cl}(S) = 0$, and (2) the property of having trivial units, i.e., $\Gamma(S, \mathcal{O}_S)^* = \mathbf{k}^*$. To begin with, we recall a well-known fact:

2.3. PROPOSITION. *Let V be a complete nonsingular algebraic variety and $U \neq \emptyset$ an open subset of V. Among the irreducible components of $V \setminus U$, let $\Gamma_1, \dots, \Gamma_r$ $(r \geq 0)$ be those of codimension one in V, and let $\overline{\Gamma}_1, \dots, \overline{\Gamma}_r$ be their images in $Cl(V)$.*

(a) $Cl(U) = 0 \iff \overline{\Gamma}_1, \dots, \overline{\Gamma}_r$ generate $Cl(V)$.

(b) $\Gamma(U, \mathcal{O}_U)^* = \mathbf{k}^* \iff \overline{\Gamma}_1, \dots, \overline{\Gamma}_r$ are linearly independent.

2.4. PRELIMINARIES. Since we will be dealing with minimal decompositions, it will be necessary to keep track of the divisors in the various blown up surfaces. Let Y_0 be any nonsingular surface and consider

$$Y_n \xrightarrow{\pi_n} Y_{n-1} \longrightarrow \cdots \xrightarrow{\pi_1} Y_0 \qquad (n \geq 1)$$

where $\pi_i : Y_i \to Y_{i-1}$ is the blowing-up of Y_{i-1} at some point P_i and let $E_i = \pi_i^{-1}(P_i) \in \operatorname{Div}(Y_i)$ $(1 \leq i \leq n)$. Given integers i, ν such that $1 \leq i \leq n$ and $0 \leq \nu \leq n$ and given $D \in \operatorname{Div}(Y_\nu)$ we define $\mu(P_i, D)$ to be the multiplicity of P_i on the appropriate strict transform of D if $i - 1 \geq \nu$, and we define it to be sero if $i - 1 < \nu$. Then we define

$$\mu(D) = \begin{bmatrix} \mu(P_1, D) \\ \vdots \\ \mu(P_n, D) \end{bmatrix} \in \mathbf{Z}^n$$

and we have the following $n \times n$ matrix:

$$\mathcal{E} = (e_{ij}) = (\mu(E_1) \ \cdots \ \mu(E_n))$$

where, of course, $e_{ij} = 0$ whenever $i \le j$. Let $D^* \in \operatorname{Div}(Y_n)$ be the total transform of $D \in \operatorname{Div}(Y_\nu)$. Then $D^* = D + a_1E_1 + \cdots + a_nE_n$ for some integers $a_1, \ldots, a_n$; let us calculate these integers. If R_i is the i^{th} row of the identity matrix I_n, define an $n \times n$ matrix $\epsilon = (\epsilon_{ij})$ by

$$\begin{aligned} (\epsilon_{11} \ \cdots \ \epsilon_{1n}) &= R_1 \\ (\epsilon_{k1} \ \cdots \ \epsilon_{kn}) &= R_k + (e_{k1} \ \cdots \ e_{k\,k-1})(\epsilon_{ij})_{\substack{1 \le i < k \\ 1 \le j \le n}} \qquad (1 < k \le n). \end{aligned}$$

So ϵ is completely determined by $\mathcal{E}$, is a lower triangular matrix with $\epsilon_{ii} = 1$ $(1 \le i \le n)$ and has $\det(\epsilon) = 1$. For $1 \le i \le n$, define

$$\begin{aligned} \epsilon_i : \mathbf{Z}^n &\longrightarrow \mathbf{Z} \\ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} &\longmapsto (\epsilon_{i1} \ \cdots \ \epsilon_{in}) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}. \end{aligned}$$

Then a straightforward argument (descending induction on ν) shows that, if we define $\epsilon_i(D) = \epsilon_i(\mu(D)) \in \mathbf{Z}$, then

$$D^* = D + \sum_{i=1}^{n} \epsilon_i(D)\, E_i \qquad (\text{in } Y_n).$$

Next, one checks that

$$\begin{aligned} \theta : \operatorname{Cl}(Y_0) \oplus \mathbf{Z}^n &\longrightarrow \operatorname{Cl}(Y_n) \\ (\overline{D}, \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}) &\longmapsto \overline{D^*} + \sum_{i=1}^{n} a_i \overline{E}_i \end{aligned}$$

is an isomorphism (where $D^* \in \operatorname{Div}(Y_n)$ is the total transform of $D \in \operatorname{Div}(Y_0)$ and E_i is the strict transform in Y_n of $E_i \in \operatorname{Div}(Y_i)$). By the above calculation, one sees that if $D \in \operatorname{Div}(Y_0)$ and if the strict transform of D in Y_n is also denoted by D, then

$$\theta^{-1}(\overline{D}) = (\overline{D}, -\epsilon\,\mu(D)).$$

Clearly, $\theta^{-1}(\overline{E}_i) = (0, K_i)$, $1 \le i \le n$, where K_i denotes the i^{th} column of the identity matrix I_n.

2.5. Definitions. Let $f : X \to Y$ be a birational morphism and write $n = n(f)$, $c = c(f)$ and $q = q(f)$. Let $\mathcal{D}$ be a minimal decomposition for f, with notation as in (1.3h). Then we define the following matrices:

$$\begin{aligned} \mu &= \mu_{\mathcal{D}} = (\mu(C_1) \ \cdots \ \mu(C_q)) \qquad (n \times q) \\ \mathcal{E} &= \mathcal{E}_{\mathcal{D}} = (\mu(E_1) \ \cdots \ \mu(E_n)) \qquad (n \times n) \\ \epsilon &= \epsilon_{\mathcal{D}} = (\epsilon_{ij}) \qquad (n \times n) \qquad \text{defined as in (2.4),} \end{aligned}$$

and we let $\epsilon' = \epsilon'_{\mathcal{D}}$ be the $c \times n$ sub-matrix of ϵ obtained by deleting the i^{th} row whenever $i \in J$.

Observe that the product $\epsilon'\mu$ is a $c \times q$ matrix; its q columns will be regarded as elements of $\mathbb{Z}^c$, even if $c = 0$ or $q = 0$. To make sense out of these extreme cases, let us *agree* that (1) the columns of a $0 \times q$ matrix generate $\mathbb{Z}^0$, and are linearly independent iff $q = 0$; (2) the columns of a $c \times 0$ matrix are linearly independent, and generate $\mathbb{Z}^c$ iff $c = 0$; and (3) the 0×0 matrix has determinant equal to 1. Without these conventions, we would have to restrict the coming bunch of results to the special case where c and q are positive; in the proofs, however, these integers will be tacitly assumed to be positive and verification of the remaining cases will be left to the reader.

2.6 **Proposition.** *Let $f : X \to Y$ be a birational morphism and $\mathcal{D}$ a minimal decomposition; let the notation be as in (2.5) and let $\delta = \delta(f)$.*

(a) If $Cl(X) = 0$, then the columns of $\epsilon'\mu$ generate $\mathbb{Z}^c$, $Cl(\operatorname{int} f(X)) = 0$, $q \geq c$ and $\delta \leq j$.

(b) If $Cl(Y) = 0$ and the columns of $\epsilon'\mu$ generate $\mathbb{Z}^c$, then $Cl(X) = 0$.

On the other hand, consider the statements:

(1) $\Gamma(X, \mathcal{O}_X)^* = \mathbf{k}^*$

(2) $\Gamma(Y, \mathcal{O}_Y)^* = \mathbf{k}^*$

(3) the columns of $\epsilon'\mu$ are linearly independent

(4) $Cl(Y) = 0$.

Then (1)$\wedge$(4) $\Longrightarrow$ (2)$\wedge$(3) $\Longrightarrow$ (1) $\Longrightarrow$ (2), and (3) implies $q \leq c$ and $\delta \leq n - q$.

Proof: Consider the minimal decomposition $\mathcal{D}$, with notation as usual. Imbed Y_0 in a complete non-singular surface $\overline{Y}_0$ and "complete the diagram":

$$\begin{tikzcd} & & Y_n \arrow[r, hook] \arrow[d, "\pi_n"] & \overline{Y}_n \arrow[d, "\overline{\pi}_n"] \\ & & \vdots \arrow[d, "\pi_1"] & \vdots \arrow[d, "\overline{\pi}_1"] \\ X \arrow[r, "f"] \arrow[uurr, hook] & Y = & Y_0 \arrow[r, hook] & \overline{Y}_0 \end{tikzcd}$$

Let the closures (in $\overline{Y}_0$) of the missing curves be denoted by $C_1, \ldots, C_q$; let the one-dimensional irreducible components of $\overline{Y}_0 \setminus Y_0$ be denoted by $L_1, \ldots, L_p$ (then the one-dimensional irreducible components of $\overline{Y}_i \setminus Y_i$ are $L_1, \ldots, L_p$ as well—recall that we use same notations for curves and their strict transforms). We have

$$\begin{aligned} \overline{Y}_0 \setminus Y_0 &= L_1 \cup \cdots \cup L_p \cup \text{ points} \\ \overline{Y}_0 \setminus \operatorname{int} f(X) &= C_1 \cup \cdots \cup C_q \cup L_1 \cup \cdots \cup L_p \cup \text{ points} \\ \overline{Y}_n \setminus X &= \bigcup_{j \in J} E_j \cup C_1 \cup \cdots \cup C_q \cup L_1 \cup \cdots \cup L_p \cup \text{ points.} \end{aligned}$$

Given $D \in \operatorname{Div}(\overline{Y}_i)$, let $\overline{D}$ be its image in $\operatorname{Cl}(\overline{Y}_i)$. Let $\theta : \operatorname{Cl}(\overline{Y}_0) \oplus \mathbb{Z}^n \to \operatorname{Cl}(\overline{Y}_n)$ be the isomorphism given in (2.4). Then

$$\begin{aligned} \theta^{-1}(\overline{L}_j) &= (\overline{L}_j, -\epsilon\mu(L_j)) = (\overline{L}_j, 0) \\ \theta^{-1}(\overline{C}_j) &= (\overline{C}_j, -\epsilon\mu(C_j)) \\ \theta^{-1}(\overline{E}_j) &= (0, K_j). \end{aligned}$$

In view of that, and by (2.3), we find

(α) $\mathrm{Cl}(Y) = 0$ (resp. Y has trivial units) iff $\overline{L}_1, \ldots, \overline{L}_p$ generate (resp. are linearly independent in) $\mathrm{Cl}(\overline{Y}_0)$;

(β) $\mathrm{Cl}(\mathrm{int}\, f(X)) = 0$ iff $\overline{L}_1, \ldots, \overline{L}_p, \overline{C}_1, \ldots, \overline{C}_q$ generate $\mathrm{Cl}(\overline{Y}_0)$;

(γ) $\mathrm{Cl}(X) = 0$ (resp. X has trivial units) iff the set

$$\{(0, K_j) \mid j \in J\} \cup \{(\overline{C}_j, -\epsilon\,\mu(C_j)) \mid 1 \le j \le q\} \cup \{(\overline{L}_j, 0) \mid 1 \le j \le p\}$$

generates (resp. is linearly independent in) the group $\mathrm{Cl}(\overline{Y}_0) \oplus \mathbf{Z}^n$.

On the other hand, it is clear that

(δ) $\{K_j \mid j \in J\} \cup \{-\epsilon\,\mu(C_j) \mid 1 \le j \le q\}$ generates (resp. is linearly independent in) $\mathbf{Z}^n$ iff the columns of $\epsilon'\mu$ generate (resp. are linearly independent in) $\mathbf{Z}^c$.

Now the reader can verify that, except for the inequalities $\delta \le j$ and $\delta \le n - q$, every assertion of the proposition is an immediate consequence of (α)–(δ). To prove the two inequalities, observe that δ is the number of zero rows in μ. Let U be the $n - \delta \times q$ sub-matrix of μ obtained by deleting the zero rows; let V be the $c \times n - \delta$ sub-matrix of ϵ' obtained by deleting the i^{th} column whenever the i^{th} row of μ is zero. Clearly, $VU = \epsilon'\mu$. The matrices U, V and $VU = \epsilon'\mu$ determine a commutative diagram of $\mathbf{Z}$-linear maps:

$$\begin{tikzcd} & \mathbf{Z}^{n-\delta} \arrow[dr, "v"] & \\ \mathbf{Z}^q \arrow[ur, "u"] \arrow[rr, "w"'] & & \mathbf{Z}^c. \end{tikzcd}$$

If the columns of $\epsilon'\mu$ generate $\mathbf{Z}^c$, i.e., w is onto, then v is onto and $\delta \le n - c = j$. If the columns of $\epsilon'\mu$ are linearly independent, i.e., w injective, then u is injective and $\delta \le n - q$.

~~2.7.~~ COROLLARY. *Let $f : X \to Y$ be a birational morphism and suppose that $Cl(Y) = 0$ and $\Gamma(Y, \mathcal{O}_Y)^* = \mathbf{k}^*$. Then*

(a) $Cl(X) = 0$ iff the columns of $\epsilon'\mu$ generate $\mathbf{Z}^c$

(b) $\Gamma(X, \mathcal{O}_X)^* = \mathbf{k}^*$ iff the columns of $\epsilon'\mu$ are linearly independent

(c) $Cl(X) = 0$ and $\Gamma(X, \mathcal{O}_X)^* = \mathbf{k}^*$ iff $\epsilon'\mu$ is a square matrix with determinant ± 1.

2.8. REMARKS.

- If the domain and codomain of f have trivial divisor class groups and trivial units then $q(f) = c(f)$.
- If we restrict ourselves to the case $j(f) = 0$ then $\epsilon' = \epsilon$ and consequently (2.6) and (2.7) are still true when all "$\epsilon'\mu$" are replaced by "μ".

2.9. COROLLARY. *Let $f : X \to Y$ be a birational morphism and supose that $\Gamma(X, \mathcal{O}_X)^* = \mathbf{k}^*$ and $Cl(Y) = 0$. Then $q_0(f) = 0$.*

PROOF: $q_0(f)$ is the number of zero columns in μ. Since the columns of $\epsilon'\mu$ are linearly independent by (2.6), $q_0(f) = 0$.

2.10. COROLLARY. *Let $f : X \to Y$ and $g : Y \to Z$ be birational morphisms and suppose that X, Y and Z have trivial divisor class groups and trivial units. Then $n(gf) = n(f) + n(g)$ and $j(gf) = j(f) + j(g) + \Delta c(f, g)$.*

PROOF: Immediate from (2.9) and (1.4).

2.11. COROLLARY. *Let $f : X \to Y$ be a birational morphism and suppose that $Cl(X) = 0$ and $\Gamma(X, \mathcal{O}_X)^* = \mathbf{k}^*$. Then $\Gamma(Y, \mathcal{O}_Y)^* = \mathbf{k}^*$ and the following are equivalent:*

(a) $Cl(Y) = 0$

(b) $c = q$

(c) $q + j = n$.

PROOF: By (2.6), Y has trivial units and (a)$\Rightarrow$(b); (b)$\Leftrightarrow$(c) is trivial, so let's prove (c)$\Rightarrow$(a). Refer to the proof of (2.6) for the notation. Let $G = \mathrm{Cl}(\overline{Y}_0) \subseteq \mathrm{Cl}(\overline{Y}_0) \oplus \mathbf{Z}^n$ and $g_i = (\overline{L}_i, 0) \in G$ $(1 \le i \le p)$. By the proof of (2.6), and since $n = q + j$, there are elements $e_1, \dots, e_n$ in $\mathrm{Cl}(\overline{Y}_0) \oplus \mathbf{Z}^n$ such that $(g_1, \dots, g_p, e_1, \dots, e_n)$ is a basis of $\mathrm{Cl}(\overline{Y}_0) \oplus \mathbf{Z}^n$. By elementary algebra, it follows that $(g_1, \dots, g_p)$ is a basis of G, i.e., $(\overline{L}_1, \dots, \overline{L}_p)$ is a basis of $\mathrm{Cl}(\overline{Y}_0)$, so $\mathrm{Cl}(Y) = 0$.

3. Factorisations.

Let $f : X \to Y$ be a birational morphism. A *factorization* of f is a pair (g, h) of birational morphisms such that $f = h \circ g$; two factorisations (g, h) and (g', h') of f are *equivalent* if there is an isomorphism u such that $g' = ug$ and $h = h'u$.

Let (g, h) be a factorisation of f, write $W = \mathrm{dom}\,(h) = \mathrm{codom}\,(g)$ and consider $h = (W \hookrightarrow Y_{n(h)} \to \cdots \to Y_0 = Y)$ determined by some minimal decomposition of h. We say that (g, h) is *good* if $q_0(g) = 0$ and if the complement of W in $Y_{n(h)}$ is a union of curves (then $n(f) = n(g) + n(h)$ by (1.4)); (g, h) is *connected* if it is good and if every connected component of $Y_{n(h)} \setminus W$ contains a missing curve of f (equivalently, of h). Hence, if Y is affine and (g, h) is connected then W is connected at infinity. We may also consider other types of factorisations by requiring that the surface $\mathrm{dom}\,(h) = \mathrm{codom}\,(g)$ have some predetermined property.

In the preceding sections we considered some numbers and matrices that give some description of a birational morphism f. All these numbers and matrices can be recovered if, for some minimal decomposition $\mathcal{D}_0$ of f, the triple $(J_{\mathcal{D}_0}, \mathcal{E}_{\mathcal{D}_0}, \mu_{\mathcal{D}_0})$ is known. In this regard, the reader should figure out an algorithm that lists all triples $(J, \mathcal{E}, \mu)$ determined by minimal decompositions of f, assuming that $(J_{\mathcal{D}_0}, \mathcal{E}_{\mathcal{D}_0}, \mu_{\mathcal{D}_0})$ is known (indeed, one can decide whether a permutation τ of $\{1, \dots, n\}$ is $\mathcal{D}_0$-allowable by looking at $\mathcal{E}_{\mathcal{D}_0}$—see (1.6)). We will now investigate the relations between the data $(J, \mathcal{E}, \mu)$ and the various types of factorisations of f. From what will be said, it will be clear that the problem of enumerating all equivalence classes of certain types of factorisations, for a given $f : X \to Y$, can be solved by simple algorithms, as long as one triple $(J, \mathcal{E}, \mu)$ is known.

3.1. DEFINITIONS. Let $f : X \to Y$ be a birational morphism and $\mathcal{D}$ a minimal decomposition of f, with notation as usual. Given a $\mathcal{D}$-closed (see (1.6)) subset A of $\{1,\ldots,n\}$, define

$$\begin{aligned} Q(\mathcal{D},A) &= \{i \mid \mu(P_j,C_i)=0, \text{ all } j\notin A\} \\ J(\mathcal{D},A) &= \{i\in J \mid \mu(P_j,E_i)=0, \text{ all } j\notin A\} \\ \#(\mathcal{D},A) &= |Q(\mathcal{D},A)|+|J(\mathcal{D},A)|. \end{aligned}$$

The next proposition says that to give an equivalence class of good factorisations of f is just the same thing as to give a $\mathcal{D}$-closed set.

3.2. PROPOSITION. *Let $f: X\to Y$ be a birational morphism and $\mathcal{D}$ a minimal decomposition of f. Then there is a unique bijection from the set of $\mathcal{D}$-closed subsets of $\{1,\ldots,n(f)\}$ to the set of equivalence classes of good factorisations of f, which satisfies the following condition: if $[\mathcal{D},A]$ is the equivalence class assigned to the $\mathcal{D}$-closed set A, $(g,h)\in[\mathcal{D},A]$, $\mathcal{D}_g$ and $\mathcal{D}_h$ are minimal decompositions of g and h respectively, $\mathcal{D}'$ is the minimal decomposition of f determined by $\mathcal{D}_g$ and $\mathcal{D}_h$ as in (1.4) and $\tau=\tau^{\mathcal{D},\mathcal{D}'}$ is the permutation defined in (1.6), then $\tau(A)=\{1,\ldots,n(h)\}$, $J_{\mathcal{D}_h}=\tau(J(\mathcal{D},A))$ and the missing curves of h are the C_i with $i\in Q(\mathcal{D},A)$. That bijection will be denoted by $[\mathcal{D},\ \ ]$. Moreover, if $\mathcal{D}''$ is any minimal decomposition of f and $A''=\tau^{\mathcal{D},\mathcal{D}''}(A)$, then $[\mathcal{D},A]=[\mathcal{D}'',A'']$.*

PROOF: For $\mathcal{D}$, let the notation be as usual. Let A be $\mathcal{D}$-closed. By (1.6), we can choose a minimal decomposition $\mathcal{D}'$ of f such that if $\tau=\tau^{\mathcal{D},\mathcal{D}'}$ then $\tau(A)=\{1,\ldots,s\}$, where $s=|A|$. Use the notations P_i', E_i', C_i', etc. for $\mathcal{D}'$. Let W be the open subset of Y_s' obtained by removing C_i, $i\in Q(\mathcal{D},A)$ and $E_{\tau i}'$, $i\in J(\mathcal{D},A)$, and let $h: W\to Y$ be the resulting birational morphism. We claim that $n(h)=s$. To see this, we use (1.3e). Indeed, let $i\in\{1,\ldots,s\}$ and consider the center P_i' of $\pi_i': Y_i'\to Y_{i-1}'$. The inverse image of P_i' in Y_s' contains an E_j' with self-intersection number equal to -1 (in Y_s'). If $E_j'\cap W=\emptyset$ then by definition of W $j\in\tau(J(\mathcal{D},A))$, so E_j' has self-intersection number -1 in Y_n'; on the other hand $j\in\tau(J(\mathcal{D},A))\subseteq\tau(J)=J_{\mathcal{D}'}$ and this contradicts (1.3i). Hence $E_j'\cap W\neq\emptyset$, and P_i' is a fundamental point of $W\hookrightarrow Y_s'\to\cdots\to Y_{i-1}'$. So $n(h)=s$ by (1.3e) and a minimal decomposition $\mathcal{D}_h$ of h is given by $W\hookrightarrow Y_s'\to\cdots\to Y_0'=Y$, together with some ordering of the set of missing curves C_i ($i\in Q(\mathcal{D},A)$) of h. By definition of W, the image of $X\hookrightarrow Y_n'\to\cdots\to Y_s'$ is contained in W. Thus we get $g: X\to W$ such that $f=hg$. We have $q_0(g)=0$ by definition of W, so (g,h) is a good factorisation of f. We define $[\mathcal{D},A]$ to be the equivalence class of (g,h); one can check that $[\mathcal{D},A]$ is independent of the choice of $\mathcal{D}'$, that $[\mathcal{D},\ \ ]$ is bijective and that any $(g,h)\in[\mathcal{D},A]$ satisfies the asserted conditions. The uniqueness of such a bijection is then trivial, and so is the last assertion: $[\mathcal{D},A]=[\mathcal{D}'',A'']$.

3.3. PROPOSITION. *Let $g: X\to W$ and $h: W\to Y$ be birational morphisms and suppose that X and Y have trivial divisor class groups and trivial units. Then W has trivial units and $q(h)+j(h)\leq n(h)$, with equality iff $Cl(W)=0$.*

PROOF: Since X has trivial units, W has trivial units by (2.6). Thus $q(h)\leq c(h)$ by (2.6), so $q(h)+j(h)\leq c(h)+j(h)=n(h)$, with equality whenever $\mathrm{Cl}(W)=0$. Conversely, if equality holds then $c(h)=q(h)$ and $c(hg)=q(hg)$; by additivity of the number $c-q$ (i.e., by (1.5)) $c(g)=q(g)$. Thus $\mathrm{Cl}(W)=0$ by (2.11) applied to g.

3.4. COROLLARY. *Let $f : X \to Y$ be a birational morphism and suppose that X and Y have trivial divisor class groups and trivial units. Let $\mathcal{D}$ be a minimal decomposition of f, A a $\mathcal{D}$-closed set and $(g,h) \in [\mathcal{D}, A]$ (see (3.2)). If $W = dom(h) = codom(g)$, then W has trivial units and $\#(\mathcal{D}, A) \le |A|$, with equality iff $Cl(W) = 0$.*

PROOF: $|A| = n(h)$ by (3.2) and $\#(\mathcal{D}, A) = q(h) + j(h)$ by (3.2) and definition (3.1). Apply (3.3).

The above fact is interesting because it suggests an algorithm. Indeed, one can decide whether $A \subseteq \{1, \dots, n\}$ is $\mathcal{D}$-closed by inspecting $\mathcal{E}_{\mathcal{D}}$; so all $\mathcal{D}$-closed sets can be enumerated. Moreover, the number $\#(\mathcal{D}, A)$ can be computed from the data $(J_{\mathcal{D}}, \mathcal{E}_{\mathcal{D}}, \mu_{\mathcal{D}})$.

The next result relates affineness of our surfaces X, W and Y to the notion of connected factorisations. Therefore, it becomes relevant to ask whether one can distinguish those $\mathcal{D}$-closed sets A that determine connected factorisations from all other $\mathcal{D}$-closed sets. We claim that it can be done. In fact, let $(J, \mathcal{E}, \mu)$ be the triple determined by $\mathcal{D}$ and fix a $\mathcal{D}$-closed set A. First, observe that $J(\mathcal{D}, A)$ and $Q(\mathcal{D}, A)$ can be obtained from $(J, \mathcal{E}, \mu)$ and A. Second, it is clear that A determines connected factorizations iff every $i \in J(\mathcal{D}, A)$ satisfies:

There exist $i_0, \dots, i_k \in J(\mathcal{D}, A)$ such that $k \ge 0$, $i_0 = i$ and such that the following intersection numbers in Y_n are positive: $E_{i_0}.E_{i_1}, \dots, E_{i_{k-1}}.E_{i_k}$, $E_{i_k}.\sum_{i \in Q(\mathcal{D},A)} C_i$,

where Y_n comes from $\mathcal{D}$, i.e., $f = (X \hookrightarrow Y_n \to \cdots \to Y_0 = Y)$. One can check that, if I_n is the $n \times n$ identity matrix and if $\mathcal{E}^t$ is the transposed of $\mathcal{E}$, then the $(i,j)^{\text{th}}$ entry of $(I_n - \mathcal{E}^t)\mu$ is $E_i.C_j$ in Y_n, and if $i > j$ then the $(i,j)^{\text{th}}$ entry of $(I_n - \mathcal{E}^t)\mathcal{E}$ is $E_i.E_j$ in Y_n. Hence the data $(J, \mathcal{E}, \mu)$ allow one to decide whether A determines connected factorizations.

3.5. PROPOSITION. *Let $f : X \to Y$ be a birational morphism and suppose that X and Y are factorial and have trivial units. Let (g,h) be a factorisation of f and write $W = dom(h) = codom(g)$. Then W has trivial units, $q(h) + j(h) \le n(h)$ and the following are equivalent:*

(a) W is factorial,

(b) $q(h) + j(h) = n(h)$ and (g,h) is a connected factorisation.

REMARK. To be factorial means to be the spectrum of a U.F.D..

PROOF: Only (a)$\Leftrightarrow$(b) requires explanations. Let us adopt the notation we used in the definition of factorisations—write $W \hookrightarrow Y_{n(h)} \to \cdots \to Y_0 = Y$. If W is factorial, then the equality holds by (3.3), $q_0(g) = 0$ by (2.9) and $Y_{n(h)} \setminus W$ is a union of curves, since W is affine. So (g,h) is good. If B is a connected component of $Y_{n(h)} \setminus W$ which doesn't contain a missing curve, then B is a nonempty union of curves E_j with $j \in J$, i.e., B is a union of *complete* curves. Since $Y_{n(h)}$ itself is not complete (for Y affine), it follows that W is not connected at infinity, which is absurd. Hence such a B doesn't exist, and (g,h) is connected.

Conversely, if (b) holds then $\mathrm{Cl}(W) = 0$ by (3.3); since (g,h) connected and Y affine, we see that W is connected at infinity. By (2.2), it's enough to show that no contracting curve of h is complete. In fact, that follows from $\mathrm{Cl}(W) = 0$: suppose that h has a complete contracting curve; then that curve is one of the E_j (= strict transform in $Y_{n(h)}$ of the exceptional curve of : $Y_j \to Y_{j-1}$) and consequently has negative self-intersection number, i.e., W contains a complete curve E with nonzero self-intersection number. On the other hand, imbed $Y_{n(h)}$ in a complete nonsingular surface S and apply (2.3) to $W \subseteq S$; then E is linearly equivalent to a divisor D supported at infinity of W, so that $E^2 = E.D = 0$, contradiction.

3.5.1. REMARK. (3.5) continues to hold if we replace condition (b) by:

(b') $q(h) + j(h) = n(h)$ and W is connected at infinity.

In fact, (a)$\wedge$(b)$\Rightarrow$(b') is trivial and (b')$\Rightarrow$(a) is precisely what the above argument proves.

Because of its relative simplicity, the case where f has *ordinary fundamental points* (i.e., f has $n(f)$ distinct fundamental points in its codomain) was studied first—see the introduction. Let us now consider the slightly more general case where $j(f) = 0$ (it is more general by (1.3i)). If the domain and codomain of such an f have trivial divisor class groups and trivial units then $q(f) = n(f)$, $\det \mu = \pm 1$ by (2.8), and all good factorisations of f are connected.

3.6. COROLLARY. *Let $f : X \to Y$ be a birational morphism with $j(f) = 0$, and suppose that X and Y are factorial and have trivial units. Let $\mathcal{D}$ be any minimal decomposition of f, let $\mu = \mu_{\mathcal{D}}$ and let r, s be positive integers such that $r + s = n = n(f)$. Then the following are equivalent:*

(a) $f = hg$ for some birational morphisms $g : X \to W$ and $h : W \to Y$ such that W is factorial and has trivial units, $n(g) = r$ and $n(h) = s$.

(b) Modulo a permutation of the columns and a permutation of the rows, μ has the form

$$\begin{bmatrix} H & B \\ O & G \end{bmatrix},$$

where H is an $s \times s$ matrix and O is the $r \times s$ zero matrix (hence G is an $r \times r$ matrix and B an $s \times r$ matrix).

PROOF: Write $\mu = (\mu_{ij})$. (a)$\Rightarrow$(b) is clear and (b)$\Rightarrow$(a) is almost clear; what has to be checked is that (b) implies the following (apparently) stronger statement:

(b′) Modulo a permutation of the columns and an *allowable* (see(1.6)) permutation of the rows, μ has the form described in (b).

Observe that if $1 \le i < n$ and $1 \le j \le n$ are such that $\mu_{ij} = 0$ and $\mu_{i+1\,j} \neq 0$, then it is allowable to interchange rows i and $i+1$. Whence (b)$\Rightarrow$(b′), and (b′)$\Rightarrow$(a) is clear by (3.5).

To conclude this section, we give a result that says that if $\delta(f)$ is the largest possible, then f factors in a nice way.

3.7. PROPOSITION. *Let $f : X \to Y$ be a birational morphism and suppose that X and Y are factorial and have trivial units. Then $\delta(f) \le j(f)$, with equality iff $f = hg$ for some birational morphisms $g : X \to W$ and $h : W \to Y$ such that W is factorial and has trivial units, $n(h) = q(h) = q(f)$ and $n(g) = j(f) = \delta(f)$ (and of course $j(h) = 0$).*

PROOF: Let $n = n(f)$, $c = c(f)$, $q = q(f)$, $j = j(f)$ and $\delta = \delta(f)$. Then $\delta \le j$ by (2.6) and we have to prove that $\delta = j$ iff f factors as specified.

Suppose $f = hg$ as specified; choose minimal decompositions $\mathcal{D}_g$, $\mathcal{D}_h$ of g, h respectively, and consider the minimal decomposition $\mathcal{D}_f$ of f obtained from $\mathcal{D}_g$ and $\mathcal{D}_h$ as in the proof of (1.4). Since each missing curve of h is a missing curve of f and $q(h) = q$, f and h have the same missing curves. Clearly, the $n(g)$

blowings-up of D_f which come from D_g have centers *i.n.* W, i.e., away from the q missing curves of f. Hence $\delta \geq n(g) = n - n(h) = n - q = n - c = j$, i.e., $\delta = j$.

Conversely, suppose that $\delta = j$. By (1.6), there exists a minimal decomposition D of f such that $\Delta_\delta = \{n - \delta + 1, \ldots, n\}$. Since $\delta = j$ and by (2.8) $c = q$, $\Delta = \{q + 1, \ldots, n\}$. With notation as usual for D, let $W = Y_q \setminus (C_1 \cup \cdots \cup C_q)$ and let $h : W \to Y$ be the corresponding birational morphism; then $n(h) = q(h) = c(h) = q$. Since the blowings-up $Y_n \to \cdots \to Y_q$ have centers away from $C_1 \cup \cdots \cup C_q$ (i.e., the centers are *i.n.* W), $f = hg$ for some $g : X \to W$. By (3.5), we conclude that W has trivial units and is factorial by (3.5.1).

REMARK. (3.7) continues to hold if every 'factorial' is replaced by 'trivial divisor class group'. The proof is the same except that, at the end, we use (3.3) instead of (3.5.1).

4. Weighted Graphs.

Because the rest of this thesis depends heavily on weighted graphs and related graph-theoretic machineries, we feel that it is appropriate to give the basic definitions and facts of the theory of weighted graphs, even if many algebraic geometers have some knowledge of it. This will help establishing our language and notations. In addition to that, we believe that not so many people are familiar with the kind of "mechanics" which is relevant to us, here. So we include the proofs of the elementary observations (4.12) and (4.13). These proofs show how one can deduce that "some branch must contract" in certain situations. Such contraction processes will be crucial in many arguments from now on. Notice that the corollaries (4.16) and (4.18) of (4.15) will be used many times in part III.

The material covered before (4.11) is approximatively what the author learned from other people (namely, from [13], from the beginning of [18] and from discussions with Russell). Everything that comes after (4.11) has been figured out by the author. However, some of these facts are simple observations that may have been noticed by many people. See also the remark after (4.15).

Graphs. Every graph that we will consider consists of finitely many vertices, some of them being connected by links, such that the links are not oriented and at most one can exist between two given vertices. So let us say that a *graph* is a pair $\mathcal{G} = (G, R)$ where G is a finite set and R is a set of subsets of G, such that every $a \in R$ contains *exactly* two elements. The elements of G are called the *vertices* of $\mathcal{G}$ and those of R are the *links* of $\mathcal{G}$. Two vertices u, v of $\mathcal{G}$ are said to be *linked* if $\{u, v\} \in R$; we also say that u is a *neighbour* of v, and vice-versa. The set of neighbours of v is denoted by $\mathcal{N}_\mathcal{G}(v)$. A vertex v of $\mathcal{G}$ is *free* (resp. *linear, a branch point*) if it has at most one (resp. at most two, at least three) neighbour(s). $|\mathcal{G}|$ will denote the number of vertices of $\mathcal{G}$.

Given vertices u, v, a *chain* from u to v is a sequence $(x_0, \ldots, x_q)$ of vertices such that $q > 0$, $u = x_0$, $v = x_q$ and $\{x_i, x_{i+1}\} \in R$ for $0 \leq i < q$. The chain is *simple* if the links $\{x_0, x_1\}, \ldots, \{x_{q-1}, x_q\}$ are distinct. It is a *loop* if it is simple and if $x_0 = x_q$. The *connected components* of $\mathcal{G}$ are defined in the obvious way. A *tree* is a connected graph without loops. A *linear tree* is a tree without branch points.

If v is a vertex of a graph $\mathcal{G}$, $\mathcal{G} \setminus \{v\}$ is the graph (G', R') where $G' = G \setminus \{v\}$ and $R' = R \setminus \{a \in R \mid v \in a\}$. If $\mathcal{G}$ is a tree then the connected components of $\mathcal{G} \setminus \{v\}$ are called the *branches* of $\mathcal{G}$ at v; clearly, the tree $\mathcal{G}$ has $|\mathcal{N}_\mathcal{G}(v)|$ branches at v.

Weighted Graphs. These are graphs with a weight (i.e., an integer) assigned to each vertex. The connection to Geometry will be explained after the basic definitions.

4.1. DEFINITION. A *weighted graph* is a triple $\mathcal{G} = (G, R, \Omega)$ where (G, R) is a graph and Ω is some set map $G \to \mathbb{Z}$. If $v \in G$, $\Omega(v)$ is called the *weight* of v.

A weighted graph can be blown up at a link or at a vertex:

4.2. DEFINITION. Let $\mathcal{G} = (G, R, \Omega)$ be a weighted graph and let x be either a link or a vertex of $\mathcal{G}$. A *blowing-up of $\mathcal{G}$ at x* is a weighted graph $\mathcal{G}' = (G', R', \Omega')$ together with an injective map $G \hookrightarrow G'$, such that if G is identified with its image in G' then $G' = G \cup \{e\}$ for some $e \notin G$ and the following conditions are satisfied:

(a) if $x = \{u, v\} \in R$ then $R' = \big(R \setminus \{\{u,v\}\}\big) \cup \big\{\{e,u\}, \{e,v\}\big\}$ and

$$\Omega'(w) = \begin{cases} \Omega(w) & \text{if } w \notin \{e,u,v\} \\ \Omega(w) - 1 & \text{if } w \in \{u,v\} \\ -1 & \text{if } w = e; \end{cases}$$

(b) if $x \in G$ then $R' = R \cup \big\{\{e, x\}\big\}$ and

$$\Omega'(w) = \begin{cases} \Omega(w) & \text{if } w \notin \{e,x\} \\ \Omega(w) - 1 & \text{if } w = x \\ -1 & \text{if } w = e. \end{cases}$$

A blowing-up of $\mathcal{G}$ at x exists and is unique, up to isomorphism (define *isomorphism* the obvious way, i.e., a bijection which preserves links and weights). So we can speak of *the* blowing-up of $\mathcal{G}$ at x, and "blowing-up" can be understood as a process, or an operation. Notice that we sometimes refer to e as *the vertex which is created in the blowing-up*; that vertex is clearly a superfluous vertex of $\mathcal{G}'$:

4.3. DEFINITION. Let $\mathcal{G}$ be a weighted graph. A *superfluous vertex* of $\mathcal{G}$ is a linear vertex e of weight -1 such that if $u, v \in \mathcal{N}_{\mathcal{G}}(e)$ then u and v are not linked.

4.4. DEFINITIONS. Let $\mathcal{G} = (G, R, \Omega)$ be a weighted graph and e a superfluous vertex of $\mathcal{G}$. A *blowing-down of $\mathcal{G}$ at e* is a weighted graph $\mathcal{G}' = (G', R', \Omega')$ together with an injective map $G' \hookrightarrow G$ such that $\mathcal{G}$ is a blowing-up of $\mathcal{G}'$ at some vertex or link and e is the vertex which is created in that blowing-up. A blowing-down of $\mathcal{G}$ at e exists and is unique, up to isomorphism; thus we can speak of *the* blowing down of $\mathcal{G}$ at e and "blowing-down" can be understood as an operation. We sometimes refer to e as *the vertex which disappears in the blowing-down.*

We say that $\mathcal{G}$ *contracts* to $\mathcal{G}''$ if either $\mathcal{G}''$ is isomorphic to $\mathcal{G}$ or if $\mathcal{G}''$ can be obtained from $\mathcal{G}$ by performing finitely many blowings-down. A weighted graph is said to be *minimal* if it has no superfluous vertex.

Two weighted graphs are *equivalent* if one can be obtained from the other by a finite sequence of blowings-up and blowings-down; that relation will be indicated by $\mathcal{G} \sim \mathcal{G}'$. Clearly, if $\mathcal{G}$ and $\mathcal{G}'$ are equivalent then $\mathcal{G}$ is connected (resp. has no loops, is a tree) iff $\mathcal{G}'$ has the same property.

Connection to Geometry. Let S be a nonsingular projective surface and let $D \in \mathrm{Div}(S)$. If D satisfies a strong version of the "normal crossings" condition, a weighted graph can be associated to the pair (S, D).

4.5. DEFINITION. Let S and D be as above. We say that D has *strong normal crossings* (*s.n.c.*) if D is effective, reduced, and if the following conditions hold:

(a) every irreducible component of D is a nonsingular curve;

(b) if C and C' are distinct irreducible components of D such that $C \cap C' \neq \emptyset$, then $C \cap C' = \{P\}$ and $(C.C')_P = 1$, for some point P;

(c) if C, C' and C'' are distinct irreducible components of D then $C \cap C' \cap C'' = \emptyset$.

Observe that the *s.n.c.* condition makes sense even if S is not complete.

4.6. DEFINITION. Let S be a nonsingular projective surface and let D be a divisor of S with *s.n.c.*. The *dual graph* $\mathcal{G}(S, D)$ associated to the pair (S, D) is the weighted graph which has the irreducible components of D as vertices, two of them linked iff they intersect in S, and such that each vertex C has weight C^2 (self-intersection number in S).

Let (S, D) be as above.
(1) If $\pi : \widetilde{S} \to S$ is the blowing-up of S at some $P \in \operatorname{supp}(D)$, $E = \pi^{-1}(P)$, $\widetilde{D}$ is the strict transform of D and $D' = \widetilde{D} + E \in \operatorname{Div}(\widetilde{S})$ then D' has *s.n.c.* and $\mathcal{G}(\widetilde{S}, D')$ is a blowing-up of $\mathcal{G}(S, D)$ in a natural way. If P belongs to two components of D then $\mathcal{G}(S, D)$ is blown up at the corresponding link; if P belongs to only one component of D then $\mathcal{G}(S, D)$ is blown up at the corresponding vertex. Clearly, E is a superfluous vertex of $\mathcal{G}(\widetilde{S}, D')$ and $\mathcal{G}(S, D)$ is the blowing-down of $\mathcal{G}(\widetilde{S}, D')$ at E.
(2) If F is a superfluous vertex of $\mathcal{G}(S, D)$ then the blowing-down of $\mathcal{G}(S, D)$ at F (which always exists) corresponds to a blowing-up morphism ($S \to \cdot$) as explained in (1), above, if and only if F is a rational curve (by Castelnuovo's criterion for contracting a curve).

When the divisor D does not have *s.n.c.* we can use the following fact:

4.7. LEMMA. *Let S be a complete nonsingular surface and suppose that $D \in Div(S)$ is reduced, effective and does not have s.n.c.. Then there exists a sequence $S_m \to \cdots \to S_0 = S$ of monoidal transformations such that, if E_i is the exceptional curve created in $S_i \to S_{i-1}$ and*

$$\begin{cases} D^0 = D \in Div(S_0) \\ D^i = (\text{strict transform of } D^{i-1}) + E_i \in Div(S_i), \quad 1 \leq i \leq m, \end{cases}$$

then $D^m \in Div(S_m)$ has s.n.c.. Moreover, if m is minimal with respect to these properties then all centers are i.n. D, $S_m \setminus supp(D^m) \cong S \setminus supp(D)$ and, if $S \setminus supp(D)$ has no loops at infinity, every E_i such that $E_i^2 = -1$ in S_m is a branch point of $\mathcal{G}(S_m, D^m)$.

Most of the facts contained in this lemma are well-known, and the reader can easily figure out the last assertions—see (5.1) for the "loops at infinity".

4.8. DEFINITIONS. Let X be a nonsingular surface. A *smooth completion* of X is an open immersion $X \hookrightarrow S$ such that S is a nonsingular projective surface and $S \setminus X = \operatorname{supp}(D)$ for some $D \in \operatorname{Div}(S)$ with *s.n.c.* (this D is then unique). The weighted graph $\mathcal{G}(S, D)$ is therefore determined by $X \hookrightarrow S$; by using the two facts mentioned immediately before (1.2), one sees that the equivalence class of $\mathcal{G}(S, D)$ depends only on X. That equivalence class will be denoted by $\mathcal{G}[X]$. Notice that smooth completions exist for any X.

We now return to pure graph theory and give a few definitions.

4.9. DEFINITIONS. An arbitrary weighted graph $\mathcal{G} = (G, R, \Omega)$ determines a bilinear form $B(\mathcal{G})$, on the real vector space $\mathbf{R}^G$ which has G as a basis, defined by

$$v_i \cdot v_i = \Omega(v_i), \qquad \text{all } i,$$
$$v_i \cdot v_j = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in R \\ 0 & \text{if } i \neq j \text{ and } \{v_i, v_j\} \notin R, \end{cases}$$

where $G = \{v_1, v_2, \ldots\}$. The *discriminant* of $B(\mathcal{G})$ is denoted by $d(\mathcal{G})$ (i.e., $d(\mathcal{G})$ is the determinant of the $|\mathcal{G}| \times |\mathcal{G}|$ matrix $(v_i \cdot v_j)$). One can check that if $\mathcal{G}'$ is a blowing-up of $\mathcal{G}$ then $d(\mathcal{G}') = -d(\mathcal{G})$. Thus the number

$$(-1)^{|\mathcal{G}|-1} d(\mathcal{G})$$

depends only on the equivalence class of $\mathcal{G}$. We define the nonnegative integer $\langle \mathcal{G} \rangle = \max \dim W$, where W runs in the set of linear subspaces $W \subseteq \mathbf{R}^G$ such that $B(\mathcal{G})(x, x) \geq 0$, all $x \in W$. One can check that $\langle \mathcal{G} \rangle$ depends only on the equivalence class of $\mathcal{G}$. The following (elementary) fact is mentioned in [**13**, p. 78]:

If $\langle \mathcal{G} \rangle \leq 1$ then there can be at most two vertices with nonnegative weights, and if there are two of them then these two vertices are linked and one of the weights is actually zero.

4.10. DEFINITION. Let $\mathcal{G} = (G, R, \Omega)$ be a weighted tree. The *fundamental group* of $\mathcal{G}$, denoted $\pi(\mathcal{G})$, is the free group on the set G, divided by the relations

(a) $v_i v_j = v_j v_i$, if $\{v_i, v_j\} \in R$,

(b) for each vertex v, if $\mathcal{N}_{\mathcal{G}}(v) = \{v_{i_1}, \ldots, v_{i_k}\}$ and $i_1 < \cdots < i_k$ then $v_{i_1} \cdots v_{i_k} = v^{-\Omega(v)}$,

where $G = \{v_1, v_2, \ldots\}$. One can prove that, up to isomorphism, $\pi(\mathcal{G})$ is independent of the ordering of G; moreover, the isomorphism class of $\pi(\mathcal{G})$ depends only on the equivalence class of $\mathcal{G}$.

REMARKS.

(a) The fact that $\pi(\mathcal{G})$ is well defined, i.e., that it is independent of the ordering of G, up to isomorphism, is clearly true in cases where it comes from topology, as is well known [**10**]. In the purely graph-theoretic situation, it doesn't seem completely trivial to us. That fact is claimed in [**18**], without proof; we don't know if there exists a published proof.

(b) The notion of fundamental group is used only once in this thesis, in the proof of (4.15.1). Moreover, only the fundamental group of a linear tree is considered, which is a somewhat trivial case. It is then clear that we could have avoided considering these groups.

4.11. DEFINITION. Given $n \in \mathbf{Z}$, the symbol $[n]$ will denote any weighted tree which has one vertex, say v, and such that v has weight n.

Now that we are done with the definitions and notations we will consider those problems, in the theory of weighted graphs, that we need to understand in order to study birational endomorphisms of $\mathbf{A}^2$. There are two such problems, as far as this thesis is concerned. The first one is to understand the behavior of the weighted trees which are equivalent to a linear tree; the second one is to find which weighted trees are equivalent to [1]. So the following considerations belong to pure graph-theory. (Notice that a weighted graph is called a (weighted) tree if the underlying graph is a tree; it is a linear tree if the underlying graph is a linear tree.)

Weighted Trees Equivalent to a Linear Tree.

4.12. LEMMA. *Let $\mathcal{G}$ be a weighted graph and v a vertex of $\mathcal{G}$. Suppose that $\mathcal{G}_0, \ldots, \mathcal{G}_k$ is a sequence of weighted graphs such that $\mathcal{G}_0 = \mathcal{G}$, $|\mathcal{G}_k| = 1$, $\mathcal{G}_i$ is either a blowing-up or a blowing-down of $\mathcal{G}_{i-1}$ $(1 \leq i \leq k)$ and none of the blowings-down is a blowing-down at v (so the vertex of $\mathcal{G}_k$ is v). Then $\mathcal{G}$ contracts to $\mathcal{G}_k$.*

Before we prove that lemma, let us state an elementary fact about contractions of weighted graphs. This fact will be used in the proof of (4.12) and at several other places without even mentioning it. Its proof is an easy inductive argument which is left to the reader.

4.12.1. LEMMA. *Let $\mathcal{G}_0, \ldots, \mathcal{G}_k$ be a sequence of weighted graphs such that $\mathcal{G}_i$ is a blowing-down of $\mathcal{G}_{i-1}$ $(1 \leq i \leq k)$. Suppose that e is a superfluous vertex of $\mathcal{G}_0$ which disappears in one of these blowings-down. Then there is a sequence $\mathcal{G}'_0, \ldots, \mathcal{G}'_k$ such that $\mathcal{G}'_0 = \mathcal{G}_0$, $\mathcal{G}'_k = \mathcal{G}_k$, $\mathcal{G}'_i$ is a blowing-down of $\mathcal{G}'_{i-1}$ $(1 \leq i \leq k)$ and $\mathcal{G}'_1$ is the blowing-down of $\mathcal{G}'_0$ at e.*

PROOF OF (4.12): If $\mathcal{G}_i$ is a blowing-down of $\mathcal{G}_{i-1}$ for $1 \leq i \leq k$, we are done. So suppose that $\mathcal{G}_i$ is a blowing-up of $\mathcal{G}_{i-1}$ and that i is maximal with respect to that property. Let e be the vertex created in that last blowing-up. Since e is distinct from v, e must disappear in a blowing-down, say from $\mathcal{G}_{j-1}$ to $\mathcal{G}_j$ (some $j > i$). By (4.12.1), we may assume that $j = i + 1$. Then $\mathcal{G}_{i-1} = \mathcal{G}_{i+1}$, i.e., we can delete $\mathcal{G}_{i-1}$ and $\mathcal{G}_i$ from the sequence $\mathcal{G}_0, \ldots, \mathcal{G}_k$, and get a shorter sequence with the same properties—and the same last term $\mathcal{G}_k$. The conclusion follows by induction.

4.13. COROLLARY. *Every minimal weighted tree equivalent to a linear tree is linear.*

PROOF: We will show that if $\mathcal{G}$ is equivalent to a linear tree and is not linear, then $\mathcal{G}$ is not minimal. Let $\mathcal{G}_0, \ldots, \mathcal{G}_k$ be a sequence of weighted trees such that $\mathcal{G}_0 = \mathcal{G}$, $\mathcal{G}_k$ is linear and $\mathcal{G}_i$ is either a blowing-up or a blowing-down of $\mathcal{G}_{i-1}$ $(1 \leq i \leq k)$. Let v be a branch point of $\mathcal{G}$. For some i, v is a linear vertex in $\mathcal{G}_i$; consequently, there is a branch B of $\mathcal{G}$ at v such that all vertices of B disappear when we go from $\mathcal{G}_0$ to $\mathcal{G}_i$. Therefore one sees that the lemma (4.12) can be applied to the sub-weighted-tree $\{v\} \cup B$ of $\mathcal{G}$, and we conclude that $\{v\} \cup B$ contracts to $\{v\}$; in particular, $\mathcal{G}$ contains a superfluous vertex.

The following is an immediate consequence of (4.12) and (4.13). It will be used many times whitout mentioning it.

4.13.1. COROLLARY. *Let $\mathcal{G}$ be a weighted tree equivalent to a linear tree, and let b be a branch point of $\mathcal{G}$. Then for some branch $\mathcal{B}$ of $\mathcal{G}$ at b, "b can absorb $\mathcal{B}$", i.e., we can contract $\mathcal{G}$ to a weighted tree $\mathcal{G}'$ such that:*

(a) $\mathcal{G}' = \mathcal{G} \setminus \mathcal{B}$ *as graphs,*

(b) $\mathcal{G}' \setminus \{b\} = \mathcal{G} \setminus (\{b\} \cup \mathcal{B})$ *as weighted graphs.*

Moreover, $\mathcal{G}'$ (i.e., the weight of b) is completely determined by $\mathcal{G}$ and $\mathcal{B}$.

Given a weighted graph $\mathcal{G}$, let the symbol $\mathcal{G} < -1$ be an abbreviation for the statement "every vertex of $\mathcal{G}$ has weight less than -1". The next fact has nothing to do with linear trees, but we include it here because its proof, which we leave to the reader, is somewhat similar to the proof of (4.12).

4.14. FACT. *If $\mathcal{G} < -1$, then $\mathcal{G}$ is the unique minimal element of its equivalence class. In particular, if $\mathcal{G}' \sim \mathcal{G}$ then $|\mathcal{G}'| \geq |\mathcal{G}|$.*

Weighted Trees Equivalent to [1]. We would like to have an algorithm that decides whether a given weighted tree $\mathcal{G}$ is equivalent to [1]. Since it is easy to contract $\mathcal{G}$ to a minimal weighted tree, we can restrict ourselves to the case where $\mathcal{G}$ is minimal. Before we state the solution of this problem, we need to introduce some notations:

(a) Given integers $\omega_1, \ldots, \omega_n$, let $[\omega_1, \ldots, \omega_n]$ be the linear weighted tree

$$\omega_1 \text{—} \omega_2 \text{—} \cdots \text{—} \omega_n,$$

where the numbers $\omega_1, \ldots, \omega_n$ are the weights. If $s_1, \ldots, s_k$ are finite sequences of integers, let $[s_1, \ldots, s_k]$ be the linear weighted tree obtained by regarding "$s_1, \ldots, s_k$" as *one* (long) sequence of integers. Moreover, if s_i has only one term, say $s_i = (\omega)$, we allow ourselves to write $[\ldots, s_{i-1}, \omega, s_{i+1}, \ldots]$ instead of $[\ldots, s_{i-1}, s_i, s_{i+1}, \ldots]$ or $[\ldots, s_{i-1}, (\omega), s_{i+1}, \ldots]$.

(b) Given $p, q \in \mathbf{Z}$ with $p \geq 0$, let R_p^q be the $p+1$-tuple $(-q-2, -2, \ldots, -2)$, and let L_p^q be the $p+1$-tuple $(-2, \ldots, -2, -q-2)$.

For instance, the tree $[L_2^1, 0, 2, R_0^2]$ is just the same as $[-2, -2, -3, 0, 2, -4]$ which is, by the way, equivalent to [1]. To see this, observe that if A, B are (possibly empty)finite sequences of integers and $a, b \in \mathbf{Z}$ then

$$[A, a, 0, b, B] \sim [A, a+i, 0, b-i, B]$$

for any $i \in \mathbf{Z}$. In our case,

$$[-2, -2, -3, 0, 2, -4] \sim [-2, -2, -1, 0, 0, -4] \sim [3, 0, -4] \sim [0, 0, -1] \sim [0, 1]$$

which is equivalent to [1]; indeed, if $n \in \mathbf{Z}$ then $[0, n] \sim [-1, -1, n] \sim [0, n+1]$ and consequently $[0, n] \sim [0, -1] \sim [1]$.

4.15. PROPOSITION. *The following is a list of all minimal weighted trees equivalent to* [1].

(a) [1]

(b) $[0, \alpha], \qquad \alpha \in \mathbf{Z} \setminus \{-1\}$

(c) $[\ldots, L_{\alpha_5}^{\alpha_4+1}, L_{\alpha_3}^{\alpha_2+1}, L_{\alpha_1}^{\alpha_0}, 0, \alpha_0+1, R_{\alpha_2}^{\alpha_1}, R_{\alpha_4}^{\alpha_3+1}, R_{\alpha_6}^{\alpha_5+1}, \ldots]$ *where $\alpha_0, \alpha_1, \ldots, \alpha_k$ is a finite sequence of nonnegative integers, with $k \geq 1$.*

REMARK. When the writer found the above list, he was unaware of the fact that it had appeared in [9, theorem 9] several years before. However, geometry (over $\mathbf{C}$) is very much involved in Morrow's result (i.e., in both the assertion and its proof) while our proposition is purely graph-theoretic so that, strictly speaking, the two results don't say the same thing. For that reason, we include a proof of (4.15). To begin with, we prove a lemma which is probably the most difficult part of the proof.

4.15.1. LEMMA. *Let $\mathcal{G} = [\omega_1, \ldots, \omega_{q+2}]$ be such that $q \geq 0$ and $\omega_i \leq -2$ $(1 \leq i \leq q)$. If $\pi(\mathcal{G})$ is trivial and $\langle \mathcal{G} \rangle = 1$ then $\mathcal{G} \sim [1]$ and one of the following holds:*

(a) $q = 0$ and $0 \in \{\omega_1, \omega_2\}$

(b) $q > 0$ and $\mathcal{G} = [-2, \dots, -2, \omega_q, 0, -\omega_q - 1]$

(c) $q > 0$ and $\mathcal{G} = [\omega_1, -2, \dots, -2, -1, -q]$.

PROOF: The vertices of $\mathcal{G}$ will be denoted by $x_1, \dots, x_{q+2}$, where the subscripts correspond to those of $\omega_1, \dots, \omega_{q+2}$.

Suppose that $q = 0$, i.e., $\mathcal{G} = [\omega_1, \omega_2]$.

Let $F(x_1, x_2)$ denote the free group on the set $\{x_1, x_2\}$, let $\langle a, b, \dots\rangle$ mean "normal subgroup generated by $a, b, \dots$" and let $[a, b]$ be the commutator of a, b. By definition of $\pi(\mathcal{G})$,

$$\pi(\mathcal{G}) = F(x_1, x_2) / \langle [x_1, x_2], x_1 x_2^{\omega_2}, x_2 x_1^{\omega_1}\rangle \cong F(x_1, x_2) / \langle x_1 x_2^{\omega_2}, x_2 x_1^{\omega_1}\rangle$$

since the latter group is abelian; one checks that this is isomorphic to $F(x_1) / \langle x_1^{1-\omega_1\omega_2}\rangle$. Since $\pi(\mathcal{G}) = 1$, we get $1 - \omega_1\omega_2 = \pm 1$.

If $1 - \omega_1\omega_2 = -1$ then $\omega_1\omega_2 = 2$; since $\langle \mathcal{G}\rangle = 1$, $\omega_1 \le 0$ or $\omega_2 \le 0$ by the fact stated at the end of (4.9), so $(\omega_1, \omega_2) = (-1, -2)$ or $(-2, -1) \implies \mathcal{G} \sim [-1]$ and $\langle \mathcal{G}\rangle = 0$, contradiction.

Hence $1 - \omega_1\omega_2 = 1$, $0 \in \{\omega_1, \omega_2\}$ and (a) holds.

Suppose that $q > 0$.

Write $a_i = -\omega_i$, $(1 \le i \le q)$. The group $\pi(\mathcal{G})$ is the free group $F(x_1, \dots, x_{q+2})$ divided by the relations $x_2 = x_1^{a_1}$ and $x_{i+2} = x_{i+1}^{a_{i+1}} x_i^{-1}$, $1 \le i \le q$ (for this is already abelian, so it's not necessary to impose the relations $x_i x_{i+1} = x_{i+1} x_i$, $1 \le i \le q+1$). If we define $f : \{1, \dots, q+2\} \to \mathbf{Z}$ by $f(1) = 1$, $f(2) = a_1$ and $f(i+2) = a_{i+1} f(i+1) - f(i)$, $1 \le i \le q$, then $x_i = x_1^{f(i)}$, $1 \le i \le q+2$. On the other hand, one can check that $\pi(\mathcal{G}) \cong F(x_1) / \langle x_1^{f(q+1) + \omega_{q+2} f(q+2)}\rangle$, and since $\pi(\mathcal{G}) = 1$, $f(q+1) + \omega_{q+2} f(q+2) = \pm 1$. From the definition of f, we then obtain

$$(1) \qquad f(q+1) - \omega_{q+2}[\omega_{q+1} f(q+1) + f(q)] = \pm 1$$

$$(2) \qquad [1 - \omega_{q+2}(\omega_{q+1} + 1)] f(q+1) + \omega_{q+2}[f(q+1) - f(q)] = \pm 1.$$

By assumption, we have $a_1, \dots, a_q \ge 2$. Using the definition of f and a straightforward inductive argument, one gets

$$(3) \qquad 1 = f(1) < f(2) < \cdots < f(q) < f(q+1)$$

$$(4) \qquad 0 < f(2) - f(1) \le \cdots \le f(q+1) - f(q) \quad \text{and}$$
$$f(i+1) - f(i) > f(i) - f(i-1) \iff a_i > 2 \qquad (2 \le i \le q).$$

In particular, $f(q+1) > 1$ so (1) implies that $\omega_{q+2} \ne 0$.

Case 1. $\omega_{q+2} > 0$.

Since $\langle \mathcal{G}\rangle = 1$, the fact stated at the end of (4.9) implies that $\omega_{q+1} \le 0$. On the other hand, $f(q+1) - f(q) > 0$ by (4) so $1 - \omega_{q+2}(\omega_{q+1} + 1) \le 0$ by (2) and consequently $\omega_{q+1} = 0$. From (1), we find

$$(5) \qquad f(q+1) - \omega_{q+2} f(q) = \pm 1.$$

If $q > 1$ then $f(q+1) = a_q f(q) - f(q-1)$ and (5) becomes $(a_q - \omega_{q+2}) f(q) - f(q-1) = \pm 1$. By (3) and (4), we see that $0 \le a_q - \omega_{q+2} \le 1$. Thus $a_q - \omega_{q+2} = 1$ (otherwise $0 f(q) - f(q-1) = \pm 1 \Rightarrow$

$f(q-1)=1 \Rightarrow q=2$, and $\mathcal{G}=[\omega_1,\omega_2,0,-\omega_2] \sim [\omega_1,0,0,0] \Rightarrow \langle\mathcal{G}\rangle > 1$ by (4.9), contradiction) and $f(q)-f(q-1)=1$. By (4), we conclude that $a_1=\cdots=a_{q-1}=2$, i.e., (b) holds.

If $q=1$ then $\mathcal{G}=[-a_1,0,\omega_3] \sim [0,0,\omega_3-a_1]$ so $\omega_3-a_1<0$ by (4.9) (for $\langle\mathcal{G}\rangle=1$). On the other hand, (5) reads $a_1-\omega_3=\pm 1$. Hence $a_1-\omega_3=1$, i.e., $\mathcal{G}=[\omega_1,0,-\omega_1-1]$, i.e., (b) holds.

Hence (b) holds whenever $\omega_{q+2}>0$. Since we know that $\omega_{q+2}\neq 0$, there remains to look at

<u>Case 2</u>. $\omega_{q+2}<0$.

By (2), we see that $1-\omega_{q+2}(\omega_{q+1}+1)\geq 0$, whence

(6) $$\omega_{q+1} \geq -1+\frac{1}{\omega_{q+2}} \geq -2.$$

On the other hand, write (1) in the form

$$[f(q+1)-\omega_{q+2}f(q)]-\omega_{q+2}\omega_{q+1}f(q+1)=\pm 1.$$

Since $f(q+1)\geq q+1$ and $f(q)\geq q$ by (3), $[q+1-q\omega_{q+2}]-\omega_{q+2}\omega_{q+1}f(q+1)\leq 1$. Whence $\omega_{q+1}<0$, i.e., $-2\leq\omega_{q+1}\leq -1$ by (6).

If $\omega_{q+1}=-2$ then $\omega_{q+2}=-1$ by (6), so (1) becomes $f(q+1)-f(q)=1$. By (4), it follows that $a_2=\cdots=a_q=2$ and $a_1=f(2)=f(1)+(f(2)-f(1))=2$ i.e., $\mathcal{G}=[-2,\ldots,-2,-1]\sim[-1]\Rightarrow\langle\mathcal{G}\rangle=0$, contradiction. Hence $\omega_{q+1}=-1$ and we proved:

(7) *If $q>0$ and $\omega_{q+2}<0$ then $\omega_{q+1}=-1$.*

Thus we have

(8) *if $q>0$ and $\omega_{q+2}<0$ then (c) holds*

by induction on q. Indeed, the case $q=1$ is proved by applying the case "$q=0$" to the blowing-down of $\mathcal{G}$ at x_2 and, similarly, the inductive step is done by considering the blowing-down of $\mathcal{G}$ at x_{q+1}. One has to observe that the weight of x_{q+2} is still negative after the blowing-down; this is because we found, just before case 1, that $q>0\Rightarrow\omega_{q+2}\neq 0$. Thus the inductive hypothesis can be applied to the blowing-down of $\mathcal{G}$.

PROOF OF (4.15): First, notice that every member of the list is minimal and equivalent to [1]. Indeed, this is trivial for (a) and (b). That every tree in (c) is equivalent to [1] can be proved by induction on k and by using the observation just before (4.15)—but this will be rather obvious once the rest of the proof is understood.

Let $\mathcal{G}$ be a minimal tree equivalent to [1]. To prove: $\mathcal{G}$ is in the list. If $|\mathcal{G}|=1$ then $\mathcal{G}=[1]$ (indeed, by considering the invariant $(-1)^{|\mathcal{G}|+1}d(\mathcal{G})$, one sees that $[m]\sim[n]\Rightarrow m=n$). If $|\mathcal{G}|=2$ then $\mathcal{G}=[0,\alpha]$ by (4.15.1), since $\langle[1]\rangle=1$ and $\pi([1])$ is trivial.

From now on, we suppose that $|\mathcal{G}|>2$.

Clearly, $\mathcal{G}$ is linear by (4.13). Let n be the number of vertices with nonnegative weight, in $\mathcal{G}$. Since $\langle\mathcal{G}\rangle=1$, we have $n\leq 2$ by (4.9); we now show that $n=2$. If $n=0$ then $\mathcal{G}<-1$ and by (4.14) $\mathcal{G}\not\sim[1]$; so $n>0$. If $n=1$ then write $\mathcal{G}=[A,\omega,B]$ where $A=(a_1,\ldots,a_\alpha)$ and $B=(b_1,\ldots,b_\beta)$ are sequences of integers less than -1, $0 \le \alpha \le \beta$ and $\omega \ge 0$. We prove that this is absurd, by induction on α. If $\alpha = 0$ then (4.15.1) is violated. Suppose $\alpha > 0$. Since we assumed $\alpha \le \beta$, we have $\beta > 0$. A tree $\mathcal{G}'$ is then obtained from $\mathcal{G}$ by performing ω blowings-up at appropriate links: $\mathcal{G}' = [A, 0, B']$ where $B' = (b'_1, \dots, b'_{\omega+\beta})$ is just B if $\omega = 0$, and if $\omega > 0$ then $B' = (-1, -2, \dots, -2, b_1 - 1, b_2, \dots, b_\beta)$. By using the trick given just before (4.15) we see that $\mathcal{G}'$ is, in any case, equivalent to

$$\mathcal{G}'' = [a_1, \dots, a_{\alpha-1}, -1, 0, b'_1 + a_\alpha + 1, b'_2, \dots, b'_{\omega+\beta}].$$

Now $\mathcal{G}''$ contracts to a minimal tree $\mathcal{G}''' = [A''', \omega''', B''']$, where $\omega''' > 0$, A''' and B''' are sequences of integers less than -1, B''' is not empty and A''' has less than α terms. In particular $|\mathcal{G}'''| > 2$ ($|\mathcal{G}'''| \ge 2$ is clear, and the equality would violate the case $|\mathcal{G}| = 2$ already proved). By the inductive hypothesis applied to $\mathcal{G}'''$, this is absurd.

Hence $n = 2$, as claimed. By (4.9), the two vertices with nonnegative weights are linked, and one of them has weight zero. Thus we can write $\mathcal{G} = [A, 0, \omega, B]$ such that $\omega \ge 0$, and $A = (x_1, \dots, x_r)$ and $B = (y_1, \dots, y_s)$ are sequences of integers less than -1. We proceed by induction on $|\mathcal{G}|$.

If $|\mathcal{G}| = 3$, or more generally if $\min(r, s) = 0$, then by (4.15.1) $\mathcal{G} = [L_{\alpha_1}^{\alpha_0}, 0, \alpha_0 + 1]$ where $\alpha_0 = \omega - 1$ and $\alpha_1 = |\mathcal{G}| - 3$ are nonnegative. Thus $\mathcal{G}$ occurs in the list.

Suppose $|\mathcal{G}| > 3$. By above, we may assume $\min(r, s) > 0$. Since $\mathcal{G} = [\dots, x_r, 0, \omega, y_1, \dots] \sim [\dots, x_r + \omega, 0, 0, y_1, \dots]$ we have $x_r + \omega \le -1$ by (4.9). We claim that equality holds. If not, $\mathcal{G} \sim [\dots, x_r + \omega, -1, -1, -1, y_1, \dots] \sim [\dots, x_{r-1}, x_r + \omega + 1, 1, y_1 + 1, y_2, \dots]$, which contracts to a minimal tree $\mathcal{G}' = [A', \omega', B']$ such that $\omega' \ge 1$, and A' and B' are sequences of integers less than -1. If A' and B' are empty then $\omega' > 1$ and $\mathcal{G}' = [\omega'] \not\sim [1]$, which is absurd. If $|\mathcal{G}'| = 2$ then $\omega' > 1$ and (4.15.1) is violated. So $|\mathcal{G}'| > 2$ and, by an earlier part of this proof, $\mathcal{G}'$ must have *two* vertices with nonnegative weights i.e., contradiction.

Hence $x_r + \omega = -1$ and consequently $\mathcal{G}$ is equivalent to $[x_1, \dots, x_{r-1}, -1, 0, 0, B]$, which contracts to a minimal tree $\mathcal{G}' = [A', \omega', 0, B]$ where $\omega' > 0$ and where A' is a sequence of r' integers less than -1, $0 \le r' < r$. Moreover, ω' is just the number of blowings-down in that contraction process, i.e., $(x_{r'+1}, \dots, x_r) = L_{\omega'-1}^{-x_r-2}$ and (if $r' > 0$) $x_{r'} < -2$. Since we assumed that $\min(r, s) > 0$, $|\mathcal{G}'| = r' + 2 + s \ge 3$. On the other hand $|\mathcal{G}'| < |\mathcal{G}|$ so, by the inductive hypothesis, $\mathcal{G}'$ has the form specified in (4.15c). It easily follows that $\mathcal{G}$ also has that form, i.e., $\mathcal{G}$ occurs in the list.

4.16. COROLLARY. *Let $\mathcal{G}$ be a minimal weighted tree equivalent to $[1]$. Then $\mathcal{G}$ is linear and:*

(a) If $|\mathcal{G}| = 1$ then $\mathcal{G} = [1]$.

(b) If $|\mathcal{G}| = 2$ then $\mathcal{G} = [0, \alpha]$, some $\alpha \in \mathbb{Z} \setminus \{-1\}$.

(c) If $|\mathcal{G}| > 2$ then $\mathcal{G}$ has exactly two vertices with nonnegative weights, these vertices are linked and exactly one of them, say u, has weight zero. Moreover, u has two neighbours, say x and y, and $\Omega(x) + \Omega(y) = -1$.

4.17. DEFINITION. Let $\mathcal{G}$ be a weighted tree and v a vertex of $\mathcal{G}$. We say that v is a *special vertex* if the number of branches B of $\mathcal{G}$ at v such that $B < -1$ (see before (4.14)) is at least two.

4.18. COROLLARY. *Let $\mathcal{G} \sim [1]$ and suppose that v is a special vertex of $\mathcal{G}$. Then*

$$\Omega(v) + |\mathcal{N}_{\mathcal{G}}(v)| \le 1.$$

PROOF: Let $n = |\mathcal{N}_{\mathcal{G}}(v)|$ and let B_1, B_2 be branches of $\mathcal{G}$ at v such that $B_1 < -1$ and $B_2 < -1$. By (4.13), we can contract $\mathcal{G}$ to a linear tree $\mathcal{G}'$ such that v doesn't disappear in that process. Clearly, the branches of $\mathcal{G}'$ at v are just B_1, B_2 (with same weights as in $\mathcal{G}$), and the weight $\Omega'(v)$ of v in $\mathcal{G}'$ satisfies $\Omega'(v) \geq \Omega(v) + n - 2$, since $n - 2$ branches of $\mathcal{G}$ at v disappeared in the contraction. By (4.16), $\mathcal{G}'$ is not minimal, i.e., $\Omega'(v) = -1$ and we get the desired inequality.

5. Other Conditions on the Domain and Codomain.

In section 2 we saw that imposing conditions on the domain and/or codomain of a birational morphism may have consequences on the structure of that morphism. We now return to such considerations, but the conditions that will be studied have a different flavor: they deal with the *graph-theoretic* structure at infinity of our surfaces.

5.1. DEFINITIONS. Let U be a nonsingular surface. We say that U is *connected at infinity* (resp. *has no loops at infinity, is a tree at infinity, is linear at infinity*) if, in the equivalence class $\mathcal{G}[U]$ of weighted graphs (see (4.8)), all graphs are connected (resp. no graph has loops, all graphs are trees, some graph is a linear tree). Of course, the new definition of "connectedness at infinity" is equivalent to the usual one.

Let us also say that U is *rational at infinity* if for some (equivalently, for every) open immersion $U \hookrightarrow \overline{U}$ such that $\overline{U}$ is a complete nonsingular surface, all curves in $\overline{U} \setminus U$ are rational.

5.2. FACTS. *Let $f : X \to Y$ be a birational morphism.*

(a) If X is rational at infinity then so is Y.

(b) If X has no loops at infinity then Y has no loops at infinity.

These facts are easily proved if f is either an open immersion or a monoidal transformation. The general case follows immediately by making use of a minimal decomposition.

5.3. FACT. *Let $f : X \to Y$ be a birational morphism. If X is rational at infinity, then all missing curves are rational.*

5.4. DEFINITION. Let Γ be a (not necessarely complete) curve. Let $\tilde{\Gamma}$ be the complete nonsingular model of Γ (i.e., the set of valuation rings of the function field of Γ over the ground field) and let $\tau : \tilde{\Gamma} \to \Gamma$ be the canonical birational transformation. Then $\tilde{\Gamma} \setminus \operatorname{dom}(\tau)$ is a finite set of closed points, called *the places of* Γ *at infinity*. Let the cardinality of $\tilde{\Gamma} \setminus \operatorname{dom}(\tau)$ be denoted by $P_\infty(\Gamma)$. We say that Γ *has* $P_\infty(\Gamma)$ *places at infinity*. Notice that if $\overline{\Gamma}$ is any complete curve which contains Γ, then τ extends to an epimorphism $\overline{\tau} : \tilde{\Gamma} \to \overline{\Gamma}$ and $\overline{\tau}^{-1}(\overline{\Gamma} \setminus \Gamma)$ is just the set of places of Γ at infinity.

5.5. LEMMA. *Let $f : X \to Y$ be a birational morphism where X has no loops at infinity. If Y has $k > 0$ connected components at infinity (i.e., an arbitrary member of $\mathcal{G}[Y]$ has k connected components), then*

$$\sum_{i=1}^{q} P_\infty(C_i) \leq k + q - 1,$$

where $C_1, \ldots, C_q$ are the missing curves of f. In particular, if Y is connected at infinity (resp. if Y is affine) then each missing curve has at most (resp. exactly) one place at infinity.

Proof: Choose a smooth completion $Y \hookrightarrow \overline{Y}$ of Y and consider the graph $\mathcal{G} = (G, R)$ given by $G = \{\overline{C}_1, \dots, \overline{C}_q, A_1, \dots, A_h\}$, where $\overline{C}_i$ is the closure of C_i in $\overline{Y}$ and $A_1, \dots, A_h$ are the connected components of $\overline{Y} \setminus Y$, and $R = \{\{\overline{C}_i, A_j\} \mid \overline{C}_i \cap A_j \neq \emptyset\}$. Since X has no loops at infinity we see that $\mathcal{G}$ doesn't have loops and that each $\overline{C}_i$ belongs to exactly $P_\infty(C_i)$ links. Thus $|R| = \sum_{i=1}^{q} P_\infty(C_i)$. On the other hand, it is a general fact that a graph $\mathcal{G}$ with no loops has at most $|\mathcal{G}| - 1$ links (exactly $|\mathcal{G}| - 1$ iff $\mathcal{G}$ is a tree). Hence we get the desired inequality. ∎

5.6. **Fact.** *Let $f : X \to Y$ be a birational morphism where X has no loops at infinity and consider a minimal decomposition of f, with notation as in (1.3h). Let C be the strict transform in Y_n of a missing curve, let $\tilde{C}$ be the complete nonsingular model of C and let $\tau : \tilde{C} \to C$ be the canonical birational transformation. Then the set map $\tau : \mathrm{dom}(\tau) \to C$ is bijective.*

5.7. **Lemma.** *Let $f : X \to Y$ be a birational morphism, where X is linear at infinity and Y is affine. Consider a minimal decomposition of f, with notation as in (1.3h). Then $Y_n \setminus X$ has $q = q(f)$ connected components, each one of the form*

$$C_i — E_{j_1} — E_{j_2} — \cdots — E_{j_k}$$

where $\{j_1, \dots, j_k\} \subseteq J$ and $C_i + E_{j_1} + \cdots + E_{j_k}$ has s.n.c. in Y_n.

Proof: Since each C_i has one place at infinity of $Y_0 = Y$ by (5.5), we can choose a smooth completion $Y_0 \hookrightarrow \overline{Y}_0$ of Y_0 such that, if L is the divisor of $\overline{Y}_0$ with s.n.c. and which satisfies $\overline{Y}_0 \setminus Y_0 = \mathrm{supp}(L)$, and if $C_1, \dots, C_q$ are the closures (in $\overline{Y}_0$) of the missing curves, then $C_1, \dots, C_q$ meet L at distinct points and $C_i.L = 1 \quad (1 \le i \le q)$. As in the proof of (2.1), let us "complete the diagram":

$$\begin{array}{ccccccc}
 & & & & Y_n & \hookrightarrow & \overline{Y}_n \\
 & & & & \downarrow \pi_n & & \downarrow \overline{\pi}_n \\
 & & & & \vdots & & \vdots \\
 & & & & \downarrow \pi_1 & & \downarrow \overline{\pi}_1 \\
X & \xrightarrow[f]{} & Y & = & Y_0 & \hookrightarrow & \overline{Y}_0
\end{array}$$

Then $\overline{Y}_n \setminus Y_n = \mathrm{supp}(L)$, and (in $\overline{Y}_n$) $C_1, \dots, C_q$ meet L at distinct points and $C_i.L = 1 \quad (1 \le i \le q)$. Since X has no loops at infinity and L is connected, $C_1, \dots, C_q$ belong to distinct connected components of $Y_n \setminus X$. On the other hand, if W is a connected component of $Y_n \setminus X$ and $\overline{W}$ is its closure in $\overline{Y}_n$, then $\overline{W}$ meets L, since X is connected at infinity; hence $\overline{W}$ contains a C_i, and there are exactly q connected components of $Y_n \setminus X$. We now show (by contradiction) that each one of these connected components has the desired properties. Let

$$D = \sum_{i=1}^{q} C_i + \sum_{i \in J} E_i + L \in \mathrm{Div}(\overline{Y}_n).$$

First, suppose that D does not have s.n.c.. By (4.7), we can consider a sequence of monoidal transformations $\overline{Y}_m \to \cdots \to \overline{Y}_n \quad (m > n)$, such that if E_i is the exceptional curve created by $\overline{Y}_i \to \overline{Y}_{i-1}$ and

$$\begin{cases} D^n = D \in \mathrm{Div}(\overline{Y}_n) \\ D^i = (\text{strict transform of } D^{i-1}) + E_i \in \mathrm{Div}(Y_i), \quad n < i \le m, \end{cases}$$

then $D^m \in \mathrm{Div}(\overline{Y}_m)$ has *s.n.c.*, all centers are *i.n.* supp $(D) \cap Y_n$, $\overline{Y}_m \setminus \mathrm{supp}(D^m) \cong X$ and if $n < i \leq m$ then

$$(*) \qquad E_i^2 = -1 \text{ in } \overline{Y}_m \Longrightarrow E_i \text{ is a branch point of } \mathcal{G}_m = \mathcal{G}(\overline{Y}_m, D^m).$$

Let $\mathcal{G}_+$ be the connected subtree of $\mathcal{G}_m$ which has $C_1, \ldots, C_q$ and the irreducible components of L as vertices. Let Σ be the set of branch points v of $\mathcal{G}_m$ such that v is not in $\mathcal{G}_+$. By (*), $E_m \in \Sigma$ so $\Sigma \neq \emptyset$. If $v \in \Sigma$, then let B_v be the branch of $\mathcal{G}_m$ at v such that B_v contains $\mathcal{G}_+$. Since $\mathcal{G}_m$ is a finite tree, we can find $v \in \Sigma$ such that, if $B_v, B_1, \ldots, B_k$ are the distinct branches of $\mathcal{G}_m$ at v ($\Rightarrow k \geq 2$) then $\Sigma \cap (B_1 \cup \cdots \cup B_k) = \emptyset$. By (*) and (1.3i), $B_i < -1 \quad 1 \leq i \leq k$ (see just before (4.14)). Since X is linear at infinity, $\mathcal{G}_m$ contracts to a linear tree; since B_1, B_2 can't disappear in that contraction, B_v must disappear. Thus we see that $B_v \sim [-1]$. Clearly, $\langle \ \rangle$ is a "nondecreasing" function, so

$$\langle \mathcal{G}(\overline{Y}_m, L) \rangle \leq \langle B_v \rangle = \langle [-1] \rangle = 0.$$

On the other hand, $\langle \mathcal{G}(\overline{Y}_0, L) \rangle > 0$ since Y_0 is affine—in the terminology of (2.1.3), supp (L) is a positive subset of $\overline{Y}_0$. Moreover, $\mathcal{G}(\overline{Y}_0, L)$ is just the same as $\mathcal{G}(\overline{Y}_m, L)$, since no blowing-up has center *i.n.* L. Hence

$$\langle \mathcal{G}(\overline{Y}_m, L) \rangle > 0,$$

contradiction. So $D \in \mathrm{Div}(Y_n)$ has *s.n.c.*.

Next, suppose that some connected component W of $Y_n \setminus X$ does not have the desired form; it means that either the dual tree $\mathcal{G}(\overline{Y}_n, F)$ is not linear or C_i is not a free vertex of it, where

$$F = C_i + E_{j_1} + \cdots + E_{j_k} \in \mathrm{Div}(\overline{Y}_n)$$

is the divisor (with *s.n.c.*) whose support is the closure $\overline{W}$ of W in $\overline{Y}_n$. In the first case, let v be a branch point of $\mathcal{G}(\overline{Y}_n, F)$; in the second case, let $v = C_i$. Let $B_v, B_1, \ldots, B_k$ be the distinct branches of $\mathcal{G} = \mathcal{G}(\overline{Y}_n, D)$ at v, where B_v is the one that contains the components of L. By (1.3i) we see that $B_i < -1 \quad (1 \leq i \leq k)$. As above, we see that v "absorbs" B_v and a contradiction follows.

II. LOCAL TREES

As was seen in part I, if S is a projective nonsingular surface and $D \in \mathrm{Div}(S)$ has *s.n.c.* (i.e., strong normal crossings, see (I.4.5)) then the pair (S, D) determines a weighted graph which carries some information about the surface $S \setminus \mathrm{supp}(D)$. In many cases, however, the divisor D with which we have to cope doesn't have *s.n.c.*. When that happens, one usually blows-up S at some points of $\mathrm{supp}(D)$, until a divisor with *s.n.c.* is obtained; then one can consider a weighted graph.

In this second part of our thesis, we present a graph theory that gives some control on the desingularisation process. To give a rough picture, let us say that a local tree is a graph theoretic device that is assigned to a singular point of an effective divisor on a surface, and that follows its desingularisation, keeping track of certain arithmetic aspects of the process. When that process terminates, we obtain a local tree from which the desired weighted tree can be recovered.

We refer the reader to the beginning of (I.4) for the terminologies and notations of graph-theory.

1. Basic Concepts.

1.1. DEFINITIONS. A *local tree* is a 4-tuple $\mathcal{T} = (T, x_0, R, \Omega)$ where:

(a) T is a finite set and $x_0 \in T$;

(b) R is a collection of subsets of T such that every $a \in R$ contains exactly two elements, and (T, R) is a tree;

(c) Ω is a set map $T \setminus \{x_0\} \to \mathbf{Z}$.

The elements of T are called the *vertices*, and those of R the *links*; x_0 is called the *root* of $\mathcal{T}$. Given $x \in T \setminus \{x_0\}$, $\Omega(x)$ is the *weight* of x. Write $R^0 = \{a \in R \mid x_0 \in a\}$ and call the elements of R^0 the *principal links* of $\mathcal{T}$. The neighbours of the root will be called the *principal vertices*.

Although it is not clear what a morphism of local trees should be, it certainly makes good sense to define an *isomorphism* of local trees to be a bijective map between their sets of vertices, preserving the root, the links and the weights.

1.2. DEFINITIONS. If $\mathcal{T} = (T, x_0, R, \Omega)$ is a local tree, a *multiplicity map for* $\mathcal{T}$ is a set map

$$\mu : R^0 \cup \{x_0\} \longrightarrow \mathbf{N}$$

(where $\mathbf{N}$ is the set of positive integers) such that $\mu(a) \geq \mu(x_0)$ for every $a \in R^0$.

A *multiplied local tree* is a pair $(\mathcal{T}, \mu)$ where $\mathcal{T}$ is a local tree and μ is a multiplicity map for $\mathcal{T}$. We will always write "m-tree" instead of "multiplied local tree". Given an m-tree $(\mathcal{T}, \mu)$, if x is either the root or a principal link the number $\mu(x)$ is called its *multiplicity*; denote by $\mathcal{N}(\mathcal{T}, \mu)$ the set $\{x \in \mathcal{N}_{\mathcal{T}}(x_0) \mid \mu(\{x, x_0\}) = \mu(x_0)\}$.

An *isomorphism* of m-trees is an isomorphism of local trees which preserves the multiplicities.

1.3. DEFINITIONS (BLOWING-UP). Let $(\mathcal{T}, \mu)$ be an m-tree, $\mathcal{T} = (T, x_0, R, \Omega)$. We are going to define three notions of blowing-up of $(\mathcal{T}, \mu)$.

(1) A *blowing-up of the first kind* of $(\mathcal{T}, \mu)$ is an m-tree $(\mathcal{T}', \mu')$, where $\mathcal{T}' = (T', x_0', R', \Omega')$, together with a root-preserving injective set map $\beta : T \to T'$, such that if we identify T with its image in T', then the following conditions hold:

(a) $\mathcal{T}' = \mathcal{T} \cup \{e\}$, for some $e \notin \mathcal{T}$

(b) There is a set A such that $\mathcal{N}(\mathcal{T},\mu) \subseteq A \subseteq \mathcal{N}_{\mathcal{T}}(x_0)$, $|\mathcal{N}_{\mathcal{T}}(x_0) \setminus A| \leq 1$ and:

(b1) $R' = \{\{e, x_0\}\} \cup (R \setminus \{\{x, x_0\} \mid x \in A\}) \cup \{\{x, e\} \mid x \in A\}$
(so that the set A is nothing else than $\{x \in \mathcal{N}_{\mathcal{T}}(x_0) \mid x \notin \mathcal{N}_{\mathcal{T}'}(x_0)\}$)

(b2) $\mu'(\{x, x_0\}) = \mu(\{x, x_0\}) - \mu(x_0)$, if $x \in \mathcal{N}_{\mathcal{T}}(x_0) \setminus A$

(c) $\mu'(\{x_0, e\}) \leq \mu(x_0)$

(d) $\Omega'(x) = \begin{cases} -1, & \text{if } x = e \\ \Omega(x), & \text{if } x \notin \{x_0\} \cup \mathcal{N}_{\mathcal{T}}(x_0) \\ \Omega(x) - 1, & \text{if } x \in \mathcal{N}_{\mathcal{T}}(x_0). \end{cases}$

A blowing-up of the first kind of $(\mathcal{T},\mu)$ will be denoted by the symbol $(\mathcal{T}',\mu') \rightharpoondown (\mathcal{T},\mu)$ or by $(\mathcal{T},\mu) \leftharpoondown (\mathcal{T}',\mu')$. Notice that the arrow goes from $(\mathcal{T}',\mu')$ to $(\mathcal{T},\mu)$ while β goes from T to T'. One should keep in mind that the symbol $(\mathcal{T},\mu) \leftharpoondown (\mathcal{T}',\mu')$ means, in particular, that a map β has been chosen.

(2) A *blowing-up of the second kind of* $(\mathcal{T},\mu)$ is a blowing-up of the first kind $(\mathcal{T},\mu) \leftharpoondown (\mathcal{T}',\mu')$ such that the set A of (1b) is $\mathcal{N}(\mathcal{T},\mu)$. That situation will be indicated either by $(\mathcal{T}',\mu') \rightarrow (\mathcal{T},\mu)$ or by $(\mathcal{T},\mu) \leftarrow (\mathcal{T}',\mu')$.

(3) A *blowing-up of the third kind of* $(\mathcal{T},\mu)$, or simply a *blowing-up of* $(\mathcal{T},\mu)$, is a blowing-up of the second kind $(\mathcal{T},\mu) \leftarrow (\mathcal{T}',\mu')$ such that equality holds in (1c). That situation will be indicated either by $(\mathcal{T}',\mu') \Rightarrow (\mathcal{T},\mu)$ or by $(\mathcal{T},\mu) \Leftarrow (\mathcal{T}',\mu')$.

1.4. LEMMA. *Let $\mathcal{T}$, $\mathcal{T}'$ be local trees and suppose that $\beta : T \to T'$ is such that there exist multiplicity maps μ_0, μ_0' such that $(\mathcal{T},\mu_0) \leftharpoondown (\mathcal{T}',\mu_0')$ with β as the underlying set map. Then, if μ' is any multiplicity map for $\mathcal{T}'$, there is a unique μ such that $(\mathcal{T},\mu) \Leftarrow (\mathcal{T}',\mu')$ with β as the underlying set map.*

PROOF: $\mathcal{T}$, $\mathcal{T}'$ and β determine the set A of (1.3, 1b). We must have:

$$\mu(x_0) = \mu'(\{e, x_0\})$$
$$\mu(\{x, x_0\}) = \begin{cases} \mu'(\{e, x_0\}) + \mu'(\{x, x_0\}), & x \in \mathcal{N}_{\mathcal{T}}(x_0) \setminus A \\ \mu'(\{e, x_0\}), & x \in A, \end{cases}$$

and this is, indeed, a multiplicity map for $\mathcal{T}$ satisfying the desired condition.

REMARK. The set of multiplicity maps for a given local tree is an additive (nonempty) semigroup. The map $\mu' \mapsto \mu$ given by (1.4) is a homomorphism of semigroups; denote it by β^*. In general, β^* is neither injective nor surjective. In particular, $\beta^*(\mu_1) = \beta^*(\mu_2) \Leftrightarrow \mu_1(a) = \mu_2(a)$, for all principal links a of $\mathcal{T}'$.

1.5. COROLLARY. *Let $\mathcal{T}$, $\mathcal{T}'$ be local trees and $\beta : T \to T'$ a root-preserving injective set map. Then the following are equivalent:*

(a) $\exists(\mu,\mu')$ such that $(\mathcal{T},\mu) \leftharpoondown (\mathcal{T}',\mu')$ with β as the underlying map.

(b) $\exists(\mu,\mu')$ such that $(\mathcal{T},\mu) \leftarrow (\mathcal{T}',\mu')$ with β as the underlying map.

(c) $\exists(\mu,\mu')$ *such that* $(\mathcal{T},\mu) \Leftarrow (\mathcal{T}',\mu')$ *with* β *as the underlying map.*

1.6. DEFINITION. Let $\mathcal{T}$, $\mathcal{T}'$ be local trees. An *identification map* is a root-preserving injective set map β $\mathcal{T} \to \mathcal{T}'$, such that the equivalent conditions of (1.5) are met. The symbol $\mathcal{T} \leftarrow \mathcal{T}'$ (or $\mathcal{T}' \to \mathcal{T}$) will be an abbreviation of the following statement:

There exists at least one identification map $T \to T'$, and a choice of such a map has been made. Moreover, T will be regarded as a subset of T' via that identification map (whenever possible).

The situation " $\mathcal{T} \leftarrow \mathcal{T}'$ " will be called a *blowing-up of local trees.*

1.7. REMARKS.

(a) If $\mathcal{T} \leftarrow \mathcal{T}'$ then $\mathcal{T}'$ has either one or two principal link(s).

(b) If $(\mathcal{T},\mu) \leftarrow (\mathcal{T}',\mu')$ then, in the notation of (1.3), $\mu(x_0) \geq \mu'(\{e,x_0\}) \geq \mu'(x_0)$.

(c) Any local tree $\mathcal{T}$ can be blown up, i.e., there exists $\mathcal{T}'$ and an identification map such that

(*) $$\mathcal{T} \leftarrow \mathcal{T}'.$$

If $|\mathcal{T}| > 1$, then there are several non-isomorphic $\mathcal{T}'$ satisfying (*) (the identification map is not fixed). If $\mathcal{T}$, $\mathcal{T}'$ are fixed, there may exist several identification maps such that (*); that's why we insist that there is a choice involved.

(d) A blowing-up of the second (or third) kind can be performed on an m-tree $(\mathcal{T},\mu)$ iff $|\mathcal{N}_{\mathcal{T}}(x_0) \setminus \mathcal{N}(\mathcal{T},\mu)| \leq 1$. If this is the case, then there is exactly one diagram (*) (up to isomorphism commuting with identification maps) such that

(**) $$(\mathcal{T},\mu) \Leftarrow (\mathcal{T}',\mu'), \text{ for some } \mu'.$$

Moreover, the restriction to R'^0 of the μ' of (**) is unique, and the possible values for $\mu'(x_0)$ are $1,\ldots,\min_{a\in R'^0} \mu'(a)$.

1.8. If $\mathcal{T}_0,\ldots,\mathcal{T}_k$ are local trees ($k \geq 1$), the symbol $\mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$ will stand for "$\mathcal{T}_0 \leftarrow \mathcal{T}_1$ and $\cdots$ and $\mathcal{T}_{k-1} \leftarrow \mathcal{T}_k$". When this is the case, k applications of (1.4) show that each multiplicity map μ_k for $\mathcal{T}_k$ determines (uniquely) $(\mu_0,\ldots,\mu_{k-1})$ such that $(\mathcal{T}_0,\mu_0) \Leftarrow \cdots \Leftarrow (\mathcal{T}_k,\mu_k)$. Moreover, if $(\mu'_0,\ldots,\mu'_k)$ is such that $(\mathcal{T}_0,\mu'_0) \Leftarrow \cdots \Leftarrow (\mathcal{T}_k,\mu'_k)$ and for some $q \in \mathbf{Q}$ we have $q\mu_k(a) = \mu'_k(a)$, all $a \in R^0_k$, then $q(\mu_0,\ldots,\mu_{k-1}) = (\mu'_0,\ldots,\mu'_{k-1})$.

1.9. DEFINITION. Given an infinite sequence $\mathcal{S}$: $(\mathcal{T}_0,\mu_0) \leftarrow (\mathcal{T}_1,\mu_1) \leftarrow \cdots$, there exists an $i \geq 0$ such that

$\mathcal{T}_i$ has at most one principal link, and if it has one then its multiplicity is $\mu_i(x_0)$;

$\forall_{j>i}$, $\mathcal{T}_j$ has exactly one principal link, say a_j, and $\mu_j(a_j) = \mu_i(x_0)$.

The least such i will be denoted by $k = k(\mathcal{S})$. Observe that $(\mathcal{T}_k,\mu_k) \Leftarrow (\mathcal{T}_{k+1},\mu_{k+1}) \Leftarrow \cdots$.

1.10. REMARKS.

(a) It is easy to construct an infinite sequence

(*) $$\mathcal{T}_0 \leftarrow \mathcal{T}_1 \leftarrow \cdots$$

where, say, all trees have two principal links. By (1.9), such a sequence does not admit multiplicity maps $\mu_0, \mu_1, \ldots$ such that

$$(**) \qquad (\mathcal{T}_0, \mu_0) \leftarrow (\mathcal{T}_1, \mu_1) \leftarrow \cdots .$$

(b) If an infinite sequence (*) admits multiplicity maps such that (**), then it admits multiplicity maps $\mu'_0, \mu'_1, \ldots$ such that

$$(\mathcal{T}_0, \mu'_0) \Leftarrow (\mathcal{T}_1, \mu'_1) \Leftarrow \cdots .$$

Indeed, define $\mu'_i = \mu_i$ for $i \geq k$ and use μ'_k and (1.8) to determine $(\mu'_0, \ldots, \mu'_{k-1})$. Notice that, by (1.8), such an infinite sequence of multiplicity maps is unique, up to multiplication by a rational number.

Arithmetic of Blowings-Up. Before we end this section, we want to give some basic facts that relate sequences of blowings-up of m-trees with the euclidean algorithm. All mathematicians who have studied blowings-up of curves are aware of such relationships; for that reason, and also because these observations are easily verified, we will omit the proofs.

REMARK. Although this is the logical place for this material to be, the reader might prefer to skip it and come back once section 2 is understood.

1.11. DEFINITIONS. Consider a sequence of local trees

$$\mathcal{S}: \qquad \mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k \qquad (k \geq 0).$$

(a) Define Mul$(\mathcal{S})$ to be the set of $k+1$-tuples $\mu = (\mu_0, \ldots, \mu_k)$ of multiplicity maps such that

$$(\mathcal{T}_0, \mu_0) \Leftarrow \cdots \Leftarrow (\mathcal{T}_k, \mu_k).$$

Then Mul$(\mathcal{S})$ is a (nonempty, additive) semigroup and (1.4) says that the projection map Mul$(\mathcal{S}) \to$ Mul$(\mathcal{T}_k)$ is an isomorphism.

(b) If $k \geq 1$ and $\mathcal{T}_0$ has one principal link a, consider the following two statements about an arbitrary element $\mu = (\mu_0, \ldots, \mu_k)$ of Mul$(\mathcal{S})$ (and notice that P(μ) implies Q(μ)).

P(μ) *Let* $r_{-1} = \mu_0(a)$ *and* $r_\nu = \mu_\nu(x_0)$ $(0 \leq \nu \leq k-1)$ *and let the euclidean algorithm of* (r_{-1}, r_0) *be written as*

$$\begin{aligned} r_{-1} &= \alpha_0 \rho_0 + \rho_1 \qquad (\textit{where } \rho_0 = r_0) \\ \rho_0 &= \alpha_1 \rho_1 + \rho_2 \\ &\vdots \\ \rho_{s-1} &= \alpha_s \rho_s . \end{aligned}$$

Then $(r_0, \ldots, r_{k-1}) = (\rho_0, \ldots, \rho_0, \rho_1, \ldots, \rho_{s-1}, \rho_s, \ldots, \rho_s)$ *where each* ρ_i *occurs exactly* α_i *times.*

Q(μ) *Let $r_{-1} = \mu_0(a)$ and $r_\nu = \mu_\nu(x_0)$ $(0 \le \nu \le k-1)$ and, given any $\nu < k-1$ such that $r_\nu \gtrless r_{\nu+1}$, let the euclidean algorithm of $(r_\nu, r_{\nu+1})$ be written as*

$$r_\nu = \alpha_0\rho_0 + \rho_1 \qquad (\text{where } \rho_0 = r_{\nu+1})$$
$$\rho_0 = \alpha_1\rho_1 + \rho_2$$
$$\vdots$$
$$\rho_{s-1} = \alpha_s\rho_s.$$

Then $\nu+\alpha_0+\cdots+\alpha_s \le k-1$ and $(r_{\nu+1}, \dots, r_{\nu+\alpha_0+\cdots+\alpha_s}) = (\rho_0, \dots, \rho_0, \rho_1, \dots, \rho_{s-1}, \rho_s, \dots, \rho_s)$ where each ρ_i occurs exactly α_i times.

1.12. LEMMA. *Let S : $\mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$ be a sequence of local trees such that $k \ge 1$ and such that $\mathcal{T}_0$ has one principal link.*

(a) The following conditions are equivalent:

- *(a1) For all ν, $\mathcal{T}_\nu$ has one principal link iff $\nu \in \{0, k\}$,*
- *(a2) $P(\mu)$ holds, for all $\mu \in Mul(S)$,*
- *(a3) $P(\mu)$ holds, for some $\mu \in Mul(S)$.*

(b) The following conditions are equivalent:

- *(b1) $\mathcal{T}_k$ has one principal link,*
- *(b2) $Q(\mu)$ holds, for all $\mu \in Mul(S)$,*
- *(b3) $Q(\mu)$ holds, for some $\mu \in Mul(S)$.*

REMARKS.

- If the conditions of (a) are met, $\mu \in \mathrm{Mul}(S)$ and if the principal links of $\mathcal{T}_0$ and $\mathcal{T}_k$ are a and a' respectively, then $\mu_k(a')$ is the g.c.d. of $\mu_0(a)$ and $\mu_0(x_0)$.
- If $k > 1$ and the conditions of (a) are met then the principal vertex of $\mathcal{T}_k$ is a branch point (for $\mathcal{T}_{k-1}$ has two principal links, while $\mathcal{T}_k$ has only one). So a branch point is created each time an euclidean algorithm terminates.

1.13. DEFINITIONS.

(a) Given S : $\mathcal{T}_0 \leftarrow \cdots \mathcal{T}_k$ such that $k \ge 1$ and both $\mathcal{T}_0$ and $\mathcal{T}_k$ have one principal link, define

$$\mathcal{J}(S) = \{j \mid 0 \le j < k,\ \mathcal{T}_j \text{ has one principal link and } \mathcal{T}_{j+1} \text{ has two}\},$$
$$\mathcal{H}(S) = \{j \mid 0 < j \le k,\ \mathcal{T}_{j-1} \text{ has two principal links and } \mathcal{T}_j \text{ has one}\}$$
$$\text{and } l = (\#\text{ of branch points of } \mathcal{T}_k) - (\#\text{ of branch points of } \mathcal{T}_0).$$

We see that $|\mathcal{J}(S)| = |\mathcal{H}(S)| = l$.
Write $\mathcal{J}(S) = \{j_0, \dots, j_{l-1}\}$, $0 \le j_0 < \cdots < j_{l-1}$,
and $\mathcal{H}(S) = \{h_1, \dots, h_l\}$, $0 < h_1 < \cdots < h_l \le k$;
then $0 \le j_0 < h_1 \le j_1 < \cdots \le j_{l-1} < h_l \le k$.

We denote by e_ν the branch point created in $\mathcal{T}_{h_\nu - 1} \leftarrow \mathcal{T}_{h_\nu}$ $(1 \le \nu \le l)$. Hence e_ν can be regarded as a vertex of $\mathcal{T}_{h_\nu}, \mathcal{T}_{h_\nu+1}, \dots, \mathcal{T}_k$.

(b) If $\mu = (\mu_0, \ldots, \mu_k) \in \mathrm{Mul}(\mathcal{S})$ then the pair $(\mathcal{S}, \mu)$ determines the following numbers (where x_0 is the root of any $\mathcal{T}_i$ and a_i is the principal link of $\mathcal{T}_i$, whenever i is such that $\mathcal{T}_i$ has one principal link):

$$\begin{cases} i_0 = \mu_{j_0}(a_{j_0}) \\ i_\nu = \mu_{h_\nu}(a_{h_\nu}) = \mu_{j_\nu}(a_{j_\nu}), \quad 0 < \nu < l \\ i_l = \mu_{h_l}(a_{h_l}) \end{cases}$$

$$\begin{cases} m_\nu = \mu_{j_\nu}(x_0), \qquad\qquad 0 \le \nu < l \\ m = m(\mathcal{S}, \mu) = m_0 + \cdots + m_{l-1}. \end{cases}$$

Then $i_0 > m_0 \geq i_1 > m_1 \geq \cdots \geq i_{l-1} > m_{l-1} \geq i_l$ and $(i_{\nu-1}, m_{\nu-1}) = i_\nu$, $\quad 1 \le \nu \le l$, by (1.12).

REMARK. These notations and facts allow us to break a sequence

$$\mathcal{S}: \qquad \mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$$

into parts that we understand. Explicitly, if $\mu = (\mu_0, \ldots, \mu_k) \in \mathrm{Mul}(\mathcal{S})$, if notations are as in (1.13) and if $r_i = \mu_i(x_0)$, $0 \le i < k$, then:

$(\mathcal{T}_0, \mu_0) \Leftarrow \cdots \Leftarrow (\mathcal{T}_{j_0}, \mu_{j_0})$ has $(r_0, \ldots, r_{j_0-1}) = (i_0, \ldots, i_0)$,

$(\mathcal{T}_{h_\nu}, \mu_{h_\nu}) \Leftarrow \cdots \Leftarrow (\mathcal{T}_{j_\nu}, \mu_{j_\nu})$ has $(r_{h_\nu}, \ldots, r_{j_\nu-1}) = (i_\nu, \ldots, i_\nu)$, $1 \le \nu < l$,

$(\mathcal{T}_{h_l}, \mu_{h_l}) \Leftarrow \cdots \Leftarrow (\mathcal{T}_k, \mu_k)$ has $(r_{h_l}, \ldots, r_{k-1}) = (i_l, \ldots, i_l)$;

and if $1 \le \nu \le l$, $(\mathcal{T}_{j_{\nu-1}}, \mu_{j_{\nu-1}}) \Leftarrow \cdots \Leftarrow (\mathcal{T}_{h_\nu}, \mu_{h_\nu})$ has $(r_{j_{\nu-1}}, \ldots, r_{h_\nu-1})$ given by the euclidean algorithm of $(i_{\nu-1}, m_{\nu-1})$, as described by the condition $\mathrm{P}(\mu)$ of (1.11) (this follows from (1.12a) and proves the assertion $(i_{\nu-1}, m_{\nu-1}) = i_\nu$ of (1.13)).

2. Relation to Geometry.

2.1. DEFINITIONS. We consider a triple (P, D, S) where

- S is a nonsingular projective surface
- $D \in \mathrm{Div}(S)$ has *s.n.c.* and $\mathcal{G}(S, D)$ (I.4.6) is a (possibly empty) tree
- $P \in \mathrm{supp}(D)$ if $D \neq 0$.

The *local tree of* (P, D, S) is $\mathcal{T} = (T, x_0, R, \Omega)$ where:

(a) $x_0 = P$, $T = \{P\} \cup \{D_1, \ldots, D_n\}$, where $D_1, \ldots, D_n$ are the distinct irreducible components of D

(b) $R = \big\{\{D_i, D_j\} \mid i \neq j \text{ and } P \notin D_i \cap D_j \neq \emptyset\big\} \cup \big\{\{P, D_i\} \mid P \in D_i\big\}$

(c) $\Omega(D_i) = D_i^2$ (self-intersection number in S).

The local tree of (P, D, S) is denoted by $\mathcal{T}(P, D, S)$.

If C is a nonzero effective divisor of S such that

- $P \in \mathrm{supp}(C)$
- C and D have no irreducible component in common,

we define *the m-tree of* (P, C, D, S) to be $(\mathcal{T}, \mu)$, where $\mathcal{T} = \mathcal{T}(P, D, S)$ and $\mu : R^0 \cup \{x_0\} \to \mathbf{N}$ is as follows:

(d) $\mu(x_0) = \mu(P, C)$ (multiplicity of P on C)

(e) $\mu(\{x_0, D_i\}) = (C.D_i)_P$ (local intersection multiplicity at P), if $\{x_0, D_i\} \in R^0$, i.e., if $P \in D_i$.

REMARK. If we fix a triple (P, D, S) satisfying the first three conditions of (2.1), the set $\mathcal{C} = \mathcal{C}(P, D, S)$ of divisors C satisfying the two other conditions is a semigroup, and the map $C \mapsto \mu$ determined by the definition (2.1) is a homomorphism of semigroups $\mathcal{C} \to \mathrm{Mul}(\mathcal{T})$.

2.2. BLOWING-UP. Let (P, C, D, S) be as in (2.1) and let $(\mathcal{T}, \mu)$ be its m-tree. Let $\pi : \widetilde{S} \to S$ be the blowing-up of S at P, $E = \pi^{-1}(P) \in \mathrm{Div}(\widetilde{S})$, let $\sim$ mean "strict transform of ... " and define $D' = \widetilde{D} + E \in \mathrm{Div}(\widetilde{S})$.

If $P' \in \mathrm{supp}(\widetilde{C})$ is *i.n.* P, then we may consider the m-tree $(\mathcal{T}', \mu')$ of $(P', \widetilde{C}, D', \widetilde{S})$. We let the reader convince himself that

$$(\mathcal{T}, \mu) \leftarrow (\mathcal{T}', \mu'),$$

where the identification map is the obvious one, and that the following claims are true.

(a) We have $(\mathcal{T}, \mu) \leftarrow (\mathcal{T}', \mu')$ iff every irreducible component Γ of D satisfies $\mathrm{supp}(\widetilde{\Gamma}) \cap \mathrm{supp}(E) \cap \mathrm{supp}(\widetilde{C}) \subseteq \{P'\}$.

(b) We have $(\mathcal{T}, \mu) \Leftarrow (\mathcal{T}', \mu')$ iff $\mathrm{supp}(E) \cap \mathrm{supp}(\widetilde{C}) \subseteq \{P'\}$.

2.3. DEFINITION. Let S, D and C be as in (2.1). If $\widetilde{P}$ is a place of C, i.e., a closed point of the nonsingular model of some irreducible component of C, then the triple $(\widetilde{P}, C, S)$ determines an infinite sequence of monoidal transformations

$$S_0 \xleftarrow{\pi_1} S_1 \xleftarrow{\pi_2} S_2 \longleftarrow \cdots$$

where $S_0 = S$, P_i = image of $\widetilde{P}$ in S_{i-1} and π_i is the blowing-up of S_{i-1} at P_i.

Let us assume that $\mathbf{P_1 \in supp(D)}$ or $\mathbf{D = 0}$.

Let $C^{(i)}$ be the strict transform of $C^{(0)} = C$ in S_i and let $E_i = \pi_i^{-1}(P_i)$; given $F \in \mathrm{Div}(S_{i-1})$, let $F^{\pi_i} = E_i +$ strict transform of F in S_i, and define $D^0 = D$, $D^i = (D^{i-1})^{\pi_i}$ $(i \geq 1)$.

Then, for $i \geq 0$, $(P_{i+1}, C^{(i)}, D^i, S_i)$ satisfies the conditions listed in (2.1) and we can consider its m-tree $(\mathcal{T}_i, \mu_i)$. By (2.2), we have

$$(\mathcal{T}_0, \mu_0) \leftarrow (\mathcal{T}_1, \mu_1) \leftarrow \cdots$$

which will be called *the infinite sequence of m-trees of* $(\widetilde{P}, C, D, S)$. The number k defined in (1.9) will be denoted by $k = k(\widetilde{P}, C, D, S)$. Observe that $(\mathcal{T}_k, \mu_k) \Leftarrow (\mathcal{T}_{k+1}, \mu_{k+1}) \Leftarrow \cdots$ and that, as far as the place $\widetilde{P}$ is concerned, the desingularization process ends with $S_{k-1} \leftarrow S_k$. What we mean, here, is that k is the least integer $i \geq 0$ which satisfies:

(a) P_{i+1} belongs to exactly one irreducible component of $C^{(i)}$,

(b) $\exists B \in \mathrm{Div}(S_i)$ with *s.n.c.* in a neighbourhood of P_{i+1}, such that $\mathrm{supp}(B) = \mathrm{supp}(C^{(i)} + D^i)$.

For these reasons, the finite sequence

$$(\mathcal{T}_0, \mu_0) \leftarrow \cdots \leftarrow (\mathcal{T}_k, \mu_k)$$

will be given special consideration; we will call it *the sequence of m-trees of* $(\widetilde{P}, C, D, S)$, and denote it by $\mu(\widetilde{P}, C, D, S)$.

2.4. LEMMA. *Let $(\widetilde{P}, C, D, S)$ be as in (2.3) and consider*

$$\mu(\widetilde{P}, C, D, S): \qquad (\mathcal{T}_0, \mu_0) \leftarrow \cdots \leftarrow (\mathcal{T}_k, \mu_k).$$

Assume $k > 0$.

(a) If $supp\,(C^{(k)} + D^k) = supp\,(B)$ for some $B \in Div\,(S_k)$ with s.n.c., then

$$(\mathcal{T}_0, \mu_0) \leftarrow \cdots \leftarrow (\mathcal{T}_k, \mu_k).$$

(b) If $\widetilde{C}$ is the disjoint union of the nonsingular models of the irreducible components of C, and if $\tau : \widetilde{C} \to supp\,(C)$ is the canonical surjective set map, then the following are equivalent:

- $(\mathcal{T}_0, \mu_0) \Leftarrow \cdots \Leftarrow (\mathcal{T}_k, \mu_k)$
- $\tau^{-1}(P_0) = \{\widetilde{P}\}$.

(c) If $supp\,(C)$ is irreducible and $S \setminus supp\,(C + D)$ is a tree at infinity (I.5.1), then

$$(\mathcal{T}_0, \mu_0) \Leftarrow \cdots \Leftarrow (\mathcal{T}_k, \mu_k).$$

PROOF: Immediate from (2.2).

3. Contraction of Local Trees.

3.1. INTRODUCTION. Given $\omega \in \mathbf{Z}$, the symbol (ω) will denote any local tree which has two vertices and such that the principal vertex has weight ω. In this section, we will study sequences

$$(\omega) = \mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$$

of local trees such that $\mathcal{T}_k$ contracts to some simple local tree, such as (ω) or a linear local tree. First, we define the necessary notions.

3.2. DEFINITION. Let $\mathcal{T} = (T, x_0, R, \Omega)$ be a local tree. We say that $\mathcal{T}$ is a *linear local tree* if it has exactly one principal link and if the tree (T, R) is linear.

3.3. DEFINITIONS. Let $\mathcal{T} = (T, x_0, R, \Omega)$ be a local tree.

(a) A *superfluous vertex* of $\mathcal{T}$ is a vertex $e \in T \setminus (\{x_0\} \cup \mathcal{N}_T(x_0))$ which is linear and which has weight -1.

(b) If e is a superfluous vertex of $\mathcal{T}$ then an *elementary contraction of $\mathcal{T}$ at e* is a local tree $\mathcal{T}' = (T', x_0', R', \Omega')$ together with a root-preserving injective set map $\beta : T' \to T$ such that, if we identify T' with its image in T, the following conditions hold:

$$T' = T \setminus \{e\}$$

$$R' = \begin{cases} \big(R \setminus \{\{e, x\} \mid x \in \mathcal{N}_T(e)\}\big) \cup \{\mathcal{N}_T(e)\}, & \text{if } |\mathcal{N}_T(e)| = 2 \\ R \setminus \{\{e, x\} \mid x \in \mathcal{N}_T(e)\}, & \text{if } |\mathcal{N}_T(e)| = 1, \end{cases}$$

$$\Omega'(x) = \begin{cases} \Omega(x) + 1, & \text{if } x \in \mathcal{N}_T(e) \\ \Omega(x), & \text{if } x \in T \setminus (\{x_0, e\} \cup \mathcal{N}_T(e)). \end{cases}$$

In other words, an elementary contraction of $\mathcal{T}$ at e can be obtained as follows: first, forget that x_0 is the root and assign an arbitrary weight to that vertex; then $\mathcal{T}$ becomes a weighted tree and e is a superfluous vertex of that tree; blow-down $\mathcal{T}$ at e; forget the weight of x_0 and remember that x_0 is the root. The local tree so obtained (together with the set map which came with the blowing-down) is an elementary contraction of $\mathcal{T}$ at e. Notice that the elementary contraction of $\mathcal{T}$ at e is unique, up to isomorphism commuting with the "underlying set maps".

(c) A *contraction* of $\mathcal{T}$ is a local tree $\mathcal{T}' = (T', x_0', R', \Omega')$ together with a set map $\beta : T' \to T$, such that either β is an isomorphism or the following condition holds:

There exist local trees $\mathcal{T}_0, \dots, \mathcal{T}_k$ and maps $\beta_1, \dots, \beta_k$ ($k \geq 1$) such that $\mathcal{T}_0 = \mathcal{T}$, $\mathcal{T}_k = \mathcal{T}'$, $(\mathcal{T}_i, \beta_i)$ is an elementary contraction of $\mathcal{T}_{i-1}$ at some superfluous vertex ($1 \leq i \leq k$), and $\beta = \beta_1 \circ \cdots \circ \beta_k$.

In particular, we see that β is a root-preserving injective map and that β restricts to a bijection of the sets of principal vertices (we say that the two trees have the same principal vertices and principal links). A contraction as above will be denoted by $\mathcal{T}' \leq \mathcal{T}$ or $\mathcal{T} \geq \mathcal{T}'$, and we will say that $\mathcal{T}$ contracts to $\mathcal{T}'$.

Observe that the set map $T' \to T$ determined by a contraction $\mathcal{T}' \leq \mathcal{T}$ allows us to identify $\{x_0'\} \cup R'^0$ with $\{x_0\} \cup R^0$. Thus we can compare multiplicity maps for the two trees:

(d) For m-trees $(\mathcal{T}, \mu)$ and $(\mathcal{T}', \mu')$, we define $(\mathcal{T}, \mu) \geq (\mathcal{T}', \mu') \iff \mathcal{T} \geq \mathcal{T}'$ and $\mu = \mu'$.

3.3.1. REMARK. We deliberately avoided the term "blowing-down" for local trees, to emphasise that the contraction is not the inverse operation of blowing-up (for blowings-up happen at the root, while contractions occur away from the root). Contractions should be thought as phenomenons that do not affect things which are local to the root, such as multiplicity maps. Indeed, if we let the notation be as in (2.1) and if E is an irreducible component of D which is a rational curve and a superfluous vertex of $\mathcal{T}$, then (by Castelnuovo's criterion for contracting a curve) the elementary contraction of $\mathcal{T}$ at E corresponds to the contraction of the curve E. More precisely, there is a monoidal transformation $\rho : S \to S'$, where S' is a nonsingular projective surface and $\rho(E)$ is a point P' of S'. Now let $\rho_* : \mathrm{Div}(S) \to \mathrm{Div}(S')$ be the homomorphism defined by $\rho_*(E) = 0$ and $\rho_*(\Gamma) = \rho(\Gamma)$ (any curve Γ other than E). Let $C' = \rho_*(C)$ and $D' = \rho_*(D)$, then (P', C', D', S') satisfies the conditions of (2.1) and determines an m-tree $(\mathcal{T}', \mu')$ such that $(\mathcal{T}, \mu) \geq (\mathcal{T}', \mu')$. Indeed, by definition of superfluous vertex, ρ is an isomorphism in a neighbourhood of P and the multiplicities are not affected by the contraction of E.

The next fact is an easy consequence of the definitions; we omit its proof.

3.4. LEMMA. *Let $\mathcal{T} = (T, x_0, R, \Omega)$, $\mathcal{T}' = (T', x_0', R', \Omega')$ and $\mathcal{T}'' = (T'', x_0'', R'', \Omega'')$ be local trees such that $\mathcal{T}' \leq \mathcal{T}$ and $\mathcal{T}'' \leq \mathcal{T}$. Then the following are equivalent:*

(a) The maps $T' \to T$ and $T'' \to T$ have the same image.

(b) There exists an isomorphism $\mathcal{T}' \cong \mathcal{T}''$ that commutes with $T' \to T$ and $T'' \to T$, i.e., the two contractions are essentially the same.

REMARK. By (3.4), we see that it is legitimate to refer to a contraction process by specifying which vertices disappear and which survive. In view of that, let us adopt the following language:

Let $\mathcal{T}$ be a local tree, v a vertex of $\mathcal{T}$ other than the root and B a branch of $\mathcal{T}$ at v, not containing the root. Suppose that $\mathcal{T} \geq \mathcal{T}' = (T', x_0', R', \Omega')$, where $T' = T \setminus B$ (after identification of T' with its image in T). We refer to that situation by saying that B *is absorbed by* v or that v *absorbs* B.

A lemma analogous to (3.4) can be proved for weighted graphs. So we can use that language for weighted graphs as well.

3.5. DEFINITION. Let ω, i, i' be positive integers. A *sequence of type* (ω, i, i') is a finite sequence of positive integers, of the form

$$m_0, \ldots, m_0, i_1, \ldots, i_1, m_1, \ldots, m_{l-2}, i_{l-1}, \ldots, i_{l-1}, m_{l-1}, \ldots, m_{l-1}, i_l, \ldots, i_l$$

where $l \geq 1$,
$m_{\nu-1}$ occurs ω times $(1 \leq \nu \leq l)$,
i_ν occurs $2n_\nu$ times, for some $n_\nu \in \mathrm{N}$ $(1 \leq \nu \leq l-1)$,
i_l occurs n_l times, for some $n_l \in \mathrm{N}$,
and such that the following conditions hold (where we define $i_0 = i$):

(a) $i_l = i'$

(b) $m_{\nu-1} = n_\nu i_\nu, \quad 1 \leq \nu \leq l$

(c) $i_{\nu-1} = \omega m_{\nu-1} + i_\nu, \quad 1 \leq \nu \leq l.$

REMARKS.

1. Consider a sequence of type (ω, i, i'), with notation as above. Then:
   (a) $i_0 > m_0 \geq i_1 > m_1 \geq \cdots \geq i_{l-1} > m_{l-1} \geq i_l$
   (b) $i_{\nu-1} = (\omega n_\nu + 1) i_\nu, \quad 1 \leq \nu \leq l$
   (c) $(i_{\nu-1}, m_{\nu-1}) = i_\nu$ (g.c.d.), $\quad 1 \leq \nu \leq l.$
2. Given a positive integer ω, let $\omega\mathrm{N} + 1$ be the set $\{\omega x + 1 \mid x \in \mathrm{N}\}$. Given any number x let $S_\omega(x)$ be the set of nonempty finite sequences $(x_1, x_2, \ldots)$ in $\omega\mathrm{N} + 1$ such that $\Pi_{i \geq 1} x_i = x$. Then $S_\omega(x) \neq \emptyset \Leftrightarrow x \in \omega\mathrm{N} + 1$. Moreover, if we fix a triple (ω, i, i') of positive integers then:
   To give a sequence of type (ω, i, i') is equivalent to giving an element of $S_\omega(i/i')$.
   In fact, if s is a sequence of type (ω, i, i'), with notation as in the definition, then s determines the following element of $S_\omega(i/i')$:
   $$(\omega n_1 + 1, \omega n_2 + 1, \ldots, \omega n_l + 1),$$
   and this is a bijection.

3.6. LEMMA. *Let ω be a positive integer and let*

$$S: \quad \mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k \quad (k \geq 0)$$

be a sequence of local trees, such that $\mathcal{T}_0$ has one principal link a. Then the following are equivalent:

(a) $\exists \mu = (\mu_0, \ldots, \mu_k) \in Mul(S)$ (see (1.11)) such that, if we write $i = \mu_0(a)$ and $r_\nu = \mu_\nu(x_0)$ $(0 \leq \nu \leq k-1)$, then $(r_0, \ldots, r_{k-1})$ is a sequence of type (ω, i, i'), for some i'.

(b) $\forall \mu = (\mu_0,\dots,\mu_k) \in \mathrm{Mul}(S)$*, if we write* $i = \mu_0(a)$ *and* $r_\nu = \mu_\nu(x_0)$ $(0 \le \nu \le k-1)$*, then* $(r_0,\dots,r_{k-1})$ *is a sequence of type* (ω, i, i')*, for some* i'*.*

Moreover, if these equivalent conditions are met then $k \ge \omega + 1$*,* T_k *has one principal link* a'*,* $\mu_k(a') = i'$ *(in the notation of (a) or (b)) and the principal vertex of* T_k *is a branch point.*

PROOF: Since $\mathrm{Mul}(S) \neq \emptyset$, (b)$\Rightarrow$(a) is trivial. If (a) holds, then $k \ge \omega + 1$ by (3.5), and the last three assertions follow from (3.5) and (1.12). Thus (b) holds, by (1.8).

3.7. DEFINITION. Let ω be a positive integer and let $S: \quad T_0 \leftarrow \cdots \leftarrow T_k$ be a sequence of local trees. We say that S *is of type* ω if T_0 has one principal link and if the equivalent conditions of (3.6) are met. When that is the case, we have in particular $k \ge \omega + 1$, T_k has one principal vertex and that vertex is a branch point of T_k.

REMARKS.

(a) If $S: \quad T_0 \leftarrow \cdots \leftarrow T_k$ is of type ω, $\mu \in \mathrm{Mul}(S)$ and if the notation of (1.13) is used for the numbers $i_0,\dots,i_l, m_0,\dots,m_{l-1}$, then the sequence $(\mu_\iota(x_0))_{\iota=0,\dots,k-1}$ looks exactly as in (3.5).

(b) If S is as in (a), then the numbers $n_1,\dots,n_l$ of (3.5) are completely determined by S. Indeed, if $\mu, \mu' \in \mathrm{Mul}(S)$ then by (1.8) there is a nonzero rational number q such that $q(\mu_0,\dots,\mu_{k-1}) = (\mu'_0,\dots,\mu'_{k-1})$.

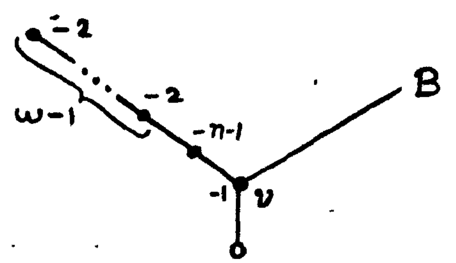
We are now ready to state the first significant result, in the theory of local trees. It gives the solution to the problem mentioned in (3.1).

3.8. THEOREM. *Let* ω *and* k *be positive integers and let* $S: \quad T_0 \leftarrow \cdots \leftarrow T_k$ *be a sequence of local trees such that* $T_0 \ge (\omega)$*,* T_k *has one principal vertex and that vertex is a branch point of* T_k*. Then the following are equivalent:*

(a) T_k *contracts to a linear local tree,*

(b) S *is of type* ω*.*

Moreover, if these conditions are satisfied then T_k *has the form*



where n *is the positive integer* n_l *of definition (3.5),* v *is the principal vertex of* T_k *and* $\mathcal{B}$ *is a branch that* v *can absorb. Moreover,* v *gets weight* 0 *after absorption of* $\mathcal{B}$*. As a consequence, if*

$$T_k \leftarrow T_{k+1} \leftarrow \cdots \leftarrow T_{k+n}$$

is the (unique) sequence such that T_{k+i} *has one principal link* $(0 \le i \le n)$*, then* $T_{k+n} \ge (\omega)$*.*

REMARKS.

- Before going through the proof, it might be a good idea to read (III.1.11), which is an application of (3.8) to Geometry.
- Since contractions do not change the number of principal links of a local tree, it certainly makes sense to assume, in (3.8), that $\mathcal{T}_k$ has one principal vertex. However, the assumption that that vertex is a branch point is there only to make the conclusion simpler; when we do have to cope with a sequence $\mathcal{S}$ such that the principal vertex of $\mathcal{T}_k$ is not a branch point, (3.8) gives a description of the nontrivial part of $\mathcal{S}$, say $\mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_{k_1}$, and $\mathcal{T}_{k_1} \leftarrow \cdots \leftarrow \mathcal{T}_k$ is trivial (i.e., every tree in it has one principal link).

Before we can prove the theorem, we need to introduce some notions and prove some facts.

3.9. DEFINITION. If $\alpha \in \mathbb{Z}$ and $\mathcal{T}$ is a local tree, let $\mathcal{T}[\alpha]$ be the weighted tree obtained from $\mathcal{T}$ by assigning the weight α to the root. We have the following properties:

(a) If $\mathcal{T} \geq \mathcal{T}'$ then $\mathcal{T}[\alpha]$ contracts to $\mathcal{T}'[\alpha]$, for all $\alpha \in \mathbb{Z}$.

(b) If $\mathcal{T} \leftarrow \mathcal{T}'$ and $|\mathcal{N}_{\mathcal{T}}(x_0)| \leq |\mathcal{N}_{\mathcal{T}'}(x_0)| = 1$ then $\mathcal{T}[\alpha] \sim \mathcal{T}'[\alpha - 1]$, for all $\alpha \in \mathbb{Z}$.

3.10. DEFINITION. A local tree $\mathcal{T}$ is *minimal* if it has no superfluous vertex.

3.11. LEMMA. *Let $\mathcal{T}$ be a local tree that contracts to a linear local tree. If $\mathcal{M}$ is a minimal local tree such that $\mathcal{M} \leq \mathcal{T}$, then $\mathcal{M}$ is linear.*

PROOF: Let $\mathcal{L}$ be a linear local tree such that $\mathcal{L} \leq \mathcal{T}$. We regard $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{T}$ as having the same root x_0 and the same principal vertex v.

Given $i \in \mathbb{Z}$, let $\mathcal{T}_i$ (resp. $\mathcal{L}_i$, $\mathcal{M}_i$) be the local tree obtained from $\mathcal{T}$ (resp. $\mathcal{L}$, $\mathcal{M}$) by increasing by i the weight of v. Then clearly $\mathcal{T}_i \geq \mathcal{L}_i$, where $\mathcal{L}_i$ is linear, and $\mathcal{T}_i \geq \mathcal{M}_i$, where $\mathcal{M}_i$ is minimal; also, $\mathcal{M}$ is linear iff $\mathcal{M}_i$ is linear, i.e., it is enough to prove that $\mathcal{M}_i$ is linear. Whence we may assume that, in $\mathcal{T}$, the weight of v is nonnegative. That assumption being in force, consider the weighted trees $\mathcal{T}[0]$, $\mathcal{L}[0]$ and $\mathcal{M}[0]$. Then $\mathcal{M}[0] \sim \mathcal{L}[0]$ by (3.9a), and $\mathcal{L}[0]$ is a linear weighted tree. Hence $\mathcal{M}[0]$ contracts to a linear weighted tree by (I.4.13). If the weighted tree $\mathcal{M}[0]$ has a superfluous vertex, then it is neither x_0 (which has weight 0) nor v (which must have nonnegative weight, by our assumption on $\Omega_{\mathcal{T}}(v)$); thus it is a superfluous vertex of the local tree $\mathcal{M}$, which is impossible. Therefore $\mathcal{M}[0]$ is a minimal weighted tree, so it is a linear weighted tree. We conclude that $\mathcal{M}$ is a linear local tree.

3.12. DEFINITION. A local tree $\mathcal{T}$ is *universally minimal* (we will write "$\mathcal{T}$ is UM") if for every sequence

$$\mathcal{T} = \mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k \qquad (k \geq 0),$$

$\mathcal{T}_k$ is minimal. Observe that if $\mathcal{T}$ is universally minimal then it is minimal, and $\mathcal{T}'$ is UM whenever $\mathcal{T} \leftarrow \mathcal{T}'$.

3.13. LEMMA. *Let $\mathcal{T}$ be a local tree. Then the following are equivalent:*

(a) $\mathcal{T}$ is UM

(b) $\mathcal{T}$ is minimal and every linear principal vertex of $\mathcal{T}$ has negative weight.

PROOF: Clearly, if $\mathcal{T}$ satisfies (b) and $\mathcal{T} \leftarrow \mathcal{T}'$ then $\mathcal{T}'$ satisfies (b); so (b)$\Rightarrow$(a) is trivial. For the converse, we prove that "not (b)" implies "not (a)". So assume (b) does not hold. If $\mathcal{T}$ is not minimal then we are done; so let's assume that $\mathcal{T}$ is minimal. Then $\mathcal{T}$ has a linear principal vertex v with weight $n \geq 0$; let a be the corresponding principal link and define a multiplicity map μ for $\mathcal{T}$ by

$$\begin{cases} \mu(x_0) = 1 = \mu(a') & (a' \in R^0 \setminus \{a\}) \\ \mu(a) = n+1. \end{cases}$$

Then we have $(\mathcal{T}, \mu) \Leftarrow (\mathcal{T}', \mu')$ for some (uniquely determined) m-tree $(\mathcal{T}', \mu')$; if $n = 0$ then v is a superfluous vertex in $\mathcal{T}'$, so we are done. Suppose $n > 0$. If $\mathcal{T}'$ is not minimal, we are done; if $\mathcal{T}'$ is minimal then v is a linear principal vertex of $\mathcal{T}'$, with weight $n-1$. So we are done by induction on n.

3.14. LEMMA. *Suppose that* $\mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$ $(k \geq 1)$ *and that* $\mathcal{T}_0 \geq \mathcal{T}_0'$. *Then there is a unique diagram*

$$\begin{array}{ccccccc} \mathcal{T}_0 & \leftarrow & \mathcal{T}_1 & \leftarrow & \cdots & \leftarrow & \mathcal{T}_k \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & & & \rotatebox{90}{$\leq$} \\ \mathcal{T}_0' & \leftarrow & \mathcal{T}_1' & \leftarrow & \cdots & \leftarrow & \mathcal{T}_k' \end{array}$$

such that the underlying diagram of set maps is commutative. (By "unique", we mean unique up to isomorphisms commuting with all maps.)

PROOF: Since all maps are injective, we may assume $k = 1$. Let T_i (resp. T_i') be the set of vertices of $\mathcal{T}_i$ (resp. $\mathcal{T}_i'$) for $i = 0, 1$.

Uniqueness. Suppose we have a diagram as in the statement (with $k = 1$). Consider the underlying diagram of set maps

$$\begin{array}{ccc} T_0 & \xrightarrow{\beta} & T_1 \\ \beta_0 \uparrow & & \beta_1 \uparrow \\ T_0' & \xrightarrow[\beta']{} & T_1' \end{array}$$

and write $T_1 = \beta(T_0) \cup \{e\}, T_1' = \beta'(T_0') \cup \{e'\}$. Since e is a principal vertex of $\mathcal{T}_1$, $e \in \beta_1(T_1')$. On the other hand, $\beta'^{-1}(\beta_1^{-1}(e)) = \beta_0^{-1}(\beta^{-1}(e)) = \beta_0^{-1}(\emptyset) = \emptyset$, so $\emptyset \neq \beta_1^{-1}(e) \subseteq T_1' \setminus \beta'(T_0') = \{e'\}$ and $\beta_1(e') = e$. Hence $\beta_1(T_1') = \beta_1(\beta'(T_0') \cup \{e'\}) = \beta(\beta_0(T_0')) \cup \{e\}$, i.e., the image of β_1 is completely determined by $\mathcal{T}_0' \leq \mathcal{T}_0 \leftarrow \mathcal{T}_1$. So uniqueness follows from (3.4).

Existence. We may assume that $\mathcal{T}_0'$ is the elementary contraction of $\mathcal{T}_0$ at some superfluous vertex v. Then, if $\beta : T_0 \to T_1$ is the identification map, $\beta(v)$ is a superfluous vertex of $\mathcal{T}_1$; let $\mathcal{T}_1'$ be the elementary contraction of $\mathcal{T}_1$ at v. Regarding T_i' as a subset of T_i, let $\beta' : T_0' \to T_1'$ be defined by $\beta'(x) = \beta(x)$. One sees that β' is an identification map.

REMARK. Whenever we have a commutative diagram as in (3.14), where the first row is denoted by $\mathcal{S}$ and the second by $\mathcal{S}'$, we have $\mathrm{Mul}(\mathcal{S}) = \mathrm{Mul}(\mathcal{S}')$ (see (1.11)).

3.15. LEMMA. *Let* $\mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$ $(k \geq 1)$ *be such that* $\mathcal{T}_{k-1}$ *has more than one principal link and* $\mathcal{T}_k$ *contracts to a linear local tree. If* $i < k$ *then* $\mathcal{T}_i$ *can't contract to a UM tree (see (3.12)).*

PROOF: Let $i < k$ be such that $\mathcal{T}_i \geq \mathcal{U}$, where $\mathcal{U}$ is UM. Construct a commutative diagram as in (3.14):

$$\begin{array}{ccccccc} & & \mathcal{T}_i & \leftarrow & \cdots & \leftarrow & \mathcal{T}_k \\ & & \rotatebox{90}{$\leq$} & & & & \rotatebox{90}{$\leq$} \\ \mathcal{U} & = & \mathcal{U}_i & \leftarrow & \cdots & \leftarrow & \mathcal{U}_k \end{array}$$

Since $\mathcal{U}_k$ is minimal, $\mathcal{U}_k \leq \mathcal{T}_k$ and $\mathcal{T}_k$ contracts to a linear local tree, (3.11) implies that $\mathcal{U}_k$ is linear. Then clearly $\mathcal{U}_{k-1}$ is linear, which is absurd since $\mathcal{T}_{k-1}$ has more than one principal link and $\mathcal{T}_{k-1} \geq \mathcal{U}_{k-1}$.

3.16. **Definitions.**

(a) A local tree $\mathcal{T}$ is a *comb* if at every vertex v there are at most two branches that don't contain the root, and at most one of them is not a linear branch. (A linear branch is a branch which contains no branch point of $\mathcal{T}$; this means more than being linear as a graph.) In particular, the root is a linear vertex.

(b) If $\mathcal{T}$ is a comb, a *tooth* of $\mathcal{T}$ is a linear branch $\mathcal{A}$ of $\mathcal{T}$, at either a branch point or the root, such that $\mathcal{A}$ doesn't contain the root. So every branch point has at least one tooth (one branch point has two teeth) and, if there are two principal links, the root has at least one tooth.

(c) $\mathcal{T}$ is a *comb with negative teeth* if it is a comb such that

(i) at every branch point there is at least one tooth $\mathcal{A}$ such that $\mathcal{A} < -1$ (I.4.14);

(ii) if $\mathcal{T}$ has two principal vertices, then one of them, say v, has negative weight and belongs to a tooth $\mathcal{A}$ such that $\mathcal{A} \setminus \{v\} < -1$.

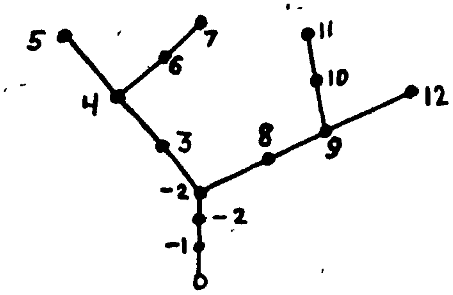
Remark. Every linear local tree is a comb with negative teeth.

3.17. **Lemma.** *Suppose that either $\mathcal{T} \leftarrow \mathcal{T}'$ or $\mathcal{T} \geq \mathcal{T}'$. If $\mathcal{T}$ is a comb (resp. a comb with negative teeth) then so is $\mathcal{T}'$.*

Proof omitted (easy).

3.18. **Definition.** We are now going to define a notation that we will use to avoid drawing pictures of local trees. We do this for practical reasons only, and we suggest that the reader reconstructs all pictures whenever he encounters these notations. To give an example, the local tree

$\mathcal{T}$:

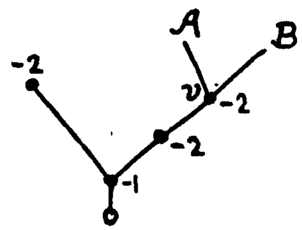


will be denoted as $\mathcal{T} = (*, -1, -2, -2, (3, 4, (5), (6, 7)), (8, 9, (10, 11), (12)))$. To formalise the notation, let $\mathcal{T}$ be either a local tree or a weighted tree with a root (i.e., a distinguished vertex), let ρ be the weight of the root (with $\rho = *$ if $\mathcal{T}$ is a local tree) and suppose that, for each vertex v, the set of branches (of $\mathcal{T}$ at v) that don't contain the root has been totally ordered. In particular, let $\mathcal{B}_1, \dots, \mathcal{B}_n$ ($n \geq 0$) be the branches of $\mathcal{T}$ at the root. Then the tree $\mathcal{T}$ will be denoted by the symbols $([\mathcal{T}])$, where $[\mathcal{T}]$ is the sequence of symbols defined by

$$[\mathcal{T}] = \begin{cases} \rho & \text{if } n = 0 \\ \rho, [\mathcal{B}_1] & \text{if } n = 1 \\ \rho, ([\mathcal{B}_1]), \dots, ([\mathcal{B}_n]) & \text{if } n > 1. \end{cases}$$

This makes sense, since each $\mathcal{B}_i$ is itself a weighted tree with a root (the root being the neighbour of the root of $\mathcal{T}$), with an ordering for each appropriate set of branches, etc. Clearly, the notation $([\mathcal{T}])$ determines the isomorphism class of $\mathcal{T}$, independently of the choices of orderings for the sets of branches.

Now that we have a well-defined notation, we will abuse it. Given a local tree

$\mathcal{T}$:  where $\mathcal{A}, \mathcal{B}$ are branches at v,

we will write $\mathcal{T} = (*, -1, (-2), (-2, -2, ([\mathcal{A}]), ([\mathcal{B}])))$ instead of "$\mathcal{T}$ is denoted by $(*, -1, (-2), (-2, -2, ([\mathcal{A}]), ([\mathcal{B}])))$". This amounts to identify $\mathcal{T}$ and $([\mathcal{T}])$; doing the same thing with $\mathcal{A}$ and $\mathcal{B}$, i.e., writing $\mathcal{A} = ([\mathcal{A}])$ and $\mathcal{B} = ([\mathcal{B}])$, we get

$$\mathcal{T} = (*, -1, (-2), (-2, -2, \mathcal{A}, \mathcal{B})).$$

PROOF OF (3.8):

Reduction to the case $\mathcal{T}_0 = (\omega)$. Let $\mathcal{T}_0' = (\omega)$ and form the commutative diagram determined by the sequence $\mathcal{S}$ and $\mathcal{T}_0' \le \mathcal{T}_0$, as in (3.14):

$$\begin{array}{ccccccccc} & & \mathcal{T}_0 & \leftarrow & \mathcal{T}_1 & \leftarrow & \cdots & \leftarrow & \mathcal{T}_k \\ & & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & & & \rotatebox{90}{$\le$} \\ (\omega) & = & \mathcal{T}_0' & \leftarrow & \mathcal{T}_1' & \leftarrow & \cdots & \leftarrow & \mathcal{T}_k' \end{array}$$

Let v be the principal vertex of $\mathcal{T}_k$; then v is the principal vertex of $\mathcal{T}_k'$ as well. Since it is a branch point of $\mathcal{T}_k$, $\mathcal{T}_{k-1}$ (which exists since $k > 0$) must have at least two principal links; now $\mathcal{T}_0$ has one principal link $\Rightarrow 0 \neq k-1 \Rightarrow \mathcal{T}_{k-2}$ exists $\Rightarrow \mathcal{T}_{k-1}$ has exactly two principal links, by (1.7a). Hence $\mathcal{T}_{k-1}'$ has exactly two principal links and (since $\mathcal{T}_k'$ has one) v must be a branch point of $\mathcal{T}_k'$; more precisely, $\mathcal{T}_k'$ has three branches at v (and the same is true for $\mathcal{T}_k$). Moreover, v has weight -1 in $\mathcal{T}_k$ (resp. in $\mathcal{T}_k'$) since it was created in $\mathcal{T}_{k-1} \leftarrow \mathcal{T}_k$ (resp. $\mathcal{T}_{k-1}' \leftarrow \mathcal{T}_k'$).

Let $\mathcal{S}'$ be the sequence $\mathcal{T}_0' \leftarrow \cdots \leftarrow \mathcal{T}_k'$ and consider the following conditions on $\mathcal{S}'$:

(a') $\mathcal{T}_k'$ contracts to a linear local tree;

(b') $\mathcal{S}'$ is of type ω.

Then (a')$\Rightarrow$(a) is trivial and (a)$\Rightarrow$(a') is an immediate consequence of (3.11), so (a)$\Leftrightarrow$(a'). By the remark which follows (3.14), (b)$\Leftrightarrow$(b') is trivial and the sequences $\mathcal{S}$ and $\mathcal{S}'$ determine the same integer n_l. Next, assume that $\mathcal{T}_k'$ has the form prescribed by the theorem, i.e., $\mathcal{T}_k' = (*, -1, \mathcal{A}', \mathcal{B}')$, where $\mathcal{A}' = (-n-1, -2, \ldots, -2)$ contains ω vertices (and $n = n_l$) and $\mathcal{B}'$ is a branch that v can absorb, the weight of v being increased by one by that contraction. By the remarks that we made immediately after the commutative diagram, $\mathcal{T}_k = (*, -1, \mathcal{A}, \mathcal{B})$, where $\mathcal{A}$ contains $\mathcal{A}'$ and $\mathcal{B}$ contains $\mathcal{B}'$ (as sets). Hence, in $\mathcal{T}_k$, v can absorb $\mathcal{B}$ and that process increases by 1 the weight of v (this is because $\mathcal{B}$ must contract to $\mathcal{B}'$ when $\mathcal{T}_k$ contracts to $\mathcal{T}_k'$). On the other hand, the fact that $\mathcal{T}_0$ has one principal link implies that one of the two branches of $\mathcal{T}_{k-1}$ at x_0 consists of vertices that were created in $\mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_{k-1}$. Whence, for some $\mathcal{X} \in \{\mathcal{A}, \mathcal{B}\}$, $\mathcal{X} < -1$. Since $\mathcal{B}$ can be absorbed by v, $\mathcal{X} \neq \mathcal{B}$. Thus $\mathcal{X} = \mathcal{A}$, so $\mathcal{A} < -1$ and no vertex of $\mathcal{A}$ can disappear in the contraction $\mathcal{T}_k \ge \mathcal{T}_k'$, which means that $\mathcal{A}$ is just $\mathcal{A}'$ (even the weights are the same), i.e., $\mathcal{T}_k$ has the desired form. This completes the proof of the reduction.

So we assume that $\mathcal{T}_0 = (\omega)$. We will prove that (a) implies both (b) and the other assertion (i.e., the description of $\mathcal{T}_k$). We believe that, after that proof, (b)$\Rightarrow$(a) will be obvious. The assertion about the sequence $\mathcal{T}_k \leftarrow \cdots \leftarrow \mathcal{T}_{k+n}$, i.e., that $\mathcal{T}_{k+n} \geq (\omega)$, is easily verified.

Suppose that (a) holds, i.e., $\mathcal{T}_k$ contracts to some linear local tree. Using the notation of (1.13), we write

$$\mathcal{J} = \mathcal{J}(S) = \{j_0, \ldots, j_{l-1}\}$$
$$\mathcal{H} = \mathcal{H}(S) = \{h_1, \ldots, h_l\}$$

where, clearly, $l \geq 1$ (for l is the number of branch points of $\mathcal{T}_k$). We proceed by induction on l.

Case l = 1. Then v is the only branch point of $\mathcal{T}_k$. Let L denote the principal vertex of $\mathcal{T}_0$. Since L is a free vertex of $\mathcal{T}_0$ it is a free vertex of $\mathcal{T}_k$. Thus $\mathcal{T}_k = (*, -1, \mathcal{A}, \mathcal{B})$ where $\mathcal{A}$ and $\mathcal{B}$ are linear branches at v and L is in $\mathcal{B}$ (say). Since $\mathcal{T}_k$ contracts to a linear tree and since, in $\mathcal{T}_k$, every vertex other than x_0, v, L has weight less than -1 we must have $\mathcal{B} = (-2, \ldots, -2, -1)$; let $n > 0$ be the number of vertices of $\mathcal{B}$. Then one easily figures out that S begins as

$$\mathcal{T}_0 = (*, \omega) \leftarrow (*, (-1), (\omega - 1)) \leftarrow \cdots \leftarrow (*, (-1, -2, \ldots, -2), (0)) = \mathcal{T}_\omega,$$

and continues

$$\begin{aligned}\mathcal{T}_{\omega+1} = (*, (-2, \ldots, -2), (-1, -1)) \leftarrow \cdots \leftarrow (*, (-n, -2, \ldots, -2), (-1, -2, \ldots, -2, -1))\\ \leftarrow (*, -1, (-n-1, -2, \ldots, -2), (-2, \ldots, -2, -1)) = \mathcal{T}_{\omega+n} = \mathcal{T}_k.\end{aligned}$$

We leave it to the reader to check that S is of type ω and that $\mathcal{T}_k$ has the desired form (with, in particular, $n = n_l = n_1$).

Inductive step. Assume $l > 1$. As in (1.13), let e_ν be the branch point created in $\mathcal{T}_{h_\nu - 1} \leftarrow \mathcal{T}_{h_\nu}$. In particular, e_{l-1} is the principal vertex of $\mathcal{T}_{h_{l-1}} = (*, -1, \mathcal{A}', \mathcal{B}')$ where $\mathcal{A}'$ and $\mathcal{B}'$ are branches at e_{l-1}. We have $h_l = k$, so e_l is the principal vertex of $\mathcal{T}_k = (*, -1, \mathcal{A}, \mathcal{B})$, where $\mathcal{A}, \mathcal{B}$ are branches at e_l and $\mathcal{B} = (b_1, \ldots, b_s, \epsilon, \mathcal{A}', \mathcal{B}')$ contains e_{l-1} (more precilely, $s \geq 0$, ϵ is the weight of e_{l-1} in $\mathcal{T}_k$ and the branches $\mathcal{A}', \mathcal{B}'$ at e_{l-1} are identical to what they were in $\mathcal{T}_{h_{l-1}}$).

Observe that, by (3.17) and the remark immediately before it, $\mathcal{T}_i$ is a comb with negative teeth ($0 \leq i \leq k$). Since $\mathcal{B}$ is not a linear branch of $\mathcal{T}_k$, it follows that $\mathcal{A}$ is a linear branch and $\mathcal{A} < -1$. Moreover, since every vertex in the simple chain $(e_l, \ldots, e_{l-1})$, except e_{l-1}, was created in $\mathcal{T}_{h_{l-1}} \leftarrow \cdots \leftarrow \mathcal{T}_k$, the weights $b_1, \ldots, b_s$ are less than -1.

Consider a sequence of elementary contractions that realises the contraction of $\mathcal{T}_k$ to a linear local tree. Since $\mathcal{A} < -1$, that contraction is nothing else than the absorption of $\mathcal{B}$ by e_l. Since $b_1, \ldots, b_s$ are less than -1, one sees that e_{l-1} must disappear before any other vertex of the simple chain $(e_l, \ldots, e_{l-1})$. Before e_{l-1} can disappear, it has to become a superfluous vertex, and in particular a linear vertex. Thus e_{l-1} must absorb either $\mathcal{A}'$ or $\mathcal{B}'$ before anything else happens. Clearly, e_{l-1} can absorb a branch in $\mathcal{T}_{h_{l-1}}$ as well, and

$$\mathcal{T}_{h_{l-1}} \text{ *contracts to a linear local tree.*}$$

Applying the inductive hypothesis to S_{l-1} : $\mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_{h_{l-1}}$, we conclude that it is a sequence of type ω and that $\mathcal{T}_{h_{l-1}}$ contracts to $\mathcal{T}'_{h_{l-1}} = (*, 0, -m-1, -2, \ldots, -2)$, where "$-2$" occurs $\omega - 1$ times and $m = n_{l-1}$. Construct the commutative diagram (see (3.14)).

$$\begin{array}{ccccc} \mathcal{T}_{h_{l-1}} & \leftarrow & \cdots & \leftarrow & \mathcal{T}_k \\ \vee\!\!/ & & & & \vee\!\!/ \\ \mathcal{T}'_{h_{l-1}} & \leftarrow & \cdots & \leftarrow & \mathcal{T}'_k \end{array}$$

Let $\alpha = j_{l-1} - h_{l-1} \geq 0$. If $\alpha = 0$ then $\mathcal{T}'_{h_{l-1}+1}$ is UM by (3.13); since $h_{l-1} + 1 = j_{l-1} + 1 < k$ (for $\mathcal{T}_{j_{l-1}+1}$ has two principal vertices by definition of j_{l-1}), this contradicts (3.15).

Hence $\alpha > 0$. Notice that $\mathcal{T}'_{j_{l-1}+1} = (*, (-1), (-2, \ldots, -2, -1, -m-1, -2, \ldots, -2))$, where the first sequence of "-2" contains α terms and the second has $\omega - 1$ terms. That contracts to $\mathcal{T}''_{j_{l-1}+1} = (*, (-1), (-1, \alpha - m - 1, -2, \ldots, -2))$ which can't be UM by (3.15). By (3.13), that tree is not minimal, and we have $\alpha = m$. We conclude that $\mathcal{T}'_{j_{l-1}} = (*, -1, -2, \ldots, -2, -1, -m-1, -2, \ldots, -2)$, where there are $\alpha - 1 = m - 1$ terms in the first sequence of "-2", and $\omega - 1$ in the second. Hence that tree contracts to (ω), and so does $\mathcal{T}_{j_{l-1}}$. Applying the inductive hypothesis (or the case $l = 1$) to S_l : $\mathcal{T}_{j_{l-1}} \leftarrow \cdots \leftarrow \mathcal{T}_k$, we see that it is of type ω and that $\mathcal{T}_k$ has the desired form. Since $\alpha = m = n_{l-1}$ and S_{l-1} is of type ω, one sees that S is of type ω. This completes the proof of (3.8).

3.19. DEFINITION. Let $\mathcal{T}$ be a local tree, v a vertex of $\mathcal{T}$ (other than the root) and $\alpha \in \mathbf{Z}$. Then $\mathcal{T}^{v,\alpha}$ denotes the local tree obtained from $\mathcal{T}$ by adding a free vertex of weight α, linked to v (and to no other vertex). That extra vertex will sometimes be called "the extra vertex". Clearly, we have the following facts:

(a) If $\mathcal{T} \leftarrow \mathcal{T}'$ then $\mathcal{T}^{v,\alpha} \leftarrow \mathcal{T}'^{v,\alpha}$

(b) If $\mathcal{T} \geq \mathcal{T}'$ and v is in $\mathcal{T}'$ then $\mathcal{T}^{v,\alpha} \geq \mathcal{T}'^{v,\alpha}$.

If $\alpha_1, \ldots, \alpha_p \in \mathbf{Z}$ $(p \geq 0)$ we can define $\mathcal{T}^{v,\alpha_1,\ldots,\alpha_p} = (\cdots(\mathcal{T}^{v,\alpha_1})\cdots)^{v,\alpha_p}$. Then assertions (a) and (b), above, are true if we replace every "α" by "$\alpha_1, \ldots, \alpha_p$".

3.20. DEFINITION. Given local trees $\mathcal{T}, \mathcal{T}'$, the symbol $\mathcal{T} \overset{+}{\leftarrow} \mathcal{T}'$ indicates that we have chosen a map β', from the set of vertices of $\mathcal{T}$ to that of $\mathcal{T}'$, satisfying the following condition:

There exist a local tree $\mathcal{T}_1$ and a blowing-up $\mathcal{T} \leftarrow \mathcal{T}_1$ such that, if e is the vertex created in that blowing-up, then $\mathcal{T}' = \mathcal{T}_1^{e,\alpha_1,\ldots,\alpha_p}$ for some $\alpha_1, \ldots, \alpha_p \in \mathbf{Z}$ $(p \geq 1)$ and β' is the composition of the identification map of $\mathcal{T} \leftarrow \mathcal{T}_1$ with the inclusion of $\mathcal{T}_1$ in $\mathcal{T}'$.

3.21. REMARK. The following comments explain how the rest of this section is related to Geometry. Suppose $(\widetilde{P}, C, D, S)$ is as in (2.3), with the following additional assumptions:

(i) $S \setminus \operatorname{supp}(C + D)$ is linear at infinity (I.5.1)

(ii) $C^{(k)} + D^k \in \operatorname{Div}(S_k)$ has *s.n.c.*.

(By (III.1.4), if $S \setminus \operatorname{supp}(C+D) \cong \mathbf{A}^2$ and C is reduced then we can choose $\widetilde{P}$ such that these conditions hold.) Then by (2.4) every blowing-up in the sequence $\mu(\widetilde{P}, C, D, S)$: $(\mathcal{T}_0, \mu_0) \leftarrow \cdots \leftarrow (\mathcal{T}_k, \mu_k)$ is (at least) of the second kind; by, say, (2.2) and (2.4), all blowings-up are of the third kind iff C is irreducible. If C is not irreducible, write $C = C_1 + \cdots + C_n$ where the labelling of the irreducible components $C_1, \ldots, C_n$ of C is such that $\widetilde{P}$ is a place of C_n. To make things simpler, we will also assume that

(iii) All components of C meet at P_1.

This assumption is really unnecessary, but it does simplify this exposition. Then each C_ν with $\nu < n$ eventually "goes away" from $\tilde{P}$, i.e., $\exists i \le k$ such that

$$P_i \in C_\nu^{(i-1)} \quad \text{in } S_{i-1}$$
$$P_{i+1} \notin C_\nu^{(i)} \quad \text{in } S_i.$$

Then $C_\nu^{(i)} + D^i \in \mathrm{Div}(S_i)$ has *s.n.c.* (because of (ii)) and $C_\nu^{(i)}$ meets E_i and no other component of D^i. Hence, if we don't want to loose C_ν, we should consider $\mathcal{T}_i^{E_i,\alpha}$ instead of $\mathcal{T}_i$, where $\alpha = (C_\nu^{(i)})^2$ in S_i. In other words $\mathcal{T}(P_{i+1}, C_\nu^{(i)} + D^i, S_i) = \mathcal{T}_i^{E_i,\alpha}$.

If $C_{\nu_1}, \ldots, C_{\nu_p}$ "go away" from $\tilde{P}$ in $S_{i-1} \leftarrow S_i$, the right tree to consider is $\mathcal{T}_i^{E_i,\alpha_1,\ldots,\alpha_p} = \mathcal{T}(P_{i+1}, C_{\nu_1}^{(i)} + \cdots C_{\nu_p}^{(i)} + D^i, S_i)$, where $\alpha_j = (C_{\nu_j}^{(i)})^2$ in S_i. The passage from $\mathcal{T}_{i-1}$ to $\mathcal{T}_i^{E_i,\alpha_1,\ldots,\alpha_p}$ is not a blowing-up; it is precisely what we denote by "$\overset{\pm}{\leftarrow}$" in (3.20).

So we are led to consider a sequence

$$\mathcal{S}^+ : \qquad \mathcal{T}_0 \leftarrow \mathcal{T}_1 \leftarrow \cdots$$

where, at appropriate places, we have $\cdots \leftarrow \mathcal{T}_{i-1} \overset{\pm}{\leftarrow} \mathcal{T}_i^{E_i,\alpha_1,\ldots,\alpha_p} \leftarrow \mathcal{T}_{i+1}^{E_i,\alpha_1,\ldots,\alpha_p} \leftarrow \cdot$ (i.e. there can be several "$\overset{\pm}{\leftarrow}$" in that sequence—see the notion of *weak sequence* (3.26)). Notice that the last term of that sequence, say $\overline{\mathcal{T}}$, is of the form

$$\overline{\mathcal{T}} = (\cdots(\mathcal{T}_k^{E_{i_1},\alpha_1^1,\ldots,\alpha_{p_1}^1})\cdots)^{E_{i_{n-1}},\alpha_1^{n-1},\ldots,\alpha_{p_{n-1}}^{n-1}}$$

and satisfies $\overline{\mathcal{T}}[\beta] \in \mathcal{G}[S \backslash \mathrm{supp}\,(C+D)]$ (see (I.4.8)), where $\beta = (C_n^{(k)})^2$ in S_k. Notice that $\overline{\mathcal{T}}$ is equivalent to a linear weighted tree, by (i).

If i_1 is the least integer such that some components $C_{\nu_1}, \ldots, C_{\nu_p}$ of C go away from $\tilde{P}$ in $S_{i_1-1} \leftarrow S_{i_1}$ then, for each C_ν that goes away at that stage, $C_\nu^{(i_1)} \cap E_{i_1}$ is some point of $C_\nu^{(i_1)}$ and a place P^ν of C_ν is determined (for $C_\nu^{(i_1)}$ is nonsingular by (ii): no further blowing-up has center *i.n.* $C_\nu^{(i_1)}$). Then $\mu(P^\nu, C_\nu, D, S)$ is contained in

$$(\mathcal{T}_0, \mu_0') \Leftarrow \cdots \Leftarrow (\mathcal{T}_{i_1-1}, \mu_{i_1-1}') \Leftarrow (\mathcal{T}, \mu')$$

for some multiplicity maps $\mu_0', \ldots, \mu_{i_1-1}', \mu'$, where $\mathcal{T} = \mathcal{T}(P^\nu, D^{i_1}, S_{i_1})$. Notice that $k(P^\nu, C_\nu, D, S) \le i_1$.

Generally speaking, the sequence $\mu(P^\nu, C_\nu, D, S)$ carries some information about C_ν, and we would like to understand it better. If, for instance, $\mathcal{T}_0 = (\omega)$ and $\mathcal{T}$ contracts to a linear local tree, then (3.8) can be used to describe the sequence. Can we use the fact that $\overline{\mathcal{T}}$ is equivalent to a linear weighted tree to deduce that $\mathcal{T}$ contracts to a linear local tree? We will see later that it is sometimes possible.

Notice that the tree $\mathcal{T}$ does not actually occur in the sequence $\mathcal{S}^+$, but is related to it as follows:

$$\begin{array}{ccc} & & \mathcal{T} \\ & & \downarrow \\ \mathcal{S}^+ : & \cdots \leftarrow \mathcal{T}_{i_1-1} \overset{\pm}{\leftarrow} & \mathcal{T}_{i_1}^{E_{i_1},\alpha_1,\ldots,\alpha_p} \leftarrow \cdots \end{array}$$

The graph-theoretic situation described by the above diagram is studied in (3.24–3.27), under the assumption that $\mathcal{T}$ does not contract to a linear local tree.

Another fact that the reader should keep in mind is that every vertex of $\mathcal{T}$ (other than the root) is a curve on S_{i_1}, and the same is true for $\mathcal{T}_{i_1}$. Further, if v is a vertex of $\mathcal{T}$ other than the root then, going back to the definitions, we see that v is actually a vertex of $\mathcal{T}_{i_1}$, and hence of $\mathcal{T}_{i_1}^{E_{i_1},\alpha_1,\dots,\alpha_p}$. This observation gives rise to the following (purely graph-theoretic) definition.

3.22. DEFINITION. Let $\mathcal{T} = (T, x, R, \Omega)$ and $\mathcal{T}_i = (T_i, x_i, R_i, \Omega_i)$ ($i = 0, 1$) be local trees and suppose that $\mathcal{T}$ has one principal link and that $\mathcal{T} \to \mathcal{T}_0 \leftarrow \mathcal{T}_1$. Let e (resp. e') be the vertex created in $\mathcal{T}_0 \leftarrow \mathcal{T}_1$ (resp. $\mathcal{T}_0 \leftarrow \mathcal{T}$). We define an injective set map $T \setminus \{x\} \to T_1$ by

$$\begin{cases} e' \mapsto e, \\ t \mapsto \beta_1(\beta^{-1}(t)), \quad t \in T \setminus \{e', x\}, \end{cases}$$

where $\beta_1 : T_0 \to T_1$ and $\beta : T_0 \to T$ are the identification maps. That map should be thought of as a natural embedding of $\mathcal{T}$ in $\mathcal{T}_1$ (or in $\mathcal{T}_1^{e,\alpha_1,\dots,\alpha_p}$, for arbitrary $\alpha_1, \dots, \alpha_p \in \mathbb{Z}$). Observe that the root of $\mathcal{T}$ is not embedded in these trees.

3.23. LEMMA. *Consider local trees $\mathcal{T} \to \mathcal{T}_0 \leftarrow \mathcal{T}_1$, where $\mathcal{T}$ has one principal link. Let e be the vertex created in $\mathcal{T}_0 \leftarrow \mathcal{T}_1$, let $\alpha \in \mathbb{Z}$ and embed $\mathcal{T}$ in $\mathcal{T}_1^{e,\alpha}$ as in (3.22). Let b be a vertex of $\mathcal{T}$, other than the root; then b has the same weight in $\mathcal{T}_1^{e,\alpha}$ as in $\mathcal{T}$. Let $B_1, \dots, B_n$ ($n \geq 0$) be the branches of $\mathcal{T}$ at b, not containing the root. Then the following are true:*

(a) If b is not the principal vertex of $\mathcal{T}$ then the branches of $\mathcal{T}_1^{e,\alpha}$ at b, not containing the root, are $B_1, \dots, B_n$—the same branches, as weighted graphs. $\mathcal{T}_1^{e,\alpha}$ has one more branch B_* at b: B_* contains the root, all principal vertices, the extra vertex $[\alpha]$ and possibly other vertices.

(b) If b is the principal vertex of $\mathcal{T}$ then one of the branches of $\mathcal{T}_1^{e,\alpha}$ at b is (of course) $[\alpha]$. Moreover:

(b1) If $\mathcal{T}_1$ has one principal link then the other branches of $\mathcal{T}_1^{e,\alpha}$ at b, not containing the root, are $B_1, \dots, B_n$, and $\mathcal{T}_1^{e,\alpha}$ has one more branch B_* at b: B_* is just the root.

(b2) If $\mathcal{T}_1$ has two principal links then the other branches of $\mathcal{T}_1^{e,\alpha}$ at b, not containing the root, are $B_1, \dots, B_{n-1}$ (if $B_1, \dots, B_n$ are suitably labelled); $\mathcal{T}_1^{e,\alpha}$ has one more branch B_* at b: B_* consists of the root, together with B_n.

PROOF: We use the notation of (3.22). The map of (3.22) restricts to a bijection $T \setminus \{x, e'\} \cong T_1 \setminus \{x_1, e\}$ which is, in fact, an isomorphism of weighted graphs $\mathcal{T} \setminus \{x, e'\} \cong \mathcal{T}_1 \setminus \{x_1, e\}$. To see that, factor the bijection as

$$T \setminus \{x, e'\} \cong T_0 \setminus \{x_0\} \cong T_1 \setminus \{x_1, e\},$$

and notice that each one of these bijections preserves links, i.e., is an isomorphism of graphs, and that if $t \in T_0 \setminus \{x_0\}$ then

$$\Omega(t) = \Omega_0(t) = \Omega_1(t), \qquad \text{if } t \text{ is not a principal vertex of } \mathcal{T}_0,$$

$$\Omega(t) = \Omega_0(t) - 1 = \Omega_1(t), \qquad \text{if } t \text{ is a principal vertex of } \mathcal{T}_0.$$

Since the principal vertex e' of $\mathcal{T}$ corresponds to the principal vertex e of $\mathcal{T}_1$ and since e (resp. e') was created in $\mathcal{T}_0 \leftarrow \mathcal{T}_1$ (resp. $\mathcal{T}_0 \leftarrow \mathcal{T}$), $b = e'$ has the same weight ($= -1$) in $\mathcal{T}_1^{e,\alpha}$ as in $\mathcal{T}$. This proves (more than) the first assertion.

This being said, the rest of the proof has nothing to do with the weights and, for the rest of this proof, we regard our trees as ordinary graphs, i.e., without weights.

Proof of (a). Let us travel via identification maps; then b corresponds to a vertex of $\mathcal{T}_0$, other than the root, and the branches of $\mathcal{T}$ at b, not containing the root, are the branches of $\mathcal{T}_0$ at b, not containing the root, and these are just the branches of $\mathcal{T}_1$ at b, not containing the root.

Proof of (b). Here, $b = e'$. The branches $B_1, \dots, B_n$ of $\mathcal{T}$ at e', not containing the root, are the branches of $\mathcal{T}_0$ at the root. Let $A \subseteq \mathcal{N}_{\mathcal{T}_0}(x_0)$ be the set determined by $\mathcal{T}_0 \leftarrow \mathcal{T}_1$, as in (1.3b); then A contains n (resp. $n-1$) vertices if $\mathcal{T}_1$ has one (resp. two) principal links. In any case, we may label $B_1, \dots, B_n$ in such a way that A contains one vertex from B_i, $1 \le i \le n-1$. We leave it to the reader to verify that all assertions of (b) are true.

3.24 PROPOSITION. *Consider local trees $\mathcal{T} \to \mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$ ($k \ge 1$) such that $\mathcal{T}$ and $\mathcal{T}_k$ have one principal link and $\mathcal{T}$ does* not *contract to a linear local tree, and suppose that for some $\alpha, \beta \in \mathbf{Z}$*

$$\mathcal{T}_k^{e,\alpha}[\beta] \text{ is equivalent to a linear weighted tree,}$$

where e is the vertex created in $\mathcal{T}_0 \leftarrow \mathcal{T}_1$. Then:

(a) $\alpha = -1$.

(b) The principal vertex of $\mathcal{T}$ is a branch point.

(c) $\mathcal{T}$ contracts to a local tree whose only branch point is its principal vertex.

(d) Given $\alpha \in \mathbf{Z}$, $\mathcal{T}[\alpha]$ is equivalent to a linear weighted tree iff $\alpha = -1$.

PROOF: If $\mathcal{T}$ has a superfluous vertex u that is not a neighbour of a principal vertex, then u is a superfluous vertex of $\mathcal{T}_0$. Let $\mathcal{T}_0'$ be the elementary contraction of $\mathcal{T}_0$ at u and form the commutative diagram:

$$\begin{array}{ccccccc} \mathcal{T} & \to & \mathcal{T}_0 & \leftarrow & \cdots & \leftarrow & \mathcal{T}_k \\ \vee\!\!/ & & \vee\!\!/ & & & & \vee\!\!/ \\ \mathcal{T}' & \to & \mathcal{T}_0' & \leftarrow & \cdots & \leftarrow & \mathcal{T}_k' \end{array}$$

Now $\mathcal{T}'$ doesn't contract to a linear local tree and, as we saw in the proof of (3.14), the vertex created in $\mathcal{T}_0' \leftarrow \mathcal{T}_1'$ is just e. So e doesn't disappear in the contraction $\mathcal{T}_k \ge \mathcal{T}_k'$ and by (3.19) we get $\mathcal{T}_k^{e,\alpha} \ge \mathcal{T}_k'^{e,\alpha}$; so $\mathcal{T}_k^{e,\alpha}[\beta] \sim \mathcal{T}_k'^{e,\alpha}[\beta]$ by (3.9a). Since (3.9a) also says that $\mathcal{T}'[-1] \sim \mathcal{T}[-1]$, it's enough to show that (a)–(d) hold for $\mathcal{T}'$. In other words, we may assume that

(*) *all superfluous vertices of $\mathcal{T}$ are neighbours of principal vertices.*

Suppose the principal vertex of $\mathcal{T}_k$ is neither a branch point nor e. Then $k-1 \ge 1$, $\mathcal{T}_{k-1}$ has one principal link and $\mathcal{T}_{k-1}^{e,\alpha}[\beta+1] \sim \mathcal{T}_k^{e,\alpha}[\beta]$ by (3.9b) (for we have $\mathcal{T}_{k-1}^{e,\alpha} \leftarrow \mathcal{T}_k^{e,\alpha}$ by (3.19)), i.e., k can be decreased. Therefore we may also assume that the principal vertex of $\mathcal{T}_k$ is either a branch point or e, so in any case it's a branch point of $\mathcal{T}_k^{e,\alpha}$, of weight -1. So:

(**) *The principal vertex of $\mathcal{T}_k^{e,\alpha}$ survives to any contraction of $\mathcal{T}_k^{e,\alpha}[\beta]$ to a linear weighted tree.*

Since $\mathcal{T}$ doesn't contract to a linear local tree, it is not a linear local tree; so $\mathcal{T}$ must have a branch point. Let b be a branch point of $\mathcal{T}$, and let $B_1, \dots, B_n$ ($n \ge 2$) be the branches of $\mathcal{T}$ at b, not containing the root. Embed $\mathcal{T}$ in $\mathcal{T}_1^{e,\alpha}$ as in (3.22).

If b is not the principal vertex of $\mathcal{T}$ then by (3.23a) the branches of $\mathcal{T}_k^{e,\alpha}$ at b are $B_1,\ldots,B_n$ and B_*, where B_* contains, in particular, the principal vertex and the root of $\mathcal{T}_k^{e,\alpha}$. Since $\mathcal{T}_k^{e,\alpha}[\beta]$ contracts to a linear weighted tree (I.4.13), b must "absorb" (see the remark following (3.4)) $n-1$ of the $n+1$ branches (of $\mathcal{T}_k^{e,\alpha}[\beta]$ at b) so it must absorb some B_i. This is impossible, because by (*) B_i contains no superfluous vertices (for b is not the principal vertex of $\mathcal{T}$). That means that (b) and (c) are satisfied; clearly, (d) is an immediate consequence of (c) (and (3.9a)).

Notice that, not only does $\mathcal{T}$ contract to a local tree whose only branch point is its principal vertex, but $\mathcal{T}$ itself is such a tree (this is because of assumption (*)). So if b is the principal vertex of $\mathcal{T}$ and $B_1,\ldots,B_n$ are the branches of $\mathcal{T}$ at at b, not containing the root, these branches are linear branches. To prove (a), there are two cases to consider. (Notice that, in $\mathcal{T}_i$ or $\mathcal{T}_i^{e,\alpha}$ ($i>0$), b is the same as e.)

Case 1. $\mathcal{T}_1$ has one principal link.

By (3.23), the branches of $\mathcal{T}_k^{e,\alpha}$ at b are $B_1,\ldots,B_n,[\alpha]$ and B_*, where B_* contains the root of $\mathcal{T}_k^{e,\alpha}$ (but B_* may not contain the principal vertex of $\mathcal{T}_k^{e,\alpha}$ since b might be that vertex). For each i, if b can absorb B_i in $\mathcal{T}_k^{e,\alpha}$ then b can absorb B_i in $\mathcal{T}$. Since $\mathcal{T}$ doesn't contract to a linear local tree, at least two B_i's can't be absorbed (in $\mathcal{T}$, hence in $\mathcal{T}_k^{e,\alpha}[\beta]$). Thus b must absorb every other branch (in $\mathcal{T}_k^{e,\alpha}[\beta]$) and, in particular, $[\alpha]$. So $\alpha=-1$.

Case 2. $\mathcal{T}_1$ has two principal links.

By (3.23), if $B_1,\ldots,B_n$ are suitably labelled then the branches of $\mathcal{T}_k^{e,\alpha}$ at b are $B_1,\ldots,B_{n-1},[\alpha]$ and B_* where, now, B_* does contain the principal vertex of $\mathcal{T}_k^{e,\alpha}$ (because $\mathcal{T}_1$ has two principal links and $\mathcal{T}_k$ has only one $\Longrightarrow k>1$ and b is distinct from the principal vertex of $\mathcal{T}_k^{e,\alpha}$). By (**), b can't absorb $B_*[\beta]$. Hence b must absorb $n-1$ branches in $B_1,\ldots,B_{n-1},[\alpha]$. Since $\mathcal{T}$ doesn't contract to a linear local tree, some B_i ($i<n$) can't be absorbed, so b absorbs $[\alpha]$ and $\alpha=-1$.

3.25. REMARK. From the above proof, it is clear that (3.24) remains true if the condition "$\mathcal{T}_k^{e,\alpha}[\beta]\sim$ to a linear weighted tree" is replaced by "$\mathcal{T}_k^{e,\alpha_1,\ldots,\alpha_p}[\beta]\sim$ to a linear weighted tree" and if conclusion (a) is replaced by "$\alpha_1=\cdots=\alpha_p=-1$". However, we do need that $p>0$ (if $p=0$ then (**) is not true). So (3.24) generalizes as mentioned above, if $p\geq 1$.

3.26. DEFINITION. A sequence $\mathcal{T}_0,\ldots,\mathcal{T}_k$ of local trees (with sets of vertices $T_0,\ldots,T_k$ respectively) is called a *weak sequence* if $k\geq 1$, $\mathcal{T}_k$ has one principal link and if there exist maps $\beta_i: T_{i-1}\to T_i$ ($1\leq i\leq k$) such that, for $i=1,\ldots,k$, either $\mathcal{T}_{i-1}\leftarrow\mathcal{T}_i$ or $\mathcal{T}_{i-1}\overset{\pm}{\leftarrow}\mathcal{T}_i$. The sequence is said to be *weak at* $\mathcal{T}_i$ if $\mathcal{T}_{i-1}\overset{\pm}{\leftarrow}\mathcal{T}_i$.

REMARK. The sequence $\mathcal{S}^+$ of (3.21) is weak.

The word "weak" is supposed to suggest that some information is lost when there are "$\overset{\pm}{\leftarrow}$" involved. We don't know if information is actually lost but we observe that, generally speaking, certain questions that can be answered for sequences of blowings-up $\mathcal{T}_1\leftarrow\cdots\leftarrow\mathcal{T}_k$ become puzzles when some "$\leftarrow$" are replaced by "$\overset{\pm}{\leftarrow}$". However, (3.24), (3.25) and the rest of this section form an attempt to recover control.

3.27. PROPOSITION. *Let $\mathcal{T}_0,\ldots,\mathcal{T}_k$ be a weak sequence of local trees, weak at $\mathcal{T}_1$ and possibly at other places. Let $\mathcal{T}_0\leftarrow\mathcal{T}$ be the blowing-up such that $\mathcal{T}$ has one principal link. Assume that $\mathcal{T}$ does not contract to a linear local tree and that there exists a linear weighted tree $\mathcal{L}$ such that*

$$\mathcal{T}_k[\beta]\sim\mathcal{L},\qquad \textit{some } \beta\in\mathbb{Z}.$$

Then every extra vertex has weight -1 in $\mathcal{T}_k$, the principal vertex of $\mathcal{T}$ is a branch point, $\mathcal{T}$ contracts to a local tree whose only branch point is its principal vertex and, given $\alpha \in \mathbb{Z}$, $\mathcal{T}[\alpha]$ is equivalent to a linear weighted tree iff $\alpha = -1$.

Moreover, if $\langle \mathcal{L} \rangle \leq 1$ then $\mathcal{T}_0$ can't contract to a local tree containing a nonprincipal vertex of nonnegative weight.

PROOF: Let's begin with the last assertion. As in the proof of (3.24), we may assume that the principal vertex of $\mathcal{T}_k$ is a branch point. Suppose $\mathcal{T}_0 \geq \mathcal{T}''$, for some local tree $\mathcal{T}''$ having a nonprincipal vertex of nonnegative weight. Let $\mathcal{T}_0 = \mathcal{T}_0' \leftarrow \cdots \leftarrow \mathcal{T}_k'$ be such that $\mathcal{T}_k$ is obtained from $\mathcal{T}_k'$ by attaching free vertices at appropriate places, i.e., $\mathcal{T}_k = (\cdots(\mathcal{T}_k'^{v_1,\alpha_1})\cdots)^{v_n,\alpha_n}$. Form the commutative diagram so determined (3.14):

$$\begin{array}{ccccccc} \mathcal{T}_0 & = & \mathcal{T}_0' & \leftarrow & \cdots & \leftarrow & \mathcal{T}_k' \\ \vee\!/ & & \vee\!/ & & & & \vee\!/ \\ \mathcal{T}'' & = & \mathcal{T}_0'' & \leftarrow & \cdots & \leftarrow & \mathcal{T}_k'' \end{array}$$

By (3.19b), $\mathcal{T}_k = (\cdots(\mathcal{T}_k'^{v_1,\alpha_1})\cdots)^{v_n,\alpha_n} \geq (\cdots(\mathcal{T}_k''^{v_1,\alpha_1})\cdots)^{v_n,\alpha_n}$, i.e., $\mathcal{T}_k$ contracts to a local tree, say $\mathcal{T}^+$, which has a vertex v of nonnegative weight, such that v is not a neighbour of the principal vertex.

Since the principal vertex of $\mathcal{T}_k$ has been assumed to be a branch point (of weight -1), the principal vertex of $\mathcal{T}^+$ is either a branch point of weight ≥ -1 or a linear vertex of weight ≥ 0. In any case, if $\mathcal{L}^+$ is a linear weighted tree to which $\mathcal{T}^+[\beta]$ contracts (such an $\mathcal{L}^+$ exists by (I.4.13), since $\mathcal{T}^+[\beta] \sim \mathcal{T}_k[\beta] \sim \mathcal{L}$), then one of the following conditions hold:

- $\mathcal{L}^+$ contains vertices u, v with positive weights;
- $\mathcal{L}^+$ contains vertices u, v with nonnegative weights and not neighbours of each other.

By (I.4.9), $\langle \mathcal{L} \rangle = \langle \mathcal{L}^+ \rangle > 1$ and the last assertion is proved.

The assertion about $\mathcal{T}[\alpha]$ is an immediate consequence of the preceding one. Let's now prove that the principal vertex of $\mathcal{T}$ is a branch point, and that $\mathcal{T}$ contracts as specified.

Let $\mathcal{T}_{k_1}, \ldots, \mathcal{T}_{k_r}$, $1 = k_1 < \cdots < k_r \leq k$, be the trees at which the sequence is weak. We proceed by induction on r.

Case $r = 1$. Since $\mathcal{T}$ does not contract to a linear local tree, we may apply (3.25). So consider the sequence of blowings-up

$$\mathcal{T} \rightarrow \mathcal{T}_0 \leftarrow \mathcal{T}_1' \leftarrow \cdots \leftarrow \mathcal{T}_k'$$

such that $\mathcal{T}_1 = (\mathcal{T}_1')^{e,\alpha_1,\ldots,\alpha_p}$ $\quad (p \geq 1)$ and $\mathcal{T}_k = (\mathcal{T}_k')^{e,\alpha_1,\ldots,\alpha_p}$ and apply (3.25).

Inductive Step. Let $r > 1$ and let $\mathcal{T}_{k_2-1} \leftarrow \mathcal{T}'$ be the blowing-up such that $\mathcal{T}'$ has one principal link. By inductive hypothesis applied to $\mathcal{T}_{k_2-1} \leftarrow \mathcal{T}'$ and the weak sequence $\mathcal{T}_{k_2-1}, \mathcal{T}_{k_2}, \ldots, \mathcal{T}_k$, we conclude that $\mathcal{T}'[-1]$ is equivalent to a linear weighted tree. Then apply the inductive hypothesis to $\mathcal{T}_0 \leftarrow \mathcal{T}$ and the weak sequence $\mathcal{T}_0, \ldots, \mathcal{T}_{k_2-1}, \mathcal{T}'$.

3.28. THEOREM. *Let ω, k be positive integers and let*

$$\mathcal{S}: \quad \mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$$

be such that $\mathcal{T}_0 \geq (\omega)$, $\mathcal{T}_k$ has one principal link and its principal vertex is a branch point. Suppose that $\mathcal{T}_k$ does not contract to a linear local tree ($\Longrightarrow k \geq 1$) and that, for some $\alpha \in \mathbb{Z}$, $\mathcal{T}_k[\alpha] \sim \mathcal{L}$ where $\mathcal{L}$ is *some linear weighted tree. Finally, suppose either that $\langle \mathcal{L} \rangle \leq 1$, or that $\mathcal{T}_{k-1}$ can't contract to a local tree having a nonprincipal vertex of nonnegative weight (and notice that the former condition implies the latter, by the proof of (3.27)).*

Then $\alpha = -1$ and $\mathcal{T}_k$ contracts to a local tree whose only branch point is its principal vertex. Let $\mu = (\mu_0, \dots, \mu_k)$ be the unique element of $Mul(S)$ such that $\mu_k(a_k) = 1$, where a_j is the principal link of $\mathcal{T}_j$ whenever $\mathcal{T}_j$ has only one principal link, and write $i = \mu_0(a_0)$ and $r_j = \mu_j(x_0)$, $0 \leq j < k$. Define integers w and p by

$$w - \sum_{j=0}^{k-1} r_j{}^2 = \alpha = -1 \qquad \text{and} \qquad p - \sum_{j=0}^{k-1} \frac{r_j(r_j - 1)}{2} = 0.$$

Then the following conditions hold, where we use the notations of (1.13), determined by S and μ:

(1) If $(i, r_0) = 1$ (i.e. $l = 1$) then

$$w = ir_0 - 1 \quad \text{and} \quad p = \frac{(i-1)(r_0 - 1)}{2}.$$

(2) If $(i, r_0) \neq 1$ (i.e. $l > 1$) then $S_h : \quad \mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_h \quad$ is of type ω, where $h = h_{l-1}$; thus $n_{l-1} = m_{l-2}/i_{l-1}$ is a positive integer. Writing $\delta = j_{l-1} - h \geq 0$, we have $n_{l-1} \geq \delta$ and

$$i^2 - w = (\omega - 1)\sum_{j=0}^{h-1} r_j^2 + [\omega n_{l-1} + 1 - \delta]i_{l-1}^2 - i_{l-1}m_{l-1} + 1$$

$$\left(1 + \frac{2}{\omega}\right) i + 2p - w - 2 = [(n_{l-1} - \delta) + 2/\omega]\, i_{l-1} - m_{l-1}.$$

Moreover, if $\omega \leq 2$ and $\mathcal{L} = [1]$ then $n_{l-1} > \delta$.

(3) If either $\omega \leq 2$ or $\mathcal{L} = [1]$, and if $(i, r_0) \neq 1$, then

$$\left(1 + \frac{2}{\omega}\right) i + 2p - w - 2 > 0.$$

Before we prove (3.28) we need some numerical lemmas. But first, let us introduce the notation

$$f(x) = \frac{x(x-1)}{2}, \qquad x \in \mathbb{Z}.$$

We have the following obvious properties:

- $f(a+b) = f(a) + f(b) + ab$
- $f(a-1) = f(a) - a + 1$
- $f(ab) = a^2 f(b) + b f(a)$.

3.29. LEMMA. *Let $i \geq \rho_0 \geq i' > 0$ be integers such that $(i, \rho_0) = i'$. If the corresponding euclidean algorithm is written as*

$$\begin{aligned} i &= \alpha_0 \rho_0 + \rho_1 \\ \rho_0 &= \alpha_1 \rho_1 + \rho_2 \\ &\vdots \\ \rho_{s-1} &= \alpha_s \rho_s \qquad (\text{where } \rho_s = i'), \end{aligned}$$

then $\alpha_0 \rho_0^2 + \cdots + \alpha_s \rho_s^2 = i\rho_0$ and $\alpha_0 \rho_0 + \cdots + \alpha_s \rho_s = i + \rho_0 - i'$.

This lemma is very easy to prove, by induction on s. We leave its proof to the reader. Together with (1.12a), this gives

3.30. COROLLARY. *Let $(\mathcal{T}_0, \mu_0) \Leftarrow \cdots \Leftarrow (\mathcal{T}_k, \mu_k)$ $(k \geq 1)$ be such that $\mathcal{T}_\nu$ has one principal link iff $\nu \in \{0, k\}$. Let a (resp. a') be the principal link of $\mathcal{T}_0$ (resp. $\mathcal{T}_k$) and write $i = \mu_0(a)$, $i' = \mu_k(a')$. Let $r_\nu = \mu_\nu(x_0)$, $0 \leq \nu \leq k-1$. Then*

$$\sum_{j=0}^{k-1} r_j^2 = ir_0, \quad \sum_{j=0}^{k-1} r_j = i + r_0 - i' \quad \text{and} \quad \sum_{j=0}^{k-1} f(r_j) = \frac{ir_0 - i - r_0 + i'}{2}.$$

3.31. LEMMA. *Let ω, i, i' be positive integers and let $(r_0, \ldots, r_{k-1})$ be a sequence of type (ω, i, i'), with notation as in (3.5). Then, if $m = m_0 + \cdots + m_{l-1}$,*

(a) $i = \omega m + i'$

(b) $\sum_{j=0}^{k-1} r_j = (\omega + 2)m - m_{l-1}$

(c) $i^2 = \omega \sum_{j=0}^{k-1} r_j^2 \;+\; (\omega n_l + 1)i'^2$

(d) $f(i-1) = \omega \sum_{j=0}^{k-1} f(r_j) \;+\; f(\omega)m + (\omega n_l + 1)f(i') - i' + 1$

PROOF:

(a)
$$\begin{aligned} i = i_0 &= \omega m_0 + i_1 \\ &= \omega m_0 + \omega m_1 + i_2 \\ &\;\vdots \\ &= \omega m_0 + \cdots + \omega m_{l-1} + i_l = \omega m + i'. \end{aligned}$$

(b)
$$\begin{aligned} \sum_{j=0}^{k-1} r_j &= \omega m_0 + 2n_1 i_1 + \cdots + \omega m_{l-2} + 2n_{l-1} i_{l-1} + \omega m_{l-1} + n_l i_l \\ &= (\omega m_0 + 2m_0) + \cdots + (\omega m_{l-2} + 2m_{l-2}) + (\omega m_{l-1} + m_{l-1}) \\ &= (\omega + 2)m - m_{l-1}. \end{aligned}$$

(c) By induction on l.
If $l = 1$ then $i = (\omega n_1 + 1)i'$ and $m_0 = n_1 i'$, so

$$\begin{aligned} i^2 &= \omega^2 n_1^2 i'^2 + 2\omega n_1 i'^2 + i'^2 = \omega(\omega m_0^2 + n_1 i'^2) + (\omega n_1 + 1)i'^2 \\ &= \omega \sum_{j=0}^{k-1} r_j^2 + (\omega n_1 + 1)i'^2. \end{aligned}$$

If $l > 1$ then define integers a, b by

$(r_0, \ldots, r_{a-1}) = (m_0, \ldots, m_{l-2}, i_{l-1}, \ldots, i_{l-1})$

$(r_a, \ldots, r_{b-1}) = (i_{l-1}, \ldots, i_{l-1})$ (where i_{l-1} occurs n_{l-1} times)

$(r_b, \ldots, r_{k-1}) = (m_{l-1}, \ldots, i_l)$. The first sequence is of type (ω, i, i_{l-1}), the last is of type (ω, i_{l-1}, i'), so we may apply the inductive hypothesis:

$$\begin{aligned} i^2 &= \omega \sum_{j=0}^{a-1} r_j^2 + (\omega n_{l-1} + 1)i_{l-1}^2 = \omega \sum_{j=0}^{a-1} r_j^2 + \omega \sum_{j=a}^{b-1} r_j^2 + i_{l-1}^2 \\ &= \omega \sum_{j=0}^{a-1} r_j^2 + \omega \sum_{j=a}^{b-1} r_j^2 + \omega \sum_{j=b}^{k-1} r_j^2 + (\omega n_l + 1)i'^2. \end{aligned}$$

(d) $$\omega\sum_{j=0}^{k-1} f(r_j) = \frac{1}{2}\left[\omega\sum_{j=0}^{k-1} r_j^2 - \omega\sum_{j=0}^{k-1} r_j\right] = \frac{1}{2}\left[i^2 - (\omega n_l + 1)i'^2 - \omega((\omega+2)m - m_{l-1})\right]$$
$$= \frac{1}{2}\left[i^2 - (\omega n_l + 1)i'^2 - 3m\omega + \omega m_{l-1}\right] - f(\omega)m$$
$$= \frac{1}{2}\left[i^2 - (\omega n_l + 1)i'^2 - 3(i - i') + \omega n_l i'\right] - f(\omega)m$$
$$= f(i) - i + \frac{1}{2}\left[-(\omega n_l + 1)i'^2 + 3i' + \omega n_l i'\right] - f(\omega)m$$
$$= f(i) - i + \frac{1}{2}\left[-(\omega n_l + 1)i'^2 + (\omega n_l + 1)i' + 2i'\right] - f(\omega)m$$
$$= f(i) - i - (\omega n_l + 1)f(i') + i' - f(\omega)m$$
$$= f(i-1) - 1 - (\omega n_l + 1)f(i') + i' - f(\omega)m$$

which is what we want.

Proof of (3.28): The first thing we have to do is to reduce to the case $\mathcal{T}_0 = (\omega)$. We leave that part to the reader. In fact, the argument is quite analogous to the corresponding reduction in the proof of (3.8). So:

We assume that $\mathcal{T}_0 = (\omega)$. Then $\mathcal{T}_\nu$ is a comb with negative teeth, $0 \le \nu \le k$, by (3.17). Let the notations of (1.13) be in force, i.e., we consider $\mathcal{J} = \{j_0, \dots, j_{l-1}\}$, $\mathcal{H} = \{h_1, \dots, h_l\}$, $e_1, \dots, e_l$, etc. By our assumptions, $\mathcal{T}_k = (*, -1, \mathcal{A}, \mathcal{B})$ where $\mathcal{A}$, $\mathcal{B}$ are the branches at e_l, not containing the root, and $\mathcal{A} < -1$ is a linear branch. So the branches of $\mathcal{T}_k[\alpha]$ at e_l are $\mathcal{A}$, $\mathcal{B}$ and $[\alpha]$. Now e_l can't absorb $\mathcal{A} < -1$, and e_l can't absorb $\mathcal{B}$ (for $\mathcal{T}_k$ doesn't contract to a linear local tree). However, $\mathcal{T}_k[\alpha] \sim \mathcal{L}$ so $\mathcal{T}_k[\alpha]$ contracts to a linear weighted tree (I.4.13) and e_l must absorb some branch, in $\mathcal{T}_k[\alpha]$. Hence e_l absorbs $[\alpha]$ and $\alpha = -1$.

If $l = 1$ then $\mathcal{T}_k$ is already a local tree whose only branch point is its principal vertex. If $l > 1$, let's prove that $\mathcal{T}_k$ contracts to such a local tree. $\mathcal{T}_k$ has three branches at e_{l-1}, say $\mathcal{A}'$, $\mathcal{B}'$ and $\mathcal{B}'_*$ where $\mathcal{A}' < -1$ and $\mathcal{B}'_*$ contains, in particular, e_l, which is a branch point of weight -1; hence e_{l-1} can absorb neither $\mathcal{A}'$ nor $\mathcal{B}'_*[-1]$ in $\mathcal{T}_k[-1]$. Since $\mathcal{T}_k[-1]$ contracts to a linear tree, e_{l-1} must absorb one of the three branches. So it absorbs $\mathcal{B}'$ and $\mathcal{T}_k$ contracts as specified.

Before we prove that conditions (1)-(3) hold, let us explain why $l = 1$ is equivalent to $(i, r_0) = 1$, as asserted in (1) and (2). We claim that $\mathcal{T}_1$ has two principal links. If not, then $\mathcal{T}_1 = (*, -1, \omega - 1)$ has a nonprincipal vertex with nonnegative weight, and so do $\mathcal{T}_2, \dots, \mathcal{T}_{k-1}$, so one of the hypotheses of the theorem is violated. Hence:

(1) $\mathcal{T}_1$ *has two principal links.*

Clearly, $\mathcal{T}_{k-1}$ has two principal links, since the principal vertex e_l of $\mathcal{T}_k$ is a branch point. Thus it is clear that $l = 1$ is equivalent to: $\mathcal{T}_\nu$ *has one principal link iff* $\nu \in \{0, k\}$, and by (1.12a), $l = 1 \iff (i, r_0) = \mu(a_k) = 1$.

Condition 1. Since $l = 1$ then, by above remarks, we may apply (3.30). Hence $\sum_{j=0}^{k-1} r_j^2 = ir_0$ and

$\sum_{j=0}^{h-1} f(r_j) = \frac{(i-1)(r_0-1)}{2}$, so

$$w = \sum_{j=0}^{h-1} r_j^2 - 1 = ir_0 - 1 \quad \text{and}$$

$$p = \sum_{j=0}^{h-1} f(r_j) = \frac{(i-1)(r_0-1)}{2}.$$

Condition 2. Suppose $l > 1$. Consider the integer $h = j_{l-1} > 0$; the branch point e_{l-1}, which absorbs the branch $\mathcal{B}'$ of $\mathcal{T}_k$, was created in $\mathcal{T}_{h-1} \leftarrow \mathcal{T}_h$; in fact, $\mathcal{T}_h = (*, -1, \mathcal{A}', \mathcal{B}')$, so $\mathcal{B}'$ can be absorbed in $\mathcal{T}_h$ as well. Hence

(2) $\qquad$ *$\mathcal{T}_h$ contracts to a linear local tree.*

So we consider the sequence $\mathcal{S}_h : \quad (\omega) = \mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_h$. By (3.8),

(3) $\qquad$ *$\mathcal{S}_h$ is of type ω, $\mathcal{T}_h = (*, -1, (-n-1, -2, \ldots, -2), \mathcal{B}')$ where $n = n_{l-1}$ and where "-2" occurs $\omega - 1$ times, and the absorption of $\mathcal{B}'$ increases by 1 the weight of e_{l-1}.*

Observe that, by definition, $\mu_h(x_0) = r_h$ and $\mu_h(a_h) = i_{l-1}$. By (3),

(4) $\qquad$ *$(r_0, \ldots, r_{h-1})$ is of type (ω, i, i_{l-1}).*

Applying (3.31) to $\mathcal{S}_h$, we deduce (where $m = m_0 + \cdots + m_{l-2}$):

(5) $$i^2 = \omega \sum_{j=0}^{h-1} r_j^2 + [\omega n + 1] i_{l-1}^2,$$

(6) $$f(i-1) = \omega \sum_{j=0}^{h-1} f(r_j) + f(\omega)m + [\omega n + 1] f(i_{l-1}) - i_{l-1} + 1,$$

(7) $$\sum_{j=0}^{h-1} r_j = (\omega + 2)m - m_{l-2},$$

(8) $$i = \omega m + i_{l-1}.$$

By definition of j_{l-1} and h, $\mathcal{T}_j$ has one principal link whenever $h \leq j \leq j_{l-1}$, so

(9) $$h \leq j < j_{l-1} \Longrightarrow r_j = i_{l-1}.$$

If we define $\delta = j_{l-1} - h \geq 0$, then $\mathcal{T}_{j_{l-1}} = (*, -1, -2, \ldots, -2, (-n-1, -2, \ldots, -2), \mathcal{B}')$, where the first sequence of "-2" contains δ terms and the second $\omega - 1$ terms. So $\mathcal{T}_{j_{l-1}}$ contracts to the following linear local tree:

$$\mathcal{L}_{j_{l-1}} = (*, 0, \delta - n - 1, -2, \ldots, -2),$$

where "-2" occurs $\omega - 1$ times. We claim that $\delta \leq n$. In fact, if $\delta > n$ then $\mathcal{L}_{j_{l-1}}$ has a nonprincipal vertex with nonnegative weight. Since by definition $j_{l-1} < k$, one can consider the commutative diagram

(3.14) determined by $\mathcal{L}_{j_{l-1}} \leq \mathcal{T}_{j_{l-1}} \leftarrow \cdots \leftarrow \mathcal{T}_{k-1}$ and deduce that $\mathcal{T}_{k-1}$ contracts to a local tree which contains a nonprincipal vertex with nonnegative weight. This contradicts one of the assumptions. So,

$$\delta \leq n. \tag{10}$$

On the other hand, we have $\mu_{j_{l-1}}(x_0) = m_{l-1}$ and $\mu_{j_{l-1}}(a_{j_{l-1}}) = i_{l-1}$ by definition, and $(i_{l-1}, m_{l-1}) = 1$. By (3.30),

$$\sum_{j=j_{l-1}}^{k-1} r_j^2 = i_{l-1} m_{l-1}, \tag{11}$$

$$\sum_{j=j_{l-1}}^{k-1} r_j = i_{l-1} + m_{l-1} - 1, \tag{12}$$

$$\sum_{j=j_{l-1}}^{k-1} f(r_j) = \frac{(i_{l-1} - 1)(m_{l-1} - 1)}{2}. \tag{13}$$

We can now check that the two equations of "condition 2" hold.

$$\begin{aligned} i^2 - w &= i^2 - \left(\sum_{j=0}^{k-1} r_j^2 - 1\right) = \left(i^2 - \sum_{j=0}^{h-1} r_j^2\right) - \sum_{j=h}^{j_{l-1}-1} r_j^2 - \sum_{j=j_{l-1}}^{k-1} r_j^2 + 1 \\ &= (\omega - 1) \sum_{j=0}^{h-1} r_j^2 + (\omega n + 1) i_{l-1}^2 - \delta i_{l-1}^2 - i_{l-1} m_{l-1} + 1, \end{aligned}$$

by (5),(9) and (11). So

$$i^2 - w = (\omega - 1) \sum_{j=0}^{h-1} r_j^2 + (\omega n + 1 - \delta) i_{l-1}^2 - i_{l-1} m_{l-1} + 1, \tag{14}$$

which is the first equation, since $n = n_{l-1}$ by (3). For the second equation, observe that

$$\begin{aligned} f(i-1) - p = f(i-1) - \sum_{j=0}^{k-1} f(r_j) &= \left(f(i-1) - \sum_{j=0}^{h-1} f(r_j)\right) - \sum_{j=h}^{j_{l-1}-1} f(r_j) - \sum_{j=j_{l-1}}^{k-1} f(r_j) \\ &= (\omega - 1) \sum_{j=0}^{h-1} f(r_j) + f(\omega) m + (\omega n + 1) f(i_{l-1}) - i_{l-1} + 1 - \delta f(i_{l-1}) - \frac{(i_{l-1} - 1)(m_{l-1} - 1)}{2}, \end{aligned}$$

by (6),(9) and (13), so

$$f(i-1) - p = (\omega - 1) \sum_{j=0}^{h-1} f(r_j) + f(\omega) m + (\omega n + 1 - \delta) f(i_{l-1}) - i_{l-1} + 1 - \frac{(i_{l-1} - 1)(m_{l-1} - 1)}{2},$$

and by multiplying that equation by 2 we obtain

$$\begin{aligned} i^2 - 3i + 2 - 2p &= (\omega - 1) \sum_{j=0}^{h-1} r_j^2 - (\omega - 1) \sum_{j=0}^{h-1} r_j + 2f(\omega) m \\ &\quad + (\omega n + 1 - \delta)(i_{l-1}^2 - i_{l-1}) - 2i_{l-1} + 2 - i_{l-1} m_{l-1} + i_{l-1} + m_{l-1} - 1 \\ &= (i^2 - w) - (\omega - 1) \sum_{j=0}^{h-1} r_j + 2f(\omega) m - (\omega n + 2 - \delta) i_{l-1} + m_{l-1} \end{aligned}$$

by (14). Therefore,

$$\begin{aligned}
3i + 2p - w - 2 &= (\omega - 1)\sum_{j=0}^{h-1} r_j \; - 2f(\omega)m + (\omega n + 2 - \delta)i_{l-1} - m_{l-1} \\
&= (\omega - 1)\left[(\omega + 2)m - m_{l-2}\right] - \omega(\omega - 1)m + (\omega n + 2 - \delta)i_{l-1} - m_{l-1} \\
&= 2(\omega - 1)m - (\omega - 1)m_{l-2} + (\omega n + 2 - \delta)i_{l-1} - m_{l-1} \\
&= 2(\omega - 1)m - (\omega - 1)m_{l-2} + (\omega - 1)ni_{l-1} + (n + 2 - \delta)i_{l-1} - m_{l-1} \\
&= 2(\omega - 1)m + (n + 2 - \delta)i_{l-1} - m_{l-1}
\end{aligned}$$

by (7) and the fact that $-m_{l-2} + ni_{l-1} = 0$, which follows from (4). Since $m = (i - i_{l-1})/\omega$ by (8), we find

$$3i + 2p - w - 2 = 2\left(1 - \frac{1}{\omega}\right)(i - i_{l-1}) + (n + 2 - \delta)i_{l-1} - m_{l-1},$$

from which the desired equation follows. The next thing we do is to prove that $n_{l-1} > \delta$ whenever $\omega \leq 2$ and $\mathcal{L} = [1]$. Suppose $n_{l-1} = \delta$ and $\mathcal{L} = [1]$. By the description of $\mathcal{L}_{j_{l-1}}$ given above, between (9) and (10), we conclude that $\mathcal{T}_{j_{l-1}} \geq \mathcal{L}_{j_{l-1}} \geq (\omega)$. Consider the local tree $\mathcal{T}$ defined by

$$\begin{array}{ccccc}
\mathcal{T}_{j_{l-1}} & \leftarrow & \cdots & \leftarrow & \mathcal{T}_h \\
\rotatebox{90}{$\leq$} & & & & \rotatebox{90}{$\leq$} \\
(\omega) & \leftarrow & \cdots & \leftarrow & \mathcal{T}
\end{array}$$

In that diagram, each tree in the lower row has the same number of principal links as the corresponding tree in the upper row; hence, in the lower row, only (ω) and $\mathcal{T}$ have one principal link (all other have two). Thus $\mathcal{T} = (*, -1, \mathcal{A}, (b_1, \ldots, b_\nu, \omega'))$, where $\mathcal{A} < -1$ is a linear branch, $\omega' < \omega$ is the weight of the vertex which *was* the principal vertex of (ω), and $\nu \geq 0$. Moreover, $b_1, \ldots, b_\nu$ are weights of vertices which have been created in $(\omega) \leftarrow \cdots \leftarrow \mathcal{T}$; so $b_i < -1$ for $1 \leq i \leq \nu$. Since $\mathcal{T}_h$ doesn't contract to a linear tree, $\mathcal{T}$ doesn't contract to a linear tree, i.e.,

$$\text{(15)} \qquad \omega' \neq -1 \;\text{ or }\; \exists_i \; b_i < -2.$$

On the other hand, since $\alpha = -1$, $\mathcal{T}[\alpha]$ contracts to the linear tree

$$\mathcal{G} = [\mathcal{A}, 0, b_1, \ldots, b_\nu, \omega'].$$

Now $\mathcal{G} \sim \mathcal{T}[\alpha] \sim \mathcal{T}_h[\alpha] \sim [1]$, so $\mathcal{G}$ must be minimal. Indeed, if $\mathcal{G}$ is not minimal, then $\omega' = -1$ and by (15) it contracts to a minimal weighted tree $\mathcal{G}' = [\mathcal{A}, 0, b_1, \ldots, b_{i-1}, b_i + 1]$ which has more than two vertices but only one nonnegative weight. Such a tree can't be equivalent to $[1]$ by (I.4.16).

So $\mathcal{G}$ is minimal. Since $|\mathcal{G}| > 2$, (I.4.16) implies that $\nu = 0$ and $\omega' > 0$. By definition of $\mathcal{T}$, we deduce that every vertex of $\mathcal{A}$ has weight -2, thus $\omega' = 1$ by (I.4.16) again. Now $\nu = 0$ implies that i_{l-1} is a multiple of m_{l-1}—recall the relation between the euclidean algorithm of (i_{l-1}, m_{l-1}) and the sequence of multiplicities of the roots in $(\omega) \leftarrow \cdots \leftarrow \mathcal{T}$. Thus $m_{l-1} = i_l = 1$ and $1 = \omega' = \omega - i_{l-1}$. Since $i_{l-1} > m_{l-1}$ by definition, we get $\omega = 1 + i_{l-1} > 2$. This proves condition 2.

Condition 3. By condition 2, it is enough to prove $(n_{l-1} - \delta + \frac{2}{\omega})i_{l-1} - m_{l-1} > 0$. This is certainly the case if $\omega \leq 2$ (for $n_{l-1} - \delta + \frac{2}{\omega} \geq \frac{2}{\omega} \geq 1$), or if $n_{l-1} > \delta$ (for $n_{l-1} - \delta + \frac{2}{\omega} > n_{l-1} - \delta \geq 1$). So we may assume that $n_{l-1} = \delta$, $\omega > 2$ and $\mathcal{L} = [1]$. The argument which proved the last assertion of condition 2 was based on the assumptions $n_{l-1} = \delta$ and $\mathcal{L} = [1]$; so we can apply it here and conclude that $m_{l-1} = 1$ and $i_{l-1} = \omega - 1$. Consequently,

$$\left(n_{l-1} - \delta + \frac{2}{\omega}\right) i_{l-1} - m_{l-1} = \frac{2}{\omega}(\omega - 1) - 1 = 1 - \frac{2}{\omega} > 0.$$

This completes the proof of the theorem.

Theorem (3.28) will be used to prove that certain local trees T_k contract to linear local trees; we will do that by proving that the numerical relations of the conclusion can not be satisfied in the case under consideration. More precisely, the statement that we will use is the following.

3.32. COROLLARY. *Let S : $T_0 \leftarrow \cdots \leftarrow T_k$ satisfy the hypothesis of theorem (3.28) and assume, in addition, that*

$$(i, r_0) = 1 \quad \text{or} \quad \omega \leq 2 \quad \text{or} \quad \mathcal{L} = [1].$$

Then no triple (d, u, v) of real numbers can satisfy one of the following conditions:

$$\text{(a)} \quad \begin{cases} u + v \leq d, \quad i = d, \quad w = d^2 - u^2 - v^2, \\ \text{and} \quad p = \dfrac{(d-1)(d-2)}{2} - \dfrac{u(u-1)}{2} - \dfrac{v(v-1)}{2} \end{cases}$$

$$\text{(b)} \quad \begin{cases} u + v + r_0 \leq d, \quad i = d + r_0, \quad w = d^2 - u^2 - v^2 + r_0^2, \\ \text{and} \quad p = \dfrac{(d-1)(d-2)}{2} - \dfrac{u(u-1)}{2} - \dfrac{v(v-1)}{2} + \dfrac{r_0(r_0-1)}{2}. \end{cases}$$

PROOF: Assume that $(i, r_0) = 1$. Then by (3.28), $w = ir_0 - 1$ and $p = \frac{(i-1)(r_0-1)}{2}$.
If (d, u, v) satisfies (a), then $d^2 - u^2 - v^2 = dr_0 - 1$ and $d^2 - 3d + 2 - u^2 - v^2 + u + v = (d-1)(r_0-1)$. These two equations imply that $(d - u - v) + (d - r_0) = 0$, whence $r_0 \geq d = i$. Thus $r_0 = i$ and $i = (i, r_0) = 1$, which is absurd. (Note that whenever the hypothesis of (3.28) is satisfied, we have $i > 1$. In fact, since the principal vertex of T_k is a branch point, we have $l \geq 1$, so $i > 1$.)
If (d, u, v) satisfies (b), then $d^2 - u^2 - v^2 + r_0^2 = (d + r_0)r_0 - 1$ and $d^2 - 3d + 2 - u^2 + u - v^2 + v + r_0^2 - r_0 = (d + r_0 - 1)(r_0 - 1)$. From these two equations, we find $(d - u - v - r_0) + d = 0$, whence $d \leq 0$ and $i = d + r_0 \leq r_0$. Thus $r_0 = i$ and $i = (i, r_0) = 1$, which is absurd.

That proves the case $(i, r_0) = 1$. Now assume that $(i, r_0) \neq 1$. Then either $\omega \leq 2$ or $\mathcal{L} = [1]$, so condition 3 of theorem (3.28) says that $B > 0$, where we define

$$A = i^2 - w - 1, \qquad B = \left(1 + \frac{2}{\omega}\right) i + 2p - w - 2.$$

Now a little calculation gives

$$(\star) \quad \begin{aligned} A &= \begin{cases} u^2 + v^2 - 1, & \text{if (a) holds,} \\ u^2 + v^2 + 2r_0 d - 1, & \text{if (b) holds,} \end{cases} \\ B &= \begin{cases} \left(-2 + \frac{2}{\omega}\right) d + u + v, & \text{if (a) holds,} \\ \left(-2 + \frac{2}{\omega}\right) d + \frac{2}{\omega} r_0 + u + v, & \text{if (b) holds.} \end{cases} \end{aligned}$$

If $\omega \geq 2$ then $-2 + \frac{2}{\omega} \leq -1$, so

$$0 < B \leq \begin{cases} -d + u + v \leq 0, & \text{if (a) holds,} \\ -d + \frac{2}{\omega} r_0 + u + v \leq -d + r_0 + u + v \leq 0, & \text{if (b) holds,} \end{cases}$$

and this is absurd (we used the fact that $d \geq 0$, if (a) or (b) holds).

If $\omega = 1$ then, by (3.23),

$$A = (n_{l-1} + 1 - \delta)i_{l-1}^2 - i_{l-1}m_{l-1} = i_{l-1}((n_{l-1} + 1 - \delta)i_{l-1} - m_{l-1}) = xy$$
$$B = (n_{l-1} - \delta + 2)i_{l-1} - m_{l-1} = i_{l-1} + (n_{l-1} - \delta + 1)i_{l-1} - m_{l-1} = x + y$$

where we define $x = i_{l-1}$ and $y = (n_{l-1} - \delta + 1)i_{l-1} - m_{l-1}$. Thus x and y are integers, $x \geq 2$ and $y \geq 1$. Whence $B^2 - 2A = x^2 + y^2 \geq 5$. On the other hand, from $(\star)$, above,

$$B^2 - 2A = \begin{cases} 2uv - u^2 - v^2 + 2 = 2 - (u - v)^2 \leq 2, & \text{if (a) holds,} \\ 4r_0(r_0 + u + v - d) - (u - v)^2 + 2 \leq 2, & \text{if (b) holds.} \end{cases}$$

Since we have already established that $B^2 - 2A \geq 5$, (d, u, v) satisfies neither (a) nor (b).

III. BIRATIONAL ENDOMORPHISMS OF THE AFFINE PLANE

1. Preliminaries on $\mathbf{A}^2$.

We will always regard $\mathbf{A}^2$ as being equipped with a fixed coordinate system. In particular, it makes sense to speak of the degree of a curve in $\mathbf{A}^2$.

Embeddings of $\mathbf{A}^2$ in complete nonsingular surfaces.

1.1. Two open immersions $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ are equivalent if they form a commutative diagram with some automorphism of $\mathbf{P}^2$. One equivalence class is "better" than the others: it consists of those embeddings that don't change the degrees of the curves. We will refer to that equivalence class by saying "embed $\mathbf{A}^2$ in $\mathbf{P}^2$ the standard way".

The following is a (trivial) consequence of the theory of "relatively minimal" rational surfaces [11].

1.2. FACT. *Let S be a rational nonsingular projective surface, $D \in Div(S)$ a reduced, effective divisor and $U = S \setminus supp(D)$. Then the following are equivalent:*

(a) $U \cong \mathbf{A}^2$.

(b) Every irreducible component of D is a rational curve, $[1] \in \mathcal{G}[U]$ and $n(D) + K_S^2 = 10$, where $n(D)$ is the number of irreducible components of D and K_S is a canonical divisor of S.

PROOF: First, observe that the number $n(D) + K_S^2$ depends only on U. Indeed, each blowing-up at a point at infinity of U decreases K^2 by 1 and increases n by 1. Since one can obtain one embedding from another by blowing-up and blowing-down at infinity of U, $n + K^2$ is an invariant of U. If $U \cong \mathbf{A}^2$ then $n + K^2 = 1 + 9 = 10$, so (a)$\Rightarrow$(b) (since $[1] \in \mathcal{G}[\mathbf{A}^2]$ and $\mathbf{A}^2$ is rational at infinity (I.5.1)). Conversely, if (b) is satisfied then $[1] \in \mathcal{G}[U]$ means that there is a smooth completion (I.4.8) $U \hookrightarrow \overline{U}$ such that $\overline{U} \setminus U$ is a nonsingular curve Γ of self-intersection 1. Since the number $n(\Gamma) + K_{\overline{U}}^2$ depends only on U, it is 10 by assumption, so $K_{\overline{U}}^2 = 9$ and $\overline{U} \cong \mathbf{P}^2$ by the theory of relatively minimal rational surfaces. Hence Γ is a line and $U \cong \mathbf{A}^2$.

1.3. COROLLARY. *Let U be an open subset of a nonsingular complete surface S. If $U \cong \mathbf{A}^2$ and $S \setminus U$ is irreducible then $S \cong \mathbf{P}^2$.*

PROOF: Repeating a part of the preceding proof, $10 = n + K_S^2 = 1 + K_S^2 \Rightarrow K_S^2 = 9 \Rightarrow S \cong \mathbf{P}^2$.

The next fact is a simple observation that turns out to be very useful.

1.4. LEMMA. *Let S be a nonsingular projective surface, $\mathbf{A}^2 \hookrightarrow S$ an open immersion and let $D \in Div(S)$ be the reduced effective divisor such that $S \setminus \mathbf{A}^2 = supp(D)$. Consider a sequence*

$$S = S_0 \xleftarrow{\pi_1} S_1 \leftarrow \cdots \xleftarrow{\pi_m} S_m \qquad (m \geq 0)$$

of monoidal transformations, where the center of π_i is a point P_i i.n. $supp(D)$ and $\pi_i^{-1}(P_i) = E_i$, such that (in the notation of (II.2.3)) $D^m \in Div(S_m)$ has s.n.c.. If m is minimum with respect to that property, then $S_m \setminus supp(D^m) \cong \mathbf{A}^2$ and:

(i) If $m \geq 2$ then $P_i \in E_{i-1}$ $(2 \leq i \leq m)$.

(b) *If $m \geq 1$ then P_1 belongs to at least two irreducible components of D.*

(c) *If $m \geq 1$ and $D = A + B$, where A and B are effective divisors and B has s.n.c. in S, then P_i belongs to the strict transform of A in S_{i-1} $(1 \leq i \leq m)$.*

PROOF: Again, use the same notation for a curve and for its strict transform in any blown up surface. Since $\mathbf{A}^2$ has no loops at infinity, it follows from (I.4.7) that (in the notation of (I.4.7))

every E_i such that $E_i^2 = -1$ in S_m is a branch point of $\mathcal{G}(S_m, D^m)$.

If (a) doesn't hold then $\mathcal{G}(S_m, D^m)$ contains two branch points u, v of weight -1 such that u, v are not neighbours of each other. Contract $\mathcal{G}(S_m, D^m)$ to a linear weighted tree $\mathcal{L}$ (I.4.13); then u and v are still in $\mathcal{L}$ and one of the following holds:

- $\mathcal{L}$ contains vertices u, v with positive weights;
- $\mathcal{L}$ contains vertices u, v with nonnegative weights and not neighbours of each other.

Thus $\langle \mathcal{L} \rangle > 1$, by (I.4.9), and that is absurd. Hence (a) holds.

We now prove (b). By the above, E_m is a branch point of $\mathcal{G}(S_m, D^m)$, of weight -1, and no other E_i has weight ≥ -1 (in S_m). If P_1 belongs to only one component of D, all components of D are in the same branch of $\mathcal{G}(S_m, D^m)$ at E_m. Thus E_m is a "special vertex" (I.4.17) and we get a contradiction with (I.4.18).

Proof of (c). Since E_m is a branch point of $\mathcal{G}(S_m, D^m)$, P_m belongs to at least three components of $D^{m-1} = A + B^{m-1}$, where we define $B^0 = B \in \operatorname{Div}(S)$ and $B^{i+1} = B^i + E_{i+1} \in \operatorname{Div}(S_{i+1})$. Since B has *s.n.c.* in S, B^i has *s.n.c.* in S_i $(0 \leq i \leq m)$ and P_m belongs to at most two components of B^{m-1}. Thus P_m belongs to (the strict transform of) A and, by (a), so do $P_1, \ldots, P_{m-1}$.

The above observations yield the following (known) fact as a byproduct:

1.5. COROLLARY. *Let U be an open subset of $\mathbf{P}^2$ such that $U \cong \mathbf{A}^2$. Then $\mathbf{P}^2 \setminus U$ is a line.*

PROOF: We know that $\mathbf{P}^2 \setminus U = \operatorname{supp}(D)$ for some reduced effective divisor D of $\mathbf{P}^2$. By (1.2), $n(D) = 1$, i.e., D is a curve. By (1.4b) D has *s.n.c.* (i.e. is nonsingular), thus $\mathbf{A}^2 \cong U \hookrightarrow \mathbf{P}^2$ is a smooth completion; the corresponding dual graph $[D^2]$ must be equivalent to $[1]$, so $D^2 = 1$ (for $[n] \sim [m] \Rightarrow n = m$, as explained at the beginning of the proof of (I.4.15)). So D is a line.

A characterization of $\mathbf{A}^2$. The following "powerful" theorem was proved by Fujita [2] and Miyanishi and Sugie [8] in characteristic zero, and generalised by Russell [17] to arbitrary characteristic.

1.6. THEOREM. *Let V be a nonsingular, factorial, rational surface with trivial units, and whose Kodaira dimension is $-\infty$. Then $V \cong \mathbf{A}^2$.*

For the notion of Kodaira dimension, see [6]; for the special case of surfaces, a simple exposition is given in [17]. From these sources, we also have

1.7. LEMMA. *Let $\phi : V' \to V$ be a dominant, separable morphism of nonsingular surfaces. Then the Kodaira dimension of V is less than or equal to that of V'.*

From these two facts, we immediately conclude

1.8. COROLLARY. *Let $f : \mathbf{A}^2 \to V$ be a birational morphism, where V is factorial (and nonsingular, as always). Then $V \cong \mathbf{A}^2$.*

(Indeed, $\hat{V}$ has trivial units by (I.2.6)). Notice that the notion of Kodaira dimension does not occur in the statement of (1.8). Notice also that (1.6) and (1.7) won't be needed in the sequel.

Lines in $\mathbf{A}^2$. Following several people, we adopt the following terminology for lines in the affine plane.

1.9. DEFINITION. Let C be a curve in $\mathbf{A}^2$.

(a) C is a *linear line* if $\deg C = 1$.

(b) C is a *coordinate line* if, modulo an automorphism of $\mathbf{A}^2$, C is a linear line. Equivalently, the polynomial $F \in \mathbf{k}[X,Y]$ determined by C satisfies the following condition: $\exists G \in \mathbf{k}[X,Y]$ such that $\mathbf{k}[F,G] = \mathbf{k}[X,Y]$.

(c) C is a *line* if $C \cong \mathbf{A}^1$ (abstractly). Equivalently, the polynomial F (as in (b)) is such that $\mathbf{k}[X,Y]/(F)$ is a polynomial algebra in one indeterminate over $\mathbf{k}$.

As is very well known, the Epimorphism Theorem of Abhyankar-Moh [1] says (in particular) that all lines are coordinate lines if $\operatorname{char}\mathbf{k} = 0$. It is also known that, in positive characteristic, not all lines are coordinate lines.

Let C be an affine plane curve with one place P at infinity (I.5.4). Embed $\mathbf{A}^2$ in $\mathbf{P}^2$ the standard way. As noted in (II.2.3), an infinite sequence of monoidal transformations is uniquely determined,

$$\mathbf{P}^2 = S_0 \overset{\pi_1}{\leftarrow} S_1 \overset{\pi_2}{\leftarrow} S_2 \overset{\pi_3}{\leftarrow} \cdots.$$

Let P_i denote the center of $\pi_i : S_i \to S_{i-1}$ and $C^{(i)}$ the strict transform on S_i of the closure in $\mathbf{P}^2$ of C. The sequence $\mu(P_1, C^{(0)}), \mu(P_2, C^{(1)}), \mu(P_3, C^{(2)}), \ldots$ is called *the multiplicity sequence of C at infinity.* That sequence is completely determined by the "embedding" of C in $\mathbf{A}^2$, i.e., is independent of the choice of an embedding of $\mathbf{A}^2$ in $\mathbf{P}^2$—as long as that embedding is "standard" (1.1).

We will now characterise the coordinate lines in terms of the multiplicity sequence at infinity.

1.10. DEFINITION. Let Γ be a curve in $\mathbf{A}^2$ with one place at infinity. We say that Γ is *graph-theoretically linear* if there is an open immersion $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ with the following property:

If $L = \mathbf{P}^2 \setminus \mathbf{A}^2$, P is the place of Γ at infinity and

$$\mu(P, \Gamma, L, \mathbf{P}^2): \qquad (\mathcal{T}_0, \mu_0) \Leftarrow \cdots \Leftarrow (\mathcal{T}_k, \mu_k),$$

then $\mathcal{T}_k$ contracts to a linear local tree.

REMARKS.

(a) Notice that $\mathcal{T}_0 = (1)$, in (1.10).

(b) See (II.2.3) for the definition of $\mu(P, \Gamma, L, \mathbf{P}^2)$. The fact that all blowings-up (in that sequence of m-trees) are of the third kind ($\Leftarrow$) is a consequence of (II.2.4b).

(c) It can be shown that if Γ is graph-theoretically linear then all open immersions $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ satisfy the condition of (1.10). We leave it to the interested reader to figure out the little argument which is needed here.

The following proposition is a corollary to (II.3.8). It will be used to show that certain curves in $\mathbf{A}^2$, with one place at infinity, are coordinate lines.

1.11. PROPOSITION. *Let Γ be a curve in $\mathbf{A}^2$, with one place at infinity. Then the following are equivalent:*

(a) Γ is graph-theoretically linear,

(b) Γ is a coordinate line.

PROOF: We first prove that (b)$\Longrightarrow$(a), which is the trivial part. Choose an open immersion $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ such that the closure in $\mathbf{P}^2$ of Γ is a projective line. Then $k(P,\Gamma,L,\mathbf{P}^2)=0$, i.e., $\mathcal{T}_k=\mathcal{T}_0=(1)$ which is already a linear local tree. Hence Γ is graph-theoretically linear.

(a)$\Longrightarrow$(b) Let Γ be graph-theoretically linear and let $j:\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ be an open immersion satisfying the condition of (1.10). Let the notation be as in (II.2.3) and consider the infinite sequence of m-trees of $(P,\Gamma,L,\mathbf{P}^2)$:

$$(\mathcal{T}_0,\mu_0) \Leftarrow \cdots \Leftarrow (\mathcal{T}_k,\mu_k) \Leftarrow \cdots .$$

By definition, $\mathcal{T}_k$ contracts to a linear local tree. If $k=0$ then $(\Gamma.L)_P=1$ in $\mathbf{P}^2$, by definition of k; hence $\Gamma.L=1$, Γ is a line in $\mathbf{P}^2$ and we are done. Assume $k>0$. Then the hypothesis of (II.3.8) is satisfied and, by the last assertion of (II.3.8), we see that $\mathcal{T}_{k+n}\geq(1)$, for some positive integer n. Writing $S_0=\mathbf{P}^2$, etc.,

$$\begin{array}{ccccccccc} S_0 & \leftarrow & \cdots & \leftarrow & S_k & \leftarrow & \cdots & \leftarrow & S_{k+n} \\ (\mathcal{T}_0,\mu_0) & \Leftarrow & \cdots & \Leftarrow & (\mathcal{T}_k,\mu_k) & \Leftarrow & \cdots & \Leftarrow & (\mathcal{T}_{k+n},\mu_{k+n}) \end{array}$$

where $\mathcal{T}_{k+n}\geq(1)$. Since all blowings-up have centers in $S_0\setminus\mathbf{A}^2$, $\mathbf{A}^2$ is naturally embedded in S_{k+n} and, in fact, $S_{k+n}\setminus\mathbf{A}^2=\operatorname{supp}(L^{k+n})$ (where the notation of (II.2.3) is used) and $(\mathcal{T}_{k+n},\mu_{k+n})$ is the m-tree of $(P_{k+n+1},\Gamma^{(k+n)},L^{k+n},S_{k+n})$. By iterating the argument of (II.3.3.1), we see that the contraction $\mathcal{T}_{k+n}\geq(1)$ corresponds to a birational morphism $\rho:S_{k+n}\to S'$ which contracts all components of L^{k+n} except E_{k+n}. Let $P'=\rho(P_{k+n+1})$, $\Gamma'=\rho_*(\Gamma^{(k+n)})$ and $L'=\rho_*(E_{k+n})$; then by (II.3.3.1) the m-tree of (P',Γ',L',S') is $((1),\mu')$, where the multiplicity μ' of the principal link of (1) is equal to the multiplicity μ_{k+n} of the principal link of $\mathcal{T}_{k+n}$, i.e., it is 1. Hence $(\Gamma'.L')_{P'}=1$ and since these two curves meet only at P', $\Gamma'.L'=1$. Now we have an embedding of $\mathbf{A}^2$ in the nonsingular projective surface S', such that the complement of $\mathbf{A}^2$ is one curve L'. As is well known (1.3), S' must be a projective plane. Since $\Gamma'.L'=1$, Γ' is a line in $S'=\mathbf{P}^2$ and we are done.

1.12. COROLLARY. *Let Γ be a curve of degree i in $\mathbf{A}^2$, with one place at infinity. Let $(r_0,r_1,\ldots)$ be the multiplicity sequence of Γ at infinity. Then the following are equivalent:*

(a) Γ is a coordinate line.

(b) Either $i=1$ or there is a positive integer k such that $(r_0,\ldots,r_{k-1})$ is a sequence of type $(1,i,1)$ ($\Longrightarrow i>1$ and $r_j=1$ if $j\geq k$).

PROOF: Clear from remark (c) after (1.10), together with (II.3.8).

REMARKS.

(a) (1.12) is not used in the sequel; that's why we didn't give a proof of remark (c), after (1.10).

(b) We mention without proof that, given any $i > 1$ and any sequence $(r_0, \ldots, r_{k-1})$ of type $(1, i, 1)$, there is a coordinate line having $(r_0, \ldots, r_{k-1}, 1, 1, \ldots)$ as its multiplicity sequence at infinity (read the proof of (1.11) backward!).

2. Some Results in the General Case.

The set of birational endomorphisms of $\mathbf{A}^2$ is a monoid, under composition of morphisms. An element f of that monoid is *trivial* if it is an automorphism of $\mathbf{A}^2$ (this is equivalent to the definition given in (I.1.3d) since any open immersion $\mathbf{A}^2 \hookrightarrow \mathbf{A}^2$ is onto by, say, (I.2.9)); it is *irreducible* if it is nontrivial and can't be written as $h \circ g$ where g and h are nontrivial,

Two birational endomorphisms f, g of $\mathbf{A}^2$ are *equivalent* if $f = v^{-1} \circ g \circ u$ for some automorphisms u, v of $\mathbf{A}^2$; we denote that by $f \sim g$. Triviality and irreducibility are properties that depend only on the equivalence class of f. As in the general case (I.1.1), we point out that $\sim$ is not compatible with the composition of morphisms.

Clearly, to give a birational endomorphism of $\mathbf{A}^2$ is equivalent to giving an endomorphism of $\mathbf{k}$-algebras

$$\phi : \mathbf{k}[X,Y] \to \mathbf{k}[X,Y]$$

such that $\mathbf{k}(\phi(X), \phi(Y)) = \mathbf{k}(X,Y)$.

To give an example, define

$$\begin{aligned} \phi_0 : \mathbf{k}[X,Y] &\to \mathbf{k}[X,Y] \\ X &\mapsto X \\ Y &\mapsto XY\,. \end{aligned}$$

The corresponding $\gamma_0 : \mathbf{A}^2 \to \mathbf{A}^2$ has one missing curve, the Y- axis, one contracting curve, the Y-axis, and one fundamental point, the origin. We leave it to the reader to verify that $n(\gamma_0) = 1$, i.e., that one can construct γ_0 by performing the following operations:

1. Blow-up $\mathbf{A}^2$ at the origin.
2. Remove, from the blown-up surface $\widetilde{\mathbf{A}^2}$, the strict transform $\widetilde{L}$ of the Y-axis L.
3. Recognise that $\widetilde{\mathbf{A}^2} \setminus \widetilde{L} \cong \mathbf{A}^2$.

The last step is to choose an isomorphism $\mathbf{A}^2 \to \widetilde{\mathbf{A}^2} \setminus \widetilde{L}$ and to obtain a birational morphism $\mathbf{A}^2 \to \widetilde{\mathbf{A}^2} \setminus \widetilde{L} \hookrightarrow \widetilde{\mathbf{A}^2} \to \mathbf{A}^2$. Depending on which isomorphism we choose, we get either γ_0 or some $\gamma \sim \gamma_0$.

2.1. DEFINITION. A birational morphism $f : \mathbf{A}^2 \to \mathbf{A}^2$ is called a *standard affine contraction in* $\mathbf{A}^2$ if the following equivalent conditions are satisfied:

(a) $f \sim \gamma_0$ (see above);

(b) $n(f) = 1$ and the missing curve of f is a coordinate line.

(Notice that, since $n(f) = 1 \Rightarrow c(f) = 1 \Rightarrow q(f) = 1$ by (I.2.8), f has one missing curve; the fundamental point of f belongs to that curve by (I.2.1), so it is true that (a)$\Leftrightarrow$(b)).

Observe that the "addition formula" $n(g \circ f) = n(f) + n(g)$ holds for birational endomorphisms of $\mathbf{A}^2$, by (I.2.10). In particular, if $n(f) = 1$ then f is irreducible.
Many questions can be asked. In particular,

- If $n(f) = 1$, is f necessarily a standard affine contraction in $\mathbf{A}^2$, i.e., must the missing curve of f be a coordinate line?
- Does there exist an irreducible f with $n(f) > 1$?
- What are the possible values of $n(f)$, $c(f)$, $q(f)$, $j(f)$, $\delta(f)$, for irreducible f? (We know $q_0(f) = 0$ by (I.2.9).)

(See the introduction for the history of these questions.)

In this section, we answer the above questions. Many other questions can be asked, but turn out to be very difficult in general; some of them will be answered in the next section, for the particular case $n(f) = 2$.

Let us now consider the main results of part I, and point out what they say about the special case "$X = Y = \mathbf{A}^2$".

2.2. COROLLARY. *Let $f : \mathbf{A}^2 \to \mathbf{A}^2$ be a birational morphism.*

(a) $q_0(f) = 0$, $q(f) = c(f)$ and $\delta(f) \le j(f)$ with equality iff f factors as $f = hg$, where g and h are birational endomorphisms of $\mathbf{A}^2$ such that $n(h) = q(h) = q(f)$ and $n(g) = j(f) = \delta(f)$.

(b) Given any minimal decomposition of f, the corresponding (square) matrix $\epsilon'\mu$ has determinant ± 1.

(c) Every missing curve of f is rational and has one place at infinity. Embed $\mathbf{A}^2$ in $\mathbf{P}^2$ the standard way; if two missing curves meet the line at infinity at distinct points then one of them is a linear line (1.9).

(d) All fundamental points of f are on the missing curves.

(e) Let the missing curves of f be $C_1, \ldots, C_q$. Let $\mathbf{A}^2 \hookrightarrow Y_{n(f)} \to \cdots \to Y_0 = \mathbf{A}^2$ be given by a minimal decomposition of f. Then $Y_{n(f)} \setminus \mathbf{A}^2$ has $q = q(f)$ connected components, each one being the support of some $D_i \in Div(Y_{n(f)})$ $(1 \le i \le q)$ with s.n.c. (in $Y_{n(f)}$), forming a linear tree

$$C_i - E_{j_1^i} - E_{j_2^i} - \cdots - E_{j_{k_i}^i},$$

where $\{j_1^i, \ldots, j_{k_i}^i\}$ is a (possibly empty) subset of J, C_i is not complete, but the $E_{j_\nu^i}$ are. Moreover, $E_{j_\nu^i}^2 < -1$ in $Y_{n(f)}$.

PROOF: $q_0(f) = 0$ by (I.2.9), $q(f) = c(f)$ by (I.2.8) and the rest of (a) by (I.3.7) and (1.8). (b) comes from (I.2.7), (d) from (I.2.1), (e) from (I.5.7) and the first two assertions of (c) from (I.5.3) and (I.5.5). We prove the rest of (c), i.e., the assertion about one of the missing curves being a linear line. Choose a minimal decomposition of f, with notation as usual, let $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ be a standard embedding (1.1), and write $L = \mathbf{P}^2 \setminus \mathbf{A}^2$. Complete the diagram:

$$\begin{tikzcd}
 & Y_n \arrow[r, hookrightarrow] \arrow[d, "\pi_n"] & \overline{Y}_n \arrow[d, "\overline{\pi}_n"] \\
 & \vdots \arrow[d, "\pi_1"] & \vdots \arrow[d, "\overline{\pi}_1"] \\
 & Y_0 \arrow[r, hookrightarrow] \arrow[d, equal] & \overline{Y}_0 \arrow[d, equal] \\
\mathbf{A}^2 \arrow[r, "f"'] \arrow[uuur] & \mathbf{A}^2 \arrow[r, hookrightarrow] & \mathbf{P}^2
\end{tikzcd}$$

and consider $D = L + C_1 + \cdots + C_q + \sum_{i\in J} E_i \in \mathrm{Div}(\overline{Y}_n)$. Then $\overline{Y}_n \setminus \mathbf{A}^2 = \mathrm{supp}(D)$ and, by (1.4), D has at most one "bad point". In particular if C_i meets L at Q_i (in $\mathbf{P}^2$), $i = 1, 2$, and if $Q_1 \neq Q_2$, then $(C_i.L)_{Q_i} = 1$ for some $i \in \{1,2\}$, i.e., $C_i.L = 1$ in $\mathbf{P}^2$ for some $i \in \{1,2\}$.

2.3. COROLLARY. *Given birational endomorphisms f, g of $\mathbf{A}^2$, $n(gf) = n(f) + n(g)$ and $j(gf) = j(f) + j(g) + \Delta c(f, g)$.*

PROOF: (I.2.10).

2.4. COROLLARY. *Let f be a birational endomorphism of $\mathbf{A}^2$, let (g, h) be any factorisation of f and write $W = dom(h) = codom(g)$. Then $q(h) + j(h) \leq n(h)$ and the following are equivalent:*

(a) $W \cong \mathbf{A}^2$

(b) $q(h) + j(h) = n(h)$ and (g, h) is a connected factorisation.

PROOF: (I.3.5) and (1.8). For the definitions, see the beginning of (I.3).

2.5. COROLLARY. *Let $f : \mathbf{A}^2 \to \mathbf{A}^2$ be a birational endomorphism with $j(f) = 0$, let D be any minimal decomposition of f, let $\mu = \mu_D$ and let r, s be positive integers such that $r + s = n = n(f)$. Then the following are equivalent:*

(a) $f = hg$, for some birational endomorphisms g, h of $\mathbf{A}^2$ such that $n(g) = r$ and $n(h) = s$.

(b) Modulo a permutation of the columns and a permutation of the rows, μ has the form

$$\begin{bmatrix} H & B \\ O & G \end{bmatrix}$$

where H is an $s \times s$ matrix and O is the $r \times s$ zero matrix.

PROOF: (I.3.6) and (1.8).

From (2.4) and the discussions of (I.3), it is clear that we have an algorithm that enumerates, for some given $f : \mathbf{A}^2 \to \mathbf{A}^2$, all equivalence classes of factorisations $\mathbf{A}^2 \to \mathbf{A}^2 \to \mathbf{A}^2$ of f. In particular, we have an algorithm that decides whether a given endomorphism is irreducible. However, such an algorithm doesn't help us to answer general questions like, say, "for which values of n are there irreducible endomorphisms f of $\mathbf{A}^2$ with $n(f) = n$?"

In fact, whether there exist irreducible birational endomorphisms with $n(f) > 1$ is a problem that remained open for some time when people began to investigate these morphisms (see the introduction). The following example settled the question.

2.6. EXAMPLE (RUSSELL). Let C_1 be an irreducible curve of degree two in $\mathbf{A}^2$, with one place at infinity (a parabola). Let P_1, P_2, P_3 be distinct points of C_1 and let C_2 (resp. C_3) be the linear line through P_1 and P_3 (resp. P_1 and P_2). Blow-up $\mathbf{A}^2$ at P_1, P_2, P_3 and remove the strict transforms of C_1, C_2, C_3 from the blown-up surface. Then the resulting open set is isomorphic to $\mathbf{A}^2$ and we obtain an irreducible birational morphism $f : \mathbf{A}^2 \to \mathbf{A}^2$ with $n(f) = 3$.

PROOF: First, we show that the surface obtained is $\cong \mathbf{A}^2$. Embed $\mathbf{A}^2$ in $\mathbf{P}^2$ the standard way and let $L = \mathbf{P}^2 \setminus \mathbf{A}^2$; let P be the place of C_1 at infinity. Blow-up $\mathbf{P}^2$ at P_1, P_2, P_3, denote the blown-up surface by $\widetilde{\mathbf{P}^2}$ and consider (i.e., make a picture of) the strict transforms of L, C_1, C_2, C_3 in $\widetilde{\mathbf{P}^2}$, with self-intersection numbers $1, 1, -1, -1$ respectively. To show: $U \cong \mathbf{A}^2$, where $U = \widetilde{\mathbf{P}^2} \setminus (L \cup C_1 \cup C_2 \cup C_3)$. By (1.2), enough to show that $[1] \in \mathcal{G}[U]$. So we blow-up until we get a divisor with *s.n.c.*; more precisely, since $(L.C_1)_P = L.C_1 = 2$, we blow-up twice at $P \in C_1 \subset \widetilde{\mathbf{P}^2}$. If the reader made the necessary pictures, he will probably agree that the resulting divisor, i.e., the reduced effective divisor at infinity of U, has *s.n.c.* and determines the dual graph (I.4.6)

$$\begin{array}{ccccc} (-1) & — & (-1) & — & (-1) \\ & & | & & \\ (-2) & — & (-1) & — & (-1), \end{array}$$

where the numbers are the weights. Now that weighted tree contracts to $[1, 0, -2]$, which is equivalent to $[1]$ by the observation just before (I.4.15). So $U \cong \mathbf{A}^2$. To prove irreducibility, consider

$$\mu = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and apply (2.5).

Remark. (1.8) was not available when Russell constructed the above example, so he couldn't use (2.5). However, he proved the following statement, which doesn't require (1.8):

If $f : \mathbf{A}^2 \to \mathbf{A}^2$ has ordinary fundamental points, then (a)$\Rightarrow$(b), in (2.5).

So, when (1.6) was discovered, Russell knew that (2.5) was true in the case of ordinary fundamental points. The generalisation to the case "$j(f) = 0$" is due to the writer.

Remark. In example (2.6), an equivalence class of birational endomorphisms $\mathbf{A}^2 \to \mathbf{A}^2$ is determined. One can show that, if $t \in \mathbf{k} \setminus \{0, 1\}$, then

$$\begin{aligned} \mathbf{k}[X, Y] &\longrightarrow \mathbf{k}[X, Y] \\ X &\longmapsto (X^2Y^2 - (t+1)XY - (t-1)^2Y + t)(X^2Y - tX - (t-1)^2) \\ Y &\longmapsto (X^2Y^2 - (t+1)XY - (t-1)^2Y + t)(X^2Y - X - (t-1)^2) \end{aligned}$$

gives an element of that equivalence class.

2.7. Example (Russell). Let $n \geq 3$, let C_1 be an irreducible curve of degree $n - 1$ in $\mathbf{A}^2$, such that

(a) C_1 has one place at infinity

(b) C_1 has a point P_1 (in $\mathbf{A}^2$) of multiplicity $n - 2$.

Clearly, such a curve exists. Choose distinct linear lines $C_2, \dots, C_n$ such that

(c) $C_i \cap C_1 = \{P_1, P_i\}$, some $P_i \in \mathbf{A}^2 \setminus \{P_1\}$ $(2 \leq i \leq n)$.

Blow-up $\mathbf{A}^2$ at $P_1, \dots, P_n$ and remove the strict transforms of $C_1, \dots, C_n$. The resulting surface is isomorphic to $\mathbf{A}^2$ and we get an irreducible $f : \mathbf{A}^2 \to \mathbf{A}^2$ with $n(f) = n$.

Verification left to the reader.

We see that irreducible endomorphisms $f : \mathbf{A}^2 \to \mathbf{A}^2$ with $n(f) = n$ exist for all $n \geq 1$. The case $n = 2$, which is not covered by the above examples, will be studied in detail in section 3.

The case $n(f) = 1$. Let f be a birational endomorphism with $n(f) = 1$. Then f is irreducible by (2.3). Since $c(f)$ is obviously 1, we have $q(f) = 1$ by (2.2a). We have $j(f) = n(f) - c(f) = 0$ and the matrix μ is the 1×1 matrix (1) by (2.2b). So $\mu(P_1, C) = 1$, where C is the missing curve and P_1 the fundamental point; by (2.2e), C is nonsingular after the blowing-up of $\mathbf{A}^2$ at P_1 (it may have a singularity at infinity), so it is already nonsingular in $\mathbf{A}^2$. Hence $C \cong \mathbf{A}^1$ by (2.2c), i.e., C is a line (1.9). To conclude that f is a standard affine contraction (2.1), all there remains to show is that C is a coordinate line. Generally, that sort of problem can be very difficult [1]. We will show that C is a coordinate line (hence that f is a standard affine contraction) by showing that it is graph-theoretically linear (1.10), (1.11). This means that a certain local tree $\mathcal{T}_k$ contracts to a linear local tree; to prove that, we assume that $\mathcal{T}_k$ does not contract, and we use our knowledge of C to exhibit a triple (d, u, v) which violates (II.3.32). Note that our argument is valid in arbitrary characteristic.

2.8. **THEOREM.** *Let f be a birational endomorphism of $\mathbf{A}^2$, with $n(f) = 1$. Then f is a standard affine contraction.*

PROOF: Embed $\mathbf{A}^2$ in $\mathbf{P}^2$ the standard way, let $L = \mathbf{P}^2 \setminus \mathbf{A}^2$, let C be (the closure in $\mathbf{P}^2$ of) the missing curve of f and let P be the place of C at infinity (of $\mathbf{A}^2$). Writing $S_0 = \mathbf{P}^2$, etc., consider

$$\begin{array}{rccccc} & S_0 & \leftarrow & \cdots & \leftarrow & S_k \\ \mu(P, C, L, \mathbf{P}^2): & (\mathcal{T}_0, \mu_0) & \Leftarrow & \cdots & \Leftarrow & (\mathcal{T}_k, \mu_k). \end{array}$$

If $k = 0$ then $C.L = \mu_0(\{P, L\}) = \mu_k(\{P, L\}) = 1$, so C is a line in $\mathbf{P}^2$, i.e., a linear line in $\mathbf{A}^2$, and we are done.

Assume $k > 0$.

Let $d = \deg C$, $u = \mu(P_1, C)$ (i.e., $u = 1$, but we don't need to know that) and

$$\alpha = d^2 - u^2 - \sum_{j=0}^{k-1} (\mu_j(x_0))^2.$$

In the notation of (II.2.3), $\mathcal{T}_k[\alpha] \cong \mathcal{G}(S_k, C^{(k)} + L^k) \sim [1]$. Clearly the principal vertex of $\mathcal{T}_k$ is a branch point. So, if $\mathcal{T}_k$ does not contract to a linear local tree, the hypothesis of (II.3.32) is satisfied. Then $(d, u, 0)$ violates (II.3.32), which is absurd. Consequently, $\mathcal{T}_k$ does contract to a linear local tree. Hence the missing curve is graph-theoretically linear, i.e., it is a coordinate line.

We now return to the general case, i.e., $n(f) \geq 1$. The above theorem generalises as follows:

2.9. **THEOREM.** *Let f be a birational endomorphism of $\mathbf{A}^2$ such that $q(f) = 1$. Then f is a composition of $n(f)$ standard affine contractions in $\mathbf{A}^2$. In particular, the missing curve and the contracting curve are coordinate lines.*

PROOF: Let C denote the missing curve of f. We proceed by induction on $n = n(f)$.

The case $n = 1$ is just (2.6), above.

Let $n > 1$ be such that the theorem holds whenever $n(f) < n$. Let f be such that $n(f) = n$. Choose a minimal decomposition of f, with notation as in (I.1.3h). Since $j(f) = n(f) - c(f) = n(f) - q(f) = n - 1$ and $n \notin J$ by (I.1.3i),

(1) $$J = \{1, \ldots, n-1\}.$$

Again by (I.1.3i),

$$(2) \qquad P_{i+1} \in E_i, \quad 1 \le i < n.$$

Thus an elementary calculation shows that

$$(3) \qquad \epsilon_{1j} \le \cdots \le \epsilon_{nj}, \quad 1 \le j \le n$$

(see (I.2.5) and (I.2.4) for definitions). Since $\epsilon_{jj} = 1$, we deduce

$$(4) \qquad \epsilon_{nj} \ge 1, \quad 1 \le j \le n.$$

On the other hand,

$$\epsilon' \mu = \left(\sum_{j=1}^{n} \epsilon_{nj} \mu(P_j, C) \right) \quad (1 \times 1 \text{ matrix})$$

so by (2.2b)

$$(5) \qquad \sum_{j=1}^{n} \epsilon_{nj} \mu(P_j, C) = 1.$$

By (4) and (5)

$$1 \ge \sum_{j=1}^{n} \mu(P_j, C) \ge \mu(P_1, C) = 1, \quad \text{so}$$

$$(6) \qquad \mu = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and consequently $\epsilon_{n1} = 1$ by (5) and (6). If $1 \le i < n$ then by (3) and (2)

$$\begin{aligned} 1 = \epsilon_{n1} \ge \epsilon_{i+1\,1} &= \sum_{k=1}^{i} \epsilon_{k1} \mu(P_{i+1}, E_k) \\ &\ge \left[\sum_{k=1}^{i-1} \mu(P_{i+1}, E_k) \right] + \mu(P_{i+1}, E_i) \ge \mu(P_{i+1}, E_i) = 1, \end{aligned}$$

whence

$$(7) \qquad P_{i+1} \in E_j \text{ in } Y_i \iff j = i, \qquad \text{all } i, j.$$

By (6) and (7), $P_n \notin (C \cup E_1 \cup \cdots \cup E_{n-2})$ in Y_{n-1}. So the image of $\mathbf{A}^2 \hookrightarrow Y_n \to Y_{n-1}$ is contained in $W = Y_{n-1} \setminus (C \cup E_1 \cup \cdots \cup E_{n-2})$. In other words, we have a factorisation (g, h) of f ($g : \mathbf{A}^2 \to W$ and $h : W \to \mathbf{A}^2$, see (I.3)). Now g has one missing curve, E_{n-1}. Since $P_n \in E_{n-1}$, $q_0(f) = 0$. On the other hand, $Y_n \setminus W$ is connected by (6) and (7), and contains C, so (g, h) is a connected factorisation of f. Clearly, $n(h) = n - 1$, $W \hookrightarrow Y_{n-1} \to \cdots \to Y_0$ gives a minimal decomposition of h and consequently $j(h) = n - 2$. Since C is a missing curve of h, $q(h) + j(h) \ge n - 1 = n(h)$. Then $W \cong \mathbf{A}^2$ by (2.4) and we may apply the inductive hypothesis to h.

REMARK. In the above argument, once (6) and (7) are proved we know that the contracting curve $E_n \cap \mathbf{A}^2$ of f has one place at infinity, and hence is a line ($\cong \mathbf{A}^1$). Then we could invoke [**15**, remark 3.4] and conclude immediately. However, that would not be a significant improvement, since the essential part of our proof is to establish (6) and (7).

Until recently, no example of an irreducible $f : \mathbf{A}^2 \to \mathbf{A}^2$ with $j(f) > 0$ was known; Russell did construct examples with infinitely near fundamental points, but they all had $j(f) = 0$. Moreover, the above theorem says that if $j(f)$ has the maximum possible value, i.e., $j(f) = n(f) - 1$, then f is reducible (unless $n(f) = 1$, of course). In fact, when the author proved that theorem, he was hoping that it was the first step in the proof that $j(f) > 0 \Rightarrow f$ reducible. However, his attempts resulted in the construction of a family of examples (2.11), showing that he was as wrong as he could possibly be. To clarify the situation, we have the following statement.

2.10. THEOREM. *Let n, j and δ be nonnegative integers. There exists an irreducible birational morphism $f : \mathbf{A}^2 \to \mathbf{A}^2$ satisfying $n(f) = n$, $j(f) = j$ and $\delta(f) = \delta$ if and only if one of the following conditions holds:*

(a) $0 = \delta = j < n$

(b) $0 \le \delta < j < n - 1$.

PROOF: Suppose there exists an irreducible f such that $n(f) = n$, $j(f) = j$ and $\delta(f) = \delta$. We have $0 \le \delta \le j < n$ by (2.2a). If $j = 0$ then (a) holds. If $j > 0$ then $n > 1$, so $\delta < j$ by (2.2a) and $j < n - 1$ by (2.9), i.e., (b) holds.

Conversely, the case (a) with $n = 1$ is realised by the standard affine contractions; the case (a) with $n > 2$ is realised by (2.7); and the case (a) with $n = 2$ is realised, as we will see in section 3. If (n, j, δ) satisfies (b), let $m = j - \delta + 1 \ge 2$ and $q = n - j \ge 2$ and choose $\delta_1 \ge 0, \dots, \delta_{q-1} \ge 0$ such that $\delta_1 + \cdots + \delta_{q-1} = \delta$. Then example (2.11) realises these numbers.

2.11. EXAMPLE. Let $m \ge 2$, $q \ge 2$, $\delta_1 \ge 0, \dots, \delta_{q-1} \ge 0$ be integers. We will construct an irreducible birational morphism $f : \mathbf{A}^2 \to \mathbf{A}^2$ with two fundamental points and satisfying

$$\begin{aligned} n(f) &= m + q - 1 + \delta_1 + \cdots + \delta_{q-1} \\ q(f) &= q \\ \delta(f) &= \delta_1 + \cdots + \delta_{q-1} \\ j(f) &= m - 1 + \delta(f). \end{aligned}$$

Choose $F_1, \dots, F_q \in \mathbf{k}[X, Y]$ such that if C_i is the affine plane curve $F_i(X, Y) = 0$ then

- C_i is a nonsingular rational curve of degree m, with one place at infinity, with multiplicity sequence at infinity: $m - 1, 1, 1, \dots$.
- There are distinct points $P_1, P_2 \in \mathbf{A}^2$ such that

$$\forall_{i \ne j} \left[C_i \cap C_j = \{P_1, P_2\},\ (C_i.C_j)_{P_1} = 1,\ (C_i.C_j)_{P_2} = m - 1 \right].$$

(For instance, $F_i = a_i Y^{m-1}(Y - 1) + X$, where $a_1, \dots, a_q$ are distinct elements of $\mathbf{k}^*$; then $P_1 = (0, 1)$ and $P_2 = (0, 0)$.)

We are going to embed $\mathbf{A}^2$ in $\mathbf{F}_m$ (one of the Nagata rational surfaces). First, embed $\mathbf{A}^2$ in $\mathbf{P}^2$ the standard way and write $\mathbf{P}^2 \setminus \mathbf{A}^2 = L$. Let C_i also denote the closure in $\mathbf{P}^2$ of the curve C_i chosen above. The curves $C_1, \dots, C_q$ all meet L at the same point P. Notice that

1) $C_i \cap L = \{P\}$, $\mu(P, C_i) = m - 1$, $C_i.L = m$, all i.
2) $C_i.C_j = m^2$, all i, j.
3) $(C_i.C_j)_P = m^2 - m$, all distinct i, j.

Blow-up $\mathbf{P}^2$ at $A_1 = P$, let D_1 be the exceptional curve, let A_2 be the point at which D_1 and L meet. Then

4) $C_i \cap L = \{A_2\} = C_i \cap D_1$, $C_i \cong \mathbf{P}^1$, $C_i^2 = 2m - 1$, $C_i.L = 1$, $C_i.D_1 = m - 1$, for all i.
5) $C_i \cap C_j = \{P_1, P_2, A_2\}$, $(C_i.C_j)_{A_2} = m - 1$, all distinct i, j.

Blow-up $m - 1$ times at the point of D_1 which is *i.n.* A_2. Call the exceptional curves so obtained $D_2, \dots, D_m$. On the resulting surface, the divisor $D_1 + \cdots + D_m + L$ has *s.n.c.*, its dual graph is the linear weighted tree

$$D_1(-m) \text{---} D_m(-1) \text{---} D_{m-1}(-2) \text{---} \cdots \text{---} D_2(-2) \text{---} L(-1)$$

where the numbers inside the "( )" are the weights, and the complement of that divisor is $\mathbf{A}^2$. Contract $L, D_2, \dots, D_{m-1}$ and let S_0 denote the complete surface obtained. We get $\mathbf{A}^2 = S_0 \setminus \operatorname{supp}(D_1 + D_m)$, where $D_1 + D_m \in \operatorname{Div}(S_0)$ has *s.n.c.* and has dual graph $\mathcal{G}(S_0, D_1 + D_m)$ as follows:

$$D_1(-m) \text{---} D_m(0).$$

In fact, $S_0 = \mathbf{F}_m$ (but we don't really need to know that).

Now $C_1, \dots, C_m$ meet D_m at distinct points and

6) $C_i \cap D_1 = \emptyset$, $C_i.D_m = 1$ and $C_i^2 = m$, all i.

We now proceed to define an equivalence class of irreducible morphisms $f : \mathbf{A}^2 \to \mathbf{A}^2$. Blow-up once at P_1; blow-up $m - 1$ times at P_2 (more precisely, always blow-up at the intersection point of (the strict transforms of) the C_i's). The last of these blowings-up makes $C_1, \dots, C_q$ pairwise disjoint. If $E_1, E_2, \dots, E_m$ are the exceptional curves so created, then on the blown up surface the divisor $E_2 + \cdots + E_m + C_1 + \cdots + C_q + D_1 + D_m$ has *s.n.c.* and its dual graph is

$$\begin{array}{c} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad C_1(0) \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad / \quad \vdots \quad \backslash \\ E_2(-2) \text{---} \cdots \text{---} E_{m-1}(-2) \text{---} E_m(-1) \text{---} C_q(0) \text{---} D_m(0) \text{---} D_1(-m). \end{array}$$

For $i = 1, \dots, q-1$, let Q_i be the intersection point of E_m and C_i.

Blow-up $\delta_1 + 1$times at Q_1,

then $\delta_2 + 1$times at Q_2,

$\vdots$

and $\delta_{q-1} + 1$times at Q_{q-1};

more precisely, always blow-up the point of E_m which is i.n. Q_i.
Denote by $E_1^1, \dots, E_{\delta_1+1}^1, E_1^2, \dots, E_{\delta_{q-2}+1}^{q-2}, E_1^{q-1}, \dots, E_{\delta_{q-1}+1}^{q-1}$ the exceptional curves so created. On the resulting surface, call it S_n, consider the divisor

$$D = E_2 + \cdots + E_m + (E_1^1 + \cdots + E_{\delta_1}^1) + \cdots + (E_1^{q-1} + \cdots + E_{\delta_{q-1}}^{q-1}) + C_1 + \cdots + C_q + D_m + D_1,$$

whose dual graph $\mathcal{G}(S_n, D)$ is

$$\begin{array}{c}
\hspace{20em} \mathcal{B}_1 \quad \cdots \quad \mathcal{B}_{q-1} \\
\hspace{20em} \diagdown \quad \diagup \\
E_2(-2) — \cdots — E_{m-1}(-2) — E_m(-q-\delta_1-\cdots-\delta_{q-1}) — C_q(0) — D_m(0) — D_1(-m)
\end{array}$$

where, for $i = 1, \dots, q-1$, $\mathcal{B}_i$ is

$$C_i(-1) — E_1^i(-2) — \cdots — E_{\delta_i}^i(-2),$$

C_i being linked to D_m. We claim that the complement of supp (D) is isomorphic to $\mathbf{A}^2$. By (1.2), enough to show that $\mathcal{G}(S_n, D) \sim [1]$. Now $\mathcal{G}(S_n, D)$ contracts to

$$\begin{aligned}
&[-2, \dots, -2, -q-\delta_1-\cdots-\delta_{q-1}, 0, \delta_1+\cdots+\delta_{q-1}+q-1, -m] \\
&\quad \sim [-2, \cdots, -2, -1, 0, 0, -m] \sim [m-1, 0, -m] \sim [-1, 0, 0] \sim [1],
\end{aligned}$$

where we use the notation for linear weighted trees defined before (I.4.15) and the fact pointed out just after that definition.

So we get an equivalence class of birational morphisms $f : \mathbf{A}^2 \to \mathbf{A}^2$. We leave it to the reader to convince himself that, if $f = h \circ g$ with $0 < n(h) < n(f)$, then h gives rise to a sub weighted tree $\mathcal{G}'$ of $\mathcal{G} = \mathcal{G}(S_n, D)$, such that $\mathcal{G}'$ contains D_1, D_m and at least one more vertex, $\mathcal{G}' \neq \mathcal{G}$ and $\mathcal{G}' \sim [1]$. We claim that $\mathcal{G}$ does not contain such a $\mathcal{G}'$. To see that, suppose $\mathcal{G}'$ exists. Then C_q is in $\mathcal{G}'$, otherwise $\mathcal{G}'$ would contract to $[p, -m]$ for some $p > 0$, and $[p, -m] \not\sim [1]$ by (I.4.16). Next, E_m is in $\mathcal{G}'$, for otherwise $\mathcal{G}'$ contracts to $[0, p, -m]$ for some $p \geq 0$, and by (I.4.16) this is not equivalent to $[1]$. So $\mathcal{G}'$ has the form

$$\begin{array}{c}
\hspace{12em} \mathcal{B}_1' \quad \cdots \quad \mathcal{B}_{q-1}' \\
\hspace{12em} \diagdown \quad \diagup \\
\mathcal{B}' — E_m(-q-\delta_1-\cdots-\delta_{q-1}) — C_q(0) — D_m(0) — D_1(-m)
\end{array}$$

where each $\mathcal{B}'$, $\mathcal{B}_i'$ is either empty or a linear branch, and

$$\begin{aligned}
\mathcal{B}_i' &= [-1, -2, \dots, -2] \quad \text{if not empty,} \\
\mathcal{B}' &= [-2, \dots, -2] \quad \text{if not empty.}
\end{aligned}$$

Notice that, if $\mathcal{B}_i'$ is not empty then the vertex of weight -1 is there and is the neighbour of D_m. Hence we see that all (nonempty) $\mathcal{B}_i'$ can be absorbed by D_m, and that the absorption of $\mathcal{B}_i'$ increases the weight of D_m by the number $|\mathcal{B}_i'|$. Let $a = |\mathcal{B}_1'| + \cdots + |\mathcal{B}_{q-1}'|$. Then $\mathcal{G}'$ contracts to the minimal weighted tree

$$\mathcal{B}' — E_m(-q-\delta_1-\cdots-\delta_{q-1}) — C_q(0) — D_m(a) — D_1(-m).$$

By (I.4.16), $a - q - \delta_1 - \cdots - \delta_{q-1} = -1$, so $|\mathcal{B}_1'| + \cdots + |\mathcal{B}_{q-1}'| = |\mathcal{B}_1| + \cdots + |\mathcal{B}_{q-1}|$, i.e., $\mathcal{B}_i' = \mathcal{B}_i$ for all i. Let $b = |\mathcal{B}'|$. Then

$$\mathcal{G}' \sim [-2, \ldots, -2, -q - \delta_1 - \cdots - \delta_{q-1}, 0, a, -m]$$
$$\sim [-2, \ldots, -2, -1, 0, 0, -m] \sim [b + 1, 0, -m]$$

where we used the observation just before (I.4.15). By (I.4.16) again, $b + 1 - m = -1$, i.e., $b = m - 2$ and $\mathcal{G}' = \mathcal{G}$. Hence f is irreducible.

3. The Case $n(f) = 2$.

In this section, we classify irreducible birational endomorphisms $f : \mathbf{A}^2 \to \mathbf{A}^2$ such that $n(f) = 2$. Observe that $j(f) = \delta(f) = 0$ by (2.10), so f has two missing curves. Moreover, it follows from (2.5) that all entries of the matrix μ (determined by any minimal decomposition of f) are nonzero.

3.1. Theorem. *Let $f : \mathbf{A}^2 \to \mathbf{A}^2$ be an irreducible birational morphism with $n(f) = 2$. Then there is a coordinate system on $\mathbf{A}^2$ such that, if $\mathbf{A}^2$ is embedded in $\mathbf{P}^2$ the standard way, then the closures of the missing curves meet the line at infinity at distinct points.*

Moreover that coordinate system is unique, up to affine automorphism of $\mathbf{A}^2$, and has the following property: if the missing curves C_1, C_2, and the fundamental points P_1, P_2 are suitably labelled, then

(a) C_1 is a rational curve of degree $2b + 1$ (for some $b \in \mathbf{N}$), with one place at infinity;

(b) $\mu(P_1, C_1) = b + 1$ and $\mu(P_2, C_1) = b$;

(c) C_2 is the linear line through P_1 and P_2 (note that P_2 is allowed to be i.n. P_1);

(d) The multiplicity sequence of C_1 at infinity begins with a sequence of type $(2, 2b+1, 1)$ and continues $1, 1, \ldots$.

3.2. Theorem. *Let C_1, C_2, P_1, P_2 satisfy the conditions (a)–(d) of (3.1). Then there exists an irreducible birational morphism $f : \mathbf{A}^2 \to \mathbf{A}^2$, with $n(f) = 2$, having C_1 and C_2 as missing curves and P_1 and P_2 as fundamental points.*

3.3. Remarks. The following comments required extensive computations that the author carried out by using methods that R. Ganong explained to him. In this regard, the author would like to express his thanks to Ganong.

(1) The condition (d) of (3.1) is not superfluous, i.e., there are curves C_1, C_2 and points P_1, P_2 satisfying (a), (b), (c) but not (d). We have the following example:

Let $\operatorname{char} \mathbf{k} = 3$,

$C_1 :\quad Y^{12} + XY^7 + X^3Y^6 - X^{15} = 0$

$C_2 :\quad Y = 0$

$P_1 = (0, 0)$ and P_2 is the unique point common to the strict transforms of C_1 and C_2 after blowing-up $\mathbf{A}^2$ at P_1 (so P_2 is i.n. P_1).

The reader can verify that (a), (b), (c) hold and that the multiplicity sequence of C_1 at infinity is

$$3, \ldots, 3, 1, 1, \ldots,$$

where "3" occurs 14 times. So (d) does not hold. The question whether such an example exists if $\operatorname{char} k = 0$ is open. No example exists such that C_1 has degree less than 15 (any characteristic).

(2) We have the following question. Given a positive integer b and a sequence $(r_0, \dots, r_{k-1})$ of type $(2, 2b+1, 1)$, do there exist C_1, C_2, P_1, P_2 satisfying (a), (b), (c) of (3.1) and such that the multiplicity sequence of C_1 at infinity begins with $r_0, \dots, r_{k-1}$ (and continues with $1, 1, \dots$)?

Now let l be as in (II.3.5). The answer is *yes* if $l = 1$ or 2. Indeed, write $(r_0, \dots, r_{k-1}) = (m_0, m_0, i_1, \dots, i_l)$ as in (II.3.5), let C_2, P_1, P_2 be as in (1), above, and let C_1 be the curve

$$\begin{cases} X^{i_0} + Y^{m_0+1} = 0 & \text{if } l = 1, \\ \left(X^{i_0/i_1} + Y^{m_0/i_1+1}\right)^{i_1} + XY^{m_0+m_1} = 0 & \text{if } l = 2. \end{cases}$$

Then the desired conditions are satisfied.

The author's opinion is that the answer is *yes* in general. In fact, he also found some examples with $l = 3$ and was beginning to understand how to go from "l" to "$l+1$", when he ran out of time and stopped thinking about that problem.

(3) From what has been said, we conclude that

(i) If f, C_1, C_2, P_1, P_2 are as in (3.1) then
$$\mu = \begin{bmatrix} b+1 & 1 \\ b & 1 \end{bmatrix}.$$

(ii) Given $b \in \mathbf{N}$, the matrix displayed in (i) can be obtained from an irreducible $f : \mathbf{A}^2 \to \mathbf{A}^2$ with $n(f) = 2$. Further, if $2b+1$ is not a prime number then, by remark 2 following (II.3.5), there is a sequence of type $(2, 2b+1, 1)$ with $l = 2$ (and there is always one with $l = 1$) so by (2), above, together with (3.2), there are nonequivalent f's realising the matrix considered above. (The "nonequivalence" comes from the discussion preceding (1.10)).

PROOF OF (3.1): We are going to construct an open immersion $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ such that the closures of the missing curves meet the line at infinity at distinct points (that is clearly equivalent to the existence of a coordinate system on $\mathbf{A}^2$ with the asserted property). All other assertions will be easily deduced from that.

Consider a minimal decomposition of f and the corresponding matrix μ:

$$\begin{tikzcd} & & Y_2 \arrow[d] \\ & & Y_1 \arrow[d] \\ \mathbf{A}^2 \arrow[uurr] \arrow[r, "f"'] & \mathbf{A}^2 \arrow[r, equal] & Y_0 \end{tikzcd} \qquad \mu = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

where we assume that $a > b \geq 1$ (this is possible because no entry of μ is zero and $\det \mu = \pm 1$). As usual, let C_1, C_2 be the missing curves and P_1, P_2 the fundamental points. Choose any open immersion $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ and let L be the line at infinity; consider the diagram

$$\begin{tikzcd} & Y_2 \arrow[r, hook] \arrow[d] & Z_2 \arrow[d] \\ & Y_1 \arrow[r, hook] \arrow[d] & Z_1 \arrow[d] \\ & Y_0 \arrow[r, hook] \arrow[d, equal] & Z_0 \arrow[d, equal] \\ \mathbf{A}^2 \arrow[uuur] \arrow[r, "f"'] & \mathbf{A}^2 \arrow[r, hook] & \mathbf{P}^2 \end{tikzcd}$$

Then $\mathbf{A}^2 = Z_2 \setminus \operatorname{supp}(C_1 + C_2 + L)$. Consider a sequence

$$Z_2 = S_0 \overset{\sigma_0}{\longleftarrow} S_1 \overset{\sigma_1}{\longleftarrow} \cdots \overset{\sigma_N}{\longleftarrow} S_N \qquad (N \geq 0)$$

where each σ_i is the blowing-up of S_{i-1} at a closed point s_i at infinity of $\mathbf{A}^2$, such that if we write $F_i = \sigma_i^{-1}(s_i)$ and

$$\begin{aligned} L^0 &= L \in \operatorname{Div}(S_0), \\ L^i &= L^{i-1} + F_i \in \operatorname{Div}(S_i) \qquad (1 \leq i \leq N), \end{aligned}$$

then $C_1 + C_2 + L^N \in \operatorname{Div}(S_N)$ has *s.n.c.*. Assume that N is minimal with respect to these properties. If $N = 0$ then $C_1 + C_2 + L$ has *s.n.c.* in S_0 and, in particular, C_1 and C_2 meet L at distinct points (in $Z_0 = \mathbf{P}^2$) and we are done. Let us assume that the missing curves meet the line at infinity at the same point. Then $N > 0$ and, by (1.4c), the center s_i of σ_i belongs to the support of $C_1 + C_2 \in \operatorname{Div}(S_{i-1})$ (all i). So the following notation makes sense:

$$\{C, C'\} = \{C_1, C_2\}, \quad \text{where} \quad \forall_i \;\; s_i \in C.$$

By (1.4b), $s_1 \in C \cap L$ in S_0. Moreover, the curve in Y_0 which corresponds to C has one place P at infinity (2.2c). So $S_0 \leftarrow \cdots \leftarrow S_N$ is the beginning of the infinite sequence of monoidal transformations determined by the triple (P, C, S_0) (see (II.2.3)). To that infinite sequence, there corresponds the infinite sequence of m-trees of (P, C, L, S_0); let

$$(\mathcal{T}_0, \mu_0) \Leftarrow \cdots \Leftarrow (\mathcal{T}_N, \mu_N)$$

be the first $N+1$ terms of that sequence. Then $\mathcal{T}_0 = (1)$ and $\mathcal{T}_N$ has one principal link. On the other hand, the curve in Y_0 which corresponds to C' has one place P' at infinity; define

$$k = \max\{j \mid 1 \leq j \leq N \text{ and } s_j \in C'\}.$$

Then $S_0 \leftarrow \cdots \leftarrow S_k$ (where the morphisms are $\sigma_1, \ldots, \sigma_k$) is the beginning of the infinite sequence of monoidal transformations determined by the triple (P', C', S_0); if we let $(\mathcal{T}'_j, \mu'_j)$ be the m-tree of (P', C', L^j, S_j), $0 \leq j \leq k$, then we have the sequence of m-trees

$$(\mathcal{T}'_0, \mu'_0) \Leftarrow \cdots \Leftarrow (\mathcal{T}'_k, \mu'_k)$$

where $(\mathcal{T}'_0 \leftarrow \cdots \leftarrow \mathcal{T}'_{k-1}) = (\mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_{k-1})$. Since σ_k is the last blowing-up which has center on C', $\mathcal{T}'_k$ must have one principal link a' and $\mu'_k(a') = 1$; whence $k \geq k(P', C', L, Z_2)$. We have the following diagram:

$$\begin{array}{ccccccccc} & & & & & & \mathcal{T}'_k & & \\ & & & & & & \downarrow & & \\ (1) = \mathcal{T}_0 & \leftarrow & \cdots & \leftarrow & \mathcal{T}_{k-1} & \overset{\pm}{\leftarrow} & \mathcal{T}_k^{s,a} & \leftarrow \cdots \leftarrow & \mathcal{T}_N^{s,a} \end{array}$$

where e is the vertex created in $\mathcal{T}_{k-1} \leftarrow \mathcal{T}_k$ and $\alpha = C'^2$ in S_N (or in S_k, since these two numbers are equal). If $\beta = C^2$ in S_N, then $\mathcal{T}_N^{e,\alpha}[\beta] = \mathcal{G}(S_N, C + C' + L^N) \sim [1]$ since $S_N \setminus \operatorname{supp}(C + C' + L^N) \cong \mathbf{A}^2$.

If $\mathcal{T}_k'$ does not contract to a linear local tree then we may apply (II.3.27) to $\mathcal{T}_k' \to \mathcal{T}_{k-1}$ and the weak sequence $\mathcal{T}_{k-1}, \mathcal{T}_k^{e,\alpha}, \ldots, \mathcal{T}_N^{e,\alpha}$. The conclusion says that

$$\mathcal{T}_0' \leftarrow \cdots \leftarrow \mathcal{T}_k'$$

satisfies the hypothesis of (II.3.28), hence that of (II.3.32). On the other hand, if we let d be the degree of C' in $Z_0 = \mathbf{P}^2$, $u = \mu(P_1, C')$ and $v = \mu(P_2, C')$ then the triple (d, u, v) satisfies condition (a) of (II.3.32), which is absurd.

Hence $\mathcal{T}_k'$ does contract to a linear local tree. As observed above, $k \geq k(P', C', L, Z_2)$; it follows that all trees in

$$\mathcal{T}'_{k(P',C',L,Z_2)} \leftarrow \cdots \leftarrow \mathcal{T}_k'$$

have one principal link and, consequently, that $\mathcal{T}'_{k(P',C',L,Z_2)}$ contracts to a linear local tree. Since the blowings-up $Z_0 \leftarrow Z_1 \leftarrow Z_2$ have centers in $\mathbf{A}^2$, hence away from L, the sequences $\mu(P', C', L, Z_0)$ and $\mu(P', C', L, Z_2)$ are just the same. Hence the appropriate local tree contracts to a linear local tree and $C' \subset \mathbf{A}^2$ is graph-theoretically linear. By (1.11), C' is a coordinate line. Since we assumed, at the beginning of this proof, that the entry a of μ was greater than 1, i.e., that C_1 was singular, C_2 is a coordinate line.

Observe that, since C_2 is nonsingular and $\det \mu = \pm 1$,

$$\mu = \begin{bmatrix} b+1 & 1 \\ b & 1 \end{bmatrix}.$$

Hence condition (b) holds, in the statement of the theorem (we will see that the number b of (a) is the same as this one).

We may assume that the open immersion $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ has been chosen such that the closure of C_2 is a line. If C_1, C_2 don't meet at infinity, we are done (with this part of the proof). So assume that $C_1 \cap C_2 \cap L \neq \emptyset$. It then follows that $L^2 \leq 0$ in S_N. Also, all that has been said, above, is still valid now.

Observe that the principal vertex of $\mathcal{T}_N^{e,\alpha}$ is a branch point (otherwise the blowing-up $S_{N-1} \leftarrow S_N$ is superfluous, i.e., N not minimal) and has weight -1. On the other hand, $\mathcal{T}_N^{e,\alpha}[\beta] \sim [1]$. From these facts, we deduce that L has negative weight in $\mathcal{T}_N^{e,\alpha}$. To see that, suppose that L has nonnegative weight; then that weight is 0, as noticed above. Then L must be a neighbour of the principal vertex in $\mathcal{T}_N^{e,\alpha}$ (otherwise, any linear weighted tree $\mathcal{L}^+$ to which $\mathcal{T}_N^{e,\alpha}[\beta]$ contracts satisfies $\langle \mathcal{L}^+ \rangle > 1$, as explained in the proof of (II.3.27)). This means that all blowings-up $(\sigma_1, \ldots, \sigma_N)$ have centers on L, so the final weight is $0 = 1 - N$ and $N = 1$. Consequently, $k = 1$, $\alpha = -2$ and $\mathcal{T}_N^{e,\alpha} = (*, -1, (0), (-2))$ in the notation of (II.3.18). Since $\mathcal{T}_N^{e,\alpha}[\beta]$ must contract to a linear tree, $\beta = -1$ and $\mathcal{T}_N^{e,\alpha}[\beta] \sim [0, 0, -2] \not\sim [1]$ (I.4.16), contradiction.

Hence L has negative weight in $\mathcal{T}_N^{e,\alpha}$. Thus at least two blowings-up have center on L, i.e., C_1 and L meet in S_1. Since $C_2.L = 1$ in $\mathbf{P}^2$, hence in S_0, they can't meet in S_1. So C_2 and C_1 are disjoint in S_1, $s_2 \notin C_2$, $k = 1$ and $\alpha = -2$. Clearly, it follows that $\mathcal{T}_1^{e,\alpha} = (*, (-1, -2), (0))$.

CLAIM. $\mathcal{T}_N^{e,\alpha} = \mathcal{T}_N^{e,-2}$ contracts to a linear local tree.

Let d be the degree of C_1 in $Y_0 = \mathbf{P}^2$ and let $r_j = \mu_j(x_0)$, $0 \le j \le N$. Define m-trees $(\mathcal{G}_0, M_0)$ and $(\mathcal{G}_1, M_1)$ by

- $\mathcal{G}_0 = (2)$, $M_0(x_0) = r_0$ and if a_0 denotes the link then $M_0(a_0) = d + r_0$;
- $\mathcal{G}_1 = (*, (-1), (1))$, $M_1(x_0) = r_0$ and if ξ and ς are the links corresponding to the weights -1 and 1 respectively, $M_1(\xi) = r_0$ and $M_1(\varsigma) = d$.

Let also $(\mathcal{G}_{j+1}, M_{j+1}) = (\mathcal{T}_j, \mu_j)$, $1 \le j \le N$.

Then $\mathcal{G}_{N+1} = \mathcal{T}_N^{e,\alpha}$, so we have to prove that $\mathcal{G}_{N+1}$ contracts to a linear local tree. Suppose it doesn't. Now,

$$(\mathcal{G}_0, M_0) \Leftarrow \cdots \Leftarrow (\mathcal{G}_{N+1}, M_{N+1})$$

where $\mathcal{G}_0 = (2)$ and $\mathcal{G}_{N+1}[\beta] \sim [1]$. Hence the sequence $\mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_{N+1}$ satisfies the hypothesis of (II.3.28) (with $\omega = 2$), and hence that of (II.3.32). Now we claim that the triple $(d, u, v) = (d, a, b)$ satisfies the second condition of (II.3.32), which is absurd. To see that, we calculate the numbers i, w and p defined in (II.3.28). First, $i = M_0(a_0) = d + r_0$. Also,

$$w = -1 + r_0^2 + \sum_{j=0}^{N-1} r_j^2,$$

$$p = \frac{r_0(r_0 - 1)}{2} + \sum_{j=0}^{N-1} \frac{r_j(r_j - 1)}{2}$$

since $\left(M_j(x_0)\right)_{j=0,\dots,N} = (r_0, r_0, r_1, \dots, r_{N-1})$. Now the very first assertion of (II.3.28) reads "$\alpha = -1$", which means $\beta = -1$ in our case. So $C_1^2 = -1$ in S_N, and

$$\begin{aligned} w &= r_0^2 + (-1 + r_0^2 + \cdots + r_{N-1}^2) = r_0^2 + (C_1^2 \text{ in } S_0) \\ &= r_0^2 + d^2 - a^2 - b^2 \\ &= r_0^2 + d^2 - u^2 - v^2. \end{aligned}$$

Similarly, since C_1 is rational,

$$\begin{aligned} p &= \frac{r_0(r_0 - 1)}{2} + (\text{arithmetic genus of } C_1 \text{ in } S_0) \\ &= \frac{r_0(r_0 - 1)}{2} + \frac{(d-1)(d-2)}{2} - \frac{a(a-1)}{2} - \frac{b(b-1)}{2}. \end{aligned}$$

For the last condition, $d - (u + v + r_0) = d - (a + b + r_0) = (C_1 . C_2)_{S_1} \ge 0$, so $u + v + r_0 \le d$. Hence (d, u, v) does satisfy the second condition of (II.3.32), and this is a contradiction. This proves the claim, i.e., that $\mathcal{T}_N^{e,\alpha}$ contracts to a linear local tree.

It follows from (II.3.8) that $\mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_{N+1}$ is of type 2 (see (II.3.7)). In the notation of (II.3.5),

$$(r_0, r_0, r_1, r_2, \dots, r_{N-1}) = (m_0, m_0, i_1, \dots, i_1, m_1, \dots, m_{l-1}, i_l, \dots, i_l).$$

We claim that $l \ge 2$. Indeed, if $l = 1$ then $i = 2m_0 + i_1 = 2r_0 + 1$, so $r_0 = d - 1$, which is impossible since C_1 is singular "at finite distance". From that, we deduce that

(*) The sequence $\left(\mu_j(x_0)\right)_{j=0,\dots,N-1}$ begins with $(m_0, i_1, \dots, i_1)$, where "i_1" occurs $2n_1$ times, and continues with $(m_1, m_1, i_2, \dots, i_l)$, which is of type $(2, i_1, 1)$.

This is really the piece of information that allows us to construct the open immersion we are after. It is now easy to see that

$$\mathcal{T}_{2n_1+1} = (*, -1, -2, \ldots, -2, (-1-n_1), (-2, \ldots, -2, -1))$$

where the first sequence of "-2" contains n_1 terms, and the second $n_1 - 1$. Hence, $\mathcal{T}_{2n_1+1} \geq (1)$.

Let $Z_0 = S'_0 \leftarrow \cdots \leftarrow S'_N$ be the beginning of the infinite sequence of monoidal transformations determined by (P, C_1, Z_0) and let $F_1, \ldots, F_N$ be the exceptional curves so created. Clearly, the beginning of the infinite sequence of m-trees of (P, C_1, L, Z_0) is just

$$(\mathcal{T}_0, \mu_0) \Leftarrow \cdots \Leftarrow (\mathcal{T}_N, \mu_N).$$

Moreover, $\mathbf{A}^2$ is naturally embedded in the surface S'_{2n_1+1}, its complement is the divisor $D = L + F_1 + \cdots + F_{2n_1+1}$, that divisor has *s.n.c.* and

(i) $C_1.D = C_1.F_{2n_1+1} = \mu_{2n_1+1}(a_{2n_1+1}) = i_1$,

(ii) $C_2.F_{2n_1+1} = 0$,

where a_{2n_1+1} denotes the principal link of $\mathcal{T}_{2n_1+1}$ and where the last assertion is a consequence of $k = 1 < 2n_1 + 1$.

Contracting the curves $L, F_2, \ldots, F_{2n_1}, F_1$, we obtain $\mathbf{P}^2$, and the line at infinity of $\mathbf{A}^2$ is now F_{2n_1+1}. Thus we have an open immersion $\mathbf{A}^2 \hookrightarrow \mathbf{P}^2$ such that C_1 and C_2 meet the line at infinity at distinct points, and this completes the first part of the proof.

It is well known that such a coordinate system on $\mathbf{A}^2$ is unique, up to a linear automorphism (or rather, an affine automorphism, since we allow translations). In fact, any automorphism which is not affine contracts the line at infinity.

Now C_2 is a linear line by (2.2c), or simply because $C_2.D = 1$ in S_{2n_1+1} and that is still true after contraction. Hence (c) holds.

We have already noticed that (b) holds; since C_1 and C_2 don't meet at infinity, $C_1 \cap C_2 \doteq \emptyset$ once we have blown-up $\mathbf{P}^2$ at P_1 and P_2 (for $\mathbf{A}^2$ has no loops at infinity, or because of (2.2e)). Hence

$$0 = (C_1.C_2)_{\mathbf{P}^2} - \mu(P_1, C_1) - \mu(P_2, C_1) = \deg C_1 - (b+1) - b,$$

i.e., $\deg C_1 = 2b + 1$ and (a) holds. From (i), we see that $\deg C_1 = i_1$, so $i_1 = 2b + 1$. Thus (d) holds, since the multiplicity sequence of C_1 at infinity is just $(m_1, m_1, i_2, \ldots)$ which begins with a sequence of type $(2, i_1, 1)$ by (*). This completes the proof of theorem (3.1).

PROOF OF (3.2): Embed $\mathbf{A}^2$ in $\mathbf{P}^2$ the standard way. Blow-up $\mathbf{P}^2$ at P_1 and P_2. Then C_1 and C_2 are disjoint and C_2 is an exceptional curve of the first kind. Contract C_2 and denote the resulting surface by S. We have $C_1 + L \in \mathrm{Div}\,(S)$ and we have to prove that $U \cong \mathbf{A}^2$, where we define $U = S \setminus \mathrm{supp}\,(C_1 + L)$. Then by (1.2), it's enough to prove that $[1] \in \mathcal{G}[U]$ (I.4.8).

Recall that $C_1 \subset \mathbf{A}^2$ has one place P at infinity. Then C_1, regarded as a curve on U, has one place at infinity: the same place P. As in (II.2.3), (P, C_1, L, S) determines a sequence of monoidal transformations and a sequence $\mu(P, C_1, L, S)$:

$$\begin{array}{ccccc} S_0 & \leftarrow & \cdots & \leftarrow & S_k \\ (\mathcal{T}_0, \mu_0) & \Leftarrow & \cdots & \Leftarrow & (\mathcal{T}_k, \mu_k) \end{array}$$

where $S_0 = S$. By definition of $k = k(P, C_1, L, S)$, $\mathcal{T}_k$ has one principal link and the multiplicity of that link is 1. In the notation of (II.2.3), this means that $C_1^{(k)}.L^k = C_1^{(k)}.E_k = 1$, where E_k is the exceptional curve created by $S_{k-1} \leftarrow S_k$ (note that $k > 0$ because $(C_1.L)_{S_0} = 2b + 1 > 1$). Now U is naturally embedded in S_k, and the complement of U is just the support of

$$D = C_1^{(k)} + L^k \in \mathrm{Div}\,(S_k).$$

Let us check that D has *s.n.c.*. By the above comments, that amounts to prove that C_1 is nonsingular. Now condition (d) of (3.1) says that

$$(r_0, \ldots, r_{k-1}) = \big(\mu_j(x_0)\big)_{j=0,\ldots,k-1} \quad \text{is of type} \quad (2, 2b+1, 1).$$

Let us use the notation $f(x) = \frac{x(x-1)}{2}$, $x \in \mathbf{Z}$, as in the numerical lemma (II.3.31). Using parts (a) and (d) of that lemma (with $\omega = 2$, $i = 2b+1$, $i' = 1$) we find that the arithmetic genus of C_1 in S_k is

$$\big(f(2b) - f(b+1) - f(b)\big) - \left(\sum_{j=0}^{k-1} f(r_j)\right) = b(b-1) - b(b-1) = 0,$$

so D has *s.n.c.*. Therefore, the dual graph $\mathcal{G}(S_k, D)$ is just $\mathcal{T}_k[\beta]$, where β is the self-intersection number of C_1 in S_k. By (II.3.31c), we get $\beta = n_l$ where n_l is determined by $(r_0, \ldots, r_{k-1})$ as in (II.3.5). On the other hand, the sequence $\mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_k$ is of type 2 (II.3.7). So theorem (II.3.8) says that $\mathcal{T}_k \geq (*, 0, -n_l - 1, -2)$, where the notation is as in (II.3.18). Hence

$$\mathcal{G}[U] \ni \mathcal{T}_k[\beta] = \mathcal{T}_k[n_l] = [n_l, 0, -n_l - 1, -2] \sim [1]$$

where we use the notation for linear weighted trees defined just before (I.4.15), and the observation which comes just after that definition. This completes the proof.

We did not use the full power of (II.3.27) in these proofs; what we used, in fact, is (II.3.24). We believe that some parts of these arguments can be generalised to $n(f) \geq 2$, by using a slightly generalised version of (II.3.27). In particular, it seems to us that the beginning of the proof of (3.1) actually shows that if $f : \mathbf{A}^2 \to \mathbf{A}^2$ has $n(f) > 1$ and if every column of μ has at most two nonzero entries then some missing curve of f is a coordinate line. However, we did not check the details, so this claim is only a conjecture. The reason why we have to limit ourselves to two nonzero entries in each column of μ is that these entries correspond to the numbers u and v of (II.3.32). We don't know how serious is that limitation. To remedy this, we could for instance try to improve (II.3.32), or to use (II.3.28) itself. Also, we haven't taken advantage of (2.2e), which is, in our opinion, an interesting and nontrivial piece of information.

On the other hand, it is not true that our arguments easily generalise to $n(f) > 2$. In particular we have the following example of an irreducible $f : \mathbf{A}^2 \to \mathbf{A}^2$ with $n(f) = 4$, all of whose missing curves are singular. Note that each one of these curves contains the four fundamental points. That example is due to Russell. He has kindly accepted to carry out the calculations, in order that actual equations be displayed here.

3.4. EXAMPLE. In $\mathbf{A}^2$, let $P_1 = (0,0)$, $P_2 = (0,1)$, $P_3 = (-1,-1)$, $P_4 = (1,2)$ and let C_1, C_2, C_3 and C_4 be the curves given by the polynomials

$$\begin{aligned}
F_1 &= Y^3 + 8X^2 - 6XY - Y^2 \\
F_2 &= Y^4 + 32X^3 - 48X^2Y + 20XY^2 - 2Y^3 + 20X^2 - 20XY + Y^2 \\
F_3 &= Y^4 - 32X^3 + 48X^2Y - 20XY^2 - 2Y^3 - 28X^2 + 20XY + Y^2 \\
F_4 &= Y^5 + 128X^4 - 288X^3Y + 224X^2Y^2 - 60XY^3 - 2Y^4 + 96X^3 - 156X^2Y + 60XY^2 + Y^3
\end{aligned}$$

respectively. Blow-up $\mathbf{A}^2$ at P_1, P_2, P_3 and P_4, and remove from the surface so obtained the strict transforms of C_1, C_2, C_3 and C_4. The resulting open set is isomorphic to $\mathbf{A}^2$, so an equivalence class of endomorphisms $f : \mathbf{A}^2 \to \mathbf{A}^2$ is determined. Notice that

$$\mu = \begin{bmatrix} 2 & 2 & 2 & 3 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix},$$

so the endomorphism is irreducible by (2.5).

In some sense, our classification of irreducible birational endomorphisms with $n(f) = 2$ is complete. However, there is a whole class of questions that we have not considered. For instance, if $(r_0, \ldots, r_{k-1})$ is a sequence of type $(2, 2b+1, 1)$ for some $b \in \mathbf{N}$, then what are all irreducible morphisms $f : \mathbf{A}^2 \to \mathbf{A}^2$ with $n(f) = 2$ such that, in the notation of (3.1), the multiplicity sequence of C_1 at infinity begins with $(r_0, \ldots, r_{k-1})$? In the simplest case, i.e., when the number l of (II.3.5) is 1, the sequence $(r_0, \ldots, r_{k-1})$ is just $(b, b, 1, \ldots, 1)$. In that case, and if we restrict ourselves to the endomorphisms f with *i.n.* fundamental points, then these morphisms are parametrised by the points of $\mathbf{A}^b$, two points $(a_1, \ldots, a_b)$ and $(a'_1, \ldots, a'_b)$ corresponding to equivalent morphisms iff

$$\exists_{\theta \in \mathbf{k}^*} \quad (a'_1, \ldots, a'_b) = (\theta a_1, \theta^2 a_2, \ldots, \theta^b a_b).$$

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