Vector Interpolation Polynomials over Finite Elements

by

Nevine Nassif, B.Eng (Honours, Electrical), M.Eng., B.A.

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Department of Electrical Engineering
McGill University
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ABSTRACT

Vector interpolation functions which approximate electromagnetic vector fields are constructed in this thesis. These vector functions are to be used when the solution of Maxwell's equations involves an irrotational or solenoidal vector field. In addition the functions are chosen so that they can easily be used in the implementation of a finite element method.

Four bases are constructed. The first two span the spaces of solenoidal or irrotational two component vector polynomials of order one in two variables whereas the other two span the spaces of solenoidal or irrotational three component vector polynomials of order one in three variables. The vector polynomials are then used within the finite element method to approximate the two component current density **J** and electric field **E** over a conducting plate and the three component current density in a three dimensional wire.

RÉSUMÉ

Des fonctions d'interpolation vectorielles servant d'approximation à des champs électromagnétiques sont construites dans cette thèse. Ces fonctions ne peuvent être utilisées que dans le cas où la solution des équations de Maxwell comprend des champs dont le rotationnel ou la divergence est nul. De plus, ces fonctions sont choisies de manière à pouvoir être utilisées dans le contexte d'une méthode d'éléments finis.

Quatre bases particulières sont construites. Deux de ces bases engendrent les espaces de dimension deux des polynômes de degré un à deux variables, dont soit le rotationnel ou la divergence est nul, tandis que des polynômes similaires mais à trois variables engendrent deux espaces de dimension trois. Ces bases de polynômes sont ensuite utilisées dans le cadre de la méthode des éléments finis pour calculer la densité de courant et le champs électrique d'une plaque conductive et la densité de courrant d'un cable tridimensionnel.

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CHAPTER 1

INTRODUCTION

Vector interpolation functions which approximate electromagnetic vector fields are constructed in this thesis. These vector functions are developed specifically to be used when the solution of Maxwell's equations involves an irrotational or a solenoidal vector field. In addition the functions are chosen so that they can easily be used in the implementation of a finite element method.

1.1 The Finite Element Method in Electromagnetics

For the last two decades the finite element method has been used effectively for the analysis of electromagnetic field problems. Several problems which had formerly been intractable became quite easy to solve computationally. At present, conferences [COMPUMAG 1976. 1978, 1981 and 1983] which report advances in the field are held regularly, books such as those by Silvester and Ferrari (1983) and Chari and Silvester (1980) have been written, and several computer packages which compute electromagnetic fields can be found on the market.

At first most of the work undertaken was concentrated mainly on determining the field in regions with translational or axial symmetry which could be reduced to a two dimensional problem. When an electromagnetic field is defined over a two dimensional region it is usually written in terms of a differential operator acting on a scalar potential or on a stream function and an approximate solution is then found for the scalar function whereas when the region is three dimensional, the vector field is formulated in terms of either a scalar potential or a vector potential. Although a formulation which expresses a three component irrotational field in terms of a scalar potential is advantageous, a formulation which expresses a three component solenoidal vector field in terms of a vector potential offers no real advantage whenever it is possible to determine the vector field directly. This is because the vector potential and the vector field both have three components whereas a scalar potential is a scalar function. Furthermore the vector potential is nonunique and one solution for the vector potential can differ from another by the gradient of a scalar function. This causes problems with the numerical approximation and attempts were made by Mohammed, Davis, Popovic, Nehl and Demerdash (1982), Kotiuga and Silvester (1982), Kotiuga (1982) and Chari, Silvester, Konrad, Csendes and Palmo (1981) to find formulations which ensure the uniqueness of the vector potential. However Friedman (1982, p.30) noted the advantages of a method where the vector field is discretized directly because the solution for the vector field is unique. In this thesis solenoidal and irrotational vector approximation functions are developed so that a method which approximated the field directly can be implemented.

1.2 The Finite Element Method Applied to Vector Fields

Often when a vector field is approximated each component of the vector is treated separately and approximated by scalar interpolation polynomials. This method is used by Chari, Silvester, Konrad, Csendes, Palmo (1981) and Demerdash, Nehl, Fouad, Mohammed (1981) in order to determine the vector potential in a three dimensional region. Webb (1982) also breaks down the electric and magnetic fields into components and performs a component by component approximation. When each component of the vector is treated alone, boundary conditions and interface conditions are difficult to impose as they are always tangential and normal to the boundary. In addition such a scheme does not allow the possibility of approximating a vector field within the space of solenoidal vector functions or the space of irrotational vector functions. A different approach would be to use vector approximation functions which could then be chosen to be either solenoidal or irrotational and which would allow the boundary and interface conditions to be easily imposed.

Finite element methods which use vector approximation functions to compute a vector field have been used extensively by numerical analysts in continuum mechanics and especially in fluid mechanics. Raviart and Thomas (1977) and Griffiths (1979) constructed two component vector approximation functions and Hecht (1981) constructed three component vector functions which are solenoidal in each element and which are continuous at the midpoints of interelement edges or faces but not along the entire edge or face. The functions required to approximate an electromagnetic solenoidal or irrotational vector field need not be continuous across an interelement face or edge but in order to satisfy Gauss' law, the normal component of a solenoidal vector field has to be continuous across an interface. Likewise to satisfy Stokes' law, the tangential component of an irrotational vector field must be continuous across an interface Hence functions tailored specifically for approximating electromagnetic fields would be desirable and are constructed in this thesis.

Okon (1982) derives vector expansion functions which can be used to approximate the current density over a two dimensional surface. His functions have either constant curl or constant divergence. He however does not attempt to find similar functions which could be used to approximate the current density in a three dimensional region.

1.3 The Vector Approximation Functions

In Chapter 2 it is shown that two types of vector approximation functions are required: solenoidal and irrotational. The solenoidal vector functions whose normal components across interfaces are continuous are used to compute the current density in a conducting region and the irrotational vector functions whose tangential components are continuous across interfaces are used to approximate the static electric field On the boundary of a region it is the normal components of a solenoidal vector field and the tangential components of an irrotational field which are prescribed. It would therefore be appropriate to find solenoidal and irrotational vector approximation functions whose undetermined coefficients are the components of the field which are normal and tangential to interfaces and to the boundary.

In Chapter 3 the two types of vector approximation functions are derived for two component vector fields in two variables and for three component vector fields in three variables. These functions are devised with the finite element method in mind. A solenoidal vector field over a given triangle or tetrahedron is approximated in the space of first order solenoidal vector polynomials. A basis for this space is constructed in such a way that when an approximation of the vector is written as a linear combination of the basis vectors, the coefficients in the linear combination are the components of the vector field normal to an edge of the triangle or to a face of the tetrahedron evaluated at a vertex. Similarly an irrotational vector field is approximated in the space of first order irrotational vector polynomials. A basis for this space is constructed in such a way that when an approximation of the vector is expressed as a linear combination of the basis vectors, the coefficients in the linear combination are the components of the vector field tangent to an edge or a face evaluated at a vertex. Having chosen such a set of basis functions, it is easy to ensure continuity of the normal component of a solenoidal vector field across interelement edges and the continuity of the tangential component of an irrotational field across these same edges. The boundary conditions are prescribed by constraining the appropriate coefficients.

Although for completeness. both solenoidal and irrotational vector approximation families should be developed, there is little practical value in using interpolation schemes for irrotational fields since these functions are easily handled by using a scalar potential. On the other hand it is impossible to describe a three component solenoidal vector field in terms of a stream function and a vector potential is often used. In this case there is great advantage in developing interpolation schemes for solenoidal vector fields.

In this thesis only functions whose components are first order polynomials are considered. It turns out that the corresponding zeroth order three and n component solenoidal vector functions were devised by McMahon (1953, 1956, 1974) and the zeroth order two component solenoidal vector polynomials were first introduced by Synge in 1952.

In Chapters 4 and 5 the two component vector approximation functions are used to determine the electric field and the current density in a two dimensional conducting plate. The finite element matrices are given and the continuity requirements and boundary conditions are imposed. The three component solenoidal vector approximation functions are then used to find the current density in a three dimensional wire whereas the electric field in the wire is determined from the approximation of the scalar potential.

Finally it is shown that whether a solenoidal or irrotational vector field is approximated with nth order vector approximation functions or whether a stream function or a scalar potential is approximated with $n+1$st order scalar polynomials, the resulting field is the same and both methods are equivalent.

CHAPTER 2

THE PROBLEM FORMULATION

Maxwell's equations often lead to boundary value problems involving irrotational or solenoidal vector fields. The purpose of this chapter is to determine the properties of these vector fields that the approximation functions which are developed in the next chapter should satisfy. To this end, two simple examples which involve solenoidal and irrotational vector fields are presented in order that the properties of these fields can be deduced.

In the following pages two problems are posed. In the first problem the resistance of a two dimensional conducting plate and in the second the resistance of a three dimensional wire is sought. The resistance is estimated by minimizing the power functional P which is expressed in terms of either the irrotational static electric field $\mathbf{E}$ or the solenoidal current density $\mathbf{J}$.

Each of the two problems is formulated in four ways. Two methods for evaluating the electric field are given. In the first the electric field is expressed as the gradient of a scalar potential and in the second the electric field is sought in the space of irrotational vector functions. Similarly there are two ways in which the current density can be computed. The two component current density can be written in terms of a stream function and the three component current density can be expressed in terms of a vector potential or both can be sought in the space of solenoidal vector functions.

When the power functional is expressed in terms of the electric field and the potential difference across the region is equal to 1, the value of the functional is always greater than the conductance of the region except at the functional's minimum where it is equal to the conductance. Similarly when the power functional is expressed in terms of the current density and the total current through the region is equal to 1, the value of the power functional is always greater than the resistance of the region except at the functional's minimum where it is equal to the resistance.

When an approximation for the minimum of the power functional is found, the approximate value of the conductance G_{app} is an upper bound for the exact conductance G of the region and the approximate value of resistance R_{app} is an upper bound for the exact resistance R of the region. The exact value of resistance lies between these two bounds.

$$\frac{1}{G_{app}} \leq \frac{1}{G} = R \leq R_{app}. \tag{2.1}$$

The above inequality is used in Chapter 5 to give an indication of the accuracy of the vector approximation functions.

2.1 A Problem Involving the Two Component E and J Vectors

The resistance of the thin conducting plate Ω of thickness d shown in Fig. 2.1(a) needs to be computed. Σ_2 and Σ_2' are equipotential surfaces between which there may be a potential difference and the tangential components of the electric field vanish on both these surfaces. Current cannot flow through the surface Σ_1 hence the normal component of $\mathbf{J}$ vanishes on this surface. The total current through the plate is I,

$$I = \int_{S_1} \mathbf{J} \cdot \mathbf{n} \, dS \tag{2.2a}$$

where S_1 is any cross-sectional surface through the plate Ω whose boundary lies in Σ_1. The plate is modeled by a two dimensional region Σ and has a nonzero constant finite conductivity σ. Σ_2 and Σ_2' reduce to two curves Γ_2 and Γ_2' and Σ_1 is replaced by the curves Γ_1 and Γ_1'. The integral given by (2.2a) becomes

$$I = d\int_{C_1} J_n \, dl. \tag{2.2b}$$

Faraday's law, the continuity equation and the constitutive relation given by Eq. (2.5) are required in order to determine either the electric field $\mathbf{E}$ or the current density $\mathbf{J}$ in the conducting plate Ω. Faraday's Law states that for any open surface S'

$$\int_{\partial S'} \mathbf{E} \cdot d\mathbf{l} = \int_{S'} -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = 0 \tag{2.3}$$

Hence the electric field integrated around any closed loop is zero if the time rate of change of the magnetic field is zero. The continuity equation states that for any three dimensional region Ω'

$$\int_{\partial \Omega'} \mathbf{J} \cdot \mathbf{n} \, dS = \int_{\Omega'} -\frac{\partial \rho}{\partial t} dV = 0. \tag{2.4}$$

Hence the net current flowing into a closed surface is zero if the time rate of change of the charge density is zero. The constitutive relation

$$\mathbf{J} = \sigma \mathbf{E} \tag{2.5}$$

relates the current density $\mathbf{J}$ to the electric field $\mathbf{E}$.

By Stokes' theorem, Faraday's law can be rewritten as

$$\int_{\partial S'} \mathbf{E} \cdot d\mathbf{l} = \int_{S'} \nabla \times \mathbf{E} \cdot d\mathbf{S} = 0 \tag{2.6}$$

from which it follows that

$$\nabla \times \mathbf{E} = 0 \qquad in\ \Omega \tag{2.7}$$

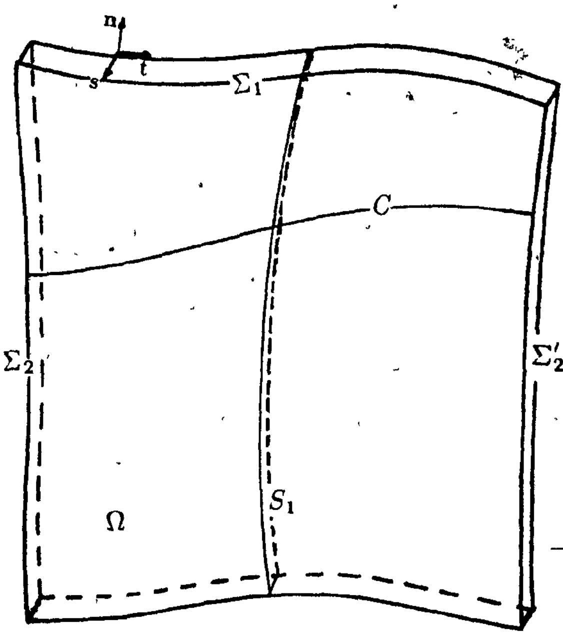

Fig. 2.1(a) A conducting plate Ω of thickness d.

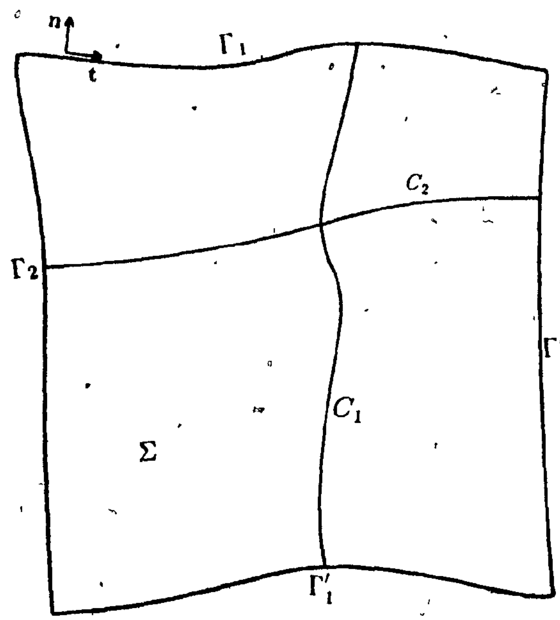

Fig. 2.1(b) A two dimensional model for the conducting plate.

and that the tangential component of the electric field E_t is continuous across interfaces (the reader can refer to a textbook on electromagnetic field theory for the derivation of the interface conditions. for example see Popovic, p.460, or Stratton, Section 1.13). Since Ω is simply connected and the curl of the electric field vanishes everywhere in Ω, $\mathbf{E}$ is a conservative field and can be written as the gradient of a scalar potential φ,

$$\mathbf{E} = \nabla\varphi \qquad in\ \Omega. \tag{2.8}$$

Because the tangential component of the electric field vanishes on Σ_2 and Σ_2', φ is a constant on Σ_2 equal to V and φ is a constant on Σ_2' equal to V'. The potential difference between Σ_2 and Σ_2' is $V - V'$. The imposed boundary conditions are $\varphi = V$ on Σ_2 and $\varphi = V'$ on Σ_2'.

By the divergence theorem, the continuity equation is rewritten as follows

$$\int_{\partial\Omega'} \mathbf{J}\cdot\mathbf{n}\,dS = \int_{\Omega'} \nabla\cdot\mathbf{J}\,dV = 0 \tag{2.9}$$

which implies that

$$\nabla\cdot\mathbf{J} = 0 \qquad in\ \Omega \tag{2.10}$$

and that the normal component of the current density J_n is continuous across interfaces.

In order to determine the electric field in the conducting plate Ω, Eqs. (2.5), (2.8) and (2.10) are combined and a solution for the second order equation

$$\nabla\cdot(\sigma\nabla\varphi) = 0 \qquad in\ \Omega \tag{2.11}$$

subject to the boundary conditions

$$\begin{aligned} \varphi &= V && on\ \Sigma_2 \\ \varphi &= V' && on\ \Sigma_2' \\ (\sigma\nabla\varphi)\cdot\mathbf{n} &= 0 && on\ \Sigma_1' \end{aligned} \tag{2.12}$$

must be found. Then Eq. (2.8) is used to compute **E**.

Next it will be shown that the extremum of the power functional

$$P = \int_\Omega \mathbf{J} \cdot \mathbf{E} \, dV = \int_\Omega \sigma \left|\mathbf{E}\right|^2 dV = \int_\Omega \sigma \left|\nabla \chi\right|^2 dV \tag{2.13}$$

subject to the principal boundary conditions

$$\begin{aligned} \chi &= V \qquad on \ \Sigma_2 \\ \chi &= V' \qquad on \ \Sigma_2' \end{aligned} \tag{2.14}$$

is a solution of the boundary value problem given by Eqs. (2.11) and (2.12), that is, the χ for which P is a minimum is φ. A necessary condition for the functional P to be stationary at φ is that the variation of P vanish for all variations of φ which vanish on Σ_2 and Σ_2'. Taking the variation of P at φ the conditions for φ to be an extremal are deduced.

The variation of P, δP, is taken assuming Ω has fixed boundaries and

$$\begin{aligned} \delta P &= \int_\Omega \sigma \delta \left|\nabla \varphi\right|^2 dV \\ &= \int_\Omega 2\sigma \nabla \delta\varphi \cdot \nabla \varphi \, dV. \end{aligned} \tag{2.15}$$

Now using the following vector identity in which **F** is a vector and g is a scalar:

$$\nabla \cdot (g\mathbf{F}) = g\nabla \cdot \mathbf{F} + \nabla g \cdot \mathbf{F} \tag{2.16}$$

and the divergence theorem, the variation of P is rewritten as

$$\delta P = \int_{\partial\Omega} 2\delta\varphi\sigma\nabla\varphi \cdot \mathbf{n} \, dS - \int_\Omega 2\delta\varphi \nabla \cdot (\sigma \nabla \varphi) dV. \tag{2.17}$$

At φ the first variation of P must vanish. Therefore setting Eq. (2.17) to zero the conditions on φ which must hold when P is stationary are determined. First of all, on the parts of the boundary of Ω where the principal boundary conditions are prescribed, the variation of φ is zero. Equation (2.17) therefore becomes

$$0 = \int_{\Sigma_1} \delta\varphi\sigma\nabla\varphi \cdot \mathbf{n}\, dS - \int_{\Omega} \delta\varphi\nabla \cdot (\sigma\nabla\varphi) dV. \tag{2.18}$$

Equation (2.18) must be an identity for any variation in φ. In particular $\delta\varphi$ can be chosen to be zero on the boundary and to take on any value within Ω, in which case

$$0 = \int_{\Omega} \delta\varphi\nabla \cdot (\sigma\nabla\varphi)\, dV. \tag{2.19}$$

Equation (2.19) holds if and only if $\nabla \cdot (\sigma\nabla\varphi) = 0$ everywhere in Ω. If $\nabla \cdot (\sigma\nabla\varphi)$ does not vanish everywhere in Ω, then $\delta\varphi$ can be chosen so that the integral in (2.19) does not vanish. However (2.19) must hold for all $\delta\varphi$ since φ is the extremal of P. Therefore, φ must satisfy the Euler-Lagrange equation

$$\nabla \cdot (\sigma\nabla\varphi) = 0 \qquad in\ \Omega. \tag{2.20}$$

Next a $\delta\varphi$ can be chosen which is non-zero on the boundary. Because Eq. (2.20) holds, Eq. (2.18) reduces to

$$\int_{\Sigma_1} \delta\varphi\sigma\nabla\varphi \cdot \mathbf{n}\, dS. \tag{2.21}$$

Eq. (2.21) holds if and only if

$$\sigma\nabla\varphi \cdot \mathbf{n} = 0 \qquad on\ \Sigma_1 \tag{2.22}$$

since $\delta\varphi$ can take on any value on the boundary. The condition given by (2.22) is a natural boundary condition and need not be imposed explicitly. Hence it has been shown that finding the function φ which makes the functional P in (2.13) stationary, subject to the principal boundary conditions (2.14), is equivalent to solving the boundary value problem described by (2.11) and (2.12).

The stationary value of the functional P is a minimum. To show this, $P(\varphi + \phi)$ is examined where $P(\varphi)$ is the stationary value.

$$\begin{aligned} P(\varphi + \phi) &= \int_\Omega \sigma \left|\nabla(\varphi + \phi)\right|^2 dV \\ &= \int_\Omega \sigma \left|\nabla \phi\right|^2 dV + \int_\Omega \sigma \left|\nabla \varphi\right|^2 dV + 2\int_\Omega \sigma \nabla\varphi \cdot \nabla\phi\, dV \\ &= P(\varphi) + P(\phi) + 2\int_\Omega \sigma \nabla\varphi \cdot \nabla\phi\, dV. \end{aligned}$$

In the above integral, ϕ can be regarded as a special instance of $\delta\varphi$ in Eq. (2.15) which means that the above integral vanishes since the variation of P vanishes at φ. Therefore $P(\varphi + \phi)$ is

$$P(\varphi + \phi) = P(\varphi) + P(\phi) \tag{2.23}$$

and since the value of P is always greater than zero, the value of $P(\varphi + \phi)$ is greater than $P(\varphi)$ for any ϕ. Therefore, $P(\varphi)$ is a minimum.

The resistance of the plate needs to be computed. The minimum of the functional can be related to the conductance of the plate. Starting once more with the functional in (2.13) and making use of the vector identity given by (2.16) and the divergence theorem, P is rewritten as

$$P = \int_{\partial\Omega} \sigma \chi \nabla \chi \cdot \mathbf{n}\, dS - \int_\Omega \chi \nabla \cdot (\sigma \nabla \chi) dV. \tag{2.24}$$

Conductance is calculated at the minimum value of power when $\chi = \varphi$. The volume integral given in (2.24) vanishes since its integrand is the Euler-Lagrange equation given in (2.20) when $\chi = \varphi$. Furthermore the surface integral vanishes on Σ_1 because of (2.22). What remains is the following:

$$P = \int_{\Sigma_2} \sigma \varphi \nabla \varphi \cdot \mathbf{n}\, dS + \int_{\Sigma_2'} \sigma \varphi \nabla \varphi \cdot \mathbf{n}\, dS. \tag{2.25}$$

From the conditions in Eq. (2.14), φ on Σ_2 and φ on Σ_2' are known constants. Since $\mathbf{J} = \sigma \nabla \varphi$ from Eqs. (2.5) and (2.8), Eq. (2.25) reduces to

$$P = V \int_{\Sigma_2} \mathbf{J} \cdot \mathbf{n} \, dS + V' \int_{\Sigma_2'} \mathbf{J} \cdot \mathbf{n} \, dS \tag{2.26}$$

Since Σ_2 and Σ_2' are both possible candidates for S_1 in Eq. (2.2a), if $\mathbf{n}$ is taken to be the outward normal to the surface Σ_2, then

$$\int_{\Sigma_2} \mathbf{J} \cdot \mathbf{n} \, dS = I$$

and

$$\int_{\Sigma_2'} \mathbf{J} \cdot \mathbf{n} \, dS = -I$$

and thus Eq. (2.26) reduces to

$$P(\varphi) = I(V - V') = (V - V')^2 G \tag{2.27}$$

since $I = (V - V')G$. The value for the conductance can thus be calculated at the minimum value of power for an imposed potential difference.

Instead of finding the electric field $\mathbf{E}$, the current density $\mathbf{J}$ can be determined. From Eq. (2.10), it is seen that the divergence of $\mathbf{J}$ vanishes everywhere in Ω. On the two dimensional surface Σ which is used to model the conducting plate Ω, $\mathbf{J}$ can be related to a stream function ψ through a differential operator as follows:

$$\mathbf{J} = \mathbf{n}' \times \nabla \psi \qquad in \ \Sigma \tag{2.28}$$

where $\mathbf{n}'$ is in the direction normal to Σ. In order to derive a second order differential equation. Eq. (2.27) is combined with Eqs. (2.5) and (2.7) to obtain

$$\mathbf{n}' \cdot \nabla \times \left[\frac{\mathbf{n}' \times \nabla \psi}{\sigma} \right] = 0 \qquad in \ \Sigma. \tag{2.29}$$

Because $J_n = 0$ on Γ_1 and Γ_1', ψ is a constant on Γ_1 equal to I_1/d and ψ is a constant on Γ_1' equal to I_2/d where $I_1 - I_2$ is equal to I, the total current. Therefore, $\mathbf{J}$ can be found by solving Eq. (2.29) for ψ subject to

$$\begin{aligned} \psi &= \frac{I_1}{d} \qquad on\ \Gamma_1 \\ \psi &= \frac{I_2}{d} \qquad on\ \Gamma_1' \\ \frac{\mathbf{n}' \times \nabla\psi}{\sigma} \cdot \mathbf{t} &= 0 \qquad on\ \Gamma_2 \\ \frac{\mathbf{n}' \times \nabla\psi}{\sigma} \cdot \mathbf{t} &= 0 \qquad on\ \Gamma_2'. \end{aligned} \tag{2.30}$$

Finding the solution to Eq. (2.29) subject to the conditions given in (2.30) is equivalent to minimizing the power functional shown in Eq. (2.13), rewritten in the following way:

$$P = \int_\Omega \mathbf{J} \cdot \mathbf{E}\, dV = d \int_\Sigma \mathbf{J} \cdot \mathbf{E}\, dS = d \int_\Sigma \frac{|\mathbf{J}|^2}{\sigma}\, dS = d \int_\Sigma \frac{|\mathbf{n}' \times \chi|^2}{\sigma}\, dS \tag{2.31}$$

subject to the principal boundary conditions

$$\begin{aligned} \chi &= \frac{I_1}{d} \qquad on\ \Gamma_1 \\ \chi &= \frac{I_2}{d} \qquad on\ \Gamma_1'. \end{aligned} \tag{2.32}$$

The χ for which P is a minimum is ψ.

A necessary condition for the functional to be stationary at ψ is that the variation of P vanish for all variations of ψ which vanish on Γ_1 and Γ_1'. Taking the variation of P at ψ and using the identity

$$\mathbf{n}' \cdot \nabla \times (g\mathbf{G}) = \mathbf{n}' \cdot g\nabla \times \mathbf{F} + \mathbf{F} \cdot \mathbf{n}' \times \nabla g, \tag{2.33}$$

the following conditions on ψ must hold when P is stationary:

$$\begin{aligned}
\mathbf{n}' \cdot \nabla \times \left[\frac{\mathbf{n}' \times \nabla\psi}{\sigma}\right] &= 0 \qquad in\ \Sigma \\
\frac{\mathbf{n}' \times \nabla\psi}{\sigma} \cdot \mathbf{t} &= 0 \qquad on\ \Gamma_2 \\
\frac{\mathbf{n}' \times \nabla\psi}{\sigma} \cdot \mathbf{t} &= 0 \qquad on\ \Gamma_2'.
\end{aligned}$$

It can also be shown that the stationary point of P is a minimum and that the resistance of the plate is calculated at that minimum. The functional P in Eq. (2.31) can be rewritten as

$$P = d \int_{\partial\Sigma} \psi \left(\frac{\mathbf{n}' \times \nabla\psi}{\sigma}\right) \cdot d\mathbf{l} - d \int_{\Sigma} \mathbf{n}' \cdot \psi \nabla \times \left(\frac{\mathbf{n}' \times \nabla\psi}{\sigma}\right) dS \qquad (2.34)$$

using the vector identity given by Eq. (2.33) and Stokes' theorem. The volume integral in (2.34) vanishes since its integrand is the Euler-Lagrange equation and the surface integral vanishes on Γ_2 and Γ_2' by virtue of the natural boundary conditions. Hence, Eq. (2.34) reduces to

$$P = d \int_{\Gamma_1} \psi \left(\frac{\mathbf{n}' \times \nabla\psi}{\sigma}\right) \cdot d\mathbf{l} + d \int_{\Gamma_1'} \psi \left(\frac{\mathbf{n}' \times \nabla\psi}{\sigma}\right) \cdot d\mathbf{l}. \qquad (2.35)$$

Because of the principal boundary conditions given in (2.32) and the constitutive relation (2.5), Eq. (2.35) can be rewritten as

$$\begin{aligned}
P &= I_1 \int_{\Gamma_1} \mathbf{E} \cdot d\mathbf{l} + I_2 \int_{\Gamma_1'} \mathbf{E} \cdot d\mathbf{l} \\
&= (I_1 - I_2)(V - V') \\
&= I(V - V') \\
&= I^2 R
\end{aligned}$$

since $V - V' = IR$. Hence resistance can be calculated from

$$\frac{P(\psi)}{I^2} = R. \tag{2.36}$$

Equations (2.27) and (2.36) express the conductance and the resistance of the plate. If an approximate value φ_{app} is found for the scalar potential, then

$$P_{appE} = P(\varphi_{app}) \geq P(\varphi)$$

for all φ_{app} as was shown by Eq. (2.23). The approximate value of conductance G_{appE} is greater than or equal to the exact value of conductance since

$$G_{appE} = \frac{P(\varphi_{app})}{(V - V')^2} \geq \frac{P(\varphi)}{(V - V')^2} = G$$

and G_{appE} is thus an upper bound for the conductance of the plate. If an approximate value ψ_{app} is found for the stream function, then

$$P_{appJ} = P(\psi_{app}) \geq P(\psi)$$

for all ψ_{app}. Therefore the approximate value of resistance R_{appJ} is greater than or equal to the exact value of resistance since

$$R_{appJ} = \frac{P(\psi_{app})}{I^2} \geq \frac{P(\psi)}{I^2} = R$$

and R_{appJ} is thus an upper bound for the resistance of the plate. A lower bound for the resistance of the plate is determined from G_{appE}. Since

$$G_{appE} \geq G$$

then

$$\frac{1}{G_{appE}} \leq \frac{1}{G}.$$

If

$$\frac{1}{G_{appE}} = R_{appE},$$

then

$$R_{appE} \leq R.$$

Thus, if approximate solutions are found for both φ and ψ, the resistance of the plate can be bounded from above and from below by

$$R_{appE} \leq R \leq R_{appJ}. \tag{2.37}$$

The difference between R_{appE} and R_{appJ} gives an indication of how close the approximation of the minimum of the power is to the actual minimum.

In the two formulations outlined in the preceding pages, it is seen that the argument of the power functional is either the potential φ or the stream function ψ. Either of these two scalars is determined when the functional is minimized but if either $\mathbf{E}$ or $\mathbf{J}$ is desired, the appropriate differential operator has to be applied to the respective scalar. An alternative approach would be to solve directly for the vector fields without the intermediate step. Rewriting the functional in Eq. (2.13) and the conditions given by (2.14) in terms of the two component electric field, the power functional

$$P = d \int_{\Sigma} \sigma \left|\mathbf{E}\right|^2 dS \tag{2.38}$$

has to be minimized subject to the following conditions:

1. the curl of the electric field $\mathbf{E}$ must vanish everywhere in the region Σ,

$$\nabla \times \mathbf{E} = 0 \qquad in\ \Sigma;$$

2.

$$E_t = 0 \qquad on\ \Gamma_2$$

$$E_t = 0 \qquad on\ \Gamma_2';$$

3.

$$\int_{C_2} \mathbf{E} \cdot d\mathbf{l} = V - V'$$

must be prescribed along any curve C_2 joining Γ_2 to Γ_2'.

When the power functional is expressed in terms of the scalar potential the first condition is satisfied when the electric field is written as the gradient of a scalar potential. The second and third conditions are satisfied when φ is prescribed on Γ_2 and Γ_2'. To find the vector field $\mathbf{E}^*$ which minimizes the power functional given in (2.38) subject to the above three conditions, the vector field $\mathbf{E}^*$ is sought within the space of irrotational two component vector functions in two variables which satisfy conditions 2 and 3. The tangential component of the electric field must be continuous across interfaces and must vanish on Γ_2 and Γ_2'. Similarly, one can also minimize the functional

$$P = d \int_\Sigma \frac{|\mathbf{J}|^2}{\sigma} dS \tag{2.39}$$

subject to the following conditions:

1. the divergence of the current $\mathbf{J}$ must vanish everywhere within the region Σ,

$$\nabla \cdot \mathbf{J} = 0 \qquad in\ \Sigma;$$

2.

$$J_n = 0 \qquad on\ \Gamma_1$$

$$J_n = 0 \qquad on\ \Gamma_1'$$

3.

$$d \int_{C_1} J_n \, dl = I$$

must be prescribed along some curve C_1 from Γ_1 to Γ_1';

To find the vector field $\mathbf{J}^*$ which minimizes the power functional given in (2.39) subject to the above three conditions, the extremal of the functional is found within the space of divergence-free two component vector functions of two variables. Furthermore, the normal component of the current density must be continuous across interfaces and must vanish on Γ_1 and Γ_1'.

Finally, the two formulations may be restated as follows:

minimize

$$P = d \int_{\Sigma} \sigma \left|\mathbf{E}\right|^2 dS$$

within the space of irrotational vector functions, subject to

$$E_t = 0 \ on\ \Gamma_2$$

$$E_t = 0 \ on\ \Gamma_2'$$

$$\int_{C_2} \mathbf{E} \cdot d\mathbf{l} = V - V'. \tag{2.40}$$

and minimize

$$P = d \int_{\Sigma} \frac{\left|\mathbf{J}\right|^2}{\sigma} dS$$

within the space the solenoidal functions, subject to

$$
\begin{aligned}
J_n &= 0 \qquad on\ \Gamma_1 \\
J_n &= 0 \qquad on\ \Gamma_1' \\
d\int_{C_1} J_n dl &= I.
\end{aligned} \tag{2.41}
$$

The conductance of the plate can be computed once more by

$$
G = \frac{P(\mathbf{E}^*)}{\left(\int_{C_2} \mathbf{E}^* \cdot d\mathbf{l}\right)^2} \tag{2.42}
$$

and the resistance of the plate is computed by

$$
R = \frac{P(\mathbf{J}^*)}{\left(d\int_{C_1} J_n^* dl\right)^2}. \tag{2.43}
$$

2.2 A Problem Involving the Three Component E and J Vectors

A problem analogous to the conducting plate problem is chosen for the three dimensional case. The resistance of a three dimensional conducting wire Ω, with nonzero constant finite conductivity σ, as pictured in Fig. 2.2, must be computed. Σ_2 and Σ_2' are equipotential surfaces between which there may be a potential difference, and the tangential components of the electric field vanish on both Σ_2 and Σ_2'. No current can flow out of the surface Σ_1 hence the component of the current density normal to Σ_1 vanishes on Σ_1. The total current through the wire is

$$
I = \int_{S_1} \mathbf{J} \cdot \mathbf{n}\, dS \tag{2.44}
$$

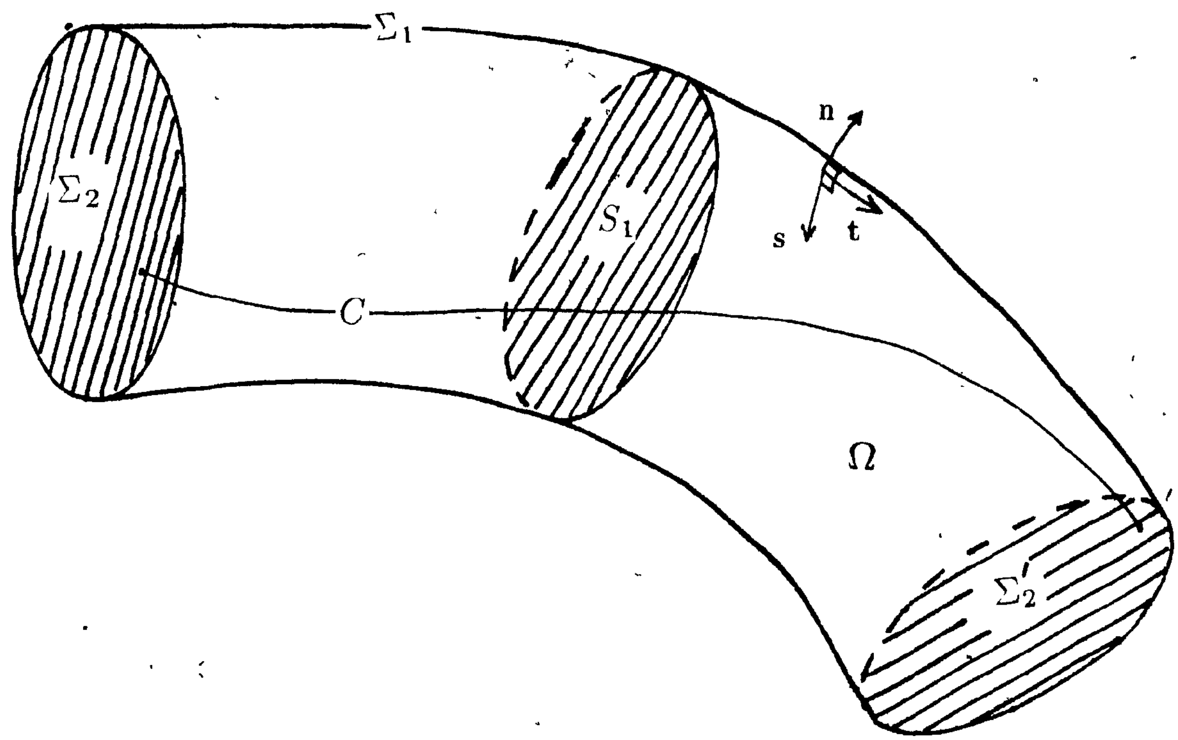

Fig. 2.2 A three dimensional wire.

where S_1 is any cross-sectional surface through the wire Ω and the boundary of S_1 lies in Σ_1.

Using the three field equations (2.3), (2.4) and (2.5) from Section 2.1, the problem is formulated in four ways. As for the two dimensional problem, **E** can be written as the gradient of a scalar function, φ. The scalar potential can then be computed by minimizing the functional

$$P = \int_{\Omega} \sigma \left|\nabla \varphi\right|^2 dV$$

subject to the principal boundary conditions

$$\varphi = V \qquad on\ \Sigma_2$$

$$\varphi = V' \qquad on\ \Sigma_2'.$$

For the above functional

$$\nabla \cdot (\sigma \nabla \varphi) = 0 \qquad in\ \Omega$$

is the Euler-Lagrange equation and

$$(\sigma \nabla \varphi) \cdot \mathbf{n} = 0 \qquad on\ \Sigma_1$$

is a natural boundary condition. The electric field is then computed from

$$\mathbf{E} = \nabla \varphi.$$

Alternatively $\mathbf{E}$ can be computed directly by minimizing the functional

$$P = \int_{\Omega} \sigma \left|\mathbf{E}\right|^2 dV \tag{2.45}$$

subject to

$$\begin{aligned} \nabla \times \mathbf{E} &= 0 \qquad in\ \Omega \\ \mathbf{E} \times \mathbf{n} &= 0 \qquad on\ \Sigma_2 \\ \mathbf{E} \times \mathbf{n} &= 0 \qquad on\ \Sigma_2' \\ \int_C \mathbf{E} \cdot d\mathbf{l} &= V - V' \end{aligned} \tag{2.46}$$

where C is any curve joining Σ_2 to Σ_2'. To find the irrotational vector field $\mathbf{E}^*$ which minimizes the power functional given in (2.45) subject to the conditions given by (2.46), the electric field $\mathbf{E}^*$ is sought within the space of irrotational three component vector functions of three variables which satisfy the principal boundary conditions. The tangential components of the field must be continuous across interfaces and must vanish on Σ_2 and Σ_2'.

A solution for the three component current density $\mathbf{J}$ can be found by writing $\mathbf{J}$ as the curl of a vector potential since the divergence of $\mathbf{J}$ vanishes everywhere in Ω,

$$\mathbf{J} = \nabla \times \mathbf{T}. \tag{2.47}$$

Since

$$\nabla \times \mathbf{E} = 0 \qquad in\ \Omega$$

and

$$\mathbf{J} = \sigma \mathbf{E}$$

the following second order equation in $\mathbf{T}$ is obtained:

$$\nabla \times \mathbf{E} = \nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = \nabla \times \left(\frac{\nabla \times \mathbf{T}}{\sigma}\right) = 0 \qquad in\ \Omega.$$

The boundary conditions which are imposed on $\mathbf{T}$ are derived from those imposed on $\mathbf{J}$. The total current through the wire is I,

$$I = \int_{S_1} \mathbf{J} \cdot \mathbf{n}\, dS = \int_{S_1} \nabla \times \mathbf{T} \cdot \mathbf{n}\, dS = \int_{\partial S_1} \mathbf{T} \cdot d\mathbf{l}.$$

Therefore

$$\int_{\partial S_1} \mathbf{T} \cdot d\mathbf{l} = I \tag{2.48}$$

must be imposed. The normal component of $\mathbf{J}$ vanishes on Σ_1. Therefore

$$(\nabla \times \mathbf{T}) \cdot \mathbf{n} = 0 \qquad on\ \Sigma_1.$$

The tangential component of the electric field vanishes on Σ_2 and Σ_2'. Hence

$$\left(\frac{\nabla \times \mathbf{T}}{\sigma}\right) \times \mathbf{n} = 0 \qquad on\ \Sigma_2$$
$$\left(\frac{\nabla \times \mathbf{T}}{\sigma}\right) \times \mathbf{n} = 0 \qquad on\ \Sigma_2'.$$

In order to determine $\mathbf{T}$ the following boundary value problem must be solved:

$$\nabla \times \left(\frac{\nabla \times \mathbf{T}}{\sigma} \right) = 0 \qquad in\ \Omega$$

subject to

$$\begin{aligned} (\nabla \times \mathbf{T}) \cdot \mathbf{n} &= 0 \qquad on\ \Sigma_1 \\ \left(\frac{\nabla \times \mathbf{T}}{\sigma} \right) \times \mathbf{n} &= 0 \qquad on\ \Sigma_2 \\ \left(\frac{\nabla \times \mathbf{T}}{\sigma} \right) \times \mathbf{n} &= 0 \qquad on\ \Sigma_2'. \end{aligned}$$

Alternatively $\mathbf{T}$ can be found by minimizing the functional P

$$P = \int_\Omega \frac{|\nabla \times \mathbf{T}|^2}{\sigma} dV \tag{2.49}$$

subject to the principal boundary conditions

$$\begin{aligned} (\nabla \times \mathbf{T}) \cdot \mathbf{n} &= 0 \qquad on\ \Sigma_1 \\ \int_{\partial S_1} \mathbf{T} \cdot d\mathbf{l} &= I. \end{aligned} \tag{2.50}$$

In order to determine the value of the tangential components of $\mathbf{T}$ on the surface Σ_1 from the above conditions, $\mathbf{T}$ is written as the gradient of any scalar function. If $\mathbf{T} = \nabla\chi$ on Σ_1, then clearly

$$(\nabla \times \mathbf{T}) \cdot \mathbf{n} = (\nabla \times \nabla\chi) \cdot \mathbf{n} = 0 \qquad on\ \Sigma_1. \tag{2.51}$$

In order for the condition (2.51) to hold χ can be any once differentiable function which has a jump discontinuity of I along any curve C which joins Σ_2 to Σ_2' †.

† This method for prescribing the tangential components of $\mathbf{T}$ on Σ_1 is suggested by Kotiuga and is discussed in the first chapter of his forthcoming Ph.D. dissertation. He cites Milani and Negro (1982) for having used a similar approach. However they restrict χ to be a harmonic function.

After computing $\mathbf{T}$, the current density $\mathbf{J}$ can be found by using Eq. (2.47). Unfortunately $\mathbf{T}$ is not unique and one solution for $\mathbf{T}$ can vary from another by the gradient of a scalar function since $\nabla \times \nabla\varphi = 0$. Therefore $\mathbf{T}' = \mathbf{T} + \nabla\varphi$ is also a valid solution. Such nonuniqueness may cause problems when $\mathbf{T}$ is computed numerically. Instead, since the solution for $\mathbf{J}$ is unique, the power functional can be written in terms of $\mathbf{J}$ and minimized. The functional

$$P = \int_\Omega \frac{|\mathbf{J}|^2}{\sigma}\, dV \tag{2.52}$$

is minimized subject to the following conditions:

$$\begin{aligned} \nabla \cdot \mathbf{J} &= 0 \qquad in\ \Omega \\ \mathbf{J} \cdot \mathbf{n} &= 0 \qquad on\ \Sigma_1 \\ \int_{S_1} \mathbf{J} \cdot \mathbf{n}\, dS &= I. \end{aligned} \tag{2.53}$$

The current density $\mathbf{J}^*$ which minimizes the functional (2.52) is sought in the space of solenoidal three component vector functions in three variables which satisfy the principal boundary conditions. In addition the normal component of J must be continuous across interfaces and vanish everywhere on Σ_1.

The conductance of the wire can be computed when the stationary value of the functional (2.45) is known from

$$G = \frac{P(\mathbf{E}^*)}{\left(\int_C \mathbf{E}^* \cdot d\mathbf{l}\right)^2} \tag{2.54}$$

and the resistance can be computed when the stationary value of the functional (2.52) is known from

$$R = \frac{P(\mathbf{J}^*)}{\left(\int_{S_1} \mathbf{J} \cdot \mathbf{n}\, dS\right)^2} \tag{2.55}$$

In this chapter a two dimensional problem has been posed and formulated in four ways. From the formulations which make use of a potential, the problem is restated in terms of the vector fields. In order to minimize the functionals which are written in terms of the vector fields, the electric field must be approximated within the space of irrotational vector functions and the current density must be approximated within the space of solenoidal vector approximation functions A basis of first order irrotational vector polynomials and a basis of first order solenoidal vector polynomials are constructed in the following chapter. A three dimensional problem has also been formulated and it has been shown that solenoidal and irrotational three component vector functions could be used to approximate the current density and the electric field respectively. The functions developed in Chapter 3 are used in Chapters 4 and 5 to compute numerically the current in the conducting plate and in the wire.

In conclusion, the problems chosen in this chapter are just one example of three possible problems in electrostatics and magnetostatics. Analogous problems which involve either the magnetic field **H** and the magnetic flux density **B** or the electric field **E** or the electric flux density **D** can be solved respectively to determine the inductance and capacitance of a region. The reader should refer to Cambrell's Ph.D. dissertation (1972) or to the paper by Hammond and Penman (1976) for more examples of dual energy principles.

CHAPTER 3

THE VECTOR INTERPOLATION POLYNOMIALS

Solenoidal and irrotational two and three component vector approximation functions are constructed in this chapter. The three component solenoidal vector interpolation functions are those of most interest but the two component vector functions are developed first because they are easier to construct and they help in providing a foundation for the construction of the three component ones. Before the vector approximation functions are developed, a finite element method which uses a Ritz minimization is described in Section 3.1 since this method will be used in the following chapters to approximate numerically the minimum of the functionals which were given in the last chapter. In Section 3.2 the solenoidal two component vector functions in two variables are developed and the irrotational vector functions are then easily derived. In Section 3.3 the three component solenoidal and irrotational vector functions are given.

3.1 The Finite Element Method using a Ritz Minimization

In order to minimize the functionals given in Chapter 2 Ritz' method which was first described in 1908 and examples of which can be found in Kantorovich (1964), is used. In this method a functional whose argument is replaced by a function of n parameters is minimized with respect to the n parameters. Often in order to simplify the minimization process, the argument of the functional is written as a linear combination of n basis functions which lie in a finite dimensional subspace of the space in which the extremal of the functional lies and which satisfy the principal boundary conditions.

For the functionals described in the preceding chapter, the vector functions $\mathbf{E}$ and $\mathbf{J}$ can be approximated by a linear combination of basis functions

$$\begin{aligned} \mathbf{E} &= \sum_{i=1}^{k} c_i \mathbf{v_i} \\ \mathbf{J} &= \sum_{i=1}^{k} d_i \mathbf{u_i} \end{aligned} \tag{3.1}$$

and substituted into the appropriate functional:

$$\begin{aligned} P &= \int_{\Omega} \sigma \left| \sum_{i=1}^{k} c_i \mathbf{v_i} \right|^2 dV \\ P &= \int_{\Omega} \frac{\left|\sum_{i=1}^{k} d_i \mathbf{u_i}\right|^2}{\sigma} dV. \end{aligned} \tag{3.2}$$

The principal boundary conditions can then be imposed by constraining p of the c_i and the d_i where p is the number of coefficients which must be constrained as a result of the principal boundary conditions. The integration appearing in the functionals in (3.2) can then be explicitly performed. This results in a function of the unconstrained d_i or c_i.The extremum of the functional can then be found by differentiating the function with respect to these c_i or d_i and setting the derivatives equal to zero. The unconstrained c_i and d_i are the Ritz parameters and the differentiation results in $k-p$ equations in $k-p$ unknowns. Since the power functional is quadratic, the $k-p$ equations are linear.

When the power functional is written in terms of basis functions

$$P = \int_{\Omega} \sigma \left(\sum_{i=1}^{k} c_i \mathbf{v_i} \right) \cdot \left(\sum_{j=1}^{k} c_j \mathbf{v_j} \right) dV$$

$$
\begin{aligned}
&= \int_\Omega \sigma \sum_{i=1}^{k} \sum_{j=1}^{k} c_i c_j \mathbf{v}_i \cdot \mathbf{v}_j \, dV \\
&= \sum_{i=1}^{k} \sum_{j=1}^{k} c_i c_j \int_\Omega \sigma \mathbf{v}_i \cdot \mathbf{v}_j \, dV,
\end{aligned} \tag{3.3}
$$

the functional is evaluated by computing the integral in Eq. (3.3). However, for all but the simplest regions it is very difficult to choose basis functions so that the above integrations are easy to perform. To simplify the integration problem, finite element methods can be used. In the finite element method a region is discretized into elements. The unknown function is approximated by shape functions whose differentiability requirements over each element and at the interelement interfaces is dictated by the original variational functional In this thesis the functions which will be used to approximate the irrotational and solenoidal vector fields are defined over elements which are triangles and tetrahedra.

3.2 The Two Component Vector Interpolation Functions

When finite element methods are used to approximate a scalar potential, the interpolation functions are generally polynomials since according to the Weierstrass[†] approximation theorem any continuous function can be approximated to any accuracy on a given interval by choosing a sufficient number of linearly independent polynomials. Furthermore, polynomials are easy to integrate, differentiate, and evaluate on a computer. The vector approximation functions which are constructed in this chapter

[†] The reader can refer to Davis (1975) for details of the Weierstrass approximation theorem.

are therefore vector functions whose components are polynomials. Furthermore, these functions are either solenoidal vector polynomials or irrotational vector polynomials.

In the problem at hand, the solution for the current density or the electric field need only be approximated within the space of solenoidal or irrotational vector fields respectively. A solenoidal field is one whose divergence is zero everywhere inside a region Ω and whose component normal to interfaces is continuous. An irrotational vector field is one whose curl is zero everywhere inside a region Ω and whose component tangent to interfaces is continuous. Hence, a solenoidal vector field is approximated by vector functions whose divergence is zero in an element and that have normal component continuity across element boundaries. Likewise, an irrotational vector field is approximated by vector functions whose curl is zero in an element and that have tangential component continuity across element boundaries

3.2.1 Solenoidal Vector Polynomials in Two Variables

In order to approximate a two component solenoidal vector field in two variables by polynomials of order n, $n^2+5n+4/2$ linearly independent solenoidal vector polynomials defined over a region Σ are needed. The above number is obtained as follows. A polynomial in two variables is of the form

$$\sum_{i=0}^{n}\sum_{j=0}^{i} a_{ij}\, x^j y^{i-j}. \tag{3.4}$$

A vector polynomial $\mathbf{u}$ therefore has the form

$$\mathbf{u} = \begin{pmatrix} \sum_{i=0}^{n}\sum_{j=0}^{i} a_{ij}\, x^j y^{i-j} \\ \sum_{i=0}^{n}\sum_{j=0}^{i} b_{ij}\, x^j y^{i-j} \end{pmatrix} \tag{3.5}$$

and the divergence of $\mathbf{u}$ in Cartesian coordinates is

$$
\begin{aligned}
\nabla \cdot \mathbf{u} &= \frac{\partial}{\partial x}\left[\sum_{i=0}^{n}\sum_{j=0}^{i} a_{ij} x^j y^{i-j}\right] + \frac{\partial}{\partial y}\left[\sum_{i=0}^{n}\sum_{j=0}^{i} b_{ij} x^j y^{i-j}\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{i} j a_{ij} x^{j-1} y^{i-j} + \sum_{i=1}^{n}\sum_{j=0}^{i-1} (i-j) b_{ij} x^j y^{i-j-1} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{i} j a_{ij} x^{j-1} y^{i-j} + (i-j+1) b_{ij-1} x^{j-1} y^{i-j} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{i} \left(j a_{ij} + (i-j+1) b_{ij-1}\right) x^{j-1} y^{i-j}.
\end{aligned}
$$

When the divergence of $\mathbf{u}$ is zero,

$$
\sum_{i=1}^{n}\sum_{j=1}^{i} \left(j a_{ij} + (i-j-1) b_{ij-1}\right) x^{j-1} y^{i-j} = 0
$$

must be true for all x, y in the region Ω. Therefore

$$
(j a_{ij} + (i-j+1) b_{ij-1}) = 0, \qquad 1 \le i \le n, \quad 1 \le j \le i \tag{3.6}
$$

holds when the divergence of $\mathbf{u}$ is zero. The number of constraints N_c that have to be imposed on the coefficients a_{ij} and b_{ij} of $\mathbf{u}$ in order that the divergence of $\mathbf{u}$ vanish everywhere in Ω is

$$
N_c = \sum_{i=1}^{n}\sum_{j=1}^{i} 1 = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{3.7}
$$

The number N of basis functions required to span the space of nth order vector polynomials defined in (3.5) is

$$
N = \sum_{i=0}^{n}\sum_{j=0}^{i} 1 = \sum_{i=0}^{n} (i+1) = (n+1)(n+2) \tag{3.8}
$$

and the number

$$N_s = \frac{(n+1)(n+4)}{2} = \frac{n^2+5n+4}{2} \qquad (3.9)$$

of basis vector functions which span the space of two component solenoidal vector polynomials is obtained by subtracting N_c from N. In the case of first order polynomials, $N_s = 5$.

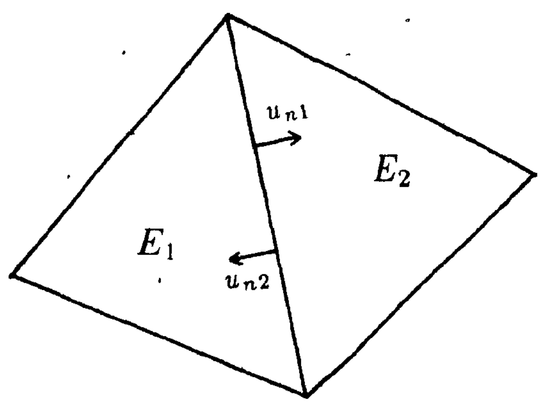

Fig. 3.1 The normal component of the vector field across an edge.

A solenoidal vector field can thus be approximated over each element by a linear combination of five solenoidal linearly independent vector functions whose components are first order polynomials. The component of the approximation of the vector field normal to interelement edges must be continuous. In other words, for the two elements E_1 and E_2 shown in Fig. 3.1, $u_{n1} = -u_{n2}$ everywhere on the edge shared by E_1 and E_2 (u_{n1} is the component of $\mathbf{u}$ in E_1 which is normal to the edge shared by E_1 and E_2). Because the vector field is approximated by first order polynomials, only two points need be chosen along the edge between E_1 and E_2 where $u_{n1} + u_{n2}$ must be set equal to zero. If there are two distinct points on the edge where $u_{n1} + u_{n2} = 0$, then $u_{n1} + u_{n2}$ is zero everywhere on the line. For example, at edge 1 of the triangle shown in Fig. 3.1, the normal component of the vector field evaluated at node 2 is d'_1 and the normal component evaluated at node 3 is d'_2. If edge 1 of one triangle is coincident with edge 1 of another triangle as shown in Fig. 3.2(a), then d'_{11} is set equal to $-d'_{22}$ and d'_{21} is set equal to $-d'_{12}$. As the normal component of the vector field must be evaluated at two points on each edge, six such points are needed on the entire triangle and the components of the vector normal to the edges and evaluated at these points are $d'_1, d'_2, d'_3, d'_4, d'_5, d'_6$. The d'_i are pictured in Fig. 3.2(b).

When a solenoidal vector field is approximated by a complete set of first order solenoidal vector polynomials, it is written as a linear combination of the basis vectors

$$\mathbf{u} = \sum_{i=1}^{5} d_i \mathbf{u}_i. \tag{3.10}$$

The d_i in the above expression have in general no relation to the d'_i defined in the preceding paragraph. However, when the continuity of the normal component of the solenoidal vector field is imposed, it is the d'_i which are equated at interelement edges. Hence the d_i must be tranformed into the d'_i. In order for the computation of the transformation to be as rapid as possible, it should contain as many zero entries as possible. This is done by constructing the basis functions in such a way that as many of the d'_i as possible are zero for each function.

The solenoidal basis vector functions are now constructed. A first order vector polynomial is of the form

$$\mathbf{u} = \begin{pmatrix} ax + by + c \\ dx + ey + f \end{pmatrix}.$$

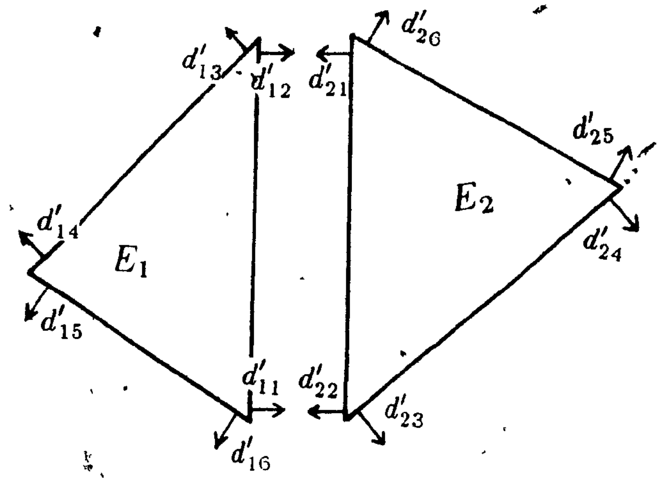

Fig. 3.2(a)

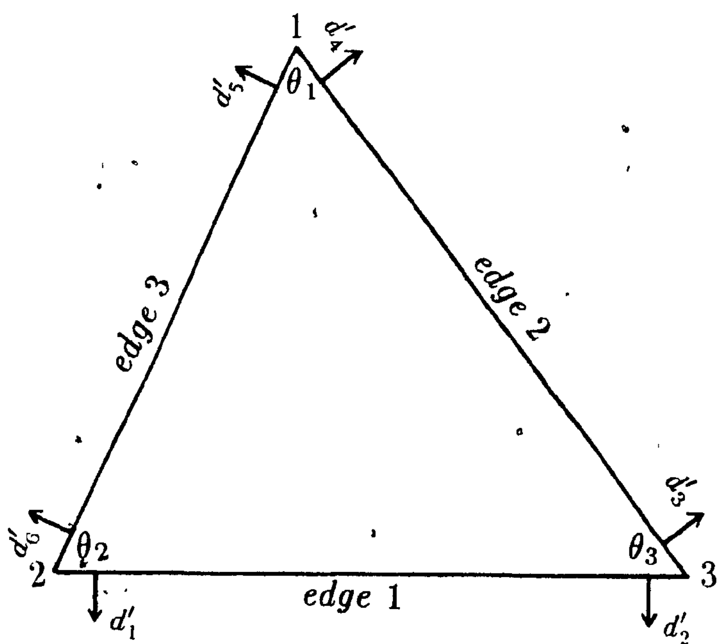

Fig. 3.2(b) The d'_i are the components of the field normal to an edge.

in Cartesian coordinates. When the divergence of $\mathbf{u}$ is zero, $a = -e$ in the above expression. Therefore a first order solenoidal vector polynomial is of the form

$$\mathbf{u} = \begin{pmatrix} ax + by + c \\ dx - ay + f \end{pmatrix}. \tag{3.11}$$

A Cartesian coordinate system whose x coordinate is parallel to one of the edges of a triangle and whose y coordinate is normal to that edge is chosen. The coordinate axes are t and n and the triangle vertices are at (t_1, n_1), (t_2, n_2), (t_3, n_3) as shown in Fig. 3.3. The normal components of $\mathbf{u}$ with respect to the edges evaluated at the vertices of the triangle are the d_i' whereas the tangential components are defined to be $c_1', c_2', \ldots c_6'$.

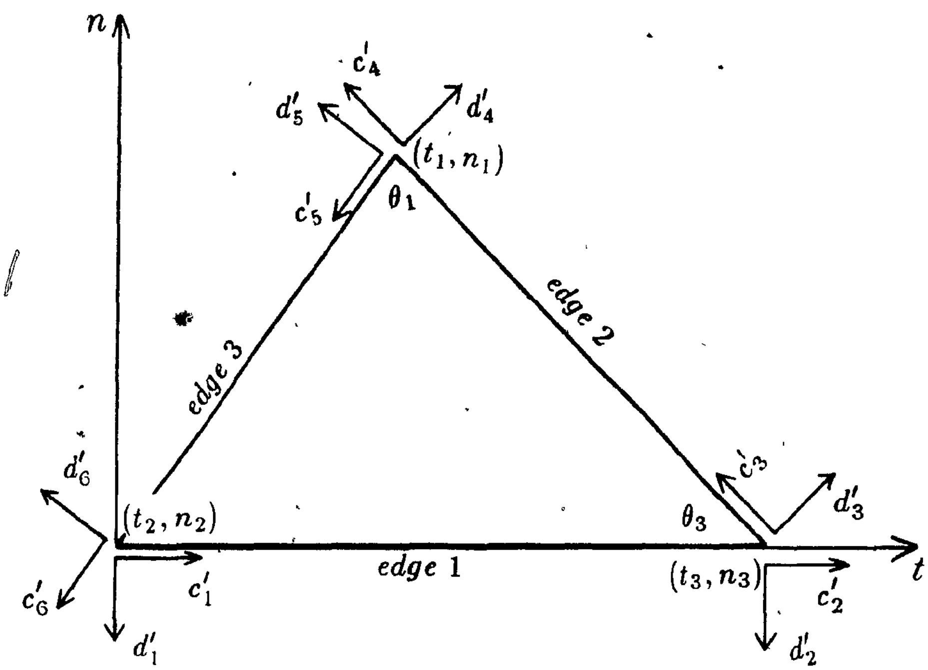

Fig. 3.3 The $t - n$ coordinate system with respect to edge 1 of the triangle.

The reader should now recall that as many of the d_i' as possible should be zero for each basis function. The simplest possibility is that only one d_i is nonzero. Hence the coefficients a, b, c, d, f of u are determined when

$$\mathbf{u} = (0,0) \qquad at\ (t_1, n_1)$$

$$\mathbf{u} = (d_1,0) \qquad at\ (t_2, n_2)$$

$$\mathbf{u} = (0,0) \qquad at\ (t_3, n_3)$$

in which case only d_6' is nonzero. The following system of equations is obtained:

$$\begin{pmatrix} t_1 & n_1 & 1 & & & \\ t_2 & n_2 & 1 & & & \\ t_3 & n_3 & 1 & & & \\ -n_1 & & & t_1 & 1 \\ -n_2 & & & t_2 & 1 \\ -n_3 & & & t_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ d_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.12}$$

From the last three equations in (3.12) it can be determined that $d = a = f = 0$ since (t_1, n_1), (t_2, n_2), (t_3, n_3) are neither coincident nor collinear. The following overdetermined set of equations for which no solution exists remains:

$$\begin{pmatrix} n_1 & 1 \\ n_2 & 1 \\ n_3 & 1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ d_1 \\ 0 \end{pmatrix}.$$

Therefore there does not exist a solenoidal vector polynomial for which $d_1', \ldots, d_5'$ are zero. Next, two d_i' are allowed to be nonzero. So

$$\mathbf{u} = (0,0) \qquad at\ (t_1, n_1)$$

$$\mathbf{u} = (d_1,0) \qquad at\ (t_2, n_2)$$

$$\mathbf{u} = (d_2,0) \qquad at\ (t_3, n_3)$$

is chosen. Here only d_3' and d_6' are nonzero. The system of equations in (3.12) is set up once more with a different right hand side. Proceeding as before it can be shown that $d = a = f = 0$, hence the following system of equations is obtained:

$$\begin{pmatrix} n_1 & 1 \\ n_2 & 1 \\ n_3 & 1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ d_1 \\ d_2 \end{pmatrix}. \tag{3.13}$$

Solving (3.13)

$$c = -bn_1$$

$$b = \frac{d_1}{(n_2 - n_1)} \qquad or \qquad b = \frac{d_2}{(n_3 - n_1)}$$

is obtained. It is seen that a solution to the above system of equations exists only if $d_1 = d_2$. In this case

$$b = \frac{d_1}{(n_2 - n_1)}$$

$$c = \frac{d_1 n_1}{(n_1 - n_2)}$$

since $n_2 = n_3$ from Fig. 3.3. Hence the solenoidal vector function $\mathbf{u}$ for which d_1', d_2', d_4' and d_5' are zero is

$$\mathbf{u} = d_1 \begin{pmatrix} \frac{1}{(n_2 - n_1)} n + \frac{n_1}{(n_1 - n_2)} \\ 0 \end{pmatrix}$$

so the first basis function is taken to be

$$\mathbf{u_1} = \begin{pmatrix} \frac{1}{(n_2 - n_1)} n + \frac{n_1}{(n_1 - n_2)} \\ 0 \end{pmatrix}. \tag{3.14}$$

$\mathbf{u_1}$ is a vector function directed parallel to edge 1. It has a constant value of 1 on edge 1 and varies linearly across the triangle to vertex 1 where it vanishes. $\mathbf{u_1}$ is pictured in Fig. 3.4. Taking the divergence of $\mathbf{u_1}$, $\nabla \cdot \mathbf{u_1} = 0$ is verified. There are two more

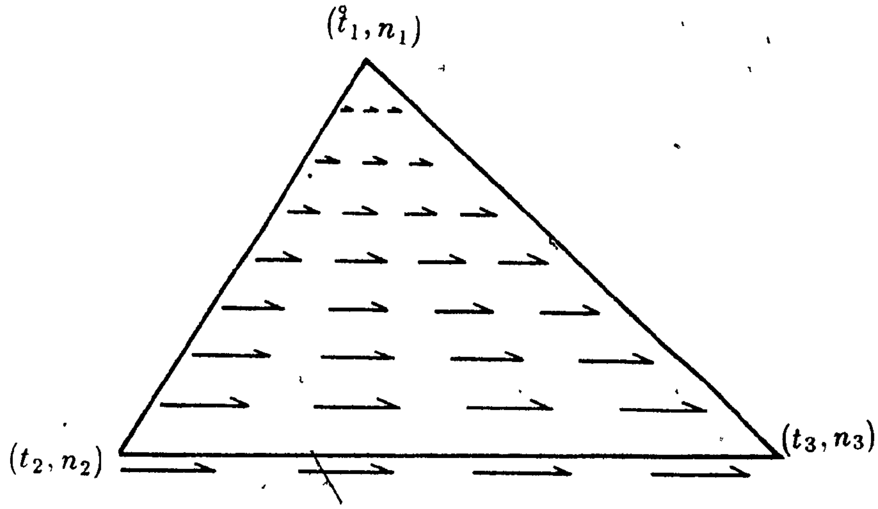

Fig. 3.4 The solenoidal vector function $\mathbf{u_1}$.

functions similar to $\mathbf{u_1}$ which are parallel to edges 2 and 3. Expressions for $\mathbf{u_2}$ and $\mathbf{u_3}$ are given later in this chapter.

A fourth vector polynomial is constructed by setting

$$\begin{aligned} \mathrm{u} &= (d_4, 0) \qquad && at\ (t_1, n_1) \\ \mathrm{u} &= (0, 0) && at\ (t_2, n_2) \\ \mathrm{u} &= (0, 0) && at\ (t_3, n_3). \end{aligned}$$

In this case d_4' and d_5' are nonzero. The right-hand-side vector of Eq. (3.12) becomes

$$(d_4,\ 0\ \ 0\ \ 0\ \ 0\ \ 0)^T.$$

Solving the system of equations

$$b = \frac{d_4}{(n_1 - n_2)}$$
$$c = \frac{d_4 n_2}{(n_2 - n_1)}$$

is obtained since $n_2 = n_3$. This new vector function therefore is

$$\mathbf{u} = d_4 \begin{pmatrix} \frac{1}{(n_1-n_2)}n + \frac{n_2}{(n_2-n_1)} \\ 0 \end{pmatrix}$$

and

$$\mathbf{u_4} = \begin{pmatrix} \frac{1}{(n_1-n_2)}n + \frac{n_2}{(n_2-n_1)} \\ 0 \end{pmatrix} \tag{3.15}$$

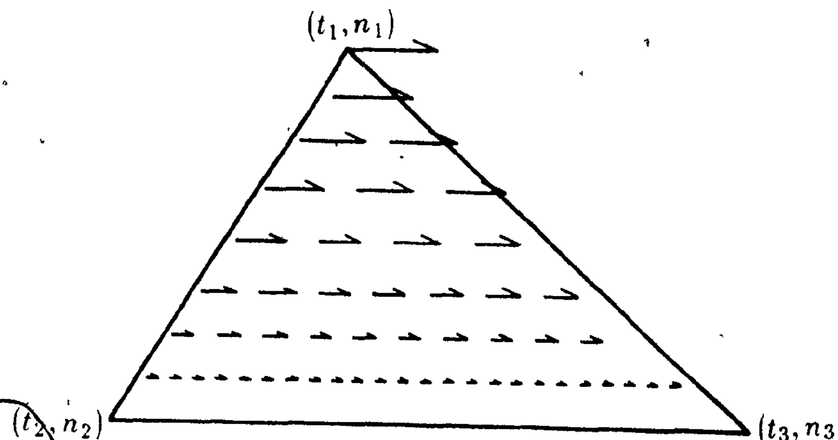

Fig. 3.5 The solenoidal vector function $\mathbf{u_4}$.

is taken as the fourth basis vector. $\mathbf{u_4}$ is parallel to edge 1 however it takes on a value of 1 at vertex 1 and vanishes everywhere on edge 1 as pictured in Fig. 3.5. Taking the divergence of $\mathbf{u_4}$, $\nabla \cdot \mathbf{u_4} = 0$ is verified. There are two other functions similar to $\mathbf{u_4}$ parallel to edge 2 and edge 3.

Six solenoidal vector polynomials of order 1 have been constructed. They however cannot be linearly independent since only 5 such functions are required to form a basis for first order solenoidal vector polynomials. Before it is determined if any choice of five $\mathbf{u}_i$ provides the necessary basis, the vector functions are rewritten using barycentric coordinates.

Barycentric[†] or simplex coordinates are coordinates which are local to every triangle. A point p in a triangle is uniquely defined by $(\varsigma_1, \varsigma_2, \varsigma_3)$ where ς_1 is the ratio of the area of triangle T_1 over the area of the entire triangle shown in Fig. 3.6. When the $\mathbf{u}_i$ are rewritten in terms of these barycentric coordinates, the t_i and n_i coordinates need not be retained.

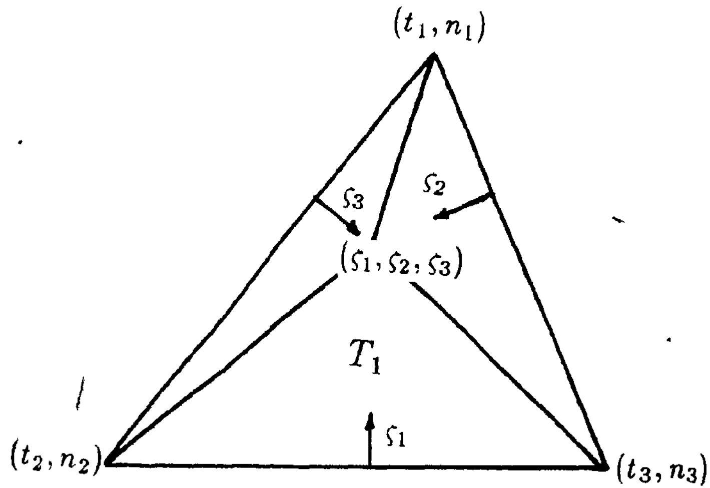

Fig. 3.6 The barycentric coordinates of a triangle.

Expressing the $\mathbf{u}_i$ in terms of barycentric coordinates, they become

$$\mathbf{u_1} = (\varsigma_2 + \varsigma_3)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

† Barycentric coordinates were first introduced by Möbius in 1827 in order to perform center of mass calculations, hence their name (see Smith (1929) for an English translation of Möbius' definition of barycentric coordinates).

$$
\begin{aligned}
\mathbf{u_2} &= (\varsigma_1 + \varsigma_3)\begin{pmatrix} -\cos\theta_3 \\ \sin\theta_3 \end{pmatrix} \\
\mathbf{u_3} &= (\varsigma_1 + \varsigma_2)\begin{pmatrix} -\cos\theta_2 \\ -\sin\theta_2 \end{pmatrix} \\
\mathbf{u_4} &= \varsigma_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\mathbf{u_5} &= \varsigma_2\begin{pmatrix} -\cos\theta_3 \\ \sin\theta_3 \end{pmatrix} \\
\mathbf{u_6} &= \varsigma_3\begin{pmatrix} -\cos\theta_2 \\ -\sin\theta_2 \end{pmatrix}
\end{aligned}
\tag{3.16}
$$

In order to show that the six functions given in (3.16) are linearly dependent, the Gram matrix is built and it is shown that its determinant is equal to zero. Each element of the Gram matrix is the inner product of two of the vector functions, i.e.

$$g_{ij} = \int_{\triangle} \mathbf{u_i} \cdot \mathbf{u_j}\, dS$$

where the domain of integration is the triangle $\triangle$. ($\triangle$ is also used to symbolize the area of the triangle.) The following matrix is obtained:

$$
G = \frac{\triangle}{12}\begin{pmatrix}
6 & -5\cos\theta_3 & -5\cos\theta_2 & 2 & -3\cos\theta_3 & -3\cos\theta_2 \\
-5\cos\theta_3 & 6 & -5\cos\theta_1 & -3\cos\theta_3 & 2 & -3\cos\theta_1 \\
-5\cos\theta_2 & -5\cos\theta_1 & 6 & -3\cos\theta_2 & -3\cos\theta_1 & 2 \\
2 & -3\cos\theta_3 & -3\cos\theta_2 & 2 & -\cos\theta_3 & -\cos\theta_2 \\
-3\cos\theta_3 & 2 & -3\cos\theta_1 & -\cos\theta_3 & 2 & -\cos\theta_1 \\
-3\cos\theta_2 & -3\cos\theta_1 & 2 & -\cos\theta_2 & -\cos\theta_1 & 2
\end{pmatrix}.
$$

Noting the symmetries in the G matrix and performing a few row and column operations, the matrix G' which is row and column equivalent to G is obtained.

$$
G' = \frac{\triangle}{12}\begin{pmatrix}
4 & -4\cos\theta_3 & -4\cos\theta_2 & 0 & -2\cos\theta_3 & -2\cos\theta_2 \\
-4\cos\theta_3 & 4 & -4\cos\theta_1 & -2\cos\theta_3 & 0 & -2\cos\theta_1 \\
-4\cos\theta_2 & -4\cos\theta_1 & 4 & -2\cos\theta_2 & -2\cos\theta_1 & 0 \\
0 & 0 & 0 & -2 & -\cos\theta_3 & -\cos\theta_2 \\
0 & 0 & 0 & -\cos\theta_3 & -2 & -\cos\theta_1 \\
0 & 0 & 0 & -\cos\theta_2 & -\cos\theta_1 & -2
\end{pmatrix}.
$$

The determinant of G' depends on the determinants of the two blocks on the diagonal. If either of these determinants vanishes, the determinant of G' vanishes, and so does the determinant of G. The determinant of the first block on the diagonal is

$$\det G_1 = 64 \begin{vmatrix} 1 & -\cos\theta_3 & -\cos\theta_2 \\ -\cos\theta_3 & 1 & -\cos\theta_1 \\ -\cos\theta_2 & -\cos\theta_1 & 1 \end{vmatrix}$$

which reduces to

$$\det G_1 = 64(1 - \cos^2\theta_1 - \cos^2\theta_2 - \cos^2\theta_3 - 2\cos\theta_1\cos\theta_2\cos\theta_3).$$

Since $\theta_1 + \theta_2 + \theta_3 = \pi$,

$$-4\cos\theta_1\cos\theta_2\cos\theta_3 = 1 + \cos 2\theta_1 + \cos 2\theta_2 + \cos 2\theta_3.$$

Also,

$$2\cos^2\theta = \cos 2\theta + 1.$$

Making use of the above two identities, $\det G = 0$ is obtained.

Eliminating any one of the $\mathbf{u_i}$ results in a set of five linearly independent solenoidal vector polynomial. If $\mathbf{u_6}$ is eliminated, the Gram determinant of the matrix G'', which is made up of the upper left five by five block of G, can be reduced to an expression of the form (see Appendix I for details)

$$\det G'' = \frac{32\triangle}{12}\left(1 - \cos^2\theta_3\right)\left(1 + \cos\theta_1\cos\theta_2\cos\theta_3\right).$$

The above expression vanishes if $\theta_3 = 0$ or if $\cos\theta_1 \cos\theta_2 \cos\theta_3 = -1$. But $|\cos\theta| \leq 1$ where the equality is true if and only if $\theta = n\pi$. Therefore the product $\cos\theta_1 \cos\theta_2 \cos\theta_3$ is equal to -1 only if

$$\theta_1 = \pi \qquad and \qquad \theta_2 = \theta_3 = 0$$

or if

$$\theta_2 = \pi \qquad and \qquad \theta_1 = \theta_3 = 0$$

or if

$$\theta_3 = \pi \qquad and \qquad \theta_1 = \theta_2 = 0$$

since $\theta_1 + \theta_2 + \theta_3 = \pi$. The determinant of G'' therefore vanishes only when the area of the triangle is zero and only then will the five solenoidal vector functions be linearly dependent.

3.2.2 Irrotational Vector Polynomials in Two Variables

In this section irrotational vector polynomials in two variables are constructed from the solenoidal vector polynomials. The curl operator in two dimensions is very similar to the divergence operator. As a matter of fact the curl operator can be written as the product of a rotation matrix and the divergence operator. If the divergence of a vector in Cartesian coordinates is written as

$$\nabla \cdot \mathbf{u} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

and the curl of a vector is written as

$$\nabla \times \mathbf{u} = \begin{pmatrix} -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

then the divergence of the vector can also be expressed as

$$\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}.$$

Hence if

$$\nabla \cdot \mathbf{u} = 0$$

then

$$\nabla \times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{u} = \nabla \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{u} = 0.$$

In other words, the irrotational vector polynomials are derived by rotating the solenoidal vector polynomials by $\frac{\pi}{2}$ or $-\frac{\pi}{2}$. Six irrotational polynomials are given below

$$\begin{aligned}
\mathbf{v_1} &= (\varsigma_2 + \varsigma_3) \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\
\mathbf{v_2} &= (\varsigma_1 + \varsigma_3) \begin{pmatrix} \sin\theta_3 \\ \cos\theta_3 \end{pmatrix} \\
\mathbf{v_3} &= (\varsigma_1 + \varsigma_2) \begin{pmatrix} -\sin\theta_2 \\ \cos\theta_2 \end{pmatrix} \\
\mathbf{v_4} &= \varsigma_1 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\
\mathbf{v_5} &= \varsigma_2 \begin{pmatrix} \sin\theta_3 \\ \cos\theta_3 \end{pmatrix} \\
\mathbf{v_6} &= \varsigma_3 \begin{pmatrix} -\sin\theta_2 \\ \cos\theta_2 \end{pmatrix}.
\end{aligned} \tag{3.17}$$

Only five of the irrotational vector polynomials given in Eq. (3.17) are linearly independent and as was the case for the solenoidal vector functions, any five can be chosen. Since inner products are invariant under orthogonal transformations and the elements of the Gram matrix are inner products, the Gram matrix for five $\mathbf{v}_i$ is the

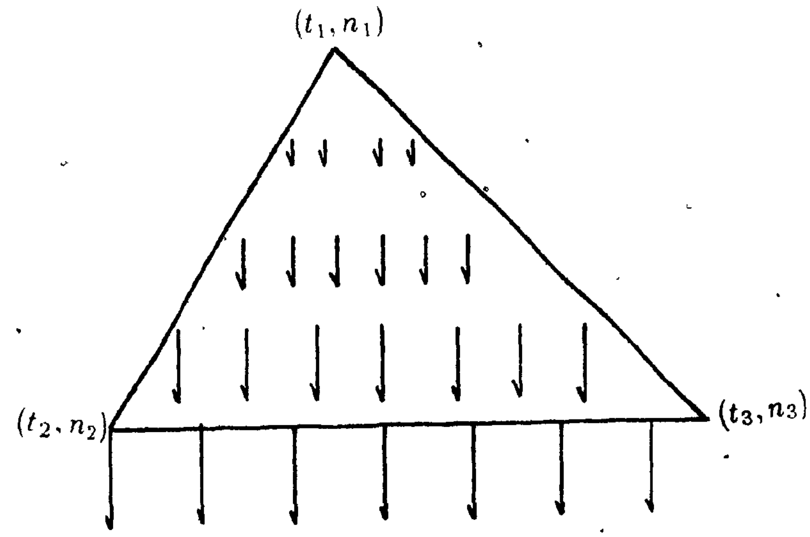

Fig. 3.7 The irrotational vector function $\mathbf{v}_1$.

same as the Gram matrix for the corresponding five $\mathbf{u}_i$. Therefore its determinant will only vanish when the area of the triangle is zero at which point the irrotational vector functions are linearly dependent. The function $\mathbf{v}_1$ is in the direction normal to edge 1 and takes on a value of 1 on that edge and vanishes at vertex 1. $\mathbf{v}_2$ and $\mathbf{v}_3$ behave in a similar fashion. $\mathbf{v}_4$ is in the direction normal to edge 1, however it vanishes on edge 1 and reaches the value of 1 at vertex 1. $\mathbf{v}_1$ and $\mathbf{v}_4$ are pictured in Figs. 3.7 and 3.8. A vector field which lies in the space of irrotational first order vector polynomials can be written as a linear combination of ~~the $\mathbf{v}_i$~~ as

$$\sum_{i=1}^{5} c_i \mathbf{v}_i. \tag{3.18}$$

In order to ensure the continuity of the component of the electric field tangent to interfaces, the c_i must be tranformed into the c_i' which are shown in Fig. 3.3. The transformation which maps the c_i into the c_i' is the same as that which maps the d_i into the d_i' because the c_i' are obtained from the d_i' by a rotation of $\frac{\pi}{2}$ and the $\mathbf{v}_i$ are obtained from the $\mathbf{u}_i$ by a rotation of $\frac{-\pi}{2}$.

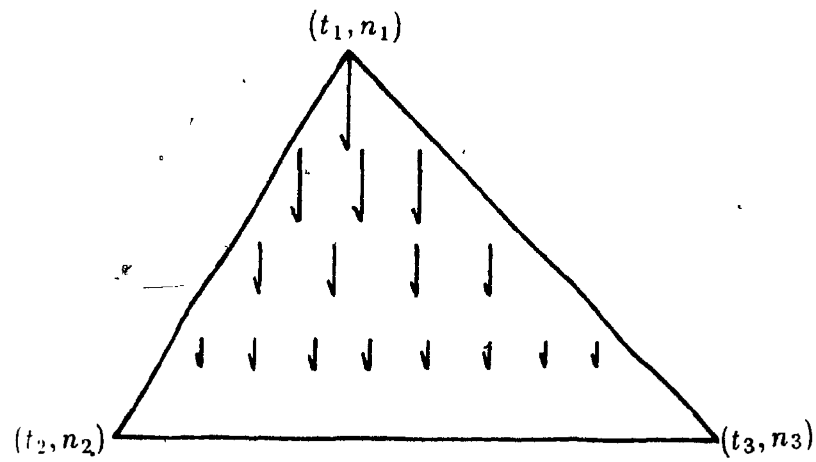

Fig. 3.8 The irrotational vector function $\mathbf{v_4}$.

3.3 The Three Component Vector Interpolation Functions

3.3.1 Solenoidal Vector Polynomials in Three Variables

The number N_3' of linearly independent functions which are required to span the space of three component vector polynomial functions in three variables over a three dimensional region is computed in the same way as for the two component functions.

A polynomial of order n in three variables is of the form

$$\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{j} a_{ijk}x^k y^{j-k} z^{i-j}$$

and a three component vector polynomial in three variables is of the form

$$\mathbf{u} = \begin{pmatrix} \sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{j} a_{ijk}x^k y^{j-k} z^{i-j} \\ \sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{j} b_{ijk}x^k y^{j-k} z^{i-j} \\ \sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{j} c_{ijk}x^k y^{j-k} z^{i-j} \end{pmatrix} \tag{3.19}$$

The divergence of the vector given by (3.19) in Cartesian coordinates is

$$\begin{aligned}
\nabla\cdot\mathbf{u} &= \frac{\partial}{\partial x}\left[\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{j} a_{ijk}x^k y^{j-k} z^{i-j}\right] + \frac{\partial}{\partial y}\left[\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{j} b_{ijk}x^k y^{j-k} z^{i-j}\right] \\
&\quad + \frac{\partial}{\partial z}\left[\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{j} c_{ijk}x^k y^{j-k} z^{i-j}\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} k a_{ijk}x^{k-1} y^{j-k} z^{i-j} + \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=0}^{j-1}(j-k) b_{ijk}x^k y^{j-k-1} z^{i-j} \\
&\quad + \sum_{i=1}^{n}\sum_{j=0}^{i-1}\sum_{k=0}^{j}(i-j) c_{ijk}x^k y^{j-k} z^{i-j-1} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} k a_{ijk}x^{k-1} y^{j-k} z^{i-j} + \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}(j-k+1) b_{ijk}x^{k-1} y^{j-k} z^{i-j} \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}(i-j+1) c_{ijk}x^{k-1} y^{j-k} z^{i-j} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}\left(k a_{ijk} + (j-k+1)\, b_{ijk-1} + (i-j+1)\, c_{ij-1k-1}\right) x^{k-1} y^{j-k} z^{i-j}
\end{aligned}$$

In order for the divergence of $\mathbf{u}$ to vanish everywhere over the region the following must hold for all x, y, z.

$$\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}\left(ka_{ijk} + (j-k+1)\,b_{ijk-1} + (i-j+1)\,c_{ij-1k-1}\right)x^{k-1}y^{j-k}z^{i-j} = 0.$$

This means that

$$\left(ka_{ijk} + (j-k+1)\,b_{ijk-1} + (i-j+1)\,c_{ij-1k-1}\right) = 0 \tag{3.20}$$

for

$$1 \le i \le n, \quad 1 \le j \le i, \quad 1 \le k \le j.$$

The number of constraints N_c' which are specified by the condition (3.20) is

$$N_c' = \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i} j = \sum_{i=1}^{n}\frac{j(j+1)}{1} = \frac{n(n+1)(n+2)}{6} \tag{3.21}$$

and the number N' of linearly independent vector polynomials which are required to span the space of three component vector polynomials of order n is

$$N' = 3\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{j} 1 = \frac{(n+1)(n+2)(n+3)}{2}. \tag{3.22}$$

Hence the number of linearly independent three component solenoidal vector polynomials of order n in three variables required to form a basis is

$$N_s' = \frac{(n+1)(n+2)(2n+9)}{6}. \tag{3.23}$$

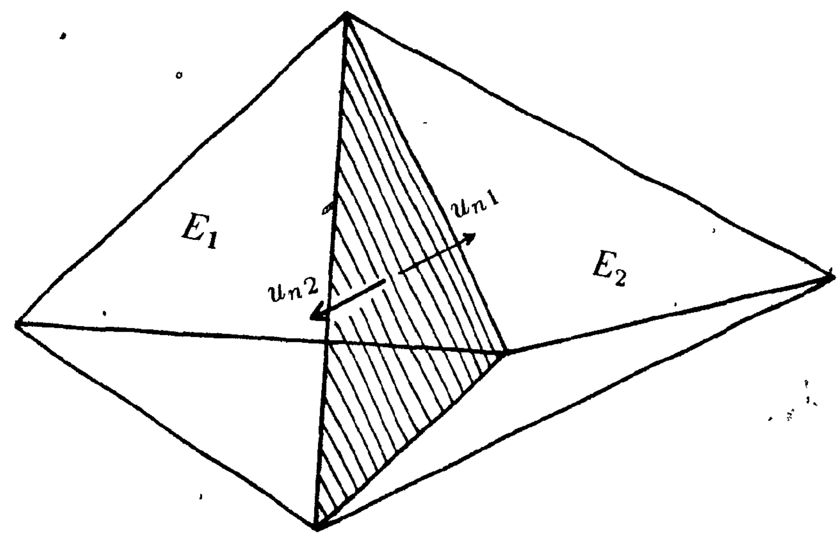

Fig.3.9 The normal component of the vector field across a face.

In the case of first order polynomials, $N_s' = 11$. A solenoidal vector field will therefore be approximated over each tetrahedron by eleven linearly independent first order vector polynomials whose divergence vanishes everywhere inside the element.

The continuity of the normal component of the field across interelement faces must be ensured by requiring that $u_{n1} = u_{n2}$ everywhere on the face shared by two elements as shown in Fig. 3.9. Since the vector field is approximated by first order polynomials, three points, where $u_{n1} + u_{n2}$ must be set equal to zero, must be chosen on each face shared by two tetrahedra. If $u_{n1} + u_{n2} = 0$ is imposed on the face shared by E_1 and E_2 at three points which are not collinear and are on that face, then $u_{n1} + u_{n2}$ is zero everywhere on the face and the component of the field normal to the face is continuous across that face. Since the normal component of the vector field must be evaluated at three points on each face, twelve points are needed for the entire element. At these points the value of the component of the field normal to a face is denoted by

$e_1', e_2', e_3', \ldots e_{12}'$. Each e_i' is the component of the field normal to a face and evaluated at a vertex of the tetrahedron.

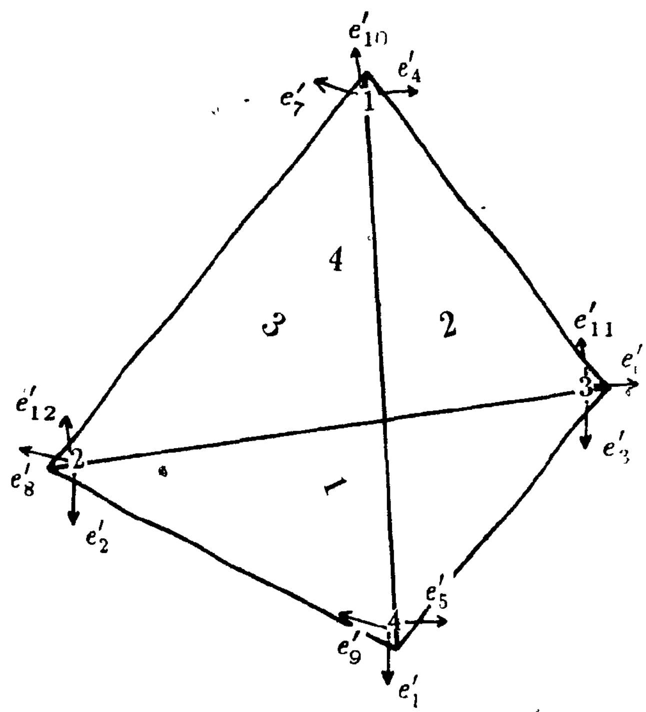

Fig. 3.10 The e_i' are the components of the field normal to an edge and evaluated at a vertex of the tetrahedron.

The eleven linearly independent three component solenoidal vector polynomials are derived in the same way the two component solenoidal vector polynomials were constructed in Section 3.2.1. They are given in (3.24) in terms of a polynomial and a unit vector directed along an edge of the tetrahedron. The polynomial is written in terms of the barycentric coordinates of the tetrahedron shown in Fig. 3.11 and the unit vectors $\mathbf{e}_i$ which are parallel to the edges of the tetrahedron are shown in Fig. 3.12.

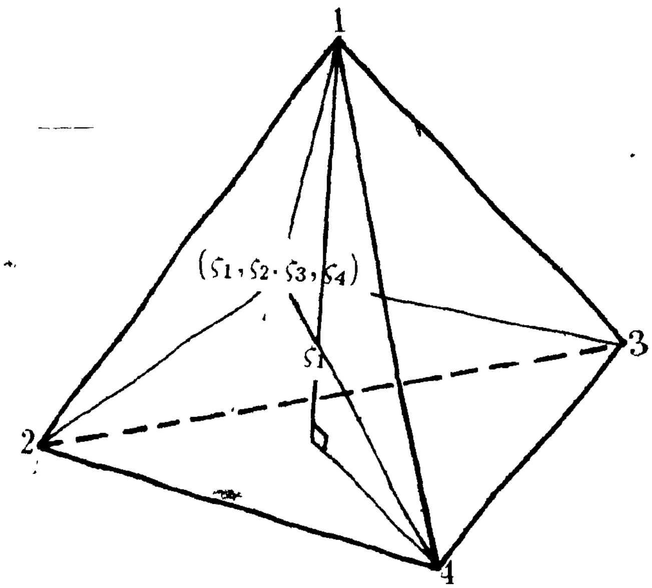

Fig. 3.11 The barycentric coordinates of a tetrahedron.

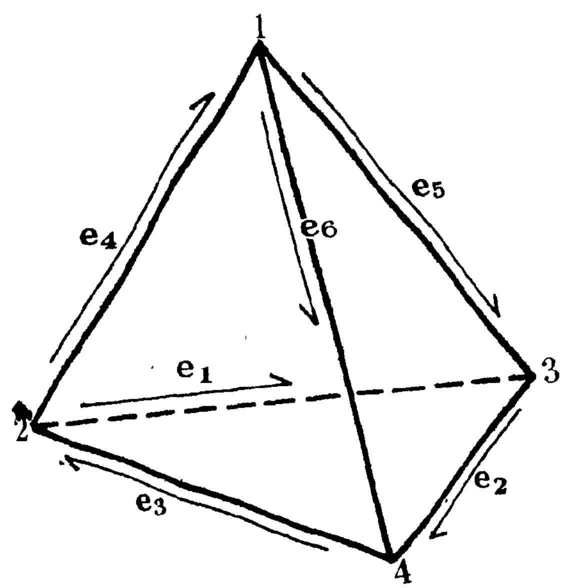

Fig. 3.12 The directions of the vectors along the edges of the tetrahedron.

$$
\begin{array}{ll}
\mathbf{u_1} = (\varsigma_2 + \varsigma_3)\mathbf{e_1} & \mathbf{u_7} = \varsigma_1\mathbf{e_1} \\
\mathbf{u_2} = (\varsigma_3 + \varsigma_4)\mathbf{e_2} & \mathbf{u_8} = \varsigma_2\mathbf{e_2} \\
\mathbf{u_3} = (\varsigma_2 + \varsigma_4)\mathbf{e_3} & \mathbf{u_9} = \varsigma_3\mathbf{e_3} \\
\mathbf{u_4} = (\varsigma_1 + \varsigma_2)\mathbf{e_4} & \mathbf{u_{10}} = \varsigma_4\mathbf{e_4} \\
\mathbf{u_5} = (\varsigma_3 + \varsigma_1)\mathbf{e_5} & \mathbf{u_{11}} = \varsigma_2\mathbf{e_5}. \\
\mathbf{u_6} = (\varsigma_1 + \varsigma_4)\mathbf{e_6} &
\end{array}
\tag{3.24}
$$

As with the two component solenoidal vector polynomials, the vector polynomials given in (3.24) are nonzero at two of the twelve e_i' and vanish at the other ten. Furthermore five vector polynomials can be picked from the eleven given in (3.24) in such a way that their projection onto a face of the tetrahedron will result in five of the two component solenoidal vector functions. For example if the three component vectors $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_7}, \mathbf{u_8}$ are projected onto face 1 of the tetrahedron pictured in Fig. 3.12, the vector polynomials which result, are the two component vectors $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}, \mathbf{u_5}$ where ς_4 is replaced by ς_1.

In the case of the two component solenoidal vector polynomials, six were found from which five had to be chosen. In the case of the three component solenoidal vector polynomials, there are eighteen of which only eleven are linearly independent. The eighteen are:

$$
\begin{array}{lll}
\mathbf{u_1} = (\varsigma_2 + \varsigma_3)\mathbf{e_1} & \mathbf{u_7} = \varsigma_1\mathbf{e_1} & \mathbf{u_{13}} = \varsigma_4\mathbf{e_1} \\
\mathbf{u_2} = (\varsigma_3 + \varsigma_4)\mathbf{e_2} & \mathbf{u_8} = \varsigma_2\mathbf{e_2} & \mathbf{u_{14}} = \varsigma_1\mathbf{e_2} \\
\mathbf{u_3} = (\varsigma_2 + \varsigma_4)\mathbf{e_3} & \mathbf{u_9} = \varsigma_3\mathbf{e_3} & \mathbf{u_{15}} = \varsigma_1\mathbf{e_3} \\
\mathbf{u_4} = (\varsigma_1 + \varsigma_2)\mathbf{e_4} & \mathbf{u_{10}} = \varsigma_4\mathbf{e_4} & \mathbf{u_{16}} = \varsigma_3\mathbf{e_4} \\
\mathbf{u_5} = (\varsigma_3 + \varsigma_1)\mathbf{e_5} & \mathbf{u_{11}} = \varsigma_2\mathbf{e_5} & \mathbf{u_{17}} = \varsigma_4\mathbf{e_5} \\
\mathbf{u_6} = (\varsigma_1 + \varsigma_4)\mathbf{e_6} & \mathbf{u_{12}} = \varsigma_3\mathbf{e_6} & \mathbf{u_{18}} = \varsigma_2\mathbf{e_6}.
\end{array}
\tag{3.25}
$$

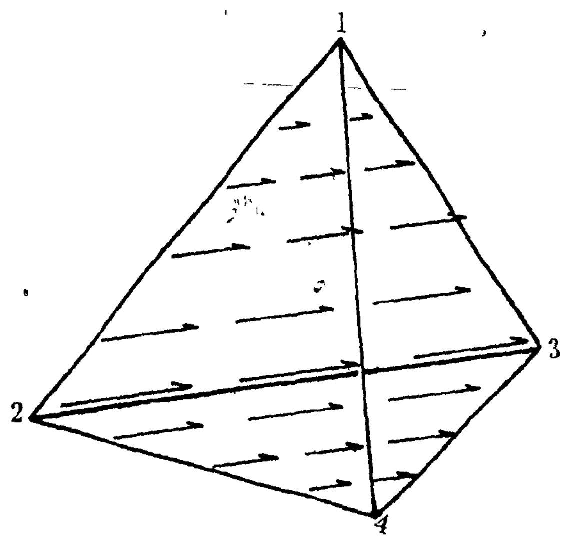

Fig. 3.13 The solenoidal three component vector function $\mathbf{u}_1$.

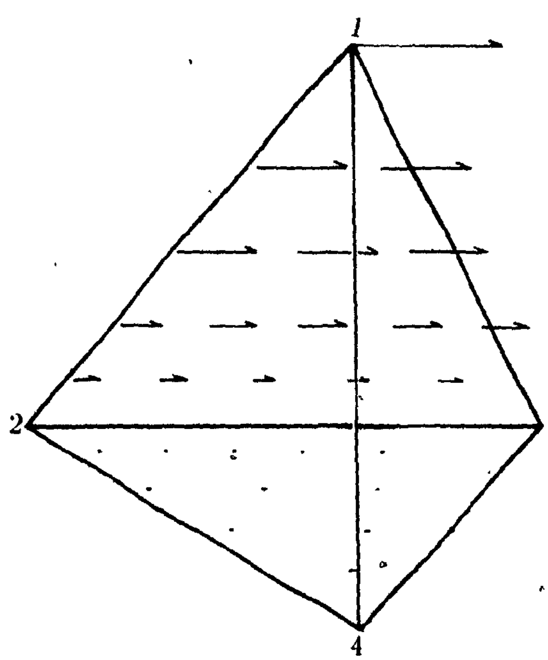

Fig. 3.14 The three component solenoidal vector function $\mathbf{u}_7$.

The first six are each parallel to one of the six edges; they are nonzero on that edge and vanish at the two vertices which do not coincide with that edge. One such ($\mathbf{u}_1$) is shown in Fig. 3.13. The remaining twelve are also parallel to the edges of the tetrahedron; they however vanish on an entire face and are nonzero at only one vertex. One vector polynomial from the second set ($\mathbf{u}_7$) is shown in Fig. 3.14. Eleven linearly independent functions must be chosen from among the eighteen and unfortunately any choice of eleven will not do. The eleven given in (3.24) are linearly independent and are chosen by ensuring that the Gram determinant is nonzero for a given tetrahedron.

3.3.2 Irrotational Vector Polynomials in Three Variables

A three component vector polynomial of first order in three variables takes on the form

$$\mathbf{v} = \begin{pmatrix} ax + by - cz + d \\ ex + fy + gz + h \\ ix + jy + kz + l \end{pmatrix}. \tag{3.26}$$

In order for $\mathbf{v}$ to be irrotational the following three conditions have to hold between the coefficients of the polynomials in the coefficients of $\mathbf{v}$

$$\begin{aligned} j - g &= 0 \\ c - i &= 0\,. \\ e - b &= 0 \end{aligned} \tag{3.27}$$

Nine linearly independent vector polynomials whose curl vanishes everywere in a three dimensional region Ω are required to span the space of irrotational three component vector polynomials of first order in three variables.

Seven of these functions are derived by choosing four vector polynomials which are normal to a face and have a value of 1 everywhere on that face and which vanish at the vertex of the tetrahedron which is not in that face. Three more are also normal to a face, vanish everywhere on the face and have a value of 1 at the vertex which is not in that face. These seven are

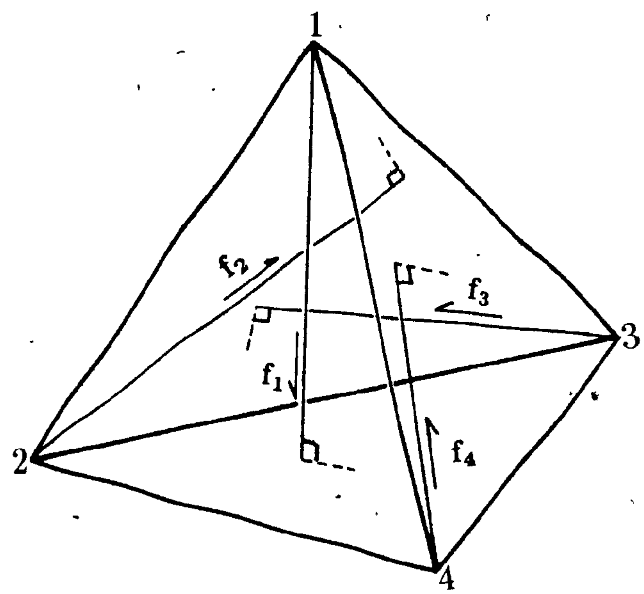

Fig. 3.15 The directions of the vectors along which the irrotational vector polynomials are directed.

$$
\begin{aligned}
\mathbf{v}_1 &= \varsigma_1 \mathbf{f}_1 & \mathbf{v}_5 &= (\varsigma_2 + \varsigma_3 + \varsigma_4)\mathbf{f}_1 \\
\mathbf{v}_2 &= \varsigma_2 \mathbf{f}_2 & \mathbf{v}_6 &= (\varsigma_1 + \varsigma_3 + \varsigma_4)\mathbf{f}_2 \\
\mathbf{v}_3 &= \varsigma_3 \mathbf{f}_3 & \mathbf{v}_7 &= (\varsigma_1 + \varsigma_2 + \varsigma_4)\mathbf{f}_3. \\
\mathbf{v}_4 &= \varsigma_4 \mathbf{f}_4 & &
\end{aligned}
\tag{3.28}
$$

The $\mathbf{f}_i$ which are shown in Fig. 3.15 are unit vectors which are each normal to one of the faces of the tetrahedron. The remaining two irrotational vector polynomials can be chosen in whichever way as long as the resulting nine vector polynomials are linearly

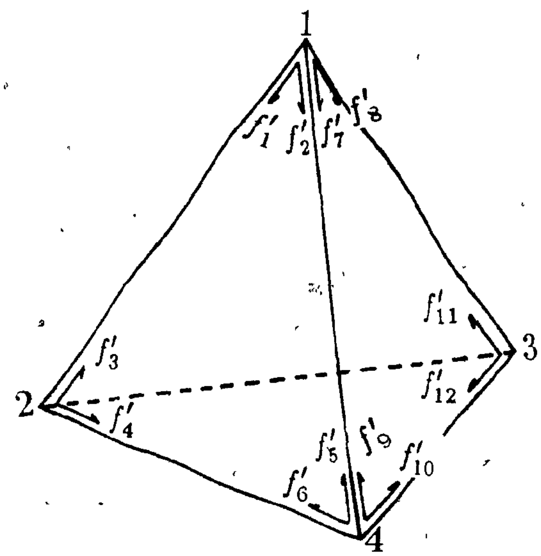

Fig. 3.16 The f_i' are the components of the field tangential to a face and evaluated at a vertex of the tetrahedron.

independent. There however does not seem to be an 'obvious' choice for these remaining two. An irrotational vector field which lies in the space of first order irrotational vector polynomials can then be written as a linear combination of the nine basis functions as

$$\sum_{i=1}^{9} f_i \mathbf{b}_i. \tag{3.29}$$

For an irrotational vector field, the tangential component of the field must be continuous across interelement interfaces. In order to impose continuity in the components of the field tangent to an interelement face, the components of the field tangent to each face of the tetrahedronare evaluated at each of the vertices and equated to the corresponding tangential components of the field in the neighbouring tetrahedra. These tangential components are the f_i' shown in Fig. 3.16. There are twenty-four f_i' which have to be equated at interelement faces in order for the tangential component of the field to be continuous. This seems to require a lot of effort and is unecessary especially when a irrotational field can be expressed as the gradient of a scalar function. Hence it is not fruitful to consider three component irrotational vector interpolation any further.

The functions which have been constructed in this chapter are used in the following two chapters to solve the two problems which were formulated in Chapter 2. The solenoidal and irrotational vector polynomials were constructed in such a way as to satisfy the requirements outlined in Chapter 2. A solenoidal vector function can be approximated by a linear combination of the solenoidal vector polynomials $\mathbf{u}_i$. The coefficients in the linear combination can be easily transformed into another set of coefficients which are the components of the vector field normal to an edge or a face and evaluated at a vertex. Similarly, an irrotational vector field can be approximated by a linear combination of irrotational vector polynomials $\mathbf{v}_i$. For two component vector fields the coefficients in the linear combination can be easily transformed into another set of coefficients which are the components of the vector field tangent to an edge and evaluated at a vertex. In the case of the three component irrotational vector fields, there is no advantage in using vector approximation polynomials and it is recommended that the field be expressed in terms of a scalar potential.

CHAPTER 4

THE FINITE ELEMENT MATRICES

The two problems posed in Chapter 2 can now be solved numerically with the help of the finite element method and the vector interpolation polynomials developed in Chapter 3. The problem region is first discretized into a finite element mesh of triangles or tetrahedra. In Section 4.1.1 the current density is approximated by a two component solenoidal vector polynomial whose component normal to interelement edges is continuous and in Section 4.1.2 the electric field is approximated by a two component irrotational vector polynomial whose component tangent to interelement edges is continuous. The method which is used to impose the continuity conditions and the boundary conditions is described in each section. In Section 4.2 the three component current density is approximated by a three component solenoidal vector polynomial in order to determine the resistance of the wire.

4.1 The Two Component Vector Interpolation Polynomials

4.1.1 The Two Component Solenoidal Vector Polynomials

The power functional which has to be minimized in order to determine the resistance of the conducting plate is

$$\begin{aligned} P &= d\int_{\Sigma} \mathbf{J}\cdot\mathbf{E}\,dS \\ &= d\int_{\Sigma} \frac{|\mathbf{J}|^2}{\sigma}\,dS. \end{aligned}$$

The current density $\mathbf{J}$ in the above functional can be approximated by a linear combination of the two component solenoidal vector polynomials of order 1 over each triangle

$$\mathbf{J} = \sum_{i=1}^{5} d_i \mathbf{u_i}$$

to obtain

$$\begin{aligned}
P_\triangle &= d \int_\triangle \frac{\left|\sum_{i=1}^{5} d_i \mathbf{u_i}\right|^2}{\sigma} dS \\
&= d \int_\triangle \frac{\left(\sum_{i=1}^{5} d_i \mathbf{u_i}\right) \cdot \left(\sum_{j=1}^{5} d_j \mathbf{u_j}\right)}{\sigma} dS \\
&= d \int_\triangle \frac{\sum_{i=1}^{5} \sum_{j=1}^{5} d_i d_j \mathbf{u_i} \cdot \mathbf{u_j}}{\sigma} dS \\
&= \sum_{i=1}^{5} \sum_{j=1}^{5} d_i d_j\, d \int_\triangle \frac{\mathbf{u_i} \cdot \mathbf{u_j}}{\sigma} dS.
\end{aligned}$$

If σ is a constant, the functional can be rewritten as

$$P_\triangle = \frac{\sum_{i=1}^{5} \sum_{j=1}^{5} d_i d_j}{\sigma'} \int_\triangle \mathbf{u_i} \cdot \mathbf{u_j}\, dS \tag{4.1}$$

where $\sigma' = \sigma/d$ and has units of conductivity/unit length. The functional $P_\triangle$ is thus approximated by

$$P_\triangle = \frac{d^T T d}{\sigma'} \tag{4.2}$$

where

$$d^T = (\, d_1 \quad d_2 \quad d_3 \quad d_4 \quad d_5 \,) \tag{4.3}$$

and each element t_{ij} of T is

$$t_{ij} = \int_\triangle \mathbf{u_i} \cdot \mathbf{u_j}\, dS.$$

The matrix T is the matrix G'' defined in Chapter 3.

$$T = \frac{\Delta}{12}\begin{pmatrix} 6 & -5\cos\theta_3 & -5\cos\theta_2 & 2 & -3\cos\theta_3 \\ -5\cos\theta_3 & 6 & -5\cos\theta_1 & -3\cos\theta_3 & 2 \\ -5\cos\theta_2 & -5\cos\theta_1 & 6 & -3\cos\theta_2 & -3\cos\theta_1 \\ 2 & -3\cos\theta_3 & -3\cos\theta_2 & 2 & -\cos\theta_3 \\ -3\cos\theta_3 & 2 & -3\cos\theta_1 & -\cos\theta_3 & 2 \end{pmatrix} \tag{4.4}$$

In order to impose the continuity requirements the d_i are transformed into the corresponding d_i' which were defined in Chapter 3. The d_i' are the components of the vector field which are normal to an edge of the triangle and evaluated at a vertex. The transformation which takes the d_i into the d_i' is the following:

$$D = \begin{pmatrix} 0 & 0 & \sin\theta_2 & 0 & -\sin\theta_3 \\ 0 & -\sin\theta_3 & 0 & 0 & 0 \\ \sin\theta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sin\theta_1 & \sin\theta_3 & 0 \\ 0 & \sin\theta_1 & 0 & -\sin\theta_2 & 0 \\ -\sin\theta_2 & 0 & 0 & 0 & \sin\theta_1 \end{pmatrix} \tag{4.5}$$

and

$$d' = Dd \tag{4.6}$$

where

$$d'^T = (\, d_1' \quad d_2' \quad d_3' \quad d_4' \quad d_5' \quad d_6' \,)\,. \tag{4.7}$$

D is not a square matrix and is therefore not invertible. However,

$$D' = \begin{pmatrix} 0 & 0 & \frac{1}{\sin\theta_3} & 0 & 0 & 0 \\ 0 & \frac{-1}{\sin\theta_3} & 0 & 0 & 0 & 0 \\ \frac{1}{\sin\theta_2} & 0 & \frac{1}{\sin\theta_1} & 0 & 0 & \frac{\sin\theta_3}{\sin\theta_1\sin\theta_2} \\ 0 & \frac{-\sin\theta_1}{\sin\theta_3\sin\theta_2} & 0 & 0 & \frac{-1}{\sin\theta_2} & 0 \\ 0 & 0 & \frac{\sin\theta_2}{\sin\theta_1\sin\theta_3} & 0 & 0 & \frac{1}{\sin\theta_1} \end{pmatrix} \tag{4.8}$$

is a transformation which takes the d' into d where the following compatibility condition must hold if Eq. (4.6) is satisfied:

$$l_1(d'_1 + d'_2) + l_2(d'_3 + d'_4) - l_3(d'_5 + d'_6) = 0 \tag{4.9}$$

where $l_\imath$ is the length of edge $\imath$ of the triangle.

It will now be shown that the condition given in (4.9) states that the integral over a closed surface of the normal component of the solenoidal vector field must vanish. Since the $\mathbf{u_i}$ are solenoidal, the integral of the divergence of $\mathbf{J}$ over a triangle vanishes:

$$\begin{aligned}
\int_\triangle \nabla \cdot \mathbf{J}\, dS &= \int_\triangle \nabla \cdot \sum_{\imath=1}^{5} d_\imath \mathbf{u_i}\, dS \\
&= \int_\triangle \sum_{\imath=1}^{5} d_\imath \nabla \cdot \mathbf{u_i}\, dS \\
&= 0.
\end{aligned}$$

By the divergence theorem

$$\int_\triangle \nabla \cdot \mathbf{J}\, dS = \int_{\partial\triangle} J_n dl$$

and the compatibility condition (4.9) is easily deduced:

$$\begin{aligned}
0 &= \int_{\partial\triangle} J_n dl \\
&= \int_{edge_1} J_n dl + \int_{edge_2} J_n dl + \int_{edge_3} J_n dl \\
&= \frac{l_1}{2}(d'_1 + d'_2) + \frac{l_2}{2}(d'_3 + d'_4) + \frac{l_3}{2}(d'_5 + d'_6).
\end{aligned}$$

The continuity of the normal component of the solenoidal field is therefore imposed by transforming the d into d' through $d = D'd'$ and then by equating the $d'_\imath$ of two neighbouring triangles across the edge that they share. For example, in the two triangles shown in Fig. 4.1,

$$\begin{aligned} d'_{13} &= -d'_{26} \\ d'_{14} &= -d'_{25} \end{aligned} \tag{4.10}$$

is imposed.

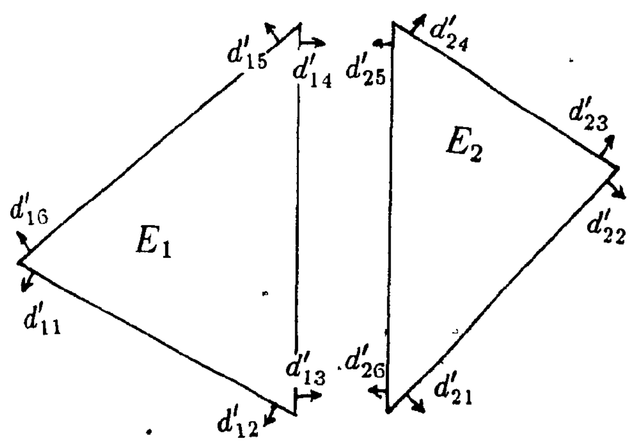

Fig. 4.1 Imposing the continuity of the normal component of **J** across an interelement edge.

Conditions such as (4.10) are satisfied over the entire mesh by building a connection matrix which equates the appropriate d'_i at all interelement boundaries. As an example, the connection matrix for the two triangles shown in Fig. 4.1 will now be given. Before any continuity conditions have been imposed the approximation of the functional P takes on the form

$$P = \frac{d_{dis}^T T_{dis} d_{dis}}{\sigma'} \tag{4.11}$$

where

$$T_{dis} = \begin{pmatrix} t_{111} & t_{112} & t_{113} & t_{114} & t_{115} & 0 & 0 & 0 & 0 & 0 \\ t_{121} & t_{122} & t_{123} & t_{124} & t_{125} & 0 & 0 & 0 & 0 & 0 \\ t_{131} & t_{132} & t_{133} & t_{134} & t_{135} & 0 & 0 & 0 & 0 & 0 \\ t_{141} & t_{142} & t_{143} & t_{144} & t_{145} & 0 & 0 & 0 & 0 & 0 \\ t_{151} & t_{152} & t_{153} & t_{154} & t_{155} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t_{211} & t_{212} & t_{213} & t_{214} & t_{215} \\ 0 & 0 & 0 & 0 & 0 & t_{221} & t_{222} & t_{223} & t_{224} & t_{225} \\ 0 & 0 & 0 & 0 & 0 & t_{231} & t_{232} & t_{233} & t_{234} & t_{235} \\ 0 & 0 & 0 & 0 & 0 & t_{241} & t_{242} & t_{243} & t_{244} & t_{245} \\ 0 & 0 & 0 & 0 & 0 & t_{251} & t_{252} & t_{253} & t_{254} & t_{255} \end{pmatrix} \tag{4.12}$$

and

$$d_{dis}^T = \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{21} & d_{22} & d_{23} & d_{24} & d_{25} \end{pmatrix}. \tag{4.13}$$

The d are then transformed into d' by

$$d = D'd' \tag{4.14}$$

for each triangle and by

$$d_{dis} = D'_{dis}d'_{dis}$$

for the entire mesh. The d' are "connected" with the help of the C matrix used in the following equation:

$$\begin{pmatrix} d'_{11} \\ d'_{12} \\ d'_{13} \\ d'_{14} \\ d'_{15} \\ d'_{16} \\ d'_{21} \\ d'_{22} \\ d'_{23} \\ d'_{24} \\ d'_{25} \\ d'_{26} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d'_{c11} \\ d'_{c12} \\ d'_{c13} \\ d'_{c14} \\ d'_{c15} \\ d'_{c16} \\ d'_{c21} \\ d'_{c22} \\ d'_{c23} \\ d'_{c24} \end{pmatrix}. \tag{4.15}$$

The functional therefore becomes

$$P = \frac{d_c'^T C^T D_{dis}'^T T_{dis} D_{dis}' C d_c'}{\sigma'} \tag{4.16}$$

where the d_c' are the variables which describe the connected problem.

Once the continuity of the normal component is ensured across every edge in the finite element mesh, the compatibility condition given by (4.9) has to be imposed. This can be done in two ways. The first method uses another connection matrix which eliminates one of the d_c' per triangle. This matrix is given below for the simple two triangle mesh shown in Fig. 4.1 however it is not clear how this matrix should be constructed for larger meshes.

$$\begin{pmatrix} d_{c11}' \\ d_{c12}' \\ d_{c13}' \\ d_{c14}' \\ d_{c15}' \\ d_{c16}' \\ d_{c21}' \\ d_{c22}' \\ d_{c23}' \\ d_{c24}' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{-l_{11}}{l_{13}} & \frac{-l_{11}}{l_{13}} & \frac{-l_{12}}{l_{13}} & \frac{-l_{12}}{l_{13}} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{l_{23}}{l_{22}} & \frac{l_{23}}{l_{22}} & 0 & \frac{-l_{21}}{l_{22}} & \frac{-l_{21}}{l_{22}} & -1 \end{pmatrix} \begin{pmatrix} d_{c11}' \\ d_{c12}' \\ d_{c13}' \\ d_{c14}' \\ d_{c15}' \\ d_{c21}' \\ d_{c22}' \\ d_{c23}' \end{pmatrix}$$

The second method uses Lagrange multipliers to impose the compatibility conditions. A condition of the type shown in (4.9) must be satisfied for every triangle in the mesh. This means that for the two triangles shown in Fig. 4.1, the two following constraints must hold:

$$l_{11}(d_{11}' + d_{12}') + l_{12}(d_{13}' + d_{14}') + l_{13}(d_{15}' + d_{16}') = 0$$

$$l_{21}(d_{21}' + d_{22}') + l_{22}(d_{23}' + d_{24}') + l_{23}(d_{25}' + d_{26}') = 0.$$

The above condition can be reexpressed in the form

$$Qd'_{dis} = QCd'_c = 0. \tag{4.17}$$

Finally the boundary conditions must be imposed. For the conducting plate, the boundary conditions given in Chapter 2 were

$$J_n = 0 \qquad on\ \Gamma_1 \tag{4.18}$$

$$J_n = 0 \qquad on\ \Gamma'_1 \tag{4.19}$$

$$\int_{C_1} J_n dl = I, \tag{4.20}$$

where C_1 is a curve which joins Γ_1 to Γ'_1. Conditions (4.18) and (4.19) are easy to impose as they require that the component of the current density normal to Γ_1 and Γ'_1 be zero. Because the current density is approximated by first order solenoidal vector polynomials over each triangle, the normal component of the current density need only be imposed at two points on an edge which makes up part of Γ_1 or Γ'_1 for the normal component of the current density to be zero everywhere on that edge. Hence, the appropriate d_c, which are the components of the vector field normal to an edge must be set to zero. The remaining coefficients are denoted by d'_f. Condition (4.20) is imposed by choosing a path through the finite element mesh and satisfying the integral constraint along that path. This path joins Γ_1 to Γ'_1 and is made up of k edges. The integral constraint reduces to a linear equation relating the d'_f which correspond to the edges which lie on the path:

$$\int_{C_1} J_n dl = \int_{k\ edges} J_n dl$$

$$
\begin{aligned}
&= \int_{edge_1} J_n dl + \ldots + \int_{edge_k} J_n dl \\
&= \frac{l_{e1}}{2}(d'_{e11} + d'_{e12}) + \ldots + \frac{l_{ek}}{2}(d'_{ek1} + d'_{ek2}) \\
&= I
\end{aligned} \tag{4.21}
$$

where l_{ei} is the length of the ith edge and d'_{ei1} is the component of the current density normal to edge i evaluated at vertex 1 of the edge. This last condition, which can be reexpressed as

$$
b^T d'_f - I = 0 \tag{4.22}
$$

can be imposed with the help of a Lagrange multiplier or by explicitly eliminating one of the d'_f which appear in (4.22).

The functional which is obtained when the functional (4.16) is constrained by the conditions (4.17) and (4.22) is

$$
L = \frac{d'^T_f T' d'_f}{\sigma'} + 2\lambda^T Q' d'_f + 2\mu(b^T d'_f - I) \tag{4.23}
$$

where $T' = C^T D'^T_{dis} T_{dis} D'_{dis} C$ and $Q' = QC$ for the d'_f. L now has to be minimized with respect to the d'_f, the λ_i, and μ. Taking the first derivatives of L with respect to the d'_f, the λ_i and μ,

$$
\begin{aligned}
\frac{\partial L}{\partial d'_f} &= \frac{2T' d'_f}{\sigma'} + 2Q'^T \lambda + 2\mu b \\
\frac{\partial L}{\partial \lambda} &= 2Q' d'_f \\
\frac{\partial L}{\partial \mu} &= 2(b^T d'_f - I).
\end{aligned}
$$

is obtained. The above can be reexpressed as

$$\begin{pmatrix} T'/\sigma' & Q'^T & b \\ Q' & 0 & 0 \\ b^T & 0 & 0 \end{pmatrix} \begin{pmatrix} d'_f \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}. \tag{4.24}$$

if

$$\frac{\partial L}{\partial d'_f} = \frac{\partial L}{\partial \lambda} = \frac{\partial L}{\partial \mu} = 0.$$

All the entries in the vector on the right hand side of (4.24) are zero except for the last one which prescribes the value of the total current I. The system of equations in (4.24) can now be solved in order to find the approximation for $\mathbf{J}$ in the space of first order solenoidal vector polynomials of order 1. Unfortunately the matrix in (4.24) is indefinite due to the 0 block on the diagonal. Had the constraints (4.17) been imposed with the help of connection matrices, the matrix in the resulting system of equations would have been positive definite. This means that a Cholesky decomposition or even an incomplete Cholesky decomposition cannot be performed on the matrix in (4.24) as it is not positive definite. However, it is not clear how a connection matrix should be constructed in order for the constraints (4.17) and the boundary condition (4.22) to be satisfied.

4.1.2 The Two Component Irrotational Vector Polynomials

In Chapter 2 the power functional was also expressed in terms of the electric field

$$P = d \int_{\Sigma} \sigma \left|\mathbf{E}\right|^2 dS$$

and the electric field $\mathbf{E}$ can be approximated by a linear combination the two component irrotational vector approximation functions over each triangle in the finite element mesh

$$\mathbf{E} = \sum_{i=1}^{5} c_i \mathbf{v}_i$$

which when substituted into the functional yields

$$P_\triangle = \sum_{i=0}^{5} \sum_{j=0}^{5} c_i c_j d \int_\triangle \sigma \mathbf{v}_i \cdot \mathbf{v}_j \, dV.$$

Since σ is assumed to be a constant, it can be taken out of the integration so that

$$P_\triangle = \sigma' c^T T c \tag{4.25}$$

is obtained where

$$\sigma' = \sigma d,$$
$$c^T = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{pmatrix} \tag{4.26}$$

and the T matrix here is the same as the one for the solenoidal vector polynomials given in (4.4).

In order to impose continuity in the tangential component of the electric field across interelement edges the c_i need to be transformed in the c_i' shown in Fig. 3.2. The c_i' are the components of the vector field tangent to an edge of the triangle and evaluated at a vertex. The transformation matrix which takes the c_i into the the c_i' is the transformation D which transformed the coefficients d_i of the solenoidal vector interpolation polynomials into the coefficients d_i'. Therefore

$$c' = Dc \tag{4.27}$$

Once more D is not invertible, however D' transforms the c' into the c as long as the compatibility condition (4.9) holds. This conditions can now be reinterpreted in terms of Stokes' theorem over one triangle. Since the electric field is approximated by irrotational vector polynomials

$$\begin{aligned}\int_\triangle \nabla \times \mathbf{E}\, dS &= \int_\triangle \nabla \times \sum_{i=1}^{5} c_i \mathbf{v}_i dS \\ &= \int_\triangle \sum_{i=1}^{5} c_i \nabla \times \mathbf{v}_i dS \\ &= 0.\end{aligned}$$

By Stokes' theorem

$$\int_{\partial\triangle} \mathbf{E} \cdot d\mathbf{l} = \int_\triangle \nabla \times \mathbf{E}\, dS$$

and the compatibility (4.9) is deduced from

$$\begin{aligned}0 &= \int_\triangle \mathbf{E} \cdot d\mathbf{l} \\ &= \int_{edge_1} E_t dl + \int_{edge_2} E_t dl + \int_{edge_3} E_t dl \\ &= \frac{l_1}{2}(c'_1 + c'_2) + \frac{l_2}{2}(c'_3 + c'_4) + \frac{l_3}{2}(c'_5 + c'_6).\end{aligned}$$

Therefore the c are transformed into the c' by

$$c = D'c' \tag{4.28}$$

on each triangle and the continuity of the tangential component of the field is ensured by equating the c' across interelement edges. This is done by connecting the c' in the same way as the d' were connected in the last section and yields the following expression for the functional

$$P = \sigma' c_c'^{} C^T D_{dis}'^T T_{dis} D_{dis}' C c_c'. \tag{4.29}$$

If the same mesh is used to obtain the expression for the functional in (4.16) and the expression for the functional in (4.29), then the C, T_{dis}, and D_{dis}' in both expressions are identical. The c_c' are the variables which describe the connected problem.

The compatibility conditions, which can be rewritten as

$$QCc_c' = Q'c_c' = 0 \tag{4.30}$$

are then imposed with the help of Lagrange multipliers. Once more for the same finite element mesh the Q in (4.30) is identical to the Q in (4.17).

Finally, the following boundary conditions have to be imposed on the tangential component of the electric field in the conducting plate:

$$E_t = 0 \quad on\ \Gamma_2 \tag{4.31}$$

$$E_t = 0 \quad on\ \Gamma_2' \tag{4.32}$$

$$\int_{C_2} \mathbf{E} \cdot d\mathbf{l} = V - V' \tag{4.33}$$

The two conditions (4.31) and (4.32) are easily imposed by setting the appropriate c_c' to zero. The integral condition (4.33) is satisfied by choosing a path C_2 from Γ_2 to Γ_2' which is made up of k triangle edges and setting the integral of the tangential component of the electric field over these edges equal to the potential difference $V - V'$.

$$\int_{C_2} \mathbf{E} \cdot d\mathbf{l} = \int_{k\ edges} E_t dl$$

$$
\begin{aligned}
&= \int_{edge_1} E_t dl + \ldots + \int_{edge_k} E_t dl \\
&= \frac{l_{e1}}{2}(c'_{11} + c'_{12}) + \ldots + \frac{l_{ek}}{2}(c'_{k1} + c'_{k2}) \\
&= V - V'.
\end{aligned}
$$

The above can be expressed as

$$a^T c'_f - (V - V') = 0 \tag{4.34}$$

and imposed using the Lagrange multiplier method. The c'_f are the coefficients which remain once conditions (4.31) and (4.32) have been imposed.

The functional which is obtained when the functional (4.29) is constrained by (4.30) and (4.34) is

$$L' = \sigma' \left(c'^T_f T' c'_f\right) + 2\lambda^T Q' c'_f + 2\mu \left(a^T c'_f - (V - V')\right). \tag{4.35}$$

The functional is minimized by taking first derivatives with respect to the c'_f, the λ_i, and μ,

$$
\begin{aligned}
\frac{\partial L}{\partial c'_f} &= \sigma' \left(2T' c'_f\right) + 2Q'^T \lambda + 2\mu a \\
\frac{\partial L}{\partial \lambda} &= 2Q' c'_f \\
\frac{\partial L}{\partial \mu} &= 2\left(a^T c'_f - (V - V')\right).
\end{aligned}
$$

Setting the above derivatives to zero, the following matrix equation is obtained:

$$
\begin{pmatrix} \sigma' T' & Q'^T & a \\ Q' & 0 & 0 \\ a^T & 0 & 0 \end{pmatrix}
\begin{pmatrix} c'_f \\ \lambda \\ \mu \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ V - V' \end{pmatrix}. \tag{4.36}
$$

4.2 The Three Component Solenoidal Vector Polynomials

As in the case of the two component current density vector, the functional for the three component current density vector can be approximated over a tetrahedron by

$$P_{\triangle} = \frac{e^T T e}{\sigma} \tag{4.37}$$

where σ is a constant,

$$e^T = (\, e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \quad e_6 \quad e_7 \quad e_8 \quad e_9 \quad e_{10} \quad e_{11}\,), \tag{4.38}$$

and T is the Gram matrix for the eleven three component solenoidal vector polynomials

$$T = \frac{\triangle}{20}\begin{pmatrix}
6e_{11} & 5e_{12} & 5e_{13} & 5e_{14} & 5e_{15} & 4e_{16} & 2e_{11} & 3e_{12} & 3e_{13} & 2e_{14} & 3e_{15} \\
 & 6e_{22} & 5e_{23} & 4e_{24} & 5e_{25} & 5e_{26} & 2e_{21} & 2e_{22} & 3e_{23} & 3e_{24} & 2e_{25} \\
 & & 6e_{33} & 5e_{34} & 4e_{35} & 5e_{36} & 2e_{31} & 3e_{32} & 2e_{33} & 3e_{34} & 3e_{35} \\
 & & & 6e_{44} & 5e_{45} & 5e_{46} & 3e_{41} & 3e_{42} & 2e_{43} & 2e_{44} & 3e_{45} \\
 & & & & 6e_{55} & 5e_{56} & 3e_{51} & 2e_{52} & 3e_{53} & 2e_{54} & 2e_{55} \\
 & & & & & 6e_{66} & 3e_{61} & 2e_{62} & 2e_{63} & 3e_{64} & 2e_{65} \\
 & & & & & & 2e_{11} & e_{12} & e_{13} & e_{14} & e_{15} \\
 & & & & & & & 2e_{22} & e_{23} & e_{24} & 2e_{25} \\
 & & & & & & & & 2e_{33} & e_{34} & e_{35} \\
 & & & & & & & & & 2e_{44} & e_{45} \\
 & & & & & & & & & & 2e_{55}
\end{pmatrix}. \tag{4.39}$$

where $e_{ij} = \mathbf{e_i} \cdot \mathbf{e_j}$ and $\triangle$ is the volume of the tetrahedron.

Before any continuity conditions have been imposed, the approximation of the functional is

$$P = \frac{e_{dis}^T T_{dis} e_{dis}}{\sigma}. \tag{4.40}$$

The e_i are transformed into the corresponding e_i' shown in Fig. 3.10 via

$$e' = Ee. \tag{4.41}$$

and the e'_i are equated across interelement faces in order to ensure that the component of the approximation to the field normal to interfaces is continuous. E is not a square matrix and is therefore not invertible. However the transformation

$$E' = \frac{1}{3\triangle}\begin{pmatrix} 0 & 0 & l_1A_4 & 0 & 0 & l_1A_3 & 0 & l_1A_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & l_2A_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -l_3A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_3A_2 & 0 & 0 & -l_3A_1 \\ 0 & 0 & 0 & l_4A_3 & 0 & 0 & l_4A_2 & 0 & 0 & 0 & l_4A_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_5A_1 & 0 \\ -l_6A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -l_1A_3 & 0 & 0 & 0 & 0 & 0 & 0 & -l_1A_1 & 0 \\ -l_2A_4 & -l_2A_4 & 0 & 0 & 0 & 0 & 0 & 0 & -l_2A_2 & 0 & 0 & -l_2A_1 \\ 0 & 0 & l_3A_4 & 0 & 0 & l_3A_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -l_4A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_4A_1 \\ 0 & 0 & 0 & l_5A_3 & 0 & 0 & l_5A_2 & 0 & 0 & l_5A_1 & l_5A_1 & 0 \end{pmatrix} \tag{4.42}$$

transforms the e'_i into the e_i, and (4.41) is satisfied if the compatibility condition

$$A_4(e'_1 + e'_2 + e'_3) + A_3(e'_4 + e'_5 + e'_6) + A_2(e'_7 + e'_8 + e'_9) + A_1(e'_{10} + e'_{11} + e'_{12}) = 0 \tag{4.43}$$

holds. The l_i in (4.41) are the length of the edges of the tetrahedron shown in Fig. 3.10 and the A_i are the areas of the four faces of the tetrahedron. The compatibility condition (4.42) can also be deduced from Gauss' as was the compatibility condition (4.9). The requirement that the component of the solenoidal vector field normal to a face be continuous is then imposed by equating the e' across interelement faces. This is done by constructing a connection matrix as in the previous section. The approximation to the functional becomes

$$P = \frac{e_c'^T C^T E_{dis}'^T T_{dis} E'_{dis} C e'_c}{\sigma} \tag{4.44}$$

where

$$e_{dis} = E'e'_{dis} = E'Ce'_c \tag{4.45}$$

and e'_c are the variables which describe the connected problem. The compatibility conditions which can be expressed as

$$Qe'_{dis} = QCe'_c = 0 \tag{4.46}$$

are once more imposed with Lagrange multipliers.

The following conditions on the normal component of the current density must also be imposed:

$$\mathbf{J} \cdot \mathbf{n} = 0 \qquad on\ \Sigma_1 \tag{4.47}$$

$$\int_{S_1} \mathbf{J} \cdot \mathbf{n}\, dS = I \tag{4.48}$$

Condition (4.47) is satisfied by setting to zero the appropriate e'_c and the remaining coefficients are denoted by e'_f. Condition (4.48) is imposed by choosing a cross-sectional surface S_1 through the finite element mesh whose boundary lies in Σ_1. This surfaces is made up of k faces and the integral constraint reduces to a linear equation relating the e'_f which correspond to the faces in S_1:

$$\begin{aligned}
\int_{S_1} \mathbf{J} \cdot \mathbf{n}\, dS &= \int_{k\ faces} J_n dS \\
&= \int_{face\ 1} J_n dS + \ldots \int_{face\ k} J_n dS \\
&= \frac{A_{f1}}{3}(e'_{f11} - e'_{f12} + e'_{f13}) + \ldots + \frac{A_{fk}}{3}(e'_{fk1} + e'_{fk2} + e'_{fk3}) \\
&= I
\end{aligned} \tag{4.49}$$

where A_{f_i} is the area of the ith face and $e'_{f_{i1}}$ is the component of the current density normal to face i and evaluated at vertex 1 of that face. The above condition can be reexpressed as

$$b^T e'_f - I = 0 \tag{4.50}$$

and can be imposed with the help of a Lagrange multiplier. The functional which must be minimized when the functional (4.44) is constrained by (4.46), (4.47) and (4.50) is

$$L = \frac{e'^T_f T' e'_f}{\sigma} + 2\lambda^T Q' e'_f + 2\mu(b^T e'_f - I) \tag{4.51}$$

where $T' = C^T E'^T_{dis} T_{dis} E'_{dis} C$ and $Q' = QC$ for the e'_f. The resulting system of equations is

$$\begin{pmatrix} T'/\sigma & Q'^T & b \\ Q' & 0 & 0 \\ b^T & 0 & 0 \end{pmatrix} = \begin{pmatrix} e'_f \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}. \tag{4.52}$$

The solenoidal vector interpolation functions are used in Chapter 5 to compute the current in a wire. However the three component vector interpolation functions are not implemented since it is still much easier to use a scalar potential when the solution for an irrotational field is sought.

CHAPTER 5

APPROXIMATIONS USING THE VECTOR POLYNOMIALS

In this chapter the resistance of the two dimensional conducting plate presented in Chapter 2 is computed. The irrotational vector functions developed in Chapter 3 are used to approximate the electric field and the solenoidal vector functions are used to approximate the current density. Next the resistance and the conductance of the three dimensional conducting wire are calculated once an approximation to the solenoidal current density and the irrotational electric field in the wire are determined.

5.1 The Electric Field in a Conducting Plate

In Chapter 2 two methods for approximating the three component electric field were given. In the first case the power functional

$$P = \int_{\Omega} \sigma \left|\nabla\chi\right|^2 dV \tag{5.1}$$

is minimized subject to the principal boundary conditions

$$\begin{aligned} \chi &= V \qquad on\ \Sigma_2 \\ \chi &= V' \qquad on\ \Sigma_2'. \end{aligned} \tag{5.2}$$

The function χ which makes P a minimum is φ. Hence the electric field is computed from

$$\mathbf{E} = \nabla\varphi. \tag{5.3}$$

In the second case,

$$P = \int_{\Omega} \sigma \left|\mathbf{E}\right|^2 dV \tag{5.4}$$

is minimized subject to

$$\nabla \times \mathbf{E} = 0 \quad in\ \Omega \tag{5.5}$$

$$E_t = 0 \quad on\ \Sigma_2 \tag{5.6}$$

$$E_t = 0 \quad on\ \Sigma_2'. \tag{5.7}$$

$$\int_{C_2} \mathbf{E} \cdot d\mathbf{l} = V - V' \tag{5.8}$$

The above two methods are equivalent. Therefore the approximate solutions obtained for the electric field using each method are the same if the approximations to **E** used in both cases span the same subspace of the space in which **E** lies. It can be shown that if polynomials of order n which are continuous and piecewise differentiable are used to approximate the scalar function φ, the electric field **E** lies in the space of irrotational vector polynomials of order $n-1$ whose tangential components are continuous across interfaces. This can be seen from Eq. (5.3). Since **E** is the gradient of φ, **E** is irrotational for any φ. If φ is a polynomial function of order n then **E** is a vector polynomial of order $n-1$. Finally, if φ is continuous across an interface, then the derivatives of φ in the direction tangential to the interface exist and are equal on both sides of the interface: E_t is therefore continuous. It can also be shown that if **E** lies in the space of irrotational vector polynomials of order $n-1$ whose tangential components are continuous across interfaces, a scalar function φ can be found such that **E** is the gradient of φ and such that φ is a continuous piecewise differentiable polynomial of order n. Since **E** is irrotational and the region Ω is simply connected, **E** is a conservative field and

$$\int_{p_0}^{p} \mathbf{E} \cdot d\mathbf{l} = \varphi(p) - \varphi(p_0) \tag{5.9}$$

where p_0 is a point of reference in Ω, p is any other point in Ω, and $\varphi(p_0)$ is a known constant. If **E** is an irrotational vector polynomial of order $n-1$, integrating **E** along a curve joining p_0 to p, a polynomial of order n is obtained for $\varphi(p)$. If the path from p_0 to p in the integral of (5.9) crosses an interface, then a $\varphi(p)$ which is continuous across the interface can be found since E_t is continuous.

In conclusion it can be said that the approximation to $\mathbf{E}$ lies in the same space whether it is written as the gradient of a continuous piecewise differentiable scalar polynomial of order n or whether it is written as an irrotational vector polynomial of order $n-1$ whose tangential components are continuous across interfaces. Therefore, whether the functional given by (5.1) or the functional given by (5.4) is minimized subject to the principal conditions, the same approximation to $\mathbf{E}$ is obtained if the approximating functions are chosen as stated above. The same can be said about the two component electric field. Whether the functional

$$P = d \int_{\Sigma} \sigma \left|\nabla \chi\right|^2 dS \tag{5.10}$$

is minimized subject to the principal boundary conditions

$$\begin{aligned} \chi &= V \qquad on\ \Gamma_2 \\ \chi &= V' \qquad on\ \Gamma_2' \end{aligned} \tag{5.11}$$

or the functional

$$P = d \int_{\Sigma} \sigma \left|\mathbf{E}\right|^2 dS \tag{5.12}$$

is minimized subject to the conditions

$$\begin{aligned} E_t &= 0 \qquad on\ \Gamma_2 \\ E_t &= 0 \qquad on\ \Gamma_2' \\ \int_{C_2} E_t dl &= V - V' \end{aligned} \tag{5.13}$$

the same approximation to $\mathbf{E}$ is obtained when the above mentioned approximating functions are chosen for the scalar potential and the electric field. A similar argument can be followed when the two component current density is written in terms of a stream function. The approximations obtained for the current density are the same whether the functional

$$P = d \int_{\Sigma} \frac{|\mathbf{n}' \times \chi|^2}{\sigma} dS \tag{5.14}$$

is minimized subject to the principal boundary conditions

$$\begin{aligned} \chi &= \frac{I_1}{d} \qquad on\ \Gamma_1 \\ \chi &= \frac{I_2}{d} \qquad on\ \Gamma_1' \end{aligned} \tag{5.15}$$

and ψ, the function which makes P stationary, is approximated by nth order polynomials which are continuous and piecewise differentiable or whether the functional

$$P = d \int_{\Sigma} \frac{|\mathbf{J}|^2}{\sigma} dS \tag{5.16}$$

is minimized subject to

$$J_n = 0 \qquad on\ \Gamma_1 \tag{5.17}$$

$$J_n = 0 \qquad on\ \Gamma_1' \tag{5.18}$$

$$d \int_{C_1} J_n dl = I. \tag{5.19}$$

and $\mathbf{J}$ is approximated by $n-$ 1st order irrotational vector polynomials whose normal component is continuous across interfaces.

The first order irrotational and solenoidal vector polynomials developed in Chapter 3 can be used to approximate the electric field $\mathbf{E}$ and the current density $\mathbf{J}$ and the solution which is obtained must be equivalent to a solution which would be obtained if the scalar function is approximated with second order polynomials. The vector functions are now used to find $\mathbf{J}$ and $\mathbf{E}$ in the two dimensional conducting plate by minimizing the power functional given in (5.12) or the functional given in (5.16).

Approximate values for conductance and resistance can then be computed from Eqs. (2.42) and (2.43) respectively. Equation (2.42) is an upper bound for the conductance or a lower bound on the resistance and Eq. (2.43) gives an upper bound on the resistance. An average of the two values can be used to approximate the resistance in the plate. In the next section conducting plates for which the resistance is known are presented. The current density and the electric field are then computed for these two dimensional plates and the computed values of resistance are compared to the exact value.

5.2 A Conducting Plate

The conducting plate shown in Fig. 5.1 is considered. This plate Σ is symmetric about the center line: Γ_1 is of the same shape as Γ_2 and Γ_1' is of the same shape as Γ_2'. Furthermore it is assumed that Γ_1, Γ_1', Γ_2, and Γ_2' are differentiable and that the four corners of the plate are right angles. In order to compute the conductance of that plate, the power functional

$$P = \sigma d \int_{\Sigma} |\nabla \chi|^2 \, dS \tag{5.20}$$

is minimized subject to

$$\begin{aligned} \chi &= V \qquad on\ \Gamma_2 \\ \chi &= V' \qquad on\ \Gamma_2' \end{aligned} \tag{5.21}$$

and G is computed from

$$G = \frac{P(\varphi)}{(V - V')^2} \tag{5.22}$$

where φ is the scalar function at which P is a minimum. In order to compute the resistance, the power functional

$$P = \frac{d}{\sigma} \int_{\Sigma} |\mathbf{n}' \times \nabla \chi|^2 \, dS \tag{5.23}$$

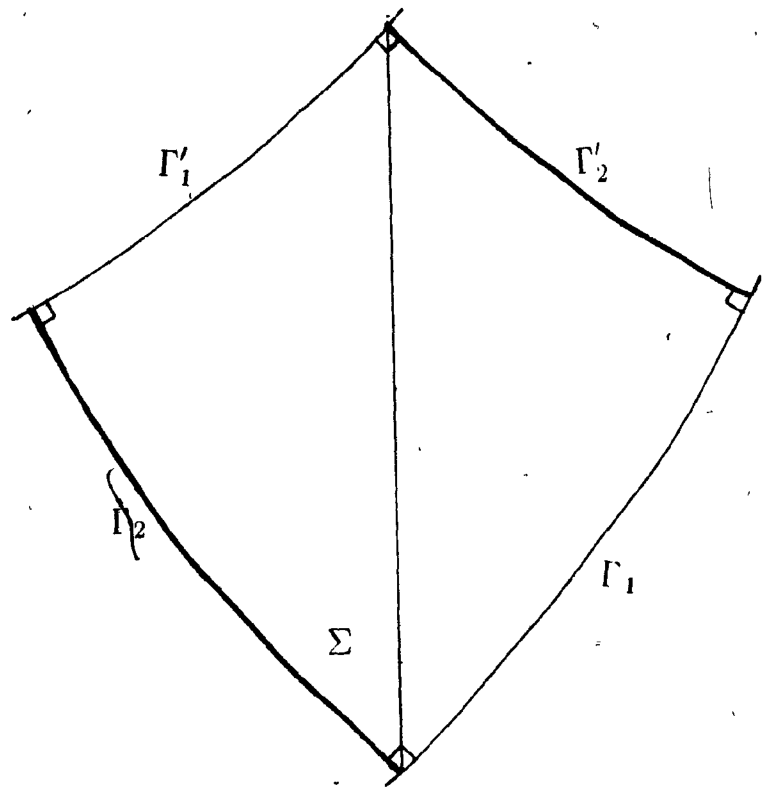

Fig. 5.1 A conducting plate Σ which is symmetric across the center line.

is minimized subject to the principal boundary conditions

$$\begin{aligned} \chi &= \frac{I_1}{d} \qquad on\ \Gamma_1 \\ \chi &= \frac{I_2}{d} \qquad on\ \Gamma_1' \end{aligned} \tag{5.24}$$

and R is computed from

$$R = \frac{P(\psi)}{(I_1 - I_2)^2} \tag{5.25}$$

where ψ is the scalar function at which P is a minimum.

It will now be shown that the functionals (5.20) and (5.23) are identical if $\sigma = 1$. The boundary conditions (5.21) and (5.24) are the same by virtue of the shape of the region Σ if V is set equal to I_1 and V' is set equal to I_2. The functional (5.23) can be rewritten as follows:

$$\begin{aligned} P &= \frac{d}{\sigma}\int_{\Sigma}(\mathbf{n}'\times\nabla\chi)\cdot(\mathbf{n}'\times\nabla\chi)dS \\ &= \frac{d}{\sigma}\int_{\Sigma}\mathbf{n}'\cdot(\nabla\chi\times(\mathbf{n}'\times\nabla\chi))dS \\ &= \frac{d}{\sigma}\int_{\Sigma}\mathbf{n}'\cdot((\nabla\chi\cdot\nabla\chi)\,\mathbf{n}'-(\nabla\chi\cdot\mathbf{n}')\,\nabla\chi)\,dS \\ &= \frac{d}{\sigma}\int_{\Sigma}\nabla\chi\cdot\nabla\chi\,dS \end{aligned} \tag{5.26}$$

since $\mathbf{n}'$ is a unit vector normal to the plate Σ. When $\sigma = 1$ the functional (5.26) is identical to the functional (5.20). Hence,

$$P(\psi) = P(\varphi). \tag{5.27}$$

If

$$(V - V') = (I_1 - I_2) = 1 \tag{5.28}$$

then

$$G = P(\varphi) = P(\psi) = R$$

or

$$\frac{1}{R} = R$$

hence

$$R = 1 \qquad or \qquad R = -1.$$

Since the resistance must be positive,

$$R = 1. \tag{5.29}$$

The conducting region shown in Fig. 5.1 therefore has a resistance and conductance of 1. In the following section the resistances of plates which possess the same symmetry† as the plate shown in Fig. 5.1 are computed. Since the exact value of the resistance of each of these plates is known, the error in the approximate value of resistance can be computed.

5.3 The Two Component Vector Approximation Functions

The exact solution for the electric field or the current density can be expressed as a linear combination of the vector polynomials constructed in Chapter 3 if the solution for **E** or **J** lies in the space of first order vector polynomials. Furthermore the approximation of the resistance of the plate is equal to the exact value of resistance since

$$R_{appJ} = P(\psi_{app}) = P(\psi) = R.$$

The resistance of the square plate shown in Fig. 5.2 is computed first. The conductivity σ and the thickness d of the plate are assumed to be 1. The current density over the plate is found by minimizing

$$P = \int_{\Sigma} |\mathbf{J}|^2 \, dS \tag{5.30}$$

subject to

$$\begin{aligned} J_n &= 0 \qquad on\ \Gamma_1 \\ J_n &= 0 \qquad on\ \Gamma_1' \\ \int_{\Gamma_2} J_n dl &= 1. \end{aligned} \tag{5.31}$$

† The reader should refer to Duffin (1959) for more on conjugate conductors.

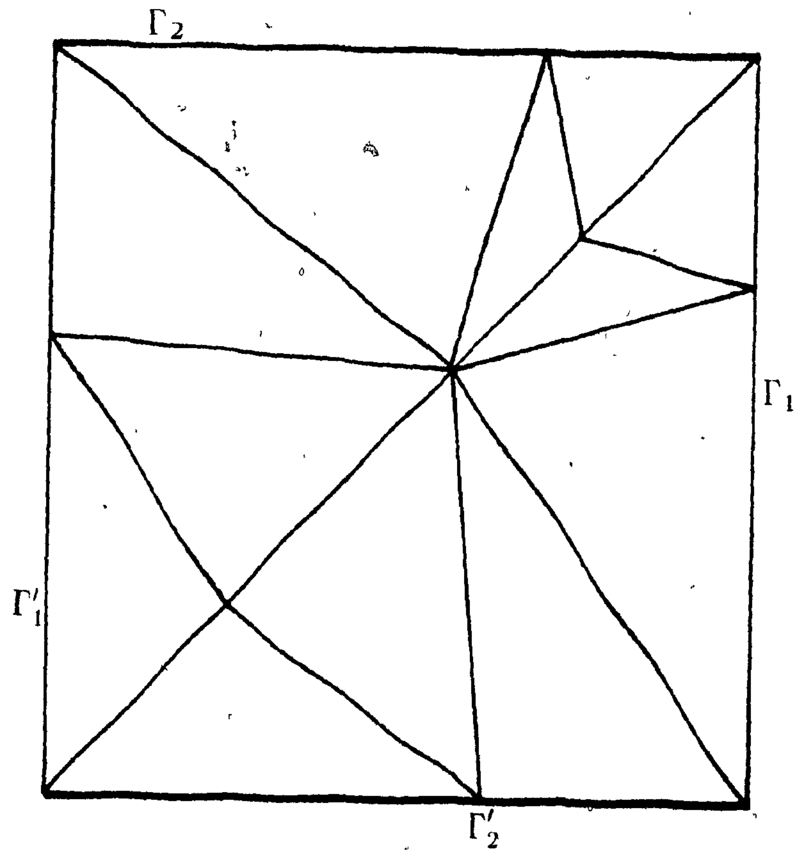

Fig. 5.2 A square conducting plate.

The square plate is triangulated as shown in Fig. 5.2. The current density is approximated over each element by a linear combination of first order solenoidal vector polynomials and the continuity of the component of the approximation to **J** normal to interelement edges is ensured. Since the solution for the current density over the square plate is a constant field, the current density can be expressed as a linear combination of first order vector polynomials. Therefore the value of resistance computed from

$$R_{appJ_0} = P(\mathbf{J}^*{}_{app}) \tag{5.32}$$

should be equal to 1. The current density is computed using the method described in Chapter 4 and the resulting resistance is found to be 1 even when the plate is triangulated as shown in Fig. 5.3.

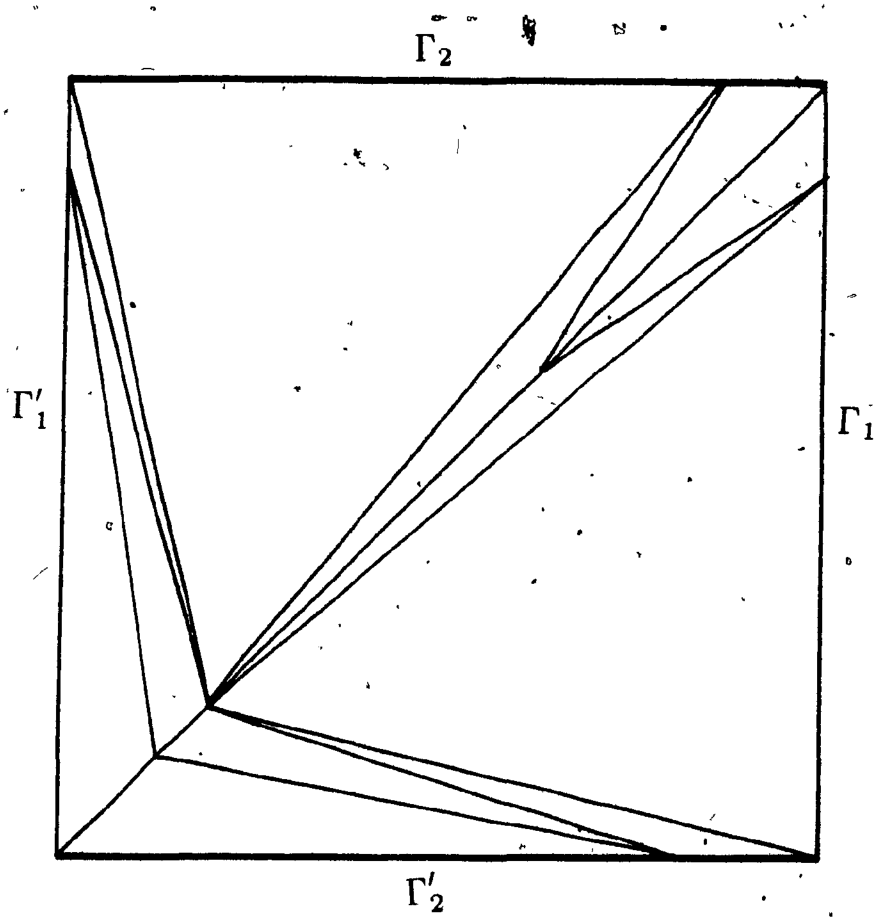

Fig. 5.3 An alternate triangulation of the square plate.

The vector approximation functions are now used to approximate the current density over the conducting plate shown in Fig. 5.4. Although the current density is not known a priori, the resistance of the plate is known to be equal to 1 when $\sigma = 1$ because the plate has the symmetry described in Section 5.2. The boundary segments Γ_1, Γ'_1, Γ_2, and Γ'_2 are approximated by continuous piecewise differentiable curves. The functional given in (5.30) is minimized subject to the principal conditions (5.31) for the conducting plate Σ shown in Fig. 5.4. The current density is approximated by a solenoidal vector polynomial of order 1 whose normal component is continuous across the interelement edges even though the exact solution is not expected to lie in that space. The resulting approximation to the current density is shown in Fig. 5.5. The

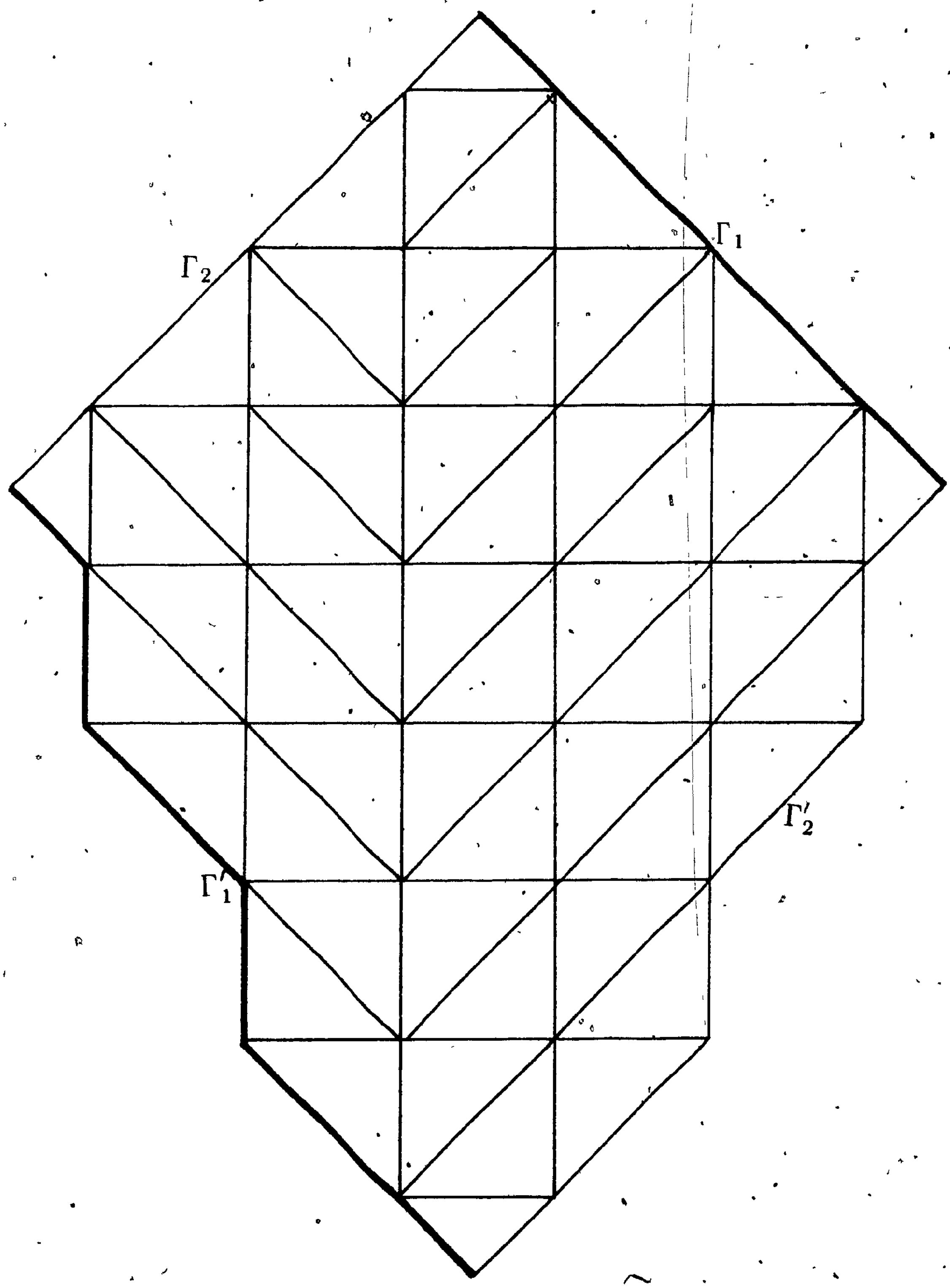

Fig. 5.4. A symmetric region which is uniformly triangulated.

resistance is computed using Eq. (5.32) and is found to be 1.002726.

The conductance in the plate is also computed. It was shown in Section 5.2 that the conductance of the plate should be equal to the resistance. In order to compute conductance, the functional

$$P = \int_{\Sigma} |\mathbf{E}|^2 \, dS \tag{5.33}$$

is minimized subject to

$$\begin{aligned} E_t &= 0 \qquad on\ \Gamma_2 \\ E_t &= 0 \qquad on\ \Gamma_2' \\ \int_{\Gamma_1} E_t dl &= 1. \end{aligned} \tag{5.34}$$

It can be seen from the (5.31) and (5.34) that the conditions on J_n are identical to those on E_t by virtue of the shape of the region, that is because Γ_1 has the same shape as Γ_2 and Γ_1' has the same shape as Γ_2'. Furthermore, as was mentioned in Section 4.1.2, the disconnected T matrices obtained when $\mathbf{E}$ is approximated by the solenoidal vector polynomials and when $\mathbf{J}$ is approximated by the irrotational vector polynomials over the same mesh are identical. Therefore it is expected that the approximation to $\mathbf{E}^*$ should be equal to the approximation to $\mathbf{J}^*$ and that the computed value for the conductance

$$G_{appE} = P(\mathbf{E}^*{}_{app}) \tag{5.35}$$

should be equal to the R_{appJ} computed by (5.32). The computation of the conductance of the plate is performed because the triangulation of Σ is not symmetric about the center line and it was thought that the asymmetry might cause the value of the conductance to differ from the value of the resistance. The computed conductance turns out to be 1.002725. The difference between the computed value of resistance and the computed value of conductance occurs only in the seventh significant digit.

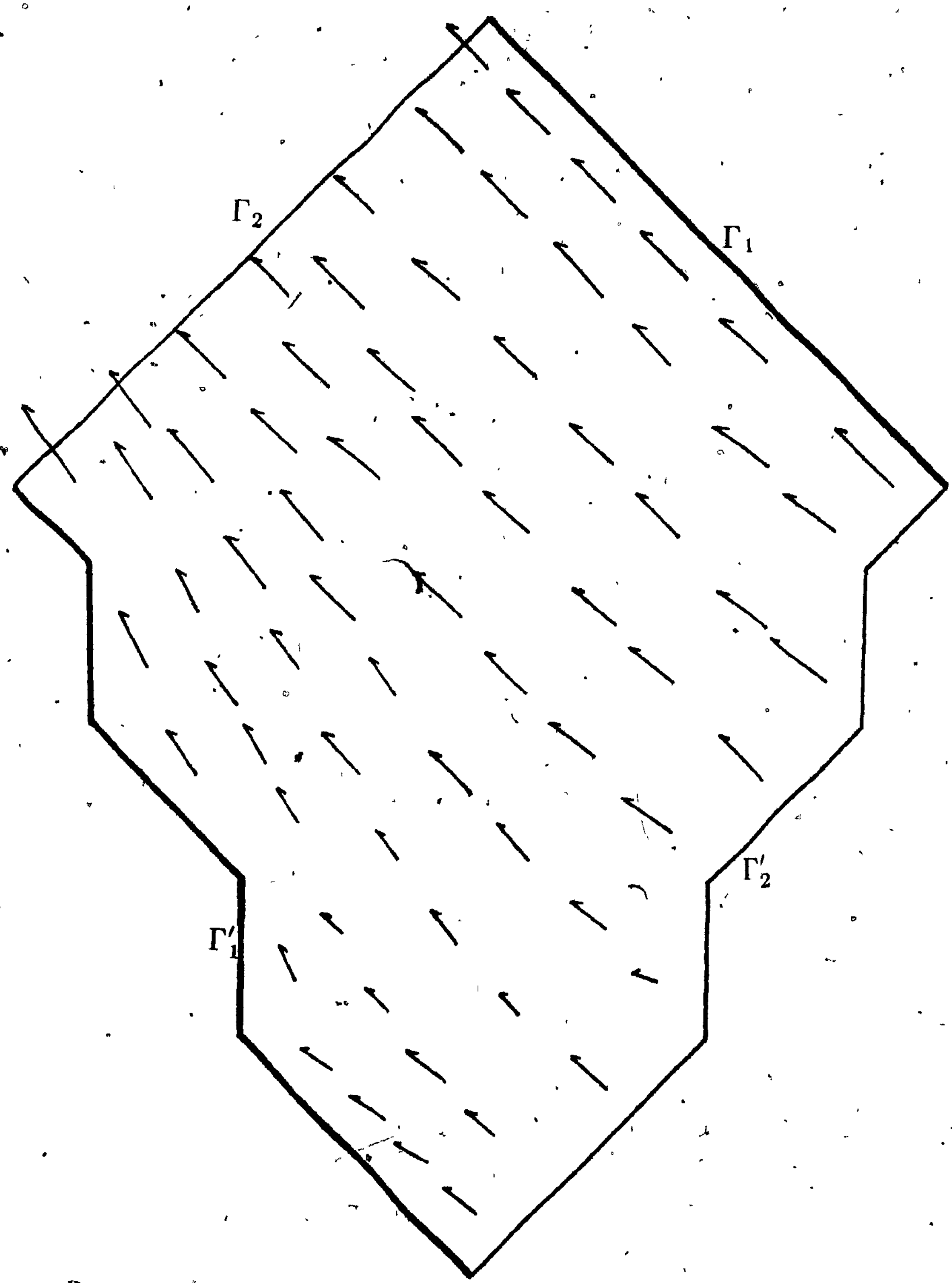

Fig. 5.5 The current density over the conducting plate shown in Fig. 5.4.

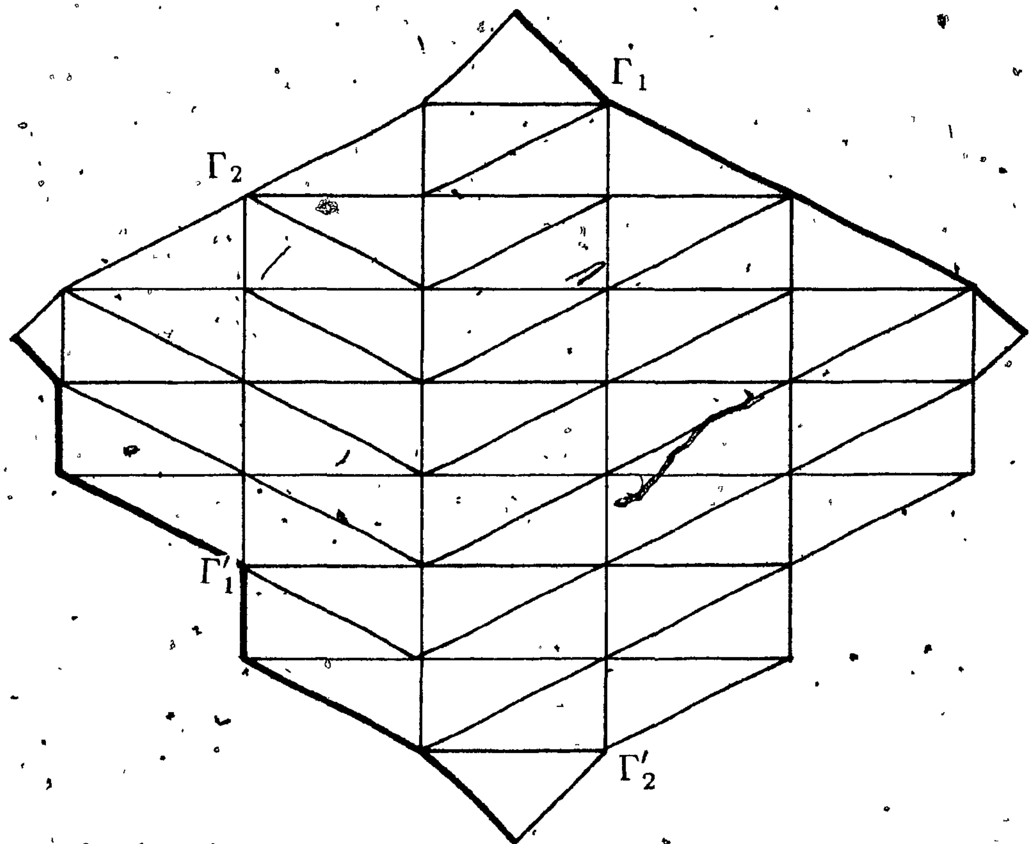

Fig. 5.6 Another conducting plate whose resistance is 1.

It is reasonable to expect that a more accurate solution for the vector fields and ultimately for the resistance and the conductance would be computed if the triangulation of the plate shown in Fig. 5.4 were refined. Such results have been shown by Ciarlet (1972,1978) and Strang (1971). It is not however clear how the approximation functions behave on triangles of different shapes. It was shown in Chapter 3 that the Gram determinant of the vector polynomials is equal to zero only when the area of the triangle is zero. The resistance of conducting plates of the type described in Section 5.2 is now computed. These conducting plates are triangulated uniformly, each with triangles of different shapes. The plates shown in Figs. 5.4 and 5.6 are examples of two of these plates. The effect of the triangulation on the error in the resistance of the plate is shown in Fig. 5.7(a). The error in the resistance is graphed versus the angle of the smallest angle in the triangle. It can be seen from the graph that the error in the resistance begins to increase rapidly once the smallest angle in the triangle is less than 14°. It is interesting to note that the error increases linearly as a function of the cotangent of the smallest angle in the triangle for angles of less than 26°. This graph is shown in Fig. 5.7(b). The conductance was also computed for all the conducting regions. The values for the resistance differed from those for the conductance only in the sixth or seventh significant digit.

5.4 The Three Component Vector Approximation Functions

The resistance of three three dimensional wires whose conductivity σ is equal to 1 is computed. The current density is computed by minimizing the functional

$$P = \int_{\Omega} |\mathbf{J}|^2 \, dV \tag{5.36}$$

subject to the conditions

$$\begin{aligned} \mathbf{J} \cdot \mathbf{n} &= 0 \quad \textit{on } \Sigma_1 \\ \int_{S_1} \mathbf{J} \cdot \mathbf{n} \, dS &= 1. \end{aligned} \tag{5.37}$$

The resistance in the wire is computed at the value $\mathbf{J}^*$ which minimizes the functional in (5.36). The current density is approximated by the three component solenoidal vector functions developed in Chapter 3 and the approximation to the resistance in the wire therefore is

$$R_{appJ} = P(\mathbf{J}^*_{app}). \tag{5.38}$$

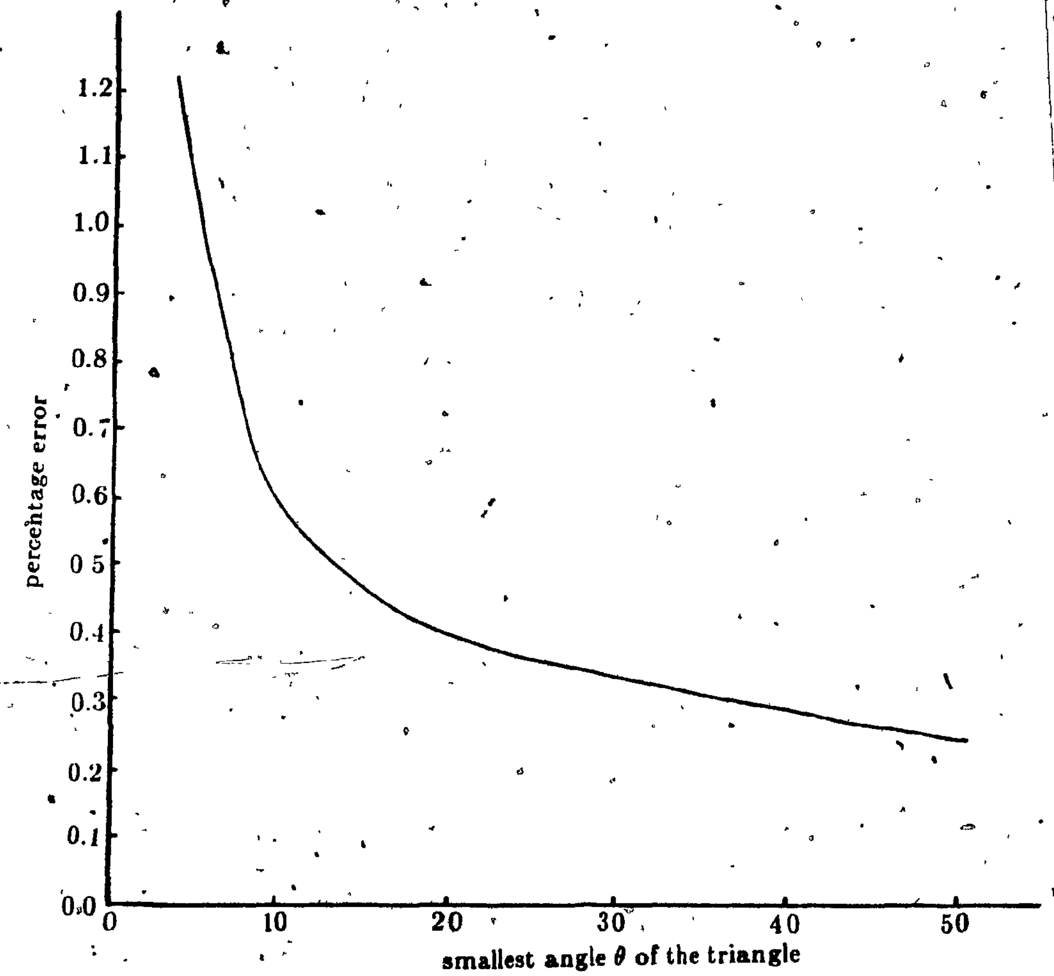

Fig. 5.7(a) The error in the resistance of the conducting plate as a function of the smallest angle of the triangles in the mesh.

If the exact solution for the current density $\mathbf{J}^{*}$ lies in the space of first order solenoidal vector functions, then the approximate value of resistance is equal to the exact value.

The resistance of a cube is determined first. The length of the side of the cube is equal to 1. The resistance of the cube can be determined analytically since

$$R = \frac{l}{\sigma A} \tag{5.39}$$

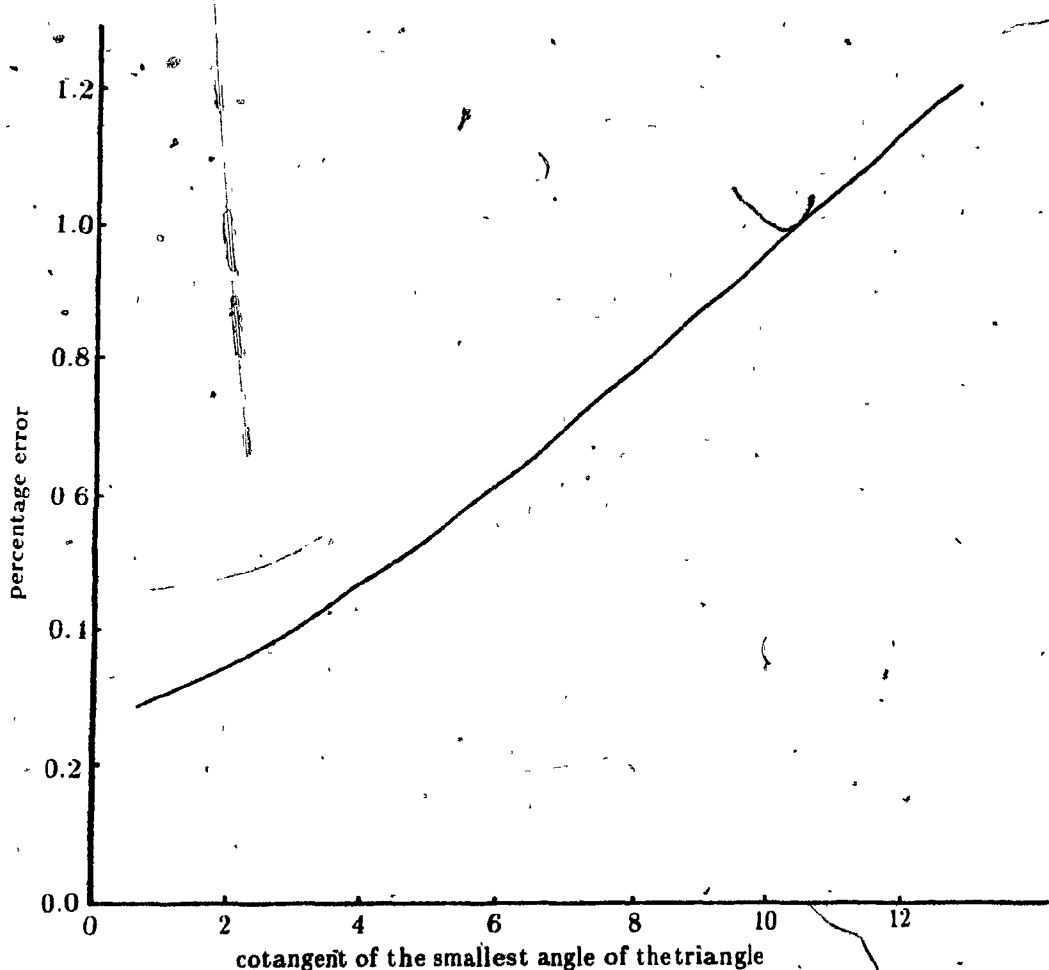

Fig. 5.7(b) The error in the resistance of the conducting plate as a function of the cotangent of the smallest angle of the triangles in the mesh.

where l is the length of the cube and A is the cross-sectional area. Since $\sigma = 1$, the resistance of the cube should be equal to 1. The solution for the current density in the wire is a constant field and can therefore be expressed as a linear combination of the first order solenoidal vector polynomials. The computed value for the resistance should therefore be equal to 1. The current density was computed using the method described in Chapter 4 for a finite element mesh of twenty-four tetrahedra and the resistance was

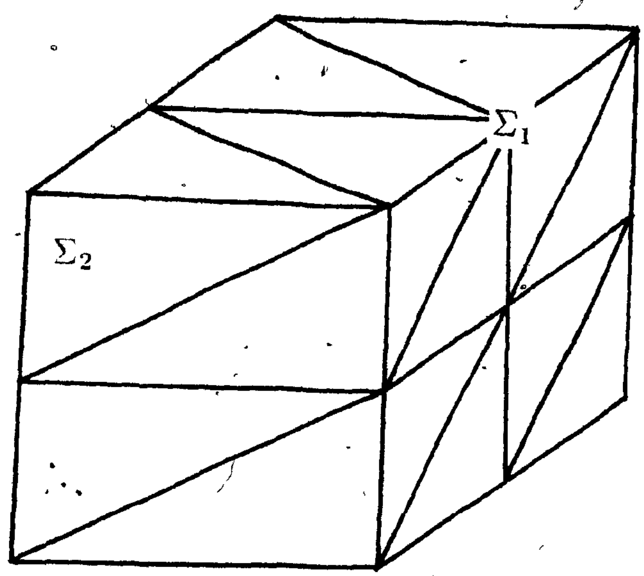

Fig. 5.8 A three dimensional wire in the shape of a cube.

found to be 0.999999.

The resistance is then calculated for the wire shown in Fig. 5.9. For the second wire the current density can also be expressed as a linear combination of the first order solenoidal vector polynomials. The approximate value of resistance should therefore be equal to the exact value of the resistance of the wire. The exact value of the resistance in the wire can be calculated from Eq. (5.39). For the geometry shown in Fig. 5.9 the resistance is 0.2666667 and the resistance which is found using Eq. (5.38) is 0.2666668.

The resistance of the third wire shown in Fig. 5.10 is computed next. The first half of the wire shown in Fig. 5.10 is identical to the first half of the wire shown in 5.9 but the second half is bent upwards. Although (5.39) cannot be used any longer to compute the resistance of that wire, it is clear that the resistance should be greater

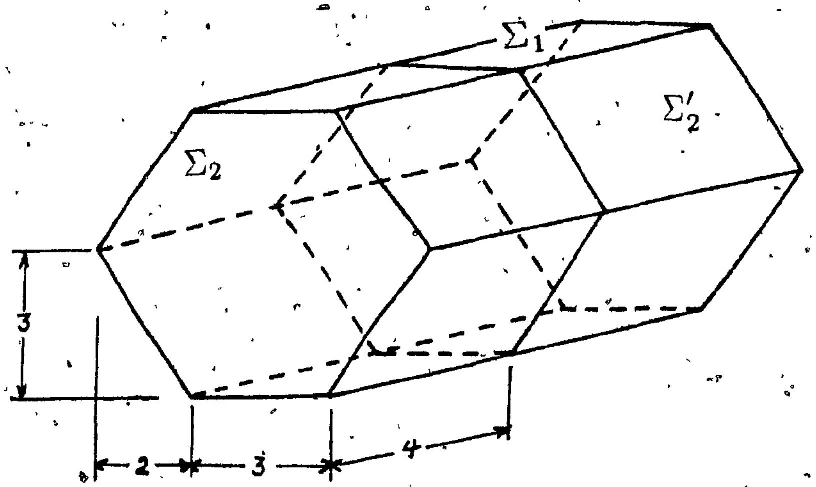

Fig. 5.9 A second three dimensional wire.

than the resistance of the wire shown in Fig. 5.9 because the length of the wire has increased and the cross-sectional area has decreased by virtue of the bend. The approximate value of resistance was found to be 0.286018 for the triangulation shown in Fig. 5.10.

The conductance of the third wire was also computed. The electric field in the region was determined by minimizing the functional in (5.1) subject to the principal boundary conditions

$$\chi = 1 \quad on\ \Sigma_2$$

$$\chi = 0 \quad on\ \Sigma_2'$$

and then using Eq. (5.3). The conductance in the wire is computed from

$$G = P(\varphi).$$

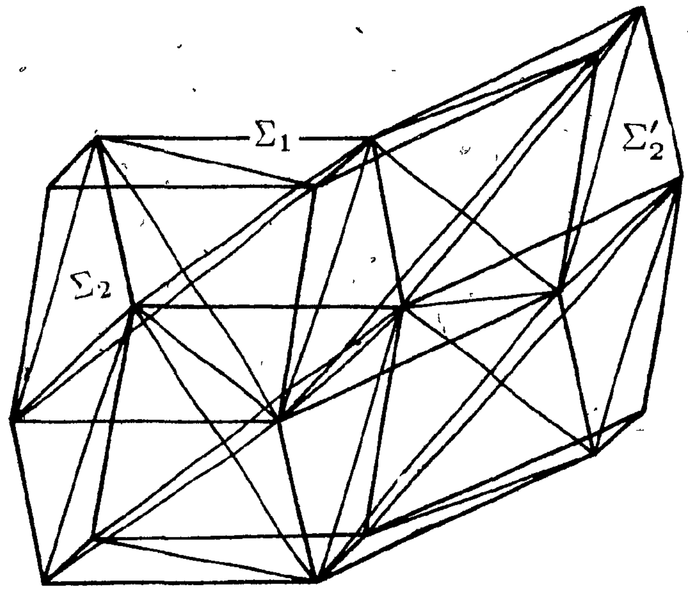

Fig. 5.10 A third three dimensional wire.

The scalar potential φ was approximated by a continuous function which is a second order polynomial in each tetrahedron. The approximation to the conductance in the wire was found to be 3.765481. Recalling Eq. (2.37),

$$R_{appE} \leq R \leq R_{appJ},$$

the value of the resistance in the wire can be estimated to be the average of R_{appE} and R_{appJ}. Since

$$R_{appE} = 0.265570$$

and

$$R_{appJ} = 0.286018,$$

the resistance in the wire is estimated to be

$$R_{app} = \frac{R_{appJ} + R_{appE}}{2} = 0.275794.$$

The percentage error in the approximation of the resistance is

$$\frac{|R_{appE} - R_{app}|}{R_{app}} \times 100 = 3.6\%$$

for a finite element mesh of twenty-four tetrahedra. It is reasonable to expect that the approximation to the resistance will improve if a finer mesh is used.

5.5 Conclusions

In this thesis solenoidal and irrotational two component first order vector polynomials defined over a triangle and three component first order vector polynomials defined over a tetrahedron were constructed. These are intended to be used when an approximation for either a solenoidal vector field or an irrotational vector field is desired. They were used in this thesis to approximate the static electric field and the current density in two dimensional conducting plates and in three dimensional wires. To the best of the author's knowledge, these vector polynomials do not appear in the literature although the zeroth order analogues were constructed by Synge and McMahon in 1952 and 1953.

Of the four families, the three component solenoidal vector polynomials are the most interesting since they offer a new method for approximating three component solenoidal vector fields. The solenoidal field need not be expressed as the curl of a vector potential and can be approximated directly. Because the solution for the field is unique, problems encountered due to the nonuniqueness of the vector potential can be avoided. In particular, the boundary conditions in terms of the field need not be translated into boundary conditions in terms of the potential and the three components of the vector field can be computed without first computing the three components of the vector potential.

When a solenoidal vector field is approximated, the coefficients in the Ritz minimization are taken to be the components of the vector field normal to the element edges or faces. Such a choice of coefficients facilitates the imposition of boundary conditions since for solenoidal vector fields the boundary conditions are stated in terms of the components of the field normal to the boundary. Similarly when an irrotational vector field is approximated the coefficients in the Ritz minimization are taken to be the components of the vector field tangent to the element edges or faces because the boundary conditions imposed on an irrotational field are stated in terms of the component of the field tangent to the boundary.

Although it was shown in Chapter 3 that the Gram determinant for the two component vector polynomials vanishes only when the area of the triangle is zero, no similar result could be proved in the case of the three component vector functions since there are eleven three component solenoidal vector polynomials and no way could be found in which to simplify the Gram determinant in order to show that it would only vanish when the volume of the tetrahedron is zero It could easily be shown that for specific tetrahedra, the Gram determinant was nonzero. However, trying to reduce the Gram determinant is not the method which should be followed in order to show that the eleven three component solenoidal vector polynomials are linearly dependent only when the volume of the tetrahedron is zero. It must be shown that if a basis for the space of nth order solenoidal vector polynomials is defined over a tetrahedron, then any affine transformation which takes that tetrahedron into another nondegenerate tetrahedron will induce a nonsingular transformation on the basis.

No attempt was made in this thesis to find vector polynomials of higher order which satisfied the constraints outlined in Chapters 2 and 3. The method used in Chapter 3 to construct the first order polynomials would be very tedious to undertake for higher order polynomials. A more general way to determine the higher order families should therefore be developed.

Finally, the vector polynomials were used to approximate the solenoidal current density and the irrotational electric field. A word of caution should be given at this point. The solenoidal vector polynomials can be used to approximate a solenoidal electromagnetic vector field but they should not be used to approximate the vector potential even when the divergence of the vector potential is zero. The solenoidal vector polynomials constructed in this thesis only ensure continuity in the normal component of the field. The vector potential however needs to have normal and tangential component continuity. First, if the divergence of the vector potential is set to zero, the vector potential must be continuous in the normal direction since it is a solenoidal field. Second, the curl of the vector potential is a solenoidal field whose normal component must be continuous which means that the tangential components of the vector potential must also be continuous across interfaces.

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APPENDIX I

For the vector polynomials $\mathbf{u}_1, \ldots, \mathbf{u}_5$ given in (3.16), the Gram matrix is

$$G'' = \frac{\triangle}{12}\begin{pmatrix} 6 & -5\cos\theta_3 & -5\cos\theta_2 & 2 & -3\cos\theta_3 \\ -5\cos\theta_3 & 6 & -5\cos\theta_1 & -3\cos\theta_3 & 2 \\ -5\cos\theta_2 & -5\cos\theta_1 & 6 & -3\cos\theta_2 & -3\cos\theta_1 \\ 2 & -3\cos\theta_3 & -3\cos\theta_2 & 2 & -\cos\theta_3 \\ -3\cos\theta_3 & 2 & -3\cos\theta_1 & -\cos\theta_3 & 2 \end{pmatrix}.$$

The rows and columns of G'' will be denoted as $R_1, R_2 \ldots R_5, C_1, C_2, \ldots C_5$. If $-R_4$ is added to R_1 and $-R_5$ is added to R_2,

$$\det G'' = \frac{\triangle}{12}\begin{vmatrix} 4 & -2\cos\theta_3 & -2\cos\theta_2 & 0 & -2\cos\theta_3 \\ -2\cos\theta_3 & 4 & -2\cos\theta_1 & -2\cos\theta_3 & 0 \\ -5\cos\theta_2 & -5\cos\theta_1 & 6 & -3\cos\theta_2 & -3\cos\theta_1 \\ 2 & -3\cos\theta_3 & -3\cos\theta_2 & 2 & -\cos\theta_3 \\ -3\cos\theta_3 & 2 & -3\cos\theta_1 & -\cos\theta_3 & 2 \end{vmatrix}.$$

Adding R_1 to $-R_4$ and R_2 to $-R_5$ followed by adding C_4 to C_1 and C_5 to C_2 results in

$$\det G'' = \frac{\triangle}{12}\begin{vmatrix} 4 & -4\cos\theta_3 & -2\cos\theta_2 & 0 & -2\cos\theta_3 \\ -4\cos\theta_3 & 4 & -2\cos\theta_1 & -2\cos\theta_3 & 0 \\ -8\cos\theta_2 & -8\cos\theta_1 & 6 & -3\cos\theta_2 & -3\cos\theta_1 \\ 0 & 0 & \cos\theta_2 & -2 & -\cos\theta_3 \\ 0 & 0 & \cos\theta_1 & -\cos\theta_3 & -2 \end{vmatrix}.$$

The determinant of the Gram matrix can then be expressed as

$$\det G'' = \frac{\triangle}{12}\cos\theta_1 \begin{vmatrix} 4 & -4\cos\theta_3 & 0 & -2\cos\theta_3 \\ -4\cos\theta_3 & 4 & -2\cos\theta_3 & 0 \\ -8\cos\theta_2 & -8\cos\theta_1 & -3\cos\theta_2 & -3\cos\theta_1 \\ 0 & 0 & -2 & -\cos\theta_3 \end{vmatrix}$$

$$+\frac{\triangle}{12}\cos\theta_3 \begin{vmatrix} 4 & -4\cos\theta_3 & -2\cos\theta_2 & -2\cos\theta_3 \\ -4\cos\theta_3 & 4 & -2\cos\theta_1 & 0 \\ -8\cos\theta_2 & -8\cos\theta_1 & 6 & -3\cos\theta_1 \\ 0 & 0 & \cos\theta_2 & -\cos\theta_3 \end{vmatrix}$$

$$+\frac{-2\triangle}{12}\begin{vmatrix} 4 & -4\cos\theta_3 & -2\cos\theta_2 & 0 \\ -4\cos\theta_3 & 4 & -2\cos\theta_1 & -2\cos\theta_3 \\ -8\cos\theta_2 & -8\cos\theta_1 & 6 & -3\cos\theta_2 \\ 0 & 0 & \cos\theta_2 & -2 \end{vmatrix}$$

and reduced to

$$\det G'' = \frac{\triangle}{12}\left((2\cos\theta_1 - \cos\theta_3\cos\theta_2)\begin{vmatrix} 4 & -4\cos\theta_3 & -2\cos\theta_3 \\ -4\cos\theta_3 & 4 & 0 \\ -8\cos\theta_2 & -8\cos\theta_1 & -3\cos\theta_1 \end{vmatrix}\right.$$

$$+ (2\cos\theta_2 - \cos\theta_1\cos\theta_3)\begin{vmatrix} 4 & -4\cos\theta_3 & 0 \\ -4\cos\theta_3 & 4 & -2\cos\theta_3 \\ -8\cos\theta_2 & -8\cos\theta_1 & -3\cos\theta_2 \end{vmatrix}$$

$$\left. + (4 - \cos^2\theta_3)\begin{vmatrix} 4 & -4\cos\theta_3 & -2\cos\theta_2 \\ -4\cos\theta_3 & 4 & -2\cos\theta_1 \\ -8\cos\theta_2 & -8\cos\theta_1 & 6 \end{vmatrix}\right).$$

Expanding the above determinants, the determinant of G'' reduces to

$$\det G'' = \frac{8\triangle}{3}\left(1 - \cos^2\theta_3\right)\left(1 + \cos\theta_1\cos\theta_2\cos\theta_3\right).$$