An Introduction to Dendritic Growth with Convection

by

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ABSTRACT

In this thesis I intend to summarize several theories dealing with dendritic growth with convection. I have also looked into a special case where the convection motion is induced by the density change in phase transition. In terms of a small parameter α, measuring the relating density change, the second order approximate solution is obtained by using regular perturbation method.

RESUME

Dans la thèse J'ai l'intention de résumer plusieurs théories sur la croissance de cristal avec la convection. Je me suit renseigné aussi sur un cas spécial où la convection est provoqué par le changement de densité dans la transition de phase. Sur le plan de un petit paramètre α, mesurant le changement relatif de densité, la deuxième-ordre solution approximatif est obtenu par utilisant la méthode de perturbation régulier.

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Chapter I

INTRODUCTION: A HISTORICAL REVIEW ON STUDY OF DENDRITIC GROWTH

Dendritic solidification is an important subject in the field of condensed matter physics. In a variety of solidification systems, solid-liquid interface becomes morphologically unstable as it grows. At the later stage of growth, experiments show that the solidification front evolves into dendrites, which have a smooth tip moving with a constant velocity, and also a growing side-branching, tree-like micro-structure. There are two central topics involved in the pattern formation of dendrite growth: (1) How to predict the growth velocity of dendrite tip; (2) How to explain the formation of the side-branching structure. Although these two topics have been studied by many investigators for more than forty years, a new reasonable approach to these problems was recently proposed, and needs to be further verified (Ref. [22] - [28]). In the following I intend to give a brief review on this progress.

For a long period of time, most investigators thought that the dendrite tip region was steady and monotonic so that it could be described by a steady state of the system. Hence, many of them have been preoccupied by the study of steady dendrite growth problem.

The first contribution to this subject was made by Ivantsov in 1947. Ivantsov considered steady dendrite growth problem with zero surface tension on the interface between the solid phase and liquid phase; and obtained an exact similarity soluti. The Ivantsov's solution shows that the dendrite is isothermal, its interface shape is paraboloidal. The Ivantsov solution, however, does not resolve the problem of dendrite growth, as it can not determine the tip radius and tip velocity separately nor explain the formation of the side-branching structure.

To select the tip velocity for a realistic dendrite growth, Nash and Glicksman studied the steady dendrite growth with the inclusion of the surface tension. They found that for any given undercooling, the tip velocity in steady dendrite growth has a maximum value. Thus they proposed that the realistic steady state of dendrite growth always selects this maximum value. They call this selection condition *the Maximum Velocity Principle* (MVP). The maximum velocity principle was soon rejected by experiments by Glicksman et al.

In 1978, based on a rudimentary linear stability analysis, Langer and Muller-Krumbhaar proposed *the Marginal Stability Hypothesis* (MSH) as the selection mechanism for the tip velocity. This hypothesis says that the natural operating point of steady dendrite growth occurs when the dendrite tip is just marginally stable and the radius of curvature at the tip is equal to the wave length of marginal stability. MSH agrees with experimental data. However, its theoretical base is not strong. Particularly, in Langer's analysis, the wave length of marginal stability at the tip adopted is taken from the planar interface case.

Due to the weakness of MSH, Langer eventually abandoned his MSH idea and with other authors, proposed a completely different theory, *the Microscopic Solvability Condition Theory* (MSC) in 1980s. MSC theory attempts to predict the tip velocity. The MSC theory states that the existence of steady state of dendrite growth is determined by the anisotropy of surface tension. Without the anisotropy, according to the MSC theory, the system has no dendritic growth type of solution.

In 1989-1990, Xu published a series of papers, concerning the selection of tip velocity and the origin of pattern formation in 3-D dendritic growth with the inclusion of surface tension. Xu's results show that (1) dendrite growth phenomenon is essentially a wave phenomenon. The solution for a realistic dendrite growth is not stable steady state, but a time-periodic global neutrally stable state near the Ivantsov solution. (2) anisotropy of surface tension is not a necessary condition for dendrite growth. A selection condition of tip velocity can be found even in the isotropic surface tension case. The condition is called *the Global Neutral Stability Condition.* (3) dendrite growth is governed by an entirely new global stability mechanism, so-called *Global Trapped Wave* (GTW) mechanism. The GTW mechanism determines a discrete set of unstable GTW modes, which connects to the dynamics of pattern formation; and the system allows an unique global neutral stable GTW mode. Xu's theory is now known as *the Interfacial Wave Theory of Solidification* (IWT), which is in good agreement with the experimental observations (See [28]).

Reviewing the above progress of the investigations of dendrite growth, we can see that these studies are focused on the thermodynamic aspect of the problem. No convection in melt is involved in the system under consideration. However, experimental observations have shown that convective motion in melt may have a profound effect on all aspects of dendritic growth. It may change the tip velocity, as well as the micro-structure of dendrite.

Convective motion in melt can be induced by a variety of sources. But the major types of convective motions can be classified as follows:

1. convection induced by the density change during phase transition;
2. convection induced by the buoyancy effect due to the presence of a body force field;
3. convection induced by external flow;

4. convection induced by other sources.

Taking into account of convection, the system becomes even more complicated and difficult. In this case the hydrodynamics must be introduced into the system. The interaction between convection and solidification becomes the major concern.

For the problem of dendrite growth with no convection, according to IWT, we have seen that the small isotropic surface tension is a determinant parameter for dendritic growth. If one uses the solution for the system with zero surface tension as a basic state, then a solution for dendrite growth with surface tension contains three parts: (1) the basic state, (2) steady regular perturbation due to the surface tension, and (3) unsteady singular perturbation also due to the surface tension.

In order to get the whole picture on the problem of dendrite growth with the inclusion of convection, it is evident that one must also solve the above three components for the solution.

Due to the difficulty of problem, these tasks are far from completion. The study on this extremely important subject has just started. The major efforts are presently only concentrated on finding the basic state solution and the regular perturbation solution. To my knowledge, only Xu considered the singular perturbation solution for the case that convection is induced by the density change (Ref. [27]). It is seen that a good progress has been made along this line; an increasing number of researchers are involved.

In this thesis, I attempt to summarize the recent results obtained for this problem. The present thesis is arranged as follows:

In chapter II, I summarize some experimental evidences of the significance of the convection effect.

In chapter III, I derive mathematical formulation for this problem.

In chapter IV, I consider dendrite growth with the convection induced by the density change and the surface tension. The second-order regular perturbation expansion solution for the problem is obtained.

In chapter V, I summarize the work by Ananth and Gill on the effect of convection induced by external flow.

In chapter VI, I summarize the work by Canright and Davis on the effect of convection induced by buoyance.

Chapter II

EXPERIMENTAL OBSERVATIONS OF CONVECTION EFFECT ON DENDRITIC GROWTH

The growth of shape preserving dendrites in a subcooled melt has been observed experimentally in detail for succinonitrile by Huang & Glicksman and for ice by Tirmizi & Gill. A number of researchers has applied the results of their experimental data to examine the theoretical results. This chapter gives a brief summary of their experiments about convection effect on dendritic growth.

2.1 Tip Growth

Fig. 2.1 is the configuration of a dendrite tip. In this picture dendrites of succinonitrile are bodies of revolution only in the neighborhood of the tip; black dots are the common parabolic curve; the dendrite is growing parallel to gravity, i.e. $\phi = 0^\circ$, where ϕ denotes the angle of gravity vector and tip growth vector. This picture shows that the configuration of the dendrite tip fits to the parabolic curve.

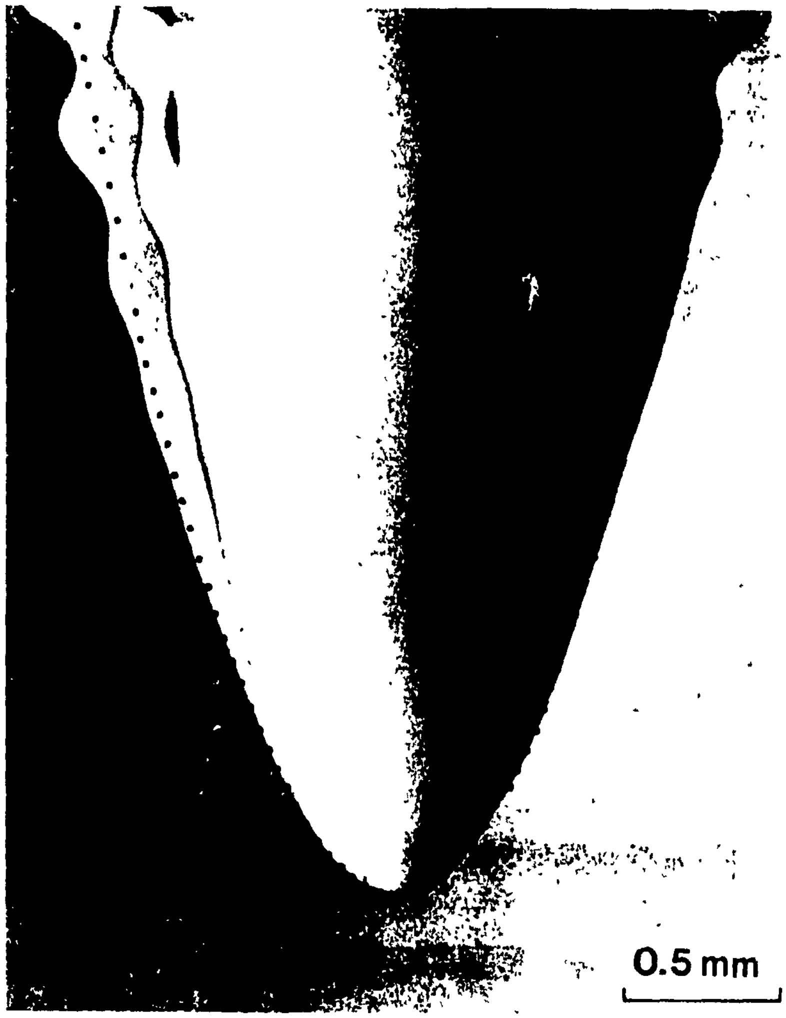

Figure 2.1: Configuration of a succinonitrile dendrite tip

2.2 Dendrite Shape

An overall view of a succinonitrile dendrite is shown by Fig. 2.2. The laboratory observations suggest that the slight anisotropy in solid-liquid interface energy plays an important role in the branching mechanism. Anisotropy in solid-liquid interface energy provides an additional source for interfacial distortion. But the anisotropy has relatively little effect in changing the growth state (V,R) of the dendrite tip, where V is the dendritic growth velocity, R is the

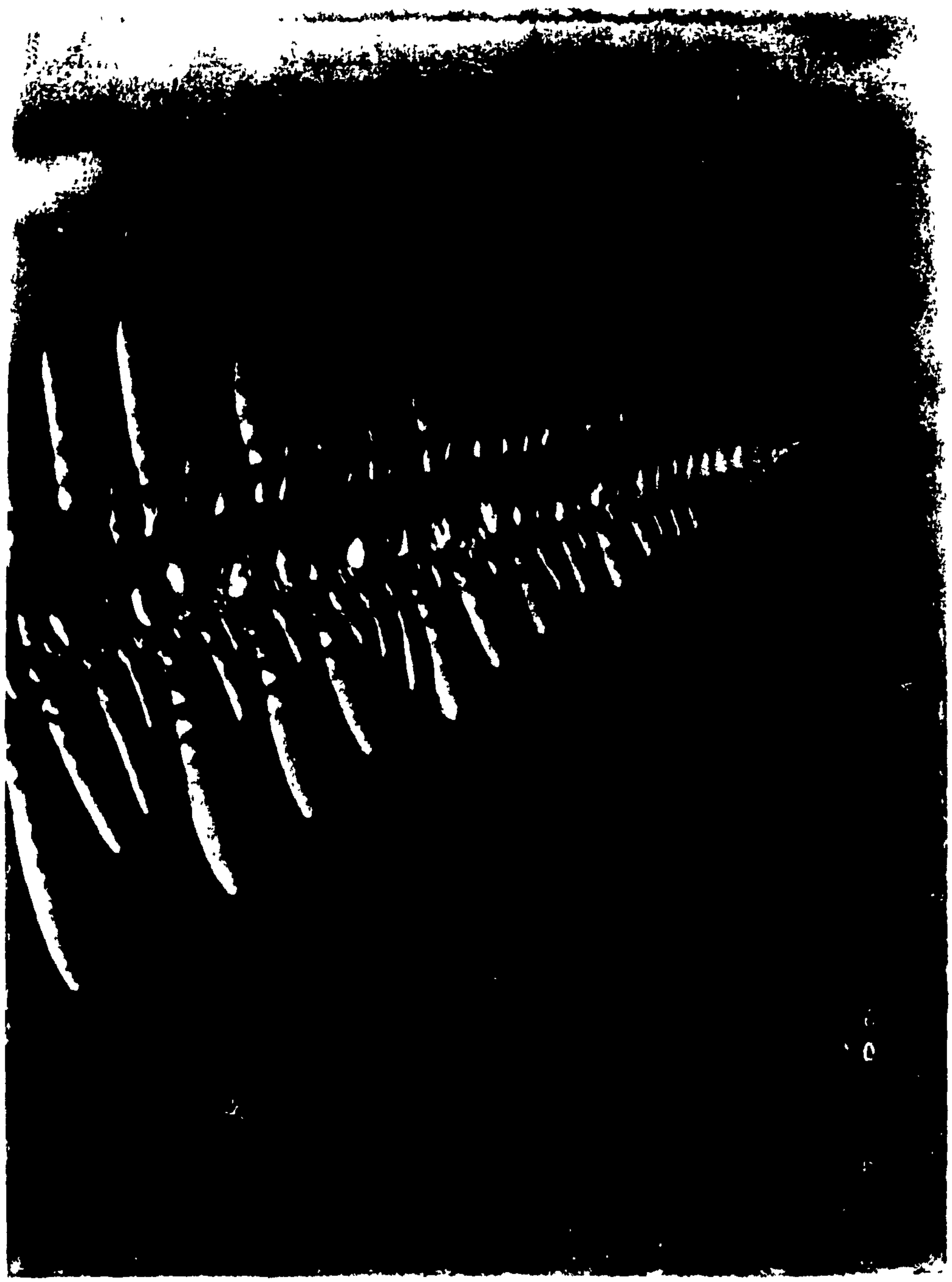

Figure 2.2: An overall view of a succinonitrile dendrite

dendritic tip radius. Moreover, anisotropy in growth kinetics may have a similar effect on the branching perturbation sequence as does the anisotropy in the solid-liquid interfacial energy (See [10]).

2.3 Effect of Natural Convection on Growth Velocities

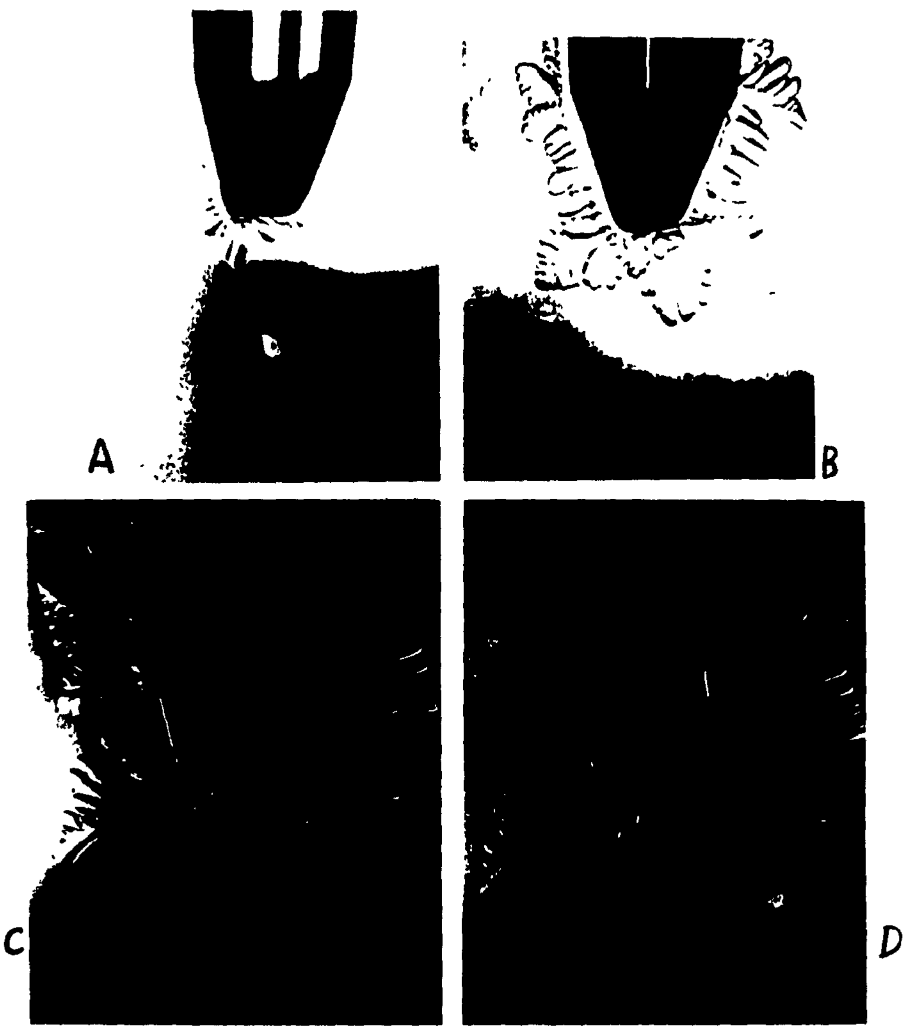

Figure 2.3: Effect of natural convection on growth velocity

The influence of natural convection on crystals which grow vertically downward is shown in the series of photographs in Fig. 2.3.[1] The crystal is ice. In

1 Tirmizi, S.H. & Gill, W.N., J. Crystal Growth, Vol. 85 (1987)

Fig. 2.3(A) t=0 min; 2.3(B) after 10 min; 2.3(C) after 12 min; 2.3(D) after 14 min. Initially, of the emerging crystals, the crystals growing vertically downwards ($\phi=0^\circ$) is bigger than its neighbors growing at angles which is not equal to 0°. As the dendritic growth process proceeds, the downward growing crystal grows progressively slower compared to the other, and finally in Fig. 2.3(D), it is much smaller than its neighboring dendrites which obviously are less influenced by natural convection.

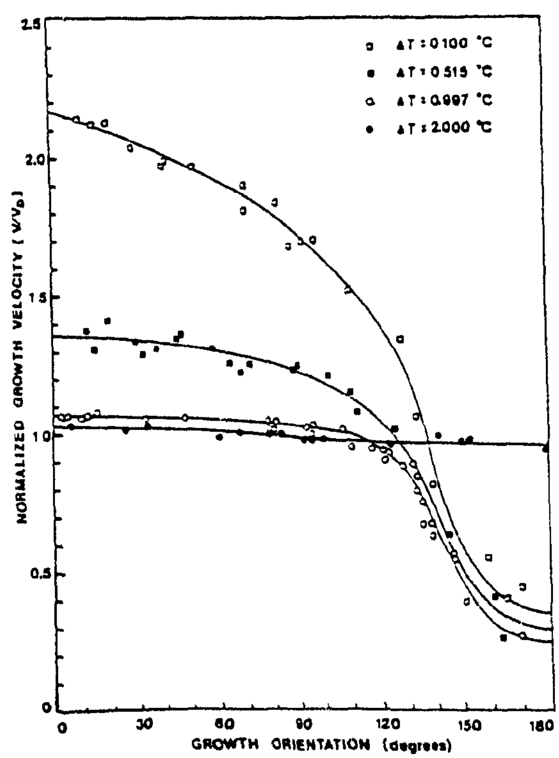

Figure 2.4: Normalized growth rate versus growth orientation

Dendritic growth velocity, V, was measured in succinonitrile as a function of growth orientation, ϕ, shown in Fig. 2.4,[2] in which the velocity is normalized

[2] Glicksman, M.E. & Huang, S.-C. , Proc. of 3rd European Sump. on Material Sciences in Space, Grenoble, ESP-142 (1979)

to the velocity, V_d, predicted for diffusion-controlled growth.

2.4 Effect of Natural Convection on Morphological Stability

The presence of buoyance-driven fluid flow causes orientation dependence of the dendrite morphology, as shown in Fig. 2.5.[3] This orientation dependence can be negligible for dendrites growing in spatial orientations within 60° of the gravity vector. For a dendrite growing upward at orientations greater than 145° to gravity, cf., Fig. 2.5(b), the shape of the tip region is, apparently, no longer symmetric.

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3 Glicksman, M.E. & Huang, S.-C. , Proc. of 3rd European Sump. on Material Sciences in Space, Grenoble, ESP-142 (1979)

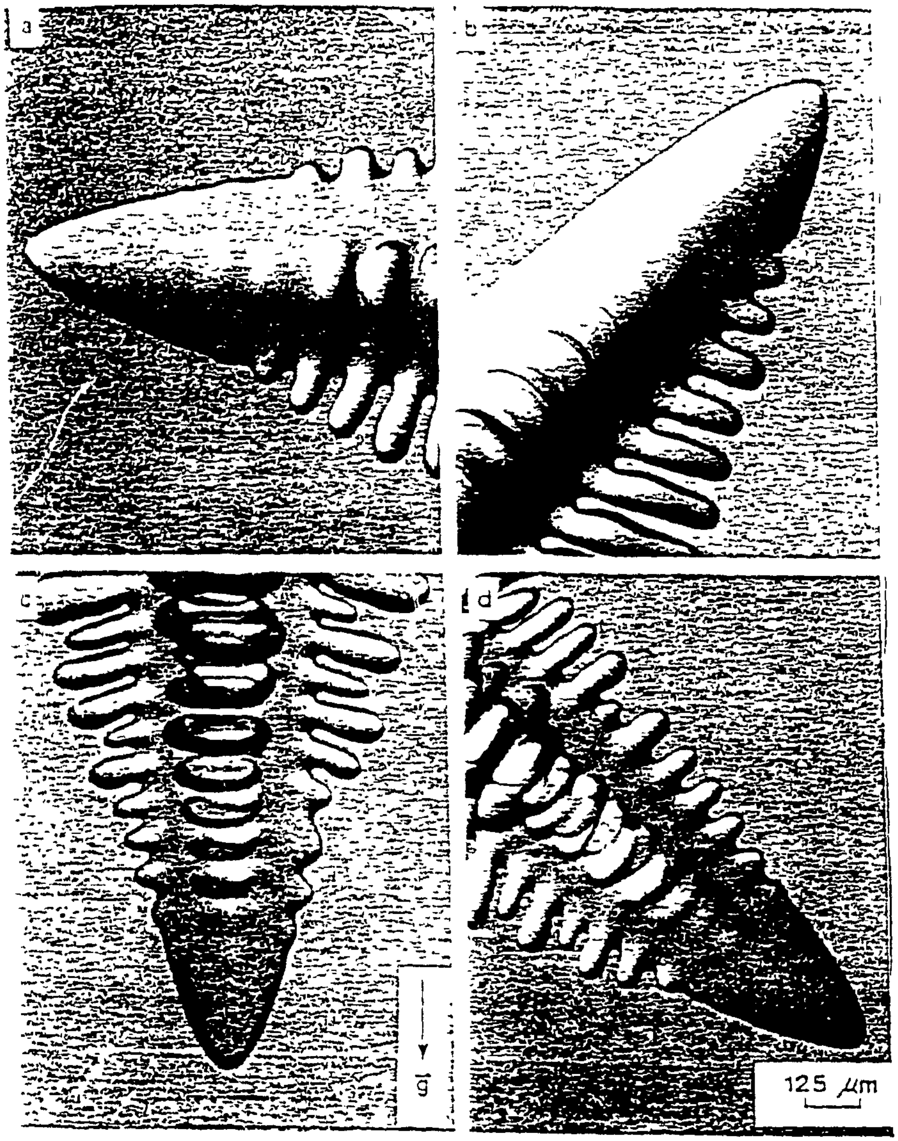

Figure 2.5: Influence of spatial orientation on dendrite tip morphology

Chapter III

THE MATHEMATICAL FORMULATION OF THE PROBLEM

3.1 The Governing Equations and Boundary Conditions

Consider a single dendrite which is growing into an undercooled pure melt in the negative z-axis direction with a constant tip velocity V. The undercooling temperature of the melt is $\overline{T}_\infty$. The melt, which is considered as incompressible viscous fluid and assumed to be infinite in extent, flows uniformly along Z-axis in the far field ahead of the tip with a constant velocity $\overline{U}_\infty$. The gravity vector is along the negative Z-axis direction. Assume that the thermal diffusivity κ_T and the heat capacity c_p of the liquid state are the same as those of the solid state, the mass density of liquid state is ρ and the mass density of solid state is ρ_s. The subscript "s" refers to the solid state in this problem. The Boussinesq approximation is applied in the melt.

The governing equations consisting of the Navier-Stokes equation and the heat conduction equation are as follows:

1. Mass-conservation equation:

$$\nabla \cdot \mathbf{U} = 0 \tag{3.1}$$

2. Momentum equation:

Applying the Boussinesq approximation, the Navier-stokes equation becomes

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{U} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{U} + \beta(\overline{T}-\overline{T}_\infty)g\mathbf{e}_Z \tag{3.2}$$

The first term in the R.H.S. of (3.2) is the pressure term; the second term is viscous stress term; the third term is the buoyance force term. Introducing the vorticity $\boldsymbol{\Omega}$ as

$$\boldsymbol{\Omega} = \nabla \times \mathbf{U} \quad , \tag{3.3}$$

then (3.2) becomes

$$\frac{\partial \Omega}{\partial t} + \nabla \times (\Omega \times \mathbf{U}) = \nu \nabla^2 \boldsymbol{\Omega} + \nabla \times [\beta(\overline{T} - \overline{T}_\infty) g \mathbf{e}_z] \tag{3.4}$$

3. Energy equation for the liquid state:

$$\frac{\partial \overline{T}}{\partial t} + \mathbf{U} \cdot \nabla \overline{T} = \kappa_T \nabla^2 \overline{T} \tag{3.5}$$

4. Energy equation for the solid state:

$$\frac{\partial \overline{T}_s}{\partial t} = \kappa_T \nabla^2 \overline{T}_s \tag{3.6}$$

where $\overline{T}$ and $\overline{T}_s$ denote the temperature fields, η_s denotes interface shape, $\mathbf{U}$ denotes the absolute velocity field of the fluid motion, t denotes the time and ν is the kinematic viscosity, β is the thermal expansion coefficient, P is the reduced pressure, g is the acceleration of gravity.

The boundary conditions are given as follows:

1. At far field:

$$\mathbf{U} = \overline{U}_\infty \mathbf{e}_Z \quad , \tag{3.7}$$

$$\overline{T} = \overline{T}_\infty \tag{3.8}$$

2. At the interface tip:

$$\mathbf{U}, \ \overline{T}, \ \overline{T}_s \ \textit{are regular} \tag{3.9}$$

3. At the interface, we assume the system is in the local thermodynamical equilibrium state. Thus we have

(i) Continuity condition of temperature:

$$\overline{T} = \overline{T}_s \tag{3.10}$$

(ii) Gibbs-Thomson condition:

$$\overline{T}_s = \overline{T}_m[1 - \frac{\gamma}{\Delta H} K\{\eta_s\}] \tag{3.11}$$

where

γ is the surface tension constant,

ΔH is the latent heat per unit volume of solid,

$\overline{T}_m$ is the melting temperature at the flat interface,

$K\{\eta_s\}$ is twice of the mean curvature of the interface.

(iii) Enthalpy conservation condition:

$$\Delta H(U_{ns} - U_{nI}) = -\Delta H U_{nI} = [(\rho\kappa_T c_p \nabla\overline{T})_{solid} - (\rho\kappa_T c_p \nabla\overline{T})_{liquid}]\cdot\mathbf{n} \tag{3.12}$$

where

U_{ns} is the velocity component of the solid state along the normal direction at the interface,

U_{nI} is the normal component of local growth velocity of the interface,

$\mathbf{n}$ is the normal vector of the interface.

(iv) Mass conservation condition:

$$\rho(U_{nl} - U_{nI}) = \rho_s(U_{ns} - U_{nI}) = -\rho_s U_{nI} \tag{3.13}$$

where

U_{nl} is the normal component of velocity of fluid at the interface.

(v) Continuity condition of tangential component of velocity:

$$(\mathbf{U}\cdot\mathbf{e}_\tau - U_{\tau I}) = (U_{\tau s} - U_{\tau I})$$

then

$$\mathbf{U}\cdot\mathbf{e}_\tau = 0 \tag{3.14}$$

where

$\mathbf{e}_\tau$ *is the unit tangential vector on the interface,*

$U_{\tau l}$ *is the tangential component of the local growth velocity of the interface,*

$U_{\tau s}$ *is the tangential component of velocity of the solid state at the interface.*

3.2 Scales and the Nondimensional System

To nondimensionalize the governing equations and their boundary conditions, we utilize the thermal length $l_T = \kappa_T / V$ as the length scale, the tip growth velocity V as the scale of the velocity, and $\Delta H/(c_p\rho)$ as the scale of the temperature, so that $T = \dfrac{\overline{T}-\overline{T}_m}{\Delta H/(c_p\rho)}$.

Let (X,Y,Z) be the laboratory frame of coordinates, (x,y,z) be a moving nondimensional coordinates fixed at the tip, **u** be the relative velocity field of fluid in the moving frame. The required transformation equation are

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \mathbf{e}_z\cdot\nabla \tag{3.15}$$

$$\frac{\mathbf{U}}{V} = \mathbf{u} - \mathbf{e}_z \tag{3.16}$$

Thus, the dimensionless governing equations are obtained immediately

1. Mass-conservation equation:

$$\nabla\cdot\mathbf{u} = 0 \tag{3.17}$$

2. Momentum equation:

$$\frac{\partial\omega}{\partial t} + (\mathbf{u}\cdot\nabla)\omega - (\omega\cdot\nabla)\mathbf{u} + \mathbf{e}_z\cdot\nabla\omega = Pr\nabla^2\omega - \frac{G}{T_\infty}\nabla\times(T\mathbf{e}_z) \tag{3.18}$$

3. Energy equation for the liquid state:

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \nabla^2 T \tag{3.19}$$

4. Energy equation for the solid state:

$$\frac{\partial T_s}{\partial t} + \mathbf{e}_z\cdot\nabla T_s = \nabla^2 T_s \tag{3.20}$$

where $Pr = \nu/\kappa_T$ is the Prandtl number; $T_\infty = -(\overline{T}_m - \overline{T}_\infty)c_p\rho/\Delta H = -St$, the parameter St is sometime called the Stefan number; $G = g\beta(\overline{T}_m - \overline{T}_\infty)\kappa_T/V^3$ is the gravitational parameter; ω is the nondimensional vorticity.

The boundary conditions are given as follows:

1. In the far field:

$$T = T_\infty \quad , \tag{3.21}$$

$$\mathbf{u} = (1 + U_\infty)\mathbf{e}_z \tag{3.22}$$

where

$U_\infty = \frac{\overline{U}_\infty}{V}$ *is the nondimensional velocity of of the external flow in the far field*

2. At the interface tip:

$$\mathbf{u},\ T,\ T_s \text{ are regular} \tag{3.23}$$

3. At the interface:

(i) Thermo-dynamical equilibrium condition:

$$T = T_s \tag{3.24}$$

(ii) Gibbs-Thomson condition:

$$T_s = -\Gamma K\{\eta_s\} \tag{3.25}$$

where

Γ is the surface tension parameter

and

$$\Gamma = \frac{l_c}{l_T} \tag{3.26}$$

l_c is called as the capillary length:

$$l_c = \frac{\gamma c_p \overline{T}_m \rho}{(\Delta H)^2} \tag{3.27}$$

(iii) Enthalpy conservation condition:

$$u_{nI} - \mathbf{e}_z \cdot \mathbf{n} = -[(1+\alpha)(\nabla T)_{solid} - (\nabla T)_{liquid}] \cdot \mathbf{n} \tag{3.28}$$

where

$$\alpha = \frac{\rho_s - \rho}{\rho} \tag{3.29}$$

(iv) Mass conservation condition:

$$\mathbf{u} \cdot \mathbf{n} = -\alpha u_{nI} + (1+\alpha)\mathbf{e}_z \cdot \mathbf{n} \tag{3.30}$$

(v) Continuity condition of tangential component of velocity:

$$\mathbf{u} \cdot \mathbf{e}_\tau = \mathbf{e}_z \cdot \mathbf{e}_\tau \tag{3.31}$$

where

u_{nI} is the normal component of local relative growth velocity of the interface.

3.3 Expression in the Paraboloidal Coordinates

In general, solving the system (3.17)-(3.31) is a very difficult problem. But adopting a paraboloidal system can make the problem easier. Let us define the moving paraboloidal coordinate system (ξ,η,θ) through the moving coordinate system (x,y,z) as (See Fig. 3.1)

$$\frac{r}{\eta_0^2} = \xi\eta \quad , \tag{3.32}$$

$$\frac{z}{\eta_0^2} = \frac{1}{2}(\xi^2 - \eta^2) \tag{3.33}$$

where

$$r^2 = x^2 + y^2$$

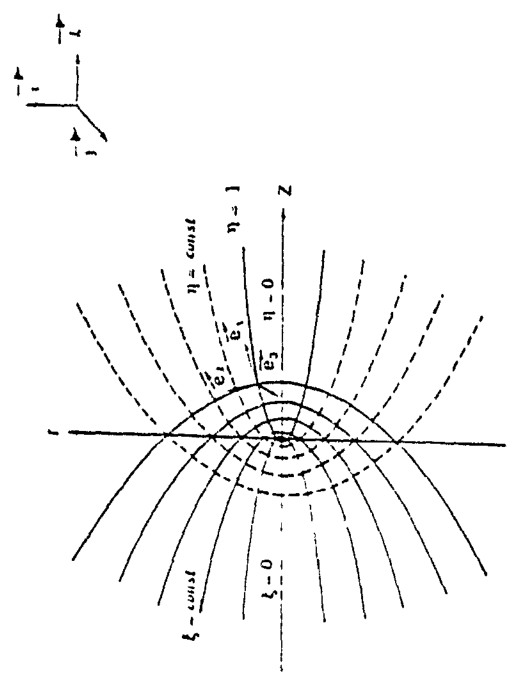

Figure 3.1: Paraboloidal coordinate system

The Lamé coefficients for this paraboloidal coordinate system are

$$H_\xi = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2} = \sqrt{\xi^2 + \eta^2}\,\eta_0^2 \tag{3.34}$$

$$H_\eta = \sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2} = \sqrt{\xi^2 + \eta^2}\,\eta_0^2 \tag{3.35}$$

$$H_\theta = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = \xi\eta\eta_0^2 \tag{3.36}$$

The transformation of unit vectors between the coordinate systems (ξ,η,θ) and (x,y,z) are

$$e_z = \frac{\xi e_\xi - \eta e_\eta}{\sqrt{\xi^2+\eta^2}} \tag{3.37}$$

and

$$\cos\theta e_x + \sin\theta e_y = \frac{\xi}{\sqrt{\xi^2+\eta^2}} e_\eta + \frac{\eta}{\sqrt{\xi^2+\eta^2}} e_\xi \tag{3.38}$$

The normal vector of the interface is found to be

$$\mathbf{n} = -\frac{\eta_s'}{\sqrt{1+\eta_s'^2}} e_\xi + \frac{1}{\sqrt{1+\eta_s'^2}} e_\eta \tag{3.39}$$

where $\eta = \eta_s(\xi,t)$ is the interface. The tangential vector at the interface is

$$e_\tau = \frac{e_\xi + \eta_s' e_\eta}{\sqrt{1+\eta_s'^2}} \tag{3.40}$$

Each point in the field can be described by the following radius vector in the moving frame:

$$\mathbf{R} = x e_x + y e_y + z e_z$$

$$= \xi\eta\cos\theta\eta_0^2 e_x + \xi\eta\sin\theta\eta_0^2 e_y + [\frac{1}{2}(\xi^2-\eta^2)\eta_0^2] e_z \tag{3.41}$$

So the interface shape can be expressed in the form:

$$\mathbf{R}(\xi,t) = \xi\eta_s\cos\theta\eta_0^2 e_x + \xi\eta_s\sin\theta\eta_0^2 e_y + [\frac{1}{2}(\xi^2-\eta_s^2)\eta_0^2] e_z \tag{3.42}$$

Then, the normal velocity of the interface can be calculated as

$$u_{nI} = \frac{\partial \mathbf{R}(\xi,t)}{\partial t} \cdot \mathbf{n} \tag{3.43}$$

We use the stream function $\Psi(\xi,\eta,t)$ and nondimensional vorticity $\boldsymbol{\omega} = (\zeta/\eta_0^2\xi\eta)\mathbf{e}_\theta$ as the basic hydro-dynamical quantities. From (3.17) we get

$$u_\xi = \frac{1}{\eta_0^1\xi\eta\sqrt{\xi^2+\eta^2}} \frac{\partial\Psi}{\partial\eta} \tag{3.44}$$

$$u_\eta = -\frac{1}{\eta_0^4\xi\eta\sqrt{\xi^2+\eta^2}} \frac{\partial\Psi}{\partial\xi} \tag{3.45}$$

where u_ξ, u_η are the nondimensional velocity components of the relative fluid motion along ξ and η directions, respectively, so that

$$\mathbf{u} = u_\xi \mathbf{e}_\xi + u_\eta \mathbf{e}_\eta$$

Substituting (3.32)-(3.45) into (3.17)-(3.31), we can express this system in the paraboloidal coordinates. Have done this, the governing equations take the following forms:

1. Kinematic equation:

$$D^2\Psi = -\eta_0^4(\xi^2 + \eta^2)\zeta \tag{3.46}$$

2. Vorticity equation:

$$PrD^2\zeta = \eta_0^4(\xi^2+\eta^2)\frac{\partial \zeta}{\partial t} + \frac{2\zeta}{\eta_0^4\xi^2\eta^2}\frac{\partial(\Psi,\eta_0^2\xi\eta)}{\partial(\xi,\eta)} - \frac{1}{\eta_0^2\xi\eta}\frac{\partial(\Psi,\zeta)}{\partial(\xi,\eta)}$$

$$- \frac{G\xi\eta\eta_0^4}{T_\infty}[\eta\frac{\partial T}{\partial \xi} + \xi\frac{\partial T}{\partial \eta}] \tag{3.47}$$

3. Energy equation for the liquid state:

$$\nabla^2 T = \eta_0^4(\xi^2+\eta^2)\frac{\partial T}{\partial t} + \frac{1}{\eta_0^2\xi\eta}(\frac{\partial \Psi}{\partial \eta}\frac{\partial T}{\partial \xi} - \frac{\partial \Psi}{\partial \xi}\frac{\partial T}{\partial \eta}) \tag{3.48}$$

4. Energy equation for the solid state:

$$\nabla^2 T_s = \eta_0^4(\xi^2+\eta^2)\frac{\partial T_s}{\partial t} + \eta_0^2\xi\frac{\partial T_s}{\partial \xi} - \eta_0^2\eta\frac{\partial T_s}{\partial \eta} \tag{3.49}$$

where

$$\nabla^2 = \{\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{1}{\xi}\frac{\partial}{\partial \xi} + \frac{1}{\eta}\frac{\partial}{\partial \eta}\} \tag{3.50}$$

$$D^2 = \{\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} - \frac{1}{\xi}\frac{\partial}{\partial \xi} - \frac{1}{\eta}\frac{\partial}{\partial \eta}\} \tag{3.51}$$

The boundary conditions are

1. As $\eta \to \infty$:

$$u_\xi \to 0 \tag{3.52}$$

$$u_\eta \rightarrow -1 - U_\infty \tag{3.53}$$

$$\zeta \rightarrow 0 \tag{3.54}$$

$$T \rightarrow T_\infty \tag{3.55}$$

2. As $\eta \rightarrow 0$:

$$\frac{\partial T_s}{\partial \eta} = 0 \tag{3.56}$$

$$T_s = O(1) \tag{3.57}$$

3. At the interface $\eta = \eta_s(\xi,t)$:

(i) Thermo-dynamical equilibrium condition:

$$T = T_s \tag{3.58}$$

(ii) Gibbs-Thomson condition:

$$T_s = -\Gamma K\{\eta_s(\xi,t)\} \tag{3.59}$$

(iii) Enthalpy conservation condition:

$$\left(\frac{\partial T}{\partial \eta} - \eta_s' \frac{\partial T}{\partial \xi}\right) - (1+\alpha)\left(\frac{\partial T_s}{\partial \eta} - \eta_s' \frac{\partial T_s}{\partial \xi}\right) + \eta_0^2(\xi\eta_s)' + \eta_0^4(\xi^2+\eta^2)\frac{\partial \eta_s}{\partial t} = 0 \tag{3.60}$$

(iv) Mass conservation condition:

$$\left(\frac{\partial \Psi}{\partial \xi} + \eta_s' \frac{\partial \Psi}{\partial \eta}\right) = \eta_0^2(\xi\eta_s)[\eta_0^2(1+\alpha)(\xi\eta_s)' + \alpha\eta_0^4(\xi^2+\eta_s^2)\frac{\partial \eta_s}{\partial t}] \tag{3.61}$$

(v) Continuity condition of tangential component of velocity:

$$\left(\frac{\partial \Psi}{\partial \eta} - \eta_s' \frac{\partial \Psi}{\partial \xi}\right) + \eta_0^4(\xi\eta_s)(\eta_s\eta_s' - \xi) = 0 \tag{3.62}$$

In the above $K\{\eta_s(\xi,t)\}$ is the curvature operator as

$$K\{\eta_s(\xi,t)\} = -\frac{1}{\eta_0^2\sqrt{\xi^2+\eta^2_s}}\left\{\frac{\eta_s''}{(1+\eta_s'^2)^{3/2}} - \frac{1}{\eta_s(1+\eta_s'^2)^{1/2}} + \frac{\eta_s'(\eta_s^2+2\xi^2)-\xi\eta_s}{\xi(\xi^2+\eta_s^2)(1+\eta_s'^2)^{1/2}}\right\} \tag{3.63}$$

The notation prime represents the derivative with respect to ξ.

The system (3.46)-(3.63) contains five parameters in total: $\{T_\infty, \Gamma, U_\infty, G, \alpha\}$, where the parameters U_∞, G, α describe the effects of convection, while Γ describes the effect of surface tension. In this thesis we shall discuss three situations separately:

(1) $\alpha \neq 0$, $U_\infty = G = 0$: the convection is induced by the density change during phase transition;

(2) $U_\infty \neq 0$, $\alpha = G = 0$: the convection is induced by external flow;

(3) $G \neq 0$, $\alpha = U_\infty = 0$: the convection is induced by the buoyancy effect.

Chapter IV

EFFECT OF FLUID MOTION INDUCED BY DENSITY CHANGE ON DENDRITIC GROWTH

4.1 Formulation

In this chapter, we will apply the regular perturbation method to solve the problem of dendritic growth in which convection is induced only by the density change during phase transition. In this case, we neglect external flow and buoyance effect, i.e. $G = 0$, $U_\infty = 0$. The surface tension parameter Γ is considered as a small quantity. Thus, system (3.46)-(3.62) is reduced to

1.

$$D^2\Psi = -\eta_0^4(\xi^2 + \eta^2)\zeta \tag{4.1}$$

2.

$$PrD^2\zeta = \eta_0^4(\xi^2+\eta^2)\frac{\partial\zeta}{\partial t} + \frac{2\zeta}{\eta_0^4\xi^2\eta^2}\frac{\partial(\Psi,\eta_0^2\xi\eta)}{\partial(\xi,\eta)} - \frac{1}{\eta_0^2\xi\eta}\frac{\partial(\Psi,\zeta)}{\partial(\xi,\eta)} \tag{4.2}$$

3.

$$\nabla^2 T = \eta_0^4(\xi^2+\eta^2)\frac{\partial T}{\partial t} + \frac{1}{\eta_0^2\xi\eta}\left(\frac{\partial\Psi}{\partial\eta}\frac{\partial T}{\partial\xi} - \frac{\partial\Psi}{\partial\xi}\frac{\partial T}{\partial\eta}\right) \tag{4.3}$$

4.

$$\nabla^2 T_s = \eta_0^4(\xi^2+\eta^2)\frac{\partial T_s}{\partial t} + \eta_0^2\xi\frac{\partial T_s}{\partial\xi} - \eta_0^2\eta\frac{\partial T_s}{\partial\eta} \tag{4.4}$$

with the boundary conditions:

1. As $\eta \to \infty$:

$$u_\xi = 0 \tag{4.5}$$

$$u_\eta = -1 \tag{4.6}$$

$$\zeta \rightarrow 0 \tag{4.7}$$

$$T \rightarrow T_\infty \tag{4.8}$$

2. As $\eta \rightarrow 0$:

$$\frac{\partial T_s}{\partial \eta} = 0 \quad , \quad T_s = O(1) \tag{4.9}$$

3. At the interface $\eta = \eta_s(\xi,t)$:

(i).

$$T = T_s \tag{4.10}$$

(ii).

$$T_s = -\Gamma K\{\eta_s(\xi,t)\} \tag{4.11}$$

(iii).

$$\left(\frac{\partial T}{\partial \eta} - \eta_s' \frac{\partial T}{\partial \xi}\right) - (1+\alpha)\left(\frac{\partial T_s}{\partial \eta} - \eta_s' \frac{\partial T_s}{\partial \xi}\right) + \eta_0^2(\xi\eta_s)' + \eta_0^4(\xi^2+\eta^2)\frac{\partial \eta_s}{\partial t} = 0 \tag{4.12}$$

(iv).

$$\left(\frac{\partial \Psi}{\partial \xi} + \eta_s' \frac{\partial \Psi}{\partial \eta}\right) = \eta_0^2(\xi\eta_s)[\eta_0^2(1+\alpha)(\xi\eta_s)' + \alpha\eta_0^4(\xi^2+\eta_s^2)\frac{\partial \eta_s}{\partial t}] \tag{4.13}$$

(v).

$$\left(\frac{\partial \Psi}{\partial \eta} - \eta_s' \frac{\partial \Psi}{\partial \xi}\right) + \eta_0^4(\xi\eta_s)(\eta_s\eta_s' - \xi) = 0 \tag{4.14}$$

According to *the Interfacial Wave Theory of Solidification* [27], we introduce the dynamical parameter of surface tension defined as

$$\epsilon = \frac{\sqrt{\Gamma}}{\eta_0^2} \tag{4.15}$$

In general, the parameter of density change α is a small quantity as ϵ. So we express

$$\alpha = \hat{\alpha}\epsilon \tag{4.16}$$

where $\hat{\alpha}$ is constant of O(1).

Furthermore, we use regular perturbation method, where the zeroth order solution should be Ivantsov's solution, and write

$$\begin{aligned} T(\xi,\eta,t) &= T_0(\eta) + \epsilon T_1(\xi,\eta) + \epsilon^2 T_2(\xi,\eta) + \cdots , \\ \zeta(\xi,\eta,t) &= \zeta_0 + \epsilon\zeta_1 + \epsilon^2\zeta_2(\xi,\eta) + \cdots , \\ \Psi(\xi,\eta,t) &= \Psi_0 + \epsilon\Psi_1(\xi,\eta) + \epsilon^2\Psi_2(\xi,\eta) + \cdots , \\ \eta_s(\xi,\eta,t) &= 1 + \epsilon\eta_1 + \epsilon^2\eta_2(\xi) + \cdots , \\ T_s(\xi,\eta,t) &= T_{s0}(1) + \epsilon T_{s1} + \epsilon^2 T_{s2}(\xi,\eta) + \cdots \end{aligned} \tag{4.17}$$

By substituting the above expansions into the system (4.1)-(4.14), one can successively derive each order approximate solution. In the following, I shall give the approximate solutions up to the second order.

4.2 The Zeroth-order Approximation Solution

The zeroth-order approximation solution is the solution for the case $\alpha = \epsilon = 0$. It turns out that this solution is just the Ivantsov's solution. To show this, let us write down the system governing the zeroth order approximation:

1.

$$D^2\Psi_0 = -\eta_0^4(\xi^2 + \eta^2)\zeta_0 \tag{4.18}$$

2.

$$PrD^2\zeta_0 = \frac{2\zeta_0}{\eta_0^4\xi^2\eta^2}\frac{\partial(\Psi_0,\eta_0^2\xi\eta)}{\partial(\xi,\eta)} - \frac{1}{\eta_0^2\xi\eta}\frac{\partial(\Psi_0,\zeta_0)}{\partial(\xi,\eta)} \tag{4.19}$$

3.

$$\nabla^2 T_0 = \frac{1}{\eta_0^2\xi\eta}\left(\frac{\partial\Psi_0}{\partial\eta}\frac{\partial T_0}{\partial\xi} - \frac{\partial\Psi_0}{\partial\xi}\frac{\partial T_0}{\partial\eta}\right) \tag{4.20}$$

4.

$$\nabla^2 T_{s0} = \eta_0^2 \xi \frac{\partial T_{s0}}{\partial \xi} - \eta_0^2 \eta \frac{\partial T_{s0}}{\partial \eta} \tag{4.21}$$

with the boundary conditions:

1. as $\eta \to \infty$:

$$\zeta_0 = 0 \quad ,$$

$$\Psi_0 = \frac{\eta_0^4 \xi^2 \eta^2}{2} \quad ,$$

$$T_0 = T_\infty$$

2. as $\eta = 1$:

$$\begin{aligned} &T_{s0} = 0 \quad , \\ &\frac{\partial \Psi_0}{\partial \xi} = \eta_0^4 \xi \quad , \\ &\frac{\partial \Psi_0}{\partial \eta} = \eta_0^4 \xi^2 \quad , \\ &T_0 = 0 \quad , \\ &\frac{\partial T_0}{\partial \eta} = -\eta_0^2 \end{aligned} \tag{4.22}$$

Suppose

$$\Psi_0 = \eta_0^4 \xi^2 f(\eta) \tag{4.23}$$

and ζ_0 be constant. By (4.3) we get

$$\frac{d^2 T_0}{d\eta^2} + \frac{1}{\eta} \frac{dT_0}{d\eta} + \frac{2\eta_0^2 f(\eta)}{\eta} \frac{dT_0}{d\eta} = 0 \tag{4.24}$$

Then the zeroth-order approximation solution is solved as:

$$T_0 = e^{\frac{\eta_0^2}{2}} \frac{\eta_0^2}{2} E_1\left(\frac{\eta_0^2 \eta^2}{2}\right) - e^{\frac{\eta_0^2}{2}} \frac{\eta_0^2}{2} E_1\left(\frac{\eta_0^2}{2}\right) \ ,$$

$$T_\infty = -e^{\frac{\eta_0^2}{2}} \frac{\eta_0^2}{2} E_1\left(\frac{\eta_0^2}{2}\right) \ ,$$

$$\Psi_0 = \frac{\eta_0^4}{2}\xi^2\eta^2 \ , \tag{4.25}$$

$$\zeta_0 = 0 \ ,$$

$$T_{s0} = 0$$

Therefore, the zeroth-order approximate solution is just the Ivantsov' solution. The vorticity is zero everywhere. The fluid motion in zeroth-order is the uniform flow.

4.3 The First-order Approximation Solution

The equations for the first-order approximation are

1.

$$Pr[\frac{\partial^2\zeta_1}{\partial\xi^2} + \frac{\partial^2\zeta_1}{\partial\eta^2} - \frac{1}{\xi}\frac{\partial\zeta_1}{\partial\xi} - \frac{1}{\eta}\frac{\partial\zeta_1}{\partial\eta}] = -\eta_0^2\eta\frac{\partial\zeta_1}{\partial\eta} + \eta_0^2\xi\frac{\partial\zeta_1}{\partial\xi} \tag{4.26}$$

2.

$$\frac{\partial^2\Psi_1}{\partial\xi^2} + \frac{\partial^2\Psi_1}{\partial\eta^2} - \frac{1}{\xi}\frac{\partial\Psi_1}{\partial\xi} - \frac{1}{\eta}\frac{\partial\Psi_1}{\partial\eta} = -\eta_0^4(\xi + \eta)\zeta_1 \tag{4.27}$$

3.

$$\frac{\partial^2 T_1}{\partial\xi^2} + \frac{\partial^2 T_1}{\partial\eta^2} + \frac{1}{\xi}\frac{\partial T_1}{\partial\xi} + \frac{1}{\eta}\frac{\partial T_1}{\partial\eta} = \eta_0^2\xi\frac{\partial T_1}{\partial\xi} - \eta_0^2\eta\frac{\partial T_1}{\partial\eta} + \frac{1}{\xi}e^{\frac{\eta_0^2}{2}-\frac{\eta_0^2\eta^2}{2}}\eta^{-2}\frac{\partial\Psi_1}{\partial\xi} \tag{4.28}$$

4.

$$\frac{\partial^2 T_{s1}}{\partial\xi^2} + \frac{\partial^2 T_{s1}}{\partial\eta^2} + \frac{1}{\xi}\frac{\partial T_{s1}}{\partial\xi} + \frac{1}{\eta}\frac{\partial T_{s1}}{\partial\eta} = \eta_0^2\xi\frac{\partial T_{s1}}{\partial\xi} - \eta_0^2\eta\frac{\partial T_{s1}}{\partial\eta} \tag{4.29}$$

the boundary conditions are

1. as $\eta \rightarrow \infty$:

$$\zeta_1 = 0 \quad , \tag{4.30}$$

$$u_1 = 0 \quad , \tag{4.31}$$

$$T_1 = 0 \tag{4.32}$$

2. at $\eta = 1$:

$$T_{s1} = 0 \quad , \tag{4.33}$$

$$\frac{\partial \Psi_1}{\partial \xi} = \eta_0^4 \xi \hat{\alpha} \quad , \tag{4.34}$$

$$\frac{\partial \Psi_1}{\partial \eta} = 0 \quad , \tag{4.35}$$

$$T_1 + \eta_1 \frac{\partial T_0}{\partial \eta} = T_{s1} = 0 \quad , \tag{4.36}$$

$$\frac{\partial T_1}{\partial \eta} + \eta_1 \frac{\partial^2 T_0}{\partial \eta^2} - \frac{\partial T_{s1}}{\partial \eta} + \eta_0^2(\eta_1 + \xi \eta_1') = 0 \tag{4.37}$$

To satisfy the equation (4.26), (4.29) and the boundary condition (4.30), (4.33), we can set

$$\zeta_1 = 0 \ ,$$

$$T_{s1} = 0$$

From (4.27), (4.31), and (4.34)-(4.35), we get

$$\Psi_1 = \frac{\eta_0^4 \hat{\alpha} \xi^2}{2}$$

Letting

$$\hat{\sigma} = \frac{\eta_0^2 \xi^2}{2} \tag{4.38}$$

$$\hat{\tau} = \frac{\eta_0^2 \eta^2}{2} \tag{4.39}$$

and

$$T_1 = T_1(\hat{\tau}) \tag{4.40}$$

we can reduce (4.28) into

$$\hat{\tau}\frac{d^2T_1}{d\hat{\tau}^2} + \hat{\tau}\frac{dT_1}{d\hat{\tau}} + \frac{dT_1}{d\hat{\tau}} = \frac{\eta_0^4\hat{\alpha}}{4\hat{\tau}}e^{\frac{\eta_0^2}{2}-\hat{\tau}} \tag{4.41}$$

Solving (4.41) , we have

$$T_1'(\hat{\tau}) = \frac{\eta_0^4 e^{\frac{\eta_0^2}{2}-\hat{\tau}}}{4\hat{\tau}}\hat{\alpha}\ln(\frac{2\hat{\tau}}{\eta_0^2}) + \hat{\alpha}cE_0(\hat{\tau}) \tag{4.42}$$

and

$$T_1(\hat{\tau}) = \hat{\alpha}\hat{T}_1(\hat{\tau})$$

where

$$\hat{T}_1(\hat{\tau}) = -\frac{\eta_0^4 e^{\frac{\eta_0^2}{2}}}{4}\int_{\hat{\tau}}^{\infty}\frac{e^{-x}}{x}\ln(\frac{2x}{\eta_0^2})\,dx - cE_1(\hat{\tau}) \tag{4.43}$$

where c is a constant; $E_n(\hat{\tau})$ are the exponential integrals defined by

$$E_n(x) = \int_1^{\infty}\frac{e^{-xs}}{s^n}ds \quad , \qquad n=0,1,2,\cdots \tag{4.44}$$

From the condition (4.36) we know that η_1 must be a constant, and from (4.36) and (4.37) we obtain

$$\eta_1 = \hat{\alpha}\hat{\eta}_1$$

where

$$\hat{\eta}_1 = [\frac{-\eta_0^4 e^{\frac{\eta_0^2}{2}}}{4}\int_{\frac{\eta_0^2}{2}}^{\infty}\frac{e^{-x}}{x}\ln(\frac{2x}{\eta_0^2})\,dx][\frac{E_0(\frac{\eta_0^2}{2})}{E_1(\frac{\eta_0^2}{2})(2\eta_0^2+\eta_0^4)+E_0(\frac{\eta_0^2}{2})\eta_0^2}] \tag{4.45}$$

and determine the constant c in (4.43):

$$c = -\frac{\hat{\eta}_1(2\eta_0^2+\eta_0^4)}{E_0(\frac{\eta_0^2}{2})} \tag{4.46}$$

Thus, the first-approximation solution is derived. We list the results as follows:

$$\zeta_1 = 0 \quad ,$$

$$\Psi_1 = \frac{\eta_0^4 \hat{\alpha} \xi^2}{2} \quad ,$$

$$T_1 = \hat{\alpha} \hat{T}_1(\hat{\tau}) \quad , \tag{4.47}$$

$$T_{s1} = 0 \quad ,$$

$$\eta_1 = \hat{\alpha} \hat{\eta}_1$$

It is seen that in the first order approximation, the vorticity is still zero everywhere; the density change affects the dendrite shape and perturbs the velocity field and temperature field.

4.4 The Second-order Approximation Solution

The governing equations for the second-order approximation are

1.

$$Pr\left[\frac{\partial^2 \zeta_2}{\partial \xi^2} + \frac{\partial^2 \zeta_2}{\partial \eta^2} - \frac{1}{\xi}\frac{\partial \zeta_2}{\partial \xi} - \frac{1}{\eta}\frac{\partial \zeta_2}{\partial \eta}\right] = -\eta_0^2 \eta \frac{\partial \zeta_2}{\partial \eta} + \eta_0^2 \xi \frac{\partial \zeta_2}{\partial \xi} \tag{4.48}$$

2.

$$\frac{\partial^2 \Psi_2}{\partial \xi^2} + \frac{\partial^2 \Psi_2}{\partial \eta^2} - \frac{1}{\xi}\frac{\partial \Psi_2}{\partial \xi} - \frac{1}{\eta}\frac{\partial \Psi_2}{\partial \eta} = -\eta_0^4(\xi^2 + \eta^2)\zeta_2 \tag{4.49}$$

3.

$$\frac{\partial^2 T_2}{\partial \xi^2} + \frac{\partial^2 T_2}{\partial \eta^2} + \frac{1}{\xi}\frac{\partial T_2}{\partial \xi} + \frac{1}{\eta}\frac{\partial T_2}{\partial \eta}$$

$$= \eta_0^2 \xi \frac{\partial T_2}{\partial \xi} - \eta_0^2 \eta \frac{\partial T_2}{\partial \eta} + \frac{1}{\xi \eta^2} e^{\frac{\eta_0^2}{2} - \frac{\eta_0^2 \eta^2}{2}} \frac{\partial \Psi_2}{\partial \xi} - \frac{\eta_0^6 \hat{\alpha}^2}{\eta^2} e^{\frac{\eta_0^2}{2} - \frac{\eta_0^2 \eta^2}{2}} \ln \eta - \eta_0^4 c \hat{\alpha}^2 E_0(\hat{\tau}) \tag{4.50}$$

4.

$$\frac{\partial^2 T_{s2}}{\partial \xi^2} + \frac{\partial^2 T_{s2}}{\partial \eta^2} + \frac{1}{\xi}\frac{\partial T_{s2}}{\partial \xi} + \frac{1}{\eta}\frac{\partial T_{s2}}{\partial \eta} = \eta_0^2 \xi \frac{\partial T_{s2}}{\partial \xi} - \eta_0^2 \eta \frac{\partial T_{s2}}{\partial \eta} \tag{4.51}$$

the boundary conditions are

(i) As $\eta \rightarrow \infty$:

$$T_2 \rightarrow 0 \tag{4.52}$$

$$u_{\xi 2} = u_{\eta 2} = 0 \tag{4.53}$$

$$\zeta_2 \rightarrow 0 \tag{4.54}$$

(ii) As $\eta \rightarrow 0$:

$$\frac{\partial T_{s2}}{\partial \eta} = 0 \quad , \quad T_{s2} = O(1) \tag{4.55}$$

(iii) At $\eta = 1$:

$$T_2 = T_{s2} + \eta_2 \eta_0^2 - \hat{\alpha} c \eta_0^2 \eta_1 E_0(\frac{\eta_0^2}{2}) - \frac{\eta_1^2}{2}(\eta_0^2 + \eta_0^4) \tag{4.56}$$

$$T_{s2} = -\frac{\eta_0^2(\xi^2+2)}{(\xi^2+1)^{\frac{3}{2}}} \tag{4.57}$$

$$\frac{\partial}{\partial \eta}(T_2 - T_{s2}) + \eta_2[\eta_0^4 + 2\eta_0^2] + \eta_0^2 \xi \frac{d\eta_2}{d\xi}$$

$$+ \eta_1[\eta_0^4 \hat{\alpha} - \eta_0^4 \hat{\alpha} c E_0(\frac{\eta_0^2}{2}) - \eta_0^2 \hat{\alpha} c E_0(\frac{\eta_0^2}{2})] - \frac{\eta_1^2}{2}[2\eta_0^2 + \eta_0^4 + \eta_0^6]$$

$$= 0 \tag{4.58}$$

$$\frac{\partial \Psi_2}{\partial \xi} = 2\hat{\alpha} \eta_0^4 \eta_1 \xi \tag{4.59}$$

$$\frac{\partial \Psi_2}{\partial \eta} = 0 \tag{4.60}$$

The solution for the above system can be split into two parts which are the solutions of two linear systems, respectively.

(1). The fist part of solution is subject to the following system:

$$Pr[\frac{\partial^2 \zeta_2^{(1)}}{\partial \xi^2} + \frac{\partial^2 \zeta_2^{(1)}}{\partial \eta^2} - \frac{1}{\xi}\frac{\partial \zeta_2^{(1)}}{\partial \xi} - \frac{1}{\eta}\frac{\partial \zeta_2^{(1)}}{\partial \eta}] = -\eta_0^2 \eta \frac{\partial \zeta_2^{(1)}}{\partial \eta} + \eta_0^2 \xi \frac{\partial \zeta_2^{(1)}}{\partial \xi} \tag{4.61}$$

$$\frac{\partial^2\Psi_2^{(1)}}{\partial\xi^2} + \frac{\partial^2\Psi_2^{(1)}}{\partial\eta^2} - \frac{1}{\xi}\frac{\partial\Psi_2^{(1)}}{\partial\xi} - \frac{1}{\eta}\frac{\partial\Psi_2^{(1)}}{\partial\eta} = -\eta_0^4(\xi^2+\eta^2)\zeta_2^{(1)} \tag{4.62}$$

$$\frac{\partial^2 T_2^{(1)}}{\partial\xi^2} + \frac{\partial^2 T_2^{(1)}}{\partial\eta^2} + \frac{1}{\xi}\frac{\partial T_2^{(1)}}{\partial\xi} + \frac{1}{\eta}\frac{\partial T_2^{(1)}}{\partial\eta}$$

$$= \eta_0^2\xi\frac{\partial T_2^{(1)}}{\partial\xi} - \eta_0^2\eta\frac{\partial T_2^{(1)}}{\partial\eta} + \frac{1}{\xi\eta^2}e^{\frac{\eta_0^2}{2}-\frac{\eta_0^2\eta^2}{2}}\frac{\partial\Psi_2^{(1)}}{\partial\xi}$$

$$-\frac{\eta_0^6\hat{\alpha}^2}{\eta^2}e^{\frac{\eta_0^2}{2}-\frac{\eta_0^2\eta^2}{2}}\ln\eta - \eta_0^4 c\hat{\alpha}^2 E_0(\hat{\tau}) \tag{4.63}$$

$$\frac{\partial^2 T_{s2}^{(1)}}{\partial\xi^2} + \frac{\partial^2 T_{s2}^{(1)}}{\partial\eta^2} + \frac{1}{\xi}\frac{\partial T_{s2}^{(1)}}{\partial\xi} + \frac{1}{\eta}\frac{\partial T_{s2}^{(1)}}{\partial\eta} = \eta_0^2\xi\frac{\partial T_{s2}^{(1)}}{\partial\xi} - \eta_0^2\eta\frac{\partial T_{s2}^{(1)}}{\partial\eta} \tag{4.64}$$

(i) As $\eta \to \infty$:

$$T_2^{(1)} \to 0 \tag{4.65}$$

$$u_{\xi 2}^{(1)} = u_{\eta 2}^{(1)} = 0 \tag{4.66}$$

$$\zeta_2^{(1)} \to 0 \tag{4.67}$$

(ii) As $\eta \to 0$:

$$\frac{\partial T_{s2}^{(1)}}{\partial\eta} = 0 \quad , \quad T_{s2}^{(1)} = O(1) \tag{4.68}$$

(iii) At $\eta = 1$:

$$T_2^{(1)} = T_{s2}^{(1)} + \eta_2^{(1)}\eta_0^2 - \hat{\alpha}c\eta_0^2\eta_1 E_0(\frac{\eta_0^2}{2}) - \frac{\eta_1^2}{2}(\eta_0^2+\eta_0^4) \tag{4.69}$$

$$T_{s2}^{(1)} = 0 \tag{4.70}$$

$$\frac{\partial}{\partial\eta}(T_2^{(1)}-T_{s2}^{(1)}) + \eta_2^{(1)}[\eta_0^4+2\eta_0^2] + \eta_0^2\xi\frac{d\eta_2^{(1)}}{d\xi}$$

$$+\eta_1[\eta_0^4\hat{\alpha} - \eta_0^4\hat{\alpha}cE_0(\frac{\eta_0^2}{2}) - \eta_0^2\hat{\alpha}cE_0(\frac{\eta_0^2}{2})] - \frac{\eta_1^2}{2}[2\eta_0^2+\eta_0^4+\eta_0^6]$$

$$= 0 \tag{4.71}$$

$$\frac{\partial \Psi_2^{(1)}}{\partial \xi} = 2\hat{\alpha}\eta_0^4\hat{\eta}_1\xi \tag{4.72}$$

$$\frac{\partial \Psi_2^{(1)}}{\partial \eta} = 0 \tag{4.73}$$

By (4.61)-(4.62), (4.66)-(4.67) and (4.72)-(4.73), the flow fields are obtained as

$$\zeta_2^{(1)} = 0 \ , \tag{4.74}$$

$$\Psi_2^{(1)} = \hat{\alpha}^2\eta_0^4\hat{\eta}_1\xi^2 \tag{4.75}$$

Letting

$$T_2^{(1)} = T_2^{(1)}(\hat{\tau})$$

(4.63) becomes

$$\hat{\tau}\frac{d^2T_2^{(1)}}{d\hat{\tau}^2} + \hat{\tau}\frac{dT_2^{(1)}}{d\tau} + \frac{dT_2^{(1)}}{d\tau} = \frac{\eta_0^4\hat{\alpha}\eta_1}{2\hat{\tau}}e^{\frac{\eta_0^2}{2}-\hat{\tau}} - \frac{\eta_0^6\hat{\alpha}^2 e^{\frac{\eta_0^2}{2}-\hat{\tau}}}{8\hat{\tau}}\ln\frac{2\hat{\tau}}{\eta_0^2} - \frac{\eta_0^2\hat{\alpha}^2 c}{2}E_0(\hat{\tau}) \tag{4.76}$$

Solving (4.76), we get

$$T_2^{(1)\prime}(\hat{\tau}) = \left[\frac{\eta_0^4\hat{\alpha}^2\hat{\eta}_1 e^{\frac{\eta_0^2}{2}}}{2} - \frac{\eta_0^2\hat{\alpha}^2 c}{2}\right]\frac{e^{-\hat{\tau}}\ln\frac{2\hat{\tau}}{\eta_0^2}}{\hat{\tau}} - \frac{\eta_0^6\hat{\alpha}^2 e^{\frac{\eta_0^2}{2}}}{16}\left(\ln\frac{2\hat{\tau}}{\eta_0^2}\right)^2\frac{e^{-\hat{\tau}}}{\hat{\tau}} + \hat{\alpha}^2 dE_0(\hat{\tau}) \tag{4.77}$$

and

$$T_2^{(1)} = \hat{\alpha}^2\hat{T}_2(\hat{\tau})$$

where

$$\hat{T}_2(\hat{\tau}) = \frac{-\eta_0^2(\eta_0^2\hat{\eta}_1 e^{\frac{\eta_0^2}{2}} - c)}{2}\int_{\hat{\tau}}^{\infty}\frac{e^{-x}\ln\frac{2x}{\eta_0^2}}{x}dx + \frac{\eta_0^6 e^{\frac{\eta_0^2}{2}}}{16}\int_{\hat{\tau}}^{\infty}\frac{e^{-x}\left(\ln\frac{2x}{\eta_0^2}\right)^2}{x}dx + dE_1(\hat{\tau}) \tag{4.78}$$

where d is a constant. By (4.64), (4.68) and (4.70) we have

$$T_{s2}^{(1)} = 0 \tag{4.79}$$

From (4.69) we know that $\eta_2^{(1)}$ must be a constant. (4.69) and (4.71) give

$$\eta_2^{(1)} = \hat{\alpha}^2 \hat{\eta}_2 \quad ,$$

$$\hat{\eta}_2 = \frac{D_1}{D_0} \quad , \tag{4.80}$$

$$d = \frac{D_2}{D_0} \tag{4.81}$$

where

$$D_0 = E_0(\frac{\eta_0^2}{2}) + E_1(\frac{\eta_0^2}{2})(\frac{2}{\eta_0^2}+1) \quad ,$$

$$D_1 = A_1 E_0(\frac{\eta_0^2}{2}) + \frac{A_2}{\eta_0^2} E_1(\frac{\eta_0^2}{2}) \quad ,$$

$$D_2 = A_2 + (2+\eta_0^2) A_1$$

and

$$A_1 = \frac{-(\eta_0^2 \hat{\eta}_1 e^{\frac{\eta_0^2}{2}} - c)}{2} \int_{\frac{\eta_0^2}{2}}^{\infty} \frac{e^{-x} \ln \frac{2x}{\eta_0^2}}{x} dx + \frac{\eta_0^4 e^{\frac{\eta_0^2}{2}}}{16} \int_{\frac{\eta_0^2}{2}}^{\infty} \frac{e^{-x} (\ln \frac{2x}{\eta_0^2})^2}{x} dx + c \hat{\eta}_1 E_0(\frac{\eta_0^2}{2}) + \frac{\hat{\eta}_1^2}{2}(1+\eta_0^2) \, ,$$

$$A_2 = -\hat{\eta}_1 [\eta_0^2 - (\eta_0^2+1) c E_0(\frac{\eta_0^2}{2})] + \frac{\hat{\eta}_1^2}{2}(2+\eta_0^2+\eta_0^4)$$

(2). The second part of the solution is subject to the following system:

$$Pr[\frac{\partial^2 \zeta_2^{(2)}}{\partial \xi^2} + \frac{\partial^2 \zeta_2^{(2)}}{\partial \eta^2} - \frac{1}{\xi}\frac{\partial \zeta_2^{(2)}}{\partial \xi} - \frac{1}{\eta}\frac{\partial \zeta_2^{(2)}}{\partial \eta}] = -\eta_0^2 \eta \frac{\partial \zeta_2^{(2)}}{\partial \eta} + \eta_0^2 \xi \frac{\partial \zeta_2^{(2)}}{\partial \xi} \tag{4.82}$$

$$\frac{\partial^2 \Psi_2^{(2)}}{\partial \xi^2} + \frac{\partial^2 \Psi_2^{(2)}}{\partial \eta^2} - \frac{1}{\xi}\frac{\partial \Psi_2^{(2)}}{\partial \xi} - \frac{1}{\eta}\frac{\partial \Psi_2^{(2)}}{\partial \eta} = -\eta_0^4 (\xi^2 + \eta^2) \zeta_2^{(2)} \tag{4.83}$$

$$\frac{\partial^2 T_2^{(2)}}{\partial \xi^2} + \frac{\partial^2 T_2^{(2)}}{\partial \eta^2} + \frac{1}{\xi}\frac{\partial T_2^{(2)}}{\partial \xi} + \frac{1}{\eta}\frac{\partial T_2^{(2)}}{\partial \eta}$$

$$= \eta_0^2 \xi \frac{\partial T_2^{(2)}}{\partial \xi} - \eta_0^2 \eta \frac{\partial T_2^{(2)}}{\partial \eta} + \frac{e^{\frac{\eta_0^2}{2} - \frac{\eta_0^2 \eta^2}{2}}}{\xi \eta^2} \frac{\partial \Psi_2^{(2)}}{\partial \xi} \tag{4.84}$$

$$\frac{\partial^2 T_{s2}^{(2)}}{\partial \xi^2} + \frac{\partial^2 T_{s2}^{(2)}}{\partial \eta^2} + \frac{1}{\xi}\frac{\partial T_{s2}^{(2)}}{\partial \xi} + \frac{1}{\eta}\frac{\partial T_{s2}^{(2)}}{\partial \eta} = \eta_0^2 \xi \frac{\partial T_{s2}^{(2)}}{\partial \xi} - \eta_0^2 \eta \frac{\partial T_{s2}^{(2)}}{\partial \eta} \tag{4.85}$$

(i) As $\eta \to \infty$:

$$\Psi_2^{(2)} \to 0 \quad , \quad \zeta_2^{(2)} \to 0 \tag{4.86}$$

$$T_2^{(2)} \to 0 \tag{4.87}$$

(ii) As $\eta \to 0$:

$$\frac{\partial T_{s2}^{(2)}}{\partial \eta} = 0 \quad , \quad T_{s2}^{(2)} = O(1) \tag{4.88}$$

(iii) At $\eta = 1$:

$$T_2^{(2)} = T_{s2}^{(2)} + \eta_2^{(2)} \eta_0^2 \tag{4.89}$$

$$T_{s2}^{(2)} = -\frac{\eta_0^2(\xi^2+2)}{(\xi^2+1)^{\frac{3}{2}}} \tag{4.90}$$

$$\frac{\partial}{\partial \eta}(T_2^{(2)} - T_{s2}^{(2)}) + \eta_2^{(2)}[\eta_0^4 + 2\eta_0^2] + \eta_0^2 \xi \frac{d\eta_2^{(2)}}{d\xi} = 0 \tag{4.91}$$

$$\frac{\partial \Psi_2^{(2)}}{\partial \xi} = 0 \tag{4.92}$$

$$\frac{\partial \Psi_2^{(2)}}{\partial \eta} = 0 \tag{4.93}$$

This system is actually the same as that discussed in Xu's paper [21]. In the following, I shall follow the approach described in [21] and give the results, some typos and algebraic errors in [21] are found, which are corrected hereby.

By (4.82)-(4.83), (4.86) and (4.92)-(4.93), the solution for flow fields is obtained as

$$\zeta_2^{(2)} = 0 \quad , \tag{4.94}$$

$$\psi_2^{(2)} = 0 \tag{4.95}$$

Letting

$$\hat{\xi} = \eta_0 \xi \tag{4.96}$$

$$\hat{\eta} = \eta_0 \eta \tag{4.97}$$

the (4.84) and (4.85) can be transformed to

$$\frac{\partial^2 T_2^{(2)}}{\partial \hat{\xi}^2} + \frac{\partial^2 T_2^{(2)}}{\partial \hat{\eta}^2} + (\frac{1}{\hat{\xi}} - \hat{\xi})\frac{\partial T_2^{(2)}}{\partial \hat{\xi}} + (\frac{1}{\hat{\eta}} + \hat{\eta})\frac{\partial T_2^{(2)}}{\partial \hat{\eta}} = 0 \quad , \tag{4.98}$$

$$\frac{\partial^2 T_{s2}^{(2)}}{\partial \hat{\xi}^2} + \frac{\partial^2 T_{s2}^{(2)}}{\partial \hat{\eta}^2} + (\frac{1}{\hat{\xi}} - \hat{\xi})\frac{\partial T_{s2}^{(2)}}{\partial \hat{\xi}} + (\frac{1}{\hat{\eta}} + \hat{\eta})\frac{\partial T_{s2}^{(2)}}{\partial \hat{\eta}} = 0 \tag{4.99}$$

By the method of separation of variables, we write

$$T_2^{(2)}(\hat{\xi},\hat{\eta}) = X(\hat{\xi})\,Y(\hat{\eta}) \quad , \tag{4.100}$$

and from (4.98) we derive

$$X'' + (\frac{1}{\hat{\xi}} - \hat{\xi})X' + \lambda_1^2 X = 0 \quad , \tag{4.101}$$

$$Y'' + (\frac{1}{\hat{\eta}} + \hat{\eta})Y' - \lambda_1^2 Y = 0 \tag{4.102}$$

Letting

$$\hat{\sigma} = \frac{\hat{\xi}^2}{2} = \frac{\eta_0^2 \xi^2}{2}$$

(4.101) becomes the Kummer equation:

$$\hat{\sigma} X''(\hat{\sigma}) + (1-\hat{\sigma})X'(\hat{\sigma}) + \frac{\lambda_1^2}{2} X = 0 \tag{4.103}$$

The fundamental solutions of (4.103) are

$$X(\hat{\sigma}) = \begin{cases} M(-\frac{\lambda_1^2}{2},1,\hat{\sigma}) \\ U(-\frac{\lambda_1^2}{2},1,\hat{\sigma}) \end{cases} \tag{4.104}$$

where M and U are the confluent hypergeometric functions. $M(-\lambda_1^2/2,1,\hat{\sigma})$ is regular at $\hat{\xi} = \hat{\sigma} = 0$ and $U(-\lambda_1^2/2,1,\hat{\sigma})$ has a logarithmic singularity at $\hat{\xi} = \hat{\sigma} = 0$.

We choose

$$X(\hat{\sigma}) = M(-\frac{\lambda_1^2}{2},1,\hat{\sigma}) \tag{4.105}$$

$M(-\lambda_1^2/2,1,\hat{\sigma})$ has following properties:

As $\hat{\sigma} \to \infty$,

1) $M(-\lambda_1^2/2,1,\hat{\sigma})$ grows algebraically when

$$\frac{\lambda_1^2}{2} = n = 0,\ 1,\ 2, \cdots$$

2) Otherwise $M(-\lambda_1^2/2,1,\hat{\sigma})$ grows exponentially.

We require that the solution does not grow too fast at the far field, therefore, we must set

$$\frac{\lambda_1^2}{2} = n = 0,\ 1,\ 2, \cdots \tag{4.106}$$

hence,

$$X(\hat{\sigma}) = M(-n,1,\hat{\sigma}) = L_n(\frac{\hat{\xi}^2}{2}) = L_n(\frac{\eta_0^2\xi^2}{2}) \tag{4.107}$$

where L_n are the Laguerre polynomials.

(i) In the liquid- phase region:

To solve (4.102), we let

$$\hat{\tau} = \frac{\hat{\eta}^2}{2} = \frac{\eta_0^2\eta^2}{2} \quad ,$$

$$Y(\hat{\eta}) = \frac{1}{\hat{\eta}} e^{-\frac{\hat{\eta}^2}{4}} Z(\hat{\tau}) \tag{4.108}$$

Then (4.102) becomes the Whittaker equation:

$$Z''(\hat{\tau}) + (-\frac{1}{4} + \frac{\lambda}{\hat{\tau}} + \frac{\frac{1}{4}-\mu}{\hat{\tau}^2})Z(\hat{\tau}) = 0 \tag{4.109}$$

where

$$\lambda = -(n + \frac{1}{2}) \quad , \quad \mu = 0 \tag{4.110}$$

We require that the solution $Y(\hat{\eta})$ vanishes exponentially as $\hat{\eta} \to \infty$. Thus, we have

$$Y(\hat{\eta}) = e^{-\frac{\hat{\eta}^2}{2}} U(n+1,1,\frac{\hat{\eta}^2}{2}) = e^{-\frac{\eta_0^2\eta^2}{2}} U(n+1,1,\frac{\eta_0^2\eta^2}{2}) \tag{4.111}$$

Finally, the solution of (4.84) is

$$T_2^{(2)}(\xi,\eta) = \sum_{n=0}^{\infty} \beta_n L_n(\frac{\eta_0^2\xi^2}{2}) \frac{U(n+1,1,\frac{\eta_0^2\eta^2}{2}) e^{-\frac{\eta_0^2\eta^2}{2}}}{U(n+1,1,\frac{\eta_0^2}{2}) e^{-\frac{\eta_0^2}{2}}} \tag{4.112}$$

where β_n are arbitrary constants to be determined.

(ii) In the solid-phase region:

We still use the method of separation of variables and write

$$T_{s2}^{(2)}(\hat{\xi},\hat{\eta}) = X(\hat{\xi}) Y(\hat{\eta}) \tag{4.113}$$

It can be shown that the solution $X(\hat{\xi})$ is the same as (4.107). But in order to solve $Y(\hat{\eta})$, we must introduce the new variable:

$$\tilde{\tau} = -\frac{\hat{\eta}^2}{2} \tag{4.114}$$

Thus, we derive

$$\tilde{\tau} Y''(\tilde{\tau}) + (1-\tilde{\tau}) Y'(\tilde{\tau}) + nY = 0 \tag{4.115}$$

The solution of (4.115) satisfying the regular condition at $\tilde{\tau} = \eta = 0$ is

$$Y(\tilde{\tau}) = L_n(-\frac{\hat{\eta}^2}{2}) = L_n(-\frac{\eta_0^2\eta^2}{2}) \tag{4.116}$$

Finally, the solution of (4.85) is

$$T_{s2}^{(2)}(\xi,\eta) = \sum_{n=0}^{\infty} \frac{\alpha_n L_n(\frac{\eta_0^2\xi^2}{2}) L_n(-\frac{\eta_0^2\eta^2}{2})}{L_n(-\frac{\eta_0^2}{2})} \tag{4.117}$$

where α_n are arbitrary constants to be determined. The solutions (4.112) and (4.117) are exactly the same as that derived in [21].

Suppose

$$\eta_2^{(2)}(\xi) = \sum_{n=0}^{\infty} \gamma_n L_n(\frac{\eta_0^2\xi^2}{2}) \tag{4.118}$$

where γ_n are constants. From the above, it is seen that the solution contains three sets of arbitrary constants: $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, n=0,1,2⋯, which should be determined by the boundary conditions (4.89), (4.90) and (4.91).

From the boundary condition (4.89) we get

$$\beta_n = \alpha_n + \eta_0^2\gamma_n \tag{4.119}$$

From the boundary condition (4.90) we get

$$\sum_{n=0}^{\infty} \alpha_n L_n(\frac{\eta_0^2\xi^2}{2}) = -\frac{\eta_0^2(\xi^2+2)}{(\xi^2+1)^{\frac{3}{2}}} \tag{4.120}$$

The constants $\{\alpha_n\}$ represent the effect of surface tension. For the zero surface tension case, we have $\alpha_n \equiv 0$, n=0,1,2,⋯. From the boundary condition (4.91) we get

$$\sum_{n=0}^{\infty}(a_n-b_n)L_n(\frac{\eta_0^2\xi^2}{2}) + (2\eta_0^2+\eta_0^4)\sum_{n=0}^{\infty}\gamma_n L_n(\frac{\eta_0^2\xi^2}{2}) + 2\sum_{n=0}^{\infty}\eta_0^2 n\gamma_n[L_n(\frac{\eta_0^2\xi^2}{2}) - L_{n-1}(\frac{\eta_0^2\xi^2}{2})]$$
$$= 0 \tag{4.121}$$

where

$$a_n = 2\beta_n[(n+1)^2 \frac{U(n+2,1,\frac{\eta_0^2}{2})}{U(n+1,1,\frac{\eta_0^2}{2})} - (n+1) - \frac{\eta_0^2}{2}] \ , \tag{4.122}$$

$$b_n = 2n\alpha_n[1 - \frac{L_{n-1}(-\frac{\eta_0^2}{2})}{L_n(-\frac{\eta_0^2}{2})}] \tag{4.123}$$

From (4.120) we get

$$\alpha_n = -\frac{\sqrt{\pi}(2n)! e^{\frac{\eta_0^2}{2}} \eta_0^4}{2n!} \{ \frac{\sqrt{2}}{\eta_0} i^{2n} erfc(\frac{\eta_0}{\sqrt{2}}) + 2(2n+1) i^{2n+1} erfc(\frac{\eta_0}{\sqrt{2}}) \} \tag{4.124}$$

where

$$i^n erfcz = \frac{2}{\sqrt{\pi}} \int_z^\infty \frac{(t-z)^n}{n!} e^{-t^2} dt \tag{4.125}$$

are repeated integrals of the error functions, which have recurrence relations:

$$i^n erfcz = -\frac{z}{n} i^{n-1} erfcz + \frac{1}{2n} i^{n-2} erfcz \ ,$$

$$i^{-1} erfcz = \frac{2}{\sqrt{\pi}} e^{-z^2} \ , \tag{4.126}$$

$$i^0 erfcz = erfcz \quad , \quad \textit{which is the error function}$$

From (4.121) we get

$$(a_n - b_n) + (2\eta_0^2 + \eta_0^4)\gamma_n + 2n\eta_0^2\gamma_n - 2(n+1)\eta_0^2\gamma_{n+1} = 0 \tag{4.127}$$

(4.127) implies

$$\gamma_{n+1} = r_n\gamma_n + s_n \tag{4.128}$$

where r_n and s_n can be obtained by (4.121)-(4.126):

$$r_n = (n+1)\frac{U(n+2,1,\frac{\eta_0^2}{2})}{U(n+1,1,\frac{\eta_0^2}{2})} \ , \quad n=0,1,2,\cdots, \tag{4.129}$$

$$s_n = -\frac{\alpha_n}{\eta_0^2}[1 + \frac{\eta_0^2}{2(n+1)} - (n+1)\frac{U(n+2,1,\frac{\eta_0^2}{2})}{U(n+1,1,\frac{\eta_0^2}{2})}] - \frac{b_n}{2(n+1)\eta_0^2} \quad , \; n=0,1,2,\cdots \tag{4.130}$$

Therefore, for any given γ_0, (4.127) will generate the series $\{\gamma_n\}$. Then the second-order approximation of the interface $\eta_2^{(2)}(\gamma_0,\xi)$ is determined. In order to determine γ_0, one can apply the far field condition proposed by Xu in [21]. Namely, we require that

$$\lim_{\xi\to\infty} \eta_2^{(2)}(\gamma_0,\xi) = 0$$

The results about α_n, β_n, γ_n are the same as that in [21], some errors involved in the formulas (3.35), (3.36), (3.39) in [21] are found and corrected hereby.

Finally, the second-order approximation is obtained as

$$\begin{aligned}
\zeta_2 &= \zeta_2^{(1)} + \zeta_2^{(2)} \quad , \\
\Psi_2 &= \Psi_2^{(1)} + \Psi_2^{(2)} \quad , \\
T_2 &= T_2^{(1)} + T_2^{(2)} \quad , \\
T_{s2} &= T_{s2}^{(1)} + T_{s2}^{(2)} \quad , \\
\eta_2 &= \eta_2^{(1)} + \eta_2^{(2)}
\end{aligned} \tag{4.131}$$

or

$$\zeta_2 = 0 \quad ,$$

$$\Psi_2 = \hat{\alpha}^2\eta_0^4\hat{\eta}_1\xi^2 \quad ,$$

$$T_2(\xi,\eta) = \sum_{n=0}^{\infty}\beta_n L_n(\frac{\eta_0^2\xi^2}{2})\frac{U(n+1,1,\frac{\eta_0^2\eta^2}{2})e^{-\frac{\eta_0^2\eta^2}{2}}}{U(n+1,1,\frac{\eta_0^2}{2})e^{-\frac{\eta_0^2}{2}}} + \hat{\alpha}^2\hat{T}_2(\frac{\eta_0^2\eta^2}{2}) \quad , \tag{4.132}$$

$$T_{s2}(\xi,\eta) = \sum_{n=0}^{\infty} \frac{\alpha_n L_n(\frac{\eta_0^2\xi^2}{2}) L_n(-\frac{\eta_0^2\eta^2}{2})}{L_n(-\frac{\eta_0^2}{2})} ,$$

$$\eta_2 = \sum_{n=0}^{\infty} \gamma_n L_n(\frac{\eta_0^2\xi^2}{2}) + \hat{\alpha}^2\hat{\eta}_2$$

In second-order approximation, the vorticity is still zero everywhere. The density change affects the interface shape, velocity field and thermal field. The interface shape and temperature field are also perturbed by the surface tension.

4.5 Discussion

In this chapter, in terms of regular perturbation method, we have obtained a second-order approximate solution for the problem. The surface tension parameter ϵ is used as the small parameter. The density change parameter α is considered as the same order of small quantity as ϵ. The dendritic growth is perturbed by both the surface tension and the convection induced by density change.

We found that the solid-liquid interface shape and the temperature field can be expressed in the form:

$$\eta = 1 + \{\alpha\hat{\eta}_1 + \alpha^2\hat{\eta}_2 + ...\} + \{\frac{\Gamma}{\eta_0^4}\sum_{n=0}^{\infty}\gamma_n L_n(\frac{\eta_0^2\xi^2}{2}) + ...\} , \tag{4.133}$$

$$T = T_0 + \{\alpha\hat{T}_1 + \alpha^2\hat{T}_2 + ...\} + \{\frac{\Gamma}{\eta_0^4} T_2^{(2)} + ...\} \tag{4.134}$$

where the first terms in the R.H.S. are the Ivantsov's solution; the second terms are the correction by the convection induced by the density change; the third terms are the correction by the surface tension.

The fluid motion is found to be irrotational flow. The velocity components are given by

$$u_\xi = \frac{\xi}{\sqrt{\xi^2+\eta^2}} + o(\alpha^2) + \dots, \tag{4.135}$$

$$u_\eta = -\frac{\eta}{\sqrt{\xi^2+\eta^2}} - \{\frac{\alpha}{\eta\sqrt{\xi^2+\eta^2}} + \frac{2\alpha^2\hat{\eta}_1}{\eta\sqrt{\xi^2+\eta^2}} + \cdots\} \tag{4.136}$$

The first term in the R.H.S. is uniform flow; the second term is the effect of the density change. The surface tension does not affect the flow field.

It can be seen, from the above analysis, that some physical effects in dendrite growth, such as density change and surface tension, can be separately considered. The total effect of these parameters equals the algebraic summation of each individual effect, provided these parameters are all small. Thus, in the following chapters, when we study dendrite growth problem with other type of convection motion, we shall put the effect of surface tension away, and only consider the case of zero surface tension.

Chapter V

SOME SPECIAL SOLUTIONS FOR DENDRITE GROWTH WITH EXTERNAL FLOW

In 1990 Ananth and Gill considered the situation in which convection is purely induced by external flow with zero surface tension (See [3]). In their papers, it is shown that, in general, the system does not allow a similarity solution. But, in several limiting cases, they do find the similarity solution for the system, the interface shape in these cases is paraboloid of revolution. In this chapter I shall summarize some of their results.

5.1 Formulation

Suppose that surface tension is zero, no density change, buoyancy effect is negligible, namely, $\alpha = 0$ and $G = 0$. The entire solid phase in this case will be isothermal with the temperature $\overline{T}_m$. Assume that the melt in the far field flows along z-axis with a constant velocity $\overline{U}_\infty$. The mathematical formulation of this problem can be reduced from (3.46)-(3.62) by setting $\alpha = 0$, $G = 0$ as follows:

1.

$$D^2\Psi = -\eta_0^4(\xi^2 + \eta^2)\zeta \tag{5.1}$$

2.

$$PrD^2\zeta = \eta_0^4(\xi^2+\eta^2)\frac{\partial\zeta}{\partial t} + \frac{2\zeta}{\eta_0^4\xi^2\eta^2}\frac{\partial(\Psi,\eta_0^2\xi\eta)}{\partial(\xi,\eta)} - \frac{1}{\eta_0^2\xi\eta}\frac{\partial(\Psi,\zeta)}{\partial(\xi,\eta)} \tag{5.2}$$

3.

$$\nabla^2 T = \eta_0^4(\xi^2+\eta^2)\frac{\partial T}{\partial t} + \frac{1}{\eta_0^2\xi\eta}(\frac{\partial\Psi}{\partial\eta}\frac{\partial T}{\partial\xi} - \frac{\partial\Psi}{\partial\xi}\frac{\partial T}{\partial\eta}) \tag{5.3}$$

The boundary conditions are given as follows:

1. As $\eta \rightarrow \infty$:

$$\mathbf{u} = -(U_\infty + 1)\mathbf{e}_\eta$$

$$\zeta \rightarrow 0 \tag{5.4}$$

$$T \rightarrow T_\infty$$

2. At the interface $\eta = \eta_s(\xi,t)$:

(i).

$$T = 0 \tag{5.5}$$

(ii).

$$(\frac{\partial T}{\partial\eta} - \eta_s'\frac{\partial T}{\partial\xi}) + \eta_0^2(\xi\eta_s)' + \eta_0^4(\xi^2+\eta^2)\frac{\partial\eta_s}{\partial t} = 0 \tag{5.6}$$

(iii).

$$(\frac{\partial\Psi}{\partial\xi} + \eta_s'\frac{\partial\Psi}{\partial\eta}) = \eta_0^4(\xi\eta_s)(\xi\eta_s)' \tag{5.7}$$

(iv).

$$(\frac{\partial\Psi}{\partial\eta} - \eta_s'\frac{\partial\Psi}{\partial\xi}) + \eta_0^4(\xi\eta_s)(\eta_s\eta_s' - \xi) = 0 \tag{5.8}$$

Suppose the stream function be following form

$$\Psi = \eta_0^4\xi^2 f(\eta) \quad , \tag{5.9}$$

$\eta = 1$ as the solid-liquid interface and $T = T(\eta)$. Then (3.44) and (3.45) give the fluid velocity components as

$$u_\xi = \frac{\xi}{\sqrt{\xi^2+\eta^2}}\frac{f}{\eta} \tag{5.10}$$

$$u_\eta = -\frac{2}{\sqrt{\xi^2+\eta^2}}\frac{f}{\eta} \tag{5.11}$$

and by (5.1) the vorticity is

$$\zeta = -\frac{\xi^2}{\xi^2+\eta^2}F \tag{5.12}$$

where

$$F = \eta\left(\frac{f}{\eta}\right)' \tag{5.13}$$

By (5.1) and (5.2) we have the differential equation for $f(\eta)$

$$\eta\left[\frac{1}{\eta^2}\left(\eta F' + \frac{2\eta_0^2}{Pr}fF\right)\right]' - \frac{4}{\xi^2+\eta^2}\left[\eta F' + \frac{\eta_0^2 F}{Pr}\left(f+\eta\frac{f}{2}\right)\right] = 0 \tag{5.14}$$

There is a contradiction to the assumption that f depends on η only because of the existence of ξ^2 in (5.14). So only in some special cases we can use (5.14) to get the exact solution. In the following sections we shall consider some important special cases.

By substituting (5.9) into the boundary conditions (5.4)-(5.8), we obtain the following formulas, respectively,

$$f(1) = \frac{1}{2} \ , \tag{5.15}$$

$$f'(1) = 1 \ , \tag{5.16}$$

$$\frac{f}{\eta}\Big|_{\eta\to\infty} = 1 + U_\infty \ , \tag{5.17}$$

$$\frac{f}{\eta^2}\Big|_{\eta\to\infty} = \frac{1}{2} + \frac{1}{2}U_\infty \ , \tag{5.18}$$

$$T(1) = 1 \ , \tag{5.19}$$

$$\frac{dT}{d\eta}(1) = -\eta_0^2 \ , \tag{5.20}$$

$$T(\infty) = 0 \tag{5.21}$$

A note should be made here: the notation used in this thesis is slightly different from that used in the papers by Ananth & Gill. With our notations, the parameters used by Ananth & Gill become: $P_G = Vl_T/\kappa_T = 1$, the growth Peclet number; $Pe = \overline{U}_\infty l_T/\kappa_T = U_\infty$, the flow Peclet number;

$Re = \overline{U}_\infty l_T/\nu = \frac{U_\infty}{Pr}$, the Reynolds number.

From (5.3) and its boundary conditions given by (5.19) and (5.21) we obtain the temperature T as

$$T = \int_\eta^\infty \frac{\eta_0^2 \exp(-\int_1^\eta \frac{2\eta_0^2 f}{\eta} d\eta)}{\eta} d\eta + T_\infty \tag{5.22}$$

where

$$T_\infty = -\int_1^\infty \frac{\eta_0^2 \exp(-\int_1^\eta \frac{2\eta_0^2 f}{\eta} d\eta)}{\eta} d\eta \tag{5.23}$$

5.2 A Solution for Dendrite Growth with Stokes Flow

Suppose $Pr \to \infty$, that means the Reynolds number Re $\to 0$ Then (5.14) is written as

$$(\frac{F}{\eta})' - \frac{4}{\xi^2+\eta^2} F = 0 \tag{5.24}$$

(5.24) is satisfied exactly for all values of ξ and η if

$$F = [\eta(\frac{f}{\eta})']' = 0 \tag{5.25}$$

Solving (5.25) with its boundary conditions (5.15) and (5.16) one gets

$$f(\eta) = \frac{1}{2}\eta^2 + s[\eta^2(\ln\eta - \frac{1}{2}) + \frac{1}{2}] \tag{5.26}$$

where s is an arbitrary constant.

Then the velocity components are given immediately by (5.10) and (5.11) as

$$u_\xi = \frac{\xi}{\sqrt{\xi^2+\eta^2}}(1 + 2s\ln\eta) \tag{5.27}$$

$$u_\eta = -\frac{\eta}{\sqrt{\xi^2+\eta^2}}(1 + 2s\ln\eta + \frac{s}{\eta^2} - s) \tag{5.28}$$

The thermal field T is given by (5.22).

5.3 A Solution for Dendrite Growth with Oseen Viscous Flow

One can use Oseen approximation to linearize (5.14) if the Reynolds number Re is very small.

Let $f = (1+U_\infty)\eta^2/2$, its far field value, and substitute it into (5.14) to get

$$\eta[\frac{1}{\eta^2}(\eta F' + \eta_0^2 \mathrm{Re}_1 \eta^2 F)]' - \frac{4}{\xi^2+\eta^2}[\eta F' + \eta_0^2 \mathrm{Re}_1 \eta^2 F] = 0 \tag{5.29}$$

where

$$\mathrm{Re}_1 = \frac{1}{Pr}(U_\infty + 1)$$

The solution of (5.29) for all values of ξ and η is

$$F' + \eta_0^2 \mathrm{Re}_1 \eta F = 0 \tag{5.30}$$

The boundary conditions of (5.30) are (5.15)-(5.18).

Solving (5.30) with its boundary conditions one obtains

$$f(\eta) = \frac{\eta f_1(\eta) - 1}{2} \tag{5.31}$$

where

$$f_1(z) = zU_\infty[1 + \frac{1}{U_\infty} + \frac{1}{U_\infty z^2} - \frac{E_1(\eta_0^2 \mathrm{Re}_1 z^2/2)}{E_1(\eta_0^2 \mathrm{Re}_1/2)}$$

$$- \frac{2}{z^2 \mathrm{Re}_1 \eta_0^2} \frac{\exp(-\eta_0^2 \mathrm{Re}_1/2) - \exp(-\eta_0^2 \mathrm{Re}_1 z^2/2)}{E_1(\eta_0^2 \mathrm{Re}_1/2)}] \tag{5.32}$$

So we can get the velocity components from (5.10) and (5.11):

$$u_\xi = \frac{U_\infty \xi}{\sqrt{\xi^2+\eta^2}}[1 + \frac{1}{U_\infty} - \frac{E_1(\eta_0^2 \mathrm{Re}_1 \eta^2/2)}{E_1(\eta_0^2 \mathrm{Re}_1/2)}] \ , \tag{5.33}$$

$$u_\eta = -\frac{U_\infty \eta}{\sqrt{\xi^2+\eta^2}}[1 + \frac{1}{U_\infty} - \frac{E_1(\eta_0^2 \mathrm{Re}_1 \eta^2/2)}{E_1(\eta_0^2 \mathrm{Re}_1/2)}$$

$$- \frac{2}{\eta_0^2\eta^2 \mathrm{Re}_1} \frac{\exp(-\eta_0^2\mathrm{Re}_1/2) - \exp(-\eta_0^2\mathrm{Re}_1\eta^2/2)}{E_1(\eta_0^2\mathrm{Re}_1/2)}] \tag{5.34}$$

By (5.22) the thermal field T is

$$T = \int_{\eta}^{\infty} \eta_0^2 \exp[-\int_1^y \eta_0^2 f_1(z)dz]\,dy - \int_1^{\infty} \eta_0^2 \exp[-\int_1^y \eta_0^2 f_1(z)dz]\,dy \tag{5.35}$$

5.4 A Solution for Dendrite Growth with Potential Flow

If $Pr \to 0$, that means the Reynolds number $\mathrm{Re} \to \infty$, the viscous terms in the Navier-Stokes equation (5.2) are negligible. Then one can obtain

$$(fF)' - \frac{2}{\xi^2+\eta^2} F(f + \frac{\eta f}{2}) = 0 \tag{5.36}$$

By observing (5.36) if

$$F = \eta(\frac{f}{\eta})' = 0 \tag{5.37}$$

then (5.36) is satisfied.

The boundary conditions of (5.37) are (5.15) and (5.16). So one can obtain the fluid field by solving (5.37) and its boundary conditions:

$$f = \frac{\eta^2}{2}(1 + U_\infty) - \frac{U_\infty}{2} \tag{5.38}$$

$$u_\xi = \frac{\xi}{\sqrt{\xi^2+\eta^2}}(1 + U_\infty) \tag{5.39}$$

$$u_\eta = -\frac{1}{\sqrt{\xi^2+\eta^2}}[(1 + U_\infty)\eta - \frac{U_\infty}{\eta}] \tag{5.40}$$

By (5.22) the temperature is

$$T = \frac{\eta_0^2}{2} e^{\frac{1}{2}\eta_0^2(1+U_\infty)} [\frac{\eta_0^2(1+U_\infty)}{2}]^{-\frac{\eta_0^2 U_\infty}{2}} \Gamma[\frac{\eta_0^2 U_\infty}{2}, \frac{\eta_0^2\eta(1+U_\infty)}{2}]$$

$$- \frac{\eta_0^2}{2} e^{\frac{\eta_0^2(1+U_\infty)}{2}} [\frac{\eta_0^2(1+U_\infty)}{2}]^{-\frac{\eta_0^2 U_\infty}{2}} \Gamma[\frac{\eta_0^2 U_\infty}{2}, \frac{\eta_0^2(1+U_\infty)}{2}] \tag{5.41}$$

where

$$\Gamma(a,x) = \int_x^\infty e^{-t} t^{a-1} dt \ , \tag{5.42}$$

the incomplete Gamma function.

5.5 Discussion

In this chapter I summarize the results derived by Ananth and Gill. We list the solutions for some special situations, and show that

1. For the limiting cases, Stokes's flow, Oseen viscous flow and potential flow, steady similarity solution of dendrite growth can be found. It is said that the theoretical results agree well with the experimental data (Ref. [3]).
2. The contradiction in (5.14) implies that the stream function does not have the form $\Psi = \eta_0^4 \xi^2 f(\eta)$, namely, the system does not allow a similarity solution.

Chapter VI

EFFECT OF FLUID FLOW DUE TO BUOYANCE

In 1990 Canright and Davis considered the effects of buoyant flow at the near-tip regions of dendrites. They presented a theoretical analysis for dendritic growth in the tip region and obtained a special solution. In this chapter I will summarize their works.

6.1 Formulation

Suppose that surface tension is zero ($\epsilon = 0$), no density change and no external flow are in this problem, namely, $\alpha = 0$ and $U_\infty = 0$, and the entire solid phase is isothermal at $\bar{T}_m$. Due to the buoyancy effect, convective motion in the melt is produced. Then the system can be reduced from (3.46)-(3.62) by setting $\alpha = 0$, $U_\infty = 0$ as follows:

1.

$$D^2\Psi = -\eta_0^4(\xi^2 + \eta^2)\zeta \tag{6.1}$$

2.

$$PrD^2\zeta = \eta_0^4(\xi^2+\eta^2)\frac{\partial\zeta}{\partial t} + \frac{2\zeta}{\eta_0^4\xi^2\eta^2}\frac{\partial(\Psi,\eta_0^2\xi\eta)}{\partial(\xi,\eta)} - \frac{1}{\eta_0^2\xi\eta}\frac{\partial(\Psi,\zeta)}{\partial(\xi,\eta)}$$

$$- \frac{G\xi\eta\eta_0^4}{T_\infty}[\eta\frac{\partial T}{\partial\xi} + \xi\frac{\partial T}{\partial\eta}] \tag{6.2}$$

3.

$$\nabla^2 T = \eta_0^4(\xi^2+\eta^2)\frac{\partial T}{\partial t} + \frac{1}{\eta_0^2\xi\eta}(\frac{\partial\Psi}{\partial\eta}\frac{\partial T}{\partial\xi} - \frac{\partial\Psi}{\partial\xi}\frac{\partial T}{\partial\eta}) \tag{6.3}$$

The boundary conditions are

1. As $\eta \to \infty$:

$$\Psi \to \frac{\eta_0^4}{2}\xi^2\eta^2 \tag{6.4}$$

$$\zeta \to 0 \tag{6.5}$$

$$T \to T_\infty \tag{6.6}$$

2. At the interface $\eta = \eta_s(\xi,t)$:

(i).

$$T = 0 \tag{6.7}$$

(ii).

$$\left(\frac{\partial T}{\partial \eta} - \eta_s'\frac{\partial T}{\partial \xi}\right) + \eta_0^2(\xi\eta_s)' + \eta_0^4(\xi^2+\eta^2)\frac{\partial \eta_s}{\partial t} = 0 \tag{6.8}$$

(iii).

$$\left(\frac{\partial \Psi}{\partial \xi} + \eta_s'\frac{\partial \Psi}{\partial \eta}\right) = \eta_0^4(\xi\eta_s)(\xi\eta_s)' \tag{6.9}$$

(iv).

$$\left(\frac{\partial \Psi}{\partial \eta} - \eta_s'\frac{\partial \Psi}{\partial \xi}\right) + \eta_0^4(\xi\eta_s)(\eta_s\eta_s' - \xi) = 0 \tag{6.10}$$

As $G << 1$, we use perturbation method to solve this problem and write

$$T = T_0 + GT_1 + \cdots ,$$

$$\Psi = \Psi_0 + G\Psi_1 + \cdots , \tag{6.11}$$

$$\zeta = \zeta_0 + G\zeta_1 + \cdots ,$$

$$\eta_s = 1 + G\eta_1 + \cdots$$

where $\eta = \eta_s$ is the interface.

6.2 A Special Tip Solution with Small Buoyancy

Assume $T_0 = T_0(\eta)$ Then, the zero-order approximation are the Ivantsov solution:

$$T_0 = e^{\frac{\eta_0^2}{2}} \frac{\eta_0^2}{2} E_1(\frac{\eta_0^2\eta^2}{2}) - e^{\frac{\eta_0^2}{2}} \frac{\eta_0^2}{2} E_1(\frac{\eta_0^2}{2})$$

$$T_\infty = -e^{\frac{\eta_0^2}{2}} \frac{\eta_0^2}{2} E_1(\frac{\eta_0^2}{2}) \tag{6.12}$$

$$\Psi_0 = \frac{\eta_0^4\xi^2\eta^2}{2}$$

$$\zeta_0 = 0$$

where E_1 is the exponential integral defined by

$$E_n(x) = \int_1^\infty \frac{e^{-xs}}{s^n} ds \ , \qquad n=0,1,2,\cdots \tag{6.13}$$

Considering the first-order approximate, we can get the equation:

$$D^2\Psi_1 = -\eta^4{}_0(\xi^2 + \eta^2)\zeta_1 \tag{6.14}$$

$$PrD^2\zeta_1 = -\eta_0^2\eta\frac{\partial\zeta_1}{\partial\eta} + \eta_0^2\xi\frac{\partial\zeta_1}{\partial\xi} - \frac{\eta_0^4\eta\xi^2}{T_\infty}\frac{dT_0}{d\eta} \tag{6.15}$$

$$\nabla^2 T_1 = \eta_0^2\xi\frac{\partial T_1}{\partial\xi} - \eta_0^2\eta\frac{\partial T_1}{\partial\eta} + \frac{1}{\eta_0^2\xi\eta}(-\frac{dT_0}{d\eta}\frac{\partial\Psi_1}{\partial\xi}) \tag{6.16}$$

with the boundary condition:

at $\eta = 1$:

$$T_1 + \eta_1 T_0' = 0 \tag{6.17}$$

$$\Psi_1 = \frac{\partial\Psi_1}{\partial\eta} = 0 \tag{6.18}$$

$$\eta_0^2(\eta_1 + \xi\frac{d\eta_1}{d\xi}) = (\frac{\partial T_1}{\partial\eta} + \eta_1 T_0'') \tag{6.19}$$

as $\eta \to \infty$:

$$T_1 \rightarrow 0 \quad , \tag{6.20}$$

$$\Psi_1 \rightarrow 0 \tag{6.21}$$

where primes denote derivatives with respect to η.

Suppose

$$\zeta_1 = \xi^2 s(\eta) \tag{6.22}$$

Then (6.15) becomes

$$Pr(\frac{d^2 s}{d\eta^2} - \frac{1}{\eta}\frac{ds}{d\eta}) + \eta_0^2 \eta \frac{ds}{d\eta} - 2\eta_0^2 s = -\frac{\eta_0^6}{T_\infty} e^{\frac{\eta_0^2}{2} - \frac{\eta_0^2 \eta^2}{2}} \tag{6.23}$$

The exponential integrals have the properties as

$$\frac{d}{dx} E_n(x) = -E_{n-1}(x) \quad , \tag{6.24}$$

$$E_{n+1}(x) = \frac{1}{n}[e^{-x} - x E_n(x)] \tag{6.25}$$

So we can obtain the solution of (6.15) as

$$\zeta_1 = \frac{\xi^2 \eta_0^2}{T_\infty (Pr-1) E_0(\frac{\eta_0^2}{2})} [E_2(\frac{\eta_0^2 \eta^2}{2}) + A E_2(\frac{\eta_0^2 \eta^2}{2Pr}) + \frac{B\eta_0^2 \eta^2}{2}] \tag{6.26}$$

where A and B are arbitrary constants.

Suppose

$$\Psi_1 = \hat{\sigma}^2 g(\hat{\tau}) + \hat{\sigma} h(\hat{\tau}) \tag{6.27}$$

Then (6.14) becomes

$$\frac{d^2 g}{d\eta^2} - \frac{1}{\eta}\frac{dg}{d\eta} = -\eta_0^4 \eta^2 s(\eta) \tag{6.28}$$

$$\frac{d^2 h}{d\eta^2} - \frac{1}{\eta}\frac{dh}{d\eta} + 8g = -\eta_0^4 \eta^2 s(\eta) \tag{6.29}$$

Solving (6.28) and (6.29) we get the stream function as

$$\Psi_1 = -\frac{2}{T_\infty (Pr-1) E_0(\tau_0)} \{\hat{\sigma}^2 [g_1(\hat{\tau}) + A Pr g_1(\frac{\hat{\tau}}{Pr}) + B\frac{\hat{\tau}^2}{2} + C\hat{\tau} + D]$$

$$+\hat{\sigma}[h_1(\hat{\tau}) + APr^2 h_1(\frac{\hat{\tau}}{Pr}) - C\hat{\tau}^2 + 2D\hat{\tau}(1-\log\hat{\tau}) + E\hat{\tau} + F]\} \tag{6.30}$$

where

$$g_1(\hat{\tau}) = E_2(\hat{\tau}) - E_3(\hat{\tau}) \quad ,$$

$$h_1(\hat{\tau}) = 2E_3(\hat{\tau}) - E_2(\hat{\tau}) \quad ,$$

$$\tau_0 = \frac{\eta_0^2}{2}$$

and C, D, E, F are arbitrary constants.

The boundary conditions for $g(\eta)$ and $h(\eta)$ are

at $\eta = 1$:

$$g = g' = 0 \tag{6.31}$$

$$h = h' = 0 \tag{6.32}$$

as $\eta \to \infty$:

$$g \to 0 \tag{6.33}$$

$$h \to 0 \tag{6.34}$$

By observing (6.30) we can not find a solution that satisfies both the interface conditions and the conditions at infinity. We choose the solution that only satisfies the interface conditions (6.31) and (6.32). So this solution is called as tip solution.

By (6.31) and (6.32) we can decide the constants with choice of B=0 and C=0:

$$A = \frac{-g_1'(\tau_0)}{g_1'(\tau_0/Pr)} \quad ,$$

$$D = -[g_1(\tau_0) + APrg_1(\frac{\tau_0}{Pr})] \quad , \tag{6.35}$$

$$E = -[h_1'(\tau_0) + APrh_1'(\frac{\tau_0}{Pr}) - 2D\log\tau_0] \quad ,$$

$$F = -[h_1(\tau_0) + APr^2 h_1(\frac{\tau_0}{Pr}) + 2D\tau_0(1-\log\tau_0) + E\tau_0] \ .$$

Suppose

$$T_1(\hat{\sigma},\hat{\tau}) = (\hat{\sigma}-1)p(\hat{\tau}) + q(\hat{\tau}) \quad , \tag{6.36}$$

$$\eta_1(\hat{\sigma}) = \tau_0[(\hat{\sigma}-1)W_0 + W_1] \tag{6.37}$$

where W_0 and W_1 are constants to be determined. Then (6.16) becomes

$$(\hat{\tau}p')' + \hat{\tau}p' - p = -T_0'\frac{g}{\eta_0^2\eta} \quad , \tag{6.38}$$

$$(\hat{\tau}q')' + \hat{\tau}q' = -T_0'(\frac{g}{\eta_0^2\eta} + \frac{h}{2\eta_0^2\eta}) \tag{6.39}$$

The boundary conditions become

at $\hat{\tau} = \tau_0$:

$$p = -\tau_0 W_0 T_0' \quad , \tag{6.40}$$

$$q = -\tau_0 W_1 T_0' \quad , \tag{6.41}$$

$$3\tau_0 W_0 = p' + \frac{W_0}{2}T_0'' \quad , \tag{6.42}$$

$$\tau_0 W_1 = q' + \frac{W_1}{2}T_0'' - 2\tau_0 W_0 \tag{6.43}$$

as $\hat{\tau} \to \infty$:

$$p \to 0 \quad , \quad q \to 0 \tag{6.44}$$

Applying variation of parameters for (6.38) and (6.39) we get the solution given by

$$p(\hat{\tau}) = \frac{-2T_\infty\{p_1(\hat{\tau}) + APrp_2(\hat{\tau}) + Dp_3(\hat{\tau}) + J[E_2(\hat{\tau}) - E_1(\hat{\tau})] + K[\hat{\tau} + 1]\}}{(Pr\text{-}1)E_1^2(\tau_0)} \tag{6.45}$$

$$q(\hat{\tau}) = \frac{-1T_\infty\{q_1(\hat{\tau}) + APrq_2(\hat{\tau}) + Dq_3(\hat{\tau}) + Eq_4(\hat{\tau}) + LE_1(\hat{\tau}) + M\}}{(Pr\text{-}1)E_1^2(\tau_0)} \tag{6.46}$$

where J, K, L, M are arbitrary constants and

$$p_1(\hat\tau) = \frac{1}{12}\{4[E_2(2\hat\tau) - 2E_1(2\hat\tau)] - 3E_1(\hat\tau)[E_2(\hat\tau) - E_1(\hat\tau)] - 2e^{-\hat\tau}[E_2(\hat\tau) - 2E_1(\hat\tau)]\} ,$$

$$p_2(\hat\tau) = \frac{1}{2}\{[E_2(\hat\tau) - E_1(\hat\tau)][2PrE_4(\frac{\hat\tau}{Pr}) - (1+2Pr)E_3(\frac{\hat\tau}{Pr}) + 2E_2(\frac{\hat\tau}{Pr}) - E_1(\frac{\hat\tau}{Pr})]$$

$$+ \frac{1}{Pr}[\hat\tau + 1][G_{22}(\hat\tau) - G_{21}(\hat\tau) - 2G_{12}(\hat\tau) + 2G_{11}(\hat\tau) + G_{02}(\hat\tau) - G_{01}(\hat\tau)]\} ,$$

$$p_3(\hat\tau) = -\{[E_2(\hat\tau) - E_1(\hat\tau)][1 - \log\hat\tau] + [\hat\tau + 1]E_1^{(2)}(\hat\tau)\} ,$$

$$q_1(\hat\tau) = \frac{1}{2}E_1^2(\hat\tau) + e^{-\hat\tau}E_1(\hat\tau) - 2E_1(2\hat\tau) , \tag{6.47}$$

$$q_2(\hat\tau) = E_1(\hat\tau)[(1-Pr)E_3(\frac{\hat\tau}{Pr}) - (2-Pr)E_2(\frac{\hat\tau}{Pr}) + E_1(\frac{\hat\tau}{Pr})]$$

$$+ \frac{1}{Pr}[(Pr-1)G_{21}(\hat\tau) + (2-Pr)G_{11}(\hat\tau) - G_{01}(\hat\tau)] ,$$

$$q_3(\hat\tau) = 2\{E_1(\hat\tau)[1 - \log\hat\tau] + e^{-\hat\tau}(\log\hat\tau - 2) - E_1^{(2)}(\hat\tau)\} ,$$

$$q_4(\hat\tau) = -e^{-\hat\tau}$$

and

$$E_1^{(2)}(\hat\tau) = \int_{\hat\tau}^{\infty}\frac{E_1(s)}{s}ds \quad , \tag{6.48}$$

$$G_{mn}(\hat\tau) = \int_{\hat\tau}^{\infty}E_m(\frac{s}{Pr})E_n(s)ds \tag{6.49}$$

From (6.44) we get

$$K = M = 0$$

From (6.40)-(6.43) the constants J and K are determined by

$$\tau_0 p'(\tau_0) - p(\tau_0) + \tau_0 p(\tau_0) = 0 \quad , \tag{6.50}$$

$$\tau_0 q'(\tau_0) + \tau_0 q(\tau_0) - p(\tau_0) = 0 \tag{6.51}$$

Then W_0 and W_1 are determined by

$$W_0 = \frac{p(\tau_0)}{2\tau_0^2} \quad , \tag{6.52}$$

$$W_1 = \frac{q(\tau_0)}{2\tau_0^2} \tag{6.53}$$

6.3 Discussion

In this chapter I summarize the results given by Canright and Davis. In this chapter the buoyancy parameter G is considered as a very small parameter. Then the perturbation method is applied to solve the dendritic growth with convection. This chapter gives a local solution in the near tip region.

The results show that the shape of dendrite can be expressed in the form:

$$\eta_s = 1 + G[(\frac{\eta_0^2\xi^2}{2} - 1)W_0 + W_1] + \ldots \tag{6.54}$$

where the first terms in the R.H.S. are the Ivantsov solution; the second terms are the correction by the convection induced by the effect of buoyance.

It is noted by Canright & Davis that their solution for this problem does not satisfy the far field conditions; their solution has a singularity at the infinity. The uniformly valid global solution for the problem of steady dendrite growth is still unknown.

Chapter VII

SUMMARY

The present thesis is mainly dealing with the regular solutions for steady dendritic growth with convection. At first, before I present the theoretical results on the problem, in chapter II, I summarize some experimental observations of convection effect on dendritic growth.

In chapter III, a complete mathematical formulation of the problem, including governing equations and their boundary conditions, is specified. Besides the undercooling parameter T_∞ and the surface tension parameter ϵ, the system contains three physical parameters that can produce convection: 1) α, measuring the density change; 2) G, measuring the buoyancy effect; 3) U_∞, measuring the external flow. We then use regular perturbation method to discuss these effects, respectively.

In chapter IV, the second-order approximate solution is obtained analytically for the case $\alpha \neq 0$, $U_\infty = G = 0$. The results in chapter IV show that the effects of the density change and the surface tension can be considered separately. The solution can be split into three parts: the first part is the Ivantsov solution; the second part is due to the effect of the density change; the third part is due to the effect of the surface tension.

In chapter V, I study the case: $G=\alpha=0$, $U_\infty \neq 0$, and summarize the works by Ananth & Gill. Ananth and Gill found the similarity solution for the system in some limiting cases, although the system, in general, does not allow a similarity solution.

In chapter VI, I summarize the work by Canright and Davis, which deals with the case $G \neq 0$ and $\alpha = U_\infty = 0$. Canright and Davis assume the parameter G is very small, and attempt to find the asymptotic expansion solution by using regular perturbation method. They found a local solution near the tip. Their solution, however, violates the far field conditions.

At the end, we conclude that the study of the dendritic growth with convection is far from completion. There are a lot of problems on this subject, which remain unsolved, and need a long term of research efforts in the future.

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