Holographic aspects of black holes in chiral conformal field theories

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Abstract

This thesis investigates holographic dualities for three-dimensional spacetimes. We first consider warped dualities, which involve gravity in a deformed anti-de Sitter space that is relevant for the description of the near-horizon region of extreme Kerr black holes. Using the covariant phase space formalism to compute surface charges, we show that the entropy calculated on the gravity side of the duality in generic theories described by Lagrangians with higher curvature terms matches with the result obtained in one of its proposed duals, the warped conformal field theory. In a different perspective, we then apply the linear functional method to constrain the spectrum of this theory by leveraging the modular transformation properties of its partition function. In particular, we derive a bound on the dimension of the lightest states dual to warped black holes, significantly improving known results. We subsequently enhance this bound by further modular analysis and prove analytically that in such a theory, consistency requires that charged primary states exist at or below the black hole threshold. We then generalize this result to a wide variety of such theories with charge, showing that any chiral conformal field theory with a current algebra contains (Virasoro + current algebra) primary states with non-trivial charge and dimension $h \leqslant c/24+1$, provided the level is bigger than a certain critical value. We find various examples of algebras for which this critical level is zero.

Abrégé

Cette thèse est dédiée à l'examen de dualités holographiques impliquant des espaces-temps tridimensionnels. Nous considérons tout d'abord les dualités distordues, qui mettent en jeu une théorie de la gravitation dans un espace anti-de Sitter déformé qui s'avère jouer un rôle dans la description de la région près de l'horizon du trou noir de Kerr extrême. En faisant usage du formalisme covariant de l'espace de phase pour calculer les charges de surface, nous montrons que l'entropie calculée du côté gravité de la dualité, dans des théories génériques dont la description fait intervenir des lagrangiens contenant des termes de courbure élevée, correspond au résultat obtenu dans une des théories duales potentielles, connue sous l'appellation de théorie conforme de champs distordue. En suivant une approche différente, nous dérivons également une borne sur la dimension des états les plus légers correspondant aux trous noirs distordus, améliorant ainsi les résultats connus de façon significative. Nous affinons ensuite cette borne par une analyse modulaire plus poussée et fournissons une preuve analytique que dans une telle théorie, la cohérence interne requiert la présence d'états primaires chargés au seuil caractérisant les trous noirs ou en-dessous de celui-ci. Nous généralisons enfin ce résultat à diverses théories de ce type avec charge, montrant toute théorie conforme de champs chirale avec une algèbre de courant contient des états primaires (sous l'algèbre de Virasoro + courant) possédant une charge non triviale et une dimension $h \leqslant c/24+1$, pour autant que le niveau soit plus grand qu'une certaine valeur critique. Nous trouvons différents exemples d'algèbres pour lesquelles ce niveau critique est égal à zéro.

Credits

The present thesis is based on original work appearing in the following publications:

- S. Detournay, L.-A. Douxchamps, G.-S. Ng and C. Zwikel, "Warped AdS_3 black holes in higher derivative gravity theories", *JHEP* **06** (2016) 014, `arXiv:1602.09089 [hep-th]`.
- K. Colville, L.-A. Douxchamps and A. Maloney, "Flavoured chiral modular bootstrap", forthcoming.

The former is the subject of Chapter 6 with Appendices A and B. The latter is confined to Chapter 7 (apart from section 7.1 which is unpublished work), together with Appendices C, D, and E.

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This solid object might survive them all. If she threw it away it would still exist somewhere or other.

Virginia Woolf – *The Years*

Life is inevitably holographic and thousands of brushes have painted us inside and out.

Jim Harrison – *Off to the Side*

Introduction

A century ago, the physics community was only just getting acquainted with the brand new theory of general relativity, which had barely been corroborated by observations of the solar eclipse made by Eddington's expeditions. There were more wonders to come, as the advent of the concept of quantum already indicated. However, as quantum mechanics was brought into existence and more bewilderment followed, no one imagined that the next hundred years would see these two theories tested more and more accurately but no solid bridge built between them. Different avenues towards a quantum theory of gravity have been convincingly explored, but none has been able to prevail so far.

New theories generally get built on old ones when some notions appear to be in need for a reinterpretation [1]. With the derivation of Einstein's equations from crystalline structure of condensed matter systems [2] or thermodynamics [3, 4], it has been realized that gravity itself could be an emergent phenomenon and that the fundamental level of explanation was perhaps not what it was initially thought to be. The local character of field theory and general relativity can also be considered a relative notion that depends on the quantum nature of the probes used [5]. This goes hand in hand with a whole new notion of spacetime; after all, absolute spacetime is a concept best suited to point-like particles. Spacetime itself is then understood in this perspective to emerge from an underlying quantum structure, with geometry merely reflecting entanglement and gravity describing the changes caused to it by matter [6]. In a less fundamental way, holographic dualities may be seen as successfully generating one dimension of space from d others, and a theory of gravity from one that has no such notion. Indeed, simply put, the idea of these dualities is that there is a certain equivalence between a theory of gravity in $d+1$ dimensions and a d-dimensional gauge theory, i.e. a field theory without gravity. Using the appropriate "dictionary" relating both sides of the duality, one can compute quantities in either one or the other. It may for example be more convenient to perform computations in a gauge theory in flat space instead of in its stringy counterpart in curved space, or to use perturbative methods in a higher-dimensional but weakly coupled classical theory of gravity instead of doing it in its strongly coupled quantum dual.

In the search for a suitable theory of quantum gravity, gauge/gravity dualities have played a prominent role. From the first hints [7, 8] to the formulation of the AdS/CFT conjecture [9, 10, 11] the particular duality relating gravity in anti-de Sitter space (AdS) to conformal field theories (CFT) gained wide attention, but more general considerations like the holographic principle, which is the statement that relevant degrees of freedom to describe any system in a given volume are "projected out" onto the surface enclosing it [12, 13], suggest that in principle duality may take a variety of forms. Throughout the years there have been various attempts to extend it beyond the original proposal, such as dS/CFT [14], which could be more directly relevant for the world we actually live in [15]. Recently, it has been suggested that a way to go about it could be to build de Sitter space from two copies of anti-de Sitter cut, warped into hemispheres and (mathematically) glued together [16]. The holographic dual would then be a pair of conformal field theories. Beyond this example, the AdS/CFT correspondence could be regarded as a toy model of quantum gravity: even if it describes a universe unlike our own, it may point in the right direction while allowing to reach a great level of understanding.

Other possible gauge/gravity dualities include Kerr/CFT [17], which states that the near-horizon region of the four-dimensional extreme Kerr black hole could be dual to a conformal field theory, the warped dualities Warped AdS/CFT [18] and Warped AdS/Warped CFT [19, 20], which bring into play deformations of anti-de Sitter space and their putative duals, and BMS/CFT [21] which relates $(d+2)$-dimensional asymptotically flat spacetimes at null infinity to conformal field theories in dimension d. The latter spacetimes have recently aroused renewed interest as they appeared to be at the center of a proposed resolution of the black hole information paradox [22]. The warped duality, for its part, is of special interest because of its connexion to a realistic model of black holes, namely extreme Kerr mentioned above.

Black holes indeed sit at the center of the attempts to formulate a consistent theory of quantum gravity. Their thoroughly gravitational nature pervaded by intrinsically quantum behaviour makes them a crucial issue in that respect, more than ever so now that a first direct experimental detection has been achieved and that a "face" has finally be put on the name (at least for M87*). In September 2015, the LIGO-Virgo collaboration observed a signal consistent with the predicted waveform corresponding to the merger of two black holes of about 30 solar masses each, colliding at nearly one-half the speed of light [23]. The equivalent of three solar masses was emitted as gravitational waves in the blink of an eye (although 1.3 billion years ago), with a peak intensity for the event greater than the combined power of all light radiated by all the stars in the observable universe. Besides this "acoustic" rendering, there is now also a "visual" one, which the Event Horizon Telescope delivered just a few months ago [24]. The Earth-sized radio telescope obtained the first image of a much larger kind of black hole, namely the galactic supermassive one sitting in the center of M87. However, much remains to be understood about this most intriguing object that Werner Israel described as "an elemental, self-sustaining gravitational field which has severed all causal connection with the material source that created it, and settled, like a soap bubble, into the simplest configuration consistent with the external constraints" [25].

Using holographic tools to shed some light on specific features of black holes will be the focus of this thesis. In particular, the sort of thing one can do is compare relevant quantities on both sides of a proposed duality. If they match, this lends some credit to the duality and means that one can use it to access previously unattainable knowledge in one of the two pictures. This is in essence what was done by Strominger and Vafa when they accounted for the entropy of a five-dimensional string theory black hole by working in the dual theory [8]. Since then, the matching of black hole entropies is generally the first step in arguing for a potential duality. However, the quantities computed in the gravity picture depend on the particular theory of which the spacetime under consideration is a solution. This strongly restrains the scope of such a check since there can be several such theories, but picking one is necessary to perform the computations. Being able to do so in more general setups is then a much better argument in favour of a duality. This is what we will do for Warped AdS_3/Warped CFT_2.

An even more natural approach to get better knowledge of a physical theory is to use the constraints imposed by its characteristic symmetries. Conformal field theories are particularly interesting in this regard since they transform in a very specific way under the action of the modular group (i.e., the isometry group of the torus). In particular, the partition function of the most basic conformal field theories is invariant under such transformations, while more general cases depart only slightly from that behaviour and the partition function always has a form of covariance. For a while, these favourable properties have been put to use to gain some insight on the spectra of conformal field theories and assess which among them are consistent, in a research program called the modular bootstrap. Non-trivial constraints on the number of independent operators with a given conformal dimension were derived using this method [26], as well as bounds on said dimension [27]. We will draw inspiration from these works as well.

Outline of the thesis

Chapter 1 is devoted to laying out general features of black holes. In particular, it has been realized that they have thermodynamic properties and that they are actually black bodies emitting thermal radiation [28]. A corresponding statistical description of black holes, however, is at odds with the field theory description in ways that cannot be simply overcome: this is the so-called "information paradox". This designation relates to the most popular formulation of the problem, where the black hole acts as a sort of black box, containing inacessible information about the star that has collapsed into it but destroying it upon evaporation via Hawking radiation, which seems to contradict the second law of thermodynamics. The very way this puzzle is put forth is tied to our understanding of black holes' nature, and this itself depends on the kind of theory of quantum gravity we have in mind. We give a brief review of the black hole information paradox, and a summary of some of its proposed resolutions.

In Chapter 2 and 3, we review both sides of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality successively before seeing how they come together. Firstly, we review three-dimensional anti-de Sitter space, and mention that despite the absence of local degees of freedom of gravity in 2+1 dimensions, it is of a certain relevance since in the negative cosmological constant case it possesses a black hole [29]. We also introduce the Brown-Henneaux boundary conditions [7] that characterize asymptotically AdS_3 spaces, and the computation of surface charges by means of the ADT method [30, 31]. Secondly, we describe the general features of conformal field theories with special attention to the two-dimensional case. We go over the double Virasoro algebra that its generators satisfy, and we construct the partition function on the torus using the corresponding characters. The modular properties of this partition function are then exploited to yield the asymptotic density of states following Cardy's formula [26]. Finally, in Chapter 4, we use the Chern-Simons formulation of three-dimensional gravity in anti-de Sitter space to show how it can be rewritten as a gauge theory, paving the way to the duality. We deliberately choose not to review this topic using any string theory, since no such notions are used in the rest of the thesis. The utility of the duality is then illustrated with the matching of entropies on both sides of $\mathrm{AdS}_3/\mathrm{CFT}_2$. We also briefly sketch the holographic principle, as an argument in favour of the search for other gauge/gravity dualities.

In this perspective, the Warped AdS_3/Warped CFT_2 duality is reviewed in Chapter 5. We explain how warped AdS spaces are obtained by means of deforming the anti-de Sitter metric [32], and what how they come up as solutions of theories of gravity that extend the classic Einstein-Hilbert action to setups with higher curvature. We then discuss how their asymptotic symmetry algebra matches the defining algebra of a warped CFT, and how the partition function is defined in such a theory. The modular properties of the latter are then investigated, and shown to reproduce an analogue of Cardy's formula [19].

In Chapter 6 we show that the entropy of the black hole calculated on the gravity side of the Warped AdS_3/Warped CFT_2 duality matches with the one computed through gauge theory methods on the field theory side. Since warped AdS_3 spaces appear only in contexts where the action contains terms involving higher-order derivatives of the Riemann tensor, this supposes to mobilize an enhanced method to compute the charges and derive the entropy. To this end, we review the covariant phase space formalism [33] and use it to prove the advertised result. This entropy matching for warped duality was previously tested for some particular cases of higher-derivative theories, and is now established to hold generically.

In Chapter 7, we adopt a different approach and derive a bound on the mass of the lightest warped black holes by putting to work the modular properties of the partition function in warped CFT. More generally, we argue that such a bound characterizes the spectrum of chiral conformal field theories with all sorts of charges. We prove our statement analytically in the simplest case, and provide numerical evidence for various other cases using a thorough analysis of the Weyl-Kac characters of affine Lie algebras.

Part I

Black holes and holography: general notions

Chapter 1

Black holes

Black holes are generally thought of as regions of spacetime exerting a gravitational pull so strong than nothing can escape from it [34]. Even if the emergence of the concept of black holes is typically associated with general relativity, this idea actually predates it by a good century. Indeed, in 1783, one John Michell imagined a star so dense that the escape velocity of a particle of light would exceed its speed [35]. This idea was cast aside with the corpuscular theory of light upon which it was built, only to resurface at the beginning of the 20th century in the form of an intriguing solution to Einstein's equations proposed by Schwarzschild [36]. In the context of general relativity, black holes are now understood as regions where spacetime curvature goes to infinity, i.e. singularities, cloaked with an event horizon which is the surface that separates the region where objects can in principle escape to infinity (the exterior of the black hole) from the region where things get trapped (the interior). Singularities were proved to be inevitable in a spacetime where trapped surfaces exist [37]. More precisely, the Penrose theorems state that under reasonable energy conditions and provided there is a Cauchy surface, a trapped surface in spacetime, such as an event horizon, implies the presence of either a Cauchy horizon (i.e. a boundary to the domain of knowledge of which is accessible from physical data in a given region of spacetime) or a singularity. Either way, spacetime is geodesically incomplete as long as such a thing as the weak energy condition holds.

In principle knowledge of the entire structure of spacetime is needed in order to define an event horizon, future history included. The highly global character of this definition, with the consequence that local measurements seem to be made irrelevant by it, led many prominent physicists to discard the notion of black hole as a "real" object. It is only when astrophysical mechanisms leading to the production of black holes were described that the idea started to gain momentum. In particular, the study of white dwarfs and neutron stars [38, 39] eventually led to the conclusion that black holes could form as a result of the collapsing of stars. If black holes could be the final stage in the life of a star, if they had anything to do with quasars and if they could provide an explanation for the phenomena happening at the galaxy's center [40], they were perhaps more than mere figments of the theorists' imagination. With observational evidence piling up, the problematic notion of the (classical) event horizon started to be regarded more and more as an idealization, even if it is one that led to powerful results. In particular, among these results, the celebrated "no-hair theorem" [41, 42] states that whatever characteristics the initial star possesses, whether topography or magnetic field for example, the final black hole into which it collapses is perfectly spherical (i.e., "bald"). The only properties the black hole owns are its mass, angular momentum and electric charge, all the others being radiated away. This is precisely what was recently observed experimentally by the LIGO-Virgo collaboration in the event of a black hole merger: the signal remarkably matched the predicted "smoothing" of both horizons into one [23].

The fate of the notion of event horizon is not settled; recently it has been argued that it is actually an artefact of a static spacetime, and that it can be dispensed with in a dynamical framework for collapsing matter [43]. This is related to the classical nature of this feature, and to the fact that a proper treatment of black holes would involve quantum corrections. But the taking into account of the quantum mechanical aspects of black holes leads to even more puzzling considerations, which we address next.

1.1. Hawking radiation and black hole information

When Hawking set out to study scalar field propagation in a non-spherical collapse geometry, nothing particularly interesting was expected to happen. Yet the result of this investigation was that there is a steady emission of particles for all frequencies, with or without rotation at the final black hole stage [28]. Furthermore, as seen by a distant observer, the emitted radiation is thermal. The origin of this flux of particles is understood to be pair creation near the horizon, with part of the particles falling into the black hole while the rest flows away to infinity, carrying with it little by little energy from the black hole. This evaporation process happens in finite time (typically of order of its cubic mass). This completed the picture of the black hole as a thermodynamic system, which we describe in the next section. It was also the beginning of one of the longest-standing conundrums in theoretical physics, namely the so-called “information paradox”. It is generally expressed in its most naïve formulation as the inconsistency between information loss resulting from the black hole’s complete evaporation and the second law of thermodynamics according to which entropy should never decrease. A more precise picture emerges when we express things in quantum mechanical terms: since the emitted quanta come from pair creation, they are entangled with the “other half” of the pairs that has been absorbed by the black hole. Hawking radiation is thus in a mixed state, and when the black hole disappears, there is no trace of the fine-grained description (i.e., pure state) this mixed state is a coarse-graining of [44]. In other words, “the Hawking process builds up entanglement between the black hole and its exterior” [45], and this contradicts unitary time evolution.

One possible way out of this situation would be to simply accept the breakdown of quantum mechanics as we know it. Perhaps in the presence of gravity, we should consider mixed states to be the fundamental entities, and their evolution in terms of density matrices would then no longer be given by a unitary S-matrix, but by a “not-S”-matrix or superscattering operator (or $-matrix) [46]. It has also been argued that unitarity and information preservation could only be expected on globally-hyperbolic spacetimes, and that only local unitarity makes sense in the evaporation spacetime [47]. In other words, pure to mixed state evolution is predicted by quantum mechanics in any situation where the final time is not a Cauchy surface, and therefore does not constitute a violation of it. In this perspective, information loss need not necessarily be a problem: one can consider that after the black hole has completely evaporated, Hawking radiation is entangled with the interior observables of the *past* black hole [48]. However, a classic objection to this position is that without unitary time evolution, the relationship between symmetries and conservation laws vanishes, but violations of conservation laws have not been witnessed [49]. One may nevertheless devise a quantum mechanical system such that its non-unitary evolution preserves energy conservation [50].

Yet the worry remains that since quantum mechanics has been tested with an extremely good precision, any small amendment to unitarity would have non-negligible consequences at the experimental level.

Another possible way out would be to consider that black holes do not evaporate completely, and even if the information is kept hidden, it remains there in the end [51]. After all, Hawking evaporation is a semi-classical process, so it may well be that this description is no longer reliable when the black hole gets really small. The problem with that conception is that an arbitrary large quantity of information would end crammed into a Planckian size remnant. If the remnant interacts with the outside world, this looks hardly compatible with particle physics and thermodynamics. In the latter case in particular, proliferation of such objects containing a large amount of states would be entropically favoured [48].

The problem may be not so much that the black hole evaporates completely or that information is lost, than that according to the field theory calculations, Hawking radiation is exactly thermal while statistical mechanics predicts it is no longer exactly thermal after Page time (i.e., halfway through the evaporation process) [47]. This discrepancy arises much earlier than the end of the black hole's life. This conflict between two descriptions of the black hole is illustrated particularly sharply by the following proposed resolution of the information paradox.

In the black hole complementarity proposal [52], it is argued that the three fundamental postulates on which a phenomenological description of the black hole by a distant observer should be based are: (i) the validity of standard quantum theory (existence of a unitary S-matrix), (ii) the validity of semi-classical gravitation theory (physics outside the horizon expressed in terms of a set of semi-classical field equations with quantum corrections), (iii) the validity of black hole thermodynamics (originating from coarse-graining of a complex but conventional quantum system). Subscribing in addition to (iv) the widespread belief that an infalling observer notices nothing in particular when she crosses the horizon, the authors claim that there is in fact a black hole description perfectly consistent with these three postulates, provided that the notion of "stretched horizon" is introduced. The stretched horizon is viewed as a visible timelike curve sitting in front of the horizon and behaving like a physical membrane. To an outside observer, it would be seen to respond to perturbation as a viscous fluid, and from the point of view of an infalling observer it would simply disappear. These two conflicting accounts are absolutely irreconciliable and must be seen as complementary, since the infalling observer's observations simply cannot be shared and compared with the distant one's. In that perspective, the fate of information is the following: it is absorbed, thermalized and re-emitted by the membrane, to be found in long-time correlations between quanta emitted at very different times. Similar views were formerly expressed in [53].

However, this was challenged by the firewall proposal [54]. The claim here is that for old black holes (with respect to Page time), postulates (i), (ii) and (iv) above are inconsistent with one another. In other words, the contradicting perspectives that were compartmentalized by complementarity can in fact become apparent to a single observer [55]. Indeed, full entanglement of recent radiation with both the older radiation (following from the purity of Hawking radiation) and the modes behind the horizon (related to the absence of drama at the horizon) violates strong subadditivity[1] of the entropy. In field theory terms, an infalling observer encounters high-energy modes (the firewall in question) at the horizon unless one relaxes the semi-classical field theory postulate. In the latter case, this would allow the novel, possibly non-local dynamics responsible for the re-emission of information to extend a macroscopic distance from the horizon (instead of being confined to Planckian distance as in the complementarity proposal). Otherwise, one must face the possibility that there simply is no spacetime inside the black hole.

But this is not necessarily a problem: in fact, this is precisely what the fuzzball proposal is about [56]. This approach simply does away with both the event horizon and the black hole interior, replacing them by a stringy picture of vibrating microstates spreaded over the transverse region where the horizon was supposed to be. There is thus no singularity for things to be destroyed at: matter falling "in" is just caught in the fuzz and radiated later with all its information. The classical intuition is preserved by a form of complementarity typical of the fuzzball: oscillations of the fuzzball happen at frequencies that are close enough to those entering the description of infall into a usual black hole. The slight difference between

[1] For a tripartite quantum system, in terms of the conditional entropy $S(A \mid B) \mathrel{\widehat{=}} S(AB) - S(B)$, the strong subadditivity theorem is the statement that

$$S(A \mid BC) \leqslant S(A \mid B) .$$

the two is what allows information to escape. It is noticeable only at low energy, and at high energy the classical picture is as good an approximation as it ever was[2]. Of course, since the "horizon" can be very large, this picture implies that quantum effects are not confined to Planckian distances [44].

A similar idea underlies another possible way out of the information problem: quantum gravity effects may indeed become relevant way before the black hole reaches Planck length. It has been argued, based on loop quantum gravity insights, that energy densities can reach Planck scale while the black hole is still macroscopic [57]. The leads to a new story for gravitationally-collapsed objects: when they enter the quantum gravity regime, they experience a new phase that takes the form of a bounce. In this phase called "Planck star", gravitational attraction is balanced by quantum pressure. Due to extreme time dilatation, from the point of view of the Planck star its lifetime is extremely short and looks more like a bounce, while as seen from a distance, it looks very long and does not contradict the black hole picture. In this perspective, the Schwarzschild horizon encloses a second trapping horizon that delimits the region where Einstein's equations no longer apply because of how compressed the Planck star is. With Hawking radiation, the outer horizon slowly shrinks as expected, and the inner one grows since the negative energy of the infalling pair partner turns positive. When they meet, they annihilate and since there is no horizon anymore all the remaining information can escape. The final stage of the collapse is a short-lived, macroscopic remnant. In the end, a black hole is nothing more than a collapsing and bouncing star seen is slow motion, which is reminiscent of Zeldovich and Novikov's old "frozen star" picture [58]. However, it has been argued that such a scenario where all the information comes out at a given time would be highly unstable, and that the energy radiated by the Planck star is different from Hawking radiation in that it is not experienced as a vacuum by an infalling observer[3] [55].

Up until now, almost every piece of the picture we painted of the black hole to begin with has been put into question in these various proposals to solve the information problem. One still remains: their baldness. Yet it is possible to conceive of an alternative solution

[2] Unfortunately, there are indications that the fuzzball would nonetheless work as a firewall and destroy an infalling observer [55].

[3] Such effects could alter the shape of the black hole's accretion halo, and hence be observable with the Event Horizon Telescope [45].

where the no-hair theorem is wrong. Indeed, it holds for stationary black holes, but if one considers time-dependent black holes, things look a bit different: the relevant symmetry group at the horizon is the Bondi-Metzner-Sachs group, which in addition to the Poincaré group symmetries contains supertranslations [22]. These are transformations that preserve the asymptotic form of the metric, and a whole collection of extra gravitational charges are associated to them. Since they affect only the subleading part of the metric, they are called “soft hair”. The hope is then that all the information associated with the initial state of the black hole is described by these charges.

The black hole information paradox is a vastly complex subject, where part of the challenge lies in the identification of the problem itself. Hopefully this brief (and far from exhaustive) review gives the reader a sense of what remains to be understood about black holes, and why it is intimately related to the sort of theory quantum gravity might turn out to be. In the meantime, Hawking radiation also contributed to paint a thermodynamical picture of black holes, which is the subject of less debate.

1.2. Black hole thermodynamics

The first hint that black holes were thermodynamical objects was the discovery by Hawking that, under the null energy condition

$$T_{ab}\, k^a\, k^b \geqslant 0\,, \tag{1.1}$$

(with T the energy-momentum tensor of infalling matter and k an arbitrary null vector) the area of the event horizon never decreases [59]. On the other hand, the uniqueness (no-hair) theorems make the black hole formation process hard to believe from a thermodynamical point of view: indeed, the collapse of a complex matter system into a comparatively simple object, characterized by only a very small number of quantities (mass, charge, angular momentum) seems to disobey the second law. The initial object entropy’s decrease as matter disappears into the black hole and gets destroyed at the singularity should be compensated by a corresponding increase in some feature of the final object, and the area of the horizon seems to be the perfect candidate to play this role. Bekenstein then introduced the concept of entropy for the black hole, and suggested that the appropriate relation to the area A should be (in natural units) [60]

$$S_{BH} = \frac{A}{4} \tag{1.2}$$

or in full units $S_{BH} = Ak_B c^3/(4G\hbar)$. The second law of thermodynamics is then understood to hold only for the total system formed by the black hole and its surroundings:

$$dS_{\text{tot}} \geqslant 0 \qquad \text{with} \quad S_{\text{tot}} = S_{BH} + S_{\text{matter}} \, . \tag{1.3}$$

This "generalized second law" [61] is widely held to be valid, but has not yet been proven [34]. A *Gedankenexperiment* allowed Bekenstein to derive from this law a bound on the entropy of the matter in the vicinity of the black hole: allegedly, the entropy of a matter system of total energy E enclosed in a sphere of radius R cannot exceed

$$S_{\text{matter}} \leqslant 2\pi E R \tag{1.4}$$

lest it form a black hole [62]. The validity of this bound is however controversial; see in particular [63].

In the same spirit, an analogue to the first law of thermodynamics for black holes can be derived from Einstein's equations [64]:

$$dM = \frac{\kappa}{8\pi}\, dA + \Omega\, dJ + \Phi\, dQ \tag{1.5}$$

where in addition to the black hole properties (mass M, charge Q and angular momentum J) the thermodynamic potentials are the angular velocity at the horizon Ω, the electrostatic potential Φ and surface gravity of the horizon κ. The latter quantity is defined by

$$\xi^b\, \nabla_b\, \xi^a = \kappa\, \xi^a \tag{1.6}$$

where ξ is the Killing vector generating the horizon (e.g. $\xi^\mu \partial_\mu = \partial_t + \Omega\, \partial_\varphi$ for a stationary, axisymmetric black hole). It is generally understood as a measure of the strength of gravity at the horizon as seen by a distant observer. For a stationary black hole, it is constant on the horizon in much the same way the temperature is constant in a body at thermodynamical equilibrium. From (1.5), one sees that temperature can be identified as

$$T = \frac{\kappa}{2\pi} \tag{1.7}$$

which is precisely the temperature of Hawking radiation [28]. For example, for a Schwarzschild black hole, the Hawking temperature is

$$T = \frac{1}{8\pi M} \tag{1.8}$$

(up to a factor of $\hbar c^3/(G k_B)$ in non-natural units) which is about $10^{-7} K$ for a black hole of one solar mass. It is therefore nothing one could measure experimentally for an astrophysical black hole. With $A \sim M^2$, and Boltzmann equation for the power emitted by a black body $P = dM/dt = A\, T^4$, one can estimate the lifetime of a black hole

$$t \ \sim \ \int dM\, M^2 \ \sim \ M^3 \tag{1.9}$$

which is long, but (as claimed in the previous section) not exponentially long.

Of course, assigning an entropy to black holes raises a number of questions. What kind of statistical underlying description could it be a coarse-graining of? What does it mean that this entropy is proportional to a characteristic area rather than to the volume of the whole object? We will come back to these deep questions in Chapter 4, where we will sketch how holography can help us understand them. In the following chapters, we will introduce some background to understand what holography is. In particular, we will review both sides of the duality in its most common incarnation, namely AdS/CFT. In the next chapter we will address the gravity side, i.e. anti-de Sitter space, and in the following one the field theory side, i.e. conformal field theory. Along the way, the reader may gather some hints of how they are going to come together in the holographic proposal.

Chapter 2

Anti-de Sitter space

Among the solutions of Einstein's equations, anti-de Sitter space is the one describing an empty universe with Lorentzian signature and constant negative curvature. To picture such a space, it is convenient to embed it in a higher-dimensional space as a slice of which we construct it. For example, embedding d-dimensional AdS in $(d+1)$-dimensional Minkowski space with two time directions yields a hyperboloid:

$$-x_0^2 - x_d^2 + \sum_{i=1}^{d-1} x_i^2 = -\ell^2 \,. \tag{2.1}$$

The parameter ℓ is the radius of curvature and is related to the cosmological constant $\Lambda = -(d-1)(d-2)/\ell^2$. A system of coordinates covering the entire space is then

$$x_0 = \ell \cosh\rho \cos\tau \,, \qquad x_d = \ell \cosh\rho \sin\tau \,, \qquad x_i = \ell \sinh\rho \, \Omega_i \quad (i = 1, \dots d-1) \tag{2.2}$$

where $\rho \in [0, \infty)$, $\tau \in [0, 2\pi)$ and Ω_i is a unit vector for the $(d-2)$-sphere such that $\sum_{i=1}^{d-1} \Omega_i^2 = 1$. One can then write the metric

$$ds^2 = \ell^2 \left(-\cosh^2\rho \, d\tau^2 + d\rho^2 + \sinh^2\rho \, d\Omega_{d-2}^2\right) \tag{2.3}$$

in these global coordinates (which in three dimensions are analogous to the Euler angles used to parametrize the sphere) or in Euclidean signature with $\theta \in [0, \pi/2)$ defined by

$\tan\theta = \sinh\rho$ as

$$ds^2 = \frac{\ell^2}{\cos^2\theta}\left(-d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega_{d-2}^2\right)\,. \tag{2.4}$$

In the latter case, the metric covers half of Einstein's static universe.

Since τ is an angle, this metric has closed timelike curves, which is an unsuitable feature for a physical model of spacetime. One generally unwinds this coordinate setting $\tau \neq \tau + 2\pi$, and works with the resulting universal cover of anti-de Sitter space. In what follows, this is what we will mean when referring to anti-de Sitter space. By replacing this unwound coordinate by t/ℓ and setting $r = \ell\sinh\rho$, one gets the familiar metric

$$ds^2 = -\left(1+\frac{r^2}{\ell^2}\right)dt^2 + \left(1+\frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2\, d\Omega_{d-2}^2\,. \tag{2.5}$$

Three-dimensional gravity is special in the sense that it does not have any local degrees of freedom. Indeed, it is defined by the Einstein-Hilbert action

$$S_{EH}[g_{\mu\nu}] = \frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\,\sqrt{-g}\,(R-2\Lambda) + B \tag{2.6}$$

with G is Newton's constant, we are working in units where $c = 1$, Λ is the cosmological constant ($= -1/\ell^2$ if the manifold $\mathcal{M}$ is anti-de Sitter space), $g_{\mu\nu}$ is the metric and R the trace of the Ricci tensor $R_{\mu\nu}$. B stands for boundary terms that are present to ensure that the variational principle is well-defined, but that will not retain our attention for now. Varying this action yields Einstein's equations in the vacuum

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} + \Lambda\,g_{\mu\nu} = 0\,. \tag{2.7}$$

Three-dimensional manifolds have the characteristic feature that their Riemann tensor is completely determined by the Ricci tensor:

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}\,R_{\nu\sigma} + g_{\nu\sigma}\,R_{\mu\rho} - g_{\nu\rho}\,R_{\mu\sigma} - g_{\mu\sigma}\,R_{\nu\rho} - \frac{1}{2}R\left(g_{\mu\rho}\,g_{\nu\sigma} - g_{\mu\sigma}\,g_{\nu\rho}\right)\,, \tag{2.8}$$

which has the consequence that their curvature is constant. As a result, any solution of pure Einstein gravity in dimension three is locally isometric to empty space, whether it is

Minkowski, de Sitter or anti-de Sitter. There are no gravitational waves or gravitons in this setup. In order to see anything non-trivial, we would need to appeal to a global analysis in which the topological properties of spacetime are relevant. We might think that this sorts out the case of three-dimensional gravity, but there is in fact more to it: namely, when the curvature is negative, it has a black hole. This is what sparked the interest for three dimensional anti-de Sitter space, as this feature responded to the need to have a simple model of a black hole, with all the important properties but not the unnecessary intrications of its higher-dimensional realistic counterpart.

2.1. The three-dimensional black hole

The black hole solution to (2.7) with $\Lambda = -1/\ell^2$ discovered by Bañados, Teitelboim and Zanelli [29], or BTZ black hole, is given by

$$ds^2 = -N^2(r)\,dt^2 + N^{-2}(r)\,dr^2 + r^2\big(N^\varphi(r)\,dt + d\varphi\big)^2 \tag{2.9}$$

with the lapse and shift functions

$$N(r) = \sqrt{-8GM + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2}}\,, \tag{2.10}$$

$$N^\varphi(r) = -\frac{4GJ}{r^2}\,, \tag{2.11}$$

and $t \in (-\infty, \infty)$, $r \in (0, \infty)$, $\varphi \in [0, 2\pi]$. It is stationary and axisymmetric, and exhibits two horizons at the points where $N(r) = 0$, namely

$$r_\pm = \ell\left[4GM\left(1 \pm \sqrt{1 - (J/M\ell)^2}\right)\right]^{\frac{1}{2}} \tag{2.12}$$

if $M > 0$ and $|J| \leqslant M\ell$. The mass M and angular momentum J of the black hole can be expressed in terms of these horizons as

$$M = \frac{r_+^2 + r_-^2}{8G\ell^2}\,, \qquad J = \frac{r_+ r_-}{4G\ell}\,. \tag{2.13}$$

For $M = J = 0$, this is "massless BTZ", although there is no black hole left so to speak since both horizons vanish. If M grows negative or $|J|$ gets too large, r_+ disappears and one is left with naked singularities, which are excluded from the physical spectrum of acceptable solutions. Thus the condition on $|J|$ plays the role of cosmic censorship in this framework. There is a notable exception though: for $J = 0$, $M = -1/8G$, the line element (2.9) turns into

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right) dt^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\varphi^2 \tag{2.14}$$

which is just (2.5) for $d = 3$. This is a little strange: one would expect the lightest BTZ black hole to be empty space, but in dimension three it is separated from anti-de Sitter space by a mass gap, since there is no way to continuously deform one into the other without running into forbidden solutions.

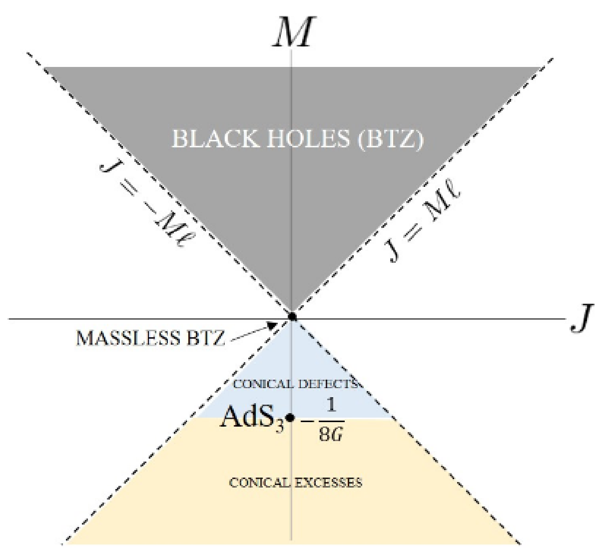

Figure 2.1 Mass gap in the BTZ black hole spectrum [65]. BTZ exists for $|J| \leqslant M\ell$, $M \geqslant 0$ and empty AdS_3 is recovered for $J = 0$, $M = -1/8G$.

Still, as we have seen above, the BTZ black hole is a solution of pure gravity in dimension three and as such it is of constant (negative) curvature, hence locally isometric to anti-de Sitter space. One can show that since they only differ by their global properties, it is possible to recover the black hole as a quotient of anti-de Sitter space. This quotient can be obtained through the use of a discrete subgroup of anti-de Sitter's symmetry group to identify points of AdS [66]. Moreover, it is easy to see that BTZ is asymptotically anti-de Sitter space: when $r \gg r_+$, the metric of the former reduces to the one of the latter.

The BTZ black hole has a number of interesting features that really make it a toy model for more realistic black holes. To begin with, it has what is called an ergosphere, which is a region outside its event horizon where a particle is subject to the influence of the black hole and is dragged along with it but still can escape its pull, possibly with more energy than it started with. The ergosphere is defined to be the region $r > r_e$ such that $g_{00}(r_e) = 0$. In the case of BTZ, it is

$$r_e = \ell\sqrt{8GM} = \sqrt{r_+^2 + r_-^2}\,. \tag{2.15}$$

This is a feature that BTZ shares with its higher-dimensional counterpart, the Kerr black hole [67]. BTZ also has this in common with the Kerr solution that it has non-trivial thermodynamical properties: its Hawking temperature is

$$T_H = \frac{\hbar^2(r_+^2 - r_-^2)}{2\pi\ell^2 r_+} \tag{2.16}$$

and its Bekenstein-Hawking entropy is

$$S_{BH} = \frac{2\pi r_+}{4\hbar G} = \frac{A}{4\hbar G} \tag{2.17}$$

with A the horizon area. It satisfies the first law of black hole thermodynamics (1.5) with $Q = 0$ and angular velocity at the horizon $\Omega = r_-/r_+\ell$.

2.2. Boundary conditions

Up to now, we have carefully ignored the boundary term introduced in (2.6). It is nevertheless crucial to the whole construction above, and we need to address it now. When the Eistein-Hilbert action is varied with respect to the metric, one gets two contributions: the first one contains the equations of motion and vanishes identically, and the second one is a boundary term. In order for the action to be well-defined, this term should vanish when the action is varied, and that depends on the theory's behavior at infinity. Therefore we need to choose appropriate boundary conditions, or resort to modifying the action with extra terms that cancel the unwanted boundary contributions.

The asymptotic symmetry algebra of a theory is the set of all the allowed gauge transformations modulo the trivial ones, that is, all the global symmetries minus the redundant ones. Symmetry properties of Riemannian spaces are characterized by Killing vectors ξ_μ, which indicate under what coordinate transformations $x'^\mu = x^\mu + \varepsilon\, \xi_\mu$ the metric is left unchanged, i.e.

$$\mathcal{L}_\xi\, g_{\mu\nu} \;=\; \xi^\lambda\, \partial_\lambda g_{\mu\nu} + \partial_\mu \xi^\lambda\, g_{\lambda\nu} + \partial_\nu \xi^\lambda\, g_{\mu\lambda} \;=\; 0 \tag{2.18}$$

with $\mathcal{L}_\xi$ the Lie derivative with respect to ξ. Equivalently, the Killing equation can be reformulated as

$$\xi_{\mu\,;\nu} + \xi_{\nu\,;\mu} = 0\,. \tag{2.19}$$

AdS_3 is a maximally symmetric spacetime, in the sense that it has the maximum number of Killing vectors, namely six. These can be written as [32]

$$\begin{aligned}
&\xi_1 = x_0\partial_3 - x_3\partial_0\,, \qquad && \xi_2 = x_1\partial_2 - x_2\partial_1\,, \qquad && \xi_3 = x_0\partial_2 + x_2\partial_0\,,\\
&\xi_4 = x_3\partial_1 + x_1\partial_3\,, && \xi_5 = x_0\partial_1 + x_1\partial_0\,, && \xi_6 = x_3\partial_2 + x_2\partial_3
\end{aligned} \tag{2.20}$$

where indices refer to the general expression (2.1). Upon combining them as

$$\begin{aligned}
\ell_0 &= \tfrac{1}{2}(\xi_1 + \xi_2)\,, & \bar{\ell}_0 &= \tfrac{1}{2}(\xi_1 - \xi_2)\,,\\
\ell_1 &= \tfrac{1}{2}(\xi_3 + \xi_4 - i\xi_5 + i\xi_6)\,, & \bar{\ell}_1 &= \tfrac{1}{2}(-\xi_3 + \xi_4 - i\xi_5 - i\xi_6)\,,\\
\ell_{-1} &= \tfrac{1}{2}(\xi_3 + \xi_4 + i\xi_5 - i\xi_6)\,, & \bar{\ell}_{-1} &= \tfrac{1}{2}(-\xi_3 + \xi_4 + i\xi_5 + i\xi_6)\,,
\end{aligned} \tag{2.21}$$

they obey the algebra

$$\begin{aligned}
i[\ell_m, \ell_n] &= (m-n)\,\ell_{m+n}\,,\\
i[\bar{\ell}_m, \bar{\ell}_n] &= (m-n)\,\bar{\ell}_{m+n}\,,\\
i[\ell_m, \bar{\ell}_n] &= 0\,,
\end{aligned} \tag{2.22}$$

for $(m,n) = 0, \pm 1$. This algebra is $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}) \simeq so(2,2)$, and we will encounter it again in the next chapter.

In order to be as general as possible, it is then natural to wonder what metrics are asymptotically anti-de Sitter. It is also preferable not to look for the largest family of such metrics possible, but only for a minimal amount of relevant metrics. The suggestion of [7] is to include conical deficits, i.e. metrics of point particles, along with pure AdS. Such metrics are given by

$$ds^2 = -\left(1+\frac{r^2}{\ell^2}\right)(dt' - A d\varphi')^2 + \left(1+\frac{r^2}{\ell^2}\right)^{-1} dr^2 + 4\omega^2 r^2 d\varphi'^2 \tag{2.23}$$

with new coordinates

$$t' = t + \frac{A}{2\omega}\varphi\,, \qquad \varphi' = \frac{\varphi}{2\omega}\,. \tag{2.24}$$

The parameters $A \in \mathbb{R}$ and $\omega > 0$ describe the identifications needed to cut cones in anti-de Sitter space, namely $(t, r, \varphi) = (t - 2\pi A, r, \varphi + 4\pi\omega)$. Pure AdS is recovered for $A = 0$ and $\omega = -1/2$.

The procedure is then to act with $so(2,2)$ generators on these metrics and look for fall-off conditions satisfied by the infinitesimally transformed metric $g_{\mu\nu} + \mathcal{L}_\xi\, g_{\mu\nu}$ with ξ a Killing vector for AdS_3 (but not the conical deficit). One obtains [7]

$$(g_{\mu\nu}) \underset{r\to\infty}{\sim} \begin{pmatrix} g_{tt} & g_{tr} & g_{t\varphi} \\ g_{rt} & g_{rr} & g_{r\varphi} \\ g_{\varphi t} & g_{\varphi r} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} -\frac{r^2}{\ell^2} + O(1) & O(r^{-3}) & O(1) \\ O(r^{-3}) & \frac{\ell^2}{r^2} + O(r^{-4}) & O(r^{-3}) \\ O(1) & O(r^{-3}) & r^2 + O(1) \end{pmatrix}. \tag{2.25}$$

These boundary conditions can actually be simplified further by making use of the gauge freedom we have. As it turns out, the subleading corrections $O(r^{-3})$, $O(r^{-4})$ in (2.25) can be set to zero simply by applying a trivial diffeomorphism [68]. We also notice that when the spatial coordinate is taken to infinity, the metric (2.14) becomes

$$ds^2 = -\frac{r^2}{\ell^2}\, dt^2 + \frac{\ell^2}{r^2}\, dr^2 + r^2 d\varphi^2 = \frac{\ell^2}{r^2}\, dr^2 - r^2\, dx^+ dx^- \tag{2.26}$$

with the light-cone coordinates

$$x^{\pm} \;\widehat{=}\; \frac{t}{\ell} \pm \varphi\,. \tag{2.27}$$

Interestingly, this is the metric of a cylinder; we will come back to that later on. An anti-de Sitter metric with the Fefferman-Graham gauge choice above and coordinates (r, x^i) with $(i = 0, 1)$ behaves asymptotically as

$$ds^2 = \frac{\ell^2}{r^2}\, dr^2 + r^2 \left(g_{ij}^{(0)} + O(1)\right) dx^i dx^j\,. \tag{2.28}$$

Brown-Henneaux boundary conditions are then expressed as

$$g_{ij}^{(0)} dx^i dx^j = -dx^+ dx^-\,. \tag{2.29}$$

This is what is generally meant by “asymptotically anti-de Sitter metrics”.

2.3. Conserved charges

We have seen in the previous chapter that black holes are characterized by just a few generic properties: mass, charge and angular momentum. Conserved charges are generally defined using the Noether procedure, but in the case of gauge theories such as general relativity, all this leads to is trivial charges. Indeed, a key feature of general relativity is to place time and space on equal footing, but without a well-defined preferred time coordinate the very notion of conservation makes no sense. In particular, there is no way to define a local energy-momentum tensor. Diffeomorphism invariance is associated to the existence of Killing vectors ξ_μ, which as we have seen above are the infinitesimal generators of isometries of the form $x'^\mu = x^\mu + \varepsilon\, \xi_\mu$, and each corresponds to a conserved quantity in the theory under consideration. The problem with general relativity is that no vector is a Killing vector for all solutions $g_{\mu\nu}$ of Einstein’s equations. Another way to put it is that Noether’s theorems overlook the non-uniqueness of conserved currents in gauge theories: indeed, the current $J^\mu + \partial_\nu k^{\mu\nu}$ for some antisymmetric $(d-2)$-form $k^{\mu\nu}$ yields terms that vanish under the variation of the action just as well as J^μ does. The associated conserved charge

$$Q_a = \int_\Sigma (d^{d-1}x)_\mu \, J_a^\mu \tag{2.30}$$

may however differ by a surface term

$$Q_k = \int_{\partial\Sigma} (d^{d-2}x)_{\mu\nu} \, k^{\mu\nu} \tag{2.31}$$

and there is no prescription for how to choose $k^{\mu\nu}$. Since in a gauge theory the main contribution vanishes, one could perfectly define the charge to be (2.31), provided that one finds a way to build $k^{\mu\nu}$ from a symmetry generator. There is an enormous body of work on this issue and its proposed solutions (see in particular [69] and references therein) which extend well beyond the scope of this thesis. One of the possible ways out for theories that admit a common asymptotic structure is to use the linearized theory

$$g^{\mu\nu} = \bar{g}^{\mu\nu} + h^{\mu\nu} \tag{2.32}$$

around a background metric $\bar{g}^{\mu\nu}$ that verifies the Killing equation $\mathcal{L}_\xi \, \bar{g}_{\mu\nu} = 0$. One way to see this is that it effectively separates the truly dynamical information ($h_{\mu\nu}$) from the coordinate information ($\bar{g}_{\mu\nu}$). The method was first elaborated for asymptotically flat spacetimes [30], then expanded to asymptotically AdS spacetimes [31] (see also [70] for a review). Since we will exclusively be concerned with the latter, we briefly review the so called "ADT method" just to give a sense of how working in the linearized theory solves our problem.

If we write our initial equations of motion as $E_{\mu\nu} = 0$, once the theory is linearized they become

$$E_{\mu\nu} \;\widehat{=}\; \bar{E}_{\mu\nu} + \delta E_{\mu\nu} = 0 + \delta E_{\mu\nu} \, . \tag{2.33}$$

General covariance requires that $\nabla_\mu \, E^{\mu\nu} = 0$, which translates upon linearization as

$$\bar{\nabla}_\mu \, \delta E^{\mu\nu} \tag{2.34}$$

at first order in perturbation, on-shell. All barred quantities are defined in terms of $\bar{g}$. With

ξ a Killing vector for the background metric $\bar{g}$, the current

$$J^\mu \;\hat{=}\; \delta E^{\mu\nu}\,\xi_\nu \qquad \text{is such that} \qquad \bar{\nabla}_\mu J^\mu = 0 \tag{2.35}$$

i.e. is conserved. We now wish to write it in terms of an antisymmetric tensor potential, independently from the equations of motion, that is

$$J^\mu = \bar{\nabla}_\nu k^{\mu\nu} \tag{2.36}$$

up to terms vanishing on-shell. Expanding $\delta E^{\mu\nu}\,\xi_\nu$ at first order in h, one gets

$$\delta E^{\mu\nu}\,\xi_\nu = \delta E_{\alpha\nu}\,\bar{g}^{\alpha\mu}\,\xi_\nu - \bar{E}^{\mu\beta}\,h_{\beta\nu}\,\xi^\nu - \bar{E}_{\alpha\nu}\,h^{\alpha\mu}\,\xi^\nu\,. \tag{2.37}$$

Since the second and third terms vanish on-shell, we can identify

$$\bar{\nabla}_\nu k^{\mu\nu} = \delta E_{\alpha\nu}\,\bar{g}^{\alpha\mu}\,\xi_\nu \tag{2.38}$$

and use it to define $k^{\mu\nu}$. The ADT charges are then given by (2.31) for a given ξ.

For example, in the case of pure gravity in anti-de Sitter space,

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(R + \frac{2}{\ell^2}\right) \tag{2.39}$$

the charge-defining 2-form is given by

$$\delta E_{\alpha\nu}\,\bar{g}^{\alpha\mu}\,\xi_\nu = \bar{g}^{\alpha\mu}\left(\delta R_{\alpha\nu} - \frac{1}{2}\bar{g}_{\alpha\nu}\delta R\right) - \frac{1}{2}h^\mu_\nu\left(R + \frac{2}{\ell^2}\right)\xi^\nu\,. \tag{2.40}$$

With a bit of work, one can show that

$$\begin{aligned}\delta E_{\alpha\nu}\,\bar{g}^{\alpha\mu}\,\xi_\nu \;&=\; \frac{1}{2}\xi^\nu\left[\nabla_\lambda, \nabla_\nu\right]h^{\lambda\mu} + \frac{1}{2}\xi^\mu R_{\sigma\lambda}h^{\sigma\lambda} + h^{\lambda[\mu}\nabla_\nu\nabla_\lambda\xi^{\nu]} - \frac{1}{2}h\nabla_\nu\nabla^\mu\xi^\nu - \frac{1}{2}h^\mu_\nu\left(R + \frac{2}{\ell^2}\right)\xi^\nu \\ &+\; \nabla_\nu\left(\xi^{[\nu}\nabla_\lambda h^{\mu]\lambda} + \xi^\lambda\nabla^{[\mu}h^{\nu]}_\lambda + \xi^{[\mu}\nabla^{\nu]}h - h^{\lambda[\nu}\nabla_\lambda\xi^{\mu]} + \frac{1}{2}h\nabla^{[\mu}\xi^{\nu]}\right)\,. \end{aligned} \tag{2.41}$$

The second line of this expression yields an explicit expression for the object of interest:

$$k^{\mu\nu} = \xi^{[\nu}\nabla_\lambda h^{\mu]\lambda} + \xi^\lambda \nabla^{[\mu} h^{\nu]}_{\lambda} + \xi^{[\mu}\nabla^{\nu]} h - h^{\lambda[\nu}\nabla_\lambda \xi^{\mu]} + \frac{1}{2} h \nabla^{[\mu}\xi^{\nu]} \,. \tag{2.42}$$

We will use a similar technique later on to compute surface charges in three-dimensional gravity with additional higher curvature terms on asymptotically anti-de Sitter spaces.

We now cross to the other side of the AdS/CFT duality and delve into conformal field theories, some features of which might look familiar to the careful reader.

Chapter 3

Conformal Field Theories

Conformal field theories[1] are relativistic quantum field theories that, on top of Poincaré invariance

$$x^\mu \to x^\mu + a^\mu \qquad \text{(translations)}\,, \tag{3.1}$$

$$x^\mu \to M^\mu{}_\nu\, x^\nu \qquad \text{(Lorentz boosts)}\,, \tag{3.2}$$

also obey invariance under

$$x^\mu \to \lambda\, x^\mu \qquad \text{(dilatations)}\,, \tag{3.3}$$

$$x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2\, b\cdot x + b^2 x^2} \qquad \text{(special conformal transformations)}\,. \tag{3.4}$$

In particular, special conformal transformations correspond to inversion-translation-inversion sequences, with the inversion being understood as $x^\mu \to x^\mu/x^2$. In other words,

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - b^\mu\,. \tag{3.5}$$

The conformal group is defined as the set of transformations $x \to x'(x)$ that act on the metric as a Weyl rescaling

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x)\, g_{\mu\nu}(x) \tag{3.6}$$

with $\Omega(x)$ arbitrary. Even if the conformal group is larger than the Poincaré group, con-

[1] This chapter is based on textbooks and reviews such as [71, 72, 73, 74].

formal field theories can help us understand quantum field theories in that the latter often exhibit invariance under such a rescaling at long distances [75]. Since quantum field theories are generally truncated by cut-offs allowing for renormalization procedures, one can think of a UV-complete theory as a renormalization group flow between conformal field theories in the UV and in the IR. Indeed, renormalization group flows are controlled by β-functions which describe the variation of the couplings with respect to energy scales. Fixed points in the renormalization group flow correspond to vanishing β-functions, and hence to scale-invariant theories [73]. Conformal field theories thus constitute a powerful tool in understanding the space of quantum field theories.

If we define the stress-energy tensor as usual through the variation of the d-dimensional action S under $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$

$$\delta S = \frac{1}{2} \int d^d x \sqrt{g}\, T^{\mu\nu}(x)\, \delta g_{\mu\nu}\,, \tag{3.7}$$

we already know that invariance of the theory under such a general transformation amounts to having conservation of $T^{\mu\nu}$:

$$T^{\mu\nu}{}_{;\mu}(x) = 0\,. \tag{3.8}$$

For (infinitesimal) conformal transformations $g_{\mu\nu} \to g_{\mu\nu} + \omega(x) g_{\mu\nu}$, the corresponding condition this gives rise to is tracelessness:

$$T^{\mu}{}_{\mu}(x) = 0\,. \tag{3.9}$$

In the following we will be exclusively concerned with conformal field theories in flat space. In that framework, given how a metric transforms by definition, (3.6) can be rewritten as

$$\eta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Omega(x)\, \eta_{\mu\nu}\,. \tag{3.10}$$

For infinitesimal transformations $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu + O(\varepsilon^2)$, this yields

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(x)\, \eta_{\mu\nu}\,. \tag{3.11}$$

Two important equations can be derived from there: first, tracing both sides, one gets $f(x) = \frac{2}{d}\,\partial\cdot\varepsilon$ and

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d}\,\partial\cdot\varepsilon\,\eta_{\mu\nu}\,. \tag{3.12}$$

Second, deriving both sides, one gets

$$(d-1)\partial^2 f = 0 \qquad \Rightarrow \qquad f(x) = A + Bx^\mu \tag{3.13}$$

which leads to

$$\varepsilon_\mu = a_\mu + b_{\mu\nu}\,x^\nu + c_{\mu\nu\rho}\,x^\nu x^\rho\,. \tag{3.14}$$

From the analysis of this expression one can retrieve all the components of the conformal group hinted at above. The generators of these infinitesimal transformations are then

$$\begin{aligned}
&P_\mu = -i\,\partial_\mu && \text{(translations)}\\
&L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) && \text{(rotations)}\\
&D = -ix^\mu\,\partial_\mu && \text{(dilatations)}\\
&K_\mu = i(x^2\partial_\mu - 2x_\mu x^\nu)\partial_\nu && \text{(special conformal transformations)}
\end{aligned} \tag{3.15}$$

and they verify the conformal algebra

$$\begin{aligned}
&[L_{\mu\nu}, P_\rho] = i\left(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu\right)\,,\\
&[L_{\mu\nu}, K_\rho] = i\left(\eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu\right)\,,\\
&[L_{\mu\nu}, L_{\rho\sigma}] = i\left(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\mu\sigma}L_{\nu\rho}\right)\,,\\
&[D, P_\mu] = iP_\mu\,,\\
&[D, K_\mu] = -iK_\mu\,,\\
&[K_\mu, P_\nu] = 2i\left(\eta_{\mu\nu}D - L_{\mu\nu}\right)\,.
\end{aligned} \tag{3.16}$$

Incidentally, a quick rewriting of these generators makes the symmetry group explicit. Indeed, taking

$$
\begin{aligned}
J_{\mu\nu} &= L_{\mu\nu}\,, \\
J_{-10} &= D\,, \\
J_{0\mu} &= \frac{1}{2}\left(P_\mu + K_\mu\right)\,, \\
J_{-1\mu} &= \frac{1}{2}\left(P_\mu - K_\mu\right)\,,
\end{aligned}
\tag{3.17}
$$

one can check that for $m, n = -1, 0, 1, \ldots, d-1$ these alternate generators J_{mn} satisfy the commutators

$$
[J_{mn}, J_{rs}] = i\left(\eta_{nr} J_{ms} - \eta_{mr} J_{ns} + \eta_{ms} J_{nr} - \eta_{ns} J_{mr}\right)\,. \tag{3.18}
$$

For $\eta_{mn} = \mathrm{diag}(-1, 1, \ldots, 1)$ this is the Lie algebra $so(d+1, 1)$. A convenient basis of fields in a conformal field theory is the one made of common eigenfunctions of $L+_{\mu\nu}$, D and K_μ. For such an operator φ evaluated at the origin,

$$
[L_{\mu\nu}, \varphi(0)] = S_{\mu\nu}\,\varphi(0)\,, \qquad [D, \varphi(0)] = -i\Delta\,\varphi(0)\,, \qquad [K_\mu, \varphi(0)] = \kappa_\mu\,\varphi(0) \tag{3.19}
$$

where $S_{\mu\nu}$ is called the spin matrix of φ and Δ is its (conformal) dimension. One can check that K_μ is a lowering operator for dimension; for this reason, one distinguishes the φ's that have $\kappa_\mu = 0$ as being primary operators from those that do not and "descend" from them, in a way analogue to the construction of a tower of states in an irreducible representation of $SU(2)$ from a highest-weight state – except here the primaries are "lowest-weight states" instead. The descendants are built by acting with P_μ, which is the corresponding raising operator, on primaries. in other words, primary fields are defined as fields $\varphi(z, \bar{z})$ that respond to a conformal transformation $z \to f(z)$ as

$$
\varphi(z, \bar{z}) \;\to\; \varphi'(z, \bar{z}) = (\partial_z f)^h (\partial_{\bar{z}} \bar{f})^{\bar{h}}\; \varphi\left(f(z), \bar{f}(\bar{z})\right)\,. \tag{3.20}
$$

Scale invariance has an unexpected side-effect: even if one considers two states infinitely apart, one can always use the dilatation operator to bring them close together again. This makes it impossible to define an S-matrix in any meaningful sense [76]. All one can do is study correlation functions of operators, but thanks to the state-operator correspondence (which we will say more about in the two-dimensional case), this contains all the information we need about the theory. The underlying idea is that any state in the theory can be created by acting with a local operator on the vacuum. Fortunately, scale invariance fixes the form of the two-point function up to a constant C:

$$\langle O_1(x_1)\, O_2(x_2)\rangle = \frac{C}{|x_1 - x_2|^{\Delta_1+\Delta_2}}\,. \tag{3.21}$$

Indeed, Poincaré invariance implies

$$\langle O_1(x_1)\, O_2(x_2)\rangle = f(|x_1 - x_2|) \tag{3.22}$$

and scale invariance under $x \to \lambda x$ implies

$$\langle O_1(x_1)\, O_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle O_1(\lambda x_1)\, O_2(\lambda x_2)\rangle \tag{3.23}$$

which, put together, yield the above constraint. The three-point function is also determined in a similar fashion [77]:

$$\langle O_1(x_1)\, O_2(x_2)\, O_3(x_3)\rangle = \frac{C_{123}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}\,|x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}\,|x_3 - x_1|^{\Delta_3+\Delta_1-\Delta_2}}\,. \tag{3.24}$$

From there, one knows everything that is needed to compute four- and higher-point functions, so a conformal field theory is completely specified by (i) its primary operators, (ii) the associated scaling dimensions, and (iii) the three-point coefficients C_{123}. This makes conformal field theories particularly easy to constrain using the so-called “bootstrap” methods. In particular, in Chapter 7, we will derive constraints on (ii) from modular transformation properties of chiral theories with charge.

The two-dimensional case

We will mostly be interested in three-dimensional gravity and its two-dimensional dual. The case of dimension two is especially interesting because there is an infinite number of generators, which means the symmetry constrains the theory enormously. Going back to (3.12), with just two spacetime coordinates (x_0, x_1) it reduces to

$$\begin{cases} \partial_0\,\varepsilon_0 - \partial_1\,\varepsilon_1 = 0 \\ \partial_0\,\varepsilon_1 + \partial_1\,\varepsilon_0 = 0 \end{cases} \tag{3.25}$$

in which we recognize the Cauchy-Riemann equations defining holomorphic functions. We then introduce the complex coordinates

$$z = x_0 + ix_1\,, \qquad \bar{z} = x_0 - ix_1\,, \tag{3.26}$$

$$\text{with} \qquad \partial \mathrel{\widehat{=}} \partial_z = \frac{1}{2}(\partial_0 + i\partial_1)\,, \qquad \bar{\partial} \mathrel{\widehat{=}} \partial_{\bar{z}} = \frac{1}{2}(\partial_0 - i\partial_1)\,,$$

and similarly for ε, $\bar{\varepsilon}$. The Cauchy-Riemann equations then become

$$\begin{cases} \bar{\partial}\varepsilon(z,\bar{z}) = 0 \qquad \text{with solution} \quad \varepsilon(z,\bar{z}) = f(z) \\ \partial\bar{\varepsilon}(z,\bar{z}) = 0 \qquad \text{with solution} \quad \bar{\varepsilon}(z,\bar{z}) = \bar{f}(\bar{z}) \end{cases} \tag{3.27}$$

with $f(z)$ arbitrary. Under an infinitesimal conformal transformation $z \to f(z) = z + \varepsilon(z)$, the metric transforms as

$$ds^2 = dz\,d\bar{z} \quad \to \quad \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}}\,dz\,d\bar{z} \tag{3.28}$$

which means the scale factor is simply $\Omega = \left|\frac{\partial f}{\partial z}\right|^2$. We just said that $f(z)$ is arbitrary, however, it is not expected to have essential singularities or branch points. In order to make sure that $f(z)$ is invertible, one requires that in addition it does not have poles of order bigger than one. This means that $f(z)$ is meromorphic and of the form

$$f(z) = \frac{az + b}{cz + d} \tag{3.29}$$

with $ad - bc \neq 0$, or $ad - bc = 1$ up to normalization.

For infinitesimal conformal transformations, $\varepsilon(z)$ can also be assumed to be meromorphic. One can then use a Laurent series expansion around $z = 0$ to rewrite the conformal transformations as

$$f(z) = z + \varepsilon(z) = z + \sum_{n\in\mathbb{Z}} c_n(-z^{n+1}) \tag{3.30}$$

and similarly for $\bar{f}(\bar{z})$. Each term of this infinite sum is generated by

$$\ell_m = -z^{m+1}\partial_z\,. \tag{3.31}$$

Together with the corresponding $\bar{\ell}_m$, these generators obey

$$\begin{aligned}[\ell_m, \ell_n] &= (m-n)\,\ell_{m+n}\,,\\ [\bar{\ell}_m, \bar{\ell}_n] &= (m-n)\,\bar{\ell}_{m+n}\,,\\ [\ell_m, \bar{\ell}_n] &= 0\end{aligned} \tag{3.32}$$

and so form two commuting copies of the Witt algebra. Since these two copies are independent, z and $\bar{z}$ are generally treated as independent variables. We see that as far as infinitesimal conformal transformations are concerned, the algebra of two-dimensional conformal field theories is indeed infinite-dimensional. We can however extract a finite-dimensional subalgebra $\{\ell_{-1}, \ell_0, \ell_1\}$ and their conjugates, which corresponds to the global conformal group as we will see shortly. These generators are singled out because they are the only ones well-defined on the Riemann sphere: from (3.31), it is clear that the ℓ_m are non-singular at $z = 0$ only for $m \geqslant -1$ and at $z = \infty$ for $m \leqslant 1$. They correspond to the previous expressions (3.15) in the following way:

$$P = i\,(\ell_{-1} + \bar{\ell}_{-1})\,, \qquad \bar{P} = \ell_{-1} - \bar{\ell}_{-1}\,, \tag{3.33}$$

$$L = -\ell_0 + \bar{\ell}_0\,, \tag{3.34}$$

$$D = i\,(\ell_0 + \bar{\ell}_0)\,, \tag{3.35}$$

$$K = i\,(\ell_1 + \bar{\ell}_1)\,, \qquad \bar{K} = -\ell_1 + \bar{\ell}_1\,. \tag{3.36}$$

Together they thus generate transformations of the form (3.29) that can also be described by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad a, b, c, d \in \mathbb{C}\,. \tag{3.37}$$

We recognize the group $SL(2, \mathbb{C})$, which is isomorphic to the Lorentz group $SO(3, 1) = SO(d+1, 1)$ with $d = 2$, as expected. Since the condition $ad - bc = 1$ is invariant under sign flip of all the variables, we can further identify the conformal group on the Riemann sphere as the Möbius group $SL(2, \mathbb{C})/\mathbb{Z}_2$, or more exactly the product $SL(2, \mathbb{C})/\mathbb{Z}_2 \times SL(2, \mathbb{C})/\mathbb{Z}_2$. The correct isomorphism is recovered when we impose the reality condition $\bar{z} = z^*$, but most of the time we will rather work in a space twice too big.

Witt algebras allow for central extensions; if we want to be completely general, we should then take that property into account. Using Jacobi identities, one can show that the central extension of the Witt algebra takes the form of a Virasoro algebra

$$[L_m, L_n] = (m - n)\, L_{m+n} + \frac{c}{12}\, m^2(m - 1)\, \delta_{m+n}\,. \tag{3.38}$$

The $\bar{L}_m$ generators obeying the same commutation relations, we end up with two copies of the Virasoro algebra, respectively characterized by central charge c and $\bar{c}$. Here again, the subalgebra $\{L_0, L_{\pm 1}, \bar{L}_0, \bar{L}_{\pm 1}\}$ generates all the global conformal transformations, as expected since finite-dimensional algebras do not have non-trivial central extensions.

To quantize a Poincaré-invariant theory, it makes sense to foliate spacetime by dividing it in surfaces of equal time. With extra scale-invariance though, one could also move from one foliation to the next using dilatations, rather than time-translations. In d dimensions, one can then build spacetime as a collection of nested S^{d-1} spheres of radius r. The evolution operator would then be $U = e^{iD\tau}$, with $\tau = \log r$ and D playing the role the Hamiltonian used to play in canonical quantization. In this radial quantization, moving from the origin to the edge of a sphere of infinite radius corresponds to going from infinite “past” to infinite “future” in terms of τ. Placing a state at a given “moment” thus amounts to inserting an operator on the sphere of radius corresponding to such a “moment”. The statement is that each state of a conformal field theory is in one-to-one correspondence with local operators.

One can then define these operators (acting at the origin) to be eigenstates $|\Delta\rangle$ of the dilatation operator with eigenvalue equal to the scaling dimension Δ, and identify:

$$\mathcal{O}_\Delta(0) \quad \leftrightarrow \quad |\Delta\rangle \ \widehat{=} \ \mathcal{O}_\Delta(0)\,|0\rangle \ . \tag{3.39}$$

In dimension two, it is convenient to define the conformal field theory in Euclidean space, with periodic boundary conditions in the space direction, in a way analogue to putting a quantum system "in a box". If (x_0, x_1) are the coordinates of Euclidean space, x_1 is then compactified on a circle of radius $R = 1$ and the theory is defined on an infinite cylinder. Radial quantization then amounts to projecting this cylinder of coordinates $w = x_0 + ix_1$ onto the complex plane via $z = e^w$. As time translations get mapped to dilatations and

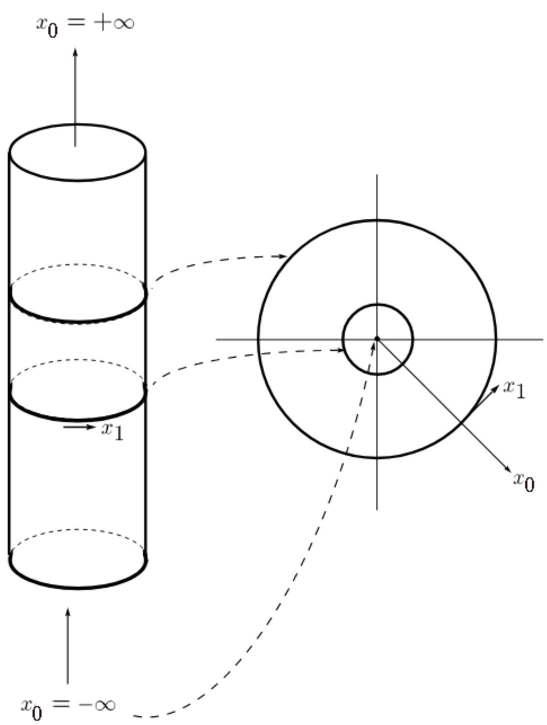

Figure 3.1 Radial quantization [74]. Euclidean time $x_0 = -\infty$ on the cylinder is mapped to the center of the circle, and $x_0 = \infty$ is mapped to the infinite radius circle.

space translations to rotations, the Hamiltonian and momentum operator are given by

$$H = L_0 + \bar{L}_0 \,, \qquad P = i(L_0 - \bar{L}_0) \,. \tag{3.40}$$

In the complex coordinates $(z, \bar{z})$, the tracelessness of the stress-energy tensor has the interesting consequence that

$$\partial_z \, T_{\bar{z}\bar{z}} = \partial_{\bar{z}} \, T_{zz} = 0 \,. \tag{3.41}$$

The non-trivial components of the stress-energy tensor are then a holomorphic (or chiral) and an anti-holomorphic (or anti-chiral) field, which we will respectively call $T(z)$ and $\bar{T}(\bar{z})$. One can then relate these to the Virasoro generators above by way of their Laurent expansion as

$$T(z) = \sum_{n \in \mathbb{Z}} L_n \, z^{-n-2} \qquad \text{with} \qquad L_n = \frac{1}{2\pi i} \oint dz \, z^{n+1} \, T(z) \,. \tag{3.42}$$

In fact the central extension of the Witt algebra into the Virasoro algebra can be seen as a quantum feature that emerges when we quantize the classical theory.

Primary states are now defined to be highest-weight states $|h\rangle$ of the Virasoro algebra's representations, such that

$$L_0 \, |h\rangle = h \, |h\rangle \,, \qquad L_m \, |h\rangle = 0 \quad \text{for} \quad m > 0 \,. \tag{3.43}$$

As for the Witt generators, the L_m play the role of annihilation operators while the L_{-m} act as creation operators, the "highest" weight states being actually the ones of lowest energy. It is indeed natural to expect the L_0 eigenvalue to be bounded from below if we are to relate this conformal field theory to any sort of physical meaning. The descendant states

$$L_{-k_1} L_{-k_2} \dots L_{-k_n} \, |h\rangle \,, \qquad k_i \geqslant 1 \tag{3.44}$$

are then eigenstates of L_0 with eigenvalue

$$h + \sum_{i=1}^{n} k_i \; \widehat{=} \; h + N \,. \tag{3.45}$$

At each level N, the number of descendants has a simple expression provided $c > 1$: it is given by the number $p(N)$ of partitions of N in distinct positive integers. For $c \leqslant 1$, things get more complicated since one has to take null states (states with vanishing norm) into account. We will not need to worry about those, since we are only interested in large central charge cases.

The Virasoro representation, made of $|h\rangle$ and all linear combinations of its descendant states, is unitary if both h and $c \geqslant 0$. Indeed, at level N the squared norm of a generic state is

$$\langle h|\, L_N L_{-N}\, |h\rangle = 2N\, h + \frac{c}{12} N(N^2 - 1)\,. \tag{3.46}$$

The condition on h comes directly from the $N = 1$ case, while the one for c springs from positivity for large N. The character of an irreducible, unitary representation of the Virasoro algebra with highest weight h and Hilbert space $\mathcal{H}$ is defined as

$$\chi_h(\tau) \;\widehat{=}\; \mathrm{Tr}_{\mathcal{H}}\left(q^{L_0 - c/24}\right)\,, \qquad q \;\widehat{=}\; e^{2\pi i \tau}\,. \tag{3.47}$$

The normalization $-c/24$ is a convention that will prove useful later on. We will see shortly what meaning can be attributed to τ in the context where we place the theory on the torus. For $h > 0$ and $c > 1$, there are no null states so the L_0 spectrum is given by the values of $h + N$ with multiplicity $p(N)$. The character then reads

$$\chi_h(\tau) = q^{h-c/24} \sum_{N \geqslant 0} p(N)\, q^N\,, \tag{3.48}$$

One can show [78] that this is equivalent to

$$\chi_h(\tau) = q^{h-c/24} \prod_{n \geqslant 1} \frac{1}{1-q^n} = \frac{q^{h-(c-1)/24}}{\eta(\tau)}\,, \tag{3.49}$$

where we have introduced the Dedekind eta function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} \left(1 - q^n\right)\,. \tag{3.50}$$

The case of the vacuum representation $h = 0$ is a bit particular. Indeed at non-zero central charge, one immediately notices that it is impossible to define a state that vanishes under the action of all Virasoro generators: such a state $|h_0\rangle$ would have to verify

$$\langle h_0| L_N L_{-N} |h_0\rangle = \frac{c}{12} N(N^2-1) \langle h_0 \mid h_0\rangle = 0 \tag{3.51}$$

which is only satisfied for $c \neq 0$ when $N = -1, 0, 1$. The best we can get is a vacuum invariant under the subalgebra generated by L_{-1}, L_0 and L_1 but not under the full Virasoro algebra. This makes the vacuum representation non-trivial as it contains all the descendant states

$$L_{-k_1} L_{-k_2} \dots L_{-k_n} |h\rangle \,, \qquad k_i \geqslant 2\,. \tag{3.52}$$

The vacuum is then defined to be the state $|0\rangle$ such that

$$L_{-1}|0\rangle = L_0|0\rangle = L_m|0\rangle = 0\,, \qquad \text{for all} \quad m > 0\,. \tag{3.53}$$

The character of the vacuum representation is then almost the same as (3.49), except for $L_{-1}|0\rangle$ that does not enter the sum anymore:

$$\chi_0(\tau) = q^{-c/24} \prod_{n \geqslant 2} \frac{1}{1-q^n}\,. \tag{3.54}$$

In particular, it is not simply the $h \to 0$ limit of the generic $h \neq 0$ character.

Theory on the torus, modular invariance and the Cardy formula

So far we have deliberately ignored the question of the relation to the physical world. We have accepted to treat z and $\bar{z}$ as distinct variables, but we know that physicality requires that we recombine our chiral and anti-chiral fields via some constraints. Geometry can help us achieving that while still being able to use the full power of complex analysis with holomorphic functions. In quantum field theories, a natural way to determine which fields the theory contains is to consider loop diagrams. Since the complex plane is topologically equivalent to the Riemann sphere, the next level of complexity (or more precisely, genus) would be the torus. Studying conformal field theories on the torus is a reasonable way to obtain the sort of constraints we are looking for.

With radial quantization, we have already placed our theory on a cylinder; in order to consider a torus, we just need to impose another periodic boundary condition along the cylinder. These two boundary conditions can be seen as two lattice vectors α_1 and α_2 that define the torus on the complex plane. The ratio of these two complex numbers $\tau = \alpha_2/\alpha_1$, or modular parameter, is actually the relevant parameter and plays a major part into what follows. A convenient choice is $(\alpha_1, \alpha_2) = (1, \tau)$ and τ is generally chosen in the upper half

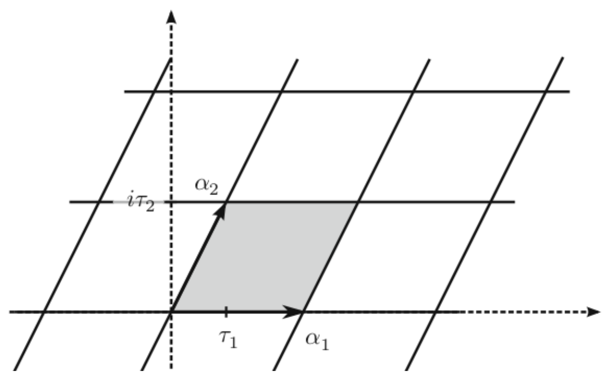

Figure 3.2 Torus lattice with $(\alpha_1, \alpha_2) = (1, \tau)$ [72].

plane. As it turns out, this lattice is invariant under transformations such as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})/\mathbb{Z}_2 \tag{3.55}$$

and so equivalent tori are obtained by way of $\tau \to \frac{a\tau+b}{c\tau+d}$. The isometry group of the torus, or modular group, is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{3.56}$$

which in particular act on τ as $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$.

One can show that under transformations $z \to f(z)$, the stress-energy tensor transforms as

$$T'(w) = (\partial_w f)^2 \, T\big(f(w)\big) + \frac{c}{12} \, S\big(f(w), w\big) \tag{3.57}$$

with the Schwarzian derivative

$$S(z,w) = \frac{1}{(\partial_w z)^2}\left((\partial_w z)(\partial_w^3 z) - \frac{3}{2}(\partial_w^2 z)^2\right) . \tag{3.58}$$

When we go from the plane to the cylinder through the map $z = e^w$, we get

$$T^{\text{cyl}}(w) = z^2 T(z) - \frac{c}{24} \tag{3.59}$$

which means the Virasoro generator L_0 gets modified as

$$L_0^{\text{cyl}} = L_0 - \frac{c}{24} \tag{3.60}$$

and similarly for the conjugate quantities. The vacuum energy on the cylider is then

$$E_0 = -\frac{c+\bar{c}}{24} . \tag{3.61}$$

We can now build a partition function for our conformal field theory. Choosing the coordinate system such that the space and time directions coincide respectively with the real and imaginary axes, we see that moving along the lattice in the time direction also makes us move a bit along the space direction. Namely, if $\tau = \tau_1 + i\tau_2$, a time step of τ_2 comes with a space step of τ_1. The partition function is then

$$Z = \text{Tr}_{\mathcal{H}}\left(e^{-2\pi\tau_2 H}\, e^{2\pi\tau_1 P}\right) \tag{3.62}$$

which, since (3.40) also works for the generators on the cylinder, we can rewrite as

$$Z = \text{Tr}_{\mathcal{H}}\left(e^{2\pi i\tau L_0^{\text{cyl}}}\, e^{-2\pi i\bar{\tau}\bar{L}_0^{\text{cyl}}}\right) = \text{Tr}_{\mathcal{H}}\left(q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - \bar{c}/24}\right) . \tag{3.63}$$

We recognize the Virasoro characters (3.47) and go on to writing

$$Z(\tau) = \sum_{h,\bar{h}} N_{h\bar{h}}\, \chi_h(\tau)\, \bar{\chi}_{\bar{h}}(\bar{\tau}) \tag{3.64}$$

with some multiplicity factor $N_{h\bar{h}}$ for each irreducible representation. This partition function is then completely independent of any Lagrangian formulation.

But its most striking feature is that it is invariant under the action of the modular group, as we have seen above. Modular invariance of a conformal field theory's partition function on the torus is precisely the feature that will allow us to constrain the theory and extract useful information without having to know it all in details. In particular, under S-transformations,

$$Z(\tau,\bar{\tau}) = Z\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) . \tag{3.65}$$

This property allows one to relate the behaviors of the theory at high and low temperatures. Indeed, remembering that in thermodynamics the parameter β that appears in the Boltzmann factor $e^{-\beta H}$ of the partition function is the inverse temperature, we can identify it with the imaginary part τ_2 of the modular parameter that multiplies H in our partition function of interest. More generally, one is led to set

$$\tau = \frac{1}{2\pi}\left(i\beta + \theta\right) \tag{3.66}$$

where the angular potential θ can also be written as $\beta\Omega$, with Ω the angular velocity. The statement above then translates to

$$Z(\beta,\bar{\beta}) = Z\left(\frac{4\pi^2}{\beta}, \frac{4\pi^2}{\bar{\beta}}\right) \tag{3.67}$$

which relates the high and low temperature regimes of the theory. For simplicity's sake, we will allow ourselves for a moment to ignore the anti-chiral sector of the theory. At low temperature the partition function is dominated by the contribution of the ground state, which we remember is $-c/24$ in this case:

$$Z(\beta) \underset{\beta\to\infty}{\to} e^{\frac{c\beta}{24}} \underset{(3.67)}{=} e^{\frac{c\pi^2}{6\beta}} . \tag{3.68}$$

The left-hand side then describes the partition function at high temperatures. Remembering the thermodynamic entropy and energy are respectively given by

$$S = (1 - \beta\, \partial_\beta) \ln Z\,, \qquad E = -\partial_\beta \ln Z \tag{3.69}$$

we can conclude

$$S = 2\pi\sqrt{\frac{c}{6}E} \qquad\qquad (\beta \to 0)\,. \tag{3.70}$$

This is in essence the Cardy formula [26], often also written in terms of the (asymptotic) density of states $\rho(E)$ defined by

$$Z(\beta) = \sum_E \rho(E)\, e^{-\beta E} \tag{3.71}$$

as

$$\rho(E) = \exp\left(2\pi\sqrt{\frac{c}{6}E}\right) \qquad\qquad (E \to \infty)\,. \tag{3.72}$$

This important result concludes our overview of conformal field theories. One may have noticed a few parallels between the material presented here and in the previous chapter: in particular, the Witt algebra coming up both as the asymptotic symmetry algebra of three-dimensional anti-de Sitter and as the defining algebra of two-dimensional conformal field theories, or the cylinder shaped boundary of the former matching the preferred setup for defining the latter. In the next chapter we will expand on these similarities and make precise the proposal that they are in fact dual to each other.

Chapter 4

AdS/CFT correspondence

In the previous chapters, we have reviewed the general features of both sides of the most well-known example of holographic duality, namely the AdS/CFT correspondence. One may have already spotted hints of why these two theories are thought to be dual. Here we propose one way to make it explicit without resorting to string theory. We then give some insight into the holographic principle, and how the correspondence was conjectured, before moving on to an illustration of the sort of results it can lead to in terms of black holes.

4.1. Chern-Simons formulation

A good way to understand gauge/gravity duality is to simply rewrite gravity as a gauge theory to begin with. Then, by carefully analyzing the asymptotic symmetries of this gauge theory, one can see some features of the dual gauge theory (which only belongs on the boundary of our spacetime) emerge from this picture. This rewriting can be done in the three-dimensional case by way of the Chern-Simons formulation [79, 80], and is valid for any sign of the cosmological constant. Instead of having the action depend on the metric $g_{\mu\nu}$, one expresses everything in terms of a frame field $e^a = e^a{}_\mu dx^\mu$ defined such that

$$g_{\mu\nu}(x) = e^a{}_\mu(x)\eta_{ab}\, e^b{}_\nu(x) \tag{4.1}$$

where η_{ab} is as usual the flat metric. This frame field or *vielbein* is not unique, but defined modulo Lorentz transformations, and it has an inverse $e^\mu{}_a$. It sort of translates "flat" quantities into "curved" ones. The role of the gauge field is played by the spin connection $\omega^{ab} = \omega^{ab}{}_\mu dx^\mu$, with $\omega^{ab} = -\omega^{ba}$. As in more familiar gauge theories like Yang-Mills, this connection is used to build a covariant derivative $D = \partial + \omega$. We can then proceed with the definition of all the usual relativistic quantities such as Christoffel symbols

$$\Gamma^\rho{}_{\mu\nu} = e^\rho{}_a \left(\partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b \, e^b{}_\nu\right) , \tag{4.2}$$

torsion

$$T^a = de^a + \omega^a{}_b \wedge e^b \,, \tag{4.3}$$

and curvature

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \,, \tag{4.4}$$

provided that the connection transforms as

$$\omega^a{}_b \quad \to \quad \Lambda^{-1}{}_c{}^a \; d\Lambda^c{}_b + \Lambda^{-1}{}_c{}^a \; \omega^c{}_d \; \Lambda^d{}_b \,. \tag{4.5}$$

Equations (4.3) and (4.4) are the first and second Cartan structure equations. The Einstein-Hilbert action (2.6) in AdS_3 can then be rewritten in these terms as [65]

$$S_{EH}[\omega] = \frac{1}{16\pi G} \int_{\mathcal{M}} \left(2e_a \wedge R_a + \frac{1}{3\ell^2} \; \varepsilon_{abc} \; e^a \wedge e^b \wedge e^c\right) \tag{4.6}$$

with

$$R_a = \frac{1}{2} \, \varepsilon_{abc} \, R^{bc} \,, \qquad \omega_a = \frac{1}{2} \, \varepsilon_{abc} \, \omega^{bc} \tag{4.7}$$

(this notation only working in dimension 3) and we have used the fact that $\det(e^a{}_\mu) = \sqrt{-g}$. One can show [80] that this is equivalent to the Chern-Simons action

$$S_{CS}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} \; A \wedge A \wedge A\right) \tag{4.8}$$

provided one identifies k as

$$k = \frac{1}{4G} \,. \tag{4.9}$$

For three-dimensional anti-de Sitter, the gauge field $A = A_\mu \, dx^\mu$ belongs to the Lie algebra $so(2,2)$

$$[J_a, J_b] = \varepsilon_{abc} \, J^c \,, \qquad [J_a, P_b] = \varepsilon_{abc} \, P^c \,, \qquad [P_a, P_b] = \varepsilon_{abc} \, J^c \,. \tag{4.10}$$

In terms of the generators J^a, P^a, the gauge field reads

$$A_\mu = \frac{1}{\ell} e^a{}_\mu P_a + \omega^a{}_\mu J_a \,. \tag{4.11}$$

Since we know that $so(2,2) \simeq sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$, it is not surprising that we can actually split the Chern-Simons action into two similar parts as

$$S_{CS}[\tilde{A}] = S_{CS}[A] - S_{CS}[\bar{A}] \tag{4.12}$$

(where we denoted our original A by $\tilde{A}$) if we set

$$A = \left(\frac{e^a}{\ell} + \omega^a \right) T_a \,, \qquad \bar{A} = \left(\frac{e^a}{\ell} - \omega^a \right) T_a \,. \tag{4.13}$$

The T_a here have nothing to do with the torsion evoked above, but are now the $sl(2,\mathbb{R})$ generators.

It has been shown [81] that the most general solution of the equations of motion with Brown-Henneaux boundary conditions (2.29) can be written

$$ds^2 = \frac{\ell^2}{r^2} \, dr^2 - \left(r \, dx^+ - \frac{\ell^2}{r} \, L(x^-) \, dx^- \right) \left(r \, dx^- - \frac{\ell^2}{r} \, \bar{L}(x^+) \, dx^+ \right) \tag{4.14}$$

where $L(x^-)$ and $\bar{L}(x^+)$ are two arbitrary chiral functions. In these terms, the Chern-Simons connections are

$$A = \begin{pmatrix} \frac{dr}{2r} & \frac{\ell}{r} \, \bar{L}(x^+) \, dx^+ \\ \frac{r}{\ell} \, dx^+ & -\frac{dr}{2r} \end{pmatrix} , \qquad \bar{A} = \begin{pmatrix} -\frac{dr}{2r} & \frac{r}{\ell} \, dx^- \\ \frac{\ell}{r} \, L(x^-) \, dx^- & \frac{dr}{2r} \end{pmatrix} . \tag{4.15}$$

It is convenient to use their r-independent, reduced version

$$a = \begin{pmatrix} 0 & \ell\,\bar{L}(x^+)\,dx^+ \\ dx^+/\ell & 0 \end{pmatrix}, \qquad \bar{a} = \begin{pmatrix} 0 & dx^-/\ell \\ \ell\,L(x^-)\,dx^- & 0 \end{pmatrix} \tag{4.16}$$

obtained by a simple gauge transformation [65]. The charges associated to asymptotic symmetries take the form [82]

$$Q_\lambda = -\frac{k}{2\pi}\int_0^{2\pi} \frac{\ell}{\sqrt{2}}\,\lambda^1\,L\,d\varphi\,, \qquad \bar{Q}_{\bar{\lambda}} = -\frac{k}{2\pi}\int_0^{2\pi} \frac{\ell}{\sqrt{2}}\,\bar{\lambda}^0\,\bar{L}\,d\varphi \tag{4.17}$$

where the λ^i are parameters describing the asymptotic symmetries

$$\delta a = d\lambda + [a,\lambda]\,, \qquad \delta\bar{a} = d\bar{\lambda} + [\bar{a},\bar{\lambda}]\,. \tag{4.18}$$

Using the Poisson brackets of these charges and defining the modes

$$L_m \;\widehat{=}\; \frac{k}{2\pi}\int_0^{2\pi} e^{im\varphi}\,L\,d\varphi\,, \tag{4.19}$$

one can compute their algebra

$$i\{L_m, L_n\} = (m-n)L_{m+n} + \frac{c}{12}\,m^3\,\delta_{m+n} \tag{4.20}$$

(and similarly for $\bar{L}$). Upon shifting the zero modes by $L_0 \to L_0 - c/24$, we get the usual factor $m(m^2-1)$ instead of m^3, and switching to ordinary commutators via $i\{.\,,\,.\} \to [.\,,\,.]$, we retrieve the Virasoro algebra (3.38) with central charges

$$c \;=\; \bar{c} \;=\; 6k \;=\; \frac{3\ell}{2G}\,. \tag{4.21}$$

Remarkably, this result was derived at a purely classical level, whereas the central extension in (3.38) typically comes as a quantum feature for the two-dimensional conformal field theory. The Brown-Henneaux central charge (4.21) was first obtained in [7], where the charges were computed in the Hamiltonian formalism via the asymptotic Killing vectors, and is generally considered to be the first hint of the AdS/CFT correspondence.

4.2. The holographic principle and AdS/CFT

The intuition at the root of the correspondence in question was first formulated as the holographic principle [12]. Considering some gas, simply constrained by the Schwarzschild limit and the Bekenstein bound (1.4), 't Hooft estimated that the number of degrees of freedom in a volume of space was given in quantum gravity in terms of the ones of the surface enclosing it. In this picture, the physical degrees of freedom are projected onto the boundary in the manner of a hologram. Of course, this statement is not to be taken too literally: "the image is somewhat blurred because of limitations of the hologram technique, but the blurring is small compared to the uncertainties produced by the usual quantum mechanical fluctuations" [12]. Similar ideas were expressed in [13]. The upshot is that the theory contains much less degrees of freedom that we would have expected from the field theory point of view, making quantum field theory a "highly redundant effective description" [63]. Contrary to the latter, the holographic picture also gives preference to unitarity over locality; the question then remains: should one retain locality and look for a gauge invariance that leaves the right number of degrees of freedom, or should locality be viewed as an emergent phenomenon?

The AdS/CFT correspondence is considered to be the most concrete realization of the holographic principle. The original proposal [9] is that $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory is dual to type IIB string theory on an asymptotically $\mathrm{AdS}_5 \times S^5$ space, in the sense that the low energy dynamics of a stack of $D3$-branes in both open and closed strings perspectives match (each one yielding one side of the duality). To be more precise, a stack of N branes couples to gravity in a way proportional to $g_s N$, with g_s the string coupling. When $g_s N$ is small, there are open strings on the branes, described at low energy by a $U(N)$ gauge theory, and closed strings away from the brane. When $g_s N$ is large, the pile of branes describes a black brane (a generalization of a black hole for extended objects) with near-horizon geometry $\mathrm{AdS}_5 \times S^5$. At low energy, the strings near the black brane decouple from the ones far away from it. The same decoupling happens for the small $g_s N$ picture between the open and the closed strings, and it is then natural to conjecture that these are the same low energy physics, but seen at different values of the coupling [83]. The conjecture is then that the equivalence persists even when the coupling is strong. The fact that the symmetry group of $\mathrm{AdS}_5 \times S^5$, which is $SO(2,4) \times SO(6)$, corresponds to the conformal group in dimension four plus the rotation symmetry of the scalars in the gauge theory, argues in favour of that conjecture. The "dictionary" relating states on both sides of a more general $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ duality was then further developed to enable the computation of n-point functions [11]. In particular, masses on the gravity side were related to conformal dimensions on the gauge side. Various checks of this conjecture have been made; among these, the Hawking-Page phase transition between thermal AdS (i.e., empty AdS plus a gas of gravitons) and the black hole was recovered in the gauge theory [84]. Following [11], gauge/gravity duality is generally understood as an equivalence of partition functions:

$$Z_{\text{gauge}}(\varphi_0) \;\equiv\; Z_{\text{gravity}}(\varphi_0) \tag{4.22}$$

where the gravitational partition function is a function of the boundary values of massless fields φ_0, which play the role of sources for the conformal field theory's correlation functions. This is the sense in which we will think of it here. Before moving on to other matters, let us conclude with a quick epistemic point: however convincing the evidence in its favour may be, holography is a conjecture and remains to be proved. In this thesis we will not discuss this aspect, but rather take the proposed duality as an assumption and see where it could lead, were it to prove valid.

4.3. Black hole entropy

As an example of how powerful thinking in terms of holography can be, let us now turn to the black hole entropy. As we have seen in Section 1.2, black holes have thermodynamical properties, including an entropy (1.2) which is proportional to the area of their horizon. Since entropy is proportional to a volume in usual thermodynamics, this suggests that black holes have fewer states that one would expect[1]. Nevertheless, this means a statistical mechanical

[1] Note however that there are arguments supporting the idea that black holes have in fact more states than those counted by the Bekenstein-Hawking entropy: the latter can indeed be seen as counting near-horizon states, whereas additional interior states would not be reflected by it as a consequence of the fact that a black hole's effect on its surroundings is independent from its interior [85]. This actually provides yet another solution to the information paradox.

description in terms of microscopic degrees of freedom should be available. Indeed, the entropy of a system is given by the number Ω of microstates this system can be in as

$$S = k_B \ \ln \Omega \,. \tag{4.23}$$

The question is of course what microstates the entropy is accounting for in the case of a black hole. Anticipating on the results of [9], in [8] they were identified exactly in the conformal field theory dual to a five-dimensional extremal black hole in string theory. The computation of the number of possible configurations allowed to recover the Bekenstein-Hawking entropy of the corresponding black hole. Without getting too much into the details of that derivation, which would go well beyond our purpose, let us just note that it relied in part on Cardy's result (3.70). In the case of $\mathrm{AdS}_3/\mathrm{CFT}_2$, this formula offers a straightforward connection between the density of states in the conformal field theory and the BTZ black hole entropy [86]. From the full Cardy formula

$$S_{\text{Cardy}} = 2\pi \sqrt{\frac{c}{6}\left(h - \frac{c}{24}\right)} + 2\pi \sqrt{\frac{c}{6}\left(\bar{h} - \frac{c}{24}\right)} \tag{4.24}$$

for a generic two-dimensional conformal field theory, one retrieves the Bekenstein-Hawking entropy (2.17) by replacing the Brown-Henneaux central charge (4.21) and the BTZ horizon radius (2.12), which can be rewritten

$$r_+^2 = 16\, G\, \ell \left(h - \frac{c}{24}\right) \,. \tag{4.25}$$

Indeed, in the case of the BTZ black hole, the mass and charge are given in terms of the Virasoro generators by [87]

$$\ell\, M = L_0 + \bar{L}_0 \,, \qquad J = L_0 - \bar{L}_0 \,, \tag{4.26}$$

where one remembers that the eigenvalues of these generators are shifted by $-c/24$. The expression above is easily obtained in the case without rotation, where $h = \bar{h}$. Incidentally, we have seen in Section 2.1 that this type of black holes satisfy a cosmic censorship condition $|J| \leqslant M\ell$. Black holes will then be interpreted as conformal field theory primary states with both h and $\bar{h}$ greater than $c/24$, or

$$\Delta = h + \bar{h} \geqslant \frac{c}{12} \,. \tag{4.27}$$

Descendants are understood to be perturbative excitations of primary states, or "boundary gravitons" in the gravity picture [88].

Realizing the full consequences of the duality with conformal field theories (through the use of Cardy's formula) has in this case made the details of the microscopic stringy description irrelevant. As elegant as it is, this result is nonetheless puzzling: indeed, the Bekenstein-Hawking entropy is valid in a semi-classical regime with large central charge, whereas Cardy's formula holds for fixed central charge, as long as the conformal dimension is large with respect to it. In other words, the Bekenstein-Hawking regime is $c \to \infty$ with h/c fixed while the Cardy regime is $h/c \to \infty$ with c fixed [89]. One could also say that what makes a conformal field theory holographic is that it sees the regime of validity of Cardy's formula extended into the Bekenstein-Hawking realm. In order to tell these conformal field theories apart from the others, one needs to understand when this extension of one regime into the other happens. It was proposed [90] that a necessary and sufficient criterion would be the sparseness of the light spectrum of the theory, by which one should understand that the density of states with conformal dimension smaller than $h - c/24$ does not grow too fast, i.e.

$$\rho(h) \lesssim \exp(2\pi h) \,. \tag{4.28}$$

A different argument was made for chiral two-dimensional conformal field theories, or more exactly families thereof parametrized with $N = c/24$, according to which just as the validity of Cardy's formula in the Cardy regime outlined above relies on modular invariance, its validity in the Bekenstein-Hawking regime is related to another symmetry between the behaviour at large N and at small N [89].

We have thus seen how holography relates theories of gravity in three-dimensional anti-de Sitter space and two-dimensional conformal field theories. This is the most common incarnation of gauge/gravity dualities, but the question remains: can we extend this idea to spacetimes more relevant for the description of the physical world, and in particular, can it be used to understand realistic models of black holes? These are the issues we address in the next chapters.

Part II

Beyond AdS/CFT

Chapter 5

Warped holography

With holography, light has been shed on the microstates of a number of supersymmetric black holes and their entropy. However, these results cannot tell us much about non-supersymmetric, non-extremal astrophysical black holes. A better candidate for a realistic model of black hole would be the Kerr black hole [67, 91]. Even in its near-extremal version, it can be related to known astrophysical black holes like the ones in the X-ray binaries GRS 1915+105 [92] or Cygnus X-1 [93]. Attempts to extend the reach of gauge/gravity dualities to this realm have been made, with in particular the Kerr/CFT proposal [17] where it was conjectured that gravity near the horizon of extreme Kerr was dual to a conformal field theory. But as we have seen, conformal field theories are associated to a full $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ isometry group, while near-horizon extreme Kerr (and actually extremal or near extremal black holes in any dimension [19, 94]) possesses an $SL(2,\mathbb{R}) \times U(1)$ isometry group [95]. In addition, the geometry near the horizon of extreme Kerr is described not by AdS_3 but by a deformation of it (squashed or stretched) [32], called Warped AdS_3 [18]. Other dualities that might be relevant for real black holes have then been explored along these lines, such as Warped AdS_3/CFT [18] or Warped AdS_3/Warped CFT [19, 20]. In this second part of the thesis, we will extend some known results of Einstein gravity to higher-curvature theories of gravity warped AdS_3 are solutions thereof. We will then build on the proximity between warped conformal field theories and chiral conformal field theories to leverage the latter's modular properties and extract information about warped black holes. We start out in this chapter with a review of Warped AdS_3/Warped CFT.

5.1. Warped anti-de Sitter

Warped AdS_3 spaces can be obtained by deforming anti-de Sitter metric (2.3) along one of its fibers. Starting from AdS_3 as a Hopf fibration $AdS_2 \times \mathbb{R}$, one multiplies the fiber by a constant factor, which effectively deforms the geometry in a way analogue to the squashing or stretching of a sphere. The conventional warping factor is $4\nu^2/(\nu^2+3)$ so that AdS_3 is recovered for $\nu = 1$. As a result, the overall ℓ^2 factor also gets rescaled, and the spacelike[1] warped AdS_3 metric is

$$ds^2 = \frac{\ell^2}{\nu^2+3}\left[-\cosh^2\rho\, d\tau^2 + d\rho^2 + \frac{4\nu^2}{\nu^2+3}\Big(du + \sinh\rho\, d\tau\Big)^2\right]. \tag{5.1}$$

The introduction of warping parameter ν results in breaking the $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ isometry group of AdS_3 down to a $SL(2,\mathbb{R}) \times U(1)$ isometry group. For $\nu^2 \neq 1$, as in the AdS_3 case, black hole solutions exist and are quotients of the warped AdS_3 metric [18]. The black hole metric can be written in the so-called warped black hole coordinates of [96] as:

$$ds^2 = dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2}(\nu^2+3) - 12mr + \frac{4j\ell}{\nu}} + d\varphi^2\left(\frac{3r^2}{\ell^2}(\nu^2-1) + 12mr - \frac{4j\ell}{\nu}\right) + dt\, d\varphi\left(\frac{-4\nu r}{\ell}\right) \tag{5.2}$$

with $r \in [0,\infty)$, $t \in (-\infty,\infty)$, $\varphi \in [0,2\pi]$, and (m,j) are parameters characterizing the black hole. This black hole possesses two horizons as long as $j < 9\ell m^2\nu/(3+\nu^2)$. They are located at

$$r_\pm = \frac{2\ell^2}{\nu^2+3}\left(3m \pm \sqrt{9m^2 - \frac{j}{\nu\ell}(\nu^2+3)}\right). \tag{5.3}$$

The warped black hole was shown to verify the first law of black hole thermodynamics [97].

We will restrict here to warp parameter $\nu^2 > 1$ (the "stretched" solution), which exhibits no pathologies such as naked closed timelike curves, unlike its "squashed" counterpart with $\nu^2 < 1$. The parameter ν, which we assume to be positive without loss of generality [18], is determined by the equations of motion of the theory under consideration in terms of its coupling constants (as is ℓ).

Warped black holes are not asymptotically AdS_3 and do not belong to the Brown-

[1] We leave aside the related timelike and null cases, where a different form of the anti-de Sitter metric is deformed to begin with.

Henneaux phase space. Instead, they satisfy the boundary conditions of asymptotically warped AdS_3 spaces [96, 98, 99, 100, 101]

$$(g_{\mu\nu}) \underset{r\to\infty}{\sim} \begin{pmatrix} 1+O(r^{-1}) & O(r^{-2}) & \frac{-2\nu r}{\ell}+O(r^0) \\ O(r^{-2}) & \frac{\ell^2}{\nu^2+3}\frac{1}{r^2}+O(r^{-3}) & O(r^{-1}) \\ \frac{-2\nu r}{\ell}+O(r^0) & O(r^{-1}) & \frac{3(\nu^2-1)}{\ell^2}r^2+O(r) \end{pmatrix}. \tag{5.4}$$

The infinitesimal diffeomorphisms leaving these boundary conditions invariant are generated by the asymptotic Killing vectors [96]

$$\begin{aligned} l_m &= \left(e^{im\varphi}+O(r^{-1})\right)\partial_t + \left(-imre^{im\varphi}+O(1)\right)\partial_r + \left(e^{im\varphi}+O(r^{-2})\right)\partial_\varphi \\ p_m &= \left(e^{im\varphi}+O(r^{-1})\right)\partial_t. \end{aligned} \tag{5.5}$$

These generators obey the following commutation relations:

$$i[l_m,l_n]=(m-n)\,l_{m+n}, \qquad i[l_m,p_n]=-n\,p_{m+n}, \qquad [p_m,p_n]=0\,. \tag{5.6}$$

The conserved charges L_m, P_m associated to these generators l_m, p_m satisfy a Virasoro-Kac-Moody algebra

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{c}{12}\,m(m^2-1)\,\delta_{m+n} \tag{5.7}$$

$$[L_m,P_n]=-n\,P_{m+n} \tag{5.8}$$

$$[P_m,P_n]=\frac{k}{2}\,m\,\delta_{m+n}\,. \tag{5.9}$$

The central charge c and level k depend once again on the particular gravitational theory in which one is working. It is important to note that warped AdS_3 spaces do not satisfy Einstein's equations: they are solutions [102, 103] of extensions of Einstein gravity, like Topologically Massive Gravity [104, 105] or New Massive Gravity [106, 107]. These extensions have been introduced to remedy the lack of propagating degrees of freedom in three-dimensional Einstein gravity and provide a stepping stone towards four-dimensional models [104]. For example, the action of Topologically Massive Gravity is

$$S_{\rm TMG} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(R+\frac{2}{\ell^2}\right)+\frac{1}{32\pi G\mu}\int d^3x\sqrt{-g}\,\varepsilon^{\lambda\mu\nu}\,\Gamma^{\rho}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}_{\rho\nu}+\frac{2}{3}\,\Gamma^{\sigma}_{\mu\tau}\Gamma^{\tau}_{\nu\rho}\right) \tag{5.10}$$

with $\varepsilon^{012}=1/\sqrt{-g}$ the Levi-Civita tensor, and the graviton mass μ is related to the warping parameter ν as $\mu = 3\nu/\ell$. In this case, the central charge and level are given by [19]

$$c_{\rm TMG} = \frac{5\nu^2+3}{\nu(\nu^2+3)}\,, \qquad k_{\rm TMG} = -\frac{\nu^2+3}{6\nu}\,. \tag{5.11}$$

The New Massive Gravity action, by contrast, is given by

$$S_{\rm NMG} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left((R-2\Lambda)+\frac{1}{\mu^2}\Big(R_{\mu\nu}R^{\mu\nu}-\frac{3}{8}R^2\Big)\right) \tag{5.12}$$

and the central charge and level of the Virasoro-Kac-Moody algebra characterizing the warped duality then become

$$c_{\rm NMG} = \frac{96\,\ell\,\nu^3}{(\nu^2+3)(20\nu^2-3)}\,, \qquad k_{\rm NMG} = -\frac{4\nu(\nu^2+3)}{\ell(20\nu^2-3)}\,. \tag{5.13}$$

In this case, warped black holes are solutions of the theory for the following couplings:

$$\mu^2 = \frac{3-20\nu^2}{2\ell^2}\,, \qquad \Lambda = \frac{4\nu^4-48\nu^2+9}{2\ell^2\,(20\nu^2-3)}\,. \tag{5.14}$$

We will not work in any of these theories specifically, but the results we derive in the next chapter for generic higher curvature theories apply to these as well.

As a final comment, let us mention that as one could have expected, warped black holes are related to BTZ black holes (2.9). More specifically, warped AdS$_3$ is related to AdS$_3$ by

$$ds^2_{\rm WAdS_3} = ds^2_{\rm AdS_3} - 2H^2\xi\otimes\xi \tag{5.15}$$

where ξ is a Killing vector[2] of AdS_3 and H is related to the warping parameter ν as $H^2 = \frac{3(1-\nu^2)}{2(\nu^2+3)}$. Similarly, warped black holes can be obtained by deforming the BTZ metric as [108]

$$ds^2_{\mathrm{WBH}} = ds^2_{\mathrm{BTZ}} - 2H^2 \xi \otimes \xi \tag{5.16}$$

with

$$\xi^\mu \partial_\mu = \sqrt{\frac{\ell}{8(M\ell - J)}}(-\partial_t + \partial_\varphi) \tag{5.17}$$

in terms of BTZ quantities and coordinates (see [108] for an explicit map).

5.2. Warped conformal field theories

As we have seen in the previous chapter, two-dimensional conformal field theories are characterized by a double Virasoro algebra. We have just seen how the analysis of the asymptotic symmetries of warped AdS_3 reveals the breaking of this symmetry: instead of a second copy of the Virasoro algebra as in (3.38), one gets a $u(1)$ algebra as in (5.7). Since this alteration follows the deformation of the AdS_3 metric, the field theory the above algebra describes is dubbed "warped conformal field theory" [19, 20]. These theories share many features with the now familiar two-dimensional conformal field theory, in particular the existence of an infinite-dimensional symmetry group, and a notion of modular covariance that allows for the derivation of a Cardy-like formula [19].

Given two coordinates (z, w) describing a plane, a general warped conformal transformation can be written as [109]

$$z \to f(z) \;, \qquad w \to w + g(w) \tag{5.18}$$

with f and g arbitrary. A warped conformal field theory, which is invariant under these, is then non-Lorentz invariant and hence non-relativistic. Such a transformation is generated by a "right-moving" momentum $P(z)$ and stress-energy tensor $T(z)$, respectively, and the

[2] The norm of this Killing (1, 0 or -1) determines which type of warped geometry we are speaking of (spacelike, null or timelike).

algebra (5.7) is retrieved through the definition of the charges

$$L_n = -\frac{i}{2\pi}\int dz\, z^{n+1}\; T(z)\,, \tag{5.19}$$

$$P_n = -\frac{1}{2\pi}\int dz\, z^{n}\; P(z)\,. \tag{5.20}$$

In particular, the transformation that takes us to the cylinder (t,φ) is

$$z = e^{-i\varphi}\,, \qquad w = t-\varphi\,. \tag{5.21}$$

Zero modes on the cylinder are then[3]

$$L_0^{\rm cyl} = L_0 - \frac{c}{24} - P_0 + \frac{k}{4}\,, \qquad P_0^{\rm cyl} = P_0 - \frac{k}{2}\,. \tag{5.22}$$

The global charges, associated respectively to energy and angular momentum, are

$$H = -i\partial_t\;, \qquad J = -i\partial_\varphi \tag{5.23}$$

or in terms of the modes on the cylinder,

$$H = P_0^{\rm cyl} + L_0^{\rm cyl}\;, \qquad J = P_0^{\rm cyl} - L_0^{\rm cyl}\,. \tag{5.24}$$

and on the plane

$$H = L_0 - \frac{c}{24} - \frac{k}{4}\;, \qquad J = 2P_0 - L_0 + \frac{c}{24} - \frac{3}{4}k\,. \tag{5.25}$$

Defining primary states $|h,p\rangle$ as eigenstates of the zero modes L_0 and P_0, descendants can be created and destroyed as usual by acting with the ladder operators L_n, P_n. Unitarity

[3] This is just the spectral flow

$$\begin{aligned} L_n &\;\to\; L_n^\alpha = L_n + 2\alpha\, P_n + \Big(\alpha^2 - \frac{c}{24}\Big)\; k\;\delta_{n0}\,,\\ P_n &\;\to\; P_n^\alpha = P_n + \alpha\, k\;\delta_{n0}\,. \end{aligned}$$

with parameter $\alpha = -1/2$.

requirements (hermiticity of the operators and positivity of norm of the states) imply [19, 109]

$$k \geqslant 0, \qquad p \in \mathbb{R}, \qquad h \geqslant p^2/k, \qquad c \geqslant 1 \,. \tag{5.26}$$

In particular, the unitarity bound $L_0 \geqslant P_0^2/k$ has the interesting consequence that in any warped conformal field theory, unlike in ordinary conformal field theory, the charge J is bounded above: indeed,

$$J \;\leqslant\; 2P_0 - \frac{P_0^2}{k} + \frac{c}{24} - \frac{3}{4}\,k \tag{5.27}$$

and one finds

$$J \;\leqslant\; J_{\max} = \frac{c}{24} - \frac{k}{4}\,. \tag{5.28}$$

Modular analysis and the partition function

To obtain the theory on the torus, one needs to define the spatial and thermal cycles. In the case at hand, without Lorentz invariance, different choices of the cycles yield different tori with their particular modular properties. In all generality, the torus should be defined by the identifications [109]

$$(t, \varphi) \sim (t - 2\pi a, \varphi + 2\pi\bar{a}) \sim (t + 2\pi z, \varphi + 2\pi\tau) \tag{5.29}$$

but it is expected that results obtained with a partition function for one particular choice can always be derived for another, since the partition functions are all related. We can thus make the "canonical" choice $(a, \bar{a}) = (0, 1)$ and work with the partition function that has "good" modular properties. The (respectively) spatial and thermal cycles can be taken to be [110]

$$(t, \varphi) \sim (t, \varphi + 2\pi) \sim (t + 2\pi z, \varphi + 2\pi\tau) \tag{5.30}$$

where τ and z are the potentials appearing in the partition function

$$Z(\tau, z) = \mathrm{Tr}\left(e^{2\pi i\tau L_0^{\mathrm{cyl}}} e^{2\pi i z P_0^{\mathrm{cyl}}}\right). \tag{5.31}$$

This can also be written in terms of plane generators L_0, P_0 as

$$\hat{Z}(\tau, z) = \mathrm{Tr}\left(e^{2\pi i\tau\left(\hat{L}_0 - \frac{c}{24}\right)} e^{2\pi i z \hat{P}_0}\right) \tag{5.32}$$

with

$$\hat{L}_0 = L_0 - 2P_0 + k\,, \qquad \hat{P}_0 = P_0 - k\,. \tag{5.33}$$

These "altered" generators result from a choice of the spectral parameter $\alpha = -1$, and they correspond to the canonical parametrization of the torus mentioned before. According to [109], this form of the partition function should be privileged since it leaves the vacuum neutral under P_0.

The transformations defining the modular group (3.56) either interchange the spatial and thermal cycles (S) or add them to one another (T). Invariance under S and T imply, respectively,

$$Z\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{-\pi i k \frac{z^2}{2\tau}}\ Z(\tau, z)\,, \tag{5.34}$$

$$Z\left(\tau + 1, z\right) = e^{-\pi i \frac{k}{2}}\ Z(\tau, z)\,. \tag{5.35}$$

Modular covariance, and not invariance, of the warped partition function is correlated to the fact that not all choices of spatial slicing of the torus are equivalent.

It was shown [19] that the asymptotic density of states in a warped conformal field theory behaves in a way similar to that of conformal field theories. The entropy is given by

$$S = 2\pi\sqrt{\frac{c}{6}\left(h - \frac{c}{24} - \frac{p^2}{k}\right)} - 4\pi i\,\frac{p\,p_0}{k} \tag{5.36}$$

where p_0 is the vacuum charge. This is a bit reminiscent of Cardy's formula; in fact, the latter can be recovered exactly provided we do some redefinitions. Indeed, through a simple Sugawara construction our Virasoro-Kac-Moody algebra can be reduced to a double Virasoro algebra as follows [100, 111]. Defining a new set of operators L_n^-, L_n^+ from the original generators L_n, P_n by

$$L_n^- \ \widehat{=}\ \frac{1}{k}\sum_m : P_{-n-m}P_m : , \qquad L_n^+ \ \widehat{=}\ L_n^- + L_n \, , \tag{5.37}$$

one can check that these now obey a pair of commuting Virasoro algebras

$$[L_m^\pm, L_n^\pm] = (m-n)L_{m+n}^\pm + \frac{c^\pm}{12}\, m^3\, \delta_{m+n} \, , \tag{5.38}$$

$$[L_m^+, L_n^-] = 0 \, . \tag{5.39}$$

with central charges $c^+ = c + 1$, $c^- = 1$. In terms of these new quantities, one can simply apply Cardy's formula to recover the entropy

$$S = 2\pi\sqrt{\frac{c^+}{6}L_0^+} + 2\pi\sqrt{\frac{c^-}{6}L_0^-} \, . \tag{5.40}$$

However, this comment serves no other purpose than to highlight the affinity between (5.36) and the original Cardy formula. In what follows, we will not use the generators (5.37) and when referring to the "warped Cardy formula", we will mean (5.36).

With this, we have reviewed all the warped holography essentials. Entropies on both sides of the potential Warped AdS$_3$/Warped CFT duality have previously been computed in some specific cases, like New Massive Gravity [111] and Topologically Massive Gravity [19] (that were both briefly evoked at the end of Section 5.1), and they were shown to match. In the next chapter, we will show that this matching of entropies holds for generic higher curvature theories admitting the warped black hole as a solution. Then in the following chapter, we will go back to partition functions on the field theory side, and see how their transformation properties under the modular group can be leveraged to shed some light on its spectrum. In particular, we will derive a bound on the mass of the lightest charged black hole in chiral conformal field theories with charge, of which warped conformal field theories are one of the simplest cases. We will make the conjecture that this bound, proved analytically in the warped CFT case, is valid for a whole range of such theories with various types of charge and provide evidence to support it.

Chapter 6

Warped black holes in higher curvature theories of gravity

In the perspective of the search for a quantum theory of gravity, it is commonly believed that Einstein gravity is just the beginning of the story. The full theory, of which general relativity would be a low-energy effective theory, is expected to contain Planck-suppressed corrections in the form of higher curvature terms in the Lagrangian. In full generality, the four-dimensional Lagrangian is

$$L = \int d^4x\sqrt{-g} \quad F\big(g_{\mu\nu}, R_{\mu\nu\sigma\rho}, \nabla_\lambda R_{\mu\nu\rho\sigma}, \nabla_{(\lambda}\nabla_{\pi)}R_{\mu\nu\rho\sigma}, \dots\big) \tag{6.1}$$

with F a possibly complicated function involving all derivative orders of the Riemann tensor. As a consequence, the computation of charges has to be modified, and the classic Bekenstein-Hawking entropy

$$S = \frac{1}{4G\hbar}\int_\Sigma V(\Sigma) \tag{6.2}$$

is replaced by the Iyer-Wald formula [112]

$$S = -\frac{2\pi}{\hbar}\int_\Sigma \frac{\delta^{\mathrm{cov}} F}{\delta R_{\mu\nu\sigma\rho}}\,\epsilon_{\mu\nu}\epsilon_{\sigma\rho}\,V(\Sigma)\,, \tag{6.3}$$

the details of which will be given below (see in particular (6.23)). As we have seen in the previous chapter, gravity in warped AdS_3 belongs to this class of higher curvature theories. In this chapter, we show that the matching of entropies on both sides of the warped duality, which has so far been verified for specific choices of such theories on the gravity side, is valid for generic theories with higher-order derivatives. In order to compare the result obtained by means of the "warped Cardy" formula to the Iyer-Wald entropy above, we will need to compute surface charges. To this end, we will use a method that relies on the covariant phase space formalism [33], instead of the ADM formalism outlined in Section 2.3. We will review it next, and see how it generalizes to higher curvature gravity frameworks.

6.1. Covariant phase space formalism

We start with the Lagrangian written as an n-form L (with n the spacetime dimension). In order to avoid clutter, we slightly abuse notation and write both a form and its Hodge dual in the same way, and we will generally omit their indices. The Lagrangian is a local functional of all fields, but we will be concerned here with the situation where it only depends on the metric tensor g. The first variation of L yields the equations of motion $E[g] = 0$ through

$$\delta L[g] = E[g]\,\delta g + d\Theta[\delta g, g]\,. \tag{6.4}$$

The $(n-1)$-form $\Theta[\delta g, g]$ is the symplectic potential form and it can always be chosen to be covariant [112]. The gauge symmetries of the theory are transformations $\delta_\xi g = \mathcal{L}_\xi g$ under which the Lagrangian transforms as

$$\delta_\xi L \;=\; \mathcal{L}_\xi L \;=\; \xi \cdot dL + d(\xi \cdot L) \;=\; d(\xi \cdot L) \tag{6.5}$$

where the dot is to be understood as the interior product, i.e. the contraction of the vector field ξ and the first index of the differential form that follows. The second equality is just the Cartan formula relating the interior, exterior and Lie derivatives. On the other hand, for a gauge transformation $\delta = \delta_\xi$, the variation (6.4) reads

$$E[g]\,\delta_\xi g = -dJ_\xi[g] \tag{6.6}$$

where the canonical Noether current is defined as

$$J_\xi[g] = \Theta[\delta_\xi g, g] - \xi \cdot L[g]\,. \tag{6.7}$$

When the equations of motion are satisfied, J is a closed form and there exists an $(n-2)$-form $Q_\xi[g]$ such that, on-shell,

$$J_\xi[g] = -dQ_\xi[g]\,. \tag{6.8}$$

$Q_\xi[g]$ is the Noether charge as defined by Wald [113]. It is not equivalent to the conserved charge H_ξ corresponding to the action of the symmetry generator ξ on the covariant phase space, but both are closely related. The latter is obtained as follows. The symplectic structure of our configuration space is defined to be[1]

$$\Omega[\delta_1 g, \delta_2 g; g] = \int_\Sigma \omega[\delta_1 g, \delta_2 g; g] \tag{6.9}$$

where Σ is a Cauchy surface, and the symplectic current ω is

$$\omega[\delta_1 g, \delta_2 g; g] = \delta_1 \Theta[\delta_2 g, g] - \delta_2 \Theta[\delta_1 g, g]\,. \tag{6.10}$$

The subscripts 1 and 2 denote a two-parameter family of field configurations [33]. In particular, the Hamiltonian H_ξ that generates the flow $g \to g + \varepsilon \delta g$ satisfies

$$\Omega[\delta_\xi g, \delta g; g] = \delta H_\xi\,. \tag{6.11}$$

One can show [112] that when the equations of motion are satisfied ($E = 0$) and with the covariant definition of Θ,

$$\omega[\delta_\xi g, \delta g; g] = -\delta J + d(\xi \cdot \Theta)\,. \tag{6.12}$$

Moreover, when ξ is a symmetry, $\omega[\delta_\xi g, \delta g; g]$ vanishes, and when the linearized equations of motion hold ($\delta E = 0$), we can substitute $\delta J = -d\,\delta Q$. We then get

$$\omega[\delta_\xi g, \delta g; g] \;=\; dk_\xi[\delta g; g] \;=\; 0 \tag{6.13}$$

with

$$k_\xi[\delta g, g] \;\hat{=}\; -\delta Q_\xi[g] - \xi \cdot \Theta[\delta g, g]\,. \tag{6.14}$$

[1] More precisely, Ω is a "pre-symplectic" form, but the correct symplectic form and phase space can be obtained by a straightforward reduction procedure detailed in [33]. Nevertheless, we will use the shortcut denomination here.

This $(n-2)$-form provides us with an expression for the conserved charge associated to ξ. Indeed, using Stokes' theorem,

$$\delta H_\xi = \int_{\partial\Sigma} k_\xi[\delta g, g]\,. \tag{6.15}$$

The charge H_ξ is then obtained through integration over configuration space. This is well-defined provided the integrability condition

$$\int_{\partial\Sigma} \delta k_\xi = 0 \tag{6.16}$$

is obeyed. The asymptotic symmetry algebra can be represented by a Dirac bracket as

$$\delta_\xi H_\zeta \;\hat{=}\; \{H_\zeta, H_\xi\} = H_{[\zeta,\xi]} + \int_{\partial\Sigma} k_\zeta[\delta_\xi g, g]\,. \tag{6.17}$$

This algebra exhibits a central extension that cannot be absorbed in a redefinition of the generators, which makes k_ξ the object of interest.

It is important to realize that the Θ defined in (6.4) is not unique, and this ambiguity reflects on the other quantities used in this formalism. Indeed, one could add an exact $(n-1)$-form to Θ, such as $\Theta \to \Theta + dY$ with Y linear, without compromising the rest of the analysis. As a result, k_ξ is defined up to the redefinitions

$$k_\xi[\delta g, g] \;\to\; k_\xi[\delta g, g] + B[\delta_\xi g, \delta g] \tag{6.18}$$

for an arbitrary boundary term B anti-symmetric in $\delta_\xi g$ and δg. In order to fix this ambiguity, an alternative definition of the symplectic potential form Θ has been proposed [114, 115]. It involves the use of a homotopy operator, the details of which will not be needed here. In essence, this operator is the inverse of the exterior derivative d (see e.g. [116] for an explicit expression). One advantage of this procedure is that it provides a definition of charges depending only on the equations of motion of the Lagrangian, and not on boundary terms. We then have

$$\tilde{k}_\xi[\delta g, g] \;=\; k_\xi[\delta g, g] + B[\delta_\xi g, \delta g; g], \tag{6.19}$$

where B is known explicitly [117]. Remark that this ambiguity is not relevant for exact symmetries but may yield distinct results in the asymptotic context (see [117] for one such example in Kerr/CFT). In the following, we will be using the Iyer-Wald charges, and explicitly check that the extra term B does not contribute.

Higher curvature generalization

The method outlined above was generalized in [117] for higher-curvature theories, which are suitable for our purposes of finding the charges for warped AdS$_3$. It has been shown [112] that a diffeomorphism-covariant Lagrangian (form) can always be put in a manifestly covariant form

$$L = F\big(g_{ab}, R_{abcd}, \nabla_{e_1} R_{abcd}, \nabla_{(e_1}\nabla_{e_2)} R_{abcd}, \ldots, \nabla_{(e_1} \ldots \nabla_{e_k)} R_{abcd}\big)\,. \tag{6.20}$$

To deal with the arbitrary number k of derivatives of the Riemann tensor, we will follow [117] and introduce auxiliary fields $\mathcal{R}$ and Z in terms of which our original Lagrangian can be rewritten without derivatives higher than second order, namely

$$\begin{aligned} L \;=\; & F\Big[g_{ab}, \mathcal{R}_{abcd}, \mathcal{R}_{abcd|e_1}, \ldots \mathcal{R}_{abcd|e_1 \ldots e_k}) + Z^{abcd}\left(R_{abcd} - \mathcal{R}_{abcd}\right) \\ & + Z^{abcd|e_1}\left(\nabla_{e_1}\mathcal{R}_{abcd} - \mathcal{R}_{abcd|e_1}\right) + Z^{abcd|e_1 e_2}\left(\nabla_{(e_2}\mathcal{R}_{abcd|e_1)} - \mathcal{R}_{abcd|e_1 e_2}\right) \\ & + \cdots + Z^{abcd|e_1\ldots e_k}\left(\nabla_{(e_k}\mathcal{R}_{abcd|e_1\ldots e_{k-1})} - \mathcal{R}_{abcd|e_1\ldots e_k}\right)\Big]\,. \end{aligned} \tag{6.21}$$

with

$$\mathcal{R}_{abcd|e_1\ldots e_s} \;=\; \nabla_{(e_1} \ldots \nabla_{e_s)} R_{abcd}\,, \tag{6.22}$$

$$Z^{abcd} \;=\; \sum_{\ell=0} (-1)^\ell \nabla_{(e_1} \ldots \nabla_{e_\ell)} \frac{\partial F}{\partial \nabla_{(e_1} \ldots \nabla_{e_s)} R_{abcd}} \;\widehat{=}\; \frac{\delta^{\mathrm{cov}}}{\delta R_{abcd}}\, F\,, \tag{6.23}$$

where $s \in [1,k]$ and the symmetrization is over the e_i indices only. These expressions come from solving the recursive equations of motion

$$\begin{aligned}\mathcal{R}_{abcd|e_1\dots e_s} &= \nabla_{(e_s}\mathcal{R}_{abcd|e_1\dots e_{s-1})}\,, &(6.24)\\ Z^{abcd|e_1\dots e_s} &= \frac{\partial F}{\partial \mathcal{R}_{abcd|e_1\dots e_s}} - \nabla_{e_{s+1}} Z^{abcd|e_1\dots e_{s+1}}\,. &(6.25)\end{aligned}$$

In those terms the charge Q_ξ admits the decomposition (in dimension three[2]) [117]

$$Q_\xi = \left(Q^{(0)}_\xi + \sum_{s\geqslant 1} Q^{(s)}_\xi\right)\varepsilon_{abc}\,dx^c \tag{6.26}$$

where

$$Q^{(0)}_\xi = \left(-Z^{abcd}\nabla_c\xi_d - 2\xi_c(\nabla_d Z^{abcd})\right)\,,\qquad Q^{(s)}_\xi = \xi_k\,A^{kab}_{(s)} \tag{6.27}$$

with

$$\begin{aligned}A^{kab}_{(s)} = -2\Big[&Z^{klcd|e_1\dots e_{s-1}a}\,\mathcal{R}^b{}_{lcd|e_1\dots e_{s-1}} + Z^{alcd|e_1\dots e_{s-1}b}\,\mathcal{R}^k{}_{lcd|e_1\dots e_{s-1}}\\ &+ Z^{alcd|e_1\dots e_{s-1}k}\,\mathcal{R}^b{}_{lcd|e_1\dots e_{s-1}} + \frac{s-1}{2}Z^{lmcd|e_1\dots e_{s-2}ka}\,\mathcal{R}_{lmcd|}{}^b{}_{e_1\dots e_{s-2}}\Big]\,.\end{aligned} \tag{6.28}$$

The boundary term obtained when varying the Lagrangian can be similarly decomposed as

$$\Theta = \frac{1}{2}\left(\Theta^{(0)} + \sum_{s\geqslant 1}\Theta^{(s)}\right)\varepsilon_{abc}\,dx^b\wedge dx^c \tag{6.29}$$

where

$$\begin{aligned}\Theta^{(0)} &= -2\left(Z^{abcd}\nabla_d\,\delta g_{bc} - (\nabla_d Z^{abcd})\,\delta g_{bc}\right)\\ \Theta^{(s)} &= 2\delta g_{ij}\,C^{ija}_{(s)} - Z^{nklm|e_1\dots e_{s-1}a}\,\delta\mathcal{R}_{nklm|e_1\dots e_{s-1}}\end{aligned} \tag{6.30}$$

[2] In higher dimension, ε_{abc} just gets replaced by $\varepsilon_{abc_3\dots c_n}$ for Q, and by $\varepsilon_{aa_2\dots a_n}$ for Θ, etc.

with

$$C^{ija}_{(s)} = 2\left(Z^{iklm|e_1\dots e_{s-1}a} + Z^{aklm|e_1\dots e_{s-1}i}\right)\mathcal{R}^{j}{}_{klm|e_1\dots e_{s-1}} - 2Z^{iklm|e_1\dots e_{s-1}j}\,\mathcal{R}^{a}{}_{klm|e_1\dots e_{s-1}}$$
$$+ (s-1)\Big(Z^{nklm|e_1\dots e_{s-2}ia}\,\mathcal{R}_{nklm|}{}^{j}{}_{e_1\dots e_{s-2}} - \frac{1}{2}Z^{nklm|e_1\dots e_{s-2}ij}\,\mathcal{R}_{nklm|}{}^{a}{}_{e_1\dots e_{s-2}}\Big)\,. \quad (6.31)$$

Both tensors $A_{(s)}$ and $C_{(s)}$ are present only when the Lagrangian has derivatives of the Riemann.

Finally, let us mention the general expression for the central terms. It has been shown [117] that it can be unwrapped as

$$\int k_\zeta[\delta_\xi g, g] = \int \left(Q_{[\zeta,\xi]} - (\mathcal{L}_\zeta Q_\xi - \mathcal{L}_\xi Q_\zeta) - \zeta\cdot\xi\cdot L\right). \quad (6.32)$$

In the case at hand, since both central charges appear in (5.7) along with a δ_{m+n}, ξ and ζ will be (l_n, l_{-n}) for the central charge c and (p_n, p_{-n}) for the level k. The first term here can be ignored [117] since one can absorb it into a shift of the Hamiltonian in (6.17). In addition, we do not need to worry about the extra term $B_c[\delta_\xi g, \delta_\zeta g; g]$ from (6.19): indeed, in our case it is [117]

$$B_c[\delta_\xi g, \delta_\zeta g; g] = \frac{1}{2}\left(-\frac{3}{2}Z^{abde}\delta_\xi g_d^f \wedge \delta_\zeta g_{fe} + 2Z^{adef}\delta_\xi g_{de}\wedge\delta_\zeta g_f^b\right)\varepsilon_{abc} \quad (6.33)$$

and for reasons we detail below, it will never contribute to the charges.

6.2. Higher curvature warped charges

Now that all the background is set up, we can move on to the particular case that we are interested in: three-dimensional warped anti-de Sitter. In this section we will show that in any higher-curvature theory, the entropy of the warped black hole (5.2) still matches the Wald entropy. We start by analyzing in detail how symmetries can be worked to simplify the expressions for the charges that we have just been reviewing.

6.2.1. Symmetries and Curvature Tensors

We recall that in dimension three, all Riemann tensors can be expressed in terms of Ricci tensors. In the case at hand, some products of Ricci tensors are fairly simple [18]:

$$\{R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu}R^{\nu\rho}R_{\rho}{}^{\mu}\} = \frac{6}{\ell^2}\left\{-1, \frac{\nu^4-2\nu^2-3}{\ell^2}, \frac{-\nu^6-3\nu^4+9\nu^2-9}{\ell^4}\right\}\,. \tag{6.34}$$

This is valid for both warped AdS_3 and its black hole, and can be easily verified for the metric (5.2). Furthermore, in a maximally symmetric spacetime, all curvature tensors (e.g. products of covariant derivatives of Riemann/Ricci tensors) can be expressed (covariantly) in terms of the metric tensor. For example, for three-dimensional Einstein gravity,

$$R_{\mu\nu\rho\sigma} = \Lambda\left(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}\right)\,. \tag{6.35}$$

One can extend this sort of argument to the warped AdS_3 case. Indeed, any tensor constructed out of the metric should respect the $SL(2,\mathbb{R})\times U(1)$ isometry. The consequences have been investigated and exploited in [117]; we will now see how it plays out in our case. In particular, any *scalar* curvature invariants constructed out of the metric are constants. As such, in the case at hand, these constants can only depend on ν and ℓ and not on the parameters of the black holes (i.e. m and j), which are parameters of the global quotients. We will see this in a example below.

For a generic tensor, let us consider what happens in the case of a symmetric-two tensor $S^{\mu\nu}$. Due to boost-invariance (which is a consequence of the $SL(2,\mathbb{R})\times U(1)$ symmetry), in a conveniently chosen *vielbein* basis $e^{\hat{i}} \equiv e^{\hat{i}}{}_{\mu}dx^{\mu}$ (with $i = 0,1,2$, explicitly given in [117]), we have

$$S^{\hat{0}\hat{0}} = -S^{\hat{1}\hat{1}}\,, \qquad S^{\hat{0}\hat{1}} = S^{\hat{0}\hat{2}} = S^{\hat{1}\hat{2}} = 0 \tag{6.36}$$

while $S^{\hat{2}\hat{2}}$ is arbitrary. This implies that any such tensor only contains two arbitrary components. In particular, we can decompose it as

$$S^{\hat{a}\hat{b}} = c_1\eta^{\hat{a}\hat{b}} + c_2 J^{\hat{a}}J^{\hat{b}}\,. \tag{6.37}$$

where the constants c_i only depend on ν and ℓ. The vector J^{μ} is most usefully chosen to be the $U(1)_R$ of the $SL(2,\mathbb{R})_L \times U(1)_R$,

$$J^\mu \partial_\mu = \partial_t = p_0\,, \tag{6.38}$$

where we work in the warped black hole coordinates in which (5.2) is written and p_0 is the $U(1)_R$ Killing vector from (5.5). Note that $J^\mu J_\mu = 1$. Translating back into spacetime indices, we obtain

$$S^{\mu\nu} = c_1 g^{\mu\nu} + c_2 J^\mu J^\nu\,. \tag{6.39}$$

Furthermore, note that

$$\nabla_\mu J_\nu = \frac{\nu}{\ell}\,\varepsilon_{\mu\nu\sigma}\,J^\sigma \tag{6.40}$$

where the convention is $\varepsilon_{tr\varphi} = \sqrt{-g} = 1$. Thus, all products of covariant derivatives of S can in turn be rewritten as products of g, ε and J. Let us give an example where $S_{\mu\nu} = R_{\mu\nu}$. Then,

$$R_{\mu\nu} = \frac{\nu^2-3}{\ell^2}\,g_{\mu\nu} + 3\,\frac{1-\nu^2}{\ell^2}\,J_\mu J_\nu\,, \tag{6.41}$$

$$\nabla_\mu R_{\nu\rho} = 3\nu\,\frac{\nu^2-1}{\ell^3}\,\big(\varepsilon_{\mu\nu\sigma}J_\rho + \varepsilon_{\mu\rho\sigma}J_\nu\big)J^\sigma\,. \tag{6.42}$$

In order to compute the Wald entropy of the warped black hole, we will need the tensor $Z^{\alpha\beta\mu\nu}$ defined in (6.23) which has the same index symmetry as the Riemann tensor and is constructed out of the metric and its derivatives. It also depends on the theory one considers. By the above argument, we can rewrite it as

$$Z^{\alpha\beta\mu\nu} = A\left[g^{\mu\alpha}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}\right] + B\left[g^{\mu\alpha}R^{\beta\nu} - g^{\nu\alpha}R^{\beta\mu} + g^{\beta\nu}R^{\alpha\mu} - g^{\beta\mu}R^{\alpha\nu}\right] \tag{6.43}$$

for some constants A and B that only depend on (ν,ℓ). In particular, when we replace this form of Z in (6.33), it is straightforward to show that $B_\varphi[\delta_\xi g, \delta_\zeta g; g]$ vanishes as we integrate over a $(t,r) =$ constant surface.

Finally, the equations of motion always take the form

$$E_{\mu\nu}[g_{\alpha\beta}] = 0 \tag{6.44}$$

where $E_{\mu\nu}[g_{\alpha\beta}]$ for a given symmetric two-tensor constructed out of the metric. For example in pure Einstein theory, it is just the Einstein tensor or the Ricci tensor. Following the above logic, evaluating $E_{\mu\nu}$ on a warped AdS or warped black hole solution, symmetries allow us to decompose $E_{\mu\nu}$ into a sum of the metric and the Ricci tensor as[3]

$$E_{\mu\nu} = E_1 R_{\mu\nu} + E_2 g_{\mu\nu} = 0\,. \tag{6.45}$$

where E_1 and E_2 are constants which only depend on (ν, ℓ) and on the couplings α_i of the theories. Note that the dependence on the couplings of the theories is linear. The equations of motion then reduce to two independent equations, i.e. setting

$$E_1(\nu, \ell, \alpha_i) = E_2(\nu, \ell, \alpha_i) = 0\,. \tag{6.46}$$

This means that as long as we have a theory with two independent couplings, such as Topologically Massive Gravity (5.10) or New Massive Gravity (5.12), we will always be able to solve the equations of motion. The couplings appear linearly in the E_i's, and for that reason we can always solve these two decoupled equations (subject to obtaining real ℓ and ν as solutions).

6.2.2. Entropy of warped black holes

For any diffeomorphism covariant theory of gravity, the Wald three-dimensional entropy formula is [112, 118, 119]

$$S_{\text{Wald}} = -2\pi \int_0^{2\pi} d\varphi\, Z^{abcd} \varepsilon_{ab}\, \varepsilon_{cd} \sqrt{g_{\varphi\varphi}}\,\Big|_{r=r_+}\,. \tag{6.47}$$

[3] In the case where the Lagrangian contains only Ricci tensors (and not covariant derivatives of Ricci tensor), we work this out very explicitly in Appendix A. For example, see (188).

In this expression, ε_{ab} is the binormal at the horizon, given by $\varepsilon_{ab} = \nabla_a \xi_b$ where ξ is the generator of the horizon with κ its surface gravity (1.6) normalized to unity. As for Z, we have encountered it already in (6.23), and we have just seen in the previous section that by general $SL(2,\mathbb{R}) \times U(1)$ symmetry arguments, it can be written as (6.43). We can then compute

$$Z^{abcd}\varepsilon_{ab}\,\varepsilon_{cd}\,\sqrt{g_{\varphi\varphi}}\,\Big|_{r=r_+} = -4\left(A + BR^a{}_b\,n^b{}_a\right)\sqrt{g_{\varphi\varphi}}\,\Big|_{r=r_+} \tag{6.48}$$

where we have used $\varepsilon^{ab}\varepsilon_{ab} = -2$ and defined $n^a{}_b \equiv -\varepsilon^{ca}\varepsilon_{cb}$. Furthermore, using

$$R^a{}_b n^b{}_a\,\Big|_{r=r_+} = \frac{2\,(\nu^2-3)}{\ell^2} \tag{6.49}$$

and putting (5.2)-(5.3) together to get $\sqrt{g_{\varphi\varphi}}\,\big|_{r=r_+} = -\Omega^{-1}$ with

$$\Omega = -\frac{\nu^2+3}{4\left(\sqrt{\ell\,\nu\,(9\ell m^2\nu - (\nu^2+3)\,j)} + 3\ell m\nu\right)} \tag{6.50}$$

the angular velocity at the horizon, one is led to

$$S_{\text{Wald}} = -\frac{16\pi^2}{\Omega}A\left(1 + \frac{B}{A}\frac{2(\nu^2-3)}{\ell^2}\right)\,. \tag{6.51}$$

We will now show how, by computing the entropy by way of the warped Cardy formula (5.36) this expression is recovered for theories with higher curvature terms.

Warped Cardy formula at leading order

In order to do that, we need the charges (6.15) appearing in the warped Cardy formula under the names h and p. Since we would rather avoid confusion with other notations in this chapter, we will rename these charges after their generators, i.e. L_0 and P_0 respectively. We first note that (5.36) can be rewritten in terms of the angular velocity (6.50) as

$$S_{\text{WCFT}} = \frac{2\pi i}{\Omega}P_0^{vac} - \frac{8\pi^2}{\beta\Omega}\left(\frac{(P_0^{vac})^2}{k} - \frac{c}{24}\right)\,, \tag{6.52}$$

where β is the inverse temperature [19].

We will first derive the expressions of the charges for the case of a Lagrangian without derivatives of the Ricci, when they depend only on $Q_\xi^{(0)}$ and $\Theta^{(0)}$, and then take a look at what sort of corrections they receive from $Q_\xi^{(s)}$ and $\Theta^{(s)}$ once we incorporate such derivatives. The exact charges are given by (6.15), i.e.

$$\delta L_0 \equiv -\delta H_{\partial_\varphi} = -\int_\infty \delta Q_{\partial_\phi}, \tag{6.53}$$

$$\delta P_0 \equiv \delta H_{\partial_t} = \int_\infty \delta Q_{\partial_t} + \int_\infty \partial_t \cdot \Theta \tag{6.54}$$

where the integral is over the $(n-2)$-dimensional sphere (with $t, r =$ constant) at spatial infinity. There is no term with $\partial_\varphi \cdot \Theta$ in δL_0 because ∂_ϕ is assumed to be tangent to this sphere.

Using the general form of Z^{abcd} (6.43), an explicit computation yields

$$\begin{aligned} L_0 &= \frac{32\pi\left(A\ell^2 + 2B\left(\nu^2-3\right)\right)}{\ell^2}\, j + \frac{24\pi\nu\left(\nu^2-1\right)\left(A\ell^2+2B\left(5\nu^2-3\right)\right)}{\ell^5}\, r^2\,, \\ P_0 &= \frac{48\pi\left(A\ell^2+2B\left(\nu^2-3\right)\right)}{\ell^2}\, m\,. \end{aligned} \tag{6.55}$$

To obtain the central terms in (6.17) as given by (6.32), it is sufficient to consider the terms proportional to n (for the level) and n^3 (for the central charge), which are

$$\begin{aligned} k &= 2i\int_\infty k_{p_n}[\mathcal{L}_{p_{-n}}g, g]\Big|_n = -\frac{32\pi\nu}{\ell}\left(A + 4B\,\frac{(2\nu^2-3)}{\ell^2}\right)\,, \\ c &= 12i\int_\infty k_{l_n}[\mathcal{L}_{l_{-n}}g, g]\Big|_{n^3} = \frac{192\pi\nu}{(\nu^2+3)}\left(A\ell + 2B\,\frac{(\nu^2-3)}{\ell}\right)\,. \end{aligned} \tag{6.56}$$

In all these expressions, A, B are the constants in terms of which Z is expressed in (6.43). Recall that picking a particular theory in which to consider the warped black hole amounts to fixing these constants so until we do so, everything we compute here is completely general. However, on general grounds, the charges associated with exact Killing vectors of a metric satisfying the equations of motion of a given theory are finite. Therefore, the r-dependence of the charges (6.55) is expected to drop out on-shell and we see that a theory admitting warped AdS_3 as a solution must have its constants A and B related in the following way:

$$B = -\frac{A\,\ell^2}{2\,(5\nu^2-3)}. \tag{6.57}$$

In other words, the coupling constants of the considered theory should satisfy the above relation. This requirement is equivalent to satisfying the equations of motion, as expected (see details in Appendix A).

Using (6.57), the charges have the following expression

$$L_0 = \frac{128\,\pi\,\nu^2}{5\nu^2-3}\,A\,j\,, \qquad P_0 = \frac{192\,\pi\,\nu^2}{5\nu^2-3}\,A\,m, \tag{6.58}$$

$$k = -\frac{32\,\pi\,\nu\,(3+\nu^2)}{\ell\,(5\nu^2-3)}\,A\,, \qquad c = \frac{768\,\ell\,\pi\,\nu^3}{(\nu^2+3)\,(5\nu^2-3)}\,A \tag{6.59}$$

with only one constant A depending on the theory under consideration.

Following [19], we pick the warped black hole ground state to be ($m = i/6$, $j = 0$) in order to match with the BTZ ground state ($M = -1/8$, $J = 0$) since both are related by (5.16). The corresponding value of P_0^{vac} is then

$$P_0^{vac} = \frac{32i\pi\nu^2}{5\nu^2-3}\,A\,. \tag{6.60}$$

We substitute these into the warped Cardy formula (6.52) and get

$$S_{\mathrm{WCFT}} = -\frac{64\pi^2 A}{\Omega}\,\frac{\nu^2}{(5\nu^2-3)} \tag{6.61}$$

which is precisely the Wald entropy (6.51) with the replacement (6.57). This proves the matching of entropies

$$S_{\mathrm{WCFT}} = S_{\mathrm{Wald}} \tag{6.62}$$

holds for higher curvature theories without derivatives of the Ricci tensor. We now set out to prove that the corrections attached to the presence of such derivatives do not alter this result.

Corrections (and how they do not contribute)

When the Lagrangian contains derivatives of the Ricci, the corrections $\Theta^{(s)}$ and $Q^{(s)}$ to the charges are activated. There will also be new terms appearing in Z^{abcd} as given by (6.43) but thanks to the symmetries, they can be absorbed in a redefinition of coefficients A and B as, say, $\tilde{A}$ and $\tilde{B}$.

Recall that the corrections in (6.26) and (6.29) were given by

$$\begin{aligned} Q^{(s)}_\xi &= \xi_k\, A^{kab}\,, \\ \Theta^{(s)} &= 2\,\delta g_{ij}\, C^{ija} - Z^{nklm|e_1\ldots e_{s-1}a}\;\delta\mathcal{R}_{nklm|e_1\ldots e_{s-1}}\,, \end{aligned} \tag{6.63}$$

where both A and C are covariant tensors constructed out of the metric, A^{kab} being anti-symmetric in (a,b) while C^{ija} is symmetric in (i,j). As it turns out, the second term in $\Theta^{(s)}$ can be incorporated to the first one: indeed, since every covariant tensor built out of a $SL(2,\mathbb{R})$ metric can be written in terms of polynomials of ε, $g_{\mu\nu}$ and J_ρ, we have

$$\mathcal{R}_{nklm|e_1\ldots e_{s-1}} = \sum_p c_p(\nu,\ell)\; t^{(p)}_{nklm|e_1\ldots e_{s-1}} \tag{6.64}$$

where t is some basis tensor built out of polynomials of ε, g and J. Crucially the coefficient c_p only depends on (ν,ℓ) and not on the black hole or quotienting parameters. Note that

$$\delta J_\mu = \delta\varepsilon_{abc} = 0, \tag{6.65}$$

since $\sqrt{-g}=1$ and $J_\mu dx^\mu = dt - (2\nu r/\ell)d\varphi$ does not contain black hole parameters. Therefore, we have

$$\delta\mathcal{R}_{nklm|e_1\ldots e_{s-1}} = \sum_p c_p(\nu,\ell)\;\delta t^{(p)}_{nklm|e_1\ldots e_{s-1}} = \sum_p c_p(\nu,\ell)\;\frac{\delta t^{(p)}_{nklm|e_1\ldots e_{s-1}}}{\delta g_{ij}}\;\delta g_{ij}, \tag{6.66}$$

since variations hit the t-tensor through $g_{\mu\nu}$ and t is a polynomial in g. It is important to note that no covariant derivatives of $g_{\mu\nu}$ appear in this tensor t. Therefore, we obtain

$$\left[Z^{nklm|e_1\ldots e_{s-1}a}\sum_p c_p\,\frac{\delta t^{(p)}_{nklm|e_1\ldots e_{s-1}}}{\delta g_{ij}}\right]\delta g_{ij}\equiv A^{ija}\delta g_{ij},\tag{6.67}$$

where $A^{ija}=A^{jia}$. Our corrections are now[4]

$$Q^{(s)}_\xi=\xi_k\,A^{kab}\,,\qquad \Theta^{(s)}=\delta g_{ij}\,C^{ija}\,.\tag{6.68}$$

Furthermore, by symmetry arguments there are only four independent non-zero components for C^{ija} and three for A^{kab} in the three-dimensional case. This means that we can decompose these tensors in the following way

$$\begin{aligned}A^{abk}&=\Big(\big(a_1 g^{ak}+a_2\,\varepsilon^{akc}J_c\big)\,J^b-(a\leftrightarrow b)\Big)+a_3\,\varepsilon^{abp}J_pJ^k\\ C^{ija}&=b_1\,g^{ij}J^a+\Big(\big(b_2J^ig^{ja}+b_3\,J^i\,\varepsilon^{jak}J_k\big)+(i\leftrightarrow j)\Big)+b_4J^iJ^jJ^a\end{aligned}\tag{6.69}$$

where a_i and b_i are constants depending only on ν and ℓ.

Let us now see how this constrains our corrections, starting with $\Theta^{(s)}$. Using the fact that the only non-zero components of δg_{ij} are δg_{rr} and $\delta g_{\varphi\varphi}$ and that $g^{ij}\delta g_{ij}=0$, we get

$$\Theta^{(s)}=\delta g_{ij}\Big(\big(b_2J^ig^{ja}+b_3J^i\varepsilon^{jak}J_k\big)+(i\leftrightarrow j)\Big)+b_4J^iJ^jJ^a\,.\tag{6.70}$$

This correction only enters the charge generated by ∂_t (see (6.53)), and when $J^\mu\partial_\mu=\partial_t$ there is no contribution from the first term in this equation since $\delta g_{tj}=0$. Moreover, since we are computing charges on a constant (t,r) surface, we are interested in the $a=r$ component[5], which implies the last term does not contribute either since $J^r=0$. Thus we have established that in the computation of δP_0, $\Theta^{(s)}$ vanishes.

[4] We also absorb a factor of 2 in the definition of C.

[5] Explicitly, we are computing

$$\int\xi^b\,\Theta^{(s)}_{bc}\,\varepsilon_{abc}\,dx^c\,,$$

with $r,t=\mathrm{cst}$ so $c=\varphi$, and since $b=t$ the only non-vanishing piece is the one with $a=r$.

Next, consider the expression for $Q^{(s)}_\xi$ which becomes

$$Q^{(s)}_\xi = \xi_k \Big(\big(2a_1 g^{k[a} + 2a_2\, \varepsilon^{pk[a} J_p\big)\, J^{b]} + a_3\, \varepsilon^{abp} J_p J^k \Big)\,. \tag{6.71}$$

For δP_0, we consider $\xi^\mu \partial_\mu = \partial_t = J^\mu \partial_\mu$. By similar arguments, we see that the first and second term vanish, so that we are left to evaluate

$$\int_\infty Q^{(s)}_\xi = \int_\infty \xi_k \left(a_3\, \varepsilon^{abp} J_p J^k\right)\, \varepsilon_{abc} dx^c = \frac{8\pi\nu\, a_3}{\ell}\; r\,, \tag{6.72}$$

where we have used $J^\mu J_\mu = 1$ and $J_\mu dx^\mu = dt - (2r\nu/\ell)d\varphi$ while setting $(a,b) = (t,r)$. Taking the variation of that expression gives us zero.

Finally, we consider the expression for $Q^{(s)}_\xi$ for $\xi^\mu\partial_\mu = \partial_\varphi$ in the computation of δL_0. Direct computation shows

$$\int_\infty Q^{(s)}_\xi = \left(\frac{16\,\pi\, a_3\, \nu^2 + 4\,\pi\, a_2\,(3+\nu^2)}{\ell^2}\right) r^2 - (48\, a_2\, m)\, r + \frac{16\, a_2\, \ell}{\nu} j\,. \tag{6.73}$$

As for the central charges, a similarly explicit computation of the corrections generated by $Q^{(s)}_\xi$ and $\Theta^{(s)}$ in (6.32) yields

$$\begin{aligned} c^{(s)} &= \frac{192\,(3\, a_2\, \pi + a_2\, \pi\nu^2 + 4\, a_3\, \pi\nu^2)\, r^2}{\ell^2} - 1152\, a_2\, m\pi r\,, \\ k^{(s)} &= -16\pi\, a_3\,. \end{aligned} \tag{6.74}$$

Thus the charges are (6.55), (6.56) with $A \to \tilde{A}$ and $B \to \tilde{B}$ plus the corrections (6.73) and (6.74). In order to have finite exact charges, the divergences must cancel. This implies

$$a_2 = 0, \qquad a_3 = -\frac{3\,(\nu^2-1)\left(\tilde{A}\ell^2 + 2\tilde{B}\,(5\nu^2-3)\right)}{2\nu\,\ell^3}. \tag{6.75}$$

The purpose of the first equation is to cancel the r-divergence. To get rid of the r^2-divergence, the second term in (6.55) for L_0 should cancel against the first term in (6.74), which yields the expression for a_3. With this, the form of the charges is left unchanged with respect to the case without derivative of the Ricci in the Lagragian. As for the central terms, finiteness of the Virasoro central charge which is a necessary condition in order to have a well-defined phase space with these symmetries, requires $a_3 = 0$. This does not change our result, but it is an additional condition on the coupling constants of the theory (though it is not excluded that this condition could be derived from requiring the metric to satisfy the equations of motion). Relation (6.57) then holds between $\tilde{A}$ and $\tilde{B}$. In any case, all corrections vanish as advertised, and the matching of entropies is preserved.

In conclusion, we have shown that the Wald entropy is always reproduced by a warped Cardy formula using the $SL(2,\mathbb{R}) \times U(1)$ symmetries and on-shell conditions. We have also found a general explicit expression for the charges.

6.3. Example

In this section, we see how all this formalism applies to a concrete Lagrangian. More examples and details can be found in Appendix B.

We take a Lagrangian with the higher curvature term

$$L_{\text{HC}} = \nabla^a R^{bc} \nabla_a R_{bc}. \tag{6.76}$$

According to (6.23), the Z-terms entering the charges are given by

$$\begin{aligned} Z^{abcd} &= \frac{\partial L}{\partial R_{abcd}} - \nabla_e \frac{\partial L}{\partial \nabla_e R_{abcd}} \\ &= -\frac{1}{2}\left(-g^{ac}\Box R^{bd} + g^{bc}\Box R^{ad} - g^{bd}\Box R^{ac} + g^{ad}\Box R^{bc}\right) \end{aligned} \tag{6.77}$$

and

$$\begin{aligned} Z^{abcd|e} &= \frac{\partial L}{\partial \nabla_e R_{abcd}} \\ &= \frac{1}{2}\left(g^{ac}\nabla^e R^{bd} - g^{bc}\nabla^e R^{ad} + g^{bd}\nabla^e R^{ac} - g^{ad}\nabla^e R^{bc}\right) . \end{aligned} \tag{6.78}$$

Using

$$\begin{aligned}
\nabla^\alpha \nabla_\mu R_{\alpha\nu} = \nabla^\alpha \nabla_\nu R_{\alpha\mu} &= -\frac{6\nu^2}{\ell^4} g_{\mu\nu} - \frac{3\nu^2}{\ell^2} R_{\mu\nu} \\
\Box R_{\mu\nu} &= \frac{12\nu^2}{\ell^4} g_{\mu\nu} + \frac{6\nu^2}{\ell^2} R_{\mu\nu} \,,
\end{aligned} \tag{6.79}$$

we can rewrite the tensor Z as (6.43),

$$\begin{aligned}
Z^{abcd} &= \frac{6\nu^2}{\ell^4} \left(-g^{ac} g^{bd} + g^{bc} g^{ad} - g^{bd} g^{ac} + g^{ad} g^{bc}\right) \\
&\quad + \frac{3\nu^2}{\ell^2} \left(-g^{ac} R^{bd} + g^{bc} R^{ad} - g^{bd} R^{ac} + g^{ad} R^{bc}\right) \,.
\end{aligned} \tag{6.80}$$

The relevant corrections to Θ and Q are

$$\begin{aligned}
\Theta^{(1)}_{bc} &= 2\left[\delta g_{ij} \left(R^{j\,i}_{\;k\,l} \nabla^a R^{kl} + R^j_{\;k} (\nabla^a R^{ik} - \nabla^i R^{ak}) - R^a_{\;k} (\nabla^j R^{ik}\right) - \delta R_{kl} \nabla^a R^{kl}\right] \varepsilon_{abc} \\
(Q^{(1)}_\xi)_c &= -2\xi_k \left[R^b_{\;l} \nabla^a R^{kl} + R^k_{\;l} \nabla^b R^{al} + R^b_{\;l} \nabla^k R^{al}\right] \varepsilon_{abc}
\end{aligned} \tag{6.81}$$

Using the relations (6.40) and (6.41) which imply $\delta R_{bc} = (\nu^2 - 3/\ell^2)\, \delta g_{bc}$, together with the decomposition of the Riemann tensor (2.8), we get

$$R^a_{\;c} R^{bc;e} = \frac{3\nu}{\ell^5} (\nu^2 - 1) \Big((\nu^2 - 3) J^b \varepsilon^{ea}_{\;\;\;s} - 2\nu^2 J^a \varepsilon^{eb}_{\;\;\;s}\Big) J^s \tag{6.82}$$

$$R^{a\,b}_{\;d\,c} R^{dc;e} = \frac{3\nu^3}{\ell^5} (\nu^2 - 1) \Big(J^b \varepsilon^{ea}_{\;\;\;s} + J^a \varepsilon^{eb}_{\;\;\;s}\Big) J^s \tag{6.83}$$

which corresponds precisely to the expected form for the corrections (6.69). We can then rewrite

$$\begin{aligned}
\Theta^{(1)}_{bc} &= \frac{6\nu}{\ell^5} (\nu^2 - 1) \delta g_{ij} \Big(\nu^2 J^i \varepsilon^{aj}_{\;\;\;s} - (2\nu^2 - 3) J^j \varepsilon^{ai}_{\;\;\;s} - (\nu^2 - 3) J^i \varepsilon^{ja}_{\;\;\;s} + 2\nu^2 J^j \varepsilon^{ia}_{\;\;\;s}\Big) J^s \varepsilon_{abc} \\
(Q_\xi)^{(1)}_c &= -\xi_k \frac{6\nu}{\ell^5} (\nu^2 - 1) \left[(\nu^2 - 3) \left(J^k \varepsilon^{ab}_{\;\;\;s} + J^a \varepsilon^{bk}_{\;\;\;s} + J^a \varepsilon^{kb}_{\;\;\;s}\right)\right. \\
&\quad \left. -2\nu^2 \left(J^b \varepsilon^{ak}_{\;\;\;s} + J^k \varepsilon^{ba}_{\;\;\;s} + J^b \varepsilon^{ka}_{\;\;\;s}\right)\right] J^s \varepsilon_{abc} \,.
\end{aligned} \tag{6.84}$$

Since we are going to integrate over φ, and we have to contract the correction to Θ with

∂_t, we need to compute $\Theta^{(1)}_{t\varphi}$ and $(Q_\xi)^{(1)}_\varphi$. We also know that δg only non-zero components are δg_{rr} and $\delta g_{\varphi\varphi}$. It is then straightforward to show that

$$\Theta^{(1)}_{t\varphi} = 0 \tag{6.85}$$

$$(Q_{\partial_t})^{(1)}_\varphi = -72\, r\, \frac{\nu^2}{\ell^6}(\nu^2-1)^2 \tag{6.86}$$

$$(Q_{\partial_\varphi})^{(1)}_\varphi = 0. \tag{6.87}$$

Even if $(Q_{\partial_t})^{(1)}_\varphi$ is non-zero, the contribution of this term to the charge vanishes upon taking the δ of it. The same results are recovered using the general method proposed in [117] (see Appendix B.5).

In this chapter, we have seen how the matching of entropies on both sides of warped holography generalizes to higher-curvature contexts. We have reviewed the covariant phase space formalism and its extension to these sorts of theories, of which warped AdS$_3$ is a solution, and we have computed the charges of the warped black hole. This allowed us to use the "Cardy-like" formula to calculate the entropy of the latter on the field theory side, recovering the result obtained by way of the Iyer-Wald formula on the gravity side and thus extending previous results derived case by case (NMG, TMG, ...) to generic theories with arbitrary higher-derivative terms. This argues in favour of the conjectured duality between warped AdS$_3$ and warped CFT.

In the next chapter, we will have a different take on the field theory dual of warped AdS$_3$. We will use the so-called "modular bootstrap program" to see how the very specific modular properties of warped CFT can be used to constrain it and extract information about black holes in warped AdS$_3$. In a more general perspective, we will then consider chiral conformal field theories with charge, and mobilize all the powerful resources of modular analysis to dig deeper into the matter.

Chapter 7

Modular bootstrap for chiral CFT

The modular bootstrap is a program that makes use of conformal field theories' modular properties to constrain the possible forms they can take. In two dimensions, consistency of a theory on arbitrary Riemann surfaces (and hence at all order of perturbation in string theory) requires two things: (1) crossing symmetry of four-point functions on the sphere, and (2) modular invariance of both the partition function and the one-point function on the torus (which corresponds to one-loop string diagrams) [120]. The conformal bootstrap program, which we will not cover here, focuses on the first one of these conditions, while the modular bootstrap imposes consistency conditions on the theory through modular invariance (or covariance). All sorts of conformal field theories have so far been investigated, with or without charge, with most of the work done along the lines of the linear functional method developed in [121] and resorting to semi-definite programming [122, 123]. This method allows to probe the part of the spectrum containing states with lower conformal dimension, and the outcome is generally a bound on these "light" states' dimension. In this chapter, we will focus on conformal field theories that are chiral, in the sense that a single copy of the Virasoro algebra generates the asymptotic symmetry group [124]. Building up on the chargeless case analysis developed in [125], we will provide a new method to derive a bound for charged conformal field theories, fully leveraging the theory's especially favourable modular properties. In the simplest case of a $u(1)$ charge, this allows us to tighten the bound to its maximum in a completely analytical way. We will thus prove that in any consistent chiral conformal field theory with such a charge, there are charged primary states with dimension at or below the black hole threshold; i.e. with

$$h \leqslant \frac{c}{24} + 1\,. \tag{7.1}$$

We then extend this analysis to cases involving generic affine Lie algebras, and our result can be stated as follows:

For any chiral conformal field theory with a current algebra $\mathfrak{g}$, with central charge c, level k, and dual Coxeter number g, there are (Virasoro + current algebra) primary states with non-trivial charge and dimension $h \leqslant \frac{c}{24} + 1$, provided $k \geqslant k_*(c,g)$. In particular, we evaluate the critical level $k_*(c,g)$ to be zero numerically for $s\hat{u}(2)$, $s\hat{u}(3)$, $s\hat{o}(10)$ and G_2, and provide an analytic proof for $u(1)$.

Before we derive this result in Sections 7.3, 7.4 and 7.5, we will engage as a preliminary exploration in applying the original modular bootstrap procedure of [121] to warped CFT, which as we will see amounts to the case of chiral conformal field theory with $u(1)$ charge. By choosing a functional wisely, we will be able to lower the bound obtained in previous works on this sort of theory, and this will be an indication that there is indeed a better result within reach.

7.1. Hellerman bound for unitary warped CFT

We begin with a non-extensive review of previous results of the modular bootstrap. In particular, in [121] a lower bound on the weight of the lowest primary state of a completely general two-dimensional conformal field theory was derived making use only of modular invariance, together with unitarity and discreteness of the spectrum. The result was that for any state,

$$\Delta = h + \bar{h} \;\leqslant\; \Delta_B \tag{7.2}$$

with

$$\Delta_B = \frac{c}{6} + 0.474 + O\left(1/c\right)\,. \tag{7.3}$$

In [120], Friedan and Keller numerically bettered that bound using semi-definite programming [122, 123], yielding

$$\Delta_B^{\mathrm{FK}} = \frac{c}{6} - 0.004 + O\left(1/c\right) \tag{7.4}$$

for bosonic conformal field theories. In [126], a similar technique was applied to conformal field theories with an extra conserved current. In that case, the analytic bound was found to be

$$\Delta_B^{\mathrm{BDFK}} = \frac{c}{6} + \frac{3}{2\pi} + O\left(1/c\right) \,, \tag{7.5}$$

which practically reproduces the Hellerman bound since $3/2\pi \simeq 0.477$. In [127], theories with spin were investigated, and numerical extrapolation lead to a refinement of the Friedan-Keller bound for large c:

$$\Delta_B^{\mathrm{CLY}} = \frac{c}{8} + \frac{1}{2} + O\left(1/c\right) \,, \qquad (c > 4) \,. \tag{7.6}$$

A similar result was found in [128] for theories involving an extra current (abelian or not):

$$\Delta_B^{\mathrm{DFX}} = \frac{c}{\alpha} + O\left(1\right) \,, \qquad \alpha > 8 \,. \tag{7.7}$$

With a somewhat different method but still relying on semi-definite programming, the authors of [129] were able to prove that

$$\frac{c}{12} \leqslant \lim_{c\to\infty} \Delta_B^{\mathrm{BLS}} < \frac{c}{9} \,. \tag{7.8}$$

Finally, in [130], a new algorithm was developed that allowed for much more computational efficiency. The numerical results were extended from up to $c \simeq 100$ to $c \simeq 1800$, and extrapolation yielded the estimate

$$\Delta_B^{\mathrm{AJHT}} = \frac{c}{9.1} + O(1) \,, \tag{7.9}$$

still well away from the black hole threshold (4.27).

In addition, in [126] (and [128]), a bound on the charge-to-mass ratio was found:

$$\frac{Q}{m} > \frac{G}{4\sqrt{\pi}} \tag{7.10}$$

which allowed the authors to make a connection with the Weak Gravity Conjecture. The idea of this conjecture [131] is that in a consistent theory of quantum gravity, black holes (including charged ones) should be able to decay via Hawking radiation without leaving remnants of any kind. There should then be sufficiently light particles in the spectrum in order for this process to take place, "light" being understood in the broader sense that not only is the mass of these particles bounded above, but also the mass-to-charge ratio. For such particles, gravitational attraction is weaker than gauge repulsion – hence the name. The Weak Gravity Conjecture being motivated by holography and string theory arguments, it is rather remarkable that an upper bound is recovered using only modular invariance. Initially, this was proposed as a criterion to discriminate effective theories of quantum gravity with matter that look consistent at low energy but cannot be completed at high energy in a consistent way (the "swampland"), from those that can (the "landscape") [132]. It was formulated in Minkowski spacetime, then extended to anti-de Sitter space [133] in order to make a connection with AdS/CFT and allow for the exploitation of conformal field theories' full potential, as in the works cited above. Interestingly, it has been argued that the Weak Gravity Conjecture could provide an explanation for weak cosmic censorhip [134]: indeed, a whole class of counterexamples to the latter conjecture could be ruled out by the very weakness of gravity. If it is not a complete answer, it is nonetheless an interesting relation between two apparently disconnected conjectures. In addition, the Weak Gravity Conjecture could also put a constraint on effective field theory models of inflation and axion inflation (see e.g. [135, 136]). We thus see that, with possible applications in cosmology, the modular bootstrap has a rather extensive reach.

Among the previous results listed above, the bootstrap method used in [126] is particularly straightforward to implement. We will now review it while extending it to warped conformal field theories. The general idea of the "Hellerman method" developed in [121] is to consider a finite set of energy levels in the spectrum, characterized by the fact that all the associated dimensions are bigger than some value h^*. Then, if we can find a functional (made of combinations of partial derivatives) such that acting on the modular transformation (5.34) gives only a positive contribution, it means there has to be states in the spectrum that do not belong to this set we have considered: indeed we need such states to get a negative contribution that balances the positive one, in order for the whole expression to vanish. These states then have dimension smaller than h^*, which gives us the desired bound. This illustrates how using only very general properties of the field theory, one is able to constrain it in a non-trivial way. Let us now see how this method works by applying it to warped conformal field theories.

7.1.1. Linear functional method for warped CFT

Recall that the partition function of warped conformal field theories can be written in terms of plane generators L_0, P_0 as (5.32). We have seen that it can also be put in a simpler form, but for now we will be satisfied with the preferred formulation of [109]. Everything we will do here will turn out to be valid for the simpler form as well. Primary states $|\hat{h}, p\rangle$ of the hatted generators (5.33) can be defined such that[1] [109]

$$\hat{L}_0 = \hat{h} + \frac{p^2}{k}\,, \qquad \hat{P}_0 = p \tag{7.11}$$

with $\hat{h} \geqslant 0$. We can then expand the partition function (5.32) over a basis of states as

$$\hat{Z}(\tau, z) = \sum_j e^{-2\pi i\tau\left(\hat{h}_j - \frac{c}{24} + \frac{p_j^2}{k}\right)} e^{2\pi i z p_j} \tag{7.12}$$

and as in [121, 126], the descendant states are included in the partition function. The modular transformation is given by (5.34).

Starting over from (7.12), where we replace $h_j = \hat{h}_j$ for more readability, the above modular transformation property can then be written, along the lines of [126], as

$$e^{-\pi i k \frac{z^2}{2\tau}}\ \hat{Z}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) - \hat{Z}(\tau, z) = \sum_j F_j(\tau, z) = 0 \tag{7.13}$$

with

$$F_j(\tau, z) = e^{-\pi i k \frac{z^2}{2\tau}}\ e^{-\frac{2\pi i}{\tau}\left(h_j - \frac{c}{24} + \frac{p_j^2}{k}\right)}\ e^{-\frac{2\pi i}{\tau} z p_j} - e^{2\pi i\tau\left(h_j - \frac{c}{24} + \frac{p_j^2}{k}\right)}\ e^{2\pi i z p_j}. \tag{7.14}$$

[1] This actually follows from the Sugawara construction (5.37).

In the search for a Hellerman bound, we focus on Z as a function of β, so taking $\tau = i\beta/2\pi$, this reads:

$$F_j(\beta, z) = e^{\frac{-4\pi^2}{\beta}\left(h_j - \frac{c}{24} + \frac{p_j^2}{k} + zp_j\right)} e^{-\pi^2 k \frac{z^2}{\beta}} - e^{-\beta\left(h_j - \frac{c}{24} + \frac{p_j^2}{k}\right) + 2\pi i z p_j} . \tag{7.15}$$

We choose a functional similar to the one in [126], which involves first and third derivatives with respect to β, as in the original Hellerman derivation:

$$\alpha(F_j) \equiv \left[\partial_\beta + A\,\partial_\beta\,\partial_z^2 + B\,\partial_\beta^3 + G\,\partial_z^2\right] F_j(z, \beta)\Big|_{z=0,\beta=2\pi} \tag{7.16}$$

with $F_j(z, \beta)$ given by (7.15). The conditions we impose on it are the following[2]:

$$\text{(i)} \quad \alpha(F_{\text{vac}}) > 0\,, \tag{7.17}$$

$$\text{(ii)} \quad \alpha(F) > 0 \qquad \text{if } p = 0\,, \tag{7.18}$$

$$\text{(iii)} \quad \alpha(F) > 0 \qquad \text{if } h > h^*\,, \tag{7.19}$$

where the vacuum, at which F_{vac} is evaluated, is chosen to be $(h, p) = (0, 0)$ since we are only considering unitary theories. This yields a polynomial of degree 6 in p. We have the freedom to choose the coefficients however we please, so after some trial and error, the best functional we get is obtained by setting

$$G = \frac{A}{2\pi}\,, \qquad A = \frac{2}{k\pi} + \frac{3B}{4k\pi}\frac{\left(c^2\pi^2 + 36(12 + c\pi)\right)}{(12\pi)^2}\,. \tag{7.20}$$

Since after acting with the functional α, B then comes as an overall factor, we can put it to

[2] which are somewhat different from the ones used in [126], where the first one is $\alpha(F_{\text{vac}}) = 1$.

1. Dropping the j indices to avoid clutter, we have

$$\alpha(F) = \frac{e^{-2\pi\left(h-\frac{c}{24}+\frac{p^2}{k}\right)}}{(24k)^3\,\pi^2}\left[p_0(h) + p^2\,P_2(h) + p^4\,P_4(h) + p^6\,P_6\right] \qquad \text{with} \tag{7.21}$$

$$\begin{aligned}
P_6 &= 2\cdot 24^3\pi^2 \\
P_4(h) &= h\left(6\cdot 24^3 k\pi^2\right) + 6\cdot 24^2 k\pi(12+c\pi) \\
P_2(h) &= h^2\,\left(6\cdot 24^3 k^2\pi^2\right) + h\,\left(-12\cdot 24^2 k^2\pi(12+c\pi)\right) + 3\cdot 24k^2\left(-12^2 + c\pi(12+c\pi)\right) \\
P_0(h) &= h^3\left(2\cdot 24^3 k^3\pi^2\right) + h^2\,\left(-6\cdot 24^2 k^3\pi(12+c\pi)\right) + h\,\left(3\cdot 24k^3\left(-12^2+c\pi(12+c\pi)\right)\right) + \alpha_0 \\
\alpha_0 &= ck^3\left(c^2\pi^2 + 36(12+c\pi)\right)
\end{aligned}$$

All terms are positive at large h. Condition (i) is verified by construction (through the choice (7.20)), and condition (iii) is verified for all p, $c \geqslant 1$, $k > 0$ with

$$h\;>\;h^* = \frac{c}{24} + \frac{3}{4\pi} + \frac{1}{8\pi}\sqrt{\frac{1}{6}\,c^2\pi^2 + 6(10+c\pi)}\,. \tag{7.22}$$

Condition (ii) turns out to be redundant. At large c, the square root term is slightly dominant:

$$h^* = \frac{c}{24}\left(1+\sqrt{\frac{3}{2}}\right) + O(\sqrt{c}). \tag{7.23}$$

This means there must be states in the warped conformal field theory that have dimension

$$h - \frac{c}{24}\;<\;\frac{c}{24}\sqrt{\frac{3}{2}} + O(\sqrt{c})\,. \tag{7.24}$$

This is a significantly better bound than the ones obtained for regular conformal theories or flavoured conformal field theories discussed above (even as these bounds need to be divided by 2 in order to translate to a chiral case such as this one). In particular, the best known bound for warped conformal field theories [110], though obtained via a simpler form of the partition function, is not as tight. Let us now comment on this alternative form of the warped partition function.

Even though in [109] it is argued that the form (7.12) of the partition function should be privileged, it is not the only way to write it. In terms of the cylinder generators, the partition function is simply

$$Z(\tau, z) = \mathrm{Tr}\left(e^{2\pi i\tau L_0^{\mathrm{cyl}}} e^{2\pi i z P_0^{\mathrm{cyl}}}\right) \tag{7.25}$$

which is the same as the partition function of a chiral conformal field theory with an extra $U(1)$ charge. This form of the warped partition function is used in [110] to argue that the bound for the full flavoured conformal field theory obtained in [128] can be straightforwardly adapted to the case of unitary warped conformal field theory. The Apolo-Song bound is then

$$h < \frac{c}{\alpha} + O(1)\,, \qquad \alpha > 16\,. \tag{7.26}$$

It is easy to check that the bound (7.24) obtained above still holds for this more tractable form of the partition function, as expected. However, if we had used it to begin with, we probably would have dismissed the functional (7.16)-(7.20) since in this simpler case the dependence in the charge p disappears entirely. Working with a slightly more complicated partition function has allowed us to go to a greater depth in the analysis.

The result (7.24) is a strong hint that one could lower the bound on charged primary states' dimension down to the black hole threshold[3]. This is precisely what we will prove in the next section, but before we do, we will expand for a bit on how the bound we have found translates into a statement on the mass of the lightest charged black hole in warped AdS$_3$.

7.1.2. Bound for the warped black hole

At this point, holography may be called into play in order to translate this result into a statement on the gravity side of the duality. The usual suspect theory dual to warped CFT is some gravity theory on the warped AdS$_3$ space reviewed in Section 5.1, but all the candidate gravity theories involve more bulk fields than required by symmetry arguments [94]. A minimal setup, similar in spirit to Einstein-Hilbert theory in more common contexts,

[3] which in the chiral case is $c/24$, instead of $c/12$.

has been built up and dubbed Lower Spin Gravity [94]. It can be written as a $(2+1)$-dimensional Chern-Simons theory such as the one reviewed in Section 4.1, in terms of an $SL(2,\mathbb{R})$ gauge field B and a $U(1)$ gauge field $\bar{B}$ as

$$S = \frac{c}{24\pi}\int \mathrm{Tr}\left(B\wedge dB + \frac{2}{3}B\wedge B\wedge B\right) + \frac{k}{8\pi}\int \ \bar{B}\wedge d\bar{B} \tag{7.27}$$

with c the Virasoro central charge and k the Kac-Moody level. This translates into geometric terms as [137]

$$S = \frac{1}{2}\int d\tau\, m\dot{x}^{\mu}\, G_{\mu\nu}\, \dot{x}^{\nu} + \frac{m}{2}\int d\tau \ + p\int d\tau\, \bar{A}_{\mu}\, \dot{x}^{\mu} \tag{7.28}$$

which is the most general action at lowest non-trivial order for a point particle of mass m and charge p coupled to the geometry described by a $SL(2,\mathbb{R})$ metric $G_{\mu\nu} = A_{\mu}A_{\nu}$ and a $U(1)$ flat connection $\bar{A}_{\mu}$. In the case of spacelike warped AdS_3, the fields B and $\bar{B}$ are related to the geometric variables A and $\bar{A}$ by [94]

$$\bar{A} \equiv A^0 \equiv \sqrt{\left|\frac{48}{c\,d^2 a}\right|}\bar{B} - \frac{2b}{d}\,\zeta^0_{\ell}\, B^{\ell}\,, \qquad A^1 \equiv \frac{\zeta^1_{\ell}\, B^{\ell}}{\sqrt{d}}\,, \qquad A^2 \equiv \frac{\zeta^2_{\ell}\, B^{\ell}}{\sqrt{d}} \tag{7.29}$$

with the ζ_i three linearly independent vectors in $SL(2)$

$$\zeta_0 = (0,-1,1)\ , \quad \zeta_1 = (0,1,1)\ . \quad \zeta_2 = (1,0,0)\ . \tag{7.30}$$

The parameters a, b, d encode, respectively, arbitrary rescalings of the time coordinate, the warping factor, and the AdS radius. More precisely, one recovers the metric of the warped black hole upon setting [138]

$$a = \frac{16}{d^2 k \ell^2}\,, \qquad b^2 = \frac{\nu^2}{2\ell^2}\,, \qquad d = \frac{\nu^2+3}{2\ell^2}\,. \tag{7.31}$$

In [137] the dictionary relating the mass m of the bulk particle to its boundary counterpart's scaling dimension was worked out, yielding the relation

$$h = \sqrt{\ell^2\, m^2 + p^2} \tag{7.32}$$

with p identified as the charge of the corresponding gauge field. In this framework, the bound (7.24) obtained above translates into a bound on the mass of the lightest bulk field in warped AdS$_3$ as

$$h > h^* \qquad \Leftrightarrow \qquad m^2 < \frac{1}{\ell^2}\left(h^{*\,2} - p^2\right) . \tag{7.33}$$

Since we expect the black holes to be able to decay via Hawking emission, this also puts a bound on the mass of the lightest warped black hole.

In this section we have used the warped CFT partition function expressed in terms of plane generators (5.32) to derive a bound *à la* Hellerman on the dimension of the lightest charged states in the theory, and we translated it into a bound on the mass of the lightest warped black hole. However, there is a more classic take on the warped CFT partition function, which is to express it in terms of cylinder generators as (5.31). When we do so, warped CFT is nothing more than a chiral conformal field theory with a $u(1)$ charge. Exploiting this similarity will lead us to an even better result, which is precisely the one hinted at by the closeness of (7.24) with $c/24$. Such a tight bound has already been derived for chiral conformal field theories by making use of the chiral partition function's especially favourable modular properties [125], leading to the conclusion that the lowest possible dimension h of primary states in such a theory needs to be such that $h \leqslant c/24 + 1$ in order to simply comply with modular invariance requirements. In the rest of this chapter, we will extend this argument to chiral conformal field theories with current algebra, proving the bound analytically for the simplest case, the one that corresponds to warped CFT, and providing evidence supporting the conjecture that it extends well beyond that. Before we do so, we need to review modular analysis into greater depth, which is the purpose of the next section.

7.2. A brief introduction to modular forms

A real surface S can be turned into a complex curve $\Sigma = \Gamma/\bar{\mathbb{H}}$ where Γ is a discrete subgroup of $SL(2,\mathbb{R})$ and $\bar{\mathbb{H}}$ is the upper half-plane with its boundary [139]:

$$\bar{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{i\infty\}\,, \qquad \mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}\,. \tag{7.34}$$

When $\Gamma = SL(2,\mathbb{Z})$, the functions that live on Σ are called modular *functions*. By definition, they are meromorphic functions $f : \bar{\mathbb{H}} \to \mathbb{C}$ with symmetry

$$f(A\tau) = f(\tau)\,, \qquad \forall \tau \in \bar{\mathbb{H}}\,, \quad A \in \Gamma \tag{7.35}$$

that have at worst a pole at $\tau = i\infty$. As we have seen before, the modular group Γ is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,. \tag{7.36}$$

Modular *forms* are objects that are almost modular functions, but not quite. Instead of (7.35), they obey

$$f(A\tau) = f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)\,, \qquad k \in \mathbb{Z}\,. \tag{7.37}$$

The integer k is referred to as the weight of the modular form [140]. A set of generators for the space of modular forms is given by the Eisenstein series of weight 4 and 6, defined respectively as[4]

$$E_4(\tau) \;=\; 1 + 240 \textstyle\sum_{n=1}^{\infty} q^n\, \sigma_3(n) \;= 1 + 240\, q + 2160\, q^2 + 6720\, q^3 + \cdots\,, \tag{7.38}$$

$$E_6(\tau) \;=\; 1 - 504 \textstyle\sum_{n=1}^{\infty} q^n\, \sigma_5(n) \;= 1 - 504\, q - 16632\, q^2 - 122976\, q^3 + \cdots\,, \tag{7.39}$$

[4] Using Lambert series, they can also be expressed as

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3\, q^n}{1 - q^n}\,, \qquad E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5\, q^n}{1 - q^n}\,.$$

where $q = e^{2\pi i\tau}$ as previously and $\sigma_i(n)$ is the divisor sigma, i.e. the sum of ith powers of the positive divisors of n. The order 2 Eisenstein series

$$E_2(\tau) \;=\; 1 - 24\sum_{n=1}^{\infty} q^n\,\sigma_1(n) = 1 - 24\,q - 72\,q^2 - 96\,q^3 - 168\,q^4 + \cdots\,, \tag{7.40}$$

for its part, is a quasi-modular function which transforms as

$$E_2(A\tau) = (c\tau+d)^2\,E_2(\tau) + \frac{6c}{\pi i}\,(c\tau+d)\,. \tag{7.41}$$

In our argument we also use the modular discriminant

$$\Delta(\tau) = \frac{E_4^3(\tau) - E_6^2(\tau)}{1728}\,, \tag{7.42}$$

a modular form of weight 12, and the J- function

$$J(\tau) = j(\tau) - 744 \qquad \text{where} \qquad j(\tau) = \frac{E_4^3(\tau)}{\Delta(\tau)} \tag{7.43}$$

which is well-studied Klein's modular function, in terms of which every modular function can be expressed [140]. The q-expansions of these modular forms are

$$\Delta(q) = q - 24\,q^2 + 252\,q^3 - 1472\,q^4 + \cdots \tag{7.44}$$

$$J(q) = q^{-1} + 196884\,q + 21493760\,q^2 + \cdots \tag{7.45}$$

Both these functions are well-known in number theory; in particular, the modular discriminant is related to Dedekind's eta function (3.50) as

$$\Delta(\tau) = (2\pi)^{12}\;\eta^{24}(\tau)\,. \tag{7.46}$$

Klein's J-function has been studied thoroughly, with special attention devoted to the coefficients c_n of its Laurent expansion

$$J(q) = q^{-1} + \sum_{n=1}^{\infty} c_n \, q^n \,. \tag{7.47}$$

The c_n are all positive integers and have been found to be bounded by [141]

$$c_n = \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}\, n^{3/4}} \left(1 - \frac{3}{32\pi\sqrt{n}} - \frac{15}{2048\pi^2 n} + \varepsilon_n\right) \qquad \text{for} \quad n \geqslant 3 \tag{7.48}$$

where $|\varepsilon_n| \leqslant \frac{3\cdot 10^{-4}}{n^{3/2}}$. The maximum value of this expression yields the asymptotic expression

$$c_n \underset{n\to\infty}{\sim} \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}\, n^{3/4}} \tag{7.49}$$

For a slightly wider range, there is also the weaker bound [141]

$$c_n \;\leqslant\; e^{4\pi\sqrt{n+1}} \qquad \text{for} \quad n \geqslant 0 \,. \tag{7.50}$$

The coefficients c_n are the starting point of Monstrous Moonshine, which unexpectedly connects the two conceptually incommensurable areas of mathematics that are group theory and number theory. Indeed, the c_n can be related to the dimensions of the smallest irreducible representations of the Monster group, the largest of the exceptional finite simple groups [142]. From there, a whole new field of research (the so-called Moonshine) opened on the ties between finite groups and modular forms [139].

Finally, let us mention a useful tool we will resort to: Hecke operators, whose action on a function $F(\tau)$ is defined as [125, 140]

$$T_m F(\tau) \;\widehat{=}\; \sum_{\delta|m} \sum_{n=0}^{\delta-1} F\left(\frac{m\tau + n\delta}{\delta^2}\right) \,. \tag{7.51}$$

In particular, such an operator acts on Klein's invariant (7.47) in the following way:

$$T_m J(q) = q^{-m} + \sum_{n=1}^{\infty} c_n^m \, q^n \,. \tag{7.52}$$

The coefficients c_n^m are related to J's coefficients $c_n = c(n)$ as [143]

$$c_n^m = \sum_{\delta|(m,n)} \frac{m}{\delta} \, c\left(\frac{mn}{\delta^2}\right) \tag{7.53}$$

where $\delta \mid (m,n)$ means that δ divides (m,n).

With this, we are now equipped to make the most of modular properties of chiral conformal field theories. We will start by reviewing briefly the analysis of the chargeless case proposed in [125], then we will see how it can be adapted for the case with charge, starting with the simplest one in the next section before getting to more generic cases in the following ones.

7.3. Chiral CFT with a $u(1)$ charge

7.3.1. Preliminary observations

Recall from Section 3 that the partition function of a chiral conformal field theory without charge on a torus of modular parameter τ is

$$Z(q) \;=\; \mathrm{Tr}\left(q^{L_0-\kappa}\right) \;=\; q^{h-\kappa}(1-\delta_{h0}\, q) \prod_{n\geqslant 1} \frac{1}{1-q^n} \,, \tag{7.54}$$

with

$$q = e^{2\pi i\tau} \,, \quad \kappa = \frac{c}{24} \tag{7.55}$$

and the Kronecker delta has allowed us to combine the vacuum and non-vacuum characters (3.54) and (3.49) in a single expression. Alternatively, this reads

$$Z(q) = \sum_{h=0}^{\infty} r_h \, q^{-\kappa+h} \tag{7.56}$$

where $r_0 = 1$ since the vacuum is unique. In [125] it is argued that since the chiral partition function is meromorphic and has a κ-th order pole at $q = 0$ (or equivalently $\tau = i\infty$), it can be written in terms of Klein's modular invariant (7.43) as

$$Z(q) = \sum_{n=0}^{\kappa} j_n \, J(q)^n \tag{7.57}$$

with $j_\kappa = 1$. For example, in the $\kappa = 2$ case, instead of writing $Z(q)$ as

$$Z(q) = q^{-2} + r_1 \, q^{-1} + r_2 + r_3 \, q + \ldots \tag{7.58}$$

we can write it as

$$\begin{aligned} Z(q) &= J^2(q) + j_1 \, J(q) + j_0 && (7.59) \\ &= q^{-2} + j_1 \, q^{-1} + j_0 + 393768 + q \; (42987520 + 196884 \, j_1) + \cdots && (7.60) \end{aligned}$$

In this case, one sees immediately that $r_1 = j_1$ and so

$$r_3 = 42987520 + 196884 \, r_1 \tag{7.61}$$

which removes some uncertainty about the coefficients. The expression (7.57) depends on $\kappa + 1$ coefficients, which is exactly the number of coefficients appearing in the polar part of $Z(q)$. One way to look at it is then to say that the partition function is completely determined by its polar part. In the expansion (7.56), which tells us which primary states (with dimension h) we have in our theory and how many (r_h) of each of those we have, everything is fixed once the first $\kappa + 1$ coefficients are. We can for instance pick the first $\kappa + 1$ coefficients such that there are no non-vacuum primaries below $h = \kappa + 1$, while remaining consistent with our original assumptions. We cannot, however, cancel the r_h with $h \geqslant \kappa + 1$: this means these states are needed in order for the chiral conformal field theory to be consistent. We could of course pick our coefficients differently and have a perfectly consistent chiral conformal field theory with some states below that threshold, but in any case, the lightest primaries in the theory will have at most $h = \kappa + 1$. In other words, the lightest primaries in the theory have $h \leqslant \kappa + 1$ or

$$h \;\leqslant\; \frac{c}{24} + 1\,. \tag{7.62}$$

With this we see how powerful a constraint we are able to obtain by simply making full use of the modular properties of the theory.

We can then extend this argument to a partition function with charge $Z(\tau, z)$, provided we show that it is some sort of modular form. Let us start with the simplest case of a single $u(1)$ charge, in which case the partition function is

$$Z(\tau, z) = \mathrm{Tr}\left(q^{L_0-\kappa} y^{J_0}\right) = \sum_{h,p} e^{2\pi i\tau(h-\kappa)} e^{2\pi i z p} \tag{7.63}$$

with

$$q = e^{2\pi i\tau}\,, \qquad y = e^{2\pi i z}\,, \qquad \kappa = \frac{c}{24}. \tag{7.64}$$

Under modular transformations[5], it obeys [126]

$$Z\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = e^{\frac{\pi i c z^2}{c\tau+d}}\, Z(\tau, z)\,. \tag{7.65}$$

In the following we replace $A\tau = \frac{a\tau+b}{c\tau+d}$. Assuming the theory is invariant under charge flip, the expansion in z has no $O\left(z\right)$ term and reads

$$\begin{aligned} Z(\tau, z) &= Z(\tau) + \frac{z^2}{2}(2\pi i)^2\, \mathrm{Tr}\left(J_0^2\, q^{L_0-\kappa}\right) + ... &(7.66)\\ &= Z(\tau) + \sum_{n=1}^{\infty} \frac{z^{2n}}{(2n)!}(2\pi i)^{2n}\, f_{2n}(\tau) &(7.67)\end{aligned}$$

[5] with this c having of course nothing to do with the central charge in κ.

where $Z(\tau)$ is the usual modular invariant partition function without charge and we define $f_{2n}(\tau) \;\widehat{=}\; \mathrm{Tr}\left(J_0^{2n}\, q^{L_0-\kappa}\right)$. In particular, in the second term in the z-expansion, $f_2(\tau)$ transforms as

$$f_2(A\tau) = (c\tau+d)^2\, f_2(\tau) + \frac{c}{2\pi i}\,(c\tau+d)\,Z(\tau). \tag{7.68}$$

If not for the second term, $f_2(\tau)$ would transform as a modular form of weight 2. This reminds us of the modular properties of the Eisenstein series of order 2, given by (7.41). Furthermore, we know that the modular discriminant (7.42) has an order q expansion (7.44), so it is easy to verify that $f_2(\tau)$ can be rewritten as [144, 145]

$$f_2(\tau) = \Delta^{-\kappa}(\tau)P_{12\kappa+2}(\tau) + \frac{1}{12}E_2(\tau)Z(\tau) \tag{7.69}$$

where $P_{12\kappa+2}$ is some modular form of weight $12\kappa+2$, that can be expressed in terms of the basis elements as

$$P_{12\kappa+2}(\tau) = \sum_{\ell=1}^{\kappa} a_\ell\, E_4^{3\ell-1}(\tau)\, E_6^{2\kappa-2\ell+1}(\tau)\,. \tag{7.70}$$

All the polar behavior is then encoded in the modular discriminant $\Delta^{-\kappa}$, so $P_{12\kappa+2}$ is finite, and this expression reproduces the transformation (7.69). One can easily work out similar transformations for all the other f_{2n} (see Appendix C), but it turns out all we need to derive our bound is the order 2 term $f_2(\tau)$ in the expansion. The coefficients a_ℓ are entirely up to us to fix as we please; in particular, we may use them to cancel as many terms as possible in the expansion of $Z(\tau, z)$. The question is indeed: how big can we make the gap between the ground state and the first charged primary state in our theory? As in the argument reviewed above, we would now like to see how modular properties of the chiral partition function with charge constrain the answer to that question.

In addition to these modular constraints, we also have to mind positivity constraints: indeed, the coefficients r_i in the expansion (7.56) of $Z(\tau)$ are counting the number of states in each cell of the spectrum; therefore they need to be positive. The same kind of argument applies to the coefficients appearing in the q-expansion of any $f_{2n}(\tau)$: they count the total J_0^{2n} charge for each cell, so they cannot be negative either. An explicit study of the cases for the first few values of κ (see Appendix C) leads to the observation that only a restrained number of terms can be made to vanish in a way that is consistent with these positivity constraints. As soon as we have used all the freedom granted by the coefficients a_ℓ, which brings the partition function from $O(q^{-\kappa})$ to $O(1)$, and picked a value for r_1 that cancels the $O(1)$ term, we have no way to pick an r_2 that cancels the $O(q)$ term in a consistent way. This hardly meets the expectation: as κ grows larger, one expects to be able to choose more and more values for the coefficients, and meet limitations only as we go to higher and higher order in the expansion of the partition function, but this is what we observe whatever the value of κ. Let us briefly take a look at an example: for $\kappa = 2$, f_2 is given by

$$\begin{aligned} f_2^{\kappa=2}(q) &= \Delta^{-2}(q)\left(a_1\, E_4^2(q)\, E_6^3(q) + a_2\, E_4^5(q)\, E_6(q)\right) + \frac{1}{12} E_2(q) Z(q) \\ &= q^{-2}\left(a_1 + a_2 + \frac{1}{12}\right) + q^{-1}\frac{1}{12}\left(-24 - 11808\, a_1 + 8928\, a_2 + r_1\right) + \frac{1}{12}\left(-72 - 24\, r_1 + r_2\right) \\ &\quad + q\,\frac{1}{3}\left(10746856 + 516720288\, a_1 - 503926368\, a_2 + 49203\, r_1 - 6\, r_2\right) + O(q^2). \end{aligned} \tag{7.71}$$

The choices

$$a_1 = -a_2 - \frac{1}{12}\,, \qquad a_2 = -\frac{1}{1728}\left(80 + \frac{r_1}{12}\right) \tag{7.72}$$

cancel the first two terms to yield

$$\begin{aligned} f_2^{\kappa=2}(q) &= \left(-6 - 2\, r_1 + \frac{r_2}{12}\right) + \left(4979664 + 32808\, r_1 - 2\, r_2\right) q \\ &+ \left(6576701472 + 4979664\, r_1 - 6\, r_2\right) q^2 + O(q^3)\,. \end{aligned} \tag{7.73}$$

We can go one step further by taking

$$r_1 = -3 + \frac{r_2}{24} \tag{7.74}$$

as long as $r_2 \geqslant 72$ for positivity. Then f_2 becomes

$$f_2^{\kappa=2}(q) = 1365\left(3576 + r_2\right) q + 207480\left(31626 + r_2\right) q^2 + O(q^3) \tag{7.75}$$

so there is no way to go further by picking a positive r_2. This case is simple, but the fact that the same logic applies when there are much more coefficient freedom is far from trivial.

In effect, we see that we run into a contradiction if we try to get rid of all the charged states with dimension below $c/24+1$. Before proving our statement, we can make this argument more precise and make explicit how the contradiction arises to get more intuition about it. In general, if we use all the freedom granted by the coefficients a_ℓ of $P_{12\kappa+2}$ to devise a chiral conformal field theory with no charged primaries up to dimension κ, we bring the partition function from $O\left(q^{-\kappa}\right)$ to $O\left(1\right)$. Then the $O(1)$ term in f_2 takes the following form:

$$f_2(q)\Big|_{O(1)} = -\alpha_0 + \alpha_\kappa r_\kappa - \sum_{n=1}^{\kappa-1} \alpha_n r_n \tag{7.76}$$

where the α_i are given by $\alpha_\kappa = \frac{1}{12}$ and[6]

$$\alpha_{\kappa-1} = 2\,, \quad \alpha_{\kappa-2} = 6\,, \quad \alpha_{\kappa-3} = 8\,, \quad \alpha_{\kappa-4} = 14\,, \quad \alpha_{\kappa-5} = 12\,, \quad \alpha_{\kappa-6} = 24\ \ldots \tag{7.77}$$

At this point, the $O(q)$ term in f_2 has a similarly repetitive form:

$$f_2(q)\Big|_{O(q)} = \gamma_0 - 2\,r_\kappa + \sum_{n=1}^{\kappa-1} \gamma_n r_n \tag{7.78}$$

where the coefficients are known numerically:

$$\begin{aligned} \gamma_{\kappa-1} &= 32808\,, \quad \gamma_{\kappa-2} = 4979664\,, \quad \gamma_{\kappa-3} = 243931152\,, \quad \gamma_{\kappa-4} = 6576635856\,, \\ \gamma_{\kappa-5} &= 120749101488\,, \quad \gamma_{\kappa-6} = 1685248831296\,, \quad \ldots \end{aligned} \tag{7.79}$$

We will see later on how to explicitly write down this sequence, but at this point we are satisfied to just notice that $\gamma_i \gg \alpha_i$ for all i. Widening the gap a bit more by cancelling the $O(1)$ term means picking

$$r_1 = \frac{1}{\alpha_1}\left(-\alpha_0 + \alpha_\kappa r_\kappa - \sum_{n=2}^{\kappa-1} \alpha_n r_n\right)\,. \tag{7.80}$$

[6] This is a known sequence: except for α_κ, this is just the sum of even divisors of $2n$, or $2\,\sigma(n)$.

Replacing (7.80) into the expression for the $O(q)$ term, we get

$$O(q) = \gamma_0 - \alpha_0 \frac{\gamma_1}{\alpha_1} + \left(\frac{\gamma_1}{\alpha_1} \alpha_\kappa - 2 \right) r_\kappa + \sum_{n=2}^{\kappa-1} \left(\gamma_n - \frac{\gamma_1}{\alpha_1} \alpha_n \right) r_n \,. \tag{7.81}$$

This is zero if

$$r_2 = -\frac{1}{\alpha_1 \gamma_2 - \alpha_2 \gamma_1} \left(\alpha_1 \gamma_0 - \alpha_0 \gamma_1 + (\alpha_\kappa \gamma_1 - 2\alpha_1)\, r_\kappa - \sum_{n=3}^{\kappa-1} (\alpha_n \gamma_1 - \alpha_1 \gamma_n)\, r_n \right) \tag{7.82}$$

which now yields[7]

$$\tilde{r}_1 = \frac{1}{\alpha_2 \gamma_1 - \alpha_1 \gamma_2} \left(-\alpha_2 \gamma_0 + \alpha_0 \gamma_2 - (\alpha_\kappa \gamma_2 - 2\alpha_2)\, r_\kappa + \sum_{n=3}^{\kappa-1} (\alpha_n \gamma_2 - \alpha_2 \gamma_n)\, r_n \right) . \tag{7.83}$$

Since the γ_i are larger and larger as i is smaller and smaller, the denominator of this latter expression is positive. Imposing the positivity constraint on $\tilde{r}_1$ is equivalent to demanding

$$\sum_{n=3}^{\kappa-1} (\alpha_n \gamma_2 - \alpha_2 \gamma_n)\, r_n \;\; \geqslant \;\; \alpha_2 \gamma_0 + (\alpha_\kappa\, r_\kappa - \alpha_0) \gamma_2 - 2\alpha_2\, r_\kappa \,. \tag{7.84}$$

No matter what r_κ is, since the α_i are small, for $\kappa \geqslant 3$ the last term on the right-hand side is negligible compared to the previous one which involves γ_2. We can then recall that positivity of (7.80) implies

$$\alpha_\kappa\, r_\kappa - \alpha_0 \;\; \geqslant \;\; \sum_{n=2}^{\kappa-1} \alpha_n\, r_n \,. \tag{7.85}$$

The condition above becomes

$$\sum_{n=3}^{\kappa-1} \alpha_n \gamma_2\, r_n - \sum_{n=3}^{\kappa-1} \alpha_2 \gamma_n\, r_n \;\; \geqslant \;\; \alpha_2 \gamma_0 + \sum_{n=2}^{\kappa-1} \alpha_n \gamma_2\, r_n - \ldots \tag{7.86}$$

[7] Of course these expressions will not mean anything for $\kappa = 1, 2$, but these cases are quite easily worked out explicitly (see Appendix C) so we can safely ignore them and focus on higher values of κ.

which boils down to

$$-\sum_{n=3}^{\kappa-1}\alpha_2\gamma_n\, r_n \;\geqslant\; \alpha_2\gamma_0+\alpha_2\gamma_2\, r_2-\dots \tag{7.87}$$

The contradiction arises as the left-hand side is negative while the right-hand side is positive. We can then conclude that for any κ, there is no way to choose r_2 such that the $O(q)$ term in the partition function of a chiral conformal field theory with $U(1)$ charge cancels out. This is a bit of a hand-waving argument, since we have not made precise how to define the sequence of γ_i, but it is a good indication that there must be a way to give a fully general proof. This is what we set out to do next.

7.3.2. Analytic proof of the bound

A convenient basis to write a modular function is the one made of Hecke operators acting on the J-function (7.52). As in [125], this will allow some properties of the partition function to become manifest. The partition function can be written in these terms as

$$Z(q)=\sum_{m=0}^{\kappa} r_{\kappa-m}\, T_m J(q)\,. \tag{7.88}$$

We will also need the Eisenstein series of order 2 (7.40), for which in the following we adopt the shorthand notation $\sigma_n = 24\,\sigma_1(n)$. Our argument will be formulated in terms of these coefficients, along with the ones of the J-function c_n and the ones from the Hecke operators c_n^m, all of which are positive. We also define $T_0 J(q)=1$ (which means all $c_n^0=0$) and $r_0=1$. We can then rewrite the second-order term (7.69) in the expansion of the partition function as

$$f_2(q)=\sum_{m=1}^{\kappa} a_m\, q\,\partial_q\, T_m J(q)+\frac{1}{12}E_2(q)Z(q)\,. \tag{7.89}$$

Indeed, one can easily verify that for a modular function $F(q)$, the expression $q\,\partial_q\,F(q)$ transforms as a modular form of weight 2 (see Appendix D). We then have

$$\begin{aligned} f_2(q) &= \sum_{m=1}^{\kappa} a_m\, q\,\partial_q \left(q^{-m} + \sum_{n=1}^{\infty} c_n^m\, q^n \right) + \frac{1}{12}\left(1 - \sum_{n=1}^{\infty} \sigma_n\, q^n\right)\left(\sum_{\ell=0}^{\kappa} r_{\kappa-\ell}\left(q^{-\ell} + \sum_{r=1}^{\infty} c_r^\ell\, q^r\right)\right) \\ &= \sum_{m=1}^{\kappa} a_m \left(-m\,q^{-m} + \sum_{n=1}^{\infty} c_n^m\, n\, q^n\right) + \frac{1}{12}\sum_{\ell=0}^{\kappa} r_{\kappa-\ell}\left(q^{-\ell} - q^{-\ell}\sum_{n=1}^{\infty}\sigma_n\, q^n + \sum_{r=1}^{\infty} c_r^\ell\, q^r \right. \\ &\qquad \left. - \sum_{r=1}^{\infty}\sum_{n=1}^{\infty}\sigma_n\, c_r^\ell\, q^{r+n}\right) . \end{aligned} \tag{7.90}$$

We proceed as in the previous subsection, trying to make the gap as big as possible by cancelling as many terms as we can. As before, we start by fixing the coefficients a_m to get rid of the polar part of f_2, which is just

$$f_2(q)\Big|_{\text{polar}} = \sum_{m=0}^{\kappa} -m\,a_m\,q^{-m} + \frac{1}{12}\sum_{\ell=0}^{\kappa} r_{\kappa-\ell}\left(q^{-\ell} - \sum_{n=1}^{\ell}\sigma_n\, q^{n-\ell}\right) . \tag{7.91}$$

Putting it to zero yields an expression for the coefficients a_m:

$$a_m = \frac{1}{12m}\left(r_{\kappa-m} - \sum_{n=m+1}^{\kappa}\sigma_n\, r_{\kappa-n-m}\right) . \tag{7.92}$$

Next we turn to the $O(1)$ term, which is

$$f_2(q)\Big|_{O(1)} = \frac{1}{12}\left(r_\kappa - \sum_{n=1}^{\kappa}\sigma_n\, r_{\kappa-n}\right) \tag{7.93}$$

(since there are no negative indices for r) and putting it to zero yields

$$r_\kappa = \sum_{n=1}^{\kappa}\sigma_n\, r_{\kappa-n} \, . \tag{7.94}$$

The $O(q)$ term is then

$$f_2(q)\Big|_{O(q)} = \sum_{m=1}^{\kappa} a_m\, c_1^m + \frac{1}{12}\left(\sum_{\ell=0}^{\kappa} r_{\kappa-\ell}\, c_1^\ell - \sum_{n=1}^{\kappa+1} \sigma_n\, r_{\kappa-n+1}\right) \tag{7.95}$$

Replacing a_m and r_κ in this expression, we get

$$\begin{aligned} 12\, f_2(q)\Big|_{O(q)} &= \sum_{m=1}^{\kappa}\left[r_{\kappa-m}\left(\frac{m+1}{m}\, c_1^m - \sigma_{m+1} - \sigma_1\sigma_m\right) - \sum_{\ell=m+1}^{\kappa} \frac{c_1^m}{m}\,\sigma_{\ell-m}\, r_{\kappa-\ell}\right] \\ &= \sum_{m=1}^{\kappa} r_{\kappa-m}\left((m+1)\, c_m - \sum_{\ell=1}^{m-1} c_\ell\, \sigma_{m-\ell} - \sigma_{m+1} - 24\,\sigma_m\right) \end{aligned} \tag{7.96}$$

where we have used (7.53) and noticed that $c_1^m = m\, c_m$. The statement is that this expression is always positive. Setting

$$F_m = (m+1)\, c_m - \sum_{\ell=1}^{m-1} c_\ell\, \sigma_{m-\ell} - \sigma_{m+1} - 24\,\sigma_m \tag{7.97}$$

it turns out that these coefficients appear in the following fairly simple expression:

$$q\,\partial_q J(\tau) + E_2(\tau)\big(J(\tau) + 24\big) = \sum_{m\geqslant 1} F_m\, q^m\,. \tag{7.98}$$

Since $(J(\tau) + 24)$ is the partition function of the theory of 24 chiral bosons on the Leech lattice, we can identify

$$q\,\partial_q J(\tau) + E_2(\tau)\big(J(\tau) + 24\big) = 12\,\mathrm{Tr}\left(J_0^2\, q^{L_0-1}\right) \tag{7.99}$$

where J_0 is now one of the $u(1)$ currents of that theory. Indeed, both sides of this equation have the same modular properties, and each of them is uniquely determined by two coefficients, which match. We then have

$$12\,\mathrm{Tr}\left(J_0^2\, q^{L_0-1}\right) = \sum_{m\geqslant 1} F_m\, q^m\,, \tag{7.100}$$

which proves the quantity (7.97) is positive. This completes the proof that the partition function of a chiral conformal field theory with $u(1)$ charge can never be of order lower than $O(q)$, which means that whatever the specifics of the chiral conformal field theory with $u(1)$ charge under consideration, the lightest charged primary state will have dimension

$$h \leqslant \frac{c}{24} + 1\,. \tag{7.101}$$

For large c, this bound coincides with the black hole threshold. This result, being completely analytic, calls for no improvement and is the definite answer as to what sort of states we can have in the light part of the spectrum. In particular, this result also applies to unitary warped conformal field theories, and betters the bounds found previously for these theories.

We have thus retrieved the chargeless case bound (7.62) in the simplest charged case. We now provide evidence that this result still holds for theories with more general symmetries.

7.4. Chiral CFT with a $s\hat{u}(2)_k$ charge

In the next two sections we are going to work with algebras of the type Virasoro plus current algebra, with the latter being now an affine Lie algebra instead of just $u(1)$. In general, an affine Lie algebra $\hat{\mathfrak{g}}$ is a central extension of a simple Lie algebra $\mathfrak{g}$. The commutation relations are

$$\left[J^a_n, J^b_m\right] = \sum_c i\, f^{abc}\, J^c_{n+m} + k\, n\, \delta^{ab}\delta_{n+m} \tag{7.102}$$

where f^{abc} are the structure constants of $\mathfrak{g}$, which is generated by J^a_0. Affine Lie algebras are then infinite-dimensional. We will consider these generic algebras in the next section, but for now let us look at the affine algebra $s\hat{u}(2)_k$ for a trial run. In this case, the commutation relations are

$$[J^3_m, J^3_n] = k\,\frac{m}{2}\,\delta_{m+n}\,, \qquad [J^3_m, J^\pm_n] = \pm J^\pm_{n+m}\,, \qquad [J^+_m, J^-_m] = k\,m\,\delta_{m+n} + 2J^3_{m+n}\,. \tag{7.103}$$

A state in a theory with this symmetry is labeled by eigenvalues of the Virasoro zero-mode L_0 and J_0^3. An important difference with the $u(1)$ case is that descendants of uncharged primary states can now carry charge: for example, using (7.103), we see that $J_{-1}^{+}\,|\text{vac}\rangle$, which is a level 1 descendant of the vacuum, has J_0^3 eigenvalue equal to 1. This means we will not be able to apply the exact same procedure as in the previous section: indeed it does not make sense to simply cancel off the polar part of the first non-trivial order f_2 of the partition function anymore. Nevertheless, it does not affect the general outcome as we will see below. But before we adjust our strategy, let us discuss the partition function and in particular the characters.

7.4.1. Characters of $s\hat{u}(2)_k$

The partition function of a chiral conformal field theory with affine $s\hat{u}(2)_k$ charge is

$$Z(\tau,z)=\sum_{\ell=0}^{k}\sum_{h\geqslant 0} r_{h\ell}\,\chi_h(\tau)\,\chi_\ell^{(k)}(\tau,z)\ , \tag{7.104}$$

with $\chi_h(\tau)$ the Virasoro character

$$\chi_h(\tau)=q^{h-\frac{c}{24}}(1-\delta_{h0}\,q)\prod_{n\geqslant 1}\frac{1}{1-q^n} \tag{7.105}$$

and the $s\hat{u}(2)_k$ characters [71]

$$\chi_\ell^{(k)}(\tau,z)=\frac{\Theta_{\ell+1,k+2}(\tau,z)-\Theta_{-\ell-1,k+2}(\tau,z)}{\Theta_{1,2}(\tau,z)-\Theta_{-1,2}(\tau,z)}\,. \tag{7.106}$$

In this expression, k is the level appearing in (7.103), $\ell=2j\in[0,k]$ is twice the spin and the generalized theta-function is

$$\Theta_{\ell,k}(\tau,z)=\sum_{n\in\mathbb{Z}+\frac{\ell}{2k}} q^{kn^2}y^{kn}\,. \tag{7.107}$$

For more readability, we set $\kappa=\frac{c}{24}$ as usual.

We will start by assuming that all primaries with $h \leqslant \kappa{+}1$ are in the spin 0 representation of $s\hat{u}(2)_k$ (i.e. uncharged), which means that our flavoured partition function takes the form

$$Z(\tau, z) = \chi_0(\tau)\chi_0^{(k)}(\tau, z) + \sum_{k=1}^{\kappa+1} r_h\ \chi_h(\tau)\chi_0^{(k)}(\tau, z) + O(q^2) \tag{7.108}$$

for some positive integers r_h which count the number of primaries of dimension h. We can then expand

$$Z(\tau, z) = Z(\tau) - 2\pi^2 z^2 f_2(\tau) + O(z^4) \tag{7.109}$$

where, as previously,

$$Z(\tau) = \mathrm{Tr}\left(q^{L_0-\kappa}\right), \qquad\qquad f_2(\tau) = \mathrm{Tr}\left(\,(J_0^3)^2 q^{L_0-\kappa}\right)\,. \tag{7.110}$$

As in the $u(1)$ case, modular properties of $Z(\tau)$ and $f_2(\tau)$ constrain them so that they are completely determined by κ coefficients (i.e. they are entirely determined by their polar parts). We will detail how it all works in the next subsection, but we can already note that this also means that $r_{\kappa+1}$ is determined by $r_1, ..., r_\kappa$. In particular one can use either $Z(\tau)$ or $f_2(\tau)$ to fix $r_{\kappa+1}$. We will show that both expressions for $r_{\kappa+1}$ are inconsistent with respect to one another for any choice of $r_1, ..., r_\kappa$. The contradiction then reveals that our assumption that all primaries with $h \leqslant \kappa + 1$ are uncharged is wrong, and that on the contrary we should have charged primaries below that threshold. In order to make this argument about uncharged primaries, we will then be mostly interested in the spin 0 character.

The character (7.106) can be rewritten

$$\chi_\ell^{(k)}(\tau, z) = q^{-\hat{c}/24}\ \left(\frac{\sum_{m\in\mathbb{Z}} q^{(k+2)m^2+m(\ell+1)}\ \sin\left(2\pi z\left[m(k+2)+\frac{\ell+1}{2}\right]\right)}{\sum_{m\in\mathbb{Z}} q^{2m^2+m}\ \sin\left(2\pi z\left[2m+\frac{1}{2}\right]\right)}\right) \tag{7.111}$$

where[8] $\hat{c} = \frac{3k}{k+2}$.

For $\ell = 0$, expanding this in z yields

$$
\begin{aligned}
q^{\hat{c}/24}\chi_0^{(k)}(\tau,z) = &\prod_{n\geq 1}\frac{1}{(1-q^n)^3}\sum_{m\in\mathbb{Z}}\Big(2m(k+2)+1\Big)q^{(k+2)m^2+m} \\
&-\frac{\pi^2 z^2}{6}\left[\prod_{n\geq 1}\frac{1}{(1-q^n)^3}\sum_{m\in\mathbb{Z}}\Big(2m(k+2)+1\Big)^3 q^{(k+2)m^2+m}\right. \\
&\left.-\prod_{n\geq 1}\frac{1}{(1-q^n)^6}\sum_{m\in\mathbb{Z}}\Big(2m(k+2)+1\Big)q^{(k+2)m^2+m}\sum_{n\in\mathbb{Z}}(4n+1)^3\,q^{2n^2+n}\right]+O(z^4)
\end{aligned}
\tag{7.112}
$$

where we have used

$$
\sum_{m\in\mathbb{Z}}(4m+1)\,q^{2m^2+m} = \prod_{n\geq 1}(1-q^n)^3 \tag{7.113}
$$

which derives from Jacobi's triple product identity.

7.4.2. Derivation of the bound

In what follows, we will work with $k > \frac{c}{24}$ for simplicity. The more general results of the next section also apply to this case, but in the meantime this allows us to sketch the argument. With $k > \frac{c}{24}$, the terms at q^{k+1} are pushed back to at least $O(q^2)$ in the partition function. It turns out we can write

$$
\sum_{m\in\mathbb{Z}}(4m+1)^3 q^{2m^2+m} = (8\,q\,\partial_q+1)\prod_{n\geq 1}(1-q^n)^3 = E_2(\tau)\prod_{n\geq 1}(1-q^n)^3 \tag{7.114}
$$

[8] We call it that way because $\hat{c}$ is in fact the central charge of the conformal field theory one can build from the current algebra using a Sugawara construction. The purpose of such a construction is to define an energy-tensor such that the generators J^a are primaries of dimension one. The modes are then given by

$$
L_m = \frac{1}{2(k+g)}\sum_{a=1}^{\dim\mathfrak{g}}\left(\sum_{l\leqslant -1}J_l^a J_{m-l}^a + \sum_{l>-1}J_{m-l}^a J_l^a\right).
$$

so that our character becomes

$$q^{\hat{c}/24}\chi_0^{(k)}(\tau,z) = \left[\prod_{n\geq 1}\frac{1}{(1-q^n)^3} + O(q^{k+1})\right] \\ - 2\pi^2 z^2\left[\frac{1}{12}\prod_{n\geq 1}\frac{1}{(1-q^n)^3}(1-E_2(\tau)) + O(q^{k+1})\right] + O(z^4)\,. \tag{7.115}$$

Putting this back into the full partition function and comparing with (7.109), we see that

$$Z(\tau) = q^{-\kappa}\left((1-q) + \sum_{h=1}^{\kappa+1} r_h\, q^h\right)\prod_{n\geq 1}\frac{1}{(1-q^n)^4} + O(q^2) \tag{7.116}$$

and

$$f_2(\tau) = \frac{1}{12}Z(\tau)\big(1-E_2(\tau)\big) + O(q^2)\,. \tag{7.117}$$

Note that $f_2(\tau)$ goes like $2q^{-\kappa+1} + O(q^{-\kappa+2})$ as expected. With a bit more work, we get

$$Z(\tau) = q^{-\kappa}\left(1 + \sum_{m=1}^{\kappa+1} q^m R_m\right) + O(q^2) \tag{7.118}$$

with

$$R_m = A_m - A_{m-1} + \sum_{\ell=0}^{m-1} r_{m-\ell}A_\ell \tag{7.119}$$

and

$$A_m = \sum_{\substack{0\leqslant n_1+n_2+n_3\leqslant m \\ n_i\geqslant 0}} p(n_1)\,p(n_2)\,p(n_3)\,p(m-n_1-n_2-n_3) \tag{7.120}$$

coming from

$$\prod_{n\geqslant 1}\frac{1}{(1-q^n)^4} \;=\; \left(\sum_{n\geqslant 0} p(n)\,q^n\right)^4 \;=\; \sum_{m=0}^{\infty} q^m A_m\,. \tag{7.121}$$

Note that each of the R_m is positive. The order q piece of $Z(\tau)$ is read off as simply

$$Z(\tau)\Big|_{O(q)} = R_{\kappa+1} \tag{7.122}$$

At this point, we have not made use of the modular properties of our chiral conformal field theory's partition function. Doing so will now allow us to constrain some of the coefficients R_m that appear in both orders of interest $Z(\tau)$ and $f_2(\tau)$ in the z-expansion of $Z(\tau, z)$, and these constraints will be the basis of the extension of our argument to the $s\hat{u}(2)_k$ case.

As in the $u(1)$ case, we can write the partition function uniquely as

$$Z(\tau) = \sum_{m=0}^{\kappa} j_{\kappa-m}\, T_m J(\tau) \tag{7.123}$$

provided we identify

$$j_0 = 1, \qquad\qquad j_m = R_m \quad \text{for } m = 1, ..., \kappa\,. \tag{7.124}$$

The Hecke operator (7.51) expands as

$$T_m J(\tau) = q^{-m} + m\, c_m\, q + O(q^2)\,. \tag{7.125}$$

With this rewriting, the modular properties of the partition function then determine the coefficient of q^1 to be

$$Z(\tau)\Big|_{O(q)} = \sum_{m=1}^{\kappa} j_{\kappa-m}\, m\, c_m = \kappa\, c_\kappa + \sum_{m=1}^{\kappa-1} R_{\kappa-m}\, m\, c_m \tag{7.126}$$

Comparing (7.122) and (7.126), this yields the requirement

$$R^{(0)}_{\kappa+1} = \kappa\, c_\kappa + \sum_{m=1}^{\kappa-1} R_{\kappa-m}\, m\, c_m\,. \tag{7.127}$$

We have added the superscript (0) to indicate this requirement comes from the $O(z^0)$ piece of the flavoured partition function. We will now examine how the modular properties of the $O(z^2)$ piece constrain the same quantity. Note that R_κ does not appear at all in the expression above; this will be important later as we can entirely use properties of $f_2(\tau)$ to constrain R_κ.

In the case at hand, the modular properties of $f_2(\tau)$ give a decomposition into the form

$$f_2(\tau) = \Delta^{-\kappa}(\tau)P_{12\kappa+2}(\tau) + \frac{k}{12}E_2(\tau)Z(\tau) \tag{7.128}$$

Comparing to (7.117) gives

$$\Delta^{-\kappa}(\tau)P_{12\kappa+2}(\tau) = \frac{1}{12}Z(\tau)\big(1-(k+1)E_2(\tau)\big) + O(q^2) \tag{7.129}$$

and putting this together with (7.118) and (7.40), we get

$$\begin{aligned}&\Delta^{-\kappa}(\tau)P_{12\kappa+2}(\tau) = \\ &q^{-\kappa}\left[-\frac{k}{12} + \sum_{\ell=1}^{\kappa+1} q^{\ell}\left(2(k+1)\,\sigma_1(\ell) - \frac{k}{12}R_\ell + 2(k+1)\sum_{j=1}^{\ell-1} R_{\ell-j}\,\sigma_1(j)\right)\right] + O(q^2)\,.\end{aligned} \tag{7.130}$$

From this expression we see that the order q piece is

$$\Delta^{-\kappa}(\tau)P_{12\kappa+2}(\tau)\Big|_{O(q)} = 2(k+1)\,\sigma_1(\kappa+1) - \frac{k}{12}R_{\kappa+1} + 2(k+1)\sum_{j=1}^{\kappa} R_{\kappa+1-j}\,\sigma_1(j)\,. \tag{7.131}$$

On the other hand, in terms of the J-function

$$\Delta^{-\kappa}(\tau)P_{12\kappa+2}(\tau) = \sum_{m=1}^{\kappa} \frac{a_{\kappa-m}}{m}\, q\,\partial_q T_m J(q) \tag{7.132}$$

$$= q^{-\kappa}\sum_{m=0}^{\kappa-1}\big(-a_m\, q^m + a_m\, c_{\kappa-m}\, q^{\kappa+1}\big) + O(q^2) \tag{7.133}$$

By comparing to our previous expression (7.129) we see that this determines the a_m as

$$a_0 = \frac{k}{12}\,, \tag{7.134}$$

$$a_m = -2(k+1)\sigma_1(m) + \frac{k}{12}R_m - 2(k+1)\sum_{j=1}^{m-1} R_{m-j}\,\sigma_1(j) \qquad \text{for } 1 \leqslant m \leqslant \kappa - 1\,.$$

Since (7.133) has no q^0 piece, we conclude from (7.130) that

$$R_\kappa = \frac{24(k+1)}{k}\left(\sigma_1(\kappa) + \sum_{j=1}^{\kappa-1} R_{\kappa-j}\sigma_1(j)\right) . \tag{7.135}$$

Finally, we also get an alternative form for the $O(q)$ piece, which is

$$\begin{aligned}\Delta^{-\kappa}(\tau)P_{12\kappa+2}(\tau)\Big|_{O(q)} &= \sum_{m=0}^{\kappa-1} a_m c_{\kappa-m} \\ &= \frac{k}{12}\,c_\kappa - \sum_{m=1}^{\kappa-1} c_{\kappa-m}\left(2(k+1)\sigma_1(m) - \frac{k}{12}\,R_m + 2(k+1)\sum_{j=1}^{m-1} R_{m-j}\sigma_1(j)\right) .\end{aligned} \tag{7.136}$$

Equating (7.131) and (7.136) then gives another expression for $R_{\kappa+1}$ which we denote

$$\begin{aligned}R^{(2)}_{\kappa+1} = \frac{24(k+1)}{k}\,\sigma_1(\kappa+1) + \frac{24(k+1)}{k}\sum_{j=1}^{\kappa} R_{\kappa+1-j}\,\sigma_1(j) - c_\kappa \\ + \sum_{m=1}^{\kappa-1} c_{\kappa-m}\left(\frac{24(k+1)}{k}\,\sigma_1(m) - R_m + \frac{24(k+1)}{k}\sum_{j=1}^{m-1} R_{m-j}\,\sigma_1(j)\right) .\end{aligned} \tag{7.137}$$

In order for the theory to be consistent, our two expressions (7.122) and (7.126) for $R_{\kappa+1}$ must be equal. We can compute

$$\begin{aligned}R^{(0)}_{\kappa+1} - R^{(2)}_{\kappa+1} = (\kappa+1)c_\kappa - f_k\,\sigma_1(\kappa+1) - f_k^2\,\sigma_1(\kappa) - f_k\sum_{m=1}^{\kappa-1} c_{\kappa-m}\,\sigma_1(m) \\ + \sum_{m=1}^{\kappa-1} R_{\kappa-m}\left((m+1)\,c_m - f_k\,\sigma_1(m+1) - f_k^2\,\sigma_1(m) - f_k\sum_{\ell=1}^{m-1} c_{m-\ell}\,\sigma_1(m)\right)\end{aligned} \tag{7.138}$$

where

$$f_k \equiv \frac{24(k+1)}{k} = 24\left(1+\frac{1}{k}\right) \tag{7.139}$$

The statement is that this quantity is never zero, hence leading to the contradiction referred to above. Note that in the limit $k\to\infty$ then this reduces precisely to the $u(1)$ case. A numerical check for m from 1 to 1000 (see Figure 7.1) shows that

$$(m+1)\, c_m - f_{m+1}\, \sigma_1(m+1) - f_{m+1}^2\, \sigma_1(m) - f_{m+1} \sum_{\ell=1}^{m-1} c_{m-\ell}\, \sigma_1(\ell) \;>\; 0 \tag{7.140}$$

which argues in favour of our conjecture that the flavoured partition of a chiral conformal field theory with $s\hat{u}(2)_k$ charge has its lightest uncharged primaries below the threshold $h \leqslant c/24 + 1$.

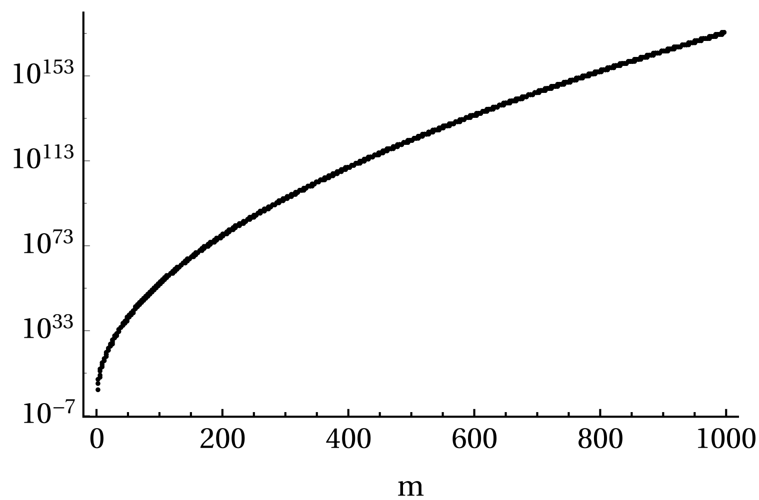

Figure 7.1 Logarithmic plot of (7.140) with respect to m with $m \in [1, 1000]$.

Recall however that this argument was valid in the regime where $k > \frac{c}{24}$, but this will be addressed in the next section. In view of the numerical evidence, we believe we can now extend this conjecture to chiral conformal field theories with more general affine symmetries.

7.5. Generic affine algebra

In this section, we will elaborate along the same lines as the argument in the $s\hat{u}(2)_k$ case. Since we will be dealing with more involved partition functions, we start with some rewriting of the characters for a generic affine Lie algebra. We will then see, as previously, how modular properties allow us to derive our bound.

7.5.1. Characters of a Lie affine algebra

In the case of generic affine Lie algebras, for a given representation of highest weight $\hat{\lambda}$, the characters are given by the Weyl-Kac character formula [146]

$$\chi_{\hat{\lambda}} = \frac{\sum_{w\in W} \epsilon(w)\, \Theta_{w(\hat{\lambda}+\hat{\rho})}}{\sum_{w\in W} \epsilon(w)\, \Theta_{w\hat{\rho}}} \tag{7.141}$$

where $\hat{\rho}$ is the Weyl vector and $\epsilon(w)$ is the signature of the element w of the Weyl group W. The theta functions evaluated at a point (ζ, τ, t) are given by

$$\Theta_{\hat{\lambda}}(\zeta,\tau,t) = e^{-2\pi i k t} \sum_{\alpha^\vee \in Q^\vee} \exp\left[-\pi i \left(2k(\alpha^\vee,\zeta) + 2(\lambda,\zeta) - \tau k \left|\alpha^\vee + \frac{\lambda}{k}\right|^2\right)\right] \tag{7.142}$$

where the $\alpha^\vee$ are the coroots of the representation of highest weight λ in the original algebra being extended. $Q^\vee$ is the (finite) coroots lattice. The hatted quantities are all affine extensions expressed in terms of the corresponding finite algebra $\mathfrak{g}$ quantities; in particular the highest weight of $\hat{\mathfrak{g}}$ is $\hat{\lambda} = (\lambda; k; 0)$ and its Weyl vector is $\hat{\rho} = (\rho; g; 0)$ with g the dual Coxeter number. As in the previous case, we will adopt an approach by contradiction, so we will be interested in the uncharged primaries ($\lambda = 0$). Generalizing the character to the non-zero case is completely straightforward and just requires replacing ρ by $\lambda + \rho$ in the expression (7.143) below. For the sake of completeness, the full expressions are given in Appendix E. Since the parameter t is of little relevance as it appears in an overall factor, it is common to write the character as a function of τ and z only [71]. We will adopt this notation here by setting $t = 0$. In the case of interest, the full character is then

$$\chi_{\hat{0}}(\zeta,\tau,0) = \frac{\sum_{w\in W} \epsilon(w) \sum_{\alpha^\vee\in Q^\vee} \exp\left[-\pi i \left(2(k+g)(\alpha^\vee,\zeta) + 2(w\rho,\zeta) - \tau(k+g)\left|\alpha^\vee + \frac{w\rho}{k+g}\right|^2\right)\right]}{\sum_{w\in W} \epsilon(w) \sum_{\alpha^\vee\in Q^\vee} \exp\left[-\pi i \left(2g(\alpha^\vee,\zeta) + 2(w\rho,\zeta) - \tau g\left|\alpha^\vee + \frac{w\rho}{g}\right|^2\right)\right]}. \tag{7.143}$$

The parameter ζ is usually taken to be

$$\zeta = \sum_{i=1}^{r} z_i \alpha_i^\vee \tag{7.144}$$

with r the rank of the algebra; here we will consider the simple case where

$$\zeta = \frac{z}{|\rho|}\rho \equiv -\frac{\tilde{z}}{2\pi i}\rho \tag{7.145}$$

where we introduce $\tilde{z}$ for ease of notation when writing formulas. This normalization is chosen so that $|\zeta|^2 = z^2$. We can rewrite the character using the fact that the argument of the exponential in Θ is proportional to

$$\begin{aligned} 2(\ell\alpha^\vee + w\rho, \zeta) - \tau\ell|\alpha^\vee + w\rho/\ell|^2 &= 2(\ell w^{-1}\alpha^\vee + \rho, w^{-1}\zeta) - \tau\ell|w^{-1}\alpha^\vee + \rho/\ell|^2 \\ &= 2(\ell\alpha^\vee + \rho, w^{-1}\zeta) - \tau\ell|\alpha^\vee + \rho/\ell|^2 \end{aligned} \tag{7.146}$$

which follows from both the invariance of the scalar product under elements of the Weyl group, and from the fact that for any fixed element w of the Weyl group the coroot lattice obeys $Q^\vee = w\,Q^\vee$. We then have

$$\chi_{\hat{0}}(\zeta,\tau,0) = \frac{\sum_{\alpha^\vee\in Q^\vee}\sum_{w\in W}\epsilon(w)\exp\left[\big(w((k+g)\alpha^\vee+\rho),\rho\big)\tilde{z}\right]e^{\pi i\tau(k+g)\left|\alpha^\vee+\frac{\rho}{k+g}\right|^2}}{\sum_{\alpha^\vee\in Q^\vee}\sum_{w\in W}\epsilon(w)\exp\left[\big(w(k\alpha^\vee+\rho),\rho\big)\tilde{z}\right]e^{\pi i\tau g\left|\alpha^\vee+\frac{\rho}{g}\right|^2}}\,. \tag{7.147}$$

We can improve on this by making use of the known property [71]

$$\begin{aligned} \sum_{w\in W}\epsilon(w)e^{(w(\ell\alpha^\vee+\rho),\tilde{z}\rho)} &= \frac{\sum_{w\in W}\epsilon(w)e^{(w(\ell\alpha^\vee+\rho),\tilde{z}\rho)}}{\sum_{w\in W}\epsilon(w)e^{(w\rho,\tilde{z}\rho)}}\sum_{w\in W}\epsilon(w)e^{(w\rho,\tilde{z}\rho)} \\ &= \left(\prod_{\alpha>0}\frac{(\alpha,\ell\alpha^\vee+\rho)}{(\alpha,\rho)}\right)\left(1+\frac{g\tilde{z}^2}{24}\left(|\ell\alpha^\vee+\rho|^2-|\rho|^2\right)+O(z^3)\right)\sum_{w\in W}\epsilon(w)e^{(w\rho,\tilde{z}\rho)} \end{aligned} \tag{7.148}$$

and we can then expand our character in z as

$$\chi_{\hat{0}}(z,\tau,0) = \frac{\sum_{\alpha^\vee}\prod_{\alpha>0}(\alpha,(k+g)\alpha^\vee+\rho)\,e^{\pi i\tau(k+g)\left|\alpha^\vee+\frac{\rho}{k+g}\right|^2}}{\sum_{\alpha^\vee}\prod_{\alpha>0}(\alpha,g\alpha^\vee+\rho)\,e^{\pi i\tau g\left|\alpha^\vee+\frac{\rho}{g}\right|^2}} \tag{7.149}$$
$$+\frac{g\tilde{z}^2}{24}\left(\frac{\sum_{\alpha^\vee}\prod_{\alpha>0}(\alpha,(k+g)\alpha^\vee+\rho)\left(|(k+g)\alpha^\vee+\rho|^2-|\rho|^2\right)e^{\pi i\tau(k+g)\left|\alpha^\vee+\frac{\rho}{k+g}\right|^2}}{\sum_{\alpha^\vee}\prod_{\alpha>0}(\alpha,g\alpha^\vee+\rho)\,e^{\pi i\tau g\left|\alpha^\vee+\frac{\rho}{g}\right|^2}}\right.$$
$$\left.-\frac{\sum_{\alpha^\vee,\beta^\vee}\prod_{\alpha>0}(\alpha,(k+g)\alpha^\vee+\rho)\prod_{\beta>0}(\beta,g\beta^\vee+\rho)\left(|g\beta^\vee+\rho|^2-|\rho|^2\right)e^{\pi i\tau(k+g)\left|\alpha^\vee+\frac{\rho}{k+g}\right|^2}e^{\pi i\tau g\left|\beta^\vee+\frac{\rho}{g}\right|^2}}{\left(\sum_{\alpha^\vee}\prod_{\alpha>0}(\alpha,g\alpha^\vee+\rho)\,e^{\pi i\tau g\left|\alpha^\vee+\frac{\rho}{g}\right|^2}\right)^2}\right).$$

In particular, one can check that this indeed gives the expansion of the $s\hat{u}(2)_k$ character we had obtained in (7.112).

Similarly, just as in the $s\hat{u}(2)_k$ case, we can simplify these characters further using known functions. In particular, the equivalent of (7.113) in the generic case would be

$$\sum_{\alpha^\vee}\prod_{\alpha>0}\frac{(\alpha,g\alpha^\vee+\rho)}{(\alpha,\rho)}e^{\pi i\tau g\left(\left|\alpha^\vee+\frac{\rho}{g}\right|^2-\left|\frac{\rho}{g}\right|^2\right)} = \phi(q)^{\dim\mathfrak{g}} \tag{7.150}$$

where $\phi(q)$ is none other than the Euler function $\prod_{n\geqslant 1}(1-q^n)$. This relation turns out to be some form of the Macdonald identities [147]. We also have a generalization of (7.114):

$$q\,\partial_q\left(\phi(q)^{-n}\right) = \frac{n}{24}\Big(1-E_2(q)\Big)\phi(q)^{-n} \tag{7.151}$$

and noting that

$$\left(|\ell\alpha^\vee+\rho|^2-|\rho|^2\right)e^{\pi i\tau\ell\left(|\alpha^\vee+\rho/\ell|^2-|\rho/\ell|^2\right)} = 2\ell\,q\,\partial_q\,e^{\pi i\tau\ell\left(|\alpha^\vee+\rho/\ell|^2-|\rho/\ell|^2\right)} \tag{7.152}$$

we can rewrite the $O(z^2)$ and $O(z^0)$ pieces of the characters in terms of one another as

$$q^{\hat{c}/24}\chi_{\hat{0}}(\tau,z) = \left(1+\frac{(2\pi i z)^2}{2}\frac{k}{12}\left(\frac{24}{\hat{c}}\,q\,\partial_q-(1-E_2)\right)+O(z^4)\right)q^{\hat{c}/24}\chi_{\hat{0}}(\tau,0) \tag{7.153}$$

where

$$q^{\hat{c}/24}\chi_{\hat{0}}(\tau,0) = \phi(q)^{-\dim\mathfrak{g}} \sum_{\alpha^\vee\in Q^\vee}\prod_{\alpha>0}\frac{(\alpha,(k+g)\alpha^\vee+\rho)}{(\alpha,\rho)}\; e^{\pi i\tau(k+g)\left(|\alpha^\vee+\rho/(k+g)|^2-|\rho/(k+g)|^2\right)}\,. \tag{7.154}$$

This expression of the characters is true for a specific combination of chemical potentials that are all proportional to the complex variable z, as we have seen before. The Sugawara central charge of the Kac-Moody algebra is now $\hat{c} = \frac{k\dim\mathfrak{g}}{k+g}$. In the next subsection, we use the partition function's modular properties to constrain these in a fashion similar to the previous case.

7.5.2. Derivation of the bound

In order to make the equations in this section simpler, we introduce a new notation for the character's q-series:

$$q^{\hat{c}/24}\chi_{\hat{0}}(0,\tau) = \sum_{n=0}^{\infty} F_n\, q^n\,. \tag{7.155}$$

By comparing with (7.154), we see that $F_0 = 1$; this substitution will be used frequently in the following.

We now want to make use of the modular properties of these objects to constrain them, as we did in the previous cases. Since we cannot simply cancel the polar part of our partition function, as already mentioned in the $s\hat{u}(2)_k$ case, we will again proceed by contradiction. Assuming contrary to our conjecture that the lightest charged primary has dimension larger than $\kappa+1$, the flavoured partition function takes the form

$$Z(\tau,z) = \sum_{h=0}^{\kappa+1} R_h\, q^{h-\kappa}\, q^{\hat{c}/24}\,\chi_{\hat{0}}(z,\tau) + O(q^2) \tag{7.156}$$

and using (7.153) and (7.155) we find

$$Z(\tau) = \sum_{h=0}^{\kappa+1} q^{h-\kappa}\sum_{\ell=0}^{h} R_{h-\ell}\, F_\ell + O(q^2)\,. \tag{7.157}$$

This uniquely determines the partition function to be

$$Z(\tau)=\sum_{m=0}^{\kappa} j_{\kappa-m}\, T_m J \tag{7.158}$$

with

$$j_m=\sum_{\ell=0}^{m} F_\ell\, R_{m-\ell}\,. \tag{7.159}$$

Using the expansion (7.125) and comparing the coefficients of q^1 in both of our expressions, we find that (7.158) constrains

$$R^{(0)}_{\kappa+1}=\sum_{m=0}^{\kappa-1}\sum_{\ell=0}^{m} R_{m-\ell}\, F_\ell\,(\kappa-m)\, c_{\kappa-m}-\sum_{j=0}^{\kappa} R_j\, F_{\kappa+1-j} \tag{7.160}$$

in terms of the coefficients c_m come of the J-function appearing in (7.158). The superscript (0) is meant to emphasize that this constraint comes from the z^0 piece of the flavoured partition function.

We now turn our attention to the function f_2 as given previously by (7.128). As in the $s\hat{u}(2)_k$ case, we focus on

$$\Delta^{-\kappa}P_{12\kappa+2}=f_2(\tau)-\frac{k}{12}E_2(\tau)Z(\tau)\,. \tag{7.161}$$

Using our ansatz for the flavoured partition function and (7.153) we find that

$$\Delta^{-\kappa}P_{12\kappa+2}=\frac{k}{12}\sum_{h=0}^{\kappa+1} q^{h-\kappa}\sum_{\ell=0}^{h} R_{h-\ell}\left(\frac{24}{\hat{c}}\,\ell-1\right) F_\ell+O(q^2)\,. \tag{7.162}$$

We can now use the fact that $\Delta^{-\kappa}P_{12\kappa+2}$ is determined entirely by its polar part to derive two constraints. The first constraint comes from the fact that $\Delta^{-\kappa}P_{12\kappa+2}$, rewritten as previously as

$$\Delta^{-\kappa}P_{12\kappa+2}=\sum_{m=1}^{\kappa} a_{\kappa-m}\, q\,\partial_q T_m J$$

has no q^0 piece. This yields

$$R_\kappa = \sum_{\ell=0}^{\kappa-1} \left(\frac{24}{\hat{c}}(\kappa - \ell) - 1 \right) R_\ell F_{\kappa-\ell}\,. \tag{7.163}$$

The second constraint comes from the $O(q)$ piece, which takes the form

$$\begin{aligned} R^{(2)}_{\kappa+1} = &\sum_{\ell=0}^{\kappa} \left(\frac{24}{\hat{c}}(\kappa + 1 - \ell) - 1 \right) R_\ell \, F_{\kappa+1-\ell} \\ &+ \sum_{m=0}^{\kappa-1} \sum_{\ell=0}^{m} \left(\frac{24}{\hat{c}}(m - \ell) - 1 \right) R_\ell \, F_{m-\ell} \, c_{\kappa-m} \end{aligned} \tag{7.164}$$

with the superscript (2) indicating this constraint comes from the $O(z^2)$ piece of the flavoured partition function.

Our claim, as before, is that these expressions are inconsistent with one another. Indeed, substituting in the values of R_κ we can calculate

$$R^{(0)}_{\kappa+1} - R^{(2)}_{\kappa+1} \;=\; \frac{24}{\hat{c}} \sum_{\ell=1}^{\kappa} R_{\kappa-\ell} \sum_{m=-1}^{\ell} \tilde{c}_m F_{\ell-m} \left(\frac{\hat{c}}{24}\,(m+1) - (\ell - m) \right) \tag{7.165}$$

where the $\tilde{c}_m$ come from the expansion

$$J(\tau) + \frac{24}{\hat{c}} F_1 = \sum_{m=-1}^{\infty} \tilde{c}_m q^m \tag{7.166}$$

i.e. $\tilde{c}_m = c_m$ for $m \neq 0$ and $\tilde{c}_0 = \frac{24}{\hat{c}}\, F_1$.

We can then see that the conjecture boils down to checking

$$\sum_{m=-1}^{\ell} \tilde{c}_m F_{\ell-m} \left[\frac{\hat{c}}{24}(m+1) - (\ell - m) \right] > 0 \tag{7.167}$$

for all integer $\ell > 0$. For various algebras and ranges of κ, and for all values of k, this constraint checks out. These results are summarized in Table 7.5.2. We explain below how it suffices to only check (7.167) for a finite number of values of k.

Algebra	Values of κ
$s\hat{u}(2)$	$1 \leq \kappa \leq 1000$
$s\hat{u}(3)$	$1 \leq \kappa \leq 1000$
$s\hat{o}(10)$	$1 \leq \kappa \leq 100$
G_2	$1 \leq \kappa \leq 1000$

Table 7.1 Algebras and range of κ checked

For example, in Figure 7.2 we have plotted the logarithm of (7.167) with respect to ℓ for the affine Lie algebra $s\hat{u}_3$ with values $\kappa \in [1, 1000]$, $k = 10$. Corresponding plots for other algebras show a similarly unambiguous growth as ℓ grows large.

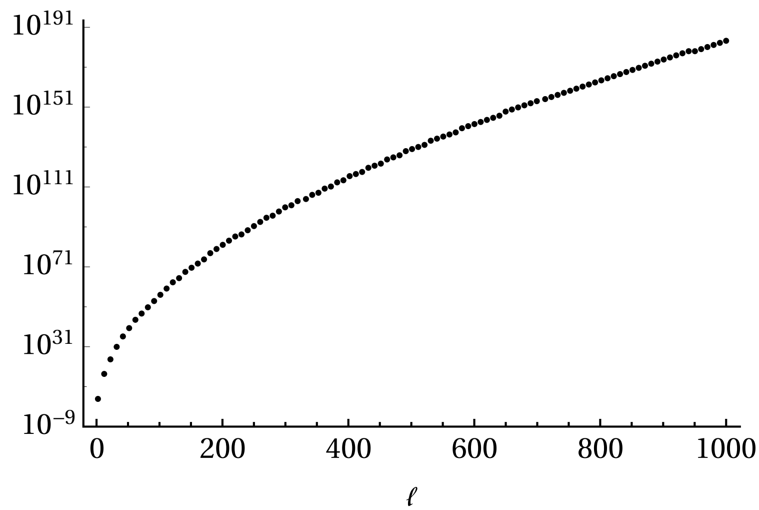

Figure 7.2 Logarithmic plot of (7.167) with respect to ℓ for $s\hat{u}_3$ with $\kappa \in [1, 1000]$, $k = 10$.

However, there is no way to make sure that the combination in (7.167) cannot be negative for some algebras. A quick examination of the situation where $\ell = 1$, $k \geqslant 2$ reveals that it can be negative for values g/k bigger than 25 approximately. This ratio of the parameters is never reached for the few cases tested, but it is for the affine algebra E_8 (which numerics confirm). Indeed, the E_8 conformal field theory at level $k = 1$ has no charged primaries. This indicates that our result may not apply for levels below a certain critical value k_*, and the

E_8 conformal field theory provides an example where $k_* > 0$. There may be a fair amount of algebras for which this critical value is zero, as in the cases we tested, but it otherwise remains to be determined.

One case that we do have analytic control over is the limit $k \to \infty$, where the constraint becomes equivalent to what we have obtained for $u(1)$. To see this we note that (7.167) is the coefficient of $q^{\ell - \hat{c}/24}$ in the combination

$$\frac{\hat{c}}{24}\,(q\,\partial_q J)\,\chi_{\hat{0}}(0,\tau) - \left(J + \frac{24}{\hat{c}}F_1\right) q\,\partial_q \chi_{\hat{0}}(0,\tau)\,. \tag{7.168}$$

In the limit $k \to \infty$ we have $\hat{c} = \dim \mathfrak{g}$ and $\chi_{\hat{0}}(0,\tau) = \eta(\tau)^{-\dim \mathfrak{g}}$ (in particular $F_1 = \dim \mathfrak{g}$). Using the property

$$q\,\partial_q \log \eta(\tau) = \frac{1}{24} E_2(\tau) \tag{7.169}$$

this reduces to the coefficient of $q^{\ell - \hat{c}/24}$ in

$$\frac{\dim \mathfrak{g}}{24}\,\eta(\tau)^{-\dim \mathfrak{g}}\big(q\,\partial_q J + E_2(J+24)\big)\,. \tag{7.170}$$

This expression clearly has positive coefficients at all orders in q as the term in brackets is simply the $u(1)$ case and η^{-1} has positive coefficients[9].

In addition, as hinted at above, for fixed κ it suffices to only check our conjecture for a finite number of k. Indeed, at fixed value of κ, (7.167) has to be checked for $\ell = 1, ..., \kappa$ and this requires computation of F_m for $m = 1, ..., \kappa + 1$. These F_m are computed from the character as

$$q^{\hat{c}/24}\chi_{\hat{0}}(0,\tau) = \sum_{m=0}^{\infty} F_m q^m \tag{7.171}$$

$$= \phi^{-\dim \mathfrak{g}} + O(q^{k+1}) \tag{7.172}$$

[9] This follows from the fact that $\eta^{-1}(q) = q^{-1/24}\phi(q)^{-1}$ and $\phi(q)^{-1}$ is the generating function for the partitions of integers.

and we see that the F_m have a fixed form for $m < k+1$.

If we consider $k \geqslant \kappa + 1$ then all of the F_m we need for the constraint come from the expansion of $\phi^{-\dim \mathfrak{g}}$. The only k dependence of (7.167) is now in the Sugawara central charge, $\hat{c} = \frac{k \dim \mathfrak{g}}{k+g}$, and this dependence is completely analytic. We can then take the derivative of the left hand side of (7.167) with respect to k and we find that

$$\frac{d}{dk} \sum_{m=-1}^{\ell} \tilde{c}_m F_{\ell-m} \left[\frac{\hat{c}}{24}(m+1) - (\ell - m) \right] = \sum_{m=-1}^{\ell} c_m F_{\ell-m} \frac{m+1}{24} \frac{g \dim \mathfrak{g}}{(g+k)^2} + 24\, F_\ell \, \frac{g}{k^2}\, \ell \quad (7.173)$$

The right hand side of this expression is a sum of positive terms and hence the derivative is always positive. Thus for $k \geq \kappa + 1$ we find that our sum is a strictly increasing function of k.

For fixed κ we then only need to check that (7.167) holds for $k = 1, ..., \kappa + 1$. The fact that after this point the sum is an increasing function of k results in the conclusion that (7.167) will also hold for all larger k as well. Together with the large k behaviour of our constraint, its monotonicity completes the proof of the advertised result:

> For any chiral conformal field theory with a current algebra $\mathfrak{g}$, with central charge c, level k, and dual Coxeter number g, there are (Virasoro + current algebra) primary states with non-trivial charge and dimension $h \leqslant \frac{c}{24} + 1$, provided $k \geqslant k_*(c, g)$.

In particular, we have evaluated $k_*(c, g)$ to be zero numerically for $s\hat{u}(2)$, $s\hat{u}(3)$, $s\hat{o}(10)$ and G_2, and analytically for $u(1)$. We have thus shown that, by harnessing the full potential of their modular properties, it is possible to put very tight constraints on the spectrum of chiral conformal field theories with charge; namely, that the “light” part of the spectrum necessarily contains charged primaries at or below the black hole threshold.

Conclusion

We have tried, at our very modest level, to shed light on some aspects of these "solid objects" that are black holes, through the prism of holography. The first part of this thesis was devoted to an overview of the latter in its most familiar version, namely $\mathrm{AdS}_3/\mathrm{CFT}_2$, as well as some general notions about black holes. We have also reviewed the information paradox in order to make explicit all the complexity surrounding the notion of black hole and its ties to the search for a theory of quantum gravity. We have then approached the problem of getting closer to such a theory by extending holography beyond its comfort zone along two distinct avenues.

Entropy of the warped black hole

We have first discussed a less common holographic scenario which has the perk of being related to astrophysical black holes without departing too much from the AdS/CFT correspondence. On the gravity side of this setup, one finds deformations of anti-de Sitter space (and in particular a black hole), and on the gauge side a field theory with symmetries characterized by a direct sum of a Virasoro algebra and a $u(1)$ Kac-Moody current algebra. We have seen that since warped AdS spaces are solutions of theories of gravity with higher curvature terms in the Lagrangian, one has to make sure to calculate all the relevant charges of these theories with the proper corresponding method. We have computed the warped black hole charges for completely general higher-derivative theories using the covariant phase space formalism, and shown that the entropy thus obtained matched the one derived from the Cardy-like formula in the warped conformal field theory. We hope this result provides further evidence that warped duality may provide insights into realistic models of black holes.

Flavoured chiral modular bootstrap

The affinity between warped CFT's algebra and a chiral conformal algebra with $u(1)$ charge then inspired us to use the powerful tools of modular analysis to constrain their spectrum. We started by deriving a tentative bound on the conformal dimension of the lightest primary states using the linear functional method. The good result obtained by trial and error indicated that an even tighter bound was attainable by making use of more number-theoretic properties. We then mobilized modular forms to rewrite the characters of the theory of interest, and showed analytically that the theory needed to have primary states with dimension below the threshold $h = c/24$ (at large central charge) in order to be consistent with its partition function's modular properties. We extended this analysis further with numerical evidence that this result applied to chiral conformal field theories with charge falling under diverse affine Lie algebras. This led us to the conclusion that any chiral conformal field theory with a current algebra contains (Virasoro + current algebra) primary states with non-trivial charge at or below the black hole threshold, provided the level is bigger than a certain critical value. We found various examples of algebras for which this critical level is zero, in addition to the simplest case in which we had been able to prove it analytically.

Weak Gravity Conjecture

Various paths forward open from this point on, the most straightfoward one being the search for an explicit expression of the critical level in terms of the quantities describing the Virasoro-Kac-Moody algebra. On the other hand, the same lines of thought that got us to obtain our bound might be used to constrain the mass-to-charge ratio, thus making contact with the Weak Gravity Conjecture. One could for example start by adding states with charge to our analysis, and see how it affects the constraint. Among the vast landscape of chiral conformal field theories, this would enable one to discriminate those that have particles allowing the lightest black holes to decay in the dual theory, and hence make sense, from those that do not. It would be equally interesting to see if our analysis can be generalized to non-chiral theories, in a way similar to what has been done for $\mathcal{N} = (2, 2)$ two-dimensional superconformal field theories [148].

Back to black holes

We have in addition attempted to translate our field theory result into a corresponding bound on the mass of the lightest warped black holes. This analysis should however be refined. An obvious way forward would then be to turn this result into a statement about extreme Kerr. However, this is far from trivial for the following reasons. First, if (self-dual) warped AdS spaces can be made to emerge seamlessly from extreme Kerr by taking the near-horizon limit and fixing one coordinate[10], it is not yet clear how to make the transition back upstream. Indeed it would require uplifting warped AdS_3 to a four-dimensional solution of pure Einstein gravity while maintaining appropriate boundary conditions. Even then, it would just yield a statement regarding near-horizon extreme Kerr and not Kerr itself. Clarifying the relation between the warped black hole and the Kerr black hole down to this depth of detail would seem like the desirable (if not effortless) next step to take.

This thesis has taken us from general relativity to number theory by way of group theory and thermodynamics. The complex interplay between these various approaches sketches the broad contours of the black hole, at least from a theoretical perspective. Many aspects of this elusive object remain to be understood, not the least of which is the way it evaporates, leaving behind only (to paraphrase Wheeler [149]) not its grin, but more captivating mysteries.

[10] That is, upon accepting that one of the remaining coordinates becomes periodic, hence the “self-dual” nuance.

Appendices

A. On-shell conditions for theories without derivatives of the Riemann tensor

In this section, we shall study the consequences of the equations of motion for the most general theory without derivatives of Riemanns. Moreover, we will show that on-shell conditions imply (6.57). Due to the fact that we are working in three dimensions, the most general action (without derivatives of Riemanns) is of the form

$$I = \int d^3x\sqrt{-g}\ f(R_{\mu\nu}, g^{\mu\nu})\,. \tag{174}$$

The equation of motion is easily derived (see [150] for example) to be

$$2\frac{\partial f}{\partial g^{\mu\nu}} - f g_{\mu\nu} = \nabla^\alpha\nabla_\nu P_{\alpha\mu} + \nabla^\alpha\nabla_\mu P_{\alpha\nu} - \Box P_{\mu\nu} - g_{\mu\nu}\nabla^\beta\nabla^\alpha P_{\alpha\beta} \tag{175}$$

where

$$P_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}\frac{\partial f}{\partial R_{\alpha\beta}}. \tag{176}$$

The object of interest $Z^{\mu\nu\alpha\beta}$ in three dimensions is

$$\begin{aligned} Z^{\mu\nu\alpha\beta} &\equiv \frac{\partial L}{\partial R_{\mu\nu\alpha\beta}} = \frac{\partial R_{\gamma\delta}}{\partial R_{\mu\nu\alpha\beta}}\frac{\partial L}{\partial R_{\gamma\delta}} \\ &= \frac{1}{4}\left[g^{\mu\alpha}\delta_\delta{}^\beta\delta_\gamma{}^\nu - g^{\alpha\nu}\delta_\gamma{}^\mu\delta_\delta{}^\beta + g^{\beta\nu}\delta_\gamma{}^\mu\delta_\delta{}^\alpha - g^{\beta\mu}\delta_\gamma{}^\nu\delta_\delta{}^\alpha\right]\frac{\partial L}{\partial R_{\gamma\delta}} \end{aligned} \tag{177}$$

which is related to $P^{\mu\nu}$ by

$$Z^{\mu\nu\alpha\beta}=\frac{1}{2}\left[g^{\mu\alpha}g^{\beta\nu}-g^{\alpha\nu}g^{\beta\mu}\right]+\frac{1}{4}\left[g^{\mu\alpha}P^{\beta\nu}-g^{\nu\alpha}P^{\beta\mu}+g^{\beta\nu}P^{\alpha\mu}-g^{\beta\mu}P^{\alpha\nu}\right]\,. \tag{178}$$

For later purposes, it is useful to note that for a locally warped AdS_3 spacetime, due to $SL(2,\mathbb{R})\times U(1)$ symmetry, we have that for $q\geq 1$,

$$(R^q_{\alpha\beta})^{\mu\nu}\equiv R_{\alpha\beta_1}R^{\beta_1}{}_{\beta_2}R^{\beta_2}{}_{\beta_3}\dots R^{\beta_{q-1}}{}_{\beta}=A_q g^{\mu\nu}+B_q R^{\mu\nu}, \tag{179}$$

with A_q and B_q constants which are dependent on ℓ and ν. For example, $A_1=0,\quad B_1=1,\quad A_2=2\nu^2\left(\nu^2-3\right)/\ell^4,\quad B_2=-(3+\nu^2)/\ell^2$. By definition, the A_q and B_q satisfy the following recursion relations

$$A_q \;=\; A_2B_{q-1}=\frac{2\nu^2(\nu^2-3)}{\ell^4}B_{q-1} \tag{180}$$

$$B_q \;=\; A_{q-1}+B_2B_{q-1}=A_{q-1}-\frac{3+\nu^2}{\ell^2}B_{q-1}\,. \tag{181}$$

As a notational convention, we will denote $\mathrm{Tr}\left(R^q_{\alpha\beta}\right)\equiv(R^q_{\alpha\beta})^{\mu}{}_{\mu}$. Moreover, it is also useful to note that

$$\begin{aligned}\nabla^\alpha\nabla_\mu R_{\alpha\nu}=\nabla^\alpha\nabla_\nu R_{\alpha\mu} &= -\frac{6\nu^2}{\ell^4}g_{\mu\nu}-\frac{3\nu^2}{\ell^2}R_{\mu\nu}\\ \Box R_{\mu\nu} &= \frac{12\nu^2}{\ell^4}g_{\mu\nu}+\frac{6\nu^2}{\ell^2}R_{\mu\nu}\,,\end{aligned} \tag{182}$$

and so

$$\nabla^\alpha\nabla_\mu R_{\alpha\nu}+\nabla^\alpha\nabla_\nu R_{\alpha\mu}-\Box R_{\mu\nu} \;=\; -\frac{24\nu^2}{\ell^4}g_{\mu\nu}-\frac{12\nu^2}{\ell^2}R_{\mu\nu} \tag{183}$$

while $\nabla_\alpha R^{\alpha\beta}=0$ using the contracted Bianchi identity and the fact that R is a constant.

For illustrative purposes, let us first consider the simple case where for some fixed $k\geq 2$,

$$f=f_k\equiv c_kR^k+b_k\;\mathrm{Tr}\left(R^k_{\mu\nu}\right)\,. \tag{184}$$

In this case,

$$P_{\mu\nu} = k\left[c_k g_{\mu\nu} R^{k-1} + b_k (R^{k-1}_{\alpha\beta})_{\mu\nu}\right] \tag{185}$$

while

$$\begin{aligned} Z^{\mu\nu\alpha\beta} &= \frac{k}{2}\left[c_k R^{k-1} + b_k A_{k-1}\right]\left[g^{\mu\alpha}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}\right] \\ &+ \frac{k}{4} b_k B_{k-1}\left[g^{\mu\alpha}R^{\beta\nu} - g^{\nu\alpha}R^{\beta\mu} + g^{\beta\nu}R^{\alpha\mu} - g^{\beta\mu}R^{\alpha\nu}\right] \end{aligned} \tag{186}$$

where we have used (179). On the other hand, the equation of motion (175) in this case reads

$$\begin{aligned} & k\left[c_k R^{k-1} R_{\mu\nu} + b_k (R^k_{\alpha\beta})_{\mu\nu}\right] - \tfrac{1}{2}\left[c_k R^k + b_k \mathrm{Tr}\left(R^k_{\alpha\beta}\right)\right] g_{\mu\nu} \\ = \; & \tfrac{1}{2} k b_k \left\{\nabla^\alpha \nabla_\nu [(R^{k-1}_{pq})_{\alpha\mu}] + \nabla^\alpha \nabla_\mu [(R^{k-1}_{pq})_{\alpha\nu}] - \Box[(R^{k-1}_{pq})_{\mu\nu}] - g_{\mu\nu}\nabla^\beta\nabla^\alpha[(R^{k-1}_{pq})_{\alpha\beta}]\right\} \\ & + k c_k \left[\nabla_\mu \nabla_\nu R^{k-1} - g_{\mu\nu}\Box R^{k-1}\right] \end{aligned} \tag{187}$$

which upon using (179) and (182)-(183) yields

$$\begin{aligned} 0 &= \left[k c_k R^{k-1} + k b_k B_k + k b_k B_{k-1}\frac{6\nu^2}{\ell^2}\right] R_{\mu\nu} \\ & -\tfrac{1}{2}\left[c_k R^k - 2k b_k A_k + b_k \mathrm{Tr}\left(R^k_{\alpha\beta}\right) - k b_k B_{k-1}\frac{24\nu^2}{\ell^4}\right] g_{\mu\nu} \,. \end{aligned} \tag{188}$$

By explicitly plugging in the metric and $R_{\mu\nu}$ for a locally warped AdS_3 metric, these equations in turn become two decoupled equations

$$0 = c_k R^{k-1} + b_k B_k + b_k B_{k-1}\frac{6\nu^2}{\ell^2}\,, \tag{189}$$

$$0 = c_k R^k - 2k b_k A_k + b_k \mathrm{Tr}\left(R^k_{\alpha\beta}\right) - k b_k B_{k-1}\frac{24\nu^2}{\ell^4}\,. \tag{190}$$

Let us now recall from (178), (179) and (185) that in this case we have

$$\begin{aligned} Z^{\mu\nu\alpha\beta} &= \frac{k}{2}\left[c_k R^{k-1} + b_k A_{k-1}\right]\left[g^{\mu\alpha}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}\right] \\ &+ \frac{k}{4} b_k B_{k-1}\left[g^{\mu\alpha}R^{\beta\nu} - g^{\nu\alpha}R^{\beta\mu} + g^{\beta\nu}R^{\alpha\mu} - g^{\beta\mu}R^{\alpha\nu}\right] \\ &\equiv A\left[g^{\mu\alpha}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}\right] + B\left[g^{\mu\alpha}R^{\beta\nu} - g^{\nu\alpha}R^{\beta\mu} + g^{\beta\nu}R^{\alpha\mu} - g^{\beta\mu}R^{\alpha\nu}\right] , \end{aligned} \tag{191}$$

where

$$A \equiv \frac{k}{2}\left[c_k R^{k-1} + b_k A_{k-1}\right] , \quad B \equiv \frac{k}{4} b_k B_{k-1} \, . \tag{192}$$

Their ratio is

$$\frac{B}{A} = \frac{1}{2}\frac{b_k B_{k-1}}{c_k R^{k-1} + b_k A_{k-1}} = \frac{1}{2}\frac{1}{-6\nu^2/\ell^2 + (A_{k-1} - B_k)/B_{k-1}} \tag{193}$$

where we have used one of the equations of motion (189). Using (181), we obtain

$$\frac{B}{A} = -\frac{\ell^2}{2(-3+5\nu^2)} \tag{194}$$

which is (6.57) as required by finiteness of charges.

One can straightforwardly extend this argument to a more general $f = f_{k;q_1,\ldots,q_n} \equiv c(k, q_1, \ldots q_n) R^k \times \mathrm{Tr}\left((R_{\mu_1\nu_1})^{q_1}\right) \times \mathrm{Tr}\left((R_{\mu_2\nu_2})^{q_2}\right) \times \ldots \times \mathrm{Tr}\left((R_{\mu_n\nu_n})^{q_n}\right)$ or even the most general action

$$f = \sum_{k,n} \sum_{q_1,\ldots,q_n} f_{k;q_1\ldots q_n} \, . \tag{195}$$

The upshot is that eventually similar arguments as above follow through and imply (6.57) as desired.

B. Corrections to charges for various higher curvature theories in 3d with $k=2$

B.1. Most general case without derivatives

We first deal with the most general Lagrangian involving no derivatives of the Ricci tensor and up to cubic order [151], namely

$$L = aR - \Lambda + bR^2 + c\, R_{\mu\nu}\, R^{\mu\nu} + m_1\, R^{\nu}_{\mu}\, R^{\rho}_{\nu}\, R^{\mu}_{\rho} + m_2\, R_{\mu\nu}\, R^{\mu\nu} R + m_3\, R^3\,. \tag{196}$$

The tensor Z is only proportional to ∂/∂Riemann. We have to consider the derivative of the Ricci scalar and Ricci tensor. After symmetrization, we get

$$\frac{\delta R}{\delta R_{abcd}} = \frac{1}{2}\left(g^{bd}g^{ac} - g^{ad}g^{bc}\right) \tag{197}$$

$$\frac{\delta (R_{\mu\nu}R^{\mu\nu})}{\delta R_{abcd}} = \frac{2}{4}\left(g^{bd}R^{ac} - g^{ad}R^{bc} - g^{bc}R^{ad} + g^{ac}R^{bd}\right)\,. \tag{198}$$

So,

$$Z^{abcd} = A\ \left(g^{bd}g^{ac} - g^{ad}g^{bc}\right) + B\ \left(g^{bd}R^{ac} - g^{ad}R^{bc} - g^{bc}R^{ad} + g^{ac}R^{bd}\right) \tag{199}$$

with

$$A = \left(\frac{a}{2} - \frac{6}{\ell}^2 b + \frac{54}{\ell^4}\, m_3 + \frac{3\nu^2(\nu^2-3)}{\ell^4}\, m_1 + \frac{3\,(3-2\nu^2+\nu^4)}{l^4}\, m_2\right)\,, \tag{200}$$

$$B = \left(\frac{c}{2} - \frac{3}{2\ell^2}\, m_2 - \frac{3(3+\nu^2)}{4\ell^2}\, m_1\right)\,. \tag{201}$$

We can make some checks against known quantities in New Massive Gravity for example, where $a = \frac{1}{16\pi}$, $b = \frac{-3}{16\pi 8\mu^2}$, $c = \frac{1}{16\pi\mu^2}$ and $m_1 = m_2 = m_3 = 0$ with $\mu^2 = \frac{3-20\nu^2}{2\ell^2}$. In this case, A and B reduce to

$$A = \frac{3-5\nu^2}{8\pi\,(3-20\nu^2)}, \qquad B = \frac{\ell^2}{16\pi\,(3-20\nu^2)}\,. \tag{202}$$

The charges obtained with these values by (6.55) and (6.56) are consistent with their expressions found by other techniques.

B.2. Lagrangian with $\Box R$

Take the Lagrangian with higher curvature terms

$$L_{\mathrm{HC}} = R - 2\Lambda + k g^{e_1 e_2} \nabla_{e_1} \nabla_{e_2} R, \tag{203}$$

then Z is given by

$$\begin{aligned} Z^{abcd} &= \frac{\partial L}{\partial R_{abcd}} + \nabla_{e_1} \nabla_{e_2} \frac{\partial L}{\partial \nabla_{e_1} \nabla_{e_2} R_{abcd}} \\ &= \frac{k}{2} \left(g^{ac} g^{bd} - g^{bc} g^{ad} \right) + k \nabla_{e_1} \nabla_{e_2} g^{e_1 e_2} \frac{1}{2} \left(g^{ac} g^{bd} - g^{bc} g^{ad} \right) \\ &= \frac{k}{2} \left(g^{ac} g^{bd} - g^{bc} g^{ad} \right) \end{aligned} \tag{204}$$

and for $s = 2$ and $s = 1$ we have:

$$Z^{abcd|e_1 e_2} = \frac{\partial L}{\partial \mathcal{R}_{abcd|e_1 e_2}} = \frac{k}{2} g^{e_1 e_2} \left(g^{ac} g^{bd} - g^{bc} g^{ad} \right) \tag{205}$$

$$Z^{abcd|e_1} = \frac{\partial L}{\partial \mathcal{R}_{abcd|e_1}} - \nabla_{e_2} Z^{abcd|e_1 e_2} = 0 \tag{206}$$

The corrections for Θ are then:

$$\Theta^{(1)}_{a_2 a_3} = \left[2 \left(Z^{ibcd|a} + Z^{abcd|i} \right) \delta g_{ij} \mathcal{R}^{j}_{bcd} - 2 Z^{ibcd|j} \delta g_{ij} \mathcal{R}^{a}_{bcd} - Z^{kbcd|a} \delta \mathcal{R}_{kbcd} \right] \varepsilon_{a a_2 a_3} = 0 \,, \tag{207}$$

$$\begin{aligned} \Theta^{(2)}_{a_2 a_3} \;&=\; \Big[2 \left(Z^{ibcd|e_1 a} + Z^{abcd|e_1 i} \right) \delta g_{ij} \mathcal{R}^{j}_{bcd|e_1} - 2 Z^{ibcd|e_1 j} \delta g_{ij} \mathcal{R}^{a}_{bcd|e_1} + Z^{kbcd|ia} \delta g_{ij} \mathcal{R}_{kbcd|}{}^{j} \\ &\quad - \frac{1}{2} Z^{kbcd|ij} \delta g_{ij} \mathcal{R}_{kbcd|}{}^{a} - Z^{kbcd|e_1 a} \delta \mathcal{R}_{kbcd|e_1} \Big] \varepsilon_{a a_2 a_3} \\ &= k \left[\left(2 \nabla^a R^{ij} + g^{ia} \nabla^j R - \frac{1}{2} g^{ij} \nabla^a R \right) \delta g_{ij} - \delta \nabla^a R + \delta g^{e_1 a} \nabla_{e_1} R + 2 \delta g^{bd} \nabla_c R_{bd} \right] \varepsilon_{a a_2 a_3} \\ &= k \left[-\frac{1}{2} g^{ij} \nabla^a R \; \delta g_{ij} - \delta \nabla^a R \right] \varepsilon_{a a_2 a_3} \end{aligned} \tag{208}$$

where we have used

$$g^{kc}g^{bd}\delta(\nabla_e R_{kbcd}) = \delta\left(\nabla_e(g^{kc}g^{bd}R_{kbcd})\right) - \delta g^{bd}\nabla_e(g^{kc}R_{kbcd}) - \delta g^{kc}\nabla_e(g^{bd}R_{kbcd})\,. \tag{209}$$

The corrections for Q are given by:

$$\begin{aligned} Q^{(1)}_{c_3} &= 0\,, & (210)\\ Q^{(2)}_{c_3} &= -2\,\xi_k\Big[Z^{klcd|e_1a}\mathcal{R}^{b}{}_{lcd|e_1} + Z^{alcd|e_1b}\mathcal{R}^{k}{}_{lcd|e_1} + Z^{alcd|e_1k}\mathcal{R}^{b}{}_{lcd|e_1} + \frac{1}{2}Z^{lmcd|ka}\mathcal{R}{}_{lmcd|}{}^{b}\Big]\varepsilon_{abc_3} & \\ &= -k\,\xi_k\,g^{ka}\,\nabla^b R\,\varepsilon_{abc_3}\,. & (211) \end{aligned}$$

If we assume warped AdS$_3$ is a solution of this theory, all the $\nabla^i R$ terms vanish and we are left with:

$$\Theta^{(1)}_{a_2a_3} = 0 \tag{212}$$

$$\Theta^{(2)}_{a_2a_3} = k\;4\nabla^a R^{ij}\,\delta g_{ij}\,\varepsilon_{aa_2a_3} \tag{213}$$

$$Q^{(1)}_{c_3} = 0 \tag{214}$$

$$Q^{(2)}_{c_3} = 0 \tag{215}$$

Computing $\nabla^a R^{ij}\,\delta g_{ij}$ explicitly gives $\Theta^{(2)} = 0$ as well.

B.3. Lagrangian with $R\Box R$

Take the Lagrangian with higher curvature terms

$$L_{\mathrm{HC}} = R - 2\Lambda + kRg^{e_1e_2}\nabla_{e_1}\nabla_{e_2}R\,. \tag{216}$$

Then Z is given by

$$Z^{abcd} = \frac{k}{2}\left(g^{ac}g^{bd} - g^{bc}g^{ad}\right)[1 + k\,\Box R] \tag{217}$$

and for $s = 2$ and $s = 1$ we have

$$Z^{abcd|e_1e_2} = \frac{k}{2}\left(g^{ac}g^{bd} - g^{bc}g^{ad}\right)g^{e_1e_2}R = R\cdot Z^{abcd|e_1e_2}_{\Box R} \tag{218}$$

$$Z^{abcd|e_1} = -\nabla_{e_2} Z^{abcd|e_1e_2} = -\frac{k}{2}\left(g^{ac}g^{bd} - g^{bc}g^{ad}\right)\nabla^{e_1}R \tag{219}$$

where the $\Box R$ subscript denotes quantities computed for the case treated in the previous section. The corrections for Θ are

$$\Theta^{(1)}_{a_2a_3} = k\, \nabla^a R(-4\delta g_{ij}R^{ij} + \delta R)\varepsilon_{aa_2a_3}\,, \tag{220}$$

$$\Theta^{(2)}_{a_2a_3} = R \cdot \Theta^{(2)}_{a_2a_3}\big|_{\Box R}\,. \tag{221}$$

The corrections for Q are given by

$$Q^{(1)}_{c_3} = 0\,, \tag{222}$$

$$Q^{(2)}_{c_3} = R \cdot Q^{(2)}_{c_3}\big|_{\Box R}\,. \tag{223}$$

We see that the same assumptions as in the the $\Box R$ case leads to the vanishing of all corrections.

B.4. Lagrangian with $\Box R \Box R$

Take the Lagrangian with higher curvature terms

$$L_{\text{HC}} = R - 2\Lambda + k(\Box R)^2\,. \tag{224}$$

Z is then given by

$$Z^{abcd} = \frac{k}{2}\left(g^{ac}g^{bd} - g^{bc}g^{ad}\right)[1 + k\,\Box(\Box R)] \tag{225}$$

and for $s = 2$ and $s = 1$ we have:

$$Z^{abcd|e_1e_2} = \frac{k}{2}\left(g^{ac}g^{bd} - g^{bc}g^{ad}\right) g^{e_1e_2}\Box R = \Box R \cdot Z^{abcd|e_1e_2}_{\Box R} \tag{226}$$

$$Z^{abcd|e_1} = -\nabla_{e_2} Z^{abcd|e_1e_2} = -\frac{k}{2}\left(g^{ac}g^{bd} - g^{bc}g^{ad}\right)\nabla^{e_1}\Box R \tag{227}$$

The corrections for Θ are

$$\begin{aligned}
\Theta^{(1)}_{a_2a_3} &= k\,\nabla^a\Box R(-4\delta g_{ij}R^{ij}+\delta R)\,\varepsilon_{aa_2a_3}\,, && (228)\\
\Theta^{(2)}_{a_2a_3} &= \Box R\cdot\Theta^{(2)}_{a_2a_3}\big|_{\Box R}\,. && (229)
\end{aligned}$$

The corrections for Q are given by:

$$\begin{aligned}
Q^{(1)}_{c_3} &= 0\,, && (230)\\
Q^{(2)}_{c_3} &= \Box R\cdot Q^{(2)}_{c_3}\big|_{\Box R}\,. && (231)
\end{aligned}$$

Again, in the warped AdS_3 case, the only potentially non-vanishing correction is $\Theta^{(1)}$; however, since $\Box R$ is zero it vanishes as well.

B.5. Lagrangian with $(\nabla^a R^{bc})^2$

Here we rework the case detailed in section 6.3 using the method in [117] directly instead of using our symmetry arguments. Recall the Lagrangian has the higher curvature term

$$L_{\text{HC}} = \nabla^a R^{bc}\nabla_a R_{bc} = g^{ap}g^{bq}g^{cr}\nabla_p R_{qr}\nabla_a R_{bc} \qquad (232)$$

and the Z-field and associated corrections are given by (6.77), (6.78) and (6.81) which we can rewrite in terms of $T^{ade} = R^{a\ d}_{\ b\ c}R^{bc;e}$ and $S^{abe} = R^a_{\ c}R^{bc;e}$ as

$$\begin{aligned}
\Theta^{(1)}_{bc} &= 2\delta g_{ij}\left[T^{jia}+S^{jia}-S^{jai}-S^{aij}-\frac{\nu^2-3}{l^2}R^{ij;a}\right]\varepsilon_{abc}\,, && (233)\\
Q^{(1)}_{c} &= -2\,\xi_k\left[S^{bka}+S^{kab}+S^{bak}\right]\varepsilon_{abc}\,. && (234)
\end{aligned}$$

Let us first compute the Θ correction. Since we know we are going to integrate over φ and we have to contract the correction to Θ with ∂_t, we need to compute the $[t\varphi]$ component of $\Theta^{(1)}$. We also know that δg only two non-zero components are $[rr]$ and $[\varphi\varphi]$, so the object we need to integrate is

$$\Theta^{(1)}_{t\varphi} = 2\left[\delta g_{rr}\,(T^{rrr}-S^{rrr})+\delta g_{\varphi\varphi}\,(T^{\varphi\varphi r}+S^{\varphi\varphi r}-S^{\varphi r\varphi}-S^{r\varphi\varphi})-\delta R_{bc}(\nabla^r+\nabla^\varphi)R^{bc}\right]\,. \qquad (235)$$

Explicit check shows that all the components of T and S entering $\Theta^{(1)}_{t\varphi}$ are zero, as is $R^{ij;a}\delta g_{ij}$. There is thus no Θ contribution to P_0.

The correction for Q we need to compute is also the φ component:

$$\begin{aligned} Q^{(1)}_{\varphi} &= -2\xi_k \left(S^{rkt} + S^{ktr} + S^{rtk} - S^{tkr} - S^{krt} - S^{trk}\right)\,, \qquad \text{i.e.} \\ &= -2\xi_k \left[R^r_{\ c}\nabla^t R^{kc} + R^k_{\ c}\nabla^r R^{tc} + R^r_{\ c}\nabla^k R^{tc} - R^t_{\ c}\nabla^r R^{kc} - R^k_{\ c}\nabla^t R^{rc} - R^t_{\ c}\nabla^k R^{rc}\right]. \end{aligned} \tag{236}$$

For $\xi = \partial_\varphi$, we get

$$(Q^{(1)}_{\partial_\varphi})_\varphi = -2\left[R^\varphi_{\ c}R^{tc;r} + R^r_{\ c}R^{tc;\varphi} - R^t_{\ c}R^{\varphi c;r} - R^t_{\ c}R^{rc;\varphi}\right] = 0 \tag{237}$$

because the terms cancel two by two explicitly. For $\xi = \partial_t$, we get

$$(Q^{(1)}_{\partial_t})_\varphi = -2\left[2R^r_{\ c}\nabla^t R^{tc} - 2R^t_{\ c}\nabla^t R^{rc}\right] = -4(S^{rtt} - S^{trt}) \tag{238}$$

$$= -72\, r\, \frac{\nu^2}{l^6}(\nu^2-1)^2. \tag{239}$$

Computations for both tensors hence match previous results (6.85)-(6.87).

C. Explicit expressions for the expansion of the flavoured chiral partition function

Comparing the partition function

$$Z(\tau, z) = Z(\tau) + \sum_{n=1}^{\infty} \frac{z^{2n}}{(2n)!}(2\pi i)^{2n} f_{2n}(\tau) \tag{240}$$

and its modular-transformed version

$$Z\left(A\tau, \frac{z}{c\tau+d}\right) = Z(\tau) + \frac{z^2}{2}\left(\frac{2\pi i}{c\tau+d}\right)^2 f_2(A\tau) + ... \tag{241}$$

$$= Z(\tau) + \frac{z^2}{2}\frac{2\pi ic}{c\tau+d}Z(\tau) + \frac{z^2}{2}(2\pi i)^2 f_2(\tau) + ... \tag{242}$$

one can derive a general expression for the $f_{2n}(\tau) \;\widehat{=}\; \mathrm{Tr}\left(J_0^{2n}\, q^{L_0-\kappa}\right)$ in terms of known modular forms Δ, E_2, E_4 and E_6. Setting

$$f_{2n}(\tau) = \Delta^{-\kappa}(\tau)P_{12\kappa+2n}(\tau) + \alpha_0\, E_2^n(\tau)Z(\tau) + \sum_{m=1}^{n-1}\alpha_m\, E_2^m(\tau)f_{2(n-m)} \tag{243}$$

where

$$P_{12\kappa+2n} = \sum_{\ell=1}^{\kappa(+1)} a_\ell\, E_4^{3\ell-n}(q)\, E_6^{2\kappa-2\ell+n}(q) \tag{244}$$

with the sum running from 1 to κ if $12\kappa+2n \equiv 2 \pmod{12}$, and to $\kappa+1$ if not[11] [140] allows to recover the proper modular transformation of each piece. In particular,

$$f_2(\tau) = \Delta^{-\kappa}(\tau)P_{12\kappa+2}(\tau) + \frac{1}{12}E_2(\tau)Z(\tau) \tag{245}$$

[11] The dimension of the space of all modular forms of weight w is $\dim M_w = \left\lfloor \frac{w}{12}\right\rfloor$ if $w \equiv 2 \pmod{12}$, and $\dim M_w = \left\lfloor \frac{w}{12}\right\rfloor + 1$ if $w \not\equiv 2 \pmod{12}$.

correctly reproduces the transformation

$$f_2(A\tau) = (c\tau + d)^2\, f_2(\tau) + \frac{c}{2\pi i}\,(c\tau + d)\, Z(\tau) \tag{246}$$

and

$$f_4(\tau) = \Delta^{-\kappa}(\tau) P_{12\kappa+4}(\tau) - \frac{1}{48} E_2^2(\tau) Z(\tau) + \frac{1}{2} E_2(\tau) f_2(\tau) \tag{247}$$

correctly reproduces the transformation

$$f_4(A\tau) = (c\tau + d)^4\, f_4(\tau) + (c\tau + d)^3\, \frac{3c}{\pi i}\, f_2(\tau) + (c\tau + d)^2\, \frac{3}{4} \left(\frac{c}{2\pi i}\right)^2\, Z(\tau)\,. \tag{248}$$

Similarly,

$$f_6(\tau) = \Delta^{-\kappa}(\tau) P_{12\kappa+6}(\tau) + \frac{5}{24^2} E_2^3(\tau) Z(\tau) - \frac{5}{16} E_2^2(\tau) f_2(\tau) + \frac{5}{4} E_2(\tau) f_4(\tau) \tag{249}$$

correctly reproduces the transformation

$$f_6(A\tau) = (c\tau + d)^6\, f_6(\tau) - \frac{15ic}{2\pi}\,(c\tau + d)^5\, f_4(\tau) - \frac{45c^2}{4\pi^2}\,(c\tau + d)^4\, f_2(\tau) + \frac{15ic^3}{8\pi^3}\,(c\tau + d)^3\, Z(\tau)\,, \tag{250}$$

etc.

Coefficients and positivity constraints

Starting with (243), we now detail the first three non-trivial orders in the $\kappa = 1$ to 6 cases. In the following, we denote the coefficients of $P_{12\kappa+2n}$ in $f_2(\tau)$ by a_i, those in $f_4(\tau)$ by b_i and those in $f_6(\tau)$ by c_i.

For $\kappa = 1$, the expansion of f_2 is

$$f_2(q) = q^{-1}\, \left(a + \frac{1}{12}\right) + \frac{1}{12}\,(r_1 - 24) + q\,\frac{1}{12}\,(r_2 - 24\, r_1) + O(q^2)\,. \tag{251}$$

Putting the $O(1/q)$ term to zero in f_2 gives us $a_1 = -\frac{1}{12}$ and the expansion in (7.117) becomes

$$\begin{aligned} f_2^{\kappa=1}(q) &= \frac{1}{12}(r_1-24)+2\,(16404-r_1)\;q+6\,(829944-r_1)\;q^2+8\,(30491394-r_1)\;q^3 \\ &+ 14\,(469759704-r_1)\;q^4+12\,(10062425124-r_1)\;q^5+24\,(70218701304-r_1)\;q^6 \\ &+ 16\,(1193358577344-r_1)\;q^7+O(q^8). \end{aligned} \tag{252}$$

Positivity of the coefficients in the expansion of f_2 puts both an upper and a lower bound on r_1:

$$24 \leqslant r_1 \leqslant 16404\,. \tag{253}$$

We can then put the $O(1)$ term to zero by picking $r_1 = 24$. This should be true also at all others orders, so we can look at how it constrains the next two orders. Replacing a_1 and r_1 into f_4 and then putting the $O(1/q)$ and $O(1)$ terms to zero yields

$$b_1 = -b_2 + \frac{1}{48}\,, \quad b_2 = \frac{5}{576}\,. \tag{254}$$

The expansion for f_4 is then

$$\begin{aligned} f_4^{\kappa=1}(q) &= 15120\,q+3265920\,q^2+197043840\,q^3 \\ &+ 6153960960\,q^4+126602028000\,q^5+O(q^6)\,. \end{aligned} \tag{255}$$

All the coefficients in this expansion are positive as they should be. f_6 can be put in a similar form using

$$c_1 = -c_2 - \frac{5}{576}\,, \quad c_2 = -\frac{25}{3456} \tag{256}$$

and we are left with an $O(q)$ expression with positive coefficients.

The $\kappa = 2$ case works the same way: we start with

$$\begin{aligned} f_2(q) &= \Delta^{-2}(q)\left(a_1\,E_4^2(q)\,E_6^3(q)+a_2\,E_4^5(q)\,E_6(q)\right)+\frac{1}{12}E_2(q)Z(q) \\ &= q^{-1}\left(80+1728\,a_2+\frac{r_1}{12}\right)+\left(-6-2\,r_1+\frac{r_2}{12}\right) \\ &\quad +q\,\left(-340215552\,a_2+16401\,r_1-2(5385528+r_2)\right)+O(q^2)\,. \end{aligned} \tag{257}$$

To get the expression in the second line we have used that the vacuum is uncharged, hence $a_1 = -a_2 - 1/12$, and the fact that in the $\kappa = 2$ case $Z(q) = J^2(q) + j_1\, J(q) + j_0$, or

$$Z(q) = q^{-2} + j_1\, q^{-1} + j_0 + 393768 + q\ (42987520 + 196884\, j_1) + \cdots \tag{258}$$

Putting the $O(1/q^2)$ and $O(1/q)$ terms to zero in f_2 gives us

$$a_1 = -a_2 - \frac{1}{12}\,, \qquad a_2 = -\frac{1}{1728}\left(80 + \frac{r_1}{12}\right) \tag{259}$$

and f_2 is given by

$$\begin{aligned} f_2^{\kappa=2}(q) \;&=\; \left(-6 - 2\,r_1 + \frac{r_2}{12}\right) + (4979664 + 32808\, r_1 - 2\, r_2)\ q \\ &+\; (6576701472 + 4979664\, r_1 - 6\, r_2)\ q^2 + 8\,(210656103912 + 30491394\, r_1 - r_2)\ q^3 \\ &+\; 14\,(13095149521032 + 469759704\, r_1 - r_2)\ q^4 \\ &+\; 12\,(958639027600872 + 10062425124\, r_1 - r_2)\ q^5 \\ &+\; 24\,(20319530222906532 + 70218701304\, r_1 - r_2)\ q^6 + O(q^7)\,. \end{aligned} \tag{260}$$

We can go one step further by taking

$$r_1 = -3 + \frac{r_2}{24} \tag{261}$$

as long as $r_2 \geqslant 72$ for positivity. Then f_2 becomes

$$\begin{aligned} f_2^{\kappa=2}(q) \;&=\; 1365\,(3576 + r_2)\ q + 207480\,(31626 + r_2)\ q^2 \\ &+\; 139230\,(12098808 + r_2)\ q^3 + O(q^4) \end{aligned} \tag{262}$$

so there is no way to go further by picking r_2. Replacing these values into f_4 and then playing the same trick yields

$$b_1 = -b_2 - b_3\,, \quad b_2 = -\frac{1}{1728}\left(-\frac{559}{16} + 3456 b_3 - \frac{r_2}{1152}\right)\,, \tag{263}$$

$$b_3 = -\frac{1}{2985984}\left(-7425 - \frac{5}{8}\,r_2\right) \tag{264}$$

and

$$\begin{aligned} f_4^{\kappa=2}(q) &= 630\,(2520 + r_2)\ q + 136080\,(22540 + r_2)\ q^2 \\ &+ 45360\,(21357000 + 181\,r_2)\ q^3 + 141120\,(864655200 + 1817\,r_2)\ q^4 + O(q^5)\,. \end{aligned} \tag{265}$$

No r_2 can be found so as to cancel the $O(q)$ term. Having replaced all of the coefficients above in f_6, we can then make the first three terms vanish using

$$c_1 = -c_2 - c_3\,, \quad c_2 = -\frac{1}{1728}\left(3456\,c_3 + \frac{5}{13824}(57528 + r_2)\right)\,, \tag{266}$$

$$c_3 = -\frac{1}{2985984}\left(\frac{25}{48}(20520 + \ r_2)\right)\,. \tag{267}$$

We then have

$$\begin{aligned} f_6^{\kappa=2}(q) &= 450\,(1800 + r_2)\ q \\ &+ 10800\,(212400 + 13\,r_2)\ q^2 \\ &+ 340200\,(2640600 + 31\,r_2)\ q^3 \\ &+ 28800\,(4568050800 + 13333\,r_2)\ q^4 + O(q^5)\,. \end{aligned} \tag{268}$$

The remaining $O(q)$ expression satisfies positivity and cannot be reduced any further.

The $\kappa = 3$ case works in a similar fashion. Writing down all the replacement rules for the coefficients would be a bit tedious, but in the end after fixing all the a_i, we find that

$$\begin{aligned} f_2^{\kappa=3}(q) &= -8 - 6\,r_1 - 2\,r_2 + \frac{r_3}{12} + (243931152 + 4979664\,r_1 + 32808\,r_2 - 2\,r_3)\ q \\ &+ 6\,(280874805216 + 1096116912\,r_1 + 829944\,r_2 - r_3)\ q^2 \\ &+ 8\,(192145590065616 + 210656103912\,r_1 + 30491394\,r_2 - r_3)\ q^3 + O(q^4)\,. \end{aligned} \tag{269}$$

We are allowed to reduce it a bit more by taking

$$r_1 = \frac{1}{6}\left(-8 - 2\,r_2 + \frac{r_3}{12}\right) \tag{270}$$

but the replacement for r_2 that would simplify this expression further is incompatible with the one for r_1. At this point we get

$$\begin{aligned} f_2^{\kappa=3}(q) &= -3640\,(-65190 + 447\,r_2 - 19\,r_3)\;q \\ &- 155610\,(-10773600 + 14056\,r_2 - 587\,r_3)\;q^2 + O(q^3)\,. \end{aligned} \tag{271}$$

Putting to zero the first four terms in f_4 by appropriate choices of b_i leaves us with

$$\begin{aligned} f_4^{\kappa=3}(q) &= -7560\,(-8400 + 70\,r_2 - 3\,r_3)\;q \\ &- 7560\,(-84937440 + 135240\,r_2 - 5653\,r_3)\;q^2 + O(q^3)\,. \end{aligned} \tag{272}$$

Again we hit a contradiction if we attempt to make the first term vanish by picking $r_2 = \frac{1}{70}\,(8400 + 3\,r_3)$. Going then to f_6, we get rid of the first four terms and are left with

$$\begin{aligned} f_6^{\kappa=3}(q) &= -900\,(-30000 + 300\,r_2 - 13\,r_3)\;q \\ &- 10800\,(-36715200 + 70800\,r_2 - 2963\,r_3)\;q^2 + O(q^3)\,. \end{aligned} \tag{273}$$

We are then left with an $O(q)$ expression that no consistent choice of r_2 can reduce.

It is actually not even necessary to ask for the numbers of states to be integer to obtain this result: asking for just positivity yields the same contradictions. Since this happens for every κ already at the level of f_2, without any need to go to higher and higher orders in the expansion of the partition function as one would have expected, it is worth taking a closer look at how exactly this works. Since the $\kappa = 1, 2$ cases have been made explicit enough, let us focus on the $\kappa = 3, 4, 5, 6$ cases where there is a priori enough parameters to make things less obvious.

We have already seen that for $\kappa = 3$, the expression for r_1 that simplifies f_2 is

$$r_1 = \frac{1}{6}\left(-8 - 2\,r_2 + \frac{r_3}{12}\right) \tag{274}$$

and from the subsequent expression for f_2 we know that we would need r_2 to be

$$r_2 = \frac{1}{447}\,(65190 + 19\,r_3) \tag{275}$$

to simplify it further. This would make r_1 become

$$\tilde{r_1} = \frac{1}{3576}\,(-178608 - \; r_3) \tag{276}$$

which is negative for any positive r_3.

For $\kappa = 4$, with all coefficients replaced f_2 is given by

$$\begin{aligned} f_2^{\kappa=4}(q) \;\; &= \;\; \left(-14 - 8\,r_1 - 6\,r_2 - 2\,r_3 + \frac{r_4}{12}\right) \\ &+ \;\; (6576635856 + 243931152\,r_1 + 4979664\,r_2 + 32808\,r_3 - 2\,r_4)\;q + O(q^2)\,. \end{aligned} \tag{277}$$

After replacing r_1 by

$$r_1 = \frac{1}{8}\left(-14 - 6\,r_2 - 2\,r_3 + \frac{r_4}{12}\right) \tag{278}$$

it becomes

$$\begin{aligned} f_2^{\kappa=4}(q) \;\; &= \;\; -\frac{4095}{2}\,(-3003544 + 86920\,r_2 + 29768\,r_3 - 1241\,r_4)\;q \\ &- \;\; 622440\,(-289799672 + 2020050\,r_2 + 676864\,r_3 - 28203\,r_4)\;q^2 + O(q^3)\,. \end{aligned} \tag{279}$$

At this point we know that there is no way to take

$$r_2 = -\frac{1}{86920}\,(-3003544 + 29768\,r_3 - 1241\,r_4) \tag{280}$$

in a way consistent with positivity constraints, but it is not obvious why. The expression for

r_1 would simply become

$$\tilde{r}_1 = \frac{1}{260760}\left(-6301644 + 1788\, r_3 - 76\, r_4\right) . \tag{281}$$

It does not seem impossible at first sight to find r_3 and r_4 such that this is positive. To get better intuition, let's take $r_4 = 1$: then r_3 needs to be bigger than 6301568/1788, so for example $r_3 = 3525$ would do. This is enough to make the term containing r_3 in r_2 largely dominant, so that it is negative. And taking r_4 larger will only make things worse, since r_3 must then be larger too with both parameters coming with opposite signs in r_1.

For $\kappa = 5$, after all the modular coefficients have been replaced, we have

$$\begin{aligned} f_2^{\kappa=5}(q) &= \left(-12 - 14\, r_1 - 8\, r_2 - 6\, r_3 - 2\, r_4 + \frac{r_5}{12}\right) \\ &+ \left(120749101488 + 6576635856\, r_1 + 243931152\, r_2 + 4979664\, r_3 + 32808\, r_4 - 2\, r_5\right) q \\ &+ O(q^2) . \end{aligned} \tag{282}$$

After taking

$$r_1 = \frac{1}{14}\left(-12 - 8\, r_2 - 6\, r_3 - 2\, r_4 + \frac{r_5}{12}\right) , \tag{283}$$

it becomes

$$\begin{aligned} f_2^{\kappa=5}(q) &= -1560\left(-73789734 + 2252658\, r_2 + 1803576\, r_3 + 602235\, r_4 - 25094\, r_5\right) q \\ &- 59280\,(-191405643336 + 1738798032\, r_2 + 1325309049\, r_3 + 441806580\, r_4 \\ &- 18408611\, r_5)\; q^2 + O(q^3) . \end{aligned} \tag{284}$$

The value of r_2 which would put the first term to zero is

$$r_2 = -\frac{1}{2252658}\left(-73789734 + 1803576\, r_3 + 602235\, r_4 - 25094\, r_5\right) , \tag{285}$$

with which r_1 would become

$$\tilde{r}_1 = \frac{1}{3003544}\left(-58795216 + 86920\, r_3 + 29768\, r_4 - 1241\, r_5\right) . \tag{286}$$

Again, the values of the remaining parameters giving a positive r_1 yield a negative r_2. If we want to see that more clearly, we can start by taking $r_5 = 0$ since it then makes it easier for $\tilde{r}_1$ to be positive. Even in this favourable case, with only two parameters to adjust, the minimal values that make $\tilde{r}_1$ positive are still too big to prevent r_2 to be negative. It seems the key fact is that most of the parameters come with opposite signs in both expressions.

In the $\kappa = 6$ case, after replacements we have

$$\begin{aligned} f_2^{\kappa=6}(q) &= \left(-24 - 12\,r_1 - 14\,r_2 - 8\,r_3 - 6\,r_4 - 2\,r_5 + \frac{r_6}{12}\right) \\ &+ 2\,(842624415648 + 60374550744\,r_1 + 3288317928\,r_2 + 121965576\,r_3 \\ &+ 2489832\,r_4 + 16404\,r_5 - r_6)\ q + O(q^2)\,. \end{aligned} \tag{287}$$

After replacing r_1 by $r_1 = \frac{1}{12}\left(-24 - 14\,r_2 - 8\,r_3 - 6\,r_4 - 2\,r_5 + \frac{r_6}{12}\right)$, we get

$$\begin{aligned} f_2^{\kappa=6}(q) &= -455\,(-3173078304 + 295158936\,r_2 + 176385648\,r_3 + 132680376\,r_4 \\ &+ 44230368\,r_5 - 1842935\,r_6)\ q + O(q^2)\,. \end{aligned} \tag{288}$$

Replacing r_2 by

$$r_2 = -\frac{1}{295158936}\,(-3173078304 + 176385648\,r_3 + 132680376\,r_4 + 44230368\,r_5 - 1842935\,r_6) \tag{289}$$

in r_1 yields

$$\tilde{r}_1 = \frac{1}{73789734}\,(-1073060640 + 2252658\,r_3 + 1803576\,r_4 + 602235\,r_5 - 25094\,r_6)\,. \tag{290}$$

In conclusion, we observe that each time, once r_1 has been chosen so as to simplify f_2, no further simplification can be achieved in a consistent way in either f_2, f_4 or f_6. Numerics allow to take this even further for higher values of κ (up to 20). This suggests that the partition function of a chiral CFT will remain of order q no matter what, which means that the spectrum needs to contain charged states of dimension at most $\kappa + 1$.

D. Writing f_2 in terms of the Hecke operator acting on J

The expression (7.89) relies on the statement that acting with $q\,\partial_q$ on a modular function $F(q)$ yields a modular form of weight 2. Indeed, expressing things in terms of τ rather than q, we start with the transformation of a modular form of weight m (7.37) and act on it with ∂_τ:

$$\begin{aligned}
\partial_\tau f\left(\frac{a\tau+b}{c\tau+d}\right) &= \partial_\tau\big((c\tau+d)^m f(\tau)\big) \qquad (291)\\
\Leftrightarrow \qquad f'\left(\frac{a\tau+b}{c\tau+d}\right)\frac{1}{(c\tau+d)^2} &= m\,c\,(c\tau+d)^{m-1}f(\tau)+(c\tau+d)^m f'(\tau)\\
\Leftrightarrow \qquad f'\left(\frac{a\tau+b}{c\tau+d}\right) &= m\,c\,(c\tau+d)^{m+1}f(\tau)+(c\tau+d)^{m+2}f'(\tau)\,.
\end{aligned}$$

For a modular function, $m=0$, yielding

$$f'\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2 f'(\tau) \qquad (292)$$

which is the transformation of a modular form of weight 2.

E. Full characters for affine Lie algebras

We consider a generic simple affine Lie algebra $\hat{\mathfrak{g}}_k$ and the character associated to the affine weight $\hat{\lambda}$ is given by

$$\chi_{\hat{\lambda}} = \frac{\sum_{w\in W} \epsilon(w)\Theta_{w(\hat{\lambda}+\hat{\rho})}}{\sum_{w\in W} \epsilon(w)\Theta_{w\hat{\rho}}} \tag{293}$$

where W is the Weyl group of $\mathfrak{g}$, $\hat{\rho}$ is the affine Weyl vector, g is the dual Coxeter number, and $\epsilon(w)$ is the signature of the element w of the Weyl group W. The theta functions evaluated at a point (ζ, τ, t) are given by [71]

$$\Theta_{\hat{\lambda}}(\zeta,\tau,t) = e^{-2\pi ikt} \sum_{\alpha^\vee\in Q^\vee} e^{-\pi i\left[2k(\alpha^\vee,\zeta)+2(\lambda,\zeta)-\tau k|\alpha^\vee+\lambda/k|^2\right]} \tag{294}$$

where the $\alpha^\vee$ are the coroots of the representation of highest weight λ in the original algebra being extended. $Q^\vee$ is the (finite) coroots lattice. The hatted quantities are all affine extensions expressed in terms of the corresponding finite algebra $\mathfrak{g}$ quantities; in particular the highest weight of $\hat{\mathfrak{g}}$ is $\hat{\lambda} = (\lambda; k; 0)$ and its Weyl vector is $\hat{\rho} = (\rho; g; 0)$.

We will start by computing the sums over the Weyl group in the expression for the character. To do the sum we note that the argument of the exponential in $\Theta_{w\hat{\lambda}}$ is proportional to

$$2k(\alpha^\vee,\zeta)+2(w\lambda,\zeta)-\tau k|\alpha^\vee+w\lambda/k|^2 = 2k(w^{-1}\alpha^\vee,w^{-1}\zeta)+2(\lambda,w^{-1}\zeta)-\tau k|w^{-1}\alpha^\vee+\lambda/k|^2 \tag{295}$$

which follows from the invariance of the scalar product under elements of the Weyl group. Our sums now take the form

$$\sum_{w\in W}\Theta_{w\hat{\lambda}} = e^{-2\pi ikt}\sum_{w\in W}\sum_{\alpha^\vee\in Q^\vee}\epsilon(w)e^{-\pi i\left[2k(w^{-1}\alpha^\vee,w^{-1}\zeta)+2(\lambda,w^{-1}\zeta)-\tau k|w^{-1}\alpha^\vee+\lambda/k|^2\right]} \tag{296}$$

$$= e^{-2\pi ikt}\sum_{w\in W}\sum_{\alpha^\vee\in Q^\vee}\epsilon(w)e^{-\pi i\left[2k(\alpha^\vee,w^{-1}\zeta)+2(\lambda,w^{-1}\zeta)-\tau k|\alpha^\vee+\lambda/k|^2\right]} \tag{297}$$

$$= e^{-2\pi ikt}\sum_{\alpha^\vee\in Q^\vee}\left(\sum_{w\in W}\epsilon(w)e^{-2\pi i(k\alpha^\vee+\lambda,w\zeta)}\right)e^{\pi i\tau k|\alpha^\vee+\lambda/k|^2} \tag{298}$$

where the middle line follows from the fact that for any fixed element of the Weyl group we have $Q^\vee = wQ^\vee$. At this point we will choose ζ proportional to ρ,

$$\zeta = -\frac{\tilde{z}}{2\pi i}\rho \tag{299}$$

for some complex variable $\tilde{z}$. Later we will see that $\tilde{z}$ is proportional to the z used throughout Chapter 7, but for now the expressions are simpler when written in terms of $\tilde{z}$. As we will see below, it is common to write the character as a function of τ and z only; we will then from here on set $t = 0$ in most expressions. In addition, recall the following identity from Lie algebras, (see e.g. [71]),

$$D_\rho = \sum_{w\in W} \epsilon(w) e^{w\rho} = \prod_{\alpha>0} \left(e^{\alpha/2} - e^{-\alpha/2}\right) \tag{300}$$

where the product is taken over the positive roots of the finite Lie algebra root lattice. Using this notation we recognize our sum over the Weyl group as simply $D_\rho(\tilde{z}(k\alpha^\vee + \lambda))$. Combining everything together back in the character the result is

$$\chi_{\hat{\lambda}}(\tilde{z}, \tau, 0) = \frac{\sum_{\alpha^\vee\in Q^\vee} \prod_{\alpha>0} \sinh\left(\frac{\tilde{z}}{2}(\alpha, (k+g)\alpha^\vee + \lambda + \rho)\right) e^{\pi i \tau (k+g)|\alpha^\vee + (\lambda+\rho)/(k+g)|^2}}{\sum_{\alpha^\vee\in Q^\vee} \prod_{\alpha>0} \sinh\left(\frac{\tilde{z}}{2}(\alpha, g\alpha^\vee + \rho)\right) e^{\pi i \tau g |\alpha^\vee + \rho/g|^2}}\,. \tag{301}$$

Expanding out as function of $\tilde{z}$ is now just a matter of expanding the hyperbolic sines that appear in our expression. Our job can be made a little simpler if we multiply and divide by $D_\rho(\tilde{z}\rho/2)$ so that the numerator and denominator both have the form (once again see e.g. [71])

$$\prod_{\alpha>0} \frac{\sinh\left(\tilde{z}(\alpha, k\alpha^\vee + \lambda + \rho)/2\right)}{\sinh\left(\tilde{z}(\alpha, \rho)/2\right)} = \left(\prod_{\alpha>0} \frac{(\alpha, k\alpha^\vee + \lambda + \rho)}{(\alpha, \rho)}\right)\left(1 + \frac{g\tilde{z}^2}{24}\left(|k\alpha^\vee + \lambda + \rho|^2 - |\rho|^2\right)\right) + O(\tilde{z}^4)\,. \tag{302}$$

Combining everything into the character gives

$$
\begin{aligned}
\chi_{\hat\lambda}(z,\tau,0) = {} & \frac{\sum_{\alpha^\vee}\prod_{\alpha>0}\frac{(\alpha,(k+g)\alpha^\vee+\lambda+\rho)}{(\alpha,\rho)}e^{\pi i\tau(k+g)|\alpha^\vee+(\lambda+\rho)/(k+g)|^2}}{\sum_{\alpha^\vee}\prod_{\alpha>0}\frac{(\alpha,g\alpha^\vee+\rho)}{(\alpha,\rho)}e^{\pi i\tau g|\alpha^\vee+\rho/g|^2}} \\
& + \frac{g\tilde z^2}{24}\Bigg[\frac{\sum_{\alpha^\vee}\left(\prod_{\alpha>0}\frac{(\alpha,(k+g)\alpha^\vee+\lambda+\rho)}{(\alpha,\rho)}\right)\left(|(k+g)\alpha^\vee+\lambda+\rho|^2-|\rho|^2\right)e^{\pi i\tau(k+g)|\alpha^\vee+(\lambda+\rho)/(k+g)|^2}}{\sum_{\alpha^\vee}\prod_{\alpha>0}\frac{(\alpha,g\alpha^\vee+\rho)}{(\alpha,\rho)}e^{\pi i\tau g|\alpha^\vee+\rho/g|^2}} \\
& - \frac{\sum_{\alpha^\vee}\sum_{\beta^\vee}\left(\prod_{\alpha>0}\frac{(\alpha,(k+g)\alpha^\vee+\lambda+\rho)}{(\alpha,\rho)}\right)\left(\prod_{\beta>0}\frac{(\beta,g\beta^\vee+\rho)}{(\beta,\rho)}\right)\left(|g\beta^\vee+\rho|^2-|\rho|^2\right)e^{\pi i\tau(k+g)|\alpha^\vee+(\lambda+\rho)/(k+g)|^2}e^{\pi i\tau g|\beta^\vee+\rho/g|^2}}{\left(\sum_{\alpha^\vee}\prod_{\alpha>0}\frac{(\alpha,g\alpha^\vee+\rho)}{(\alpha,\rho)}e^{\pi i\tau g|\alpha^\vee+\rho/g|^2}\right)^2}\Bigg] \\
& + O(\tilde z^4)\,.
\end{aligned}
\tag{303}
$$

This expression can be simplified significantly by noting that

$$
\begin{aligned}
&\left(|k\alpha^\vee+\lambda+\rho|^2-|\rho|^2\right)e^{\pi i\tau k\left(|\alpha^\vee+(\lambda+\rho)/k|^2-|(\lambda+\rho)/k|^2\right)} = \\
&\qquad\left(2k\,q\frac{d}{dq}+|\lambda+\rho|^2-|\rho|^2\right)e^{\pi i\tau k\left(|\alpha^\vee+(\lambda+\rho)/k|^2-|(\lambda+\rho)/k|^2\right)}
\end{aligned}
\tag{304}
$$

and that the expression in each of the denominators is related to a MacDonald identity [147]:

$$
\phi(q)^{\dim\mathfrak g} = \sum_{\alpha^\vee\in Q^\vee}\prod_{\alpha>0}\frac{(\alpha,g\alpha^\vee+\rho)}{(\alpha,\rho)}e^{\pi i\tau g\left(|\alpha^\vee+\rho/g|^2-|\rho/g|^2\right)}
\tag{305}
$$

where

$$
\phi(q) = \prod_{n=1}^{\infty}(1-q^n)\,.
\tag{306}
$$

The last thing we need is the relation between $\tilde z$ and z, which will follow from the transformation properties of the characters. Under a general modular transform we have[71]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\zeta, \tau, t) \mapsto \left(\frac{\zeta}{c\tau + d}, \frac{a\tau + b}{c\tau + d}, t + \frac{c|\zeta|^2}{c\tau + d} \right) . \tag{307}$$

If we now consider an S transformation then the characters transform as [71]

$$\chi_{\hat{\lambda}} \left(\frac{\zeta}{\tau}, -\frac{1}{\tau}, t + \frac{|\zeta|^2}{2\tau} \right) = \sum_{\hat{\mu}} S_{\hat{\lambda}\hat{\mu}} \chi_{\hat{\mu}}(\zeta, \tau, t) \tag{308}$$

where we have omitted the precise details of the S-matrix. As we can already see from the above expression, the transformation property of t closely resembles the anomalous transformation properties of the flavoured partition function. It is common in the literature to work with characters that are independent of t, usually by setting $t = 0$. Equivalently we can note that

$$\chi_{\hat{\lambda}}(\zeta, \tau, t) = e^{-2\pi i k t} \chi_{\hat{\lambda}}(\zeta, \tau, 0) \tag{309}$$

so that a t-independent character can be defined by

$$\bar{\chi}_{\hat{\lambda}}(\zeta, \tau) = e^{2\pi i k t} \chi_{\hat{\lambda}}(\zeta, \tau, t) \,. \tag{310}$$

Then using (308) we see that

$$\bar{\chi}_{\hat{\lambda}} \left(\frac{\zeta}{\tau}, -\frac{1}{\tau} \right) = e^{\pi i k |\zeta|^2/\tau} \sum_{\hat{\mu}} S_{\hat{\lambda}\hat{\mu}} \bar{\chi}_{\hat{\mu}}(\zeta, \tau) \tag{311}$$

where the factor of $e^{\pi i k |\zeta|^2/\tau}$ accounts for the anomalous transformation of the flavoured partition function. In order to use the technology developed for $u(1)$ and $s\hat{u}(2)$, we want to normalize everything so that $|\zeta|^2 = z^2$. We have chosen ζ proportional to ρ so that this choice of normalization demands

$$\zeta = \frac{z}{|\rho|} \rho = -\frac{\tilde{z}}{2\pi i} \rho \,. \tag{312}$$

The expansion of characters only have even powers of z so the relation we need can be nicely summarized as

$$\tilde{z}^2 = \frac{(2\pi i z)^2}{|\rho|^2} = \frac{12(2\pi i z)^2}{g \dim \mathfrak{g}} \tag{313}$$

where we have used the standard result $|\rho|^2 = \frac{g \dim \mathfrak{g}}{12}$.

Combining all of this yields a relation between the $O(z^2)$ and $O(z^0)$ pieces of the characters which is given by

$$q^{-m_\lambda}\chi_{\hat{\lambda}}(z,\tau,t) = \left[1 + \frac{(2\pi i z)^2}{2}\left(\frac{2(k+g)}{\dim \mathfrak{g}} q\frac{d}{dq} + \frac{(\lambda, \lambda+2\rho)}{\dim \mathfrak{g}} - \frac{k}{12}(1-E_2)\right) + O(z^4)\right] q^{-m_\lambda}\chi_{\hat{\lambda}}(0,\tau,t) \tag{314}$$

where

$$q^{-m_\lambda}\chi_{\hat{\lambda}}(0,\tau,t) = e^{-2\pi i k t}\phi(q)^{-\dim \mathfrak{g}} \sum_{\alpha^\vee \in Q^\vee} \prod_{\alpha>0} \frac{(\alpha, (k+g)\alpha^\vee + \lambda + \rho)}{(\alpha, \rho)} e^{\pi i \tau (k+g)\left(|\alpha^\vee + (\lambda+\rho)/(k+g)|^2 - |(\lambda+\rho)/(k+g)|^2\right)} \tag{315}$$

and we have defined, following [71], the combination

$$m_\lambda = \frac{|\lambda+\rho|^2}{2(k+g)} - \frac{|\rho|^2}{2g} \tag{316}$$

$$= h_\lambda - \frac{\hat{c}}{24}\,. \tag{317}$$

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