Spectral Theory and Eigenfunctions

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I would like to dedicate my thesis-in loving memory to my father, who instilled in me an appreciation for the beauty of mathematics beyond the scientific realm of practical utility. Many of the childhood memories I cherish most revolve around seeing my dad ardently pursue his passion for math and taking time to explain and transfer some of his knowledge and insight to yours truly; his youngest daughter. Even in his absence, he has always guided me forward.

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ABSTRACT

In this thesis, we review spectral theory of Laplace-Beltrami operartor on closed manifolds and manifolds with boundary, concentrating on the properties of eigenfunctions in the high energy limit, including asymptotic distribution of eigenfunctions, as well as results about nodal sets. We also discuss results about isospectrality. Finally, we discuss "Quantum Ergodicity" type results for metrics with jump-like discontinuities, including the so-called branching billiards.

ABRÉGÉ

Dans ce mémoire, nous passons en revue la théorie spectrale de l'opérateur de Laplace-Beltrami sur une variété fermée et sur une variété à bord, en nous concentrant sur les propriétés des fonctions propres dans la limite des hautes énergies, ce qui inclut la distribution asymptotique des fonctions propres de même que des résultats sur les ensembles nodaux. Nous discutons aussi de certains résultats d'isospectralité. Enfin, nous discutons de certains résultats de type "ergodicité quantique" pour des métriques avec discontinuités à saut, ce qui inclut les billards à ramifications.

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CHAPTER 1
Introduction

In the thesis, we give an introduction to the spectral theory of the Laplace operator on Romanian manifolds (possibly with boundary), and discuss various properties of the corresponding eigenvalues and eigenfunctions.

In Chapter 2, we define the Laplacian on (closed) Riemannian manifolds, state Weyl's law describing the asymptotic distribution of eigenvalues, and discuss applications to several important equations in Mathematical Physics (The Heat equation, the Wave equation and the Schrödinger equation).

In Chapter 3 we define basic boundary value problems (Dirichlet and Neumann), discuss domain monotonicity results for eigenvalues, and state basic results about the remainder in Weyl's law.

In Chapter 4 we discuss basic constructions of isospectral manifolds and domains, with an emphasis on Sunada's method.

In Chapters 5 and 6 we discuss concentration of high energy eigenfunctions. In Chapter 5 we consider their L^p norms and describe their quasi-symmetry properties; in Chapter 6 we state basic results about uniform distribution of squares of high energy eigenfunctions (in the weak sense) on manifolds with ergodic geodesic flow, the so-called *quantum ergodicity.* Next, we discuss the corresponding results for manifolds with boundary, and describe Hassell's counterexample to the *Quantum Unique Ergodicity* conjecture. In Chapter 7 we describe the corresponding results for metrics with jump discontinuities.

In Chapter 8 we describe basic results about nodal sets of Laplace eigenfunctions (including the Courant nodal domain theorem); state Yau's conjecture about the size of the nodal set; discuss convexity of nodal sets. We also describe several recent results about nodal sets with prescribed topology, and survey results about nodal topology of random linear combinations of Laplace eigenfunctions, and of random spherical harmonics.

CHAPTER 2
Spectral Theory of the Laplacian

In this chapter we shall partially follow the notes "Analysis on Manifolds via the Laplacian" by Y. Canzani, [Can].

2.1 Definitions

Laplace-Beltrami operator in Euclidean space $\mathbb{R}^n$ is define by the formula $\Delta f = \operatorname{div} \operatorname{grad} f$; in local coordinates, it is given by the formula

$$\Delta f = -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

On a compact n-dimensional Riemannian manifold M endowed with a Riemannian metric $g = (g_{ij})$, the Laplace operator Δ is given in local coordinates by

$$\Delta = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right),$$

where $\sqrt{g} = \sqrt{\det(g_{ij})}$. The operator Δ is self-adjoint. We denote by $L^2(M)$ the space of square-integrable functions on M (with respect to the volume form induced by the metric g). If the manifold is compact, the Spectral Theorem implies that there exists an orthonormal $\{\phi_j\}$ basis of $L^2(M)$ consisting of Laplace eigenfunctions ϕ_j with eigenvalues λ_j,

$$\Delta \phi_j + \lambda_j \phi_j = 0.$$

2.2 Weyl's law

We begin by summarizing the spectral theory of the Laplace-Beltrami operator on closed manifolds ($\partial M = \emptyset$). It is well-known ([Cha]) that on a compact connected smooth Riemannian manifold M without boundary, the

Laplacian Δ has a discrete spectrum

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

and there exists a basis of $L^2(M)$ consisting of eigenfunctions of Δ:

$$\Delta\phi_j + \lambda_j\phi_j = 0 \tag{2.1}$$

Let us denote by $N(\lambda)$ the number of eigenvalues λ_j satisfying $\lambda_j < \lambda$.

The following result is (various versions of which are due to Weyl, Polya, Avakumovic, Levitan, Hörmander and others) is known as *Weyl's law:*

Theorem 2.2.1. *On an n-dimensional closed manifold M*

$$N(\lambda) \asymp c_n \text{vol}(M)\lambda^{n/2}, \qquad \lambda \to \infty,$$

where c_n is a constant depending only on n.

Example: Weyl's law for the torus $\mathbf{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ A basis of (complex-valued) eigenfunctions is given $\phi_\xi(x) = e^{i(x,\xi)}$, where $\sigma \in \mathbb{Z}^n$ and $x \in \mathbf{T}^n$. Accordingly, the number of eigenvalues is given by

$$\#\{\xi \in \mathbb{Z}^n : |\xi|^2 < \lambda\},$$

i.e. the number of lattice points in the ball of radius $\sqrt{\lambda}$ centered at the origin. It is well-known that that number is asymptotic to the volume of the ball and grows like $C\lambda^{n/2}$ as $\lambda \to \infty$.

We remark that for a general $f \in L^2(\mathbf{T}^n)$, expansion into series of eigenfunctions of Δ coincides with the n-dimensional Fourier series.

2.3 Applications to PDE

The study of the Laplacian is motivated by the fact that it arises in some of the most fundamental partial differential equations (PDE) in Mathematical

Physics, including the heat equation, the wave equation and the Schrödinger equation.

2.3.1 The Heat equation.

The heat equation decribes the propagation of heat in a solid body. Denote the temperature at a point x in a domain $\Omega \in \mathbb{R}^n$ and at time t by $u(x,t)$. Denote by c the conductivity of the material comprising Ω. Then $u(x,t)$ satisfies

$$\Delta u(x,t) = -\frac{1}{c}\frac{\partial}{\partial t}u(x,t).$$

We first look for *product solutions*

$$u(x,t) = \phi(x)\alpha(t).$$

Substituting into the heat equation, we find that

$$\frac{\Delta\varphi(x)}{\varphi(x)} = -\frac{\alpha'(t)}{\alpha(t)}, \qquad x \in \Omega, t > 0.$$

This shows that there must exist a $\lambda \in \mathbb{R}$ such that

$$\lambda' = -\lambda\alpha$$

and

$$\triangle\varphi = \lambda\varphi$$

Therefore φ must be an eigenfunction of the Laplacian with eigenvalue λ and $\alpha(t) = e^{-\lambda t}$. Once you have these particular solutions $u_k = e^{-\lambda t}\varphi_k$ you use the superposition principle to write general solution

$$u(x,t) = \sum_k a_k e^{-\lambda_k t}\varphi_k(x)$$

where the coefficients a_k are chosen depending on the initial conditions. You could do the same with the wave equation or with the Schrödinger equation and you will also find particular solutions of the form $u_k(x,t) = \alpha_k(t)\varphi_k(x)$

with

$$\triangle\varphi_k = \lambda_k\varphi_k$$

and $\alpha_k(t) = e^{-\lambda_k t}$.

2.3.2 The Wave equation.

The wave equation describes the motion of the waves on the surface of a fluid.

$$\Delta u(x,t) = -\frac{1}{c}\frac{\partial}{\partial t^2}u(x,t)$$

where $\sqrt{c}$ is the speed of sound in the fluid, and $u(x,t)$ denotes the height of the surface of a fluid above the point x at time t.

The same equation arises when you study the vibrations of a drum (or another musical instrument). Let the domain Ω represent the vibrating surface of a drum, where $\partial\Omega$ would be fixed. The wave equation $\Delta u(x,t) = -\frac{\partial^2}{\partial^2 t}u(x,t)$ describes the vertical displacement of the vibrating surface above the point x at time t, and fixing $\partial\Omega$ corresponds to the so-called *Dirichlet* boundary condition $u(x,t) = 0$ for all points $x \in \partial\Omega$.

We shall solve the wave equation on a closed manifold M by expressing the solutions using the spectral theory of the Laplacian on M. We shall look for product solutions of the form

$$u(x,t) = \sum_j f_j(t)\Phi_j(x). \tag{2.2}$$

Let the initial data be given by

$$\begin{aligned} u(x,0) &= \sum_j a_j\Phi_j(x), \\ \frac{\partial}{\partial t}u(x,0) &= \sum_j b_j\Phi_j(x). \end{aligned} \tag{2.3}$$

Take $u(x,t) = \sum_j f_j(t)\Phi_j(x)$ into $\frac{\partial^2 u}{\partial t^2} = -\triangle_x u$. It follows that $\sum_j f_j''(t)\Phi_j(x) = \sum_j f_j(t)(-\triangle_x \Phi_j(x)) = \sum_j \lambda_j f_j(t)\Phi_j(x)$. Therefore $f_j''(t) = \lambda_j f(t)$, which we solve to get $f_j(t) = \alpha_j \cos(\sqrt{\lambda_j} t) + \beta_j \sin(\sqrt{\lambda_j} t)$.

The initial data is given by $f_j(0) = \alpha_j$ and $\frac{\partial f_j}{\partial t}(0) = \beta_j(\sqrt{\lambda_j})$

So from (2.3) we get $\sum_j \alpha_j \Phi_j(x) = \sum_j a_j \Phi_j(x)$, therefore $\lambda_j = a_j$ and $\sum_j \beta_j \sqrt{\lambda_j}\Phi_j(x) = \sum_j b_j \Phi_j(x)$, therefore $\beta_j = \frac{b_j}{\sqrt{\lambda_j}}$.

2.4 The Schrödinger equation

Consider a quantum particle on a manifold (or a domain) Ω; we assume that there are no external forces. Then the particle is described by a solution of $u(x,t)$ the Schrödinger equation, where $x \in \Omega$ and $t \in \mathbb{R}$:

$$\frac{h^2}{2m}\Delta u(x,t) = ih\frac{\partial}{\partial t}u(x,t).$$

Here h is Planck's constant and m is the mass of the free particle.

Normalizing u so that $\| u(\cdot\,,t) \|_{L^2(\Omega)} = 1$ one interprets $u(x,t)$ as probability density. That is, if $A \subset \Omega$ then the probability that your quantum particle be inside A at time t is given by $\int_A | u(x,t) |^2 dx$.

We shall look for particular solutions of the form $u_k(x,t) = \alpha_k(t)\varphi_k(x)$ with

$$\triangle \varphi_k = \lambda_k \varphi_k.$$

Separating variables, we find that $\alpha_k' = \frac{-ih\lambda_k}{2m}\alpha_k$, hence a general solution is given by the formula

$$u(x,t) = \sum_k e^{-ih\lambda_k/(2m)t}\varphi_k(x).$$

CHAPTER 3
Manifolds with boundary

3.1 Dirichlet and Neumann boundary conditions

We follow the presentation in [Cha, Ch. 1] Let M be a manifold with boundary ∂M. We assume that $\overline{M}$ is compact and connected. We also assume that ∂M is piecewise C^∞, and discuss several boundary value problems on M. The standard boundary value problems considered in the literature include *Dirichlet*, *Neumann* and *mixed* problems.

Dirichlet boundary value problem is defined as follows:

Definition 3.1.1. Find all real numbers λ such that there exists a nontrivial eigenfunction $\phi \in C^2(M) \cap C^0(\overline{M})$ satisfying

$$\Delta\phi + \lambda\phi = 0 \quad \textit{on } M; \qquad \phi = 0 \quad \textit{on } \partial M.$$

Neumann boundary value problem is defined as follows:

Definition 3.1.2. Find all real numbers λ such that there exists a nontrivial eigenfunction $\phi \in C^2(M) \cap C^1(\overline{M})$ satisfying

$$\Delta\phi + \lambda\phi = 0 \quad \textit{on } M; \qquad \partial_n\phi = 0 \quad \textit{on } \partial M,$$

where $\partial_n\phi$ denotes the normal derivative of ϕ.

Mixed boundary value problem is defined as follows: let N be an open submanifold of ∂M.

Definition 3.1.3. Find all real numbers λ such that there exists a nontrivial eigenfunction $\phi \in C^2(M) \cap C^1(M \cup N) \cap C^0(\overline{M})$ satisfying

$$\Delta\phi + \lambda\phi = 0 \ \ \textit{on } M; \ \ \phi = 0 \ \ \textit{on } \ \partial M \setminus N; \ \ \partial_n\phi = 0 \quad \textit{on } N.$$

Below we shall consider examples of solutions to the boundary value problems 3.1.1, 3.1.2 and 3.1.3.

3.1.1 Examples

Interval: Dirichlet boundary conditions

Consider an interval $[0, \ell]$ with Dirichlet boundary conditions:

$$-\varphi_{xx} = \lambda^2\varphi, \varphi(0) = \varphi(\ell) = 0$$

The eigenfunctions are

$$\varphi_k(x) = \sin(\frac{k\pi}{\ell}x) for k \geq 1,$$

with eigenvalues $\lambda_k = (\frac{k\pi}{\ell})^2$ for $k \geq 1$.

Interval: Neumann boundary conditions

Neuman boundary conditions: $\varphi'(0) = \varphi'(\ell) = 0$

It is easy to see solving ODE

The eigenfunctions are $\varphi_k(x) = \cos(\frac{k\pi}{\ell}x)$ for $k \geq 1$

We take real part, because imaginary part doesn't satisfy boundary condition

with eigenvalues $\lambda_k = (\frac{k\pi}{\ell})^2$ for $k \geq 0$

Parallelipiped

Consider the case of n-dimensional parallelipiped $\Omega = [0, l_1] \times ... \times [0, l_n]$ with Dirichlet boundary conditions. It is known that one can choose a basis of eigenfunctions on Ω consisting of product eigenfunctions $\phi_{1,k_1}(x_1) \cdot \phi_{2,k_2}(x_2) \times \ldots \times \phi_{n,k_n}(x_n)$, where ϕ_{j,k_j} satisfies $-\partial^2/(\partial x_j^2)\phi_{j,k_j} = \lambda_{j,k_j}\phi_{j,k_j}$ and each ϕ_{j,k_j} satisfies Dirichlet boundary conditions $\phi_{j,k_j}(0) = \phi_{j,k_j}(l_j) = 0$.

From section 3.1.1, we find that the eigenvalues of Ω are

$$\pi^2 \left(\frac{{k_1}^2}{{l_1}^2} + \ldots + \frac{{k_n}^2}{{l_n}^2} \right), \qquad k_1, \ldots, k_n \in \mathbb{Z}_+.$$

3.2 Domain monotonicity of Eigenvalues

We first formulate a result about domain monotonicty for Dirichlet data:

Theorem 3.2.1. *Let M be a compact connected manifold. We assume that ∂M is piecewise C^∞, and consider a boundary value problem on M.*

Let $\Omega_1 ... \Omega_k$ be piecewise C^∞ subsets of M with disjoint interiors. Assume that the boundaries $\partial\Omega_j$ always intersect ∂M transversally. For each $1 \le j \le k$ we have Dirichlet boundary conditions on Ω_j, where we require vanishing Dirichlet data on $\partial\Omega_j \cap M$, and leaving unchanged the original data on the intercection $(\partial M \cap \partial\Omega_j)$. Let λ_k denote the eigenvalues on M and let $0 \le \mu_1 \le \mu_2 \le ...$ denote the union (with possible repitions) of the spectra of all the Ω_j's. Then $\lambda_k \le \mu_k$.

Proof. Let $\varphi_i, 1 \le i \le k-1$ be eigenfunctions with the eigenvalues λ_i on M; we choose $\{\varphi_i\}$ to be orthonormal: $\langle\varphi_i, \varphi_j\rangle = \delta_{ij}$. Let $\psi_i, 1 \le i \le k$ be the eigenfunctions corresponding to the eigenvalues μ_i on Ω_i. Set $\psi_i \equiv 0$ on $M \backslash \Omega_i$. Then we can normalize ψ to be orthonormal in $L^2(M)$; in addition, they lie in the space of admissible functions on which the quadratic form $D[f, f]$ defined by Δ is well-defined.

We can construct

$$\Phi = \sum_1^k a_i \psi_i \in L^2(M)$$

such that Φ is orthogonal to all ψ_j-s:

$$\Phi = \sum_1^k a_i \psi_i \in L^2(M) \bigcap (span\{\varphi_1, ..., \varphi_{k-1}\})^\perp,$$

since we have $k-1$ linear equations $\sum_{i=1}^k a_i \langle\psi, \phi_j\rangle = 0$, and k unknowns $a_1, a_2, \ldots, a_k$.

Then we have

$$\lambda_k||\Phi||^2 \leq D[\Phi, \Phi] = \sum_{j=1}^{k} a_j^2 \leq \nu_k||\Phi||^2,$$

which implies the result. In the proof, we used admissibility of ψ_j-s, integration by parts, and the assumption that their supports are disjoint by construction.

□

We next formulate a result about domain monotonicty for Neumann data; the proof is similar, so we do not include it.

Theorem 3.2.2. *Let M and Ω_j be as in 3.2.1, and assume also that $\overline{M} = \cup_j \overline{\Omega_j}$. For each Ω_j we require vanishing Neumann data on $\partial\Omega_j \cap M$, and leave unchanged the original data on the intersection $(\partial M \cap \partial\Omega_j)$. Let λ_k denote the Neumann eigenvalues on M and let $0 \leq \mu_1 \leq \mu_2 \leq ...$ denote the union (with possible repitions) of the spectra of all the Ω_j's. Then $\lambda_k \geq \mu_k$.*

3.3 Remainder in Weyl's law

In this section we consider a bounded domain in $\Omega \subset \mathbb{R}^n$ with piecewise smooth boundary. Let $N(\lambda)$ be the eigenvalue counting function for either Dirichlet or Neumann boundary value problem. In 1911, H. Weyl in [Weyl] proved an asymptotic formula (2.2.1) for domains, using monotonicity results for Dirichlet and Neumann eigenvalue problems discussed earlier.

The behaviour of the remainder in Weyl's law was open for many years, until V. Ivrii in [Iv2] established thew following two-term asymptotic formula:

$$N(\lambda) = \frac{\mathrm{vol}(\Omega)}{(4\pi)^{n/2}\Gamma(n/2+1)}\lambda^{n/2} \pm \frac{\mathrm{vol}(\partial\Omega)}{2^{n+1}\pi^{(n-1)/2}\Gamma((n+1)/2)}\lambda^{(n-1)/2} + R(\lambda), \tag{3.1}$$

where the sign + in front of the second term corresponds to Neumann boundary conditions, and the sign − to Dirichlet boundary conditions. We refer to [Iv2, SV, Vas1, Vas2, PS, EPT, Mel] for further details and proofs.

According to the Weyl conjecture, $R(\lambda) = o(\lambda^{(n-1)/2})$. Ivrii ([Iv2]) proved this conjecture under the *nonperiodicity* assumption: the measure of the set of periodic trajectories of the billiard flow is equal to zero. Ivrii conjectured that this condition holds for all Euclidean billiards.

Lower bounds for the remainder were considered in [EPT], bulding on previous work by Jakobson, Polterovich and Toth. Previous results for translation billiards were established by L. Hillairet in [Hil].

CHAPTER 4
Isospectral manifolds and domains

The exposition in this section closely follows the paper [Br] by R. Brooks. The study of isospectral manifolds and domains was motivated by the famous question of M. Kac "Can you hear the shape of a drum." The first example of isospectral and non-isometric manifolds was given by J. Milnor in [Mil].

4.1 The Sunada Method

A systematic way of constructing pairs of isospectral manifolds and domains was given by T. Sunada in [Sun]. His method was based on an interpretation of isospectrality in terms of finite groups. The simplicity and elegance of the Sunada method led to a period of many significant developments of the isospectral manifolds and related problems.

Definition 4.1.1. Let G be a finite group, and H_1, H_2 two subgroups of G. Then the triple (G, H_1, H_2) satisfies the *Sunada condition* if

$$\forall g \in G, \{[g] \cap H_1\} = \{[g] \cap H_2\}, \tag{4.1}$$

where $[g]$ denotes the conjugacy class of g in G.

Theorem 4.1.2. *Let (G, H_1, H_2) satisfy* (4.1)*, and let $\phi : \pi_1(M) \to G$ be a surjective homomorphism. If M^{H_1} and M^{H_2} are the coverings of M with $\pi_1(M^{H_i}) = \phi^{-1}(H_i)$, then M^{H_1} is isospectral to M^{H_2}.*

The proof uses the following result:

Lemma 4.1.3. *The Sunada condition* (4.1) *is equivalent to the following condition:*

$$ind_{H_1}^{G}(1_{H_1}) \text{ is } G-equivalent \text{ to } ind_{H_2}^{G}(1_{H_2}) \tag{4.2}$$

where "1_{H_i}" denotes the trivial representation of the group H_i, and "ind" denotes the induced representation.

Proof. We give a proof from the paper [Pes] by Hubert Pesce:

Let λ denote an eigenvalue of M^G, and E_λ the associated eigenspace. Then E_λ is a representation space of G, and the multiplicity of λ in $spec(M^{H_i})$ is just the dimension of the H_i-invariant subspace of E_λ. Writing this last as $[1_{H_i} : Res^G_{H_i}$, where Res denotes the restriction of the representation and $[V : W]$ denotes the multiplicity of the representation V in W, Frobenius reciprocity says that

$$[1_{H_i} : Res^G_{H_i}(E_\lambda)] = [ind^G_{H_i}(1_{H_i})] : E_\lambda$$

This last is independent of i, since, by the equivalence of (4.1) and (4.2), we have that $ind^G_{H_1}(1_{H_1})$ is G-equivalent to $ind^G_{H_2}(1_{H_2})$. □

C. Gordon and R. Wilson in [GW] constructed first examples of isospectral deformations of metrics. Very nice and explicit families of examples in the plane were constructed in [GWW] and [BCDS].

Related constructions were also studied in spectral graph theory, and in scattering theory.

CHAPTER 5
L^p norms of high energy eigenfunctions

In this chapter we review results about bounds on L^p norms of high energy eigenfunctions, their quasi-symmetry properties and related questions.

5.1 General L^p bounds

Given an orthonormal basis $\{\phi_j\}$ of $L^2(M)$ of Laplace Given an orthonormal basis $\{\phi_j\}$ of $L^2(M)$ of Laplace eigenfunctions, the spectral function of the Laplacian is given by the formula:

$$e(x, y, \lambda) = \sum_{\lambda_j < \lambda} \phi_j(x)\overline{\phi_j(y)}$$

Avakumovic and Levitan showed ([Av, Lev]; see also [Hor] for general elliptic operators) that

$$\sum_{\lambda_j < \lambda} |\phi_j(x)|^2 = e(x, x, \lambda) = \frac{1}{(2\pi)^2} |B^n| \lambda^{\frac{n}{2}} + R(\lambda, x), \tag{5.1}$$

where $R(\lambda, x) = O(\lambda^{\frac{n-1}{2}})$ uniformly in x, a result called the local Weyl law. On negatively curved manifolds, a logarithmic improvement was obtained by Bérard in [Ber].

The following result follows easily from (5.1):

Theorem 5.1.1. *Let (M, g) be a compact n-dimensional C^∞ Riemannian manifold. Then if ϕ, with $\triangle\phi + \lambda\phi = 0$, is an eigenfunction of the Laplacian,*

$$\frac{\|\phi_\lambda\|_\infty}{\|\phi_\lambda\|_2} \leq c_1 \lambda^{\frac{n-1}{4}}$$

For the proofs of the following basic theoems about L^p norms, we refer to [Sog, SeSo]. The key estimate is for the "critical" exponent $p_n = \frac{2(n+1)}{n-1}$; the other results are obtained by interpolating between $p = 2$ and $p = p_n$, and between $p = p_n$ and $p = \infty$.

Theorem 5.1.2. *Let $\triangle\phi + \lambda\phi = 0$ be an eigenfunction of Laplacian on a smooth compact n-dimensional manifold M. Let $p_n = \frac{2(n+1)}{n-1}$. Then for $p_n \leq p \leq +\infty$*

$$\|\phi_\lambda\|_p \leq \lambda^{\frac{n}{2}(\frac{1}{2}-\frac{1}{p})-\frac{1}{4}}$$

For $2 \leq p \leq p_n$

$$\|\phi_\lambda\|_p \leq \lambda^{\frac{n-1}{4}(\frac{1}{2}-\frac{1}{p})}$$

On negatively-curved manifolds, improvements were obtained by Hassell and Tacy.

For "generic" negatively-curved metrics g on Riemannian manifolds, Sarnak made the following L^∞ conjecture (consistent with the "random wave" conjectures about eigenfunction behavior):

Conjecture 5.1.3. *Let g be a* generic *negatively curved metric on a compact smooth manifold M, and let $\Delta_g\phi_\lambda + \lambda\phi_\lambda = 0$ be an orthonormal basis of $L^2(M)$. Then for every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that $\|\phi_\lambda\|_{L^\infty} \leq C_\epsilon\lambda^\epsilon$ as $\lambda \to \infty$*

We remark that this conjecture does not hold on ceratin *arithmetic* hyperbolic manifolds: Rudnick and Sarnak have constructed examples of eigenfunctions with L^p norms growing polynomially at Heegner points on certain arthmetic hyperbolic 3-manifolds. Accordingly, the assumption of genericity of a metric g seems necessary.

5.2 Quasi-symmetry properties of eigenfunctions

In view of the predictions of the random wave conjecture of quantum chaos it seems natural to investigate the relationship between positive and negative parts of real eigenfunctions on Riemannian manifolds. In particular, it seems natural to consider the ratio $\|\varphi_+\|_p / \|\varphi_-\|_p$ of the L^p norms of the positive part

φ_+ and the negative part φ_- of an eigenfunction φ. In this section we review results from [JN02].

It was proved earlier by Nadirashvili that $\|\varphi_+\|_\infty/\|\varphi_-\|_\infty$ is bounded away from zero and infinity by constants that depend only on the manifold (and do not depend on the eigenvalue); more information about the precise value of the constant was obtained by Kröger (see the references in [JN02]). Also, on closed manifolds eigenfunctions with $\lambda > 0$ are L^2-orthogonal to constant functions, hence $\int_M \phi_\lambda = 0$ and thus $\|\varphi_+\|_1 = \|\varphi_-\|_1$. In [JN02], the authors showed that for $1 < p < \infty$, there exist constant $0 < c_p < C_p < \infty$ depending only on p and the Riemannian metric, such that for any nonconstant eigenfunction $\phi = \phi_\lambda$,

$$c_p < \frac{||\phi_+||_p}{||\phi_-||_p} < C_p$$

One can show that zonal spherical harmonics provide an example of a sequence of eigenfunctions with $\|\varphi_+\|_\infty/\|\varphi_-\|_\infty > C > 1$.

Another interesting questions concerns the *volume* of the domains $M_+ = \{x \in M : \varphi(x) > 0\}$ and $M_- = \{x \in M : \varphi(x) < 0\}$. Nadirashvili and Donnelly-Fefferman showed that for real-analytic M, the ratio $\frac{Vol(M_+)}{Vol(M_-)}$ is bounded away from zero and infinity by constants depending only on the Riemannian metric g on M. Nazarov, Polterovich and Sodin obtained further improvements on surfaces, [NPS]. Blum, Gnutzman and Smilanky in [BGS] obtained a random wave theory prediction for the rate of decay of the variance $(Vol(M_+) - Vol(M_-))^2$ as $\lambda \to \infty$ and compared it with numerical results for ergodic planar billiards.

CHAPTER 6
Quantum Ergodicity

In this chapter we discuss various results related to *Quantum Ergodicity*, or uniform distribution of eigenfunctions.

6.1 Closed manifolds

Let M be a compact, connected smooth manifold. Let $\Delta\phi_j + \lambda_j\phi_j = 0$ denote the spectrum of the Laplacian on M. Below we shall discuss limits of eigenfunctions. To an eigenfunction ϕ_j with $||\phi_j||_2 = 1$ one can associate a measure $d\mu_j = |\phi_j(x)|^2 d\text{vol}(x)$ on M with the density; the corresponding measure in phase space is a distribution $d\omega_j$ on the unit cosphere bundle S^*M (defined below), projecting to $d\mu_j$ on M. It is defined as follows: to a smooth function $a \in C^\infty(S^*M)$ one associates a *quantization* $Op(a)$ that is a pseudo-differential operator of order 0 with principal symbol a. Then $d\omega_j$, sometimes also called a *Wigner measure* is defined by

$$\langle a, d\omega_j\rangle := \int_M (A\phi_j)(x)\overline{\phi_j(x)} d\text{vol}(x) = \langle A\phi_j, \phi_j\rangle.$$

By compactness, sequences of of measures $d\mu_j$ and $d\omega_j$ will have limit points as $\lambda_j \to \infty$; a natural question is to classify those limit points.

The classification depends on the properties of the geodesic flow G^t on S^*M: if G^t is *completely integrable*, then $d\omega_j$ concentrate on the set of invariant tori (Liouville tori) that satisfy the so-called *quantization condition*; see e.g. [CdV77].

Various versions of the the following *Quantum Ergodicity* theorem were established in [CdV85, HMR, Shn74, Shn93, Zel87, Zel96]

Theorem 6.1.1. *If G^t is ergodic, then it was shown in [CdV85, HMR, Shn74, Shn93, Zel87, Zel96] that*

$$\lim_{\lambda\to\infty}\frac{1}{N(\lambda)}\sum_{\lambda_j\le\lambda}\left|\langle A\phi_j,\phi_j\rangle-\int_{S^*M}\sigma_A d\omega\right|^2=0, \tag{6.1}$$

where $d\omega$ denotes the Liouville measure on S^*M.

It follows that for a subsequence ϕ_{j_k} of the full desnity, $d\omega_j \to d\omega$, and projecting on M we find that $d\mu_j \to d\text{vol}$, e.g. almost all high energy eigenfunctions become equidistributed on the manifold and in phase space.

Rudnick and Sarnak conjectured that on negatively curved manifolds, the conclusion of Theorem 6.1.1 holds without averaging; this is sometimes called quantum unique ergodicity (QUE). This conjecture has been proved for some arithmetic hyperbolic manifolds by Lindenstrauss [Lin], with further progress by Soundararajan and Holowinsky.

6.1.1 Example of QE

Let $M = S^1$. Choose a basis of eigenfunctions consisting of

$$\{1,\sqrt{2}\sin(nx),\sqrt{2}\cos(nx),n\ge 1\}.$$

Let $f(x)$ be a continuous function on S^1 that describes our observable. Then

$$\int_0^{2\pi} f(x)\cdot 2\sin^2(nx)dx=\int_0^{2\pi} f(x)dx-\int_0^{2\pi} f(x)\cos(2nx)dx.$$

The second integral converges to 0 as $n\to\infty$ by Riemann-Lebesgue lemma, establishing the result in that case; the computation for $\cos(nx)$ is completely analogous and will not be presented here.

6.2 QUE for arithmetic hyperbolic surfaces

In this section, we give an exposition of results in [Lin]. Rudnick and Sarnak formulated the following *Quantum Unique Ergodicity* conjecture:

Conjecture 6.2.1. *Let M be a compact Riemannian manifold with negative sectional curvatures, and let the measures $d\mu_j = |\phi_j(x)|^2 dx$ and $d\omega_j$ be defined as before. Then the Liouville measure $d\omega$ is the unique possible limit of $d\omega_j$-s, and $d\mathrm{vol}(x)$ is the only possible limit of $d\mu_j$-s.*

In other words, the conclusion of Theorem 6.1.1 holds *without averaging* on such manifolds.

This conjecture was proved for compact arithmetic hyperbolic surfaces $\Gamma\backslash\mathbb{H}^2$ by Elon Lindenstrauss in [Lin]. Recall that a surface is called *arithmetic* if the commensurator of Γ in dense in $\mathrm{SL}(2,\mathbb{R})$.[1] Elements of the commensurator give rise to the so-called *Hecke operators*, which correspond to averages of isometries of finite degree covers of $\Gamma\backslash\mathbb{H}^2$. Those operators are self-adjoint, and commute with hyperbolic Laplacian and with each other. For arithmetic surfaces, there exist infinitely many such operators. Lindenstrauss considered joint eigenfunctions of the hyperbolic laplacian and those Hecke operators. Except for some harmless obvious multiplicities, the spectrum of Laplacian is generally expected to be simple, in which case establishing results for Hecke eigenfunctions gives no great restrictions.

Theorem 6.2.2. *Let $M = \Gamma\backslash\mathbb{H}^2$ be a compact arithmetic hyperbolic surface. Then 6.2.1 holds for joint eigenfunctions of the hyperbolic Laplacian and Hecke operators on M.*

Previously, T. Watson proved QUE for such functions by assuming the Generalized Riemann Hypothesis (GRH). Under that strong assumption, he also obtained an optimal rate of convergence.

[1] An element g belongs to the commensurator $Com(\Gamma)$ of Γ if $\Gamma \cap g\Gamma g^{-1}$ has finite index in both groups.

For noncompact arithmetic hyperbolic surfaces, QUE for the continuous spectrum (for $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$) was established by Luo, Sarnak and Jakobson. For square-integrable eigenfunctions ϕ_j, Lindenstrauss showed that any limit of $d\omega_j$-s must be a multiple of the Liouville measure $d\omega$; projecting, one obtains that any limit of $d\mu_j$-s must be equal to $c \cdot dxdy/y^2$ (where x, y denote the usual coordinates in the upper-half plane model of $\mathbb{H}$). It followed from later work by Soundararajan [Sou] that $c = 1$; in other words, no mass "escapes to infinity." QUE for *holomorphic* eigenfunctions was established by Holowinsky and Soundararajan in [HS].

6.3 Manifolds with boundary

Quantum Ergodicity theorem for manifolds with boundary was established in [GL, ZZ]. Quantum Ergodicity for eigenfunction restrictions has been considered in [HZ, TZ1, TZ2, DZ, CTZ]. A natural question is whether one can establish QUE type results for manifolds with boundary. It has long been conjectured in the physics literature that the Bunimovich stadium would serve as a counterexample to a QUE type result.

6.3.1 Stadium Domains that are not QUE

Bunimovich stadium is a region bounded by two parallel sides of a rectangle and two semicircles whose diameters are formed by the other two sides of the rectangle; in particular, this billiard is *convex* but not *strictly convex*. It is well-known from the work of Laztukin that for any strictly convex domain with sufficiently smooth boundary. However, despite being ergodic, the billiard flow for the Bunimovich stadium is not as strongly ergodic as (say) for the Sinai billiard (a square with the circle removed). In particular, there exists a one-parameter family of "bouncing ball" orbits (trajectories orthogonal to the two parallel sides). The flow-invariant measure supported on that family of orbits is a natural candidate for a limit measure for an "exceptional" sequence of eigenfunctions, and it has been conjectured by physicists that such exceptional sequences would exist on the Bunimovich stadium.

A. Hassell ([Ha]) has shown that on the Bunimovich stadium billiard, there exist exceptional sequences of eigenfunctions concentrating on the *bouncing ball* orbits, so the analogue of the QUE conjecture does not hold for all billiards.

Theorem 6.3.1. *Let S_t be the Bunimovich stadium with rectangle with sides* 1 *and t. For almost every value of $t \in [1, 2]$ the Dirichlet Laplacian on the stadium S_t is non-QUE.*

The proof starts by constructing "quasi-modes" (linear combinations of eigenfunctions) that concentrate on bouncing ball orbits. The problem is to prove that certain the number of Laplace eigenvalues in certain energy intervals that support those quasi-modes is uniformly bounded above (which implies that some of the corresponding eigenfunctions cannot become uniformly distributed). This was proved by Hassell for almost every value of t.

The results of Hassell were generalized in [HM].

Quantum ergodicity results for eigenfunction rstrictions to ∂M, and more generally to hypersurfaces inside M, were studied by Hassell, Zelditch, Toth, Dyatlov, Zworki and Christianson. We refer the reader to [CTZ] references therein for formulation and proofs of those results.

CHAPTER 7
QE for metrics with jump discontinuities

The aim of this chapter is to discuss a version of Quantum Ergodicity theorem for quantum systems with discontinuities. We consider metrics that are allowed to have jump discontinuities along hypersurfaces. Such metrics model wave propagation in layered materials, e.g. air-water, different layers in Earth's crust, composite materials, semiconductors, liquids that do not mix etc. Waves propagating in such materials experience both *reflection* and *refraction* when passing through the interface between different materials: part of the wave gets reflected back, and another part is refracted (according to Snell's law) into another material. For that reason, such materials are often called *ray-splitting* or *branching*. Ray splitting not only occurs in the quantum systems we consider but also happens in situations described by systems of partial differential equations and higher order equations. Moreover, ray-splitting occurs in a natural way in quantum graphs. The semiclassical theory for such systems was studied in [JSS], where Quantum Ergodicity theorem for eigenfunctions was proved; here we give a quick review of selected results in that paper.

Ray-splitting billiards have been studied extensively in the Physics literature, see e.g. [BYNK, BAGOP1, BAGOP2, BKS, COA, KKB, TS1, TS2] and references therein. The emphasis has been on spectral statistics, trace formulae, eigenfunction localization ("scarring"), and the behaviour of periodic orbits. In the mathematical literature, the emphasis has been on the propagation of singularities [Iv1] and spectral asymptotics [Iv2, Saf2].

The simplest example of a ray-splitting billiard involves two isotropic materials touching at a hypersurface. The metric in each layer is then given by $g_i = n_i(x)^2 g_{eucl}$, where $n_i(x)$ is the refraction index in layer i and g_{eucl} denotes the Euclidean metric. Wave propagation in these media is described by the wave equation

$$(\partial^2/\partial t^2 + \Delta)\phi(x,t) = 0$$

where Δ denotes is the Laplacian for the corresponding metric, and the solutions satisfy transmissive boundary conditions ([JSS, §2]). The high energy limit of such a systems shows properties that do not remind of classical mechanics: because of branching, there is no classical flow on phase space that describes the high energy limit. Moreover, the naive generalization of Egorovs theorem fails in this situation. In [JSS] it was shown that after forming an average over the eigenstates the quantum dynamics relates to a certain probabilistic dynamics that takes into account the different branches of geodesics emerging in this way. Our main result establishes a quantum ergodicity theorem in the case where this classical dynamics is ergodic. The proof relies on a precise symbolic calculus for Fourier integral operators associated with canonical transformations ([JSS, §4]) and on a local Weyl law for such operators [JSS, Thm 6.2]. A local parametrix for the wave kernel was constructed in [JSS]. It consisted of a sum of such Fourier integral operators ([JSS, §5.1]) and apply to them the above results. The usual proof of quantum ergodicity is based on the consideration of the positive operator obtained by squaring the average of the time-evolution of a pseudodifferential operator. Egorovs theorem plays an important role in this construction. Since it does not hold for ray-splitting billiards, the standard proof cannot cannot be modified directly. Instead in [JSS] the authors used local Weyl law for an operator that is not necessarily positive but whose expectation value with respect to any eigenfunction is positive.

In standard situations where Quantum Ergodicity theorem was established, ergodic properties of classical dynamics have long been studied (in some cases, for more than 100 years). Ergodic properties of branching billiards are completely unknown. The first example of a class of ergodic branching billiards was constructed by Yves Colin de Verdiere. Due to abundance of inhomogeneous materials in nature, further understanding of ergodic properties of branching billiards seems a very interesting open problem in Dynamical Systems.

CHAPTER 8
Nodal sets

In this chapter we discuss some of the main results and conjectures about nodal sets of eigenfunctions of the Laplacian on manifolds (with or without boundary).

Definition 8.0.2. The nodal set $\mathcal{N}(f)$ of a function $f : M \to \mathbb{R}$ is the set $f^{-1}(0) = \{x \in M : f(x) = 0\}$.

For a smooth metric g on an n-dimensional manifold M, nodal set of an eigenfunction ϕ is an $(n-1)$-dimensional subset $N = \mathcal{N}(\phi)$ of M. Most results concern the *size* of N and the *topology* of N and its complement $M \setminus N$.

8.1 Yau's conjecture

One of the main problems about nodal sets is estimating their size. Lower bounds for the size of nodal sets of eigenfunctions on surfaces were obtained by Brüning in [Bru]. Shortly afterwards, S.T. Yau formulated his famous conjecture about the size of nodal sets:

Conjecture 8.1.1. *For a smooth metric g on a compact manifold M, there exist constants $0 < c < C < \infty$ such that $c\sqrt{\lambda_j} \leq vol(N_{\varphi_j}) \leq C\sqrt{\lambda_j}$ as $j \to \infty$*

Yau's conjecture has only been proven for real-analytic metrics g. This result was proved by Donelly and Fefferman in [DF1]. Upper bounds for the size of nodal sets in dimension two of size $C\lambda^{3/4}$ was proved in [DF2], but even in dimension two there is a large gap between upper and lower bounds. Related questions were also studied in [NPS].

Lower bounds in dimension three and higher for general smooth metrics were studied in [SZ1, SZ2, HezS, CM, Man, Stein]. The following theorem was proved in [CM].

Theorem 8.1.2. *Given a closed n-dimensional Riemannian manifold M of dimension $n \geq 3$, there exists C so that $H^{n-1}(u = 0) \geq C\lambda^{(3-n)/4}$*

Related questions for Dirichlet and Neumann eigenfunctions on manifolds with boundary were considered in [Ar]. The best lower bound is of order $\lambda^{(3-n)/4}$, so there is a very large gap between known lower and upper bounds on the size of nodal sets.

8.2 Courant's Nodal Domain Theorem

Definition 8.2.1. A *nodal domain* of an eigenfunction ϕ_λ is a connected component of the complement $M \setminus \mathcal{N}(\phi_\lambda)$ of the nodal set of ϕ_λ.

A fundamental result about nodal domains is *Courant's nodal domain theorem.*

Theorem 8.2.2. *Let M be a connected compact smooth Riemannian manifold without boundary, and let g be a Riemannian metric on M. Let $\Delta_g \phi_k + \lambda_k \phi_k = 0$ be the eigenfunctions of Δ_g that form a complete orthonormal basis of $L^2(M, \mathrm{dvol}_g)$; we let $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ denote the corresponding eigenvalues of of Δ_g. Then the number of nodal domains of ϕ_k is less or equal to k.*

Note that the constant eigenfunction with eiegnvalue 0 has one nodal domain.

Remark 8.2.3. Consider a sequence of eigenfunctions $\phi_n(x, y) = sin(nx - y)$ on of the Laplacian for the flat metric on $\mathbf{T}^2$. It is easy to show that each of them has exactly *two* nodal domains. This example shows that in general one cannot obtain a growing *lower* bound on the number of nodal domains of ϕ_k.

Proof. We give a proof from [Cha]. We call a connected manifold A with compact closure and nonempty piecewise C^∞ boundary a *normal domain.* We will outline a proofs in an easier cases when all the nodal domains of ϕ_k are normal domains. We shall argue by contradiction. Assume that ϕ_k has at least $k+1$ nodal domains; denote them by $G_1, ...G_k, G_{k+1}, \ldots$

For each $1 \leq j =\leq k$, define $\psi_j(x) = \phi_k(x)$ if $x \in G_j$, and $\psi_j(x) = 0$ is $x \in M \setminus G_j$. As in the proof of eigenvalue monotonicity theorems, one can construct a nontrivial linear combination $f = \sum_{j=1}^{k} a_j \psi_j$ satisfying $0 = (f, \phi_1) = ... = (f, \phi_{k-1})$. One verifies that $\psi_j \in D(M)$ for each j. Then Rayleigh's theorem, the max-min method, and the divergence theorem imply that

$$\lambda_k \leq D[f, f]/\|f\|^2 \leq \lambda_k.$$

Therefore, f is an eigenfunction of λ_k that vanishes identically on G_{k+1}. This contradicts the maximum principle for Laplace eigenfunctions, finishing the proof of the Courant's nodal domain theorem. □

Remark 8.2.4. It follows easily from 8.2.2 that ϕ_1 always constant sign and that λ_1 is always simple (on connected manifolds!) Also, ϕ_2 has precisely 2 nodal domains. Note that λ_2 can be multiple, e.g. it will have multiplicity $(n+1)$ for the round metric on S^n. Finally, if a normal domain Ω in M is a nodal of eigenfunction of some eigenvalue λ, then λ is the lowest eigenvalue for the eigenvalue problem of Ω with original boundary data on $\partial\Omega \cap \partial M$, and vanishing Dirichlet boundary data on $\partial\Omega \cap M$.

The following result was proved by Pleijel in [Ple].

Theorem 8.2.5. *Consider M with either the closed or Dirichlet eigenvalue problem. Assume that for any domain Ω in M we have the isoperimetric inequality*

$$\{\lambda^\star(\Omega)\}^{n/2} vol\Omega > (2\pi)^n/\omega_n.$$

Then, letting n_k denote the number of nodal domains of $\lambda_k(M)$, we have

$$\limsup(n_k/k) < 1.$$

Here, $\lambda^\star(\Omega)$ denotes lowest Dirichlet eigenvalue of Ω.

Thus, equality in Courant's theorem can be achieved for only a finite number of eigenvalues. Eigenfunctions that attain the upper bound in Courant's nodal domain theorem were investigated by Helffer, Hoffmann-Ostenhof and Terracini in [HHT] and other papers. The geometric interpretation of the discrepancy between the actual number of nodal domains and Courant's bound was given in the papers by Berkolaiko, Smilansky and Kuchment, see e.g. [BKuS] and references therein.

8.2.1 Convexity of nodal sets

Payne conjectured in [Pay] in a convex domain $\Omega \in \mathbb{R}^n$, the level sets of the lowest Dirichlet eigenfunction are convex. It was settled affirmatively by Brascamp and Lieb in [BrL], and reinvestigated, using more elementary arguments, by Caffarelli and Spruck in [CaSp].

8.3 Prescribing nodal sets of first eigenfunctions

Here we summarize some recent results from the papers [EnPS13, EnPS14, EHP, EPSS] by A. Enciso, D. Peralta-Salas, D. Hartley and S. Steinerberger.

The main result of the paper [EnPS14] (following some previous results in [EnPS13]) asserts that, given an n-manifold M and any closed hypersurface $\Sigma \subset M$, there is a metric g on M such that Σ is a connected component of the nodal set of the eigenfunction u_1. Moreover, if Σ separates, then one can show that the nodal set does not have any other connected components. More precisely, we have the following statement. It is assumed that the hypersurface Σ is connected, $n \geq 3$ and all objects are of class C^∞.

Theorem 8.3.1. *Let Σ be a closed orientable hypersurface of M. Then there exists a Riemannian metric on M such that Σ is a connected component of the nodal set ${u_1}^0$. If Σ separates, then the nodal set is exactly Σ.*

The following even stronger result was proved in [EPSS]:

Theorem 8.3.2. *Let M be a d-dimensional manifold ($d \geq 3$) endowed with a Riemannian metric g_0 and let Σ be a separating hypersurface. Then, given any $\delta > 0$, there is a metric g in M conformally equivalent to g_0 and with the same volume such that its first eigenvalue λ_1 is simple and the nodal set of its first eigenfunction u_1 is $\Phi(\Sigma)$, where Φ is a diffeomorphism of M whose distance to the identity in $C^0(M)$ is at most δ.*

It follows from the previous theorem that the following result holds:

Theorem 8.3.3. *Let M be a d-manifold endowed with a Riemannian metric g_0 with $d \geq 3$, and let N be a positive integer. Then there is a metric g on M, conformally equivalent to g_0 and with the same volume, such that its first nontrivial eigenfunction u_1 has at least N non-degenerate critical points.*

Thus, the number of critical points cannot be uniformly bounded above.

In [EHP] the authors prove the following result:

Theorem 8.3.4. *Let M be a compact 3-manifold, and let γ be a knot in M. Then there exists a Riemannian metric g on M such that for the first nontrivial eigenvalue of Δ_g on M, there exists a* complex-valued *eigenfunction u whose nodal set $u^{-1}(0)$ has a connected component given by γ.*

The authors also consider higher-dimensional versions of that result.

8.4 Topology of Nodal sets

In this section we survey some results about the *topology* of nodal sets of high energy eigenfunctions.

The following Theorem was proved by Eremenko, Jakobson and Nadirashvili in [EJN]:

Theorem 8.4.1. *Let $0 < m \leq n$, and let $n - m$ be even. For every set of m disjoint closed curves on the 2-sphere, whose union E is invariant with respect to the antipodal map, there exists an spherical harmonic (Laplace eigenfunction for the round metric on S^2) of large anough degree n whose zero set is equivalent to E.*

This seems to be the strongest known result about *high energy* eigenfunctions.

More is known about nodal sets of *random* spherical harmonics. Note that certain problems simplify when suitably randomized. For example, P. Berard has proved a version of Yau's conjecture about the *size* of nodal sets for *random linear combinations* of Laplace eigenfunctions. Fundamental results on the topology of nodal sets of random spherical harmonics were obtained by Nazarov and Sodin in [NS]. They studied the number $N(f_n)$ of nodal domains of a random spherical harmonic f_n of degree n (whose Laplace eigenvalue is of order n^2). They showed that as $n \to \infty$, the mean of $N(f_n)/n^2$ tends to a positive constant a, and the random variable $N(f_n)/n^2$ concentrates exponentially around a. Their study was motivated by a very intriguing conjecture of Bogomolny and Schmit about the number of nodal domains. Their results were generalized by Gayet, Welschinger, Sarnak, Wigman, Canzani ([SaWi, CSar]) and others. Below we review some of those generalizations.

8.4.1 Results in [SaWi]

Sarnak and Wigman in [SaWi] consider linear combinations of Laplace eigenfunctions on an n-dimensional compact Riemannian manifold, with eigenvalues t_j^2 lying in a certain small enough "energy window" $[\alpha T, T]$, where $T \to \infty$. They call those comibantions *monochromatic random waves,* and denote the collection of those functions by $E_{M,\alpha}(T)$.

They let $H_1(n-1) \cup \infty$ denote the one-point compactification of the descrete countable set of diffeomorphism classes of compact connected manifolds dimension $n-1$. Similarly, they let $B_1(n) \cup \infty$ be the one-point compactification of discrete countable set of diffeomorphism classes of n-dimensional compact connected manifolds with boundary, and $\mathcal{T} \cup \infty$ be the one-point compactification of the (discrete countable) set of connected rooted finite graphs (i.e. graphs together with a marked node, referred to as the "root"). Given a random function f as above, they denote its nodal set by $V(f)$, decomposed as a union $\cup_{c \in \mathcal{C}(f)} c$ of its connected components. They denote the disjoint union of nodal domains of f by $\cup_{\omega \in \Omega(f)} \omega$. Note that each $c \in C(f)$ and $\omega \in \Omega(f)$ clearly determine points in $H_1(n-1)$ and $B_1(n)$, which are denoted by $t(c)$ and $t(\omega)$ respectively.

On of the main results in [SaWi] asserts that as $T \to \infty$ and for typical f in $E_{M,\alpha}(T)$, the above measures converge ω-star to universal measures which depend only on n and α (and not on M). Let $H(n-1)$ consist of all elements of $H_1(n-1)$ which can be embedded in $\mathbb{R}^n$, and similarly let $B(n)$ consist of all elements of $B_1(n)$ which can be embedded in $\mathbb{R}^n$ and let $\mathcal{T}$ is the set of all finite rooted trees.

Theorem 8.4.2. *There are probability measures $\mu_{C,n,\alpha}, \mu_{\Omega,n,\alpha}, \mu_{X,n,\alpha}$ supported on $H(n-1) \cup \infty, B(n) \cup \infty, T(n) \cup \infty$ respectively, such that for any given $H \in H(n-1), B \in B(n)$ and $G \in T$ and $\varepsilon > 0$. Then $Prob\{f \in E_{M,\alpha}(T)\}$: $max(|\mu_{C(f)}(H) - \mu_{C,n,\alpha}(H)|, |\mu_{\Omega(f)}(B) - \mu_{\Omega,n,\alpha}(B)|, |\mu_{X(f)}(G) - \mu_{X,n,\alpha} G|) > \varepsilon\} \to 0$ as $T \to \infty$.*

These results were extended in [CSar].

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