SOME RESULTS CONCERNING INTUITIONISTIC LOGICAL CATEGORIES

by

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ABSTRACT

Intuitionistic logical categories are defined as a modification of the logical categories of Volger [2] and [3], intended to represent intuitionistic first-order theories. An intuitionistic prelogical category contains certain morphisms to be thought of as formulas of a theory, and operations analogous to propositional connectives and existential quantification are defined on these morphisms. An intuitionistic logical category has, in addition, an operation analogous to universal quantification. It is shown that the category $\underset{\sim}{S}$ of sets is intuitionistic logical, that $\underset{\sim}{S}$ can be made prelogical in a number of ways, and that any category of form $\underset{\sim}{S}^{\underset{\sim}{P}}$, where $\underset{\sim}{P}$ is partially ordered, is intuitionistic logical. A premodel of a prelogical category is a structure-preserving functor to $\underset{\sim}{S}$; a Kripke model of an intuitionistic logical category is a logic-preserving functor to $\underset{\sim}{S}^{\underset{\sim}{P}}$ for a certain partially ordered class $\underset{\sim}{P}$. A completeness theorem is proved for prelogical categories, which states that any "nontheorem" of the category can be separated from "truth" by an appropriate premodel.

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RÉSUMÉ

On définit les catégories "logiques intuitionistes" comme des modifications des catégories logiques de Volger (voir [2] et [3]), au but de généraliser la notion de "théorie intuitioniste du premier ordre". Une catégorie prélogique-intuitionniste contient certains morphismes qu'on peut considérer comme des formules bien-formées d'une théorie, et on définit pour ces morphismes des analogues aux connectives propositionnels, et à la quantification existentielle. Une catégorie logique-intuitioniste a, de plus, une opération pareille à la quantification universelle. On démontre que la catégorie Ens des ensembles est logique-intuitioniste, que Ens peut être prélogique de plusieurs façons, et puis que toute catégorie de la forme $\underline{\text{Ens}}^{P}$, où P est partiellement ordonné, est logique-intuitioniste. Un prémodèle d'une catégorie prélogique C est un foncteur à valeurs dans Ens qui préserve la structure de C; un modèle de Kripke d'une catégorie logique-intuitioniste est un tel foncteur à valeurs dans $\underline{\text{Ens}}^{P}$, avec un certain ensemble P partiellement ordonné. On démontre un théorème de complétitude pour les catégories prélogiques; ce théorème dit que pour toute formule, qui n'est pas un théorème de C, il existe un prémodèle qui la distinque de la "verité".

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CONTENTS

List of Symbols

The following notation is for category-theoretic items:

$|\underline{C}|$ and $Ob(\underline{C})$ will be used interchangeably to denote the class of objects of $\underline{C}$

$\underline{C}^{\underline{D}}$ category of functors from $\underline{D}$ to $\underline{C}$

$\underline{C}|_X$ category of objects over X in $\underline{C}$ category

$\underline{S}$ category of sets

$\underline{S}_o$ category of finite cardinals

$X \times Y$ product of X and Y

$Z \xrightarrow{<f,g>} X \times Y$ product map of $Z \xrightarrow{f} X$ and $Z \xrightarrow{g} Y$

$X + Y$ coproduct of X and Y

$X + Y \xrightarrow{\binom{f}{g}} Z$ coproduct map of $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$

$\Delta_X : X \to X \times X$ diagonal map of X

$\nabla_X : X + X \to X$ codiagonal map of X

id_X and X will be used interchangeably to denote the identity morphism of $X \in |\underline{C}|$

I terminal object

$!_X : X \to I$ unique map

$Pb(f,g)$ pullback of f and g

$X \cong Y$ X is isomorphic to Y

f^{-1} inverse of the isomorphism f

The symbols which follow are with respect to sets X and Y, a function $X \xrightarrow{f} Y$ which is not necessarily an isomorphism, and elements $x \in X$, $y \in Y$.

N set of natural numbers

sup X supremum of X

inf X infimum of X

$\partial^0(f)$ and dom f will be used interchangeably to denote the domain of f

$\partial^1(f)$ codomain of f

rge(f) range of f

$f^{-1}(y)$ set of $x \in X$ such that $f(x) = y$.

BACKGROUND AND INTRODUCTION

It has been known for some time that the set of formulas of a first-order theory T (more precisely, this set modulo the relation of "provable bi-implication") can be viewed as a lattice by interpreting the logical connectives "and" and "or" as the lattice meet and join respectively. When this is done, the axioms of the classical predicate calculus dictate that the lattice will be a Boolean algebra; the equivalence class of tautologies is the unit or greatest element and the equivalence class of contradictory statements is the zero. The logical connectives $\neg$ ("not") and $\Rightarrow$ ("implies") coincide with the Boolean complement and relative complement respectively. The existential and universal quantification correspond to infinite joins and meets of the form $\bigcup_{\xi} \beta(\xi)$ and $\bigcap_{\xi} \beta(\xi)$ respectively, where the notation $\beta(\xi)$ means that ξ is a variable occurring freely in the formula β. This Boolean algebra is denoted $U(T)$. The theorems or deducible propositions of the theory T form a filter of $U(T)$. For any formulas α, β, $\alpha \leq \beta$ iff $\alpha \Rightarrow \beta$ is a theorem.

Rasiowa and Sikorski in [1] show that many logical concepts and theorems can be expressed and proved in this algebraic formulation. In particular, the completeness theorem for first-order theories can be stated and proved in this way.

The R-S definition of a <u>realization</u> R of a theory T (what logicians usually call a <u>structure</u> for T) is as follows:

(i) R specifies a set J (the <u>universe</u>) to which variables may be mapped and a complete Boolean algebra A (the truth-value algebra).

(ii) R associates to each n-ary function symbol f in $\mathcal{L}(T)$ (the language of T) a function $f_R: J^n \to J$.

(iii) R associates to each m-ary predicate symbol ρ in $\mathcal{L}(T)$ a function $\rho_R: J^m \to A$.

In particular, it is clear that if A is the algebra 2, then this definition gives precisely the usual logical notion of a structure. R-S call this special case a <u>semantic realization</u>.

Notions of <u>satisfiability</u>, <u>validity</u> and <u>model</u> are defined, by interpreting logical connectives as the operations of A, in such a way that an R-S <u>semantic model</u> is precisely a classical model, (i.e. all axioms of T are "true") and a formula is R-S <u>valid</u> iff it is valid in the usual sense of "true in all models", where the variable symbols are considered as variables ranging over J.

An important special case is the <u>canonical realization</u> R^o determined by a homomorphism h: $U(T) \to A$, where A is any complete Boolean algebra. This is the realization whose universe is just the set T of terms of $U(T)$ itself, and the operation on formulas is "substitution", i.e. composition with h. The exact definition is as follows:

(i) The universe of R^o is T = {all terms of $U(T)$}.
The truth value algebra is A.

(ii) A function symbol $f \mapsto f_{R^o}$ defined by

$$f_{R^o}(\tau_1,\ldots,\tau_n) = f(\tau_1,\ldots,\tau_n) \in T.$$

(iii) A predicate symbol $\rho \mapsto \rho_{R^o}$ given by

$\rho_{R^o}(\tau_1,\ldots,\tau_m) = h(\|\rho(\tau_1,\ldots,\tau_m)\|) \in A$, where $\|\beta\|$ denotes the equivalence class in $U(T)$ of a formula β

R-S note that such a realization will be a model for T if h also preserves infinite joins and meets.

The completeness theorem, that a formula which is valid in all models must be a theorem, is then proved in R-S [1] by constructing, for every non-theorem α, a canonical semantic model in which α is false. A sketch of their argument follows:

A theory T is said to be <u>rich</u> iff for every formula $\exists x\, \beta(x)$ in $U(T)$ there is a term τ in $U(T)$ such that "$\exists x\, \beta(x) \Rightarrow \beta(\tau)$" is a theorem of T. Using quite a simple and equational argument, R-S prove that a rich theory can "inherit" models from its quotient algebras in a logically faithful way, i.e. that:
If T is rich and F a maximal filter of $U(T)$ (in particular the maximal filter containing a certain irrefutable α), then the natural homomorphism h: $U(T) \xrightarrow[\text{onto}]{} U(T)/F = 2$ preserves infinite operations and therefore determines a canonical semantic model for T (in which α is satisfiable). But it is shown in turn that every consistent theory can be extended conservatively to a rich theory by adding constants, and hence that every consistent theory has models of the required kind.

The notion of a first-order theory has been translated into categorical language in the following definition, suggested by F.W. Lawvere and modified by H. Volger in [2] and [3].

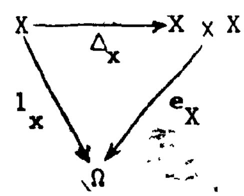
A category $\underline{T}$ is called an <u>elementary theory</u> iff :
(1) $\underline{T}$ has two distinguished objects V and Ω such that, for every X $X \in \text{Ob}(\underline{T}) \ni X \neq \Omega$, X can be specified as a product V^n for some finite n, and unless n = 0, $\underline{T}(\Omega, X) = \emptyset$.
$\underline{T}$ has all objects V^n and hence in particular a terminal object, denoted I.

(2) Ω is a Boolean algebra object, i.e. for every $X \in Ob(\underline{T})$, $\underline{T}(X,\Omega)$ is a Boolean algebra and for every $X \xrightarrow{f} Y$ in $\underline{T}$, $\underline{T}(f,\Omega)$ is a Boolean homomorphism. The greatest and least elements of $\underline{T}(X,\Omega)$ are denoted $1_x, 0_x$ respectively.

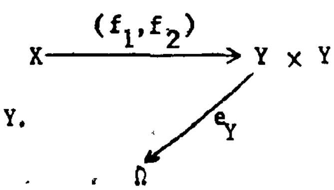
(3) For every $f:X \to Y$ in $\underline{T}$ there exists a functor $\exists_f[\ ]: \underline{T}(X,\Omega) \to \underline{T}(Y,\Omega)$ which is left adjoint to the "substitution" functor $\underline{T}(f,\Omega) : \underline{T}(Y,\Omega) \to \underline{T}(X,\Omega)$. $\exists_f[\ ]$ is called <u>existential quantification along f.</u>

(4) Two technical conditions (see [3], 1.4.2 and 1.4.3) are given regarding a sort of commutativity of $\exists_f[\ ]$ with certain pullbacks

(5) Define the <u>equality predicate</u> on each $X \in |\underline{T}|$ as $e_x = \exists_{\Delta_x}[1_x]$, where Δ_x as usual denotes the diagonal.



(a) For any f_1, f_2 such that $e_x < f_1, f_2 > = 1_x$, then $f_1 = f_2 : X \to Y$.



(b) e_Ω is $\Leftrightarrow$ (Boolean bi-implication.)

The motivation for condition (3) will be discussed below, but it should be clear that (1) and (2) are natural conditions if V is thought of as representing the set of variable symbols of the theory and Ω as "representing" the algebra of truth values in the following sense:

A model F for $\underline{T}$ will specify (or consist of):

A set V to which V is mapped ;

the set 2 to which Ω is mapped;

for every morphism $f: V^n \to V$ in $\underline{T}$, a function $f_F: \mathcal{V}^n \to \mathcal{V}$, which might be called a term;

for every morphism $g: V^n \to \Omega$ in $\underline{T}$, a function $g_F: \mathcal{V}^n \to 2$, i.e. precisely an n-ary relation or predicate on $\mathcal{V}$, giving rise to "formulas"; these form a Boolean algebra in the usual way.

In particular, morphisms $\rho: I \to \Omega$ in $\underline{T}$ are sent to maps: $1 \to 2$. i.e. these are the sentences of the theory, which contain no free variables and are therefore simply either "true" or "false".

Now in the category $\underline{S}$ of sets, logical quantification takes the following form:

(We shall use Volger's notation of "#" for the 1-1 corresponding between maps $X \xrightarrow{\varphi} 2$ and subsets $X' \xrightarrow{\varphi^\#} X$, which exists since 2 is a subobject classifier in $\underline{S}$. φ is called the characteristic map of $\varphi^\#$).

Consider a formula $\beta(v_1, \ldots, v_n, v)$ in the theory, i.e. corresponding to a map $\beta_F: \mathcal{V}^{n+1} \to 2$, which has free variables $v_1, \ldots, v_n, v$ ranging over $\mathcal{V}$.

Let p denote the projection $\mathcal{V}^n \times \mathcal{V} \xrightarrow{p} \mathcal{V}^n$ and let

$\exists_p[\ ]: \underline{S}(\mathcal{V}^{n+1}, 2) \to \underline{S}(\mathcal{V}^n, 2)$

be given by $(\exists_p[\varphi])^\# = p(\varphi^\#)$ [call this equation I]

i.e. $\exists_p[\ ]$ corresponds to the "direct image" map of the representing subsets. One can see that $\exists_p[\beta_F]$ is the formula "there exists $v \in \mathcal{V} \ni \beta(v_1, \ldots, v_n, v)$ holds". I.e. existential quantification of variables is performed by taking direct image under a projection. (For more general maps $\mathcal{V}^n \xrightarrow{f} \mathcal{V}^m$, $\exists_f[\varphi]$ is true for $y \in \mathcal{V}^m$ iff there exists a pre-image of y under f such that φ holds, so that there is still an "existential" meaning.)

Furthermore, let $\forall_p[\ ]: \underline{S}(V^{n+1},2) \to \underline{S}(V^n,2)$ be defined by

$$(\forall_p[\varphi])^{\#} = \{y \in V^n \mid p^{-1}(y) \subseteq \varphi^{\#}\} \qquad \text{[equation II]}$$

Again one can see that $\forall_p[\beta_F]$ is the formula "for all $v \in V$, $\beta(v_1,\ldots,v_m,v)$ holds", i.e. II describes universal quantification. And in general $\forall_f[\varphi]$ holds for y iff φ holds for all pre-images of y under f. Now for every map $X \xrightarrow{f} Y$ in $\underline{S}$, we have the usual substitution map $\underline{S}(f,2): \underline{S}(Y,2) \to \underline{S}(X,2)$ given by $\underline{S}(f,2)(\varphi) = \varphi f$. One can see that this is also expressed by taking the inverse image, under f, of the corresponding subset, since $(\varphi f)^{\#} = f^{-1}(\varphi^{\#})$. It is well known that direct image is left adjoint in $\underline{S}$ to inverse image, i.e. that

$$f(X') \subseteq Y' \quad \text{iff} \quad X' \subseteq f^{-1}(Y') \quad \text{for all } X' \subseteq X,\ Y' \subseteq Y .$$

It is also clear that the functor defined by II is right adjoint to inverse image in $\underline{S}$, since

$$f^{-1}(Y') \subseteq X' \quad \text{iff} \quad Y' \subseteq \{ y \in Y \mid f^{-1}(y) \subseteq X' \}$$

Condition (3) of the definition of an elementary theory thus generalizes the known fact that in $\underline{S}$, existential quantification (given by direct image) is left adjoint to substitution (given by inverse image). The corresponding definition and property of universal quantification can, in this classical (i.e. non-intuitionistic) situation be obtained by Boolean dualization of $\exists$. Conditions (4) generalize similar properties in $\underline{S}$ of direct image and inverse image.

Following definition I of $\exists$ in $\underline{S}$, consider $\exists_{\Delta_X}[1_X]$.

We have $(\exists_{\Delta_X}[1_X])^{\#} = \Delta_X(1_X^{\#}) = \Delta_X$ (all of X)

$$= \{(x,x) \mid x \in X\} = \{(x,y) \mid x = y \in X\} .$$

i.e. $\exists_{\Delta_X}[1_X]$ is precisely the predicate of equality on a set. This motivates the definition of "equality" in an object given in condition (5) of the elementary-theory definition. Condition 5(a) stipulates that this generalized equality must be strict, i.e. does not "identify" distinct morphisms; 5(b) indicates that "equal" truth values are exactly those which imply each other.

Thus it is seen that the definition of an elementary theory embodies the relevant properties of a classical first-order theory.

Volger later generalizes condition (1) to admit all objects of the form $V^n \times \Omega^m$, and finally dispenses with such representations altogether, in favour of the concept of a logical category. It may be noted that a logical category bears more resemblance to a multi-sorted theory whereas an elementary theory corresponds to a theory with a single type.

A category $\underline{C}$ is called logical in [2] and [3] iff

(1) $\underline{C}$ has finite products. The terminal object (which must exist by (1)) is denoted I and the unique morphism $X \to I$ is denoted $!x$ for each $X \in |\underline{C}|$.

(2) $\underline{C}$ has a specified Boolean algebra object Ω . The greatest and least elements of $\underline{C}(X,\Omega)$, $X \neq \Omega$ and of $\underline{C}(\Omega,\Omega)$ are denoted $1_X, 0_X$ and 1,0 respectively. The Boolean complement is denoted $\sim : \Omega \to \Omega$.

(3)(4)(5) For each $X \xrightarrow{f} Y$ in $\underline{C}$, conditions (3),(4),(5) for an elementary theory are satisfied.

A model of a logical category $\underline{C}$ is a functor $F: \underline{C} \to \underline{S}$ (sets) which preserves finite products, sends Ω to the set 2, sends 0,1, ~, $\wedge$ to the logical "false", "true", "not" and "and" respectively, and sends $\exists_f[\ ]$ to "direct image". More generally if $(\underline{C},\Omega,\exists,\ldots)$ and $(\underline{C}',\Omega'\exists'\ldots)$ are any two logical categories, and $F:\underline{C}\to\underline{C}'$ a functor, F is called a <u>logical functor</u> iff F preserves finite products, sends Ω to Ω', preserves the Boolean-algebra structure of each $\underline{C}(X,\Omega)$, and sends $\exists$ to $\exists'$.

It should be noted that $\underline{S}$ is of course a logical category: $\underline{S}$ has (cartesian) products, 1 (singleton) is terminal, 2 is the Boolean-algebra object in the well-known way, and the discussion on pages 5-6 shows that the direct image map defines an appropriate quantification. So a model for $\underline{C}$ is really just a logical functor: $\underline{C}\to\underline{S}$.

A logical functor $\underline{C}\to\underline{C}'$ is sometimes called a <u>$\underline{C}'$-model for $\underline{C}$</u> i.e. at least formally, any logical category can be a "model-recipient". $\underline{S}$ is the most intuitively appropriate to embody the classical notion of a model. But, as will be discussed in the present work other categories such as toposes of form $\underline{S}^P$ can be appropriate model-recipients for intuitionistic theories.

The functor $\underline{C}$ (I,): $\underline{C}\to\underline{S}$ may or may not be logical. When it is, it is called in [3] <u>the canonical (semantic) model for $\underline{C}$</u>.

To see the motivation for this terminology, consider first that the objects of $\underline{C}$ contain, in a sense, all that remains of the intuitive variables of the theory. Remembering that, in the case where $\underline{C}$ is an elementary theory, an object X is a product V^n of V, the class-of-variables object, we continue to think of X, even in the general cases, as a class of terms.

Volger notes that the canonical functor $\mathcal{C}(I,\ )$ will be a model for $\mathcal{C}$ iff $\mathcal{C}$ is <u>maximally consistent</u>, i.e. $\mathcal{C}(I,\Omega) = \{0,1\}$, and $\mathcal{C}$ is <u>rich</u>, i.e. for every $X \xrightarrow{\varphi} \Omega$ in $\mathcal{C}$ with $\exists_X [\varphi] = 1$, there is a $I \xrightarrow{k} X$ in $\mathcal{C}$ such that $\varphi k = 1$.

This definition of maximally consistent translates as: "every closed formula of $\mathcal{C}$ is either provable or refutable" which is just the usual logical definition. Likewise, since maps $I \to X$, i.e. $1 \to X$ in $\mathcal{S}$ are just specified elements of X, we call a map $I \xrightarrow{k} X$ in $\mathcal{C}$ a <u>constant in $\mathcal{C}$</u>, and richness has the usual logical meaning.

Thus the canonical model $\mathcal{C}(I,\ )$ has as its "universe" the set of maps $I \to X$, i.e. intuitively the terms of a certain type, and each "formula" $X \xrightarrow{f} \Omega$ is sent to f ( ), i.e. is mapped by substitution.

A comparison with the discussion on pages 1-2 shows that this gives the canonical semantic model (in the sense of R-S) determined by the identity Boolean homomorphism : $\mathcal{C}(I,\Omega) \to 2$. The condition of richness is also natural in the categorical case since $\exists$ in $\mathcal{S}$ is real logical existence.

The categorical statement of the completeness theorem is that given two distinct morphisms in $\mathcal{C}$, there exists a model for $\mathcal{C}$ which separates them. This implies the classical completeness theorem, since in particular for $X \xrightarrow{f} \Omega$ with $f \neq 1_X$, i.e. f not a theorem, the image of f will be false in the model for some k. It is proved in [2,3] by first extending to a rich category by adding constants, then reducing by an ultrafilter to a maximally consistent category, and iterating these two constructions, alternately, countably many times to produce a rich and maximally consistent extension. This extension is faithful, and so composing with the canonical model gives the required result for (classical) logical categories.

Following the lines of Volger's treatment, the present work will develop a semantics for intuitionistic logical categories, and will prove a partial completeness theorem.

Since work on this thesis was begun, several alternative formulations of this topic have been done. A. Joyal has proved a completeness theorem for certain categories which generalize intuitionistic first-order logic in a way different from ours. His definition of a model is also more general and the method of proof is different from ours. Joyal outlined the proof of his theorem at a meeting in 1972. More recently, two other proofs of such a theorem were given at the Seminaire d'Ete de l'Universite de Montreal in July, 1974: one by G. Reyes and M. Makkai (unpublished) and one by P. Freyd (there are mimeographed notes available).

CHAPTER 1: BASIC PROPERTIES

1.1 Definition. A category $\mathcal{C}$ is called an intuitionistic prelogical category (or simply prelogical) iff:

(a) $\mathcal{C}$ has finite products.

The terminal object is denoted I; for each $X \in |\mathcal{C}|$ The unique map is denoted $X \xrightarrow{!_X} I$.

(b) $\mathcal{C}$ has a specified Heyting algebra object ; i.e. for each $X \in |\mathcal{C}|$ and each morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$, $\mathcal{C}(X,\Omega)$ is a Heyting algebra and $\mathcal{C}(f,\Omega)$ is a Heyting homomorphism.

We use the notation 0_X, 1_X, $\vee$, $\wedge$, $\Rightarrow$ and $\neg$ for the least element, greatest element, join, meet, relative pseudo-complement and pseudo-complement, respectively, of $\mathcal{C}(X,\Omega)$. The special symbols 0 and 1 denote 0_I and $1_I : I \to \Omega$.

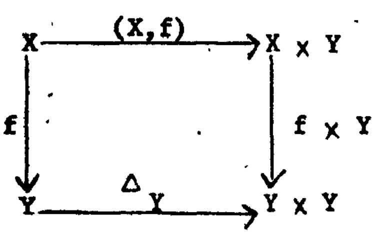
(c) For every $X \xrightarrow{f} Y$ in $\mathcal{C}$, the functor $\mathcal{C}(f,\Omega): \mathcal{C}(Y,\Omega) \to \mathcal{C}(X,\Omega)$ (often denoted ( ) f) has a left adjoint denoted $\exists_f[\ ]$;

i.e. $\exists_f[\varphi] \leq \psi$ iff $\varphi \leq \psi f$ for all $X \xrightarrow{\varphi} \Omega$, $Y \xrightarrow{\psi} \Omega$ in $\mathcal{C}$.

$\exists_f[\ ]$ is called existential quantification along f.

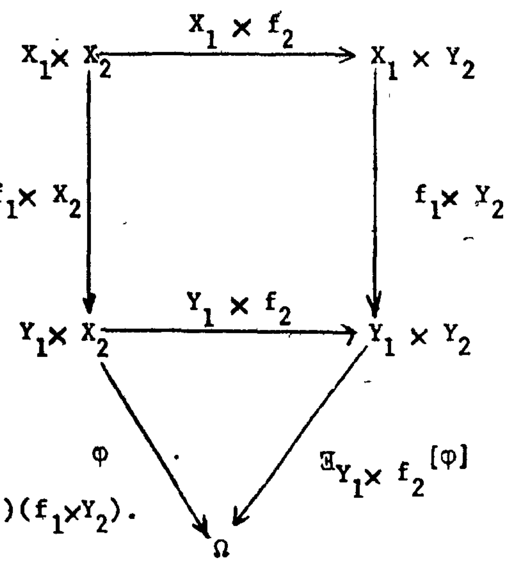
(d) Beck conditions for existential quantification:

(i) Given $X \xrightarrow{f} Y$ and the pullback shown, then



$$\exists_{(X,f)}[1_Y f] = (\exists_{\Delta_Y}[1_Y])(f \times Y) .$$

(ii) Given any $X_1 \xrightarrow{f_1} Y_1$, $X_2 \xrightarrow{f_2} Y_2$, $Y_1 \times X_2 \xrightarrow{\varphi} \Omega$, and the diagram shown (the square is automatically a pullback), then



$$\exists_{X_1 \times f_2}[\varphi(f_1 \times X_2] = (\exists_{Y_1 \times f_2}[\varphi])(f_1 \times Y_2).$$

(e) For every $Y \in |\mathcal{C}|$ define equality on Y as the morphism $Y \times Y \xrightarrow{e_Y} \Omega$ given by $e_Y = \exists_{\Delta_Y}[1_Y]$. This e_Y must satisfy:

(i) If $f_1, f_2 \colon X \to Y$ are such that $e_Y < f_1, f_2 > = 1_X$, then $f_1 = f_2$.

(ii) $e_\Omega = <=>$, i.e. for all $f, g \colon \Omega \to \Omega$,

$e_\Omega < f, g > = (f \Rightarrow g) \wedge (g \Rightarrow f)$.

1.2 Definition. A category $\mathcal{C}$ is called an intuitionistic logical category iff:

(a) - (e) $\mathcal{C}$ is prelogical.

(f) For every $X \xrightarrow{f} Y$ in $\mathcal{C}$, the functor $\mathcal{C}(f, \Omega)$ has a right adjoint, denoted $\forall_f[\ ]$.

i.e. $\forall_f[\varphi] \geq \psi$ iff $\varphi \geq \psi f$ for all $X \xrightarrow{\varphi} \Omega$, $Y \xrightarrow{\psi} \Omega$ in $\mathcal{C}$.

$\forall_f[\ ]$ is called universal quantification along f.

<u>1.3</u> <u>Proposition</u>. The definition <u>1.1(c)</u> of existential quantification is equivalent to conditions (<u>a</u>) and (<u>b</u>):

(<u>a</u>) $\varphi \wedge (\exists_f[\varphi])f = \varphi$

(<u>b</u>) $\psi \wedge \exists_f[\varphi] = \exists_f[\psi f \wedge \varphi]$

for all $X \xrightarrow{f} Y$, $X \xrightarrow{\varphi} \Omega$ and $Y \xrightarrow{\psi} \Omega$ in $\mathcal{C}$.

We remark moreover that

(<u>c</u>) Since $\exists_f[\ ]$, $\forall_f[\ ]$ are defined as adjoints, each is unique.

(<u>d</u>) Since $\exists_f[\ ]$, $\forall_f[\ ]$ are both functors between ordered categories, $\therefore$ each is order-preserving.

<u>Proof</u>: Assume the adjointness condition <u>1.1(c)</u>. Certainly $\exists_f[\varphi] \leq \exists_f[\varphi]$. Call $\psi = \exists_f[\varphi]$; $\therefore$ $\exists_f[\varphi] \leq \psi$.

$\therefore$ By adjointness, $\varphi \leq \psi f = (\exists_f[\varphi])f$. i.e. $\varphi \wedge (\exists_f[\varphi])f = \varphi$: i.e. (<u>a</u>) holds. Now recall that the functor $(\ ) \wedge \varphi : \mathcal{C}(X,\Omega) \longrightarrow \mathcal{C}(X,\Omega)$ is left adjoint to $\varphi \Rightarrow (\ )$; i.e. $(\varphi \wedge \chi_1) \leq \chi_2$ iff $\chi_1 \leq (\varphi \Rightarrow \chi_2)$.

Since each $\mathcal{C}(f,\Omega)$, i.e. $(\ )f$, is a Heyting homomorphism, we have $(\psi_1 \Rightarrow \psi_2)f = \psi_1 f \Rightarrow \psi_2 f$. i.e. if we fix any $Y \xrightarrow{\psi} \Omega$, then

$$\Big(\psi f \Rightarrow (\ )\Big)\Big((\ )f\Big) = \Big((\ )f\Big)\Big(\psi \Rightarrow (\ )\Big).$$

Take left adjoints of this expression, remembering that a composite of adjoints, in reverse order, give the adjoint of a composition.

$$\therefore \quad \Big(\exists_f[\ ]\Big)\Big(\psi f \wedge (\ )\Big) = \Big(\psi \wedge (\ )\Big)\Big(\exists_f[\ ]\Big).$$

i.e. for any $X \xrightarrow{\varphi} \Omega$, $\exists_f[\psi f \wedge \varphi] = \psi \wedge \exists_f[\varphi]$;

i.e. (<u>b</u>) holds.

Conversely, assume that (a) and (b) hold.

Suppose $\exists_f[\varphi] \leq \psi$. Since ( )f is a Heyting homomorphism,

$\therefore (\exists_f[\varphi])f \leq \psi f$. $\therefore \varphi \leq (\exists_f[\varphi])f \leq \psi f$ by (a).

On the other hand suppose $\varphi \leq \psi f$, i.e. $\varphi \wedge \psi f = \varphi$.

$\therefore \exists_f[\varphi \wedge \psi f] = \exists_f[\varphi]$. But $\therefore \exists_f[\varphi] = \exists_f[\varphi \wedge \psi f] = \exists_f[\varphi] \wedge \psi$ by (b) ; i.e. $\exists_f[\varphi] \leq \psi$. The adjointness is therefore proved.

1.4 Proposition. Let $\mathcal{C}$ be a prelogical category. Then the following conditions follow from 1.1(a) (b),(c) and (e); they do not require the Beck conditions. (Unless otherwise stated in the individual conditions, the symbol f,φ,ψ refer to morphisms $X \xrightarrow{\varphi}$, $X \xrightarrow{f} Y$, and $Y \xrightarrow{\psi} \Omega$ in $\mathcal{C}$.)

(a) (i) $\varphi \leq (\exists_f[\varphi])f$ and (ii) $\psi \geq \exists_f[\psi f]$.

(b) (i) $\exists_f\left[(\exists_f[\varphi])f\right] = \exists_f[\varphi]$

and (ii) $(\exists_f[\psi f])f = \psi f$.

(c) (i) $\exists_{1d_X}[\varphi] = \varphi$ and (ii) $(\exists_{gf}[\varphi]) = \exists_g\left[\exists_f[\varphi]\right]$

where $Y \xrightarrow{g} Z$ is any morphism.

(d) (i) $\exists_f[\varphi] = 0_Y$ iff $\varphi = 0_X$ and (ii) $\left(\exists_f[1_X]\right)f = 1_X$.

(e) If f is an isomorphism, then $\exists_f[\varphi] = \varphi f^{-1}$.

(f) If f is epi, then $\exists_f[\psi f] = \psi$.

(g) $e_Y \Delta_Y = 1_Y$ and hence $e_Y < f,f > = 1_X$.

(h) f is epi iff $\exists_f[1_X] = 1_Y$.

(i) $\neg$ is order-reversing, i.e. if $\varphi_1 \leq \varphi_2$ then $\neg\varphi_1 \geq \neg\varphi_2$.

(j) $\neg\,\exists_f[\neg(\varphi_1 \Leftrightarrow \varphi_2)] \leq \exists_f[\varphi_1] \Leftrightarrow \exists_f[\varphi_2]$.

(k) Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ and $Y \xrightarrow{h} Z$.

Then $e_Y < f,g > \leq e_Z < hf,hg >$.

(l) $\exists_{\Delta_X}[\varphi] = e_X \wedge \varphi p_1$, where p_i are the projections of $X \times X$.

(m) $e_Y \wedge \varphi q_1 \leq \varphi q_2$, where q_i are the projections of $Y \times Y$

and φ is $Y \xrightarrow{\varphi} \Omega$.

Proof: (a) The first statement is just 1.3 (a) and has been proved.

As for the second, we have $\psi f \leq \psi f$. Call $\varphi = \psi f$; $\therefore \varphi \leq \psi f$.

$\therefore$ By adjointness, $\exists_f[\varphi] \leq \psi$, i.e. $\exists_f[\psi f] \leq \psi$.

(b) Take $\exists_f[\ ]$ of (a)(i).

This gives $\exists_f[\varphi] \leq \exists_f\left[\exists_f[\varphi]\, f\right]$, since $\exists_f$ is order preserving.

Call $\psi = \exists_f[\varphi]$ and call $\chi = \psi f$. Certainly $\chi \leq \psi f$;

$\therefore$ by adjointness $\exists_f[\chi] \leq \psi$. i.e. $\exists_f\left[\exists_f[\varphi] f\right] \leq \exists_f[\varphi]$, proving (b)(i).

Take $\mathcal{C}(f,\Omega)$ of (a)(ii).

This gives $(\exists_f[\psi f])f \leq \psi f$ since $\mathcal{C}(f,\Omega)$ is order preserving. Call $\chi = \psi f$.

Then by first result of (a), $\exists_f[\chi]f \geq \chi$ as required for (b)(ii).

(c) $\exists_X[\varphi] \leq \varphi$ since $\varphi \leq \varphi X$. But $\exists_X[\varphi] = \exists_X[\varphi]X \geq \varphi$ by 1.4(a)(i).

$\therefore \exists_X[\varphi] = \varphi$.

Moreover $\mathcal{C}(gf,\Omega) \cong \mathcal{C}(f,\Omega)\, \mathcal{C}(g,\Omega)$. The adjoint to this composition

($\exists_{gf}[\ ]$) is the composition of the adjoints in reverse order,

i.e. $\exists_g\left[\exists_f[\ ]\right]$.

(d) $0_X \leq \forall f$ for every $\forall$. Hence $\exists_f[0_X] \leq \forall$ for every $\forall$;

i.e. $\exists_f[0_X] = 0_Y$. Conversely let $\exists_f[\varphi] = 0_Y$. By (a)(i),

$\varphi \leq (\exists_f[\varphi])f = 0_Y f = 0_X$; i.e. $\varphi = 0_X$. Thus (d)(i) is proved.

Moreover $(\exists_f[1_X])f \geq 1_X$ by (a)(i), and so (d)(ii) holds.

(e) Since f is an isomorphism we have easily that $\varphi f^{-1} \leq \forall$ iff $\varphi \leq \forall f$, i.e. that $(\ )f^{-1}$ is left adjoint to $(\ )f$. Thus since adjoints are unique, $\varphi f^{-1} = \exists_f[\varphi]$.

(f) We quote Volger, [3], result 1.8.7: Since f is epi, $(\ )f$ is full and faithful and therefore the back adjunction is an isomorphism.

(g) $e_Y \Delta_Y = (\exists_{\Delta_Y}[1_Y])\Delta_Y = 1_Y$ by (d)(ii). Hence $e_Y < f,f > = e_Y \Delta_Y f = 1_X$, because $(\ )f$ is a Heyting homomorphism.

(h) If f is epi, then by (f) $\exists_f[1_X] = \exists_f[1_Y f] = 1_Y$. Suppose conversely, $\exists_f[1_X] = 1_Y$ and let $Y \overset{h_1}{\underset{h_2}{\rightrightarrows}} Z$ be such that $h_1 f = h_2 f$. By (g),

$e_Y < h_1 f, h_2 f > = 1_X$; i.e. $e_Y < h_1, h_2 > f$ = call $\forall f = 1_X$. Since $1_X \leq \forall f$, by adjointness $\exists_f[1_X] \leq \forall$. i.e. $1_Y = \exists_f[1_X] \leq e_Y < h_1, h_2 >$. Hence by 1.1(e)(i) $h_1 = h_2$ and so f is epi.

(i) Since $\varphi_1 \leq \varphi_2$, $\therefore$ $\neg\varphi_2 \wedge \varphi_1 \leq \neg\varphi_2 \wedge \varphi_2 = 0$. By definition $\neg\varphi_1$ is the largest element $\forall$ such that $\forall \wedge \varphi_1 = 0$. Hence $\neg\varphi_2 \leq \neg\varphi_1$.

(j) $\left(\neg \exists_f\left[\neg(\varphi_1 \Rightarrow \varphi_2)\right]\right) \wedge \exists_f[\varphi_1] = \exists_f\left[\left(\neg \exists_f[\neg(\varphi_1 \Rightarrow \varphi_2)]\right)(f) \wedge \varphi_1\right]$

(by 1.3(b) with $\forall = \neg \exists_f[\neg(\varphi_1 \Rightarrow \varphi_2)]$)

$$= \exists_f\left[\neg\left(\exists_f[\neg(\varphi_1 \Rightarrow \varphi_2)]f\right) \wedge \varphi_1\right]$$ (because ( )f is a Heyting homomorphism)

$$\leq \exists_f\left[(\varphi_1 \Rightarrow \varphi_2) \wedge \varphi_1\right]$$ By 1.4(i)

(since $\exists_f[\neg(\varphi_1 \Rightarrow \varphi_2)]f \geq \varphi_1 \Rightarrow \varphi_2$ by 1.3(a))

$$\leq \exists_f[\varphi_2] \, .$$

i.e. By adjointness of $\Rightarrow$ and $\wedge$,

$\neg\,\exists_f\left[\neg(\varphi_1 \Rightarrow \varphi_2)\right] \leq \exists_f[\varphi_1] \Rightarrow \exists_f[\varphi_2]$. Call this inequality (j)(i)

Now $\varphi_1 \Leftrightarrow \varphi_2 \leq \varphi_1 \Rightarrow \varphi_2$. $\therefore$ $\neg(\varphi_1 \Leftrightarrow \varphi_2) \geq \neg(\varphi_1 \Rightarrow \varphi_2)$ by 1.4(i) ;

$\therefore$ $\exists_f[\neg(\varphi_1 \Leftrightarrow \varphi_2)] \geq \exists_f[\neg(\varphi_1 \Rightarrow \varphi_2)]$.

$\neg\,\exists_f\left[\neg(\varphi_1 \Leftrightarrow \varphi_2)\right] \leq \neg\,\exists_f[\neg(\varphi_1 \Rightarrow \varphi_2)] \leq \exists_f\,\varphi_1 \Rightarrow \exists_f[\varphi_2]$ by (j)(i)

Call that inequality (j)(ii) .

Similarly we obtain $\neg\,\exists_f[\neg(\varphi_2 \Rightarrow \varphi_1)] \leq \exists_f[\varphi_2] \Rightarrow \exists_f[\varphi_1]$ which we denote (j)(iii), and hence

$\neg\,\exists_f\left[\neg(\varphi_1 \Leftrightarrow \varphi_2)\right] \leq \neg\,\exists_f[\neg(\varphi_2 \Rightarrow \varphi_1)] \leq \exists_f[\varphi_2] \Rightarrow \exists_f[\varphi_1]$,

denoted (j)(iv).

Inequalities (j)(ii) and (j)(iv) give

$$\neg\,\exists_f\left[\neg(\varphi_1 \Leftrightarrow \varphi_2)\right] \leq \left(\exists_f[\varphi_1] \Rightarrow \exists_f[\varphi_2]\right) \wedge \left(\exists_f[\varphi_2] \Rightarrow \exists_f[\varphi_1]\right)$$

$$= \exists_f[\varphi_1] \Leftrightarrow \exists_f[\varphi_2]$$ as required.

(k) $1_Y = 1_Z h = e_Z \Delta_Z h$ by (g)

$= e_Z (h \times h)\Delta_Y$. Call $\psi = e_Z(h \times h)$. Thus $1_Y \leq \psi\Delta_Y$;

$\therefore$ By adjointness $\exists_{\Delta_Y}[1_Y] \leq \psi = e_Z(h \times h)$. i.e. $e_Y \leq e_Z(h \times h)$.

Compose both sides with $< f,g >$ to get inequality (k).

(l) Note that $p_1\Delta_X = id_X$.

$\therefore \quad \exists_{\Delta_X}[\varphi] = \exists_{\Delta_X}[1_X \wedge \varphi] = \exists_{\Delta_X}[1_X \wedge \varphi p_1\Delta_X]$

$= \exists_{\Delta_X}[1_X] \wedge \varphi p_1$ (by 1.3(b))

$= e_X \wedge \varphi p_1$.

(m) Since $q_2\Delta_Y = id_Y$, $\therefore \varphi \leq \varphi = \varphi q_2\Delta_Y$ = call $\psi\Delta_Y$.

$\therefore$ By adjointness, $\exists_{\Delta_Y}[\varphi] \leq \psi = \varphi q_2$. Substitute by (l)

$\therefore \quad e_Y \wedge \varphi q_1 \leq \varphi q_2$.

The following properties of universal quantification can now be proved.

1.5 Proposition . Let $\underline{C}$ be an intuitionistic logical category and let

$$X \underset{h}{\overset{f}{\rightrightarrows}} Y, \quad Y \xrightarrow{g} Z, \quad X \xrightarrow{\varphi} \Omega \text{ and } Y \xrightarrow{\psi} \Omega \text{ in } C .$$

(a) (i) $\varphi \geq (\forall_f[\varphi])f$ and (ii) $\psi \leq \forall_f[\psi f]$

(b) (i) $\forall_f[(\forall_f[\varphi])f] = \forall_f[\varphi]$ and (ii) $(\forall_f[\psi f])f = \psi f$

(c) (i) $\forall_{id_X}[\varphi] = \varphi$ and (ii) $\forall_{gf}[\varphi] = \forall_g[\forall_f[\varphi]]$

(d) (i) $\forall_f[\varphi] = 1_Y$ iff $\varphi = 1_X$ and (ii) $(\forall_f[0_X])f = 0_X$.

(e) $\forall_{!_X}[e_Y < f,h >] = 1$ iff $f = h$.

This last is often called an extensionality condition; in $\underline{S}$ it says intuitively that morphisms are equal iff they are equal at every point of the domain.

Proof: (a) - (d) These are categorical-dual to the conditions 1.4(a)-(d) , which were proved using the properties of left adjoints. Since $\forall$ is defined as a right adjoint, one need only dualize the arguments used to prove 1.4(a)-(d). For example, the verification of (d)(i) is as follows:

Let $\varphi = 1_X$. $\therefore$ For every $\psi: Y \longrightarrow \Omega$, $\varphi \geq \psi f$. $\therefore$ By right adjointness $\forall_f[\varphi] \geq \psi$, every ψ. i.e. $\forall_f[\varphi] = 1_X$.

Conversely let $\forall_f [\varphi] = 1_Y$. $\therefore$ $\varphi \geq (\forall_f[\varphi])f = 1_Y f = 1_X$ by ~~1.5~~ (a)(i).

(e) Let $f = h$, $\therefore$ By 1.4(g), $e_Y < f,h > = 1_X$.

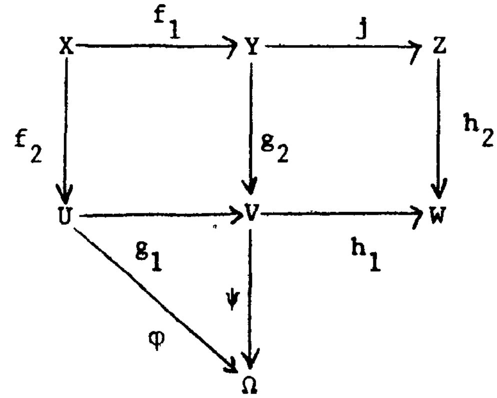
$\therefore$ $\forall_{!_X}[e_Y < f,h >] = 1$ by 1.5(d)(i).

Conversely let $\forall_{!_X}[e_Y < f,h >] = 1$. $\therefore$ $e_Y < f,h > = 1_X$ (by 1.5(d)(i)); $\therefore$ $f = h$ (by 1.1(e)(i)), and so the extensionality is proved.

The Beck conditions 1.1(d) give rise to several similar properties:

1.6 Proposition. Let $\mathcal{C}$ be an intuitionistic prelogical category. Then the following conditions on quantification hold:

(a) Given any two pullback diagrams as shown, with the properties that for any $U \xrightarrow{\varphi} \Omega$ and $V \xrightarrow{\psi} \Omega$,



(i) $\exists_{f_1}[\varphi f_2] = (\exists_{g_1}[\varphi])g_2$

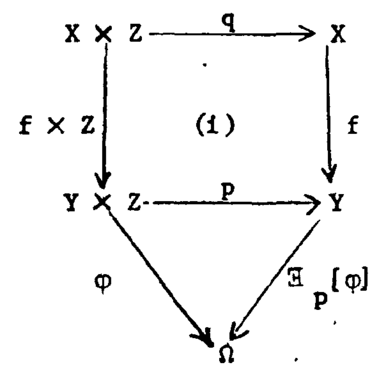
and

(ii) $\exists_j[\psi g_2] = (_{h_1}[\psi])h_2$,

then

$$\exists_{jf_1}[\varphi f_2] = (\exists_{h_1 g_1}[\varphi])h_2 .$$

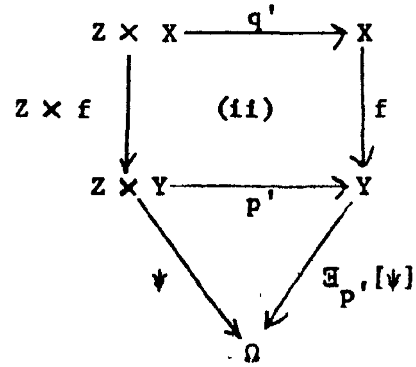
(b) Given $X \xrightarrow{f} Y$ and projections $X \times Z \xrightarrow{q} X$, $Y \times Z \xrightarrow{p} Y$, and any $Y \times Z \xrightarrow{\varphi} \Omega$, $Z \times Y \xrightarrow{\psi} \Omega$, then



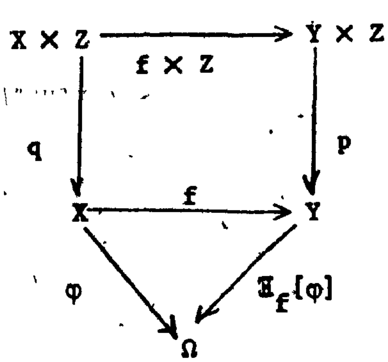
(i) $\exists_q[(\varphi)(f \times Z)] = (\exists_p[\varphi])f$

and

(ii) $\exists_{q'}[(\psi)(Z \times f)] = (\exists_{p'}[\psi])f$.



(c)



Given the diagram at the left,

Given $X \xrightarrow{f} Y$ and projections as in (b) and any $X \xrightarrow{\varphi} \Omega$,

then $\exists_{f \times Z}[\varphi q] = (\exists_f[\varphi])p$.

(d) Given $X_1 \xrightarrow{h} Y_1$,

$Y_1 \times X_2 \xrightarrow{f} Y_2$, projections

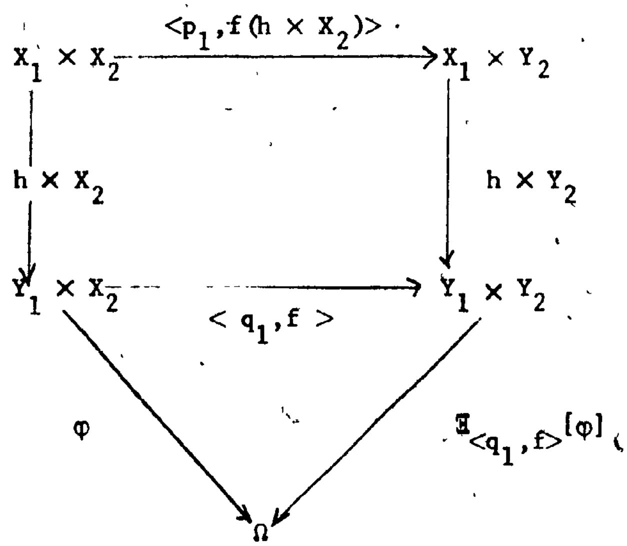
$X_1 \times X_2 \xrightarrow{p_1} X_1$

$Y_1 \times Y_2 \xrightarrow{q_1} Y_1$

and any map $Y_1 \times X_2 \xrightarrow{\varphi} \Omega$

then $\exists_{\langle p_1, f(h \times X_2)\rangle}[\varphi(h \times X_2)]$

$= (\exists_{\langle q_1, f\rangle}[\varphi])\ (h \times Y_2)$.



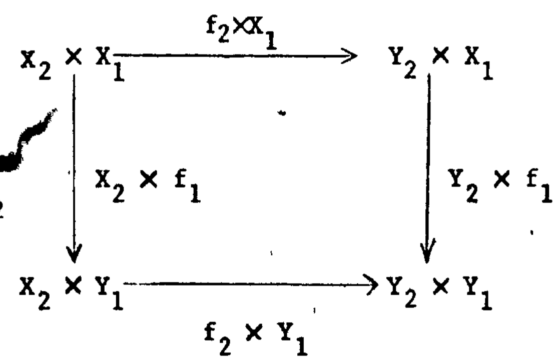
(e) The condition 1.1(d)(ii) on quantification is equivalent to the condition :

Given $X_1 \xrightarrow{f_1} X_1$, $X_2 \xrightarrow{f_2} Y_2$

and $X_2 \times Y_1 \xrightarrow{\varphi} \Omega$,

then $\exists_{f_2 \times X_1}[\varphi(X_2 \times f_1)] =$

$(\exists_{f_2 \times Y_1}[\varphi])(f_1 \times Y_2)$.



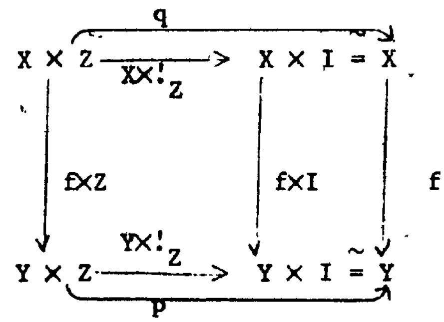
Proof: (a) $\exists_{jf_1}[\varphi f_2] = \exists_j[\exists_{f_1}[\varphi r_2]]$ (by 1.4(c)(ii)

$= \exists_j[\exists_{g_1}[\varphi]g_2]$ (by (i))

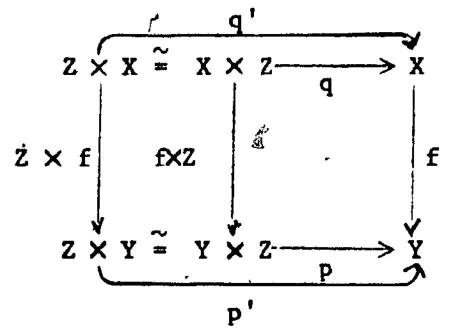
$= (\exists_{h_1}[\exists_{g_1}[\varphi]])h_2$ (by (ii), where ψ here is $\exists_{g_1}[\varphi]$)

$= (\exists_{h_1 g_1}[\varphi])h_2$ (by 1.4(c)(ii).)

(b) It may be seen that diagram (i) is just a special case of the diagram in 1.1(d)(ii), with $X_1 = X$, $X_2 = Z$, $Y_1 = Y$, $Y_2 = I$, $f_1 = f$ and $f_2 = !_Z$.

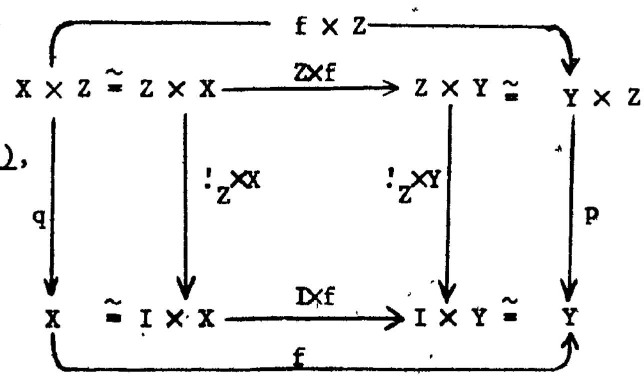


Similarly, 1.6(b)(ii) is just 1.6(1)(i) with the product isomorphisms added, as shown.



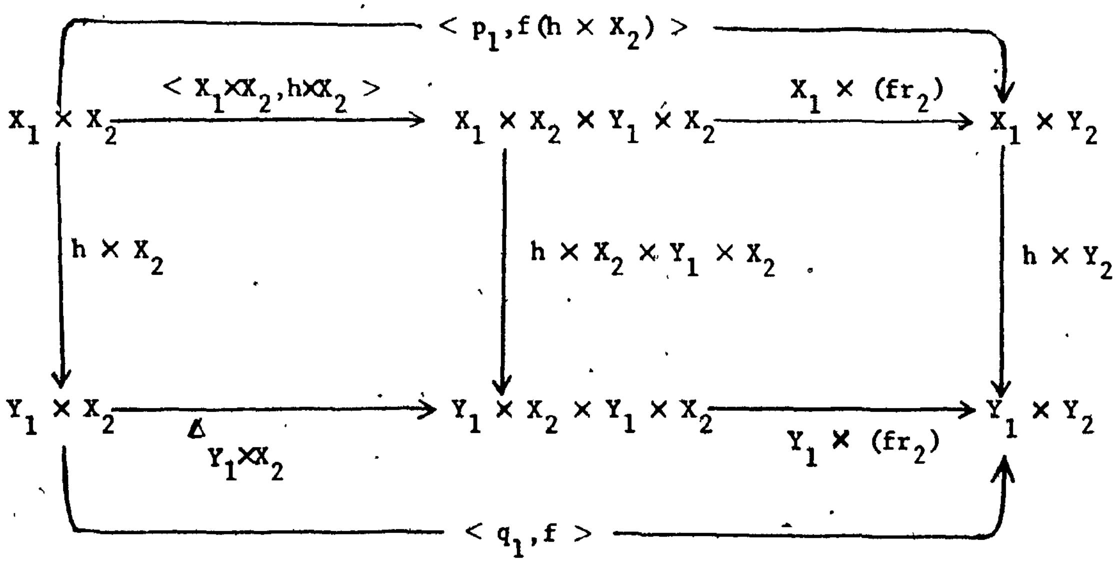
Thus by 1.6(a) one has the required result.

(c) The centre square of this diagram is a special case of 1.1(d)(ii), with $X_1 = Z$, $X_2 = X$, $Y_1 = I$, $Y_2 = Y$, $f_1' = !_Z$ and $f_2 = f$,



Adding the product isomorphisms gives the outer square 1.6(c).

(d) We quote Volger, [3], result 1.4.6, which says that the present 1.6(d) is a consequence of 1.1(d)(i) and (ii). It may be remarked that the diagram of 1.6(d) can be obtained by the composition shown below:



where h, f, p_1, q_1 have been defined,

and r_2 refers to the projection $X_2 \times Y_1 \times X_2 \xrightarrow{r_2} Y_1 \times X_2$.

The left-hand square is of the form of diagram <u>1.1(d)(i)</u>,

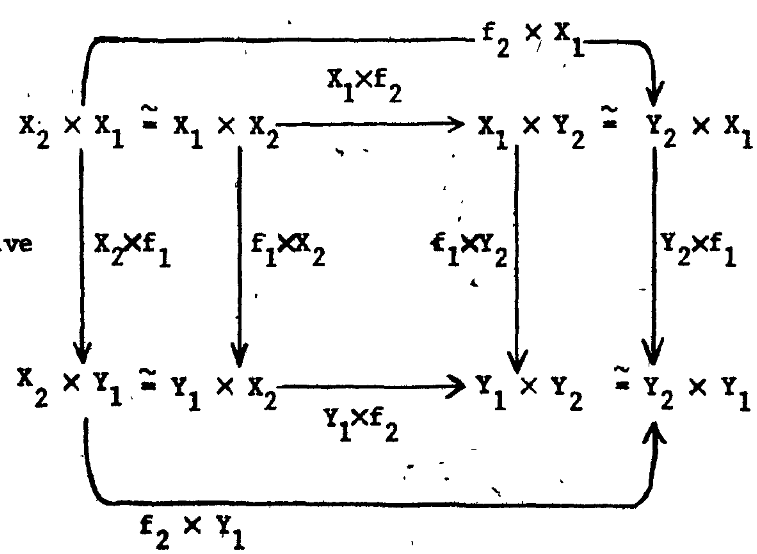
with $X = X_1 \times X_2$, $Y = Y_1 \times X_2$, and $f = h \times X_2$.

The right-hand square is of the form of the diagram <u>1.1(d)(ii)</u>,

with $X_1 = X_1$, $X_2 = X_2 \times Y_1 \times X_2$, $Y_1 = Y_1$, $Y_2 = Y_2$,

$f_1 = h$ and $f_2 = fr_2$.

(e) Assume <u>1.1(d)(ii)</u>, and compose with the product isomorphisms to give the first diagram shown. Then by <u>1.6(a)</u> the stated condition holds.



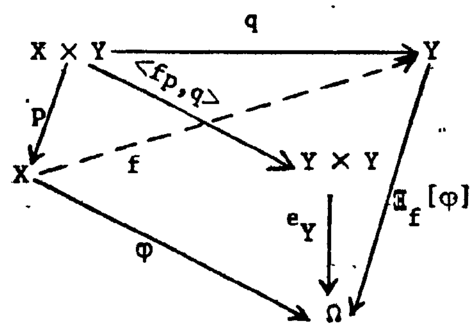
Conversely, assume the condition (e), and compose with the same isomorphisms to give the second diagram. Then by 1.6(a), the condition 1.1(d)(ii) must hold.

$$\begin{array}{ccccc} X_1\times X_2 \cong X_2\times X_1 & \overset{X_1\times f_2}{\underset{f_2\times X_1}{\longrightarrow}} & Y_2\times X_1 \cong X_1\times Y_2 \\ f_1\times X_2 \quad X_2\times f_1 & & Y_2\times f_1 \quad f_1\times Y_2 \\ Y_1\times X_2 \cong X_2\times Y_1 & \overset{f_2\times Y_1}{\underset{Y_1\times f_2}{\longrightarrow}} & Y_2\times Y_1 \cong Y_1\times Y_2 \end{array}$$

A number of useful properties follow from the various "Beck conditions".

1.7 Proposition. Let $\underline{C}$ be prelogical. Then the following conditions hold:

(a) Given $X \xrightarrow{f} Y$ and projections $X\times Y \xrightarrow{p} X$, $X\times Y \xrightarrow{q} Y$ and any $X \xrightarrow{\varphi} \Omega$, then $\exists_f[\varphi] =$



$$= \exists_q\left[(\varphi p) \quad (e_Y \langle fp, q\rangle)\right].$$

(b) Let the following be morphisms in $\underline{C}$: $X_1 \xrightarrow{f_1} Y_1$, $X_2 \xrightarrow{f_2} Y_2$, $X_1\times X_2 \xrightarrow{p_i} X_i$, $Y_1\times Y_2 \xrightarrow{q_i} Y_i$ and $X_i \xrightarrow[\varphi_i]{} \Omega$; then $\exists_{f_1\times f_2}[(\varphi_1 p_1)\wedge(\varphi_2 p_2)]$

$$= (\exists_{f_1}[\varphi_1]q_1) \wedge (\exists_{f_2}[\varphi_2]q_2).$$

(c) Let $X \overset{f_1}{\underset{g_1}{\rightrightarrows}} Y_1$, $X \overset{f_2}{\underset{g_2}{\rightrightarrows}} Y_2$.

Then $e_{Y_1\times Y_2} < f_1,f_2,g_1,g_2 > = (e_{Y_1} < f_1,g_1 >) \wedge (e_{Y_2} < f_2 g_2 >)$

i.e. if we call

$t = < p_1,p_3,p_2,p_4 >: Y_1 \times Y_2 \times Y_1 \times Y_2 \to Y_1 \times Y_1 \times Y_2 \times Y_2$,

and call $Y_1 \times Y_1 \times Y_2 \times Y_2 \xrightarrow{q_1} Y_1 \times Y_1$,

then $e_{Y_1\times Y_2} = (e_{Y_1} q_1 \wedge e_{Y_2} q_2)\, t$.

(d) Let $Z \times Y \times Y \xrightarrow{\psi} \Omega$ and call the projections of

$Z \times Y \times Y \times Y$, $Z \times Y \times Y \times Y \xrightarrow{t_1} Z \times Y$, $Z \times Y \times Y \times Y \xrightarrow{t_2} Y$, $Z \times Y \times Y \times Y \xrightarrow{t_3} Y$.

Then $e_Y < t_2,t_3 > \wedge \psi < t_1,t_2 > \leq \psi < t_1,t_3 >$.

(e) Let $Z \times Y \times Y \xrightarrow{\psi} \Omega$, $X \xrightarrow{f_1} Z \times Y$ and $X \overset{f_2}{\underset{f_3}{\rightrightarrows}} Y$.

Then $e_Y < f_2,f_3 > \wedge \psi < f_1,f_2 > \leq \psi < f_1,f_3 >$.

(f) Let $X \xrightarrow{f_1,\, f_2,\, f_3} Y$. Then $e_Y < f_2,f_3 > \wedge e_Y < f_1,f_2 >$

$\leq e_Y < f_1,f_3 >$.

(g) Let $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ and call the projections $X \times Y \xrightarrow{p} X$, $X \times Y \xrightarrow{q} Y$

Then $e_Y < fp,gp > \leq e_Y < fp,q > \Leftrightarrow e_Y < gp,q >$.

(h) Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, $X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \Omega$, and projections $X \times Y \xrightarrow{p} X$, $X \times Y \xrightarrow{q} Y$

Then $\neg\, \exists_q \Big[\neg\big(e_Y \langle fp, gp \rangle\big) \wedge (\varphi p \Leftrightarrow \psi p)\big)\Big]$

$$\leq \exists_f[\psi] \Leftrightarrow \exists_g[\psi] .$$

Proof:

(a) Note that $q(X,f) = f$ and

$p(X,f) = X$. $\therefore$ $\varphi p(X,f) = \varphi = \varphi \wedge 1_X$

and $(fp,q) = f \times Y$.

$\therefore$ $\exists_f[\ ] = \exists_{q(X,f)}[1_X \wedge \varphi p(X,f)]$

$= \exists_q[\exists_{(X,f)}[1_X \wedge \varphi p(X,f)]]$ by 1.4(c)

$= \exists_q[\exists_{(X,f)}[1_X] \wedge \varphi p]$ by 1.3(b)

$= \exists_q[(\exists_{\Delta_Y}[1_X])(f \times Y) \wedge \varphi p]$ by 1.1(d)(i)

$= \exists_q[e_Y(fp,q) \wedge \varphi p]$ (by definition of e_Y and above note),

which proves (a).

(b) Let r_1, r_2 denote the projections of $Y_1 \times X_2$.

Then $(\exists_{f_1}[\varphi_1]q_1) \wedge (\exists_{f_2}[\varphi_2]q_2)$

$= (\exists_{f_1}[\varphi_1]q_1) \wedge (\exists_{Y_1 \times f_2}[\varphi_2 r_2])$ (by 1.6(c), with $X = X_2$, $Y = Y_2$, $Z = Y_1$, $f = f_2$, $\varphi = \varphi_2$).

$= \exists_{Y_1 \times f_2}\Big[\big(\exists_{f_1}[\varphi_1]q_1\,(Y_1 \times f_2)\big) \wedge (\varphi_2 r_2)\Big]$

(by 1.3(b) with $f = Y_1 \times f_2$, $\varphi = \varphi_2 r_2$ and $\psi = \exists_{f_1}[\varphi_1]q_1$)

$= \exists_{Y_1 \times f_2}[(\exists_{f_1 \times X_2}[\varphi_1 p_1]) \wedge (\varphi_2 r_2)]$ (by 1.6(c) with $X = X_1$, $Y = Y_1$, $Z = X_2$, $f = f_1$, $\varphi = \varphi_1$, and note $r_1 = q_1(Y_1 \times f_2)$.

$= \exists_{Y_1 \times f_2}[\exists_{f_1 \times X_2}[(\varphi_1 p_1) \wedge (\varphi_2 r_2)(f_1 \times X_2)]]$

(by 1.3(b) with $f = f_1 \times X_2$. $\varphi = \varphi_1 p_1$ and $\psi = \varphi_2 r_2$)

$= \exists_{f_1 \times f_2}[(\varphi_1 p_1) \wedge (\varphi_2 r_2)(f_1 \times X_2)]$ (by 1.4(c))

$= \exists_{f_1 \times f_2}[(\varphi_1 p_1) \wedge (\varphi_2 p_2)]$ as required.

(c) Call the projections $Y_1 \times Y_2 \xrightarrow{r_i} Y_i$, and note that $1_{Y_1 \times Y_2} = 1_{Y_1} r_1 = 1_{Y_2} r_2$. Note also that $\Delta_{Y_1} \times \Delta_{Y_2} = t(\Delta_{Y_1 \times Y_2})$ and hence (since t is an isomorphism) that $\Delta_{Y_1 \times Y_2} = t^{-1}(\Delta_{Y_1} \times \Delta_{Y_2})$, where $t^{-1} = \langle s_1, s_3, s_2, s_4 \rangle$ and s_i are the projections of $Y_1 \times Y_1 \times Y_2 \times Y_2$.

$\therefore \quad e_{Y_1 \times Y_2} = \exists_{\Delta_{Y_1 \cdot Y_2}}[1_{Y_1 \times Y_2}]$

$= \exists_{t^{-1}(\Delta_{Y_1} \times \Delta_{Y_2})}[1_{Y_1} r_1 \wedge 1_{Y_2} r_2]$ (by the two notes just mentioned)

$= \exists_{t^{-1}}[\exists_{\Delta_{Y_1} \times \Delta_{Y_2}}[1_{Y_1} r_1 \wedge 1_{Y_2} r_2]]$ (by 1.4(c))

$= \exists_{t^{-1}}[(\exists_{\Delta_{Y_1}}[1_{Y_1}] q_1) \wedge (\exists_{\Delta_{Y_2}}[1_{Y_2}] q_2)]$ (by 1.7(b))

$$= \exists_{t^{-1}}[(e_{Y_1}q_1) \wedge (e_{Y_2}q_2)]$$

$$= \big((e_{Y_1}q_1) \wedge (e_{Y_2}q_2)\big)\, t \quad (\text{by } \underline{1.4(e)}).$$

Thus 1.7(c) is proved.

(d) Note that we shall retain the notation p_i, q_i, t, r_i from (c), and also denote by u_1, u_2 the projections $Z \times Y \times Y \times Z \times Y \times Y \rightrightarrows Z \times Y \times Y$, and by s_i the projections of $Z \times Y \times Z \times Y \times Y \times Y$ to $Z \times Y$, $Z \times Y$, Y and Y respectively.

Now $(\psi < t_1,t_2>) \wedge e_Y < t_2,t_3 > = (\psi < t_1,t_2 >) \wedge (1_{Z\times Y\times Y\times Y}) \wedge (e_Y < t_2,t_3 >)$

$$= (\psi < t_1,t_2 >) \wedge (e_{Z\times Y} < t_1,t_1 >)\,(e_Y < t_2,t_3 >) \quad (\text{by } \underline{1.4(g)})$$

$=$ call that expression (d)(i).

By the usual properties of products,

$$(e_{Z\times Y} < t_1,t_1 >) \wedge (e_Y < t_2,t_3 >) = \big((e_{Z\times Y}q_1) \wedge (e_Y q_2)\big)\, t < t_1,t_2,t_1,t_3 >$$

$$= \big((e_{Z\times Y} < s_1,s_2 >) \wedge (e_Y < s_3,s_4 >)\big) < p_1,p_3,p_2,p_4 > < t_1,t_2,t_1,t_3 >,$$

hence $= e_{Z\times Y\times Y} < t_1,t_2,t_1,t_3 >$ by 1.7(c).

Thus $\underline{d(i)} = \psi < t_1,t_2 > \wedge \big((e_{Z\times Y}q_1 \wedge e_Y q_2)t < t_1,t_2,t_1,t_3 >\big)$

$$= \psi < t_1,t_2 > \wedge (e_{Z\times Y\times Y} < t_1,t_2,t_1,t_3 >) \quad (\text{by } \underline{1.7(c)})$$

$$= (\psi u_1 < t_1,t_2,t_1,t_3 >) \wedge (e_{Z\times Y\times Y} < t_1,t_2,t_1,t_3 >)$$

$$= (\forall u_1 \wedge e_{Z\times Y\times Y}) < t_1, t_2, t_1, t_3 >$$

$$\leq (\forall u_2) < t_1, t_2, t_1, t_3 > \quad \text{(by } \underline{1.4(m)}\text{)} .$$

$$= \forall < t_1, t_3 > \quad \text{as required,}$$

thus completing the proof of (d).

(e) Compose inequality 1.7(d) with the product map

$$X \xrightarrow{< f_1, f_2, f_3 >} Z \times Y \times Y \times Y \quad \text{to obtain (e).}$$

(f) This is a special case of (e), with $Z = I$ and $\forall = e_Y$.

(g) For convenience call $A = e_Y < fp, gp >$, $B = e_Y < fp, q >$, and $C = e_Y < gp, q >$. Then by (f) we obtain $A \wedge B \leq C$ and $A \wedge C \leq B$.

By adjointness of $\Rightarrow$ and $\wedge$, the former inequality gives $A \leq B \Rightarrow C$ and the latter, $A \leq C \Rightarrow B$.

Hence $A \leq (B \Rightarrow C) \wedge (C \Rightarrow B) = B \Leftrightarrow C$, which is the required result.

(h) Recall first that if $\alpha, \beta, \gamma, \delta$ are elements of any Heyting algebra, then

(h)(i) $\big((\alpha \Leftrightarrow \beta) \wedge (\gamma \Leftrightarrow \delta)\big) \leq \big((\alpha \wedge \gamma) \Leftrightarrow (\beta \wedge \delta)\big)$.

Call $A = (e_Y < fp, gp >) \wedge (\varphi p \Leftrightarrow \forall p)$

Then $A \leq (e_Y < fp, q > \Leftrightarrow e_Y < gp, q >) \wedge (\varphi p \Leftrightarrow \forall p)$ (by 1.7(g))

$$\leq \big((e_Y < fp, q >) \wedge \varphi p\big) \Leftrightarrow \big((e_Y < gp, q >) \wedge \forall p\big) \quad \text{(by (h) (i).)}$$

Call $\varphi_1 = (e_Y < fp,q >) \wedge \varphi p$ and $\varphi_2 = (e_Y < gp,q >) \wedge \psi p$.

$\therefore$ We have the inequality $A \leq \varphi_1 \Leftrightarrow \varphi_2$. Now since 7 is order reversing and $\exists$ order preserving, $\therefore$ $7\, \exists_q [7(\ )]$ is order preserving.

$$\therefore\ 7\, \exists_q [7A] \leq 7\, \exists_q [7\, (\varphi_1 \Leftrightarrow \varphi_2\,)]$$

$$\leq \exists_q [\varphi_1] \Leftrightarrow \exists_q [\varphi_2] \quad \text{(by 1.4(j))}$$

$$= \exists_f [\varphi] \Leftrightarrow \exists_q [\psi] \quad \text{(by 1.7(a)).}$$

This completes the proof of 1.7 .

1.8 Definition . A prelogical category $\mathcal{C}$ is called fair iff every map $X \xrightarrow{!_X} I$ is epi in $\mathcal{C}$.

Remarks: (a) We note that by 1.4(h) this implies $\exists_{!_X} [1_X] = 1$ for every X, but it does not imply that there are any points $I \xrightarrow{k} X$ in $\mathcal{C}$ unless C is also rich.

(b) Moreover if C is fair, then all projection maps in C are epi. For, let $X \times Y \xrightarrow{p} X$ and $X \times Y \xrightarrow{q} Y$ be projections in C. Because $(\ )q$ is a Heyting homomorphism, $1_{X \times Y} = 1_Y q$. Applying 1.4(h) twice and 1.6(c) once we have

$$\exists_p [1_{X \times Y}] = \exists_p [1_Y q] = (\exists_{!_Y} [1_Y])!_X = 1\, !_X = 1_X ,$$ and so p is epi.

CHAPTER 2: TWO IMPORTANT EXAMPLES

2.1 Proposition. The category $\underline{S}$ of sets is an intuitionistic logical category with 2 taken as the Heyting algebra object.

Proof and Description: (a) $\underline{S}$ has cartesian products; the terminal object is the singleton set, usually denoted 1.

(b) The set 2 is a Boolean algebra object, a fortiori, Heyting algebra object, in the following well-known way:

(Let $X \in |\underline{S}|$ and let $\varphi^{\#}$ denote the subset of X whose characteristic map is $X \xrightarrow{\varphi} 2$).

The map $X \xrightarrow{0_X} \{0,1\} = 2$ is the characteristic map of the empty subset $\emptyset \subseteq X$; i.e. $0_X(x) = 0$ for all $x \in X$.

1_X is the characteristic map of $X \subseteq X$; i.e. $1_X(x) = 1$ for all $x \in X$.

Given $\varphi, \psi \in \underline{S}(X,2)$, $\varphi \vee \psi$ is defined by:

$$(\varphi \vee \psi)(x) = \begin{cases} 1 & \text{if } \varphi(x) = 1 \text{ or } \psi(x) = 1 \\ 0 & \text{otherwise} \end{cases}$$

i.e. $= \varphi(x) \vee \psi(x)$ using $\vee$ in the Boolean algebra 2.

Equivalently, $(\varphi \vee \psi)^{\#} = \varphi^{\#} \cup \psi^{\#}$.

$\varphi \wedge \psi$ is defined by $(\varphi \wedge \psi)(x) = 1$ iff $\varphi(x) = 1$ and $\psi(x) = 1$, i.e. $(\varphi \wedge \psi)(x) = \varphi(x) \wedge \psi(x)$, i.e. $(\varphi \wedge \psi)^{\#} \cap \psi^{\#}$.

$7(\varphi)$ is given by $(7\varphi)(x) = 7(\varphi(x)) = 1$ iff $\varphi(x) = 0$; $\varphi \Rightarrow \psi$ is similarly $(\varphi \Rightarrow \psi)(x) = \psi(x) \Rightarrow \psi(x)$.

It follows easily that each $\underline{S}(f,2)$ will preserve the above Heyting structure: Let $\psi_1, \psi_2 : Y \longrightarrow 2$ and $X \xrightarrow{f} Y$. Then, for example,

$$((\psi_1 \vee \psi_2)f)(x) = (\psi_1 \vee \psi_2)(f(x)) = \psi_1(f(x)) \vee \psi_2(f(x))$$
$$= (\psi_1 f)(x) \vee (\psi_2 f)(x) \quad \text{for every } x \in X .$$

The other operations are checked similarly.

($\underline{c}$) Recall from the introduction that an appropriate quantification is given by the direct image map on corresponding subsets, i.e. $(\exists_f[\varphi])^{\#} = f(\varphi^{\#})$.

Equivalently, $(\exists_f[\varphi])(y) = 1$ in $\underline{S}$ iff there exists a pre-image $x \in f^{-1}(y)$ such that $\varphi(x) = 1$.

($\underline{d}$) The Beck conditions hold for every pullback in $\underline{S}$;

i.e. if
$$\begin{array}{ccc} D & \xrightarrow{h} & A \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$
is any pullback in $\underline{S}$ and $B \xrightarrow{\varphi} 2$ any map, then $(\exists_g[\varphi])f = \exists_h[\varphi k]$

This is verified by ordinary diagram chasing: Recall that the pullback of f and g is the equalizer of

$$A \times B \underset{gp_B}{\overset{fp_A}{\rightrightarrows}} C .$$ In S this is given by

$D = \{(x,y) \in A \times B \mid f(x) = g(y)\}$, e the inclusion $D \overset{e}{\hookrightarrow} A \times B$,

$h = p_A e$ and $k = p_B e$. Thus in particular $h(x,y) = x$ and $k(x,y) = y$ for all $(x,y) \in D$.

Now suppose for some $a \in A$ that $((\exists_g[\varphi])f)(a) = 1$,

$\therefore (\exists_g[\varphi])(f(a)) = 1$. $\therefore$ By the description of $\exists$ in $\underline{S}$, there exists a pre-image $b \in g^{-1}(f(a))$ such that $\varphi(b) = 1$. Then $g(b) = f(a)$ and therefore $(a,b) \in D$ and $b = k(a,b)$. $\therefore 1 = \varphi(b) = \varphi k(a,b)$ and $h(a,b) = a$.

Thus there exists a pre-image $(a,b) \in h^{-1}(a)$ such that $(\varphi k)(a,b) = 1$; i.e. precisely $(\exists_h[\varphi k])(a) = 1$.

We have thus shown that $((\exists_g[\varphi])f)^{\#} \subseteq (\exists_h[\varphi k])^{\#}$. The reverse inclusion can be verified similarly.

(e) Properties (e)(i) and (ii) are satisfied trivially since e_X in $\underline{S}$ is just the predicate of equality.

(f) For each $X \xrightarrow{f} Y$ in $\underline{S}$, $\forall_f[\ ]$ is given by

$[(\forall_f[\varphi])^{\#} = \{y \in Y | y = f(x) \text{ implies } x \in \varphi^{\#}\}$; equivalently,

$(\forall_f[\varphi])(y) = 1$ iff for all pre-images $x \in f^{-1}(y)$, $\varphi(x) = 1$.

As noted in the introduction, this map is right adjoint to $(\ )f$ as required.

Thus $\underline{S}$ is intuitionistic logical.

We remark that the extensionality condition takes a particularly simple form in $\underline{S}$: It states that any two maps are equal if they coincide pointwise for all points. To be explicit, let $f,g: X \to Y$ in $\underline{S}$ be such that $\forall_{!_X}[e_Y < f,g>] = 1: 1 \to 2$. By (f) above this says that for every pre-image $x \in (!_X)^{-1}(i)$ (that is, simply every $x \in X$), $e_Y < f(x), g(x) > = 1$, i.e. that $f(x) = g(x)$ for all $x \in X$.

2.2 Proposition. The category $\underline{S}$ of sets can be made into a prelogical category with any complete Heyting algebra taken as the Heyting algebra object.

Proof and Description: (a) Take cartesian products as before.

(b) Any Heyting algebra can be made into a Heyting algebra object in $\underline{S}$ by inheritance, as follows: Let Ω be the Heyting algebra and let $0, 1, \vee, \wedge, \Rightarrow, 7$ denote its operations. Let $X \in |\underline{S}|$ and $\varphi, \psi : X \to \Omega$.

Define $X \xrightarrow{0_X} \Omega$ by $0_X(x) = 0$ for all $x \in X$;

$X \xrightarrow{1_X} \Omega$ by $1_X(x) = 1$ for all $x \in X$;

$X \xrightarrow{\varphi \wedge \psi} \Omega$ by $(\varphi \wedge \psi)(x) = \varphi(x) \vee \psi(x)$ for all $x \in X$;

$\varphi \wedge \psi$ by $(\varphi \wedge \psi)(x) = \varphi(x) \wedge \psi(x)$;

$\varphi \Rightarrow \psi$ by $(\varphi \Rightarrow \psi)(x) = \varphi(x) \Rightarrow \psi(x)$;

and 7φ by $(7\varphi)(x) = 7(\varphi(x))$.

As they did in the case where $\Omega = 2$, these operations make $\underline{S}(X, \Omega)$ into a Heyting algebra.

Likewise, given $X \xrightarrow{f} Y$, we will have exactly as before that $(\varphi \vee \psi)f = \varphi f \vee \psi f$, $(\varphi \wedge \psi)f = \varphi f \wedge \psi f$, and so on; that is, $\underline{S}(f, \Omega)$ preserves the Heyting structure.

(c) Let $X \xrightarrow{f} Y$, $X \xrightarrow{\varphi} \Omega$, and define $Y \xrightarrow{\exists_f[\varphi]} \Omega$ by $(\exists_f[\varphi])(y) = \sup\{\varphi(x) \mid x \in X \text{ and } f(x) = y\}$ for each $y \in Y$.

We shall now verify that $\exists_f[\ ]$ is adjoint to $(\ )f$, i.e. that for every $Y \xrightarrow{\psi} \Omega$ in $\underline{S}$,

$$(\exists_f[\varphi])(y) \leq \psi(y) \text{ for all } y \in Y \text{ iff } \varphi(x) \leq (\psi f)(x) \text{ for all } x \in X.$$

Suppose that for all $y \in Y$, $(\exists_f[\varphi])(y) \leq \psi(y)$. By definition of a supremum, this says that for all $y \in Y$,

$$\varphi(x) \leq \psi(y) \text{ for each } x \in f^{-1}(y).$$

But $X = \text{dom } f$; $\therefore$ every $x \in X$ lies in $f^{-1}(y)$ for some $y = f(x)$.

Thus for each $x \in X$, $\varphi(x) \leq \psi(f(x))$.

Conversely suppose that for all $x \in X$, $\varphi(x) \leq (\psi f)(x)$; i.e. for all $x \in X$, $\varphi(x) \leq \psi(f(x))$. i.e. for each $y \in \text{rge}(f)$, $\varphi(x) \leq \psi(y)$ for all $x \in f^{-1}(y)$. Therefore, for each $y \in \text{rge}(f)$, $\sup_{x \in f^{-1}(y)} \varphi(x) \leq \psi(y)$; i.e. for each $y \in \text{rge}(f)$, $(\exists_f[\varphi])(y) \leq \psi(y)$.

Now for each $y \in Y\text{-rge}(f)$, $(\exists_f[\varphi])(y) = \sup_{x \in f^{-1}(y)} \varphi(x) = \sup \emptyset = 0$

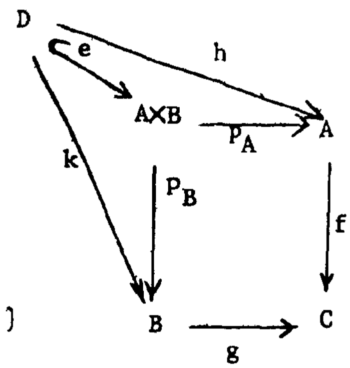
Hence certainly for each $y \in Y\text{-rge}(f)$, $(\exists_f[\varphi])(y) \leq \psi(y)$.

Thus for all $y \in Y$, $(\exists_f[\varphi])(y) \leq \psi(y)$.

Therefore $\exists_f[\ ]$ is a left adjoint as required.

We note in particular that $(\exists_f[\varphi])(y) = 1$ iff there exists some $x \in X$ such that $\varphi(x) = 1$ and $f(x) = y$. Thus in the case where $\Omega = 2$, 2.2(c) defines the same operation as the direct image map in 2.1(c).

(d) The Beck conditions hold for every pullback.



i.e. Let $(h,k) = pb(f,g)$ be any pullback and $B \xrightarrow{\varphi} \Omega$ any map;
Then $\exists_h[\varphi k] = (\exists_g[\varphi])f$.

For, recall that $D = \{(x,y) \in A \times B \mid f(x) = g(y)\}$ and that for given $a_o \in A$,
$h^{-1}(a_o) = \{(a_o,b) \in D\} = \{(a_o,b) \mid b \in B \text{ and } f(a_o) = g(b)\}$.
Fix $a \in A$. Then $(\exists_h[\varphi k])(a) = \sup\{(\varphi k)(a,b) \mid (a,b) \in h^{-1}(a)\}$
$= \sup\{\varphi(b) \mid b \in B \text{ and } f(a) = g(b)\}$. [Call that equation <u>(d)(i)</u>.]
Moreover $((\exists_g[\varphi])f)(a) = (\exists_g[\varphi])(f(a)) = \sup\{\varphi(b) \mid b \in g^{-1}(f(a))\}$

$= \sup\{\varphi(b) \mid b \in B \text{ and } f(a) = g(b)\}$ = R.H.S. <u>(d)(i).</u>

Thus $(\exists_h[\varphi k])(a) = (\exists_g[\varphi]f)(a)$, as required.

(<u>e</u>) For any $Y \in |\underline{S}|$ and any pair $(x,z) \in Y \times Y$,

$e_Y(x,z) = (\exists_{\Delta_Y}[1_Y])(x,z) = \sup\{1_Y(y) \mid y \in \Delta_Y^{-1}(x,z)\}$.

If $x \neq z$ this gives $\sup \emptyset = 0$; if $x = z$ it gives $\sup\{1\} = 1$.
Thus the equality predicate resulting from the quantification defined by <u>2.2(c)</u> is simple equality. As has been noted in <u>2.1(e)</u>, this predicate satisfies conditions <u>1.1(e)(i)</u> and <u>(ii).</u>

<u>2.3</u> <u>Remark.</u> Given any partially ordered set P, the topos $\underline{S}^P$ is an intuitionistic logical category with Ω_p, the subobject classifier, taken as the Heyting algebra object.

The full proof will not be given here, as it makes extensive use of the theory of toposes which is nowhere used in the present work. However, the interested reader may find the details in Kock and Wraith [6], using the following description of $\underline{E}$:

It is known that $\underline{P}$ is embedded in $\underline{S}^{P}$ by

$$p \longmapsto \underline{P}(p,\ ), \quad p \xrightarrow{\eta} q \longmapsto \underline{P}(\eta,\ ) \ ;$$

this is usually called the contravariant Yoneda representation of $\underline{P}$. Also, given any $X \in \underline{S}^{P}$ and $p \in |\underline{P}|$, one has the Yoneda correspondence (which we shall denote $\ulcorner\ \urcorner$) between elements $x \in X(p)$ and natural transformations $\underline{P}(p,\ ) \xrightarrow{\ulcorner x \urcorner} X$, defined as follows:

For $x \in X(p)$ and $q \in |\underline{P}|$, $\underline{P}(p,q) \xrightarrow{\ulcorner x \urcorner_q} X(q)$ is given by

$$\ulcorner x \urcorner_q \ (p \xrightarrow{\eta} q) = (X(\eta))(x) \ .$$

This correspondence is a set isomorphism ; its inverse (denoted $\llcorner\ \lrcorner$) is: Let $\underline{P}(p,\ ) \xrightarrow{\nu} X$; then $\llcorner \nu \lrcorner = \nu_p(id_p) \in X(p)$.

A topos structure, in the sense of [6], page 5, can be put on $\underline{S}^{P}$ as follows:

As is well known, $\underline{S}^{P}$ has finite limits and exponentation .

The object Ω_P of $\underline{S}^{P}$ is defined by

$$\Omega_P(p) = \{\text{subobjects } R \rightarrowtail \underline{P}(p,\ ) \text{ in } \underline{S}^{P}\ \}$$

and $\Omega_P(p \xrightarrow{\eta} q) = \eta^*$ given by $\eta^*(R \overset{j}{\rightarrowtail} \underline{P}(p,\ )) = \text{pullback}\ (j, \underline{P}(\eta,\ ))$ in $\underline{S}^{P}$.

Define the map $I \xrightarrow{\text{TRUE}} \Omega_P$ in $\underline{S}^P$ by :

$(\text{TRUE})_p : 1 \longrightarrow \Omega_P(p)$ sends $i \in 1$ to the identity transformation $P(p, \ ) \rightarrowtail\!\!\twoheadrightarrow P(p, \ )$.

For any subobject $R \overset{j}{\rightarrowtail} X$ in $\underline{S}^P$, define $X \xrightarrow{\chi^j} \Omega_P$, the characteristic map of j, as:

$$X(p) \xrightarrow{\chi^j_p} \Omega_P(p) \quad \text{is given by} \quad \rightarrowtail \xrightarrow{\chi^j_p(x)} \underline{P}(p, \ ) = (\bar{x})^*(j)$$

for every $x \in X(p)$, i.e. by the pullback

$$\begin{array}{ccc} \cdot & & \cdot \\ \big\downarrow{\scriptstyle \chi^j_p(x)} & & \big\downarrow{\scriptstyle j} \\ \underline{P}(p) & \xrightarrow{\bar{x}} & X \end{array} \qquad \text{in } \underline{S}^P$$

It can be verified (cf. [10], or [8] and [9]) that for every $R \overset{j}{\rightarrowtail} X$ in $\underline{S}^P$ and χ^j thus defined, the diagram shown is a pullback in $\underline{S}^P$, i.e. that Ω_P is a subobject classifier of $\underline{S}^P$, and hence $\underline{S}^P$ is a topos.

$$\begin{array}{ccc} R & \xrightarrow{!_R} & I \\ \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle \text{TRUE}} \\ X & \xrightarrow[\chi^j]{} & \Omega_P \end{array}$$

Taking $\underline{S}^P$, Ω_P as the $\underline{E}, \Omega$ of [6], it can now be concluded that $\underline{S}^P, \Omega_P$ is an intuitionistic logical category. The Heyting algebra structure of $\underline{S}^P(X, \Omega_P)$ is defined in [6], pages 28-30. (Notice that the notion of Heyting algebra object as discussed in [6] pages 34-35 is stronger than ours.) The adjoints $\exists_f$ and $\forall_f$ are developed in [6] 1.11, 1.12 and pages 30-32. The Beck conditions on $\exists_f$ follow from [6] 1.36.

CHAPTER 3: LOGICAL FUNCTORS AND PREMODELS

3.1 Definition. Let $\underline{C}$ and $\underline{C}'$ be intuitionistic prelogical categories with Heyting-algebra objects Ω, Ω' respectively. We say that a functor $F: \underline{C} \longrightarrow \underline{C}'$ is prelogical iff:

(a) F preserves finite products (and hence in particular $F(I) = I'$);

(b) $F(\Omega) \cong \Omega'$;

(c) For every $X \in Ob(\underline{C})$, F together with the isomorphism in (b) preserve 0_X, 1_X, $\vee$ and $\wedge$ of $\underline{C}(X,\Omega)$;

(d) F preserves existential quantification.

3.2 Definition. A prelogical functor $\underline{C} \xrightarrow{F} \underline{C}'$ is called an extension of $\underline{C}$ iff F is bijective on objects. In such a case we will often abuse the terminology slightly and say that $\underline{C}'$ is an extension of $\underline{C}$.
$\underline{C} \xrightarrow{F} \underline{C}'$ is called an enriching extension of $\underline{C}$ iff, for every $X \xrightarrow{\varphi} \Omega$ in $\underline{C}$ such that $\exists_X [\varphi] = 1$, there is some $I \xrightarrow{x} FX$ in $\underline{C}'$ such that $(F\varphi)x = 1$.
An extension $\underline{C} \xrightarrow{F} \underline{C}'$ is said to be conservative iff $F(\alpha) = 1$ in $\underline{C}'$ implies $\alpha = 1$ in $\underline{C}$.

Remarks to Definition 3.2 : (a) In what follows we shall often think of an extension as having $Ob(\underline{C}') = Ob(\underline{C})$ and F the identity functor on objects. This involves no loss of generality since, because F must be logic-preserving and bijective on objects, the class of such extensions will contain isomorphic copies of all other extensions. Accordingly, given $\underline{C} \xrightarrow{F} \underline{C}'$ and $I \xrightarrow{\beta} \Omega$ in $\underline{C}$, we will often refer to $F(\beta)$ as β. Analogously in non-categorical logic, one usually defines an extension T' of a theory T to be a theory T' whose language $L(T')$ includes $L(T)$. It would add nothing fundamentally new to modify this definition to include theories T'' such that there is a logic-preserving functor: $L(T) \longrightarrow L(T'')$ which is bijective on types, and the usual definition is more convenient because it enables one to speak of the "same" sentence or formula in both theories.

(b) The definition here given of a conservative extension is also motivated by the usage in ordinary logic: $T' \geq T$ is a conservative extension iff no nontheorems of T become theorems in T'.

3.3 Remark: Any composite of prelogical functors is prelogical; A composite of extensions is an extension, and a composite of conservative extensions is conservative. This is clear because all the required properties are equational, and hence preserved by composition.

3.4 Definition. Let $\underline{C}$ be an intuitionistic prelogical category and Ω' any Heyting algebra. A functor $F: \underline{C} \longrightarrow \underline{S}$ is called a premodel for $\underline{C}$ iff F is prelogical, where $\underline{S}$ is made prelogical with Ω' according to 2.2.

In particular, if $\Omega' = 2$ we call F a semantic premodel, following the usage of [1].

3.5 Definition. Let $\underline{C} : \underline{C}'$ be intuitionistic logical categories. A functor $F: \underline{C} \longrightarrow \underline{C}'$ is said to be intuitionistic logical (abbreviated int.log.) iff :

(a) F is prelogical ;

(b) For every $X \in Ob(\underline{C})$, F together with the isomorphism in 3.1(b) preserves $\Rightarrow$ and $\neg$ of $\underline{C}(X,\Omega)$;

(c) F preserves universal quantification.

3.6 Definition. Let $\mathcal{C}$ be a prelogical category and let $I \xrightarrow{\alpha} \Omega$ be any sentence in $\mathcal{C}$.

(a) $\mathcal{C}$ is said to be consistent if $0 \neq 1 : I \to \Omega$.

(b) $\mathcal{C}$ is α-consistent if it is consistent and $\alpha \neq 1$,

(c) $\mathcal{C}$ is maximally consistent if it is consistent and $\mathcal{C}(I,\Omega) = \{0,1\}$.

(d) $\mathcal{C}$ is prime iff, for any $I \overset{\varphi}{\underset{\psi}{\rightrightarrows}} \Omega$, if $\varphi \vee \psi = 1$ then either $\varphi = 1$ or $\psi = 1$.

(e) $\mathcal{C}$ is strongly prime iff for any $X \in \mathrm{Ob}\,\mathcal{C}$ and any $X \overset{\varphi}{\underset{\psi}{\rightrightarrows}} \Omega$, if $\varphi \vee \psi = 1_X$ then $\varphi = 1_X$ or $\psi = 1_X$.

(f) $\mathcal{C}$ is called rich iff the identity functor: $\mathcal{C} \to \mathcal{C}$ is an enriching extension of $\mathcal{C}$, i.e. iff, for every $X \xrightarrow{\varphi} \Omega$ in $\mathcal{C}$ such that $\exists_X [\varphi] = 1$, there is $I \xrightarrow{x} X$ in $\mathcal{C}$ such that $\varphi x = 1$.

(g) $\mathcal{C}$ is called saturated if it is consistent, prime and rich.

(h) $\mathcal{C}$ is called α-saturated iff it is α-consistent, prime and rich.

3.7 (a) Proposition. If a prelogical category $\mathcal{C}$ is maximally consistent, then $\mathcal{C}$ is prime.

Proof: Let $\varphi, \psi : I \to \Omega$ be such that $\varphi \vee \psi = 1$. Since $\mathcal{C}$ is consistent $((\varphi \vee \psi) \Rightarrow 0) = 0 \neq 1$. Using the theorem for Heyting algebras (cf[1]) that $(\beta \Rightarrow \alpha) \wedge (\gamma \Rightarrow \alpha) \leq (\beta \vee \gamma) \Rightarrow \alpha$, we have

(i) $(\varphi \Rightarrow 0) \wedge (\psi \Rightarrow 0) \leq (\varphi \wedge \psi) \Rightarrow 0 = 0$. This forces either $(\varphi \Rightarrow 0) \neq 1$ or $(\psi \Rightarrow 0) \neq 1$, since otherwise (i) contradicts consistency. Equivalently, since $\mathcal{C}$ is maximally consistent, either

$$(\varphi \Rightarrow 0) = 0 \quad \text{or} \quad (\psi \Rightarrow 0) = 0 .$$

By definition of $\Rightarrow$ this says that either $\varphi = 1$ or $\psi = 1$.

3.7.(b) Corollary. Maximally consistent and rich categories are saturated.

3.8 Theorem: Let $\mathcal{C}$ be a prelogical category. Then $\mathcal{C}$ is maximally consistent and rich iff $\mathcal{C}(I, \ ): \mathcal{C} \to \mathcal{S}$ is a semantic premodel.

Proof: ($\Rightarrow$) As is well known, $\mathcal{C}(I, \ )$ is a functor and preserves finite products, since

$$\mathcal{C}(I, X \times Y) \cong \mathcal{C}(I,X) \times \mathcal{C}(I,Y) \qquad \text{for all } X,Y \in \mathrm{Ob}(\mathcal{C}) ,$$

$$\text{and } \langle f,g \rangle (\ ) = \langle f(\ ), g(\ ) \rangle \text{ for all } Z \xrightarrow{f} X,\ Z \xrightarrow{g} Y,$$

by definition of products.

In particular, $\mathcal{C}(I,I) = 1$ and $\mathcal{C}(I, !_X) = !_{\mathcal{C}(I,X)}$ for all X.

Since $\mathcal{C}$ is maximally consistent, $\mathcal{C}(I,\Omega) \cong 2$; i.e. $\mathcal{C}(I, \ )$ preserves Ω .

For every $I \xrightarrow{x} X$ and $X \xrightarrow{\varphi} \Omega$ in $\mathcal{C}$,

$$\mathcal{C}(I,\varphi)\ (x) = \mathcal{C}(x,\Omega)\ (\varphi) = \varphi x$$

and $\mathcal{C}(x,\Omega)$ is a Heyting homomorphism by 1.1(b).

i.e. $(\varphi_1 \vee \varphi_2)x = \varphi_1 x \vee \varphi_2 x$,

$(\varphi_1 \Rightarrow \varphi_2)x = \varphi_1 x \Rightarrow \varphi_2 x$,

$(1_X)x = 1$, and so on;

i.e. $\mathcal{C}(I, \ )$ preserves all the Heyting operations of each $\mathcal{C}(X,\Omega)$.

We shall make use of the reduction formula 1.7(a) to show that $\underline{C}(I,\ )$ preserves existential quantification.

First let $X \in Ob(\underline{C})$ and $I \overset{x_1}{\underset{x_2}{\rightrightarrows}} X$ in $\underline{C}$. By 1.1(e) (i) and 1.4(g),

$$e_X < x_1, x_2 > = 1 \text{ iff } x_1 = x_2 .$$

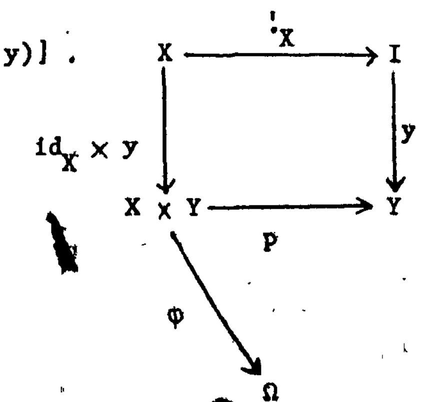
i.e exactly, $e_X(\ )$ is the equality predicate (in $\underline{S}$) on $\underline{C}(I,X) \times \underline{C}(I,X)$.

Thus $\underline{C}(I,\ )$ preserves equality.

Let $X \times Y \xrightarrow{p} Y$ and $X \times Y \xrightarrow{\varphi} \Omega$ in $\underline{C}$. We wish to show that $(\exists_p[\varphi])(\ ) = \exists_{p(\ )}[\varphi(\ )]$ as maps: $\underline{C}(I,Y) \to 2$.

Let $I \xrightarrow{y} Y$ in $\underline{C}$ and suppose that $(\exists_p[\varphi])y = 1$. By 1.6(b) (i),

$$1 = (\exists_p[\varphi])y = \exists_{!_X}[\varphi(id_X \times y)] .$$



Since $\underline{C}$ is rich, the latter implies that there is $I \xrightarrow{x} X$ in $\underline{C}$ such that

$$\varphi(id_X \times y)x = 1 \text{ in } \underline{C} .$$

i.e. $\varphi < x,y > = 1$.

i.e. there is a preimage (x,y) for y under $p(\ )$ which satisfies $\varphi(\ )$.

i.e. precisely that $(\exists_{p(\ )}[\varphi(\ )])(y) = 1$ in $\underline{S}$.

Conversely suppose $(\exists_{p(\ )}[\varphi(\ )])(y) = 1$, i.e. there is such an x.

Note that $\varphi \leq (\exists_p[\varphi])p$ by 1.3(a). Then

$$1 = \varphi < x,y > \leq (\exists_p[\varphi])\ p\ < x,y > = (\exists_p[\varphi])\, y$$

and $\therefore (\exists_p[\varphi])\, y = 1.$

Thus $\mathcal{C}(I,\ )$ preserves quantification along projections and therefore by 1.7(a) preserves all existential quantification.

This shows that $\mathcal{C}(I,\ )$ is a semantic premodel for $\mathcal{C}$.

(<=) Conversely assume that $\mathcal{C}(I,\ )$ is a semantic premodel.
Since $\mathcal{C}(I,\Omega) \cong 2$, $\therefore \mathcal{C}$ is maximally consistent.
Let $\exists_{!X}[\varphi] = 1$ in $\mathcal{C}$.
Since $\mathcal{C}(I,\ )$ preserves quantification, $\therefore \exists_{!\mathcal{C}(I,X)}[\varphi(\ )] = 1$ in $\mathcal{S}$.
i.e. there is an $x \in \mathcal{C}(I,X)$ such that $\varphi x = 1$;
i.e. precisely, $\mathcal{C}$ is rich.
This completes the proof of 3.8.

3.9 Remark: For categories which are not maximally consistent we have the following weaker property for the canonical functor: If $\mathcal{C}$ is any rich prelogical category, then $\mathcal{C}(I,\ )$ preserves products and Heyting operations, and furthermore

$$\mathcal{C}(I,\ )(\exists_f[\varphi]) = 1 \quad \text{iff} \quad \exists_{\mathcal{C}(I,f)}[\mathcal{C}(I,\varphi)] = 1 .$$

This fact follows directly from the proof of 3.8 . The preservation of products and Heyting operations was proved with no reference to maximality. Likewise $\mathcal{C}(I,\ )$ preserves equality: it has been noted in 2.2 that the equality predicate in $\mathcal{S}$ is always the usual, two-valued equality, even when $\mathcal{S}$ is made prelogical with an Ω' larger than 2. The proof that $(\exists_p[\varphi])\ (\ ) = 1$ iff $\exists_{p(\ )}[\varphi\ (\ )] = 1$ goes through for any rich category; the only use of maximal consistency was to conclude from this that $(\exists_p[\varphi])\ (\ ) = \exists_{p(\ )}[\varphi\ (\ )]$ for all values.

CHAPTER 4: COMPLETENESS THEOREM FOR PRELOGICAL CATEGORIES

The main work of this chapter is to extend appropriate α-consistent prelogical categories to maximally consistent α-saturated categories. In order to do this we must first develop three general kinds of operations which can be performed on prelogical categories: extension by constants, reduction modulo an equivalence relation, and limit of a chain of extensions.

As the terminology suggests, one may think of the construction in 4.0 - 4.4 as representing the addition of new constants (in the logical sense) to the language of a theory. We emphasize here that no new axiom schemata are added to the theory; however one does acquire some new axioms, for example by substitution of the new constants into axiom schemata already present. In the present categorical setting, the procedure is more complicated but the effect similar. The definition 4.1 (following [2], 2.7) is designed to ensure that the new "constants" will indeed behave like constants of a theory. For example, the above note regarding substitution into existing axiom schemata becomes the following condition: For any new constant $I \xrightarrow{k} X$ added, $1_X k = 1$.

4.0 **Definition**. Let $\underline{C}$ be a prelogical category, and $\underline{K}$ any category. $\underline{K}$ is called a category of constants for $\underline{C}$ iff

(a) $Ob(\underline{K}) = Ob(\underline{C})$.

(b) For each $X \neq I$ and every Y, $\underline{K}(X,Y) = \emptyset$.

4.1 Definition. Let $\underline{C}$ be a small prelogical category and $\underline{K}$ a category of constants for $\underline{C}$. We construct the extension of $\underline{C}$ by the constants in $\underline{K}$ as follows:

Let K be the set $K = \bigcup_{X \in |\underline{K}|} \underline{K}(I,X)$

Let $\underline{S}_o$ denote the category of finite cardinals, whose morphisms are all set mappings.

4.1(a) Define $\underline{K}^{\#} = \underline{S}_o|_K$; i.e. $\underline{K}^{\#}$ is the category of finite sequences in K.

So an object of $\underline{K}^{\#}$ is a pair (n,c) with $n \in Ob(\underline{S}_o)$ and c a map: $m \to K$ in $\underline{S}_o$, and a morphism $(n,c) \xrightarrow{s} (m,d)$ in $\underline{K}^{\#}$ is a set mapping $n \xrightarrow{s} m$ such that $ds = c$.

4.1(b) Define the (contravariant) functor $A: \underline{K}^{\#} \to \underline{C}$ by

$A(n \xrightarrow{c} K) = \partial^1(c(0)) \times \partial^1(c(1)) \times \dots \times \partial^1(c(n-1))$,

where as usual ∂^1 denotes the co-domain (in $\underline{C}$) ;

$A(c \xrightarrow{s} d)$ is given by $(p_k)(A(s)) = q_{s(k)}$ for each $k \in n$,

where p_k denotes the k'th projection of A(c) and q_j denotes the j'th projection of A(d).

i.e. A(s) is the product map $< q_{s(0)}, q_{s(1)}, \dots, q_{s(n-1)} >$.

4.1(c) We now define $\underline{C}[\underline{K}]$, the extension of $\underline{C}$ by the constants in $\underline{K}$, by

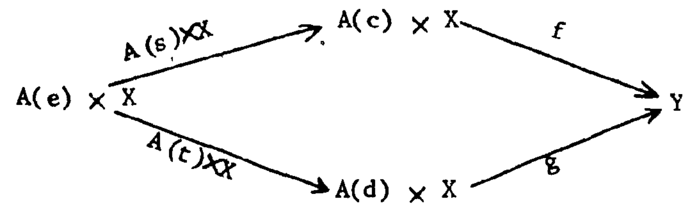
(i) $Ob(\underline{C}[\underline{K}]) = Ob(\underline{C})$

(ii) For given fixed $X,Y \in Ob(\underline{C})$, call

$M_{X,Y}$ = {all pairs ⟦ f,c ⟧ , where $c \in \underline{K}^{\#}$ and f is a map $A(c) \times X \xrightarrow{f} Y$ in $\underline{C}$ } .

Define the equivalence relation R on $M_{X,Y}$ by

(⟦ f,c ⟧, ⟦ g,d ⟧) $\in$ R iff there exist maps $c \xrightarrow{s} e$ and $d \xrightarrow{t} e$ in $\underline{K}^{\#}$ such that the diagram on the next page commutes.



Now define $\underline{C}[\underline{K}]\ (X,Y) = M_{X,Y}/R$, and denote the equivalence class of $\{\!|\, f,c \,|\!\}$ under R by $\{\!|\, f|c \,|\!\}$.

4.2 Proposition. (a) A as defined above is a contravariant functor.

(b) A is faithful and takes finite coproducts to products.

(c) $\underline{C}[\underline{K}]$ as defined above is a category

Moreover $\underline{K}^{\#}$, A and $\underline{C}[\underline{K}]$ satisfy the following properties:

(d) If c in $\underline{K}^{\#}$ is monic and $c \neq \emptyset$ (i.e. $n \neq \emptyset$), then for every pair of maps $c \overset{s}{\underset{t}{\rightrightarrows}} d$ in $\underline{K}^{\#}$ there exists $d \xrightarrow{u} c$ in $\underline{K}^{\#}$, such that $us = ut = id_c$.

(e) If $\{\!|\, f|c \,|\!\} = \{\!|\, g|c \,|\!\}$ with c monic, then $f = g$.

(f) If $\{\!|\, f|c \,|\!\} = \{\!|\, f|d \,|\!\}$ with f monic, then $c = d$.

(g) Given any $A(d) \times X \xrightarrow{f} Y$ in $\underline{C}$ and any $d \xrightarrow{s} c$ in $\underline{K}^{\#}$,

$\{\!|\, (f)(A(s) \times X)|c \,|\!\} = \{\!|\, f|d \,|\!\}$.

Proof: (a) Note first that A is well-defined and contravariant. For any object (n,c) of $\underline{K}^{\#}$ and any $i \in n$, c(i) is an element of K, i.e. a morphism $I \xrightarrow{c(i)} X_i$ for some object X_i in $\underline{C}$. Since n is finite, the product $A(c) = X_0 \times \dots \times X_{n-1}$ will be an object of $\underline{C}$. For any $c \xrightarrow{s} d$ in $\underline{K}^{\#}$ and any $k \in n$, $s(k)$ is $\in m$ and $q_{s(k)}$ is a projection $A(d) \xrightarrow{q_{s(k)}} \partial^1(d(s(k)))$ in $\underline{C}$.

But $ds = c$ by definition of morphisms in $\underline{K}^{\#}$; thus $q_{s(k)}$ is a map $A(d) \xrightarrow{q_{s(k)}} \partial'(c(k))$ in $\underline{C}$. Therefore the product map

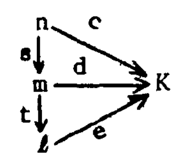
$A(s) = \langle q_{s(0)}, q_{s(1)}, \ldots, q_{s(n-1)} \rangle$ will be a map

$$A(d) \xrightarrow{A(s)} A(c) = \partial^1(c(0)) \times \ldots \times \partial^1(c(n-1)) \text{ in } \underline{C}, \text{ as required.}$$

It is clear that for every c in $\underline{K}^{\#}$,

$$A(id_c) = \langle p_{id_c(0)}, \ldots, p_{id_c(n-1)} \rangle = \langle p_o, \ldots, p_{n-1} \rangle = id_{A(c)}.$$

To verify that A preserves composition, let the diagram at the right be any diagram in $\underline{K}^{\#}$ and let r_i denote the i'th projection of A(e). This gives rise to the diagram below in $\underline{C}$:



$$A(e) \xrightarrow{A(t) = \langle r_{t(0)}, \ldots, r_{t(m-1)} \rangle} A(d) \xrightarrow{A(s) = \langle q_{s(0)}, \ldots, q_{s(n-1)} \rangle} A(c)$$

$$A(e) \xrightarrow{A(ts) = \langle r_{ts(0)}, \ldots, r_{ts(n-1)} \rangle} A(c)$$

For each $k \in n$, $(p_k)(A(s))(A(t)) = q_{s(k)}A(t)$ (by definition of $A(s)$)

$= r_{t(s(k))}$ (by definition of $A(t)$).

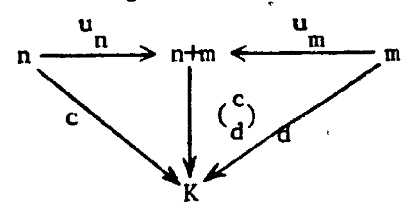
Thus $A(s)\,A(t) = \langle r_{t(s(0))}, \ldots, r_{t(s(n-1))} \rangle$

$$= \langle r_{ts(0)}, \ldots, r_{ts(n-1)} \rangle = A(ts)$$

as required, and so A is a functor.

(b) Suppose that one has maps $c \overset{s}{\underset{t}{\rightrightarrows}} d$ in $\underline{K}^{\#}$ such that $A(s) = A(t)$. If q_j denote the projections of $A(d)$, then for each $k \in n$, $p_k A(s) = p_k A(t)$; i.e. $q_{s(k)} = q_{t(k)}$. This forces $s(k) = t(k)$ for all $k \in n$, i.e. $s = t$. Therefore A is faithful.

We note that coproducts in $\underline{K}^{\#}$ take the following form: Let $(n,c),(m,d)$ be arbitrary objects of $\underline{K}^{\#}$. Since $\underline{S}_o$ has finite coproducts, there is the coproduct $n + m$ in $\underline{S}_o$, and the "factoring map" $n+m \xrightarrow{\binom{c}{d}} K$ in $\underline{S}_o$ given by



$$\binom{c}{d}(i) = \begin{cases} c(i) & \text{if } i = 0,1,\ldots,n-1 \\ d(i-n) & \text{if } i = n,n+1,\ldots,n+m-1. \end{cases}$$

The coproduct $c+d$ in $\underline{K}^{\#}$ is precisely this map $\binom{c}{d}$, which we shall usually denote by $\hat{cd}$. The injection maps $c \longrightarrow \hat{cd}$ and $d \longrightarrow \hat{cd}$ in $\underline{K}^{\#}$ are just the injections $n \xrightarrow{u_n} n+m$ and $m \xrightarrow{u_m} n+m$ in $\underline{S}_o$.

Now $A(c+d) = A\binom{c}{d} = \partial^1(\binom{c}{d}(0)) \times \ldots \times \partial^1(\binom{c}{d}(n+m-1))$

$= \partial^1(c(0)) \times \ldots \times \partial^1(c(n-1)) \times \partial^1(d(0)) \times \ldots \times \partial^1(d(m-1))$

$= A(c) \times A(d)$. Thus A takes coproducts to products.

(c) For any $X \in Ob(\underline{C})$, the identity morphism of X in $\underline{C}[\underline{K}]$ is $\# \, id_X \, | \, \emptyset \, \#$, where for convenience we let $\emptyset$ stand for both the empty set $\emptyset \in Ob(\underline{S}_o)$ and the empty map $\emptyset \xrightarrow{\emptyset} K \in Ob(\underline{K}^{\#})$.

Composition in $\underline{C}[\underline{K}]$ is defined as follows:

Let $X \xrightarrow{\# g|d \#} Y$ and $Y \xrightarrow{\# f|c \#} Z$ be maps in $\underline{C}[\underline{K}]$, i.e. $A(d) \times X \xrightarrow{g} Y$ and $A(c) \times Y \xrightarrow{f} Z$ are maps in $\underline{C}$.

<u>4.2(c)(i).</u> Define $\{\!\!\{ f|c \}\!\!\}\{\!\!\{ g|d \}\!\!\}$ in $\underline{C}[\underline{K}]$ to be $\{\!\!\{ (f)(A(c) \times g)|\widehat{cd} \}\!\!\}$.

One sees from the diagram

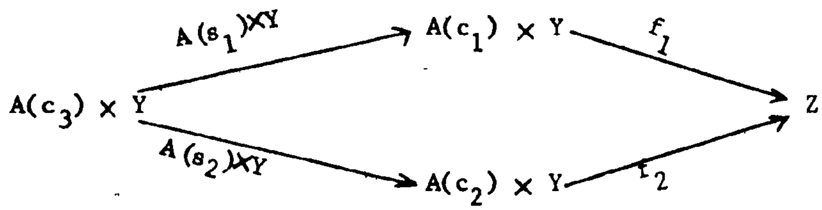
$$A(\widehat{cd}) \times X = A(c) \times A(d) \times X \xrightarrow{A(c) \times g} A(c) \times Y \xrightarrow{f} Z$$

that the definition does give a morphism : $X \to Z$ in $\underline{C}[\underline{K}]$.

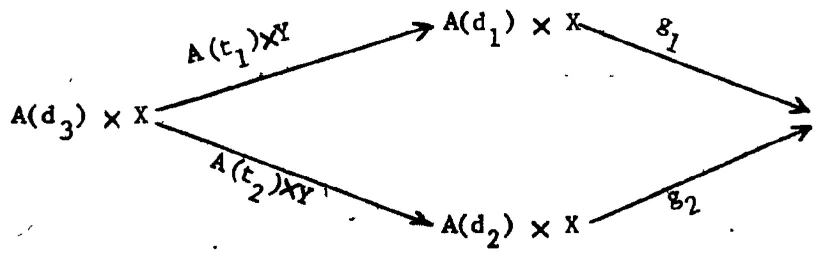
To verify that the composition is well-defined,

Suppose $\{\!\!\{ f_1|c_1 \}\!\!\} = \{\!\!\{ f_2,c_2 \}\!\!\}$ and $\{\!\!\{ g_1|d_1 \}\!\!\} = \{\!\!\{ g_2|d_2 \}\!\!\}$

via the commutative diagrams

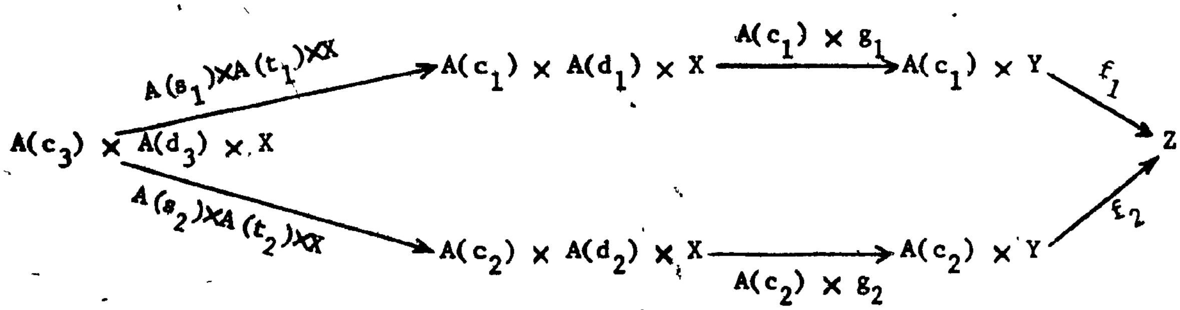


and



in $\underline{C}$.

Then the following diagram must commute in $\underline{C}$.



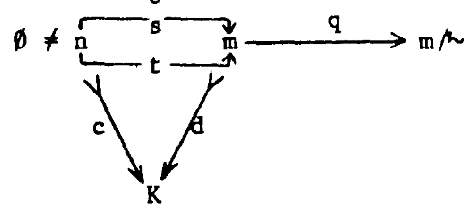
Thus $\llbracket (f_1)(A(c_1) \times g_1) | c_1 \hat{} d_1 \rrbracket = \llbracket (f_2)(A(c_2) \times g_2) | c_2 \hat{} d_2 \rrbracket$;

i.e. $\llbracket f_1 | c_1 \rrbracket \llbracket g_1 | d_1 \rrbracket = \llbracket f_2 | c_2 \rrbracket \llbracket g_2 | d_2 \rrbracket$

and so composition is well-defined.

It is clear that this composition is associative, and that the morphisms $\llbracket id_X | \emptyset \rrbracket$ act as identity maps. Thus $\underline{C}[\underline{K}]$ is a category.

($\underline{d}$) Let q denote the coequalizer of (s,t) in $\underline{S}_o$. Recall that such a coequalizer is constructed in $\underline{S}_o$ using the following equivalence relation ~ :



(j,k) ∈ ~ iff there exist $i_1, \ldots, i_M \in m$ such that $k = s(i_M)$, $j = s(i_1) = t(i_2)$, and for each $h = 1, \ldots, M-1$, $s(i_h) = t(i_{h+1})$.

Since $ds = dt = c$ and c is monic, it follows that s and t are monic and furthermore that $s(i) = t(j)$ implies $i = j$ (because then $ds(i) = dt(j)$, i.e. $c(i) = c(j)$).

Hence ~ must consist exactly of pairs $(s(i), t(i))$ with $i \in n$, pairs $(t(i), s(i))$ with $i \in n$, and pairs (j,j) with $j \in m$.

Define $u: m \to n$ by : $u(s(i)) = u(t(i)) = i$, and $u(j) = 0 \in n$ if j is not in the range of s nor of t. From the above discussion one sees that u is well-defined and is the required map.

($\underline{e}$) By definition of the equivalence R, there exist maps $c \overset{s}{\underset{t}{\rightrightarrows}} d$ in $\underline{K}^*$ such that

($\underline{i}$) $(f)(A(s) \times X) = (g)(A(t) \times X)$.

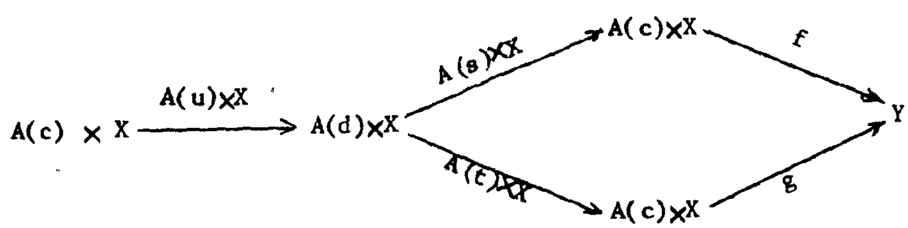
But since c is monic, by 4.2(d) there exists $d \xrightarrow{u} c$ such that $us = ut = id_c$. $\therefore$ $A(us) = A(ut) = A(id_c)$. i.e. since A is a contravariant functor,

$$A(s)A(u) = A(t)A(u) = id_{A(c)} . \quad \text{Therefore}$$

(ii) $(f)(A(s) \times X)(A(u) \times X) = (f)(id_{A(c)}) = f$, and

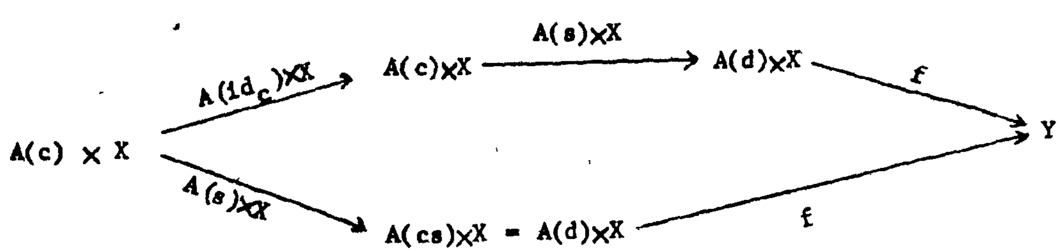
(iii) $(g)(A(t) \times X)(A(u) \times X) = (g)(id_{A(c)}) = g$.

But by (i) , L.H.S. (ii) = L.H.S. (iii). Hence f = g as required.



(f) By definition of R there exist $c \xrightarrow{s} e$ and $d \xrightarrow{t} e$ in $\underline{K}^{\#}$ such that $(f)(A(s) \times X) = (f)(A(t) \times X)$. Since f is monic this implies $A(s) = A(t)$; Hence (since A is faithful) $s = t$. But since s,t are morphisms of $\underline{K}^{\#}$, $\therefore$ $es = c$ and $et = d$. Thus $c = d$.

(g) The morphism $c \xrightarrow{id_c} c$ and $d \xrightarrow{s} c$ in $\underline{K}^{\#}$ give the following commutative diagram in $\underline{C}$, which proves the required equivalence.



This completes the proof of 4.2 .

4.3 Lemma. $\underline{C}$ and $\underline{K}$ are embedded in $\underline{C}[\underline{K}]$ by the functors

(a) $J_1: \underline{C} \to \underline{C}[\underline{K}]$ where $J_1(f) = \{\!\!\{ f|\emptyset \}\!\!\}$ for all maps f in $\underline{C}$;

(b) $J_2: \underline{K} \to \underline{C}[\underline{K}]$ where $J_2(I \xrightarrow{k} X) = \{\!\!\{ id_X|k \}\!\!\}$,

using the notation $\hat{k}$ for the map $1 \xrightarrow{\hat{k}} K$ given by $\hat{k}(0) = k$.

Proof: Given $X \xrightarrow{f} Y$ in $\underline{C}$, $J_1(f) = I \times X \xrightarrow{f} Y$, since $A(\emptyset) = I$. Because $\emptyset\hat{\emptyset} = \emptyset$; composition will be just composition in $\underline{C}$ and J_1 is a functor.

Suppose that $\{\!\!\{ f|\emptyset \}\!\!\} = \{\!\!\{ g|\emptyset \}\!\!\}$. Then precisely

$$I \times X \xrightarrow{f} Y = I \times X \xrightarrow{g} Y \text{ in } \underline{C} ,$$

i.e. $f = g$ in $\underline{C}$ and so J_1 is faithful.

Note that in particular for $f = id_X$, $J_1(f) = I \times X \xrightarrow{id_X} X$,

i.e. that J_1 is the identity functor on objects.

Since k is a map $I \xrightarrow{k} X$ in $\underline{K}$, $\therefore$ $A(\hat{k}) = \partial^1(\hat{k}(0)) = \partial^1(k = X$.

Thus $\{\!\!\{ id_X|\hat{k} \}\!\!\} = X \times I \xrightarrow{id_X} X$.

Suppose $J_2(k_1) = J_2(k_2)$ for some $I \overset{k_1}{\underset{k_2}{\rightrightarrows}} X$.

i.e. $\{\!\!\{ id_X|\hat{k}_1 \}\!\!\} = \{\!\!\{ id_X|\hat{k}_2 \}\!\!\}$. Since id_X is monic, by 4.2(f) this implies $\hat{k}_1 = \hat{k}_2$.

$\therefore$ $\hat{k}_1(0) = \hat{k}_2(0)$; i.e. $k_1 = k_2$, proving that J_2 is faithful.

<u>4.4 Theorem.</u> Let $\underline{C}$ be a prelogical category and $\underline{K}$ a category of constants for $\underline{C}$. Then

(a) $\underline{C}[\underline{K}]$ is a prelogical category and J_1 is a prelogical functor.

(b) If $\underline{C}$ and $\underline{K}$ are small, then $\underline{C}[\underline{K}]$ is small.

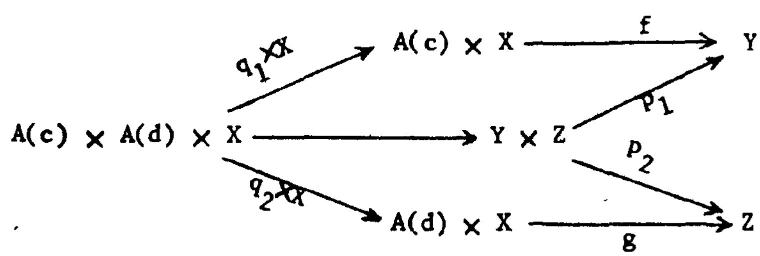
(c) If $\underline{C}$ is fair, then $\underline{C}[\underline{K}]$ is fair.

Proof: (a) Let ~~$X,Y,Z \in Ob(\underline{C})$ and~~ let $X \xrightarrow{[\![f|c]\!]} Y$, $X \xrightarrow{[\![g|d]\!]} Z$ by morphisms in $\underline{C}[\underline{K}]$. Let q_1, q_2 denote the projections of $A(c) \times A(d)$ in $\underline{C}$.

<u>4.4(a)(i).</u> Then the product map in $\underline{C}[\underline{K}]$,

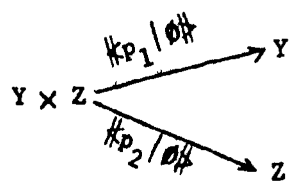
$$< [\![f|c]\!] , [\![g|d]\!] > \text{, is } [\![<(f)(q_1 \times X), (g)(q_2 \times X)> | \hat{cd}]\!] ,$$

making use of the product map in $\underline{C}$.



The projections of $Y \times Z$ in $\underline{C}[\underline{K}]$ are $[\![p_1 | \emptyset]\!]$ and $[\![p_2 | \emptyset]\!]$, where p_1, p_2 are the projections of $Y \times Z$ in $\underline{C}$.

Clearly the map defined by (i) will inherit from $\underline{C}$ the universal property required to make



a product in $\underline{C}[\underline{K}]$.

In particular, consider $I \times X \xrightarrow{f} Y$ and $I \times X \xrightarrow{g} Z$;

i.e. consider the case where $[\![f|c]\!] = [\![f|\emptyset]\!] = J_1(f)$ and $[\![g|d]\!] = J_1(g)$. By the usual isomorphisms $I \times I \times X \cong I \times X \cong X$, we obtain immediately

$$J_1(\langle f,g \rangle) = [\![\langle f,g \rangle \mid \emptyset]\!] = [\![\langle (f)(q_1 \times X),(g)(q_2 \times X) \rangle \mid \emptyset\emptyset]\!]$$

$$= \langle [\![f|\emptyset]\!], [\![g|\emptyset]\!] \rangle = \langle J_1(f), J_1(g) \rangle$$

That is, J_1 preserves products.

With respect to the Heyting algebra structure, the following properties of $\underline{C}[K]$ should be noted:

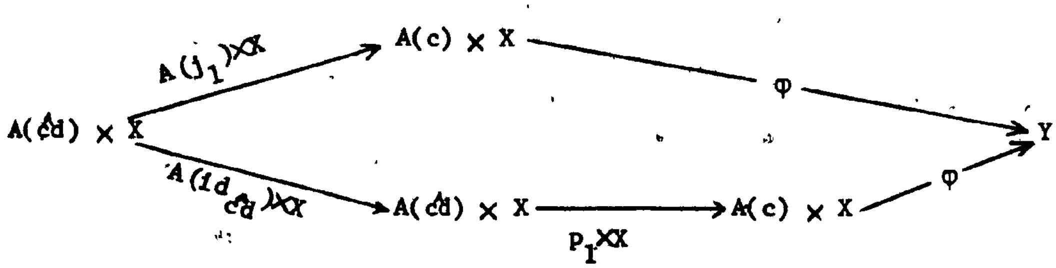
4.4 (a)(ii). First, given any maps $[\![\varphi|c]\!], [\![\psi|d]\!]: X \to Y$ in $\underline{C}[\underline{K}]$,

then $[\![\varphi|c]\!] = [\![(\varphi)(p_1 \times X)|\hat{cd}]\!]$

and $[\![\psi|d]\!] = [\![(\psi)(p_2 \times X)|\hat{cd}]\!]$,

where now p_i denote the projections of $A(c) \times A(d)$ in $\underline{C}$.

Letting j_1 denote the injection map $c \xrightarrow{j_1} \hat{cd} = c + d$ in $\underline{K}^*$ and remarking that $A(j_1) = p_1$, we verify (ii) by the commutative diagram shown below for φ, and by a similar diagram for ψ.



Let $\not\{\varphi|c\not\}$, $\not\{\psi|d\not\} : X \longrightarrow \Omega$ in $\underline{C}[\underline{K}]$. Making use of the Heyting algebra operations of $\underline{C}(A(\hat{cd}) \times X,\Omega)$, we define

<u>4.4 (a)(iii)</u> $\not\{\varphi|c\not\} \wedge \not\{\psi|d\not\} = \not\{((\varphi)(p_1\times X)) \wedge ((\psi)(p_2\times X))|\hat{cd}\not\}$;

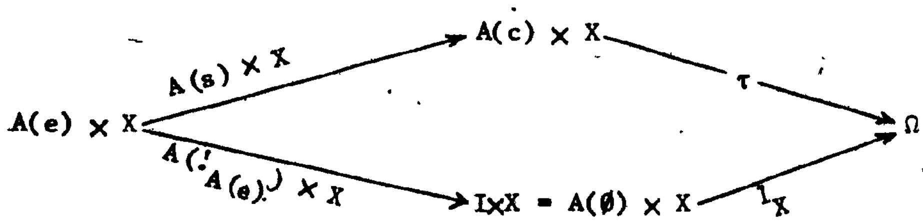
$\not\{\varphi|c\not\} \Rightarrow \not\{\psi|d\not\} = \not\{((\varphi)(p_1\times X)) \Rightarrow ((\psi)(p_2\times X))|\hat{cd}\not\}$;

and $\not\{\varphi|c\not\} \leq \not\{\psi|d\not\}$ iff $\not\{\varphi|c\not\} \wedge \not\{\psi|d\not\} = \not\{\varphi|c\not\}$.

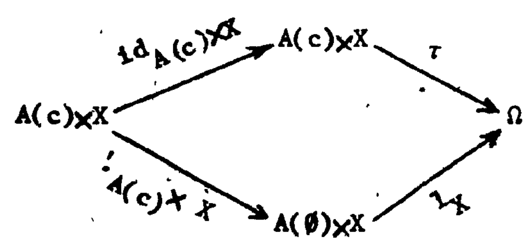
Clearly this last will be the case iff

$$(\varphi)(p_1 \times X) \leq (\psi)(p_2 \times X) \text{ in } \underline{C}(A(\hat{cd}) \times X,\Omega)$$

<u>4.4(a) (iv)</u>. Secondly, note that the map $X \xrightarrow{\not\{1_X|\emptyset\not\}} \Omega$ in $\underline{C}[\underline{K}]$ (i.e. the equivalence class of $\not\{1_X,\emptyset\not\}$) consists of all maps $A(c) \times X \xrightarrow{\tau} \Omega$ in $\underline{C}$ such that there exists $c \xrightarrow{s} e$ in $\underline{K}^{\#}$ making the first diagram below commute in $\underline{C}$.



In particular, $\not\{1_X|\emptyset\not\}$ includes all maps $A(c)\times X \xrightarrow{\tau} \Omega$ in $\underline{C}$ such that $(\tau)(id_{A(c)\times X}) = (1_X)(!_{A(c)}\times X)$



That is (since composition preserves the Heyting structure in $\underline{C}$), all maps $\tau = 1_Z$ for any objects $Z \to A(c) \times X$ in $\underline{C}$.

This class $\# 1_X | \emptyset \#$ will be seen to be the greatest element of $\underline{C}[\underline{K}]$ (X,Ω).

Similarly, $\# 0_X | \emptyset \# = \# 0_{A(c)\times X} | c \#$ for every $c \in Ob(\underline{K}^{\#})$ will be shown to be the least element.

In order to prove that 4.4(a)(iii) and (iv) make $\underline{C}[\underline{K}]$ (X,Ω) into a Heyting algebra, we must show that:

(v) It is an infimum-semilattice (i.e. that $\wedge$ is commutative, associative and idempotent).

(vi) => defines a relative pseudo-complement (i.e that

$$\# \chi | e \# \leq (\# \varphi | c \# \Rightarrow \# \psi | d \#) \text{ iff}$$

$$\# \chi | e \# \wedge \# \varphi | c \#) \leq \# \psi | d \#).$$

(vii) $\# 0_X | \emptyset \#$ is a least element (i.e. that for all $\# \varphi | c \#$, $\# 0_X | \emptyset \# \leq \# \varphi | c \#$).

Let $\# \varphi | c \#$, $\# \psi | d \#$ and $\# \chi | e \#$ be any maps: $X \to \Omega$ in $\underline{C}$, and for convenience denote the Heyting algebra $\underline{C}(A(c\hat{d}\hat{}e) \times X)$ by A.

The properties (v) of $\wedge$ are inherited directly from A. For example, to verify commutativity:

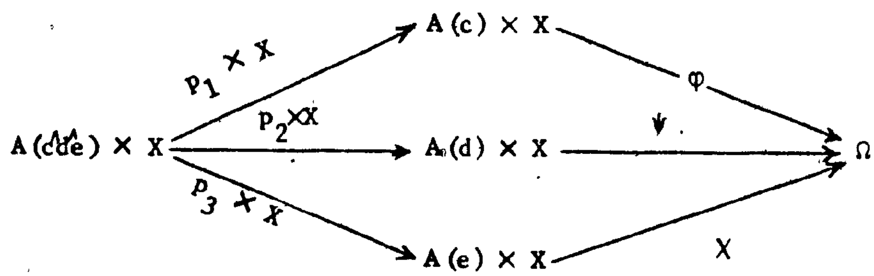
$$\# \varphi | c \# \wedge \# \psi | d \# = \# (\varphi)(p_1 \times X) \wedge (\psi)(p_2 \times X) | c\hat{d} \#$$

$$= \# (\psi)(p_2 \times X) \wedge (\varphi)(p_1 \times X) | c\hat{d} \# \quad \text{(by commutativity in } A).$$

$$= \# \psi | d \# \wedge \# \varphi | c \#.$$

The other two condtions for (v) follow similarly.

Next let p_1, p_2, p_3 denote the projections of $A(cde) = A(c) \times A(d) \times A(c)$ in $\underline{\mathcal{C}}$, and consider the diagram shown.



We have by <u>4.4(a)(iii)</u> that

$$\Vert \chi | e \Vert \leq (\Vert \varphi | c \Vert \Rightarrow \Vert \psi | d \Vert) \quad \text{in} \quad \underline{C}[\underline{K}](X,\Omega)$$

$$\text{iff } (\chi)(p_3 \times X) \leq ((\varphi)(p_1 \times X) \Rightarrow (\psi)(p_2 \times X) \quad \text{in } A,$$

which, by the adjointness of $\Rightarrow$ and $\wedge$ on A, occurs iff

$$((\chi)(p_3 \times X) \wedge (\varphi)(p_1 \times X)) \leq (\psi)(p_2 \times X) \quad \text{in } A,$$

$$\text{i.e. iff } (\Vert \chi | e \Vert \wedge \Vert \varphi | c \Vert) \leq \Vert \psi | d \Vert \quad \text{in } \underline{C}[\underline{K}](X,\Omega).$$

Thus <u>4.4(a)(vi)</u> Has been proved.

Since $0_{A(c) \times X} \leq \varphi$ in $\underline{C}(A(c) \times, \Omega)$

$$\therefore \quad \Vert 0_X | \emptyset \Vert = \Vert 0_{A(x) \times X} | c \Vert \leq \Vert \varphi | c \Vert \quad \text{in } \underline{C}[\underline{K}](X,\Omega).$$

Thus $\Vert 0_X | \emptyset \Vert$ is a least element, making $\underline{C}[\underline{K}](X,\Omega)$ a Heyting algebra.

Moreover given any $X \overset{\varphi'}{\underset{\psi'}{\rightrightarrows}} \Omega$ in $\underline{C}$, the definitions <u>4.4(a)(iii)</u> and <u>(iv)</u> dictate precisely that

$$J_1(\varphi' \wedge \psi') = J_1(\varphi') \wedge J_1(\psi'); \; J_1(\varphi' \Rightarrow \psi') = J_1(\varphi') \Rightarrow J_1(\ ');$$

$J_1(1_X) = 1_X$ of $\underline{C}[\underline{K}](X,\Omega)$; and $J_1(0_X) = 0_X$.

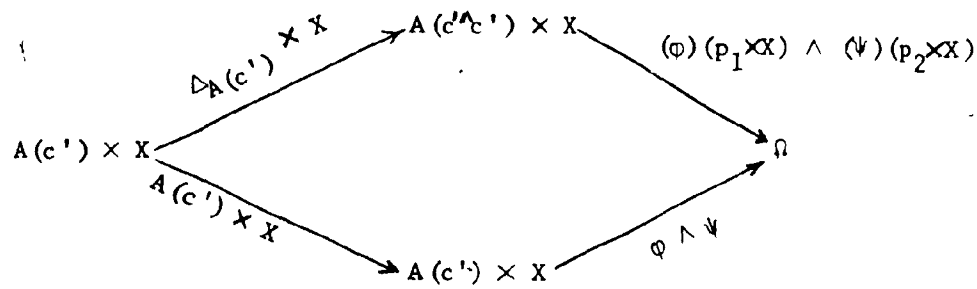
That is, J_1 preserves the Heyting algebra structure of every $\underline{C}(X,\Omega)$.

In order to conclude that Ω is a Heyting algebra object of $\underline{C}[K]$, it remains to be shown that $\underline{C}[K]\,(\llbracket f|e \rrbracket,\Omega)$ is a Heyting homomorphism for every map $\llbracket f|e \rrbracket$ in $\underline{C}[\underline{K}]$.

Note that one consequence of 4.4(a)(iii) is that for any $c' \in |\underline{K}^{\#}|$ and any $A(c') \times X \overset{\varphi}{\underset{\psi}{\rightrightarrows}} Y$,

4.4(a)(viii) $\llbracket \varphi|c' \rrbracket \wedge \llbracket \psi|c' \rrbracket = \llbracket \varphi \wedge \psi|c' \rrbracket$.

The equivalence is verified by the following diagram:



Now let $A(c) \times Y \xrightarrow{\varphi} \Omega$, $A(d) \times Y \xrightarrow{\psi} \Omega$, and $A(e) \times X \xrightarrow{f} Y$; let p_i denote the projections $A(c) \times A(d)$, and r_1 the projection $A(c\hat{d}e) \underset{r_1}{\to} A(\hat{c}e)$.

Then $(\llbracket \varphi|c \rrbracket \wedge \llbracket \psi|d \rrbracket)(\llbracket f|e \rrbracket)$

$= \llbracket (\varphi)(p_1 \times Y) \wedge (\psi)(p_2 \times Y)|c\hat{d} \rrbracket\, \llbracket f|e \rrbracket$ by 4.4(a)(iii)

$= \llbracket ((\varphi)(p_1 \times Y) \wedge (\psi)(p_2 \times Y))\,(A(c\hat{d}) \times f)|\hat{c}\hat{d}e \rrbracket$

by 4.2(c)(i)

$= \llbracket (\varphi)(p_1 \times Y)(A(cd) \times f) \wedge (\psi)(p_2 \times Y)(A(cd) \times f) \,|\, cde \rrbracket$

because composition in $\underline{C}$ preserves all Heyting operations

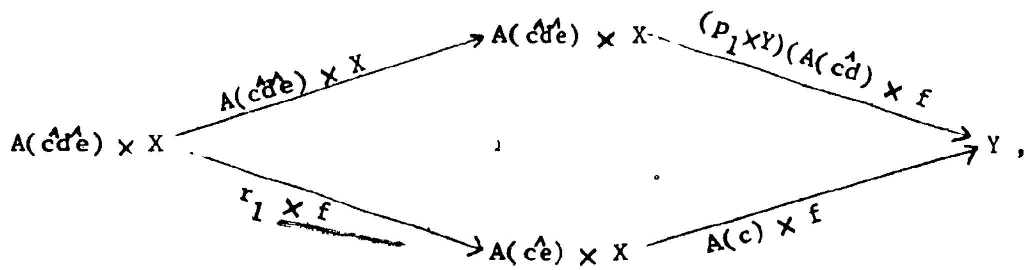
$= \llbracket (\varphi)(p_1 \times Y)(A(cd) \times f \,|\, cde \rrbracket \wedge \llbracket (\psi)(p_2 \times Y)(A(cd) \times f) \,|\, cde \rrbracket$

by <u>4.4(a)(viii)</u>

$=$ call (ix) .

But $\llbracket (\varphi)(p_1 \times Y)(A(\hat{c}\hat{d}) \times f) \,|\, \hat{c}\hat{d}\hat{e} \rrbracket = \llbracket (\psi)(A(c) \times f) \,|\, c\hat{e} \rrbracket$

by the following diagram:



and $\llbracket (\psi)(p_2 \times Y)(A(cd) \times f) \,|\, cde \rrbracket = \llbracket (\psi)(A(d) \times f) \,|\, de \rrbracket$

by similar diagram.

Therefore (<u>ix</u>) $= \llbracket (\varphi)(A(c) \times f) \,|\, ce \rrbracket \wedge \llbracket (\psi)(A(d) \times f) \,|\, de \rrbracket$

$= (\llbracket \varphi \,|\, c \rrbracket \llbracket f \,|\, e \rrbracket) \times (\llbracket \psi \,|\, d \rrbracket \llbracket f \,|\, e \rrbracket)$

precisely by <u>4.2(c)(i)</u>.

Thus composition with $\llbracket f \,|\, e \rrbracket$ preserves $\wedge$.

It is clear that the same arguement can be used for each of the Heyting operations, and so Ω is a Heyting algebra object in $\underline{C}[K]$.

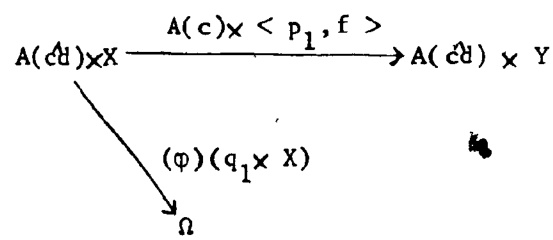
Existential quantification is defined in $\underline{C}[K]$ by :

<u>4.4(a)(x)</u> Let $A(d) \times X \xrightarrow{f} Y$ and $A(c) \times X \xrightarrow{\varphi} \Omega$ in $\underline{C}$ be given and let p_1, q_1 denote the projection maps $A(d) \times X \xrightarrow{p_1} A(d)$ and $A(cd) \xrightarrow{q_1} A(c)$ in $\underline{C}$.

Then define

$$\exists_{\# f|d \#}[\# \varphi|c \#] = \# \exists_{A(c)\times <p_1, f>}[(\varphi)(q_1 \times X)]|cd \#$$

We remark that this does define a morphism of $\underline{C}[K]$, since the quantification on the right-hand side is performed in $\underline{C}$ on the diagram shown.



We show next that (x) defines a left adjoint to substitution. By <u>1.3</u>, it is sufficient to prove that (<u>xi</u>) and (<u>xii</u>) hold for all $A(c) \times X \xrightarrow{\varphi} \Omega$, $A(d) \times X \xrightarrow{f} Y$ and $A(e) \times X \xrightarrow{\psi} \Omega$ in $\underline{C}$

(<u>xi</u>) $\# \varphi|c \# \leq (\exists_{\# f|d \#}[\# \varphi|c \#])\# f|d \#$.

(<u>xii</u>) $\exists_{\# f|d \#}[\# \varphi|c \# \wedge (\# \psi|e \# \; \# f|d \#)]$

$$= \exists_{\# f|d \#}[\# \varphi|c \#] \wedge \# \psi|e \# .$$

To prove (xii) we shall make use of the sublemma (xiii)

(xiii) Given any $A(d)\times X \xrightarrow{f} Y$, $A(\hat{c}d)\times X \xrightarrow{\varphi'} \Omega$ and projections $A(d)\times X \xrightarrow{p_1'} A(d)$ and $A(\hat{c}\hat{d}d) \xrightarrow{q_1'} A(\hat{c}d)$,

then the definition (x) reduces to

$$\exists_{\# f|d \#}[\#\varphi'|\hat{c}d\#] = \#\exists_{A(c)\times < p_1',f>}[\varphi']|\hat{c}d\#$$

Proof: $\exists_{\# f|d \#}[\#\varphi'|\hat{c}d\#]$

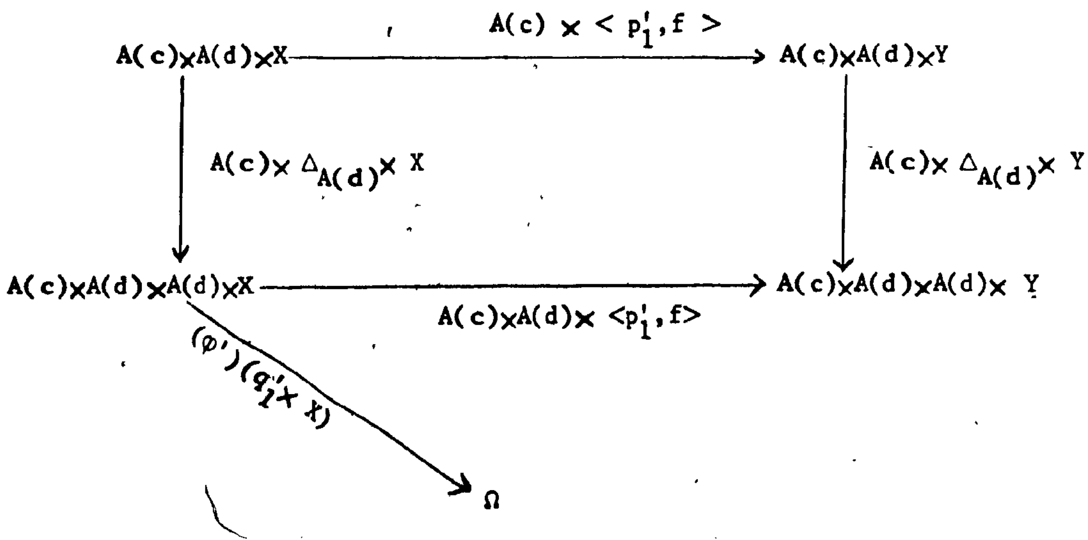
$= \#\exists_{A(\hat{c}d)\times < p_1',f >}[(\varphi')(q_1'\times X)]\,|\hat{c}\hat{d}d\#$ by (x)

$= \#\left(\exists_{A(\hat{c}d)\times < p_1',f >}[(\varphi')(q_1'\times X]\right)\left(A(c)\times\Delta_{A(d)}\times Y\right)|\hat{c}d\#$

by 4.2.(g), where $s = c + \nabla_d : \hat{c}\hat{d}d \longrightarrow \hat{c}d$ in $\underline{K}^{\#}$,

= call (xiv).

Now the diagram below is of the form of 1.6(d), with $X_1 = A(c)\times A(d) = A(\hat{c}d)$, $X_2 = X$, $Y_1 = A(\hat{c}\hat{d}d)$, $Y_2 = Y$, $h = A(c)\times\Delta_{A(d)} = A(s)$ and $f = fr_2'$; where r_2' denotes the projection $A(\hat{c}\hat{d}d)\times X \xrightarrow{r_2'} A(d)\times X$



Thus by 1.6(d) one has

(xiv) $\quad = \llbracket \exists_{A(c)\times \langle p_1', f\rangle} [\varphi'(q_1' \times X)(A(c) \times \Delta_{A(d)} \times X)] \mid \hat{cd} \rrbracket$

which $\quad = \llbracket \exists_{A(c)\times \langle p_1', f\rangle} [\varphi'] \mid cd \rrbracket$, thus proving (xiii).

Now the proof of (xi) proceeds as follows: For any $A(d) \times X \xrightarrow{f} Y$ and $A(c) \times X \xrightarrow{\varphi} \Omega$ and projections $A(d) \times X \xrightarrow{p_1} A(d)$, $A(\hat{cd}) \xrightarrow{q_1} A(c)$, one has

$$(\exists_{\llbracket f|d \rrbracket} [\llbracket \varphi|c \rrbracket])(\llbracket f|d \rrbracket) = \llbracket \exists_{A(c)\times \langle p_1, f\rangle} [\varphi(q_1 \times X)] \mid \hat{cd} \rrbracket \llbracket f|d \rrbracket$$

by (x)

$$(\exists_{\llbracket f|d \rrbracket} \llbracket \varphi|c \rrbracket \; (\llbracket f|d \rrbracket)$$

$$= \llbracket (\exists_{A(c)\times \langle p_1, f\rangle} [\varphi(q_1 \times X)]) (A(\hat{cd}) \times f) \mid \hat{cd}\hat{d} \rrbracket \quad \text{by } 4.2(c)$$

$$= \llbracket (\exists_{A(c)\times \langle p_1, f\rangle} [\varphi(q_1 \times X)]) (A(\hat{cd}) \times f)\; A(c) \times \Delta_{A(d)} \times X) \mid \hat{cd} \rrbracket$$

by 4.2(e), using $s = c + \nabla_d : \hat{cdd} \to \hat{cd}$,

$$= \llbracket (\exists_{A(c)\times \langle p_1, f\rangle} [\varphi(q_1 \times X)]) (A(c) \times \langle p_1, f \rangle) \mid \hat{cd} \rrbracket$$

$$\geq \llbracket \varphi(q_1 \times X) \mid \hat{cd} \rrbracket \quad \text{by } 1.3(a) \text{ and } 4.4(a)(iii)$$

$$= \llbracket \varphi|c \rrbracket \quad \text{by } 4.4(a)(ii).$$

Thus 4.4(a)(xi) has been proved.

To prove (<u>xii</u>) let f, φ, p_1, q_1 be as above; let $A(e) \times Y \xrightarrow{\psi}$ be any map, and we adopt the following notation for projections:

$A(\hat{c}\hat{e}\hat{d}) \times Y \xrightarrow{\tilde{q}_1} A(\hat{c}d) \times Y$, $A(\hat{c}\hat{e}\hat{d}) \times Y \xrightarrow{\tilde{q}_2} A(\hat{e}d) \times Y$,

$A(\hat{c}d) \times X \xrightarrow{r_1} A(c) \times X$ (Thus $r_1 = q_1 \times X$), $A(\hat{e}d) \times Y \xrightarrow{s_2} A(e) \times Y$,

$A(\hat{c}\hat{e}\hat{d}) \times X \xrightarrow{t_1} A(\hat{c}d) \times X$, and $A(\hat{c}\hat{e}\hat{d}) \times X \xrightarrow{t_2} A(\hat{e}d) \times X$.

Then

$$\exists_{\# f|d \#} [\# \varphi|c \# \wedge (\# \psi|e \# \# f|d \#)] = \exists_{\# f|d \#}[\# \varphi|c \# \wedge \# \psi(A(e) \times f)|\hat{e}d \#]$$

by <u>4.2(c)(i)</u>

$$= \exists_{\# f|d \#} [\# \varphi|c \# \wedge (\# \psi\, s_2(A(e) \times < p_1, f >)|\hat{e}d \#]$$

$$= \exists_{\# f|d \#} [\# \varphi r_1 t_1 \wedge (\psi\, s_2(A(e) \times < p_1, f>)\, t_2)|\hat{c}\hat{e}\hat{d} \#]$$

by <u>4.4(a)(iii)</u>

$$= \# \exists_{A(\hat{c}e) \times <p_1, f>} [\varphi\, r_1 t_1 \wedge (\psi\, s_2(A(e) \times <p_1, f>)\, t_2)] | \hat{c}\hat{e}\hat{d} \#$$

by <u>4.4(a)(xiii)</u>

$$= \# \exists_{A(\hat{c}e) \times <p_1, f>} [\varphi r_1 t_1 \wedge (\psi s_2 \tilde{q}_2\, (A(\hat{c}e) \times < p_1, f>))] \,|\hat{c}\hat{e}\hat{d} \#$$

$$= \# \exists_{A(\hat{c}e) \times <p_1, f>} [\varphi r_1 t_1] \wedge \psi s_2 \tilde{q}_2 |\, \hat{c}\hat{e}\hat{d} \#$$ by <u>1.3(b)</u> in <u>C</u> .

$= \langle\!\langle \exists_{A(c^\wedge e)\times\langle p_1, f\rangle}[\varphi\, r_1 t_1] \mid c^\wedge e^\wedge d \rangle\!\rangle \wedge \langle\!\langle \psi\, s_2 \tilde{q}_2 \mid c^\wedge e^\wedge d \rangle\!\rangle$ by 4.4(a)(iii)

$= \langle\!\langle \exists_{A(c^\wedge e)\times\langle p_1, f\rangle}[\varphi r_1 t_1] \mid c^\wedge e^\wedge d \rangle\!\rangle \wedge \langle\!\langle \psi \mid e \rangle\!\rangle$ by 4.4(a)(ii)

$= \langle\!\langle \exists_{A(c)\times\langle p_1, f\rangle}[\varphi r_1]) \tilde{q}_1 \mid c^\wedge e^\wedge d \rangle\!\rangle \wedge \langle\!\langle \psi \mid e \rangle\!\rangle$ by 1.6(c) with

$X = A(c^\wedge d) \times X,\ Z = A(e),\ Y = A(c^\wedge d) \times Y$, etc.

$= \langle\!\langle \exists_{A(c)\times\langle p_1, f\rangle}[\varphi\, r_1] \mid c^\wedge d \rangle\!\rangle \wedge \langle\!\langle \psi \mid e \rangle\!\rangle$ by 4.2(g), using

s = injection: $c^\wedge d \to c^\wedge e^\wedge d$.

$= (\exists_{\langle\!\langle f \mid d \rangle\!\rangle}[\langle\!\langle \varphi \mid c \rangle\!\rangle]) \wedge \langle\!\langle \psi \mid e \rangle\!\rangle$ by 4.4(a)(x)

Thus 4.4(a)(xii) has been proved, and therefore the quantification in $\underline{C}\,[\underline{K}]$ is an adjoint as required.

In particular, since adjoints are unique, this tells us that the quantification is well-defined , i.e. does not depend on the choice of representative morphisms.

We note also that by 4.3(a) and 4.4(a)(x), for any $X \xrightarrow{f} Y$ and $X \xrightarrow{\varphi} \Omega$ in $\underline{C}$,

$$\exists_{J_1(f)}[J_1(\varphi)] = \exists_{\langle\!\langle f \mid \emptyset \rangle\!\rangle}[\langle\!\langle \varphi \mid \emptyset \rangle\!\rangle] = \langle\!\langle \exists_f[\varphi] \mid \emptyset \rangle\!\rangle = J_1(\exists_f[\varphi]) .$$

That is, J_1 preserves the existential quantification, and therefore is an intuitionistic prelogical functor.

The Beck conditions on $\underline{C}[\underline{K}]$ take the following form:

4.4(a)(xv) Given any $A(d) \times X \xrightarrow{f} Y$ $\underline{C}$, then

$$\exists_{\langle \sharp X|\emptyset \sharp, \sharp f|d \sharp\rangle}[\sharp 1_Y|\emptyset \sharp \sharp f|d \sharp]$$

$$= \left(\exists_{\sharp \Delta_Y|\emptyset \sharp}[\sharp 1_Y|\emptyset \sharp]\right)\left(\sharp f|d \sharp \times \sharp Y|\emptyset \sharp\right)$$

(xvi) Given any $A(d_1) \times X_1 \xrightarrow{f_1} Y_1$, $A(d_2) \times X_2 \xrightarrow{f_2} Y_2$

and any $A(c) \times Y_1 \times Y_2 \xrightarrow{\varphi} \Omega$ in $\underline{C}$, then

$$\exists_{\sharp X_1|\emptyset \sharp \times \sharp f_2|d_2 \sharp}\left[\left(\sharp \varphi|c \sharp\right)\left(\sharp f_1|d_1 \sharp \times \sharp X_2|\emptyset \sharp\right)\right]$$

$$= \left(\exists_{\sharp Y_1|\emptyset \sharp \times \sharp f_2|d_2 \sharp}[\sharp \varphi|c \sharp]\right)\left(\sharp f_1|d_1 \sharp \times \sharp Y_2|\emptyset \sharp\right).$$

By 1.6(e) it is equivalent to (xvi) that given any f_1, f_2 as above and any $A(c) \times X_2 \times Y_1 \xrightarrow{\varphi} \Omega$, then

$$\exists_{\sharp f_2|d_2 \sharp \times \sharp X_1|\emptyset \sharp}\left[\left(\sharp \varphi|c \sharp\right)\left(\sharp X_2|\emptyset \sharp \times \sharp f_1|d_1 \sharp\right)\right]$$

$$= \left(\exists_{\sharp f_2|d_2 \sharp \times \sharp Y_1|\emptyset \sharp}[\sharp \varphi|c \sharp]\right)\left(\sharp Y_2|\emptyset \sharp \times \sharp f_1|d_1 \sharp\right).$$

We first prove (xv):

$$\exists_{\langle \sharp X|\emptyset \sharp,\ \sharp f|d \sharp \rangle} \left[\sharp 1_Y|\emptyset \sharp < f|d \sharp\right]$$

$$= \exists_{< \sharp X|\emptyset \sharp,\ \sharp f|d \sharp} \left[\sharp 1_Y f|d \sharp\right] \quad \text{by } \underline{4.2(c)(i)}$$

$$= \exists_{\sharp < p_2, f >|d \sharp} \left[\sharp 1_Y f|d \sharp\right] \quad \text{by } \underline{4.4(a)(i)}$$

$$= \sharp\ \exists_{< p_1, < p_2, f >>} [1_Y f] \big| d \sharp \quad \text{by } \underline{4.4(a)(xiii)}$$

$$= \sharp\ \exists_{< A(d) \times X, f >} \left[1_Y f\right] | d \sharp = \sharp \left(\exists_{\Delta_Y} [1_Y]\right) \left(f \times Y\right) | d \sharp \quad \text{by } \underline{1.1(d)(i)}$$

$$= \left(\sharp\ \exists_{\Delta_Y} [1_Y] | \emptyset \sharp\right) \left(\sharp f \times Y | d \sharp\right) \quad \text{by } \underline{4.2(c)(i)}$$

$$= \left(\sharp\ \exists_{\Delta_Y} [1_Y] | \emptyset \sharp\right) \left(\sharp f|d \sharp \times \sharp Y|\emptyset \sharp\right)$$

$$= \left(\exists_{\sharp \Delta_Y|\emptyset \sharp} [\sharp 1_Y|\emptyset \sharp]\right) \left(\sharp f|d \sharp \times \sharp Y|\emptyset \sharp\right)$$

as required for (xv) .

We next verify 4.4(a)(xvi) as follows:

Adopt the projection notations $A(d_1) \times A(d_2) \times X_2 \times X_1 \xrightarrow{r} A(d_1) \times A(d_2)$

$A(c) \times A(d_2) \times X_2 \times A(d_1) \times X_1 \xrightarrow{q_2'} A(c) \times X_2 \times A(d_1) \times X_1$,

$A(d_2) \times X_2 \xrightarrow{p} A(d_2)$, $A(c) \times A(d_2) \times X_2 \times Y_1 \xrightarrow{q_2} A(c) \times X_2 \times Y_1$, and

$A(d_2) \times X_2 \times Y_1 \xrightarrow{u} A(d_2)$.

$$\exists_{\# f_2|d_2 \# \times \# X_1|\emptyset \#}\left[(\# \varphi|c \#)(\# X_2|\emptyset \# \times f_1|d_1 \#)\right]$$

$$= \exists_{\# f_2 \times X_1|d_2 \#}\left[\# \varphi|c \# \;\; \# X_2 \times f_1|d_1 \#\right]$$

$$= \exists_{\# f_2 \times X_1|d_2 \#}\left[\# \varphi(A(c) \times X_2 \times f_1)|\hat{c}d_1 \#\right] \quad \text{by } \underline{4.2(c)(i)}$$

$$= \# \exists_{A(c) \times \langle r, f_2 \times X_1 \rangle}\left[\varphi(A(c) \times X_2 \times f_1)q_2'\right]|\hat{c}\hat{d}_2\hat{d}_1 \#$$

by $\underline{4.4(a)(x)}$

$$= \# \exists_{A(c) \times \langle p, f_2 \rangle \times A(d_1) \times X_1}\left[\varphi q_2(A(c) \times A(d_2) \times f_1)\right]|\hat{c}\hat{d}_2\hat{d}_1 \#$$

= call <u>4.4(a)(xvii)</u>.

Moreover, $\left(\exists_{\# f_2|d_2 \# \times \# Y_1|\emptyset \#}[\# \varphi|c \#]\right)(\# Y_2|\emptyset \#) \times (\# f_1|d_1 \#)$

$$= \left(\exists_{\# f_2 \times Y_1|d_2 \#}[\# \varphi|c \#]\right)\left(\# Y_2 \times f_1|d_1 \#\right)$$

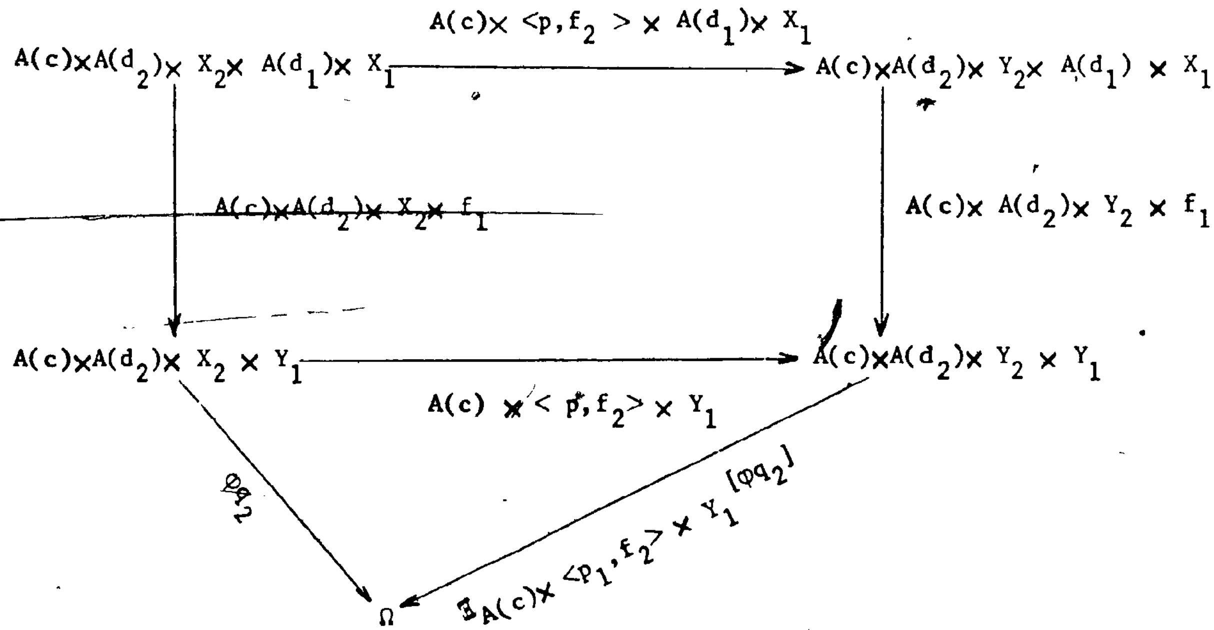
$$= \left(\# \exists_{A(c) \times \langle u, f_2 \times Y_1 \rangle}[\varphi q_2]|\hat{c}d_2 \#\right)\left(\# Y_2 \times f_1|d_1 \#\right) \quad \text{by } \underline{4.4(a)(x)}$$

$$= \# \left(\exists_{A(c) \times \langle p, f_2 \rangle \times Y_1}[\varphi q_2]\right)\left(A(c) \times A(d_2) \times Y_2 \times f_1\right)|\hat{c}\hat{d}_2\hat{d}_1 \#$$

by <u>4.2 (c)(i)</u>

= call <u>(xviii)</u>

Now the diagram shown



is of form 1.6(e) with $X_1 = A(d_1) \times X_1$, $X_2 = A(c) \times A(d_2) \times X_2$,

$Y_1 = Y_1$, $Y_2 = A(c) \times A(d_2) \times Y_2$, $f_1 = f_1$, $f_2 = A(c) \times < p,f_2 >$,

and $\varphi = \varphi\, q_2$.

Thus by 1.6(e) we have (xvii) = (xviii), and so 4.4(a)(xvi) has been proved. That is, the Beck conditions hold in $\underline{C}[K]$.

The remaining properties necessary to make $\underline{C}[K]$ an intuitionistic prelogical category are the conditions 1.1(e) regarding equality.

Suppose one has $A(d_1) \times X \xrightarrow{f_1} Y$, $A(d_2) \times X \xrightarrow{f_2} Y$

such that $\left(\exists_{\sharp \Delta_Y | \emptyset \sharp} [\, \sharp\, 1_Y | \emptyset\, \sharp] \right) \left\langle \sharp\, f_1 | d_1\, \sharp, \sharp\, f_2 | d_2\, \sharp \right\rangle = \sharp\, 1_X | \emptyset\, \sharp$

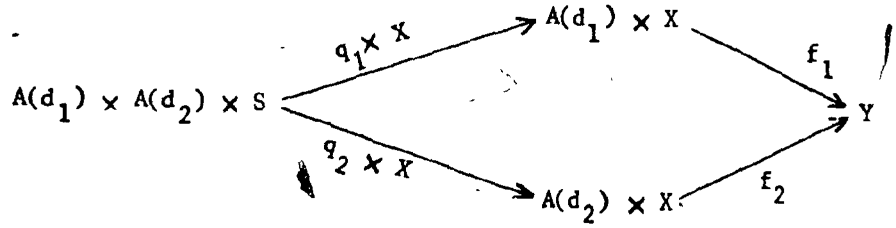
That is, by 4.4(a)(x) and 4.4(a)(i),

$$\sharp\ \exists_{\Delta_Y}[1_Y]\,|\,\emptyset\ \sharp\ \ \langle\!\langle (f_1 \langle (q_1 \times X), f_2 \rangle\, (q_2 \times X))\,|\,d_1\hat{}d_2 \rangle\!\rangle = \sharp\ 1_X\,|\,\emptyset\ \sharp$$

where one has the projections q_1, q_2 of $A(d_1) \times A(d_2)$ as shown below.

i.e $\langle\!\langle e_Y \langle (f_1)(q_1 \times X), (f_2)(q_2 \times X) \rangle\,|\,d_1\hat{}d_2 \rangle\!\rangle = \sharp\ 1_{A(d_1) \times A(d_2) \times X}\,|\,d_1\hat{}d_2\ \sharp$

by 4.4(a)(i) and the note 4.4(a)(iv).



Since $\underline{C}$ satisfies 1.1(e)(i), this implies that

(xix) $\quad f_1(q_1 \times X) = f_2(q_2 \times X)$.

Since $q_1 = A(n_1)$, $q_2 = A(n_2)$ for the coproduct ~~injections~~

$$d_1 \xrightarrow{n_1} d_1\hat{}d_2\ , \quad d_2 \xrightarrow{n_2} d_1\hat{}d_2 \ \text{ in } \underline{K}^{\#} ,$$

∴ (xix) says precisely that $\sharp\ f_1|d_1\ \sharp = \sharp\ f_2|d_2\ \sharp$, and hence that $\underline{C}[\underline{K}]$ satisfies 1.1(e)(i) .

To verify 1.1(e)(ii), Let $A(d) \times \Omega \xrightarrow{f} \Omega$, $A(e) \times \Omega \xrightarrow{g} \Omega$ be any maps and let q_1, q_2 denote $A(d) \times A(e) \xrightarrow[q_1]{} A(d)$, $A(d) \times A(e) \xrightarrow[q_2]{} A(e)$.

Then $\sharp\, e_\Omega | \emptyset \,\sharp \; \langle \sharp\, f|d \,\sharp, \sharp\, g|e \,\sharp \rangle$

$= \sharp\, e_\Omega \langle f(q_1 \times \Omega), g(q_2 \times \Omega) \rangle | de \,\sharp$ by 4.2(c)(i) and 4.4(a)(i) ,

$= \sharp\, [(f(q_1 \times \Omega)) \Rightarrow (g(q_2 \times \Omega))] \wedge [(g(q_2 \times \Omega)) \Rightarrow (f(q_1 \times \Omega))] \| de \,\sharp$

because $\underline{C}$ satisfies 1.1(e)(ii)

$= (\sharp\, f|d \,\sharp \Rightarrow \sharp\, g|e \,\sharp) \wedge (\sharp\, g|e \,\sharp \Rightarrow \sharp\, f|d \,\sharp)$ by 4.4(a)(iii)

so that $\underline{C}[\underline{K}]$ satisfies 1.1(e)(ii) and therefore is prelogical .

This completes the proof of 4.4(a).

4.4(b) is obvious from the definition of $\underline{C}[\underline{K}](X,Y)$: Each $M_{X,Y}$ can be at most card($\underline{K}^\sharp$) times as large as $\underline{C}(X,Y)$.

4.4(c) Suppose that $\underline{C}$ is fair, and note that for each $Y \times Z \xrightarrow[p_1]{} Y$ in $\underline{C}$,

$$J_1(p_1) = \sharp\, p_1 | \emptyset \,\sharp = Y \times Z \xrightarrow{p_1} Y,$$

which must certainly be epi iff p_1 is.

But let $Y \times Z$ be any product in $\underline{C}\,[\underline{K}]$ and let p_1, p_2 denote the projections in $\underline{C}$; the projections of $Y \times Z$ in $\underline{C}[\underline{K}]$ are precisely $J_1(p_1)$ and $J_1(p_2)$, i.e. $\sharp\, p_1 | \emptyset \,\sharp$ and $\sharp\, p_2 | \emptyset \,\sharp$.

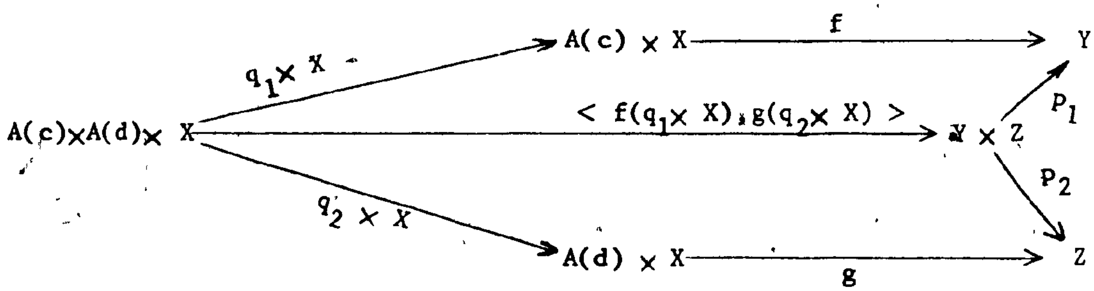
For: Let $X \xrightarrow{\sharp\, f|c \,\sharp} Y$, $X \xrightarrow{\sharp\, g|d \,\sharp} Z$ in $\underline{C}[\underline{K}]$. Then one has

$$p_1 \langle f(q_1 \times X), g(q_2 \times X) \rangle = f(q_1 \times X)$$

where q_1, q_2 are the projections of $A(c) \times A(d)$ in $\underline{C}$.

$$\# p_1|\emptyset \# \left\langle \# f|c \#, \# g|d \# \right\rangle = \# f(q_1 \times X)|cd \# = \# f|c \#$$

Thus all projections in $\underline{C}[K]$ are epi, i.e. $\underline{C}[K]$ is fair.



Theorem 4.4, the main theorem on extension by constants, has therefore been proved.

The next few results develop the idea of a prelogical quotient category with respect to a given filter on $\underline{C}(I,\Omega)$. Intuitively, one may think of such a procedure as tracing all the logical consequences of a new set of axioms added to a theory. Reducing $\underline{C}(I,\Omega)$ modulo a filter results in some sentences α being made equivalent to 1, i.e. in new (sentence) theorems. This in turn causes the reduction of the $\underline{C}(X,\Omega)$, i.e. creates new (open-formula) theorems of type X. In particular one acquires more theorems of form $e_Y < f,g >$, thus identifying certain terms of the theory.

4.5 Definition. Let $\underline{C}$ be a prelogical category.

By a prelogical congruence relation $\underline{R}$ on $\underline{C}$ we shall mean(s) a system $\{R_{X,Y}\}_{X,Y \in Ob(\underline{C})}$, of equivalence relations $R_{X,Y}$ on the classes $\underline{C}(X,Y)$ satisfying the following properties:

4.5(b) If $(f_1, f_2) \in R_{Y,Z}$ and $(g_1, g_2) \in R_{X,Y}$, then $(f_1 g_1, f_2 g_2) \in R_{X,Z}$

We sometimes describe this property by saying that F is compatible with (or, closed under) composition.

4.5(c) If $(f_1, g_1) \in R_{X,Y_1}$ and $(f_2, g_2) \in R_{X,Y_2}$,

then $(< f_1, f_2 >, < g_1, g_2 >) \in R_{X, Y \times Y_2}$.

where as usual $<,>$ denotes the product map.

This property is also referred to as "R is compatible with (or, closed under) products".

4.5(d) If $(f_1, f_2) \in R_{X,Y}$ and $(\varphi_1, \varphi_2) \in R_{X,\Omega}$ then

$$(\exists_f [\varphi_1] \ , \ \exists_{f_2} [\varphi_2]) \in R_{Y,\Omega}$$

Equivalently we say that R is compatible with (or, closed under) existential quantification.

4.5(e) If $(\varphi_1, \varphi_2) \in R_{X,\Omega}$ and $(\psi_1, \psi_2) \in R_{X,\Omega}$ then

$$(\varphi_1 \vee \psi_1, \varphi_2 \vee \psi_2) \in R_{X,\Omega} \quad \text{and} \quad (\varphi_1 \wedge \psi_1, \varphi_2 \wedge \psi_2) \in R_{X,\Omega}$$

That is, R is a lattice congruence relation.

Note: As a consequence of (e) we have that $\|\varphi\|_R \leq \|\psi\|_R$ iff $\varphi \leq \psi$, where $\| \ \|_R$ denotes the equivalence class under R.

4.5(f) If $(\varphi_1, \varphi_2) \in R_{X,\Omega}$ and $(\psi_1, \psi_2) \in R_{X,\Omega}$ then $(\varphi_1 \Rightarrow \psi_1, \varphi_2 \Rightarrow \psi_2) \in R_{X,\Omega}$

i.e R is a Heyting-algebra congruence.

4.5(g) If $(e_Y < f,g >, 1_X) \in R_{X,\Omega}$, then $(f,g) \in R_{X,Y}$.

We sometimes refer to this property as "R is strict".

4.6 Definition Let $\underline{C}$ be a consistent prelogical category and let V be a filter on $\underline{C}(I,\Omega)$. Define a system F^V of filters F_X on $\underline{C}(X,\Omega)$ (i.e. F^V is the system $F = (F_X | X \in Ob(\underline{C}))$ by:

Let $X \xrightarrow{\varphi} \Omega$ in $\underline{C}$.

Then $\varphi \in F_X$ iff $\neg \exists_{!_X}[\neg\varphi] \in V$.

4.7. Proposition (a) If V is proper and $\underline{C}$ is fair, then each F_X is proper.

(b) For each $\varphi \in F_Y$ and $X \xrightarrow{f} Y$ in $\underline{C}$, φf is in F_X.

(c) For each $\varphi \in F_X$ and $X \xrightarrow{f} Y$ in $\underline{C}$, $\neg \exists_f[\neg\varphi]$ is in F_Y .

Proof: (a) We shall show that $0_X \notin F_X$. In any Heyting algebra $\neg 0 = 1$; $\therefore \neg 0_X = 1_X$. Since $\underline{C}$ is fair, $(\exists_{!_X}[\neg 0_X])\, !_X = \neg 0_X = 1_X$ by 1.4(f)

$\therefore \neg(\exists_{!_X}[\neg 0_X]\, !_X) = \neg 1_X = 0_X$; i.e. (since ( ) $!_X$ is Heyting homomorphism) $(\neg \exists_{!_X}[\neg 0_X])\, !_X = 0_X = 0\, !_X$. But $!_X$ is epi (by fairness).

$\therefore \neg \exists_{!_X}[\neg 0_X] = 0$, and $0 \notin V$ since V is proper. i.e. $0_X \notin F_X$ as required.

(b) Since $\varphi \in F_Y$, $\neg \exists_{!_Y}[\neg\varphi] \in V$.

$\exists_{!_X}[\neg(\varphi f)] = \exists_{!_Y} f[\neg(\varphi f)] = \exists_{!_Y}[\exists_f[\neg(\varphi f)]]$ (by 1.4(c)(ii))

$= \exists_{!_Y}[\exists_f[(\neg\varphi)f)]]$ (because ( ) f is a Heyting homomorphism)

$\leq \exists_{!_Y}[\neg\varphi]$ (by 1.4(a)(ii), since $\exists_{!_Y}$ is order-preserving).

Since $\neg$ is order-reversing (1.4(a)(i)) $\therefore \neg \exists_{!_X}[\neg(\varphi f)] \geq \neg \exists_{!_Y}[\neg\varphi] \in V$.

$\therefore \neg \exists_{!_X}[\neg(\varphi f)] \in V$ i.e. $\varphi f \in F_X$, as required.

(c) Since $\varphi \in F_X$, we have $7\,\exists_{!_X}\,[7\varphi] \in V$. Again using (1.4(c)(ii)) we have

$\exists_{!_X}\,[7\varphi] = \exists_{!_Y f}\,[7\varphi] = \exists_{!_Y}\,[\,\exists_f[7\varphi]]\quad [\exists_{!_Y}\,[77\,\exists_f\,[7\varphi]]$.

∴ Since 7 is order-reversing,

$7\,\exists_{!_Y}\,[77\,\exists_f[7\varphi]] \geq 7\,\exists_{!_X}\,[7\varphi] \in V$;

∴ $7\,\exists_{!_Y}\,[77\,\exists_f[7\varphi]] \in V$. i.e. $7\,\exists_f[7\varphi] \in F_Y$, as required.

4.8. Definition. Let C be a prelogical category and V a filter on C (I,Ω). Define a system R^V of relations $R_{X,Y}$ on C(X,Y) i.e. $R^V = \bigcup\,(R_{X,Y}\,|X,Y \in Ob(C))$ by :

Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in C . Then

$(f,g) \in R_{X,Y}$ iff $e_Y < f,g > \in F_X^V$

i.e. iff $7\,\exists_{!_X}\,[7\,e_Y < f,g >\,] \in V$.

R is called the relation on C generated by V .

4.9 Theorem R^V is a prelogical congruence relation on C.

Moreover if V is proper and C is consistent and fair, then R is proper.

Proof: We verify first that each $E_{X,Y}$ is an equivalence relation on C(X,Y).

The reflexive property follows from 1.4(g) since $1_X \in F_X$.

With respect to the symmetric property, notice that the isomorphism $Y \times Y \xrightarrow{<p_2,p_1>} Y \times Y$ has the properties that

$< g,f > = < p_2,p_1 > < f,g >$ for any $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ and that

$< p_2,p_1 > \Delta_Y = \Delta_Y$.

$\therefore \; e_Y < p_2, p_1 > = (\exists_{\Delta_Y} [1_Y]) < p_2,p_1 >$

$= (\exists_{<p_2,p_1>} \; \Delta_Y [1_Y]) < p_2,p_1 >$ (by above note).

$= \left(\exists_{< p_2,p_1 >} [\exists_{\Delta_Y} [1_X]\right) < p_2,p_1 >$ (by 1.4(c)(ii))

$\geq \exists_{\Delta_Y} (1_Y]$ (by 1.4(a)(i)

$= e_Y$

Thus if we suppose $(f,g) \in \overline{R}_{X,Y}$ then we have

$e_Y <g,f> = e_Y <p_2,p_1> <f,g> \; \geq \; e_Y <f,g > \in V$,

since ( ) <f,g> is a Heyting homomorphism. i.e. we have

$e_Y <g,f > \in V$; i.e. $(g,f) \in R_{X,Y}$.

To verify the transitive property, Let $(f,g) \in R_{X,Y}$ and $(g,h) \in R_{X,Y}$.

i.e. $e_Y <f,g > \in F_X$ and $e_Y <g,h> \in F_X$. By 1.7(f),

$e_Y <f,h> \; \geq \; e_Y <f,g > \; \wedge \; e_Y <g,h> \in F_X$.

$\therefore \; e_Y <f,h> \in F_X$ i.e. $(f,h) \in R_{X,Y}$.

Thus $R_{X,Y}$ is transitive and therefore is an equivalence relation.

To show that R is compatible with composition,

Let $(g_1, g_2) \in R_{X,Y}$ and $(f_1, f_2) \in R_{Y,Z}$.

Therefore $e_Z \langle f_1, f_2 \rangle \in F_Y$ and $e_Y \langle g_1, g_2 \rangle \in F_X$.

∴ By <u>4.7(b)</u> $(e_Z) \langle f_1, f_2 \rangle (g_2) \in F_X$; i.e. $e_Z \langle f_1 g_2, f_2 g_2 \rangle \in F_X$.

Moreover we have $e_Y \langle g_1, g_2 \rangle \in F_X$ giving

$$e_Z \langle f_1 g_1, f_1 g_2 \rangle \geq e_Y \langle g_1, g_2 \rangle \in F_X \quad \text{by } \underline{1.4(k)}$$

Now $e_Z \langle f_1 g_1, f_2 g_2 \rangle \geq e_Z \langle f_1 g_1, f_1 g_2 \rangle \wedge e_Z \langle f_1 g_2, f_2 g_2 \rangle$

<u>by 1.7(f).</u>

Hence $e_Z \langle f_1 g_1, f_2 g_2 \rangle \in F_X$; i.e. $(f_1 g_1, f_2 g_2) \in R_{X,Z}$ as required.

We next verify that R is compatible with products.

Suppose $(f_1, g_1) \in R_{X,Y_1}$ and $(f_2, g_2) = R_{X,Y_2}$ ∴ By hypothesis $e_Y \langle f_1, g_1 \rangle \in F_X$ and $e_{Y_2} \langle f_2, g_2 \rangle \in F_X$.

But by <u>1.7(c)</u>,

$$e_{Y_1 \times Y_2} \langle \langle f_1, f_2 \rangle , \langle g_1, g_2 \rangle \rangle = e_{Y_1} \langle f_1, g_1 \rangle \wedge e_{Y_2} \langle f_2, g_2 \rangle \in F_X$$

i.e. $(\langle f_1, f_2 \rangle, \langle g_1, g_2 \rangle) \in R_{X,Y_1 \times Y_2}$ as required.

Now let $(f_1, f_2) \in R_{X,Y}$ and $(\varphi_1, \varphi_2) \in R_{X,\Omega}$

We have by hypothesis that $e_Y \langle f_1, f_2 \rangle \in F_X$ and that $e_\Omega \langle \varphi_1, \varphi_2 \rangle \in F_X$ i.e. $\varphi_1 \Leftrightarrow \varphi_2 \in F_X$ (by <u>1.1(e)(ii)</u>) . Let p,q denote the projections. $X \times Y \xrightarrow{p} X$ and $X \times Y \xrightarrow{q} Y$.

Then by <u>4.7(b)</u> $e_Y \langle f_1, f_2 \rangle p \in F_{X \times Y}$ and $(\varphi_1 \Leftrightarrow \varphi_2) p \in F_{X \times Y}$.

i.e. $e_Y < f_1p, f_2p > \in F_{X\times Y}$ and $(\varphi_1 p) \Leftrightarrow (\varphi_2 p) \in F_{X\times Y}$

(since ( )p preserves Heyting operations) .

Thus $(e_Y < f_1p, f_2p >) \wedge (\varphi_1 p \Leftrightarrow (\varphi_2 p))$ = call χ, is $\in F_{X\times Y}$.

∴ By 4.7(c) $\neg \exists_q [\neg\chi] \in F_Y$.

But $\neg \exists_q [\neg\chi]$ is precisely L.H.S. 1.7(h); ∴ by 1.7(h),

$\exists_q[\varphi_1] \Leftrightarrow \exists_{f_2}[\varphi_2] \geq \neg \exists_q[\neg\chi] \in F_X$.

i.e $e_\Omega \left\langle \exists_{f_1}[\varphi_1], \exists_{f_2}[\varphi_2] \right\rangle \in F_Y$ by 1.1(e)(ii) ;

i.e. $\left(\exists_{f_1}[\varphi_1], \exists_{f_2}[\varphi_2] \right) \in R_{Y,\Omega}$.

Thus we have shown that R is compatible with quantification .

In verifying that R is a Heyting algebra congruence, we shall make use of the fact that for $\varphi_1, \varphi_2 \in \underline{C}(X,\Omega)$, $(\varphi_1,\varphi_2) \in R_{X,\Omega}$ iff $e_\Omega < \varphi_1, \varphi_2 > \in F_X$ iff $\varphi_1 \Leftrightarrow \varphi_2 \in F_X$ by 1.1(e)(ii).

Let $(\varphi_1,\varphi_2) \in R_{X,\Omega}$ and $(\varphi_1',\varphi_2') \in R_{X,\Omega}$;∴$\varphi_1 \Leftrightarrow \varphi_2 \in F_X$ and $\varphi_1' \Leftrightarrow \varphi_2' \in F_X$.

Call $\psi = (\varphi_1 \Leftrightarrow \varphi_2) \wedge (\varphi_1' \Leftrightarrow \varphi_2')$

$= (\varphi_1 \Rightarrow \varphi_2) \wedge (\varphi_2 \Rightarrow \varphi_1) \wedge (\varphi_1' \Rightarrow \varphi_2') \wedge (\varphi_2' \Rightarrow \varphi_1') \in F_X$,

and notice that for compatibility with $\vee$ it is sufficient to show that

4.9(a) $\psi \leq (\varphi_1 \vee \varphi_1') \Rightarrow (\varphi_2 \vee \varphi_2')$ and

(b) $\psi \leq (\varphi_2 \vee \varphi_2') \Rightarrow (\varphi_1 \vee \varphi_1')$.

By adjointness of $\Rightarrow$ and $\wedge$, (a) is equivalent to

$$\psi \wedge (\varphi_1 \vee \varphi_1') \leq \varphi_2 \vee \varphi_2' ,$$

i.e to $(\psi \wedge \varphi_1) \vee (\psi \wedge \varphi_1') \leq \varphi_2 \vee \varphi_2'$ by distributivity .

But $\psi \wedge \varphi_1 \leq (\varphi_1 \Rightarrow \varphi_2) \wedge \varphi_1 \leq \varphi_2$, and

$\psi \wedge \varphi_1' \leq (\varphi_1' \Rightarrow \varphi_2') \wedge \varphi_1' \leq \varphi_2'$, and thus we do have

$(\psi \wedge \varphi_1) \vee (\psi \wedge \varphi_1') \leq \varphi_2 \vee \varphi_2'$ i.e. we have (a).

(b) is verified similarly, and therefore R is compatible with $\vee$.

For compatibility with $\wedge$, it is sufficient to show that

$\psi \leq (\varphi_1 \wedge \varphi_1') \Leftrightarrow (\varphi_2 \wedge \varphi_2')$ i.e. that

(c) $\psi \leq (\varphi_1 \wedge \varphi_1') \Rightarrow (\varphi_2 \wedge \varphi_2')$ and

(d) $\psi \leq (\varphi_2 \wedge \varphi_2') \Rightarrow (\varphi_1 \wedge \varphi_1')$

Now $\psi \wedge \varphi_1 \wedge \varphi_1' \leq (\varphi_1 \Rightarrow \varphi_2) \wedge \varphi_1 \wedge (\varphi_1' \Rightarrow \varphi_2') \wedge \varphi_1' \leq \varphi_2 \wedge \varphi_2'$.

i.e. by adjointness (c) holds.

(d) follows similarly, to give R compatible with $\wedge$.

To show that R is compatible with $\Rightarrow$ it is sufficient to show that

(e) $\psi \leq (\varphi_1 \Rightarrow \varphi_1') \Rightarrow (\varphi_2 \Rightarrow \varphi_2')$ and

(f) $\psi \leq (\varphi_2 \Rightarrow \varphi_2') \Rightarrow (\varphi_1 \Rightarrow \varphi_1')$

Now $\psi \wedge (\varphi_1 \Rightarrow \varphi_1') \wedge \varphi_2 \leq (\varphi_2 \Rightarrow \varphi_1) \wedge (\varphi_1' \Rightarrow \varphi_2') \wedge (\varphi_1 \Rightarrow \varphi_1') \wedge \varphi_2$

$$\leq \; ((\varphi_2 \Rightarrow \varphi_1) \wedge \varphi_2) \wedge (\varphi_1 \Rightarrow \varphi_1') \wedge (\varphi_1' \Rightarrow \varphi_2')$$

$$\leq \; \varphi_1 \wedge (\varphi_1 \Rightarrow \varphi_1') \wedge (\varphi_1' \Rightarrow \varphi_2')$$

$$\leq \; \varphi_1' \wedge (\varphi_1' \Rightarrow \varphi_2') \; \leq \; \varphi_2' \; .$$

i.e. by adjointness $\psi \wedge (\varphi_1 \Rightarrow \varphi_1') \leq (\varphi_2 \Rightarrow \varphi_2')$;

i.e. by adjointness (e) holds.

Similarly (f) can be verified, and thus R is a Heyting congruence.

It remains only to show that R is strict.

Suppose $(e_Y \langle f,g\rangle, 1_X) \in R_{X,\Omega}$; i.e. $(e_Y \langle f,g\rangle \Leftrightarrow 1_X$.

$\therefore$ $1_X \Rightarrow (e_Y \langle f,g\rangle \in F_X$; i.e. $e_Y \langle f,g\rangle \in F_X$, since in any Heyting algebra $1 \Rightarrow \alpha = \alpha$. But this says precisely that $(f,g) \in R_{X,Y}$ as required; therefore R is a prelogical congruence relation.

Moreover, if V is proper and C is fair, then $0 \notin F_X$ by 4.7(a).

But $e_\Omega \langle 0,1 \rangle = (0 \Rightarrow 1) \wedge (1 \Rightarrow 0) = 0$, if C is consistent.

$\therefore e_\Omega \langle 0,1 \rangle \notin F_X$; ie $(0,1) \notin R_{I,\Omega}$.

This completes the proof of Theorem 4.9.

4.10 Definition. Let C be any small prelogical category and R any prelogical congruence relation on C

(a) Define the category C/R by

$Ob(C/R) = Ob(C)$ and $C/R(X,Y) = C(X,Y)/R$.

(b) Define the functor $Q: C \to C/R$ by :

Q is the identity functor on objects, and

$$Q(X \xrightarrow{f} Y) = \|f\|_{R_{X,Y}} \; .$$

C/R is called the quotient category of C with respect to R and Q is called the canonical projection functor.

4.11 Theorem: (a) C/R is a small prelogical category and Q is a prelogical extension.

(b) If C is fair then C/R is fair

(c) If C is consistent and R is proper then C/R is consistent.

Proof: (a) Given $X \in Ob(C) = Ob(C/R)$, define id_X in C/R as $\|id_X\|$.

(a)(i) Given $X \xrightarrow{\|f\|} Y$, $Y \xrightarrow{\|g\|} Z$ in C/R, define $\|g\| \, \|f\|$ as $\|gf\|$.

Property 4.5(b) and the definition of Q guarantee that this composition will be well-defined and associative, that $\|id_X\|$ does act as an identity morphism (since $\|f\| \, \|id_X\| = \|fid_X\| = \|f\|$), and that Q will preserve identities and composition.

Since each C(X,Y) is a set, clearly C/R(X,Y) is a set, ie. C/R is small. Thus C/R is a small category and Q is a functor. It should also be noted that by definition of C/R, Q is full.

Given $X \xrightarrow{\|f\|} Y$, $X \xrightarrow{\|g\|} Z$ in C/R, Define the product map in C/R by:

4.11(a)(ii) $< \|f\|, \|g\| > = \|< f,g >\|$

By 4.5(c) we have that this is well-defined, has the universal property required for a product map and is preserved by Q. If we denote by p_1, p_2 the projections of $Y \times Z$ in C, then clearly $\|p_1\|, \|p_2\|$ are the projections of $Y \times Z$ in C/R.

Given $X \overset{\|\varphi\|}{\underset{\|\psi\|}{\rightrightarrows}} \Omega$ in C/R, we define the lattice operations by:

(a)(iii) $\|\varphi\| \vee \|\psi\| = \|\varphi \vee \psi\|$ and $\|\varphi\| \wedge \|\psi\| = \|\varphi \wedge \psi\|$.

By 4.5(e), $\vee, \wedge$ are well-defined, are preserved by Q, and make $\underline{C}/R(X,\Omega)$ into a distributive lattice. This in turn gives us that

(iv) $\|\varphi\| \leq \|\psi\|$ iff $\varphi \leq \psi$, since then $\|\varphi\| \wedge \|\psi\| = \|\varphi \wedge \psi\| = \|\varphi\|$.

Thus in particular $\|0_X\| \leq \|\varphi\| \leq \|1_X\|$ for every φ, giving the least and greatest elements. We further define

(v) $\|\varphi\| \Rightarrow \|\psi\| = \|\varphi \Rightarrow \psi\|$

Then 4.5(f) and 1.1(b) ensure that this will be well-defined, and make $\underline{C}/R(X,\Omega)$ into a Heyting algebra.

All this, plus 4.11(a)(i) and 1.1(b), give that every $\underline{C}/R(\|f\|,\Omega)$ will be a Heyting homomorphism.

Thus $\underline{C}/R$ has a Heyting algebra object, whose structure is preserved by Q.

4.11 (a)(v) Given $X \xrightarrow{\|\varphi\|} \Omega$ and $X \xrightarrow{\|f\|} Y$ in $\underline{C}/R$, define the quantification by

$$\exists_{\|f\|}[\|\varphi\|] = \| \exists_f[\varphi] \| .$$

By 4.5(d), this is well-defined and preserved by Q.

Now suppose $\|\varphi\| \leq \|\psi\| \|f\| = \|\psi f\|$ for some $Y \xrightarrow{\psi} \Omega$.

$\therefore$ $\varphi \leq \psi f$ by 4.11(a)(iv);

$\therefore$ $\exists_f[\varphi] \leq \psi$ by adjointness of $\exists$ in $\underline{C}$.

i.e. $\|\exists_f[\varphi]\| \leq \|\psi\|$ by 4.11 (a)(iv).

Similarly we obtain the reverse implication.

Thus $\exists_{\|f\|}[\ ]$ is adjoint to $(\ )\|f\|$.

The Beck conditions are forced to hold because they are equational conditions involving only compositions, products, and $\exists$. For example, 1.1(d)(i) results from:

$$\exists_{<\|id_X\|,\|f\|>}[\|1_Y\|\ \|f\|] = \|\exists_{<id_X,f>}[1_Y f]\| \quad \text{by } \underline{4.11(a)(i)-(v)}$$

$$= \|(\exists_{\Delta_Y}[1_Y])(f \times id_Y)\| \quad \text{by } \underline{1.1(d)(i)} \text{ in } \underline{C}$$

$$= (\exists_{\|\Delta_Y\|}[\|1_Y\|])(\|f\| \times \|id_Y\|) \quad \text{by } \underline{4.11(a)(i)-(v)}$$

Thus $\underline{C}/R$ has an appropriate quantification, and Q is a prelogical functor.

It remains to show that $\underline{C}/R$ satisfies conditions 1.1(e). Suppose $\|e_Y\| < \|f\|,\|g\| > = \|1_Y\|$ for some $X \overset{f}{\underset{g}{\rightrightarrows}} Y$.
i.e. $\|e_Y <f,g>\| = \|1_X\|$; i.e. $(e_Y <f,g>, 1_X) \in R_{X,\Omega}$.

Therefore $(f,g) \in R_{X,Y}$ since R is strict. i.e. $\|f\| = \|g\|$, and so 1.1(e)(i) holds in $\underline{C}/R$.

Moreover $\|e_\Omega\| < \|\varphi\|, \|\psi\| > = \|e_\Omega <\varphi,\psi>\| = \|\varphi <\Rightarrow> \psi\|$ by 1.1(e)(ii) in $\underline{C}$ $= \|\varphi\| <\Rightarrow> \|\psi\|$ by 4.11 (a)(ii)-iv) .

Thus $\underline{C}/R$ satisfies 1.1(e)(ii) and therefore is a prelogical category.

Since Q is prelogical and is the identity on objects, it is clearly an extension .

(b) Now suppose that $\underline{C}$ is fair, i.e. by 1.8(b) all projections in $\underline{C}$ are epi.

Let $X \times Y \xrightarrow{\|p\|} Y$ be any projection map in $\underline{C}/R$; $\therefore$ p is epi in $\underline{C}$.

$\therefore \quad \exists_{\|p\|}\,[\|1_{X\times Y}\|] = \|\exists_p\,[1_{X\times Y}]\| = \|1_Y\|$ by 1.4(h)

Thus by 1.4(h) $\|p\|$ is epi, and so $\underline{C}/R$ is fair.

(c) If $\underline{C}$ is consistent and R proper, we have $0 \neq 1$ in $\underline{C}$, $\therefore$ $(0,1) \notin R$. $\therefore$ $\|0\| \neq \|1\|$; i.e. $\underline{C}/R$ is consistent. This completes the proof of 4.11.

We next define, and give several properties of, the category which arises as the intuitive limit of an infinite sequence of extensions.

4.12 Definition. Let $\underline{C}_o \xrightarrow{F_{0,1}} \underline{C}_1 \xrightarrow{F_{1,2}} \cdots \longrightarrow \underline{C}_i \xrightarrow{F_{i,i+1}} \underline{C}_{i+1} \longrightarrow \cdots$

be a countable chain of prelogical extensions $F_{i,i+1}$, where each $\underline{C}_i$ is a small prelogical category, with $Ob\,(\underline{C}_i) = Ob\,(\underline{C}_o)$, and each $F_{i,i+1}$ is the identity functor on objects.

Define $F_{0,2} = F_{1,2}\,F_{0,1}$ and generally .

$F_{i,j} = F_{j-1,j}F_{j-2,j-1} \cdot \cdot F_{i,i+1}$. We adopt the notation $F_{i,i}$ for the identity functor $\underline{C} \to \underline{C}_i$. Define the category $\hat{\underline{C}}$ (also denoted $R \lim_{i\to\infty} \underline{C}_i$) as follows: $Ob\,(\hat{\underline{C}}) = Ob(\underline{C}_o)$

$\hat{\underline{C}}(X,Y) = \bigcup_{i\in N} \underline{C}_i(X,Y) \Big/ \equiv$ where $\equiv$ is the relation defined by $f_{i_1} \equiv f_{i_2}$ iff there exists $j \in N$ that $F_{i_1,j}(f_{i_1}) = F_{i_2,j}(f_{i_2})$

Define the functor $F^i : \underline{C}_i \to \hat{\underline{C}}$ for each i by: F^i is the identity functor on objects and $F^i(X \xrightarrow{f} Y) = \|f\|_{\equiv}$

Remarks to Definition 4.12: (a) The $F_{i,j}$ are all prelogical extensions by 3.3 .

(b) The equivalence $\equiv$ says roughly that two morphism are equivalent in C iff they are eventually identified. A special case of this is the situation where $i_1 \geq i_2$ and $f_{i_1} = F_{i_2,i_1}(f_{i_2})$. Note also that if two morphisms are equivalent with respect to $\equiv$, then their images throughout the chain are also equivalent. Explicitly, if $f_{i_1} \equiv f_{i_2}$ and j,k are any indices, then $F_{i_1,j}(f_i) \equiv F_{i_2,k}(f_{i_2})$.

4.13 Theorem. C so defined is a small prelogical category and each F^i is a prelogical extension.

Proof: The relation $\equiv$ is clearly an equivalence .

Moreover $\equiv$ is compatible with composition:

Let $Y \xrightarrow{f_1} Z$ in C_{i_1}, $Y \xrightarrow{f_2} Z$ in C_{i_2} $\ni$ $f_1 \equiv f_2$

and let $X \xrightarrow{g_1} Y$ in C_{i_1}, $X \xrightarrow{g_2} Y$ in C_{i_2} $\ni$ $g_1 \equiv g_2$.

Hence there exist $j,k \in N \ni F_{i_1,j}(f_1) = F_{i_2,j}(f_2)$ = call $Y \xrightarrow{f_j} Z$ in C_j; and $F_{i_1,k}(g_1) = F_{i_2,k}(g_2)$ = call $X \xrightarrow{g_k} Y$ in C_k.

Call $\ell = \max(j,k)$ and call $f = F_{j,\ell}(f_j)$, $g = F_{k,\ell}(g_k)$. Now certainly fg = fg

i.e. $F_{j,\ell}(f_j)F_{k,\ell}(g_k) = F_{j,\ell}(f_j)F_{k,\ell}(g_k)$ i.e.

i.e. $F_{j,\ell}(F_{i_1,j}(f_1))F_{k,\ell}(F_{i_1,k}(g_1)) = F_{j,\ell}(F_{i_2,j}(f_2))F_{k,\ell}(F_{i_2,k}(g_2))$

i.e by definition of the $F_{i,j}$ $F_{i_1,\ell}(f_1)F_{i_1,\ell}(g_1) = F_{i_2,\ell}(f_2)F_{i_2,\ell}(g_2)$

But $F_{i_1,\ell}$, $F_{i_2,\ell}$ are prelogical and therefore, preserve composition.

$\therefore$ $F_{i_1,\ell}(f_1g_1) = F_{i_2,\ell}(f_2g_2)$ i.e precisely $f_1g_1 \equiv f_2g_2$, as required.

We use this fact to define composition in $\hat{\underline{C}}$ as follows:

Let $X \xrightarrow{\|g\|} Y \xrightarrow{\|f\|} Z$, where g is in $\underline{C}_{i_2}$ f is in $\underline{C}_{i_1}$. Call $j = \max(i_1, i_2)$ and call $f_j = F_{i_1,j}(f)$, $g_j = F_{i_2,j}(g)$. (Thus $f_j \equiv f$ and $g_j \equiv g$)

Define $\|f\| \, \|g\| = \|f_j g_j\|$. The compatibility with composition ensures that this operation is well-defined, and that for composable morphisms f',g' in any $\underline{C}_i$,

<u>4.13 (a)</u> $\|f'g'\| = \|f'\| \, \| \, \|g'\|$. In particular it follows that $\|f\| \, \|id_Y\| = \|f \, id_Y\| = \|f\|$ and $\|id_Y\| \, \|g\| = \|g\|$.

To verify that the composition is associative, let $W \xrightarrow{\|h\|} X$ in $\hat{\underline{C}}$, where h is in $\underline{C}_{i_3}$. Call $j' = \max(i_1, i_2, i_3)$ and call $f_{j'} = F_{i_1,j'}(f)$, etc. Then we have $f_{j'} \equiv f$, $g_{j'} \equiv g$, $h_{j'} \equiv h$; moreover $(f_{j'})(g_{j'} \, h_{j'}) = (f_{j'} g_{j'}) h_{j'}$ because composition in $\underline{C}_{j'}$ is associative.

$\therefore \quad \|f_{j'} \, (g_{j'} h_{j'})\| = \|(f_{j'} g_{j'}) h_{j'}\|$.

$\therefore$ By <u>4.13(a)</u> $\|f_{j'}\| \, (\|g_{j'}\| \, \|h_{j'}\|) = (\|f_{j'}\| \, \|g_{j'}\|) \, \|h_{j'}\|$;

i.e that $\|f\| \, (\|g\| \, \|h\|) = (\|f\| \, \|g\|) \, \|h\|$.

Thus $\hat{\underline{C}}$ is a category and each F^i is a functor.

By the same method as was just used for composition, it can be verified that $\equiv$ is strict and is compatible with products, Heyting operations, and quantification.

For example, compatibility with $\vee$ is proved as follows: Let $\varphi_{i_1} \equiv \varphi_{i_2}$ and $\psi_{i_1} \equiv \psi_{i_2}$ by $F_{i_1,j}(\varphi_{i_1}) = F_{i_2,j}(\varphi_{i_2})$, $F_{k_1,\ell}(\psi_{i_1}) = F_{k_2,\ell}(\psi_{i_2})$.

Call $m = \max(j,\ell)$ and call $\varphi_m = F_{i,m}(\varphi_{i_1}) = F_{i_2,m}(\varphi_{i_2})$ and ψ_m = etc. Thus $\varphi_m \vee \psi_m = F_{i_1,m}(\varphi_{i_1}) \vee F_{i_1,m}(\psi_{i_1}) = F_{i_1,m}(\varphi_{i_1} \vee \psi_{i_1})$ because $F_{i,j}$ are prelogical. Similarly $\varphi_m \vee \psi_m = F_{i_2,m}(\varphi_{i_2} \vee \psi_{i_2})$ $\therefore$ $(\varphi_{i_1} \vee \psi_{i_1}) \equiv (\varphi_{i_2} \vee \psi_{i_2})$. The other compatibility properties, i.e. 4.5(c)-(g) are verified similarly. Thus $\equiv$ satisfies 4.5(b)-(g).

We then define product maps, Heyting operations and quantification in $\hat{\underline{C}}$ by means of representative elements as was done for composition, and it can be shown that these operations are well-defined, are preserved by Q, and make $\hat{\underline{C}}$ into a prelogical category. In each case the required property follows from the compatibility of $\equiv$ and the corresponding property of the $\underline{C}_i$, by a process of tracing to a $\underline{C}_j$ which contains all the morphisms involved.

To illustrate the procedure we shall carry out the definition and proof for $\exists$:

Let $X \xrightarrow{\|f\|} Y$, $X \xrightarrow{\|\varphi\|} \Omega$ and $Y \xrightarrow{\|\psi\|} \Omega$ in $\hat{\underline{C}}$, where f is in $\underline{C}_i$, φ is in $\underline{C}_j$ and ψ is in $\underline{C}_k$. Call $\ell = \max(i,j,k)$ and call $f_\ell = F_{i,\ell}(f)$, $\varphi_\ell = F_{j,\ell}$. Note that thus $\|f_\ell\| = \|f\|$, $\|\varphi_\ell\| = \|\varphi\|$ and $\|\psi_\ell\| = \|\psi\|$. Define $\exists_{\|f\|}[\|\psi\|] = \|\exists_{\|f_\ell\|}[\psi_\ell]\|$

To see that this is well-defined, let $f' \equiv f$ and $\varphi' \equiv \varphi$ where f' is in $\underline{C}_{i'}$, φ' in $\underline{C}_{j'}$, $F_{i',k'}(f') = F_{i,k}(f)$ and $F_{j',\ell'}(\varphi') = F_{j,\ell}(\varphi)$.

Call $m = \max(k',\ell')$, call $f'_m = F_{i',m}(f)$ and so on.

Thus $\|f'_m\| = \|f'\| = \|f\| = \|f_\ell\|$ and $\|\varphi'_m\| = \|\varphi'\| = \|\varphi\| = \|\varphi_\ell\|$.

Since $\equiv$ is compatible with $\exists$; $\therefore \|\exists_{f'_m}[\varphi'_m]\| = \|\exists_{f_\ell}[\varphi_\ell]\|$; i.e

$\exists_{\|f\|}\|\varphi\| = \exists_{\|f'\|}\|\varphi'\|$.

This fact gives us a fortiori: that for any morphism $X \xrightarrow{h} Y$,

$X \xrightarrow{\chi} \Omega$ in the same category $\underline{C}_n$,

<u>4.13(b)</u> $\exists_{\|h\|}[\|\chi\|] = \|\exists_h[\chi]\|$ i.e precisely that each F^i preserves existential quantification.

To show that $\exists_{\|f\|}[\ ]$ is adjoint to $(\ )\|f\|$, let i,j,k,ℓ be as before, call $\psi_\ell = F_{k,\ell}(\psi)$ and note the following two facts:

$\|\psi_\ell f_\ell\| = \|\psi_\ell\|\|f_\ell\| = \|\psi\|\ \|f\|$ by <u>4.13(a)</u>, and $\leq$ is defined by

$\|\varphi_\ell\| \leq \|\psi_\ell\|$ if $\varphi_\ell \leq \psi_\ell$.

Suppose $\exists_{\|f\|}[\|\psi\|] \leq \|\psi\|$; i.e. $\|\exists_{f_\ell}[\varphi_\ell]\| \leq \|\psi_\ell\|$; i.e $\exists_{f_\ell}[\varphi_\ell] \leq \psi_\ell$

$\therefore$ By adjointness of $\exists$ in $\underline{C}_\ell$, $\varphi_\ell \leq \psi_\ell f_\ell$; i.e. $\|\varphi_\ell\| \leq \|\psi_\ell f_\ell\|$.

i.e. $\|\varphi\| \leq \|\psi\|\ \|f\|$.

Conversely suppose $\|\varphi\| \leq \|\psi\|\ \|f\|$, and retracing the argument gives $\exists_{\|f\|}[\|\varphi\|] \leq \|\psi\|$.

All of the properties <u>1.1</u> of a prelogical category can be verified in $\hat{\underline{C}}$ by the same method. Likewise one obtains the preservation of the prelogical operations by each $\underline{C} \xrightarrow{F^i} \hat{\underline{C}}$.

Moreover, $\hat{\underline{C}}$ is small because $\bigcup_i \underline{C}_i(X,Y)$ is a union of sets, hence is a set.

4.14 Proposition (a) If each $\underline{C}_i$ is consistent, then $\hat{\underline{C}}$ is consistent

(b) If each $\underline{C}_i$ is maximally consistent then so is $\hat{\underline{C}}$.

(c) If each $\underline{C}_i$ is rich, then $\hat{\underline{C}}$ is rich

(d) If each $\underline{C}_i$ is fair, then $\hat{\underline{C}}$ is fair.

(e) If each $F_{i,i+1}$ is conservative, then each F^i is conservative.

Proof: Note first that in a chain of the kind described, given any $\underline{C}_i \xrightarrow{F_{i,i+1}} \underline{C}_{i+1}$; $F_{i,i+1}$ must be the identity functor on certain universal morphisms. Namely, for all objects, X,Y of $\underline{C}_i$, $F_{i,i+1}$ must preserve the morphisms $id_X, 0_X, 1_X, \Delta_X, !_X$ and the projections $X \times Y \xrightarrow{p_1} X$, $X \times Y \xrightarrow{p_2} Y$, simply by virtue of being prelogical (recall remark 3.2(a)). This fact will be used implicitly in the proof of the proposition.

(a) This is obvious since, given any $i,j \in N$, since $F_{i,j}$ is prelog - $F_{i,j}(0) = 0$ and $F_{i,j}(1) = 1$. If each $\underline{C}_i$ is consistent, $\therefore 0 \neq 1$. $\therefore$ $F_{i,j}(0)$ never $F_{i,j}(1)$.

$$F_{i,j}(0) \text{ never } = F_{k,j}(1) \text{ (for any } k\text{). i.e. } 0 \neq 1 .$$

(b) By hypothesis each $\underline{C}_i(I,\Omega) = \{0,1\}$. $\therefore \bigcup_i \underline{C}_i(I,\Omega) = \{0,1\}$.

$\therefore$ by consistency, $\therefore$ $\underline{C}(I,\Omega) = \{0,1\}$.

(c) Let $\exists_{\|!_X\|}[\|\varphi\|] = \|1\|$ in $\hat{\underline{C}}$, where φ is in $\underline{C}_i$. Without loss of generality we may say $!_X$ and 1 are morphisms of $\underline{C}_i$.

$\therefore \; \exists_{\|!_X\|} \|\varphi\|] = \|1\|$; $\therefore$ there is $j \geq i$ such that

$$F_{i,j}(\exists_{!_X}[\varphi]) = F_{i,j}(1) \text{ in } \underline{C}_j .$$

i.e. since $F_{i,j}$ is prelogical , $\exists_{!_X}[F_{i,j}(\varphi)] = 1$ in $\underline{C}_j$.

But $\underline{C}_j$ is rich; $\therefore$ there is $I \xrightarrow{x} X$ in $\underline{C}_j$ such that

$(F_{i,j}(\varphi))x = 1$. $\therefore \|F_{i,j}(\varphi)\| \, \|x\| = \|1\|$; i.e. $\|\varphi\|\|x\| = \|1\|$,

and so $\hat{\underline{C}}$ is rich.

($\underline{d}$) Assume that each $\underline{C}_j$ is fair; i.e. all projections are epi in $\underline{C}_j$.

Let $X \times Y \xrightarrow{\|p\|} X$ be any projection map in $\hat{\underline{C}}$, where p will then

be a projection $X \times Y \xrightarrow{p} X$ in some $\underline{C}_i$.

Now $\exists_{\|p\|}[\|1_{X \times Y}\|] = \| \exists_p[1_{X \times Y}] \| = \|1_Y\|$ by $\underline{1.4(h)}$ since p is

epi in $\underline{C}_i$.

Thus by $\underline{1.4(h)}$ $\|p\|$ is epi in $\hat{\underline{C}}$, and so $\hat{\underline{C}}$ is fair.

(e) Suppose that each $F_{i,i+1}$ is conservative. Let $i \in N$ and

let $I \xrightarrow{\alpha} \Omega$ in $\underline{C}_i$ such that $F^i(\alpha) = 1$ in $\hat{\underline{C}}$ $\therefore \|\alpha\| = \|1\|$;

i.e. there is $j \geq i$ such that $F_{i,j}(\alpha) = 1$ in $\underline{C}_j$.

Since $F_{i,j}$ is conservative by $\underline{3.3(c)}$, $\alpha = 1$ in $\underline{C}_i$, and so F^i

is conservative.

__Remark__: The definition of R lim $\underline{C}_i$ could be generalized to any

countable chain of small prelogical categories where the functors

$F_{i,i+1}$ are prelogical and 1-1 on objects. The definition of Ob($\hat{\underline{C}}$)

would then be modified as follows:

$Ob(\hat{\underline{C}}) = \bigcup_{i \in N} Ob(\underline{C}_i)/\equiv'$, where $\equiv'$ is defined by $X_{i_1} \equiv' X_{i_2}$ iff there exists j such that $F_{i_1,j}(X_{i_1}) = F_{i_2,j}(X_{i_2})$;

$\hat{\underline{C}}(X,Y)$ unchanged.

It can be verified, although tediously, that 4.13, 4.14 and 4.15 would hold for this more general $\hat{\underline{C}}$. However, for the present purposes we require only the definition 4.12. Indeed, it may be remarked by 3.2(a) that in the case where the $F_{i,j}$ are extensions (i.e. are bijective on objects) then every prelogical chain is naturally isomorphic to some chain of the kind in 4.12.

4.15. Proposition. Let... $\underline{C}_i \xrightarrow{F_{i,i+1}} \underline{C}_{i+1}$... be a chain of prelogical extensions as in 4.12, and consider the categories $\underline{D}_j = \underline{C}_{2j}$ and $\underline{E}_j = \underline{C}_{2j+1}$.
The prelogical extensions $G_{j,j+1} = F_{2j,2j+2}$ and $H_{j,j+1} = F_{2j+1,2j+3}$ clearly form chains

$$\ldots \underline{D}_j \xrightarrow{G_{j,j+1}} \underline{D}_{j+1} \ldots \quad \text{and} \quad \underline{E}_j \xrightarrow{H_{j,j+1}} \underline{E}_{j+1}$$

of the kind in 4.12 also.
Call $\hat{\underline{D}} = \underset{j \to \infty}{R \lim} \underline{D}_j = \underset{i \to \infty}{R \lim} \underline{C}_{2i}$ and $\hat{\underline{E}} = \underset{j \to \infty}{R \lim} \underline{E}_j = \underset{i \to \infty}{R \lim} \underline{C}_{2i+1}$.
Then $\hat{\underline{D}} = \hat{\underline{E}} = \hat{\underline{C}}$.

Proof: Clearly it is sufficient to show that $\hat{\underline{D}} = \hat{\underline{C}}$. Now $Ob(\hat{\underline{D}}) = Ob(\hat{\underline{C}})$ by definition and for each X,Y, $\bigcup_i \underline{C}_i(X,Y) = \bigcup_i \underline{C}_{2i}(X,Y)$. So we require only to show that the equivalence relations are the same. Denote the equivalences as $\equiv_C$ and $\equiv_D$.

Suppose $X \xrightarrow{f} Y$ in $\underline{D}_j$, $X \xrightarrow{h} Y$ in $\underline{D}_k$ are such that $f \equiv_D h$. Thus there exists $\ell > j,k$ such that $G_{j,\ell}(f) = G_{k,\ell}(h)$; i.e. $F_{2j,2\ell}(f) = F_{2k,2\ell}(h)$ giving $f \equiv_c h$.

Conversely let $X \xrightarrow{f} Y$ in $\underline{C}_{i_1}$, $X \xrightarrow{h} Y$ in $\underline{C}_{i_2}$ such that $f \equiv_c h$. ∴ There exists $j \geq i_1, i_2$ such that $F_{i_1,j}(f) = F_{i_2,j}(h)$ = call g in $\underline{C}_j$. Let k be any even number $\geq j$. Then $F_{i_1,k}(f)$, $F_{i_2,k}(h)$ are maps in $D_{k/2}$ and $G_{k,k}(f) = G_{k,k}(h)$ ($= F_{j,k}(g)$). Thus $f \equiv_D h$, and so the relations are equivalent.

In the following three theorems we make specific application of the general constructions developed so far, in order to construct conservative saturated extensions for all appropriate prelogical categories.

<u>4.16</u> <u>Theorem</u>. Let $\underline{C}$ be a small prelogical category which is consistent and fair. Then there exists a small, fair, consistent category $\underline{C}'$ with $Ob(\underline{C}') = Ob(\underline{C})$, and a conservative enriching extension $\underline{C} \xrightarrow{H} \underline{C}'$ which is the identity functor on objects.

<u>Proof</u>: Define the category $\underline{K}$ by:

$Ob(\underline{K}) = Ob(\underline{C})$.

For each $X \in Ob(\underline{C})$, $\underline{K}(I,X)$ and $\underline{C}(X,\Omega)$ are isomorphic sets; denote the element of $\underline{K}(I,X)$ corresponding to a given $X \xrightarrow{\Phi} \Omega$ in $\underline{C}$ by $I \xrightarrow{\overline{\Phi}} X$.

For each $X,Y \in Ob(\underline{C})$ with $X \neq I$, $\underline{K}(X,Y) = \emptyset$.

Clearly $\underline{K}$ is a category of constants for $\underline{C}$ and so we have, by 4.4, the faithful extension $\underline{C} \xrightarrow{J_1} \underline{C}[\underline{K}]$ with $\underline{C}[\underline{K}]$ small, fair and consistent, and the embedding $\underline{K} \xrightarrow{J_2} \underline{C}[\underline{K}]$.

Consider the set $W \subseteq \underline{C}[\varphi]\ (I,\Omega)$ defined by

$$W = \{ J_1(\varphi)\ J_2(\bar{\varphi}) \quad \text{all } \varphi \in \underline{C}(X,\Omega) \text{ such that } \exists_X [\varphi] = 1 \text{ in } \underline{C}$$

Note that $0 \notin W$ because:

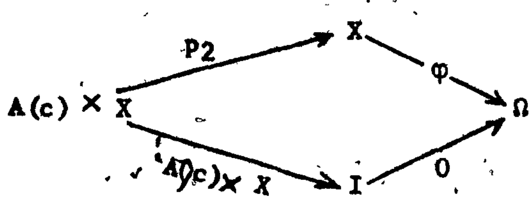
For each $X \xrightarrow{\varphi} \Omega$, recall that $J_1(\varphi) = \nparallel \varphi | \emptyset \nparallel$ (i.e. $I \times X \xrightarrow{\varphi} \Omega$) and $J_2(\bar{\varphi}) = \nparallel id_X | \bar{\varphi} \nparallel$ (i.e. $A(\bar{\varphi}) \times I = X \times I \xrightarrow[id_X]{} X$).

Thus $J_1(\varphi)\ J_2(\bar{\varphi}) = < \varphi | \bar{\varphi} >$ (i.e. $I \times X \times I \xrightarrow{id_X} I \times X \xrightarrow{\varphi} \Omega$;

i.e. $X \xrightarrow{\varphi} \Omega$).

Suppose that $J_1(\varphi) J_2(\bar{\varphi}) = \theta$ in $\underline{C}[\ ]$; i.e. $\nparallel \varphi | \bar{\varphi} \nparallel = \nparallel 0 | \emptyset \nparallel$.

By definition 4.1 there must then exist some $A(c) = Y \in Ob(\underline{C})$ such that the diagram



commutes.

i.e. $\varphi\, p_2 = 0_{Y \times X}$. $\therefore$ $\exists_{p_2} [\varphi\, p_2] = 0_X$ by 1.4(d)(i).

But since p_2 is epi (by fairness), $\therefore$ $\exists_{p_2} [\varphi\, p_2] = \varphi$ by 1.4(f)

$\therefore$ $\varphi = 0_X$; $\therefore$ $\exists_X [\varphi] = 0 \neq 1$.

Thus by definition of W, $J_1(\varphi)\ J_2(\bar{\varphi}) \notin W$; and so $0 \notin W$.

Let V be the filter on $\underline{C}[K](I,\Omega)$ generated by W, and let R be the congruence relation on $\underline{C}[K]$ generated by V as in 4.8.

Call $\underline{C}' = \underline{C}[K]/R$.

Define $\underline{C} \xrightarrow{H} \underline{C}'$ by $H = Q\,J_1$ where $Q: \underline{C}[K] \to \underline{C}'$ denotes the canonical projection.

We have $\underline{C}'$ small, fair, prelogical with $Ob(\underline{C}') = Ob(\underline{C})$ and Q an extension, by 4.11.

Moreover by the construction we have, for each $X \xrightarrow{\varphi} \Omega$ with $\exists_X [\varphi] = 1$ in $\underline{C}$, the constant $k = Q(J_2(\bar{\varphi}))$ in $\underline{C}'$ such that

$$H(\varphi)k = Q(J_1(\varphi))\,Q(J_2(\bar{\varphi})) = Q(J_1(\varphi)J_2(\bar{\varphi}))$$

$$= \|J_1(\varphi)J_2(\bar{\varphi})\|_R = \|1\| \text{ since } J_1(\varphi)J_2(\bar{\varphi}) \in W \subseteq V .$$

i.e. precisely that $\underline{C} \xrightarrow{H} \underline{C}'$ enriches $\underline{C}$.

It remains only to prove that H is conservative.

Let $I \xrightarrow{\alpha} \Omega$ in $\underline{C}$ such that $H(\alpha) = 1$; i.e. $J_1(\alpha) \in V$.

Since V is generated by W, there must exist a finite number of maps $(X_i \xrightarrow{\varphi_i} \Omega)_{i=1,\ldots,n}$ in $\underline{C}$ such that $\exists_{X_i} [\varphi_i] = 1$ for each i and

$$\bigwedge_{i=1}^{n} J_1(\varphi_i)J_2(\bar{\varphi}_i) \leq J_1(\alpha) \text{ in } \underline{C}[K](I,\Omega).$$

i.e.
$$\bigwedge_{i=1}^{n} J_1(\varphi_i)J_2(\bar{\varphi}_i) \wedge J_1(\alpha) = \bigwedge_{i=1}^{n} J_1(\varphi_i)J_2(\bar{\varphi}_i) .$$

i.e. by definition of J_1, J_2 and $\wedge$ in $\underline{C}[K]$,

$$\# \bigwedge_{i=1}^{n} \varphi_i p_i \wedge \alpha\, !_X \mid \hat{\overline{\varphi_1}} \wedge \hat{\overline{\varphi_2}} \wedge \dots \wedge \hat{\overline{\varphi_n}} \#$$

$$= \# \bigwedge_{i=1}^{n} \varphi_i p_i \mid \hat{\overline{\varphi_1}} \wedge \dots \wedge \hat{\overline{\varphi_n}} \# ,$$

where $X = X_1 \times \dots \times X_n$ with projections p_i.

But constants are monic (trivially, since $Z \overset{f}{\underset{g}{\rightrightarrows}} I$ must $= !_Z$);

$\therefore$ by 4.2(e), $\bigwedge_{i=1}^{n} \varphi_i p_i \wedge \alpha !_X = \bigwedge_{i=1}^{n} \varphi_i p_i$ in $\underline{C}$.

i.e. $\bigwedge_{i=1}^{n} \varphi_i p_i \leq \alpha !_X$ in $\underline{C}$.

$\therefore$ by adjointness of $\wedge$, $\Rightarrow$, we have $\bigwedge_{i=2}^{n} {}_i p_i \leq \varphi_1 p_1 \Rightarrow \alpha\, !_X$

$$= (\varphi \Rightarrow \alpha\, !_X) p_1 .$$

Again by adjointness, $\bigwedge_{i=3}^{n} \varphi_i p_i \leq \varphi_2 p_2 \Rightarrow (\varphi \Rightarrow \alpha !_X) p$.

Iterating this process n times would give

4.16(a) $1_X = \bigwedge \emptyset = \bigwedge_{i=n}^{n} \varphi_i p_i \leq \varphi_n p_n \Rightarrow \dots \Rightarrow \varphi_1 p_1 \Rightarrow \alpha !_X$.

To make the procedure explicit, we introduce the following notation:

Define $Y_o, \dots, Y_n \in \mathrm{Ob}(\underline{C})$ by $Y_o = I$, $Y_j = X_j \times Y_{j-1}$. That is ,

$$Y_o = I,\ Y_1 = X_1 \times I = X_1,\ Y_2 = X_2 \times X_1, \dots, Y_n = X_n \times \dots \times X_1 = X.$$

Denote the projections of the Y_j by $Y_j \xrightarrow{r_j} X_j$, $Y_j \xrightarrow{q_j} Y_{j-1}$.

Define $Y_j \xrightarrow{\alpha_j} \Omega$ by $\alpha_o = \alpha$ and $\alpha_j = (\varphi_j r_j \Rightarrow \alpha_{j-1} q_j)$.

that is $\alpha_o = \alpha$, $\alpha_1 = (\varphi_1 r_1) \Rightarrow (\alpha q_1) = \varphi_1 \Rightarrow \alpha\, !_X$, $\alpha_2 = (\varphi_2 r_2) \Rightarrow$ $(\alpha_1 q_2) = (\varphi_2 r_2) \Rightarrow (\varphi_1 \Rightarrow \alpha\, !_X) q_2$, and so on.

Then <u>4.16(a)</u> says that $1_X = \alpha_n$.

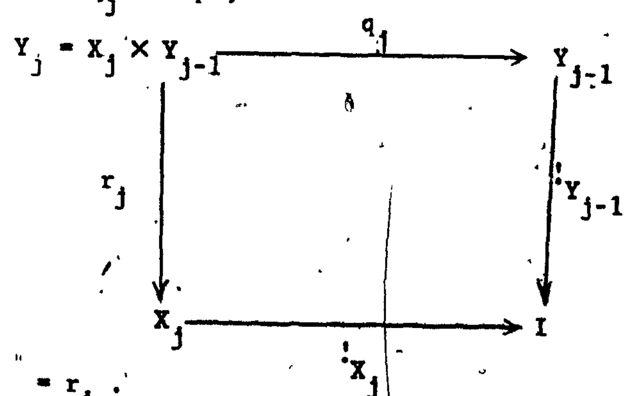
We shall now use this fact to conclude by induction that $\alpha = 1$. The descending induction step is as follows:

Suppose $\alpha_j = 1_{Y_j}$ for some $j \geq 1$; i.e. $\varphi_j r_j \Rightarrow \alpha_{j-1} q_j = 1_{Y_j}$;

i.e. $\varphi_j r_j \leq \alpha_{j-1} q_j$.

We remark that since <u>C</u> is fair, each q_j is epi,

and also that each diagram as shown is a pullback of the form of <u>1.6(b)(ii)</u> with



$X = Y_{j-1}$, $Y = I$, $Z = X_j$,

$f = !_{Y_{j-1}}$, and $Z \times f = id_{X_j} \times !_{Y_{j-1}} = r_j$.

$\therefore$ by <u>1.6 (b)(ii)</u>, $(\exists_{!_{X_j}} [\varphi_j])\, !_{Y_{j-1}} = \exists_{q_j} [\varphi_j r_j]$

$\leq \exists_{q_j} [\alpha_{j-1} q_j]$ (since $\exists_{q_j}$ is order-preserving),

$= \alpha_{j-1}$ by <u>1.4(f)</u>, because <u>C</u> is fair.

But by hypothesis each $\exists_{X_j}[\varphi_j] = 1$. Thus $\alpha_{j-1} \geq 1_{Y_{j-1}} = 1_{Y_{j-1}}$.

Hence by induction downward on i we obtain $\alpha_0 = 1$, i.e. $\alpha = 1$, and so H is conservative. This completes the proof of 4.16.

4.17 Theorem. Let $\underline{C}$ be a small prelogical category which is consistent and fair. Then there exists a small, fair, consistent and rich $\underline{C}^*$ with $Ob(\underline{C}^*) = Ob(\underline{C})$, and a conservative extension $\underline{C} \xrightarrow{F} \underline{C}^*$, which is the identity functor on objects.

Proof: Call $\underline{C}_o = \underline{C}$ and apply 4.16 to obtain $\underline{C}_o \xrightarrow{H} \underline{C}'$ small, fair, consistent, enriching and conservative. Call $\underline{C}' = \underline{C}_1$ and call $H = F_{0,1}$. Apply the same procedure to $\underline{C}$ to produce $\underline{C}_1 \xrightarrow{F_{1,2}} \underline{C}_2$ and iterate countably many times.

We thus have a chain of extensions

$$\cdots \; \underline{C}_i \xrightarrow[F_{i,i+1}]{} \underline{C}_{i+1} \; \cdots$$

of the kind described in 4.12, and with the following properties.
Each $\underline{C}_i$ is small, fair, and consistent, and
each $F_{i,i+1}$ is conservative and enriching.
Call $\underline{C}^* = \hat{\underline{C}}_i = R \lim \underline{C}_i$ and call $\underline{C} \xrightarrow{F} \underline{C}^*$, $F = F^o$ as defined in 4.12.
By 4.13 and 4.14, $\underline{C}^*$ is small, fair and consistent, $Ob(\underline{C}^*) = Ob(\underline{C})$, and F is a conservative extension which is the identity on objects.

Moreover $\underline{C}^*$ is rich: Let $X \xrightarrow{\|\varphi\|} \Omega$ in $\underline{C}^*$ such that $\exists_{\|!_X\|}(\|\varphi\|) = \|1\|$ in $\underline{C}^*$. is in $\underline{C}_i$ for some i; and since $F_{i,i+1}$ enriches $\underline{C}_i$, there is a constant $I \xrightarrow{k} X$ in $\underline{C}_{i+1}$ such that $F_{i,i+1}(\varphi)k = 1$ in $\underline{C}_{i+1}$.

$\therefore$ $\|F_{i,i+1}(\varphi)k\| = \|1\|$; i.e. $\|F_{i,i+1}(\varphi)\| \; \|k\| = \|1\|$ in $\underline{C}^*$, and so k is the required constant.

Thus $\underline{C} \xrightarrow{F} \underline{C}^*$ is the desired rich extension.

4.18 Thecrem. Let $\underline{C}$ be a small prelogical category which is consistent and fair, and let $I \xrightarrow{\alpha} \Omega$ in $\underline{C}$ be such that $7\alpha \neq 0$. then there exists a rich and maximally consistent prelogical category $\overline{\underline{C}}$ and an extension $\underline{C} \xrightarrow{G} \overline{\underline{C}}$ such that $G(\alpha) = 0$ in $\overline{\underline{C}}$.

Proof: By hypothesis $7\alpha \neq 0$. (Notice that in particular this implies $\alpha \neq 1$, since $71 = 0$.)

Let V_o be an ultra filter on $\underline{C}(I,\Omega)$ containing 7α (by the prime ideal theorem there always exists such a proper ultrafilter.)

Call $R_o = R^{V_o}$, the prelogical congruence generated by V_o, Call $\underline{C}_o = \underline{C}$, call $\underline{C}_1 = C_o/R$ and call $G_{0,1} = Q$.

Then $\underline{C}_1$ is small, fair and consistent, and $G_{0,1}$ is an extension, by 4.11.

Since V_o is an ultrafilter, $\therefore$ $\underline{C}_1(I,\Omega) = \underline{C}(I,)/V_o = \{0,1\}$; i.e $\underline{C}_1$ is maximally consistent. Also note that $\alpha \notin V_o$, since if it were, we would have $0 = 7\alpha \wedge \alpha \in V_o$, contradicting the fact that V_o is proper. Hence $G_{0,1}(\alpha) = \|\alpha\| \neq 1$ in $\underline{C}_1$, and of course $\mathrm{Ob}(\underline{C}_1) = \mathrm{Ob}(\underline{C})$ and $G_{0,1}$ is the identity on objects.

Apply 4.17 to $\underline{C}_1$; call $\underline{C}_2 = \underline{C}_1^*$ and call $G_{1,2} = F$ [illegible] .˙. We obtain

$$\underline{C}_0 \xrightarrow[G_{0,1}]{} \underline{C}_1 \xrightarrow[G_{1,2}]{} \underline{C}_2$$ with $\underline{C}_2$ rich. Note that

$G_{1,2} = G_{0,1}(\alpha) \neq 1$ because $G_{1,2}$ is conservative.

We now alternate these two steps countably many times, thus constructing a chain

.... $\underline{C}_i \xrightarrow[G_{i,i+1}]{} \underline{C}_{i+1} \longrightarrow \dots$ of the kind assumed in 4.12.

with the properties that:

(a) All the $\underline{C}_i$ are small and consistent;

(b) For every $i, G_{0,i}(\alpha) \neq 1$;

(c) For every $j \geq 1$, $\underline{C}_{2j-1}$ is maximally consistent, and

(d) For every $j \geq 1$, $\underline{C}_{2j}$ is rich.

By 4.15 $R \lim_{i \to \infty} \underline{C}_i = R \lim_{i \to \infty} \underline{C}_{2j} = R \lim_{i \to \infty} \underline{C}_{2j+1}$ = call $\overline{\underline{C}}$

Because of (c) above and 4.14, $\overline{\underline{C}} = R \lim_{j \to \infty} \underline{C}_{2j}$ is maximally consistent.
Because of (d) above and 4.14 $\underline{C} = R \lim_{j \to \infty} \underline{C}_{2j}$ is rich .

Define $\underline{C} \xrightarrow{G} \overline{\underline{C}}$ as $G = G^{\circ}$; by 4.13 G is a prelogical extension.
Moreover $G(\alpha) = \|\alpha\| \neq 1$ by (b) above. .˙. Since $\underline{C}$ is maximally consistent, .˙. $G(\alpha) = 0$.

Therefore $\underline{C} \xrightarrow{G} \overline{\underline{C}}$ is the required extension.

The completeness theorem now follows from the final results of this chapter and the previous one.

4.19 Theorem (Completeness of Prelogical Categories)

Let $\underline{C}$ be a small prelogical category which is consistent and fair, and let $I \xrightarrow{\alpha} \Omega$ in $\underline{C}$ be such that $7\alpha \neq 0$. Then there is a semantic premodel $\underline{C} \xrightarrow{F} \underline{S}$ such that $F(\alpha) = 0$.

Proof: By 4.18 there is an extension $\underline{C} \xrightarrow{G} \overline{\underline{C}}$ such that $\overline{\underline{C}}$ is rich and maximally consistent and $G(\alpha) = 0$.

By 3.8 the canonical functor $\overline{\underline{C}} \xrightarrow{\overline{C}(I,\ )} \underline{S}$ is a semantic premodel for $\overline{\underline{C}}$.

Call $F = (\overline{\underline{C}}(I,\ ))G$. By 3.3 F is a premodel for $\underline{C}$, and clearly F is semantic.

Moreover $F(\alpha) = \overline{\underline{C}}(I, G(\alpha)) = \overline{\underline{C}}(I,0) = 0$.

Thus F is the required premodel, and the completeness is proved.

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