Inspection of the Exterior of Polygons under Infinite and Limited Visibility Models

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DEDICATION

This thesis is dedicated to Saquib, for always believing in me.

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ABSTRACT

We study the problem of externally guarding or inspecting a polygonal environment with a mobile guard (watchman) in both unlimited and limited visibility models. A survey of the literature on guarding problems is presented for both stationary and mobile guards, as well as for both interior and exterior workspaces.

Our research concentrates on the external inspection of a single, convex or simple, polygon. In particular, we study the relationship between the interior angles of convex polygons and the length of an external inspection route under unlimited visibility. We then propose a method for computing the shortest external inspection route for convex polygons under limited visibility, and also an approximate solution for simple polygons. Finally, we present experimental work which was performed on random convex polygons to evaluate the results and validate the findings under both types of visibility assumptions.

ABRÉGÉ

Nous examinons le problème de l'inspection externe dun environnement polygonal avec un garde mobile selon des modèles de visibilité illimitée et limitée. Un aperçu de la littérature sur des problèmes d'inspection est présenté pour des gardes immobiles et gardes mobiles, ainsi que pour l'intérieur et l'extérieur des environnements polygonaux.

Notre principal sujet d'étude est l'inspection externe d'un polygone convexe ou simple. En particulier, nous étudions le rapport entre les angles intérieurs des polygones convexes et la longueur d'un itinéraire externe d'inspection avec visibilité illimitée. Nous proposons alors une méthode pour obtenir l'itinéraire d'inspection externe le plus court pour les polygones convexes avec visibilité limitée, ainsi qu'une solution approximative pour les polygones simples. Finalement, nous présentons le travail expérimental ayant été effectué sur des polygones convexes aléatoires pour valider les résultats selon les deux types de modèles de visibilité.

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CHAPTER 1
Introduction

Guarding problems have been a topic of interest in the computational geometry community for a considerably long time. Given a polygonal region, the optimal placement of stationary guards or cameras so that the entire region is visible has been a popular problem of study, referred to as the Art Gallery problem. With the passage of time and the development of technology, as the idea of moving robots with sensing or vision capabilities became more plausible, the obvious next question that arose is how mobile robots can be utilized for the same task instead of a set of stationary guards. However using a mobile robot would mean mapping a path for the robot to follow along which the entire region is visible. This problem is referred to as the Watchman Route problem.

Originally the definition of visibility in computational geometry and motion planning problems stated that two points are visible to each other as long as the straight line connecting them does not intersect the exterior of the region in which they are contained. However, real-life robots are limited in their capacity of sensing and this definition of visibility is not appropriate. Most practical robots would have a maximum viewing distance, outside of which the sensing measurements may be inaccurate and details in the environment less or not at all visible. This necessitates casting the original art gallery and watchman route problems under the more restricted definition of visibility to make the solutions more usable in practical applications.

While the problem of inspecting the interiors of polygonal regions by mobile robots have been studied extensively over the years, a related problem that has not received as much scrutiny is that of inspecting the exterior of polygons using mobile robots. An area that has undergone even less analysis is what occurs if the inspection route is constrained by the mobile robot having a limited vision or sensing range. Hence, a major area of interest in the work presented in this thesis is composed of this topic.

1.1 Problem Statement

This thesis is mainly concerned with the problem of finding a shortest route for a robot that aims to inspect the exterior of a polygonal region. We consider the problem under both the classical definition of visibility, which we will refer to as unlimited visibility, as well as the more realistic limited visibility definition, where the robot's maximum viewing distance is some fixed constant.

An obvious solution to the external inspection route problem may seem to just be a route that follows the boundary of the polygonal region, undoubtedly ensuring that each point on the external boundary is seen by the robot, and hence also saving computational time. However, in the case of some polygonal environments, the optimal external inspection route may be significantly shorter than the route following the boundary of the polygon, especially if the visibility range is relatively large. Reducing the distance travelled by the robot may be an important consideration when the consumption of energy, fuel or other such resources needs to be minimized and controlled.

We look into the cases where the best solution is to follow the boundary and what properties of the polygon determine this factor. We also investigate the converse case to determine what, if any, properties allow the optimum route to be shorter than the route following the boundary of a polygon.

We study the computation of the shortest route for convex and simple polygonal environment exteriors under varying visibility ranges. We also examine how limiting the visibility range of the robot affects the length of the shortest external inspection route.

1.2 Approach

We study the external inspection of a polygonal region using two visibility models: i) the *unlimited visibility model* assumes that the mobile guard is equipped with a camera or sensor with infinite visibility along an unobstructed line of sight, and ii) the *limited visibility model* assumes that the camera or sensor has a finite visibility range, fixed at a certain constant.

For both models we assume that the viewing can be done omni-directionally; there are no constraints on the guard's field of view, which is assumed to be 360°. We also assume that the guard has a priori information of the map of the environment. The external inspection is limited to guarding a single polygon in this work, and the polygonal size, shape, angle and distance information is available to the guard from the start.

Our approach is to begin quite naturally with the unlimited visibility assumption, or in other words, an ideal situation. We study the solution presented for computing external watchman routes for convex polygons given in [64] under unlimited visibility. We attempt to improve on some proofs given in [64] by introducing some new lemmas and proofs that support the claims in the paper and by elaborating on details for increased clarity.

We then examine what properties of the structure of convex polygons affect the length of the external inspection route under unlimited visibility, and propose a few conjectures that the size of the interior angles of the polygon may be a factor. We narrow down the investigation by specifically showing the effects of the size of the interior angles of convex quadrilaterals on an external watchman route, and disprove the initial conjecture made. We also briefly discuss the difference between computing external watchman routes under unlimited visibility on simple polygons from that of convex polygons.

After the thorough investigation of the unlimited visibility model, we move on to incorporating a limited visibility range on the robot. We study the effect of reducing the visibility range from an infinite to a finite range, down to zero visibility. We examine how this causes the length of the route to increase on convex polygons. We also propose an approximate solution for computing a short external route for inspecting simple polygons under limited visibility, by using disks of radius equal to the visibility range.

In the experimental work, we generate random convex polygons and use these to both validate the findings on the unlimited visibility model, as well as to test the effect of different values of the visibility range on specific convex polygons under the limited visibility model. We use the experimental evidence to support a theory that the optimal solution for a convex polygon in the unlimited visibility model leads to the optimal solution in the limited visibility model.

1.3 Applications

In this era of modern technology, the application of mobile robots for particular inspection tasks that cannot be easily carried out by humans has become a necessity in several scenarios. Robots do not have many of the limitations of the human body, which makes certain environments often inaccessible for human inspection. Thus, inspections that need to be performed in outer-space, underwater or even hazardous environments such as nuclear plants, may be made more safely and efficiently by utilizing mobile robots with vision sensors.

External inspection is often a required function when endeavoring to inspect the outer rim of sea-going vessels, space stations, nuclear generators or oil pipelines, to name a few examples. The AERcam (Autonomous Extra-vehicular Robotic Camera), which is a free-flying inspection robot in space, described in [22] and [23], can be an example of one of the motivations of this work. This robot could be used to inspect the exterior of a space station for damage, leaks and other such problems. Similarly, a collaborative project developed at the Universities of McGill, York and Dalhousie is the AQUA robot described in [30]. This is an amphibious legged robot capable of swimming and visually navigating underwater. One of the many applications of this robot can be underwater inspection tours of the hulls of ships or submerged oil rigs.

Cost and efficiency may be a major factor of consideration in such projects, for which reason shorter inspection routes would be greatly useful. Also, limited visibility becomes an even greater concern in surroundings such as underwater, where visibility is significantly reduced.

1.4 Outline

This thesis deals with the external inspection problem of polygonal regions under unlimited and limited visibility. In Chapter 2, related work is discussed, by first describing the previous work that assumed unlimited visibility in art gallery and watchman route problems, together with their many variations, and then the work where limited visibility has been considered. Other problems where limited visibility has been addressed is also briefly discussed here. Chapter 3 describes the external watchman route problem for convex and simple polygons with the assumption of unlimited visibility, while Chapter 4 discusses the problem under the limited visibility range constraint. We present some experimental work and the results on convex polygons for both the limited and unlimited visibility models in Chapter 5. We conclude in Chapter 6, where we discuss the significance of the results and present ideas for future work and improvements on this topic.

1.5 Statement of Originality

I present an extensive study of the literature related to the art gallery, watchman route, and inspection problems in this thesis, together with previous work in some other related areas. All the references are cited in the text, mostly in Chapter 2, with a few in other chapters where they are relevant. An exposition of some of the work presented in [64] is given in the beginning of Chapter 3 and Section 3.1, in my own words, for increased simplicity and clarity, and some new lemmas and proofs are introduced, as original work, to support claims made without adequate proof in [64]. The conjectures and their arguments (Section 3.1.1 and 3.1.2), the work on the limited visibility problem (Chapter 4) and the experimental work (Chapter 5) presented are also my own original contributions, inspired by joint discussions with my thesis supervisor, Professor Whitesides.

CHAPTER 2
Related Work

Art gallery and watchman route problems have been a well-known, well-researched topic of study for computer scientists involved in the fields of combinatorial and computational geometry, motion planning and also computer graphics, to name only a few areas where it has sparked interest. The problems have generated an enormous amount of literature in past and recent years. The scope has spread into several related problems such as polygon decomposition, visibility problems and several variations on the original problem.

In this chapter, the original formulation of the art gallery problem is described and some of the significant results that subsequently came out. Section 2.1 deals with the traditional version of the problem, where it is assumed that the guard, typically a camera or robot with vision sensors in practice, has unlimited visibility, i.e. a point is visible if the straight line between the point and the guard lies entirely inside the polygonal environment. Several variations of the art gallery problem, that spawned from the originally posed question, are described in Section 2.1.1. In Section 2.1.2, some of the literature dealing with the watchman route problem is discussed, followed by the variations of the problem that have been studied over time, but all with the assumption of unlimited visibility along an unobstructed line of sight.

Section 2.2 goes into the literature related to the art gallery and watchman route problems where the work assumes a limited visibility constraint, so that the guard or watchman has a visibility range within which the surrounding details can be viewed clearly, and outside of which visibility may be poor. A brief description of other problems related to limited visibility has also been introduced here.

To study the properties of the solution to the external watchman route problem on convex polygons, it was necessary to generate random convex polygons for the experimental work. We defer the discussion of the previous work related to random convex polygon generation to Chapter 5.

2.1 Unlimited Visibility

2.1.1 Art Gallery Problems

The original art gallery problem was posed in 1973 by Klee in conversation with Chvátal. His question was:

> *What is the smallest number of guards necessary to guard an art gallery of* n *walls?*

An art gallery can be considered a polygon in the plane, with the walls as edges of the polygon. Thus, the problem is to find the smallest number of guards needed to cover or supervise any polygon of n vertices and n edges.

In this original formulation of the problem, the *guards* are considered stationary points, and they *cover* a polygon if every point in the polygon is visible from at least one of the guards. A point is said to be *visible* from another point if the line segment joining them is completely contained in the polygon. The guards are also assumed to see in every direction, or have a $360°$ field of view. Hence, visibility of the guards is not considered to have any physical constraints, and visibility is blocked only if there is an obstruction or wall in view. Therefore it can be said that it is assumed that the guards have *unlimited visibility* in this sense.

Soon after the classical art gallery question was posed, Chvátal showed that $\lfloor n/3 \rfloor$ guards are always sufficient and sometimes necessary to guard a polygon of n vertices [24]. This result has come to be known as *Chvátal's Art Gallery Theorem.* Chvátal established this theorem using a relatively complex inductive proof on triangulation graphs of polygons. Later, Fisk proved the theorem using a much simpler, concise method [36]. He showed this by first triangulating the polygon, and then 3-coloring the vertices of the triangulation, ensuring that no two adjacent vertices have the same color. Choosing any of the three color sets results in a set of vertices from which the whole polygon is visible. The smallest of the three color sets can contain no more than $\lfloor n/3 \rfloor$ vertices.

Such guards, placed at the vertices of a polygon, are called *vertex guards.* Guards that can be placed anywhere in the interior of the polygon are known as *point guards.*

Lee and Lin showed that finding the minimum number of vertex guards necessary to guard a given polygon is an NP-hard problem [59]. Aggarwal showed that the problem for point guards is also NP-hard [1]. However, Avis and Toussaint do find a polynomial time ($O(nlogn)$) guard placement algorithm in [6], by decomposing a polygon into star-shaped pieces, since the region seen by each point guard has the property of being star-shaped. Tarjan and van Wyk, in [81], came up with an $O(nloglogn)$ time algorithm for triangulating polygons, and since the guard placement algorithm depends mainly on the triangulation time, this improved on the previously achieved running time. However, Chazelle's linear time algorithm for polygon triangulation [17] finally gave way to an $O(n)$ algorithm for the art gallery theorem.

In 1987, J. O' Rourke published a book dedicated to studying the art gallery theorem and algorithms [66]. The tremendous interest in this topic also gave rise to several more papers and surveys in the area, including Shermer's comprehensive survey paper on the problem and its variations [73].

Art gallery problems are often referred to as *illumination problems* where instead of placing guards, the problem requires placing light sources that will illuminate the entire polygonal region. Some of these problems assume that the illumination may come from both direct rays and reflected rays. There are many variants of the illumination problem including polygonal illumination [33], illuminating families of convex sets ([82]), and floodlight illumination problems ([10], [34], and [84]).

The classical art gallery theorem branched to give rise to several related questions, including problems focused on different types of guards, or with different classes of polygons for the art gallery. While the traditional problem considered *vertex guards* (guards positioned at the vertices of the polygon) or *point guards*, in 1981 Toussaint introduced *edge guards*. Such guards are allowed to move along the edges of a polygon rather than being stationary at one point. O' Rourke introduced *mobile guards*, which are guards allowed to move along closed line segments inside the polygon [65]. Over the years, several forms of the art gallery problem using these guards have been studied.

Polygon classes

The classical art gallery problem considered only simple polygons as the transformation of art galleries to a geometric shape. Following this, varied scenarios for the form of art galleries were studied, as described below.

Rectangular art galleries: In [84] and [26] it was proved that exactly $\lfloor n/2 \rfloor$ guards can cover any rectangular art gallery with n rooms. Here, a more realistic form of an art gallery is assumed, housed in a rectangular building, divided into rectangular rooms, where any two adjacent rooms have a door connecting them, and guards can be placed in doorways as well as inside rooms.

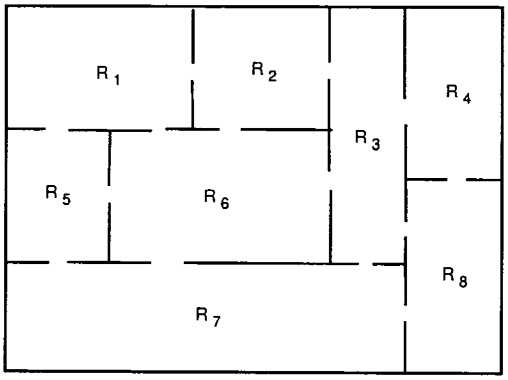


Figure 2–1: A rectangular art gallery

Orthogonal polygons: Kahn et al [51] showed that an orthogonal n-gon can always be guarded by $\lfloor n/4 \rfloor$ vertex guards and this is sometimes necessary (which means that this is the maximum number of vertex guards necessary). This led to studying the problem of breaking down an orthogonal polygon into convex quadrilaterals ([68], [70] and [60]). Culberson and Reckhow showed in [25] that partitioning an orthogonal polygon with a rectangular cover is NP-complete. It was also proved in [71] that both the minimum vertex guard and point guard problems for orthogonal polygons are NP-hard. Guarding rectilinear art galleries was also studied in detail in [31], [44] and [34].

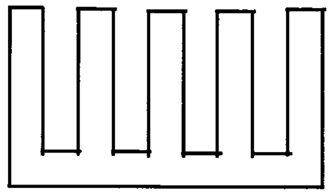


Figure 2–2: An orthogonal polygon where $\lfloor n/4 \rfloor$ guards are necessary

Polygons with holes: A polygon P with a set of h disjoint polygons $P_1, P_2, ..., P_h$ contained inside it is called a polygon with holes. O' Rourke showed that polygons with holes can always be guarded by $\lfloor (n+2h)/3 \rfloor$ vertex guards [66]. In [9] it was shown that in the case of point guards, $\lceil (n+h)/3 \rceil$ guards are always sufficient and sometimes necessary to guard a polygon with n vertices and h holes.

Orthogonal polygons with holes: O' Rourke proved that $\lfloor (n+2h)/4 \rfloor$ guards can always cover any orthogonal polygon of n vertices and h holes [66]. In [47] it was shown that $\lfloor n/4 \rfloor$ point guards are always sufficient to guard any orthogonal n-vertex, h-hole polygon. More work on orthogonal polygons with holes has been done in [48].

Edge and Mobile guards

The edge guard problem can be defined as: Given a polygon P (with holes allowed) with n vertices, find a smallest subset S of edges of P such that every point on the boundary of the polygon P can be seen from at least one point on an edge in S. The edges in S are called *edge-guards*. The art gallery problem for edge guards is also NP-hard, even for polygons without holes [59].

Mobile guards were first studied in [65] and [1]. Mobile guards are generalized versions of edge guards, since they can move along diagonals or edges of a polygon. O'

Rourke [65] proved by induction that the minimum number of mobile guards necessary and sufficient to guard a polygon is $\lfloor n/4 \rfloor$. He also showed that the minimum number of mobile guards necessary and sufficient to guard an orthogonal polygon is $\lfloor (3n+4)/16 \rfloor$.

A polygon that can be guarded with one edge guard is called a *weakly edge visible* polygon. In [69] Sack and Suri gave a linear time algorithm to determine if a polygon is weakly edge visible. An algorithm for computing the shortest edge guard that can guard a polygon is given in [18]. And an $O(nlogn)$ algorithm for determining if a polygon is guardable by a one line-segment guard is given [53].

Fortress problem

The *fortress problem* is a variation of the art gallery problem where it is necessary to guard the outside of the art gallery, rather than the inside. Hence the fortress problem may be stated as the following question:

How many guards are needed to see the exterior of a polygon?

O' Rourke and Wood showed that $\lceil n/2 \rceil$ vertex guards are necessary and sufficient to see the exterior of any polygon with n vertices [65]. In [1] it was shown $\lceil n/4+1 \rceil$ vertex guards are necessary and sufficient to see the exterior of any orthogonal polygon with n vertices. These proofs give linear time algorithms for guarding the polygon exterior.

Aggarwal and O' Rourke showed that $\lceil n/3 \rceil$ point guards are sufficient and sometimes necessary to cover the exterior of polygons with n vertices [66]. Shermer's proof of this is given in [65].

Further study of the fortress problem was done in [87], where the problem is studied with edge guards, and in [86] vertex guards were studied.

Prisonyard Problem

The *prisonyard problem* arises when it is required to guard both the inside and the outside of the art gallery simultaneously. Hence the fortress problem asks:

How many guards are needed to see the interior and the exterior of a polygon with n vertices?

In [65] it was proved that $min(\lceil n/2 \rceil + 2, \lfloor (n + \lceil h/2 \rceil)/2 \rfloor, \lfloor 2n/3 \rfloor)$ guards are always sufficient, where r is the number of reflex vertices and h is the number of convex vertices of the polygon.

The prisonyard problem is studied at length in [38] and it was shown that $\lceil n/2 \rceil$ vertex guards are always sufficient and sometimes necessary to guard both the interior and exterior of a simple polygon simultaneously. It can be seen that this number is the same as that for the fortress problem.

Polyhedral Terrains

The 2D polygonal version of the art gallery problem has also been extended to guarding 3D polyhedral terrains as a relatively new computational geometry problem. This problem is also known to be NP-hard. Bounds on the number of guards needed and algorithms for vertex and edge guards were given in [12]. Further study of this topic can be seen in [35] and [11]. Approximation algorithms have also been proposed in [32].

2.1.2 Watchman Route Problems

While the art gallery problem deals with placing a set of stationary guards in a way so that they see the entire gallery, the watchman route problem is a variation that aims to

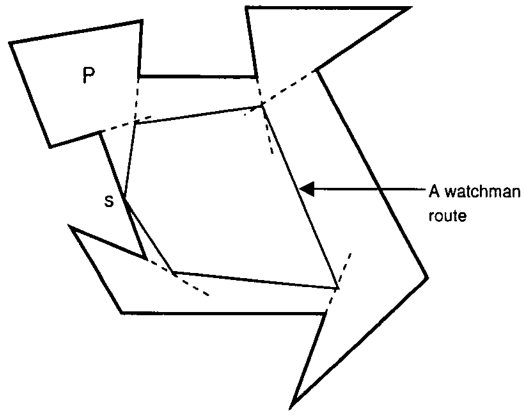


Figure 2–3: A watchman route for polygon P

achieve the same goal, but by placing one moving guard that takes a route in the gallery along which it sees everything.

Formally, a *watchman route* for a polygon can be defined as a closed walk, curve or polygonal chain within the polygon with the property that every point in the polygon is visible from some point along the route [19]. Thus, the shortest watchman route problem is to find the shortest curve that does this for a given polygon. This problem has been called a hybrid visibility problem [73], since it not only involves visibility but also combines this with other geometric and conceptual properties.

Chin and Ntafos showed that the problem is NP-hard for polygons with holes, even if the polygon and holes are convex and orthogonal. They also showed that it is NP-hard for 3-dimensional simple polyhedra. They do give a linear time algorithm for orthogonal polygons without holes [19]. In [20] they gave an $O(n^4)$ algorithm for simple polygons if a starting point s is specified. Tan et al. improved on this bound to give an $O(n^3)$ algorithm using incremental techniques [78] and further improved this to $O(n^2)$ in [79] using a divide-and-conquer approach. However Hammar and Nilsson showed that these time bounds have errors in them and presented a modified $O(n^2)$ time algorithm [45]. This solution itself was shown to contain some inconsistencies by Tan, and he subsequently proposed a correct $O(n^4)$ time algorithm [80]. Dror et al. recently showed that the results for touring a sequence of polygons in [29] implies a simpler algorithm for the fixed watchman route that runs in $O(n^3 logn)$ time. However to achieve a faster, more practical solution, Tan also presented a simple, linear time approximation algorithm that outputs a watchman route at most $\sqrt{2}$ times the length of the shortest watchman route [76].

All the above algorithms work only when the route is forced through a specified starting point. Such routes are referred to as *fixed watchman routes* [45]. This restriction can pose a significant hindrance in certain cases when it causes a route to be arbitrarily longer than the shortest watchman route with no restrictions, referred to as the *floating watchman route* [45].

Carlsson et al. gave the first polynomial time algorithm, with an $O(n^6)$ bound, for finding the shortest watchman route without forcing the route through a given starting point ([15], [61]). The complexity of the algorithm did prompt them to develop an $O(n^4)$ time approximation algorithm for the problem, which finds a watchman route at most a constant factor longer than the shortest watchman route [14]. However, Tan later came up with an $O(n^5)$ algorithm for the unrestricted shortest watchman route problem, by showing that the floating shortest watchman route requires time $O(n)$ times that of the fixed watchman route [74], which improved on the previous $O(n^6)$ bound.

External Watchman Routes

The *external watchman route* can be defined as the problem of determining a watchman route where it is necessary to guard the exterior of a polygon. Ntafos and Gewali gave an $O(n^4)$ algorithm for finding the shortest external watchman route without a given starting point [64]. They also provided linear time algorithms for convex, star-shaped, monotone and orthogonal polygons, which belong to a class called *weakly externally visible polygons*. This concept as well as details on external watchman routes will be discussed in subsequent chapters. A significant portion of this thesis has been dedicated to the results from this paper ([64]) and its extensions.

Gewali and Stojmenovic studied the computation of shortest external watchman routes using parallel algorithms as well; they gave an $O(logn)$ time algorithm with the help of $O(n/logn)$ processors in CREW-PRAM models. They also showed that for a convex polygon the shortest external watchman route can be found in $O(logn)$ time using $O(n)$ processors on a hypercube [39].

In [40] the problem of finding the shortest external watchman route for a pair of convex polygons, with a total of n vertices, is studied and an $O(n^2)$ time algorithm is described.

Computing Vision Points on Watchman Routes

The problem of computing vision points is a variation of the watchman route problem in which the watchman is restricted to remain on a closed curve in the polygon P and surveys the polygon only at some selected points (or *vision points*) of the curve, which together should see all of P.

Carlsson et al. [16] showed that finding the minimum number of vision points along a shortest watchman route is NP-hard. The problem is studied at length in [61], including some restricted polygon classes such as spiral, doughnut, histogram and monotone polygons.

The Robber Route Problem

The robber route problem is that of finding a route R, given a starting point s on the boundary of polygon P, that sees every point on a set of partial edges S of the polygon, but such that R is not visible from any point on a set of threat points T. This problem was introduced by Ntafos [62]. It was shown to be solvable in $O(n^4)$ time for arbitrary polygons and in linear time for orthogonal polygons.

Zookeeper and Safari Routes

Another variant of the watchman route is finding a route that has to visit a number of subpolygons in the polygon P. The *Zookeeper Route* is restricted to touch but not enter the subpolygons, much like a zookeeper might visit each animal cage at the zoo. The *Safari Route* is, however, allowed to enter the subpolygons, just as visitors might go inside the animal habitats in a safari trip. An example of these routes is illustrated in Figure 2–4.

Chin and Ntafos showed that both these problems are NP-hard in general [21], but can be solved in polynomial time if the subpolygons inside P are restricted to be convex and attached to the boundary of P. They gave an $O(n^2)$ algorithm to solve the fixed version of the zookeeper problem, with a specified starting point, which was later reduced to $O(nlog^2n)$ in [46]. This was further reduced to an $O(nlogn)$ algorithm in [7]. Tan also used his previous linear-time approximation scheme for finding a shortest watchman route

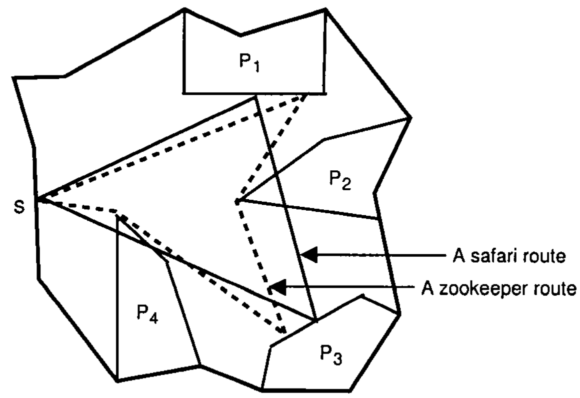


Figure 2–4: A safari and zookeeper route

[76] to approximate the shortest zookeeper route that has to go through a specified starting point in $O(n)$ time.

For the floating zookeeper route problem, Tan gave an $O(n^2)$ time algorithm for finding the shortest zookeeper route for arbitrary polygons with no specified starting point [75].

In the case of safari routes, Ntafos gave an $O(n^3)$ time algorithm to solve the safari route problem when the subpolygons are attached to the boundary of P [63]. However, in [77] it is shown that this time bound was erroneous and a corrected $O(n^3)$ algorithm is presented. Recently, Dror et al. suggested using their results in [29] to solve the fixed safari route problem in $O(n^2 logn)$ time. For the floating safari route problem, Tan and Hirata removed the specified starting point restriction and gave an $O(n^4)$ time algorithm for the problem [77].

2.2 Limited Visibility

The traditional definition of simple straight-line visibility is not a realistic model of the sensors used in applications of computer graphics or robot vision. All the algorithms described so far in this chapter have assumed ideal conditions, where the physical limitations of real sensors are not taken into account. Hence, when these algorithms are applied using real cameras or robots with limited sensing capabilities, the algorithms do not suffice and frequently fail at the task.

Recently the focus has changed from trying to find exact algorithms for idealized situations to trying to find reasonably accurate solutions in the presence of practical constraints. In this section, we discuss the solutions of the problems we have introduced in the last section, into which additional considerations have been incorporated, including a limited visibility range, to make the algorithms work better in practice. Since the limited visibility version of the problems are known to be NP-hard, all the algorithms to be described give approximate solutions.

For the art gallery or watchman route problems under unlimited visibility, a set of guards or a route that sees the whole boundary of a polygon also sees every point in the interior of the polygon. However, for limited visibility, this is no longer true. Hence some of the solutions to be discussed only inspect the boundary of the polygon while others inspect both the boundary and the interior, with the latter algorithms having higher computational complexity.

2.2.1 Art Gallery Problems

Ntafos introduced the notion of *d-visibility*, where two points are d-visible if they are visible to each other and are at most a distance d apart [63]. If a guard has d-visibility capabilities, this means that it can see as far as a disk of radius d around it. Covering an art gallery with such guards requires covering a polygon with disks of radius d. This can be linked to the disk-cover problem, where a set of n points has to be covered with the smallest number of disks of radius d, which is known to be an NP-complete problem [50].

González-Baños and Latombe studied the problem of using a mobile robot, equipped with range sensors, to acquire range-images so as to automatically build a visual representation of an environment [41]. The problem requires minimizing the number of sensing operations which means solving an extended version of the art gallery problem. They developed two randomized algorithms that compute a near-optimal number of sensing locations for scanning the workspace. However, they used a model in which incidence and visibility range limitations are taken into account, i.e. the angle between the line-of-sight from a guard location and a surface cannot be greater than a specified angle, and the distance between them cannot be greater than a range d. The upper bound for the running times of the two algorithms given are $O(nm^2)$ and $O(n_g n log n_g n)$ respectively, where n is the number of edges of the polygon, m is the number of random samples taken, and n_g is the number of guard locations found by the algorithm [41].

Danner and Kavraki also studied a randomized solution to the inspection problem of computing a short path for a robot with vision sensors so that the entire boundary of the workspace is visible [27]. Here again, the model included the constraints of maximum angle of incidence and maximum viewing distance. In the solution they presented they used the same randomized approach described in the first incremental algorithm in [41] to select art-gallery guards that represent a set of potential sensing locations. In their method, the region each sample guard can see is clipped to both the incidence and the visibility range constraints at each iteration of the algorithm. In this way, the set of samples that see the most new length of border is kept and the algorithm is repeated until the entire boundary is covered by the limited-visibility sensors [27].

Another paper where a sensor placement strategy uses a randomized algorithm to solve a variant of the art gallery problem is [42]. The strategy tries to compute a set of sensing locations that might be most effective to build a 3D model of the workspace. The set of guards found satisfies both incidence and range constraints, similar to the previous two papers described. The idea is to use a random sampling scheme to transform the problem to a set coverage problem that can be computed using a greedy approach, leading to an approximate solution. The algorithm gives a set of sensing locations that with high probability is at most a factor $O(log(n + h).log(clog(n + h)))$ from the optimal size c, where n is number of vertices and h is the number of holes in the polygonal workspace [42].

A different approach to the ones previously described is used in [52] to find a small number of guard positions that can visually inspect a 2D workspace. The computational time required by the algorithm depends on whether the whole workspace requires inspection or only the boundary. The guards are assumed to have cameras with a $360°$ field of view, but a predefined limited-visibility capability. The method uses a decomposition algorithm from [72] to divide the workspace into a number of convex polygons. Then each of these convex polygons are further divided into smaller polygons, using a divide-and-conquer strategy, so that each of these can be inspected by only one guard. Hence in this way, a suboptimal yet fast and efficient solution is given for finding a set of limited-visibility guard locations to inspect a 2D region [52].

2.2.2 Watchman Route Problems

The first paper that addressed the watchman route problem under visibility range constraints was by Ntafos [63]. Two versions of the watchman route problem are discussed. One is to find the shortest route so that each point in the boundary of a given polygon is d-visible from it, referred to as the *d-watchman problem.* The other is to find the shortest route so that each point in the whole polygon is d-visible from it, referred to as the *d-sweeper problem.*

The d-watchman problem is equivalent to finding a shortest route that visits a set of disks of radius d centered at the vertices of the polygon. In this paper [63], the authors approximated the circles by inscribing regular convex polygons of k sides inside them, and they described an $O(k^2n^3)$ algorithm that finds the safari route that visits the set P' of k-gons attached to the boundary of the polygon P, where n is the total number of vertices in P and P'.

The d-sweeper problem is equivalent to sweeping a polygon with a circular disk of radius d such that the entire polygon is covered with minimum travel distance of the disk. This is related to the Traveling Salesman Problem on simple grids. The authors gave an approximate solution to the TSP on grids problem, which also provides an approximate solution to the d-sweeper problem [63].

A problem equivalent to the d-sweeper problem is further studied in [5]. Here it is referred to as a *lawnmowing* or *milling* problem, where the shortest route has to be computed for a square or circular cutter of radius d, so that every point on a given region is covered by the cutter. The lawnmowing problem allows the cutter to exit the polygonal region whereas the milling problem restricts the cutter to stay inside. Hence the watchman route under limited visibility is more related to the milling version of the problem with a circular cutter. The authors showed that the problems are NP-hard in general and gave a 2.5-factor approximation algorithm for the milling problem [5].

2.2.3 Other Problems Related to Limited Visibility

Besides art gallery and watchman route problems, several other problems have also been studied under limited visibility in recent years. Several well-known problems have been recast under limited visibility to make them more practical and realistic in different applications.

In [55] Kim studied the notion of d-visibility when trying to find the kernel of a polygon P, which is the set of points in P from which every point in P is visible. He gave an $O(n)$ algorithm for finding the *d-kernel* of P, which is the set of points from which every point in P is d-visible [55].

Kim et al. also studied the problem of finding the edge visibility polygon under limited visibility [56]. The *d-visibility polygon* from an edge e is defined as the set of all points in P that are d-visible from e. They presented a linear time algorithm for computing the d-visibility polygon from an edge e of a given polygon P [56].

Pursuit-evasion problems have been a popular subject of study in computational geometry, motion planning, and game theory, where one or more hunters seek to capture one or more preys on a graph or polygonal region. Isler et al. [49] studied the version of this problem on a graph when the prey is assumed to have limited visibility. Here, however, the notion of limited visibility in [49] is slightly different from d-visibility: the prey can only see nodes on the graph that are adjacent to the current location of the prey. The authors showed that two hunters are sufficient to capture a prey with limited visibility, and they presented polynomial time algorithms for the problem [49].

Several problems in robotics have been researched under limited visibility and limited visibility problems continue to be of enormous interest in the field, as most sensors have limited sensing capabilities. A few references where a limited visibility range has been considered include [85] for mobile robot navigation, [8] and [13] for robot exploration problems, [54] for localization, and [2], [3], [4], [37] and [43] for formation, convergence and gathering problems of multiple robots with limited visibility.

CHAPTER 3
External Watchman Routes: The Unlimited Visibility Model

Internal watchman routes, where the watchman with unlimited visibility has to take a route in the interior of a polygon so that every point in the polygon is visible, has been studied extensively as mentioned in Chapter 2. The algorithm with the lowest computational time found to date is $O(n^3 logn)$ for the shortest fixed watchman route on simple polygons with n vertices, where a starting point is specified [29]. The algorithm with the lowest running time for the shortest unrestricted floating watchman route is $O(n^5)$ [74].

On the other hand, external watchman routes, where the watchman has to patrol the exterior of a polygon, has mainly been studied in [64], under the unlimited visibility assumption. Their algorithm for the shortest external watchman route on simple polygons has the same computational complexity as that for the internal watchman route problem, which they prove by converting the external problem to a set of internal problems.

The internal watchman route can be reduced to the external watchman route; therefore, the external problem can be said to be at least as hard as the internal one. The reduction is done in the following way [64]: In the internal watchman route problem, a polygon P and a starting point s on the boundary of P is given, and it is required to find the shortest route through s that sees all of P. To transform the problem, P can be enclosed in a rectangle, or even a triangle, as shown in Figure 3–1. Then a narrow corridor can be added by connecting s to the enclosing rectangle, producing a new polygon P', which

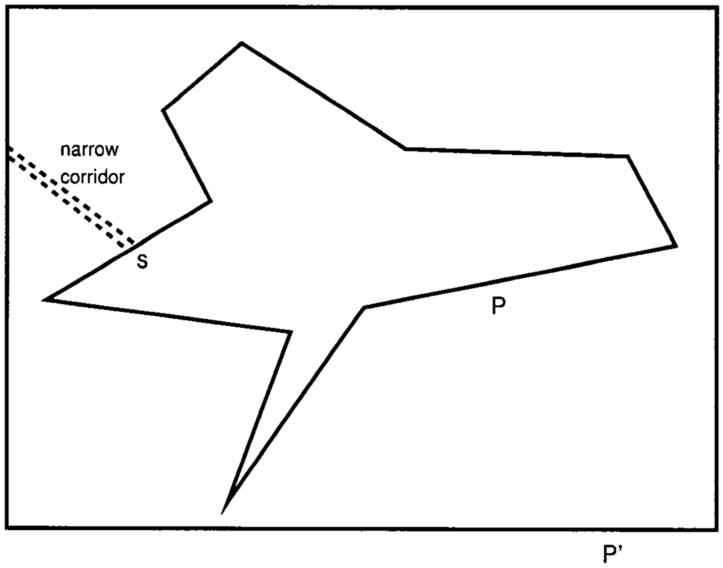


Figure 3–1: Reduction of the internal watchman route problem to the external watchman route problem

has the polygon P as an inner cave. Now if a shortest external watchman route W' can be computed for P', this also solves the internal watchman route W for P as a part of the route W'.

In the following sections of this chapter, we study the external watchman route problem under the unlimited visibility model, specifically on convex polygons, in detail.

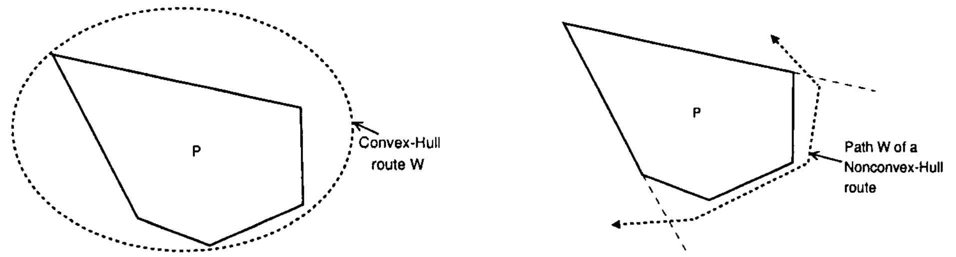
3.1 Convex Polygons

Although the external watchman route is very similar to the internal one, in certain ways it is very different. For instance, the shortest internal watchman route for a convex polygon P would be the starting point s. In fact, any point in the kernel of a polygon would suffice, and since the entire polygon can be seen from any point in a convex polygon, the specified starting point s would constitute the shortest watchman route. However, the external watchman route for the same polygon P would not be as simple, and can be viewed more like a problem where the boundary of a polygon hole P has to be inspected in an internal watchman route problem.

A linear time algorithm for finding the shortest external watchman route for convex polygons was presented in [64]. This work by Ntafos and Gewali was one of the only references we found for the external inspection problem, and it has been the basis and starting point for our work. In this chapter, we study the solution presented there in more detail and provide an exposition of part of the paper, in areas where we found it lacked full clarity and precision of details. We also introduce a few lemmas and proof sketches of our own to support the claims in [64], and endeavor to extend the solution by making some conjectures.

To begin, we note that there are two types of external watchman routes for convex polygons:

- *Convex-Hull Routes* which are closed curves that wrap around the whole polygon P, so that they contain P. Since the route need not be simple, we consider that it encloses P if P belongs to some finite region created by the route. Thus, any ray outgoing from any point on the boundary of P will intersect with this route. A simple example is shown in Figure 3–2(a).
- *Nonconvex-Hull Routes* are all other routes, which are curves that do not wrap around the whole polygon P, i.e. P belongs to the infinite region created by the route. An example of this can be seen in Figure 3–2(b), where the watchman travels back and forth on W between the tips of the arrows.



(a) A Convex-Hull Route (b) A Non-Convex-Hull Route

Figure 3–2: Illustration of a Convex-Hull Route and a Nonconvex-Hull Route

In [64], the shortest convex-hull route for any convex P is claimed to be the route following the boundary of the polygon. Furthermore, all shortest nonconvex-hull routes are claimed to be of the form shown in Figure 3–2(b), called a *2-leg route*. By definition, 2-leg routes have two extreme points and consist of an inner path W, along which the entire exterior of P is visible, and a return path R, which completes the route. Both the inner path and return path of the route are said to connect the two extreme points of the 2-leg route. (In the figure, $W = R$.)

However, the claim that these are the only two types of shortest external watchman routes possible for convex polygons has not been adequately justified in [64]. Here we show that these two types are the only forms of shortest external watchman routes by introducing two new lemmas. But first, we explain a few key concepts.

For an external watchman route to see the entire exterior of a polygon, it has to see all the edges. So, if unlimited visibility is assumed, we can say that for every edge in P, an external watchman route has to either contact that edge or the extension of the edge, or belong to the half-space from which the entire edge is visible, to be able to see every point on the boundary of P. Thus, *any* external watchman route determines an ordered sequence of intersection points of the path with an edge or extension of an edge. Note that not all edge extensions need to be intersected, and that the *shortest* external watchman route must satisfy the property that the path between any such two intersection points is as short as possible.

Now, if we consider any arbitrary external watchman route for a convex polygon P, we can note that the route divides the plane into regions or faces. Only two cases are possible.

i) P lies in a bounded region.

ii) P lies on the infinite region.

We now use each of these two cases, treated in separate lemmas, to sketch a proof that shortest routes have only the two forms claimed in [64], namely, in the case of a convex-hull route, the watchman follows the boundary of the convex polygon, and in the other case, the watchman follows a 2-leg route with $W = R$.

Lemma 1 : *The shortest external convex-hull watchman route for a convex polygon P is the route following the boundary of P.*

Proof : Let W be an external watchman route for a convex polygon P that bounds P but does not follow the boundary of P. By assumption, the polygon P lies in a bounded region of the curve, as illustrated in Figure 3–3, and any outgoing ray from a point on the boundary of P will intersect the curve, ensuring that every point on P is seen from W. Now, extend any edge e of P on both sides until it touches W on either side of e at points x and y, as shown in Figure 3–3. Now define route W' to be the route taken by traversing the subpath of W between x and y which wraps around P, but replace the other span of

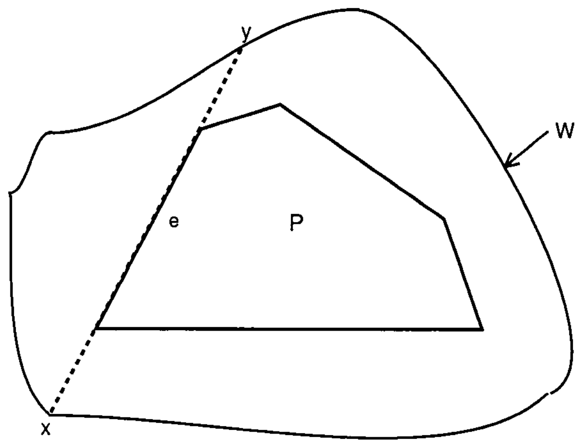


Figure 3–3: Proof of Lemma 1 - Convex-hull route case

W with the straight line segment xy, containing the edge of P. So W' is still an external watchman route and is shorter than W.

It is clear that this procedure can be applied repeatedly to the obtained shorter route, by extending every edge on both sides and cutting off the regions not containing P, until we achieve the shortest route possible in this case: the route following the convex-hull or boundary of P. $\diamond$

Now we turn to the second case, where the route does not enclose the polygon.

Lemma 2 : *The shortest external watchman route for a convex polygon P that lies on an infinite region created by the route is a 2-leg route.*

Proof : Suppose W is a curve from which the entire boundary of P can be seen, but does not enclose the polygon. We know from earlier descriptions that to see all edges, some edges are visible if the route lies in the half-space from which it can be seen, and some edges have to be seen by the route touching their extensions. Consider the shortest subpath W' of the curve from which the entire exterior of P is visible. By minimality, this subpath must start from an intersection point of the curve with the extension of an edge (one of the edge extensions that have to be touched in order for the edge to be seen). We travel along the curve W' until every edge has been seen and stop at another intersection point of the curve with an edge extension. At this point, the subpath we traced out sees all of the exterior of P. The extreme points of this path must lie on edge extension lines, and these lines must be distinct (by minimality of the path).

Consider these two edge extensions from which we start and stop (refer to them as $l(e_1)$ and $l(e_2)$). The polygon and path lie in the same half-space determined by the each of the two extensions; otherwise the path could be shortened, as it would not need to travel to at least one of $l(e_1)$, $l(e_2)$). Therefore, the polygon and the path lie in the intersection of half-spaces determined by $l(e_1)$ and $l(e_2)$). Clearly this intersection is not a strip, with $l(e_1)$ parallel to $l(e_2)$), since the path sees all of P. Therefore, both P and the path lie in a cone, or wedge, formed by $l(e_1)$ and $l(e_2)$). Furthermore, the vertex of the cone or wedge (i.e. the apex) is a vertex of the polygon; if not, the whole boundary would not be visible from W'.

So now we have a path W' as illustrated in Figure 3–4. (The path W' shown here is in the form of an arc for ease of illustration.) The path ends at two extreme points, the starting and ending points, x and y, that lie on the two extensions of the adjacent edges e_1 and e_2. And the rays from point $q = l(e_1) \cap l(e_2)$ containing e_1 and e_2 create the cone or wedge that contains the polygon P and the path W'.

If the path W' does not touch P, it can be moved or translated towards the point q (the vertex of the wedge), along the bisecting line L of this wedge, until it eventually contacts

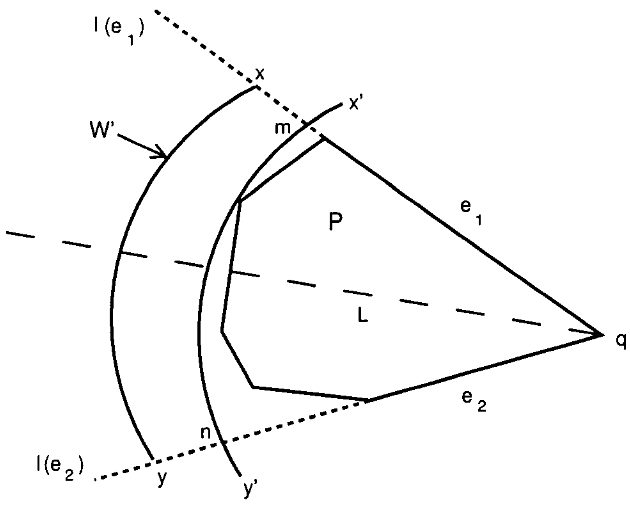


Figure 3–4: Proof of Lemma 2 - Non-convex-hull route case

the boundary of P, either at a vertex or an edge. Now the points that touch the extensions of the edges e_1 and e_2 are m and n, and since all other edge extensions cross W', the subpaths $m \to x'$ and $n \to y'$ (as shown in the figure) do not see anything that is not already seen from the path $m \to n$. Hence we can replace $x' \to y'$ by $m \to n$ to make a shorter path that sees all of P, contradicting the minimality of W'. Hence W' contacts P. Thus, W' travels across the wedge from a point on $l(e_1)$ to a point on $l(e_2)$, and touches P.

Note that any path that touches P (but not its interior) and that travels across the wedge from $l(e_1)$ to $l(e_2)$ sees all of P. Hence, by minimality, W' behaves like a rubber band stretched across P between m and n. It therefore follows along the boundary of P, and drops to m (or n) with a segment perpendicular to $l(e_1)$ (or $l(e_2)$) if m (or n) is not a vertex of P.

To turn W' into a closed watchman route, we have to go back to the starting point, and since there is no shorter path between the two points, we take the same path back, thus completing the route.

Hence, all nonconvex-hull routes are in the form of 2-leg routes. $\diamond$

We define a *leg* in an external watchman route as a segment connecting a vertex or edge of a polygon P to an extension of an edge of P. Thus, a 2-leg route consists of at most two legs, which contact extensions of two adjacent edges of the polygon, and a middle part of the route, which follows the polygon boundary. However, note that a 2-leg route includes routes with zero, one, or two legs, and may or may not contain a middle body.

Thus we can now characterize the shortest external watchman route for convex polygons by rewriting *Theorem 1* [64] more elaborately.

Theorem 1 : *A shortest external watchman route W for a convex polygon P can be one of two types:*

i W follows the boundary of P to make a convex hull route

ii W is a two-leg route, in which the two legs are the shortest segments (or perpendicular segments) onto the extensions of two adjacent edges of P. W may consist of three parts: the middle body of the route W_m which follows part of the boundary of P, and two legs, W_1 and W_2. Any of these parts W_m, W_1 or W_2 may be empty in the route W. And the return path R, which completes the route back to the starting point, is such that $R = W$.

Proof : The proof of this follows from the statements of Lemma 1 and Lemma 2 presented earlier. ◇

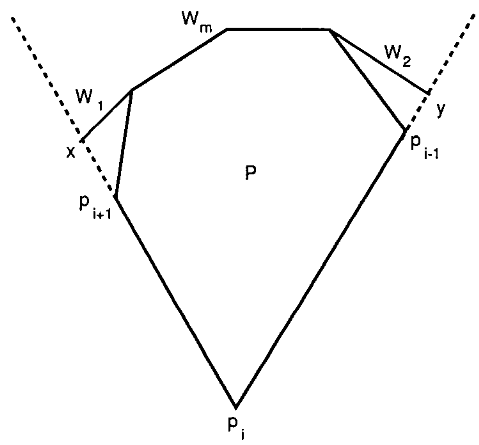


Figure 3–5: Shortest two-leg route for polygon P

As can be seen from Figure 3–5 , the legs W_1 and W_2 are always perpendicular to the extensions of edges (p_i, p_{i+1}) and (p_i, p_{1-1}) to make them the shortest links to the extensions. However, if the interior angle of the vertices p_{i+1} or p_{i-1} is less than 90°, then the perpendicular segment falls inside the interior of P. Thus, in such cases, the shortest route would consist only of the inner body W_m, and one or no legs.

It was shown in [64] that the shortest external watchman route for convex polygons can be constructed in linear time. For a polygon P with n edges, computing the shortest 2-leg route for a given pair of adjacent edges (or *wedge*) takes $O(n)$ time. Using binary search on the boundary of P, the legs can be computed in $O(logn)$. Once a 2-leg route for one of the wedges is computed, the rest can be computed by traversing around P, i.e.

for the next wedge, the contacts of the legs on the polygon advance around the polygon. All the 2-leg routes can be found in $O(n)$ time in this way, since it takes $O(1)$ time per edge during the advances, and each edge is only considered a constant number of times. Computing the convex-hull length also takes $O(n)$ time. Once all the possible 2-leg routes are found, the shortest one of these is compared to the convex-hull length, and the route with the shortest length is selected. Thus the total computational time required is linear [64].

3.1.1 Our Conjectures

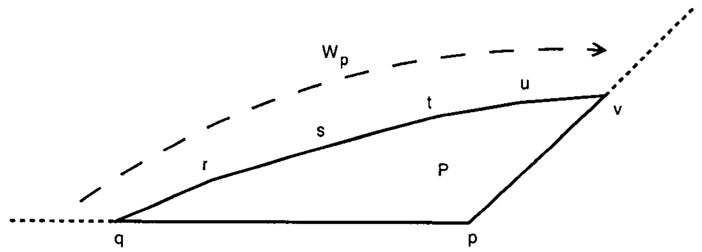
In the previous section, it was seen that finding the optimal watchman route for a convex polygon, when unlimited visibility is assumed, entails determining whether the convex-hull route for that polygon P or the shortest two-leg route among all other two-leg routes of P, is shorter. In other words, if the pair of adjacent edges, or the wedge, for which the two-leg route is the shortest among all others can be determined, then the process is simply to see if this gives a shorter route than the convex-hull route.

When the shortest external watchman route for a convex polygon P is a 2-leg route, it means there exists a certain wedge, which we can refer to as the *best wedge*, for which the 2-leg route is as short or shorter than all others. This also means $W(P)+R(P) \leq CH(P)$, where $W(P)$ is the watchman path between the two extreme points of the 2-leg route that sees all of the exterior of P, $R(P)$ is the return path that completes the route, and $CH(P)$ is the length of the convex hull of the given polygon P. According to *Theorem 1* (refer to section 3.1), $R = W$ for the shortest external watchman route; therefore the above equation simply means $2W(P) \leq CH(P)$ for the 2-leg route to be the shortest external watchman route for P. Thus the path W is required to be less than half the length of the convex hull of P to qualify it as the optimal path for the shortest route.

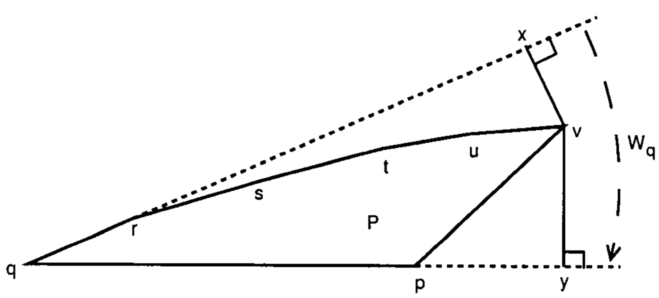
If it were possible to determine what properties make a wedge the best one for a 2-leg watchman route, the algorithm for finding the shortest external watchman route for convex polygons could immediately identify the wedge, compute its 2-leg route and compare it with the convex-hull route, instead of finding the 2-leg route for every wedge of P. Although this may not improve the computational complexity of the overall solution from the linear time already given in [64], it would however simplify the solution as well as the programming complexity of the algorithm.

Looking at the form of a 2-leg route (example Fig 3–5) under unlimited visibility for a polygon P, one may conjecture that the shortest 2-leg route may always come from the wedge where the sum of the lengths of the adjacent edges of the wedge is maximum. If the sum of the adjacent edge lengths is greater than all other adjacent pairs in the polygon, it may mean that the route W has to travel smaller lengths to see the entire exterior of polygon P.

However such a conjecture can be easily disproved using the following counterexample. As can be seen from Figure 3–6(a), edges pq and pv, which are incident on the vertex p, have the sum of lengths greater than all pairs of adjacent edges of P. However the route W_p (the route corresponding to the wedge incident on vertex p) has to travel over the span of every other edge of P to see the entire exterior of P, from point q to point v. However in Figure 3–6(b), the route W_q for the wedge on vertex q, whose adjacent edge lengths give a smaller sum, allows a shorter route for the same polygon P, starting from x to the vertex v and over the leg vy to point y.



(a) A 2-Leg Route on the wedge with the largest sum of adjacent edges



(b) A 2-Leg Route on the wedge that does not have the largest sum of adjacent edges

Figure 3–6: Counterexample for wedges with the greatest sum of adjacent edge lengths giving the shortest 2-leg route

This example however can lead to a new conjecture. As it can be seen from Figure 3–6, the sum of the adjacent edge lengths pq and vp is the largest. However, the interior angle of the wedge p is quite wide. In the wedge q, the sum of the pair of adjacent edge lengths is smaller, yet the interior angle is much smaller. This may intuitively lead to the question whether shorter external watchman routes come from narrower wedges. Hence we make the following conjecture, which we will follow up on and attempt to prove or disprove in the subsequent sections.

Conjecture 1 : *The best wedge, giving the shortest 2-leg route among all other 2-leg routes for a convex polygon P, is the one with the smallest interior angle in P.*

This conjecture can be seen to clearly work for the base case, where $n = 3$ and the polygon P is a triangle. As can be seen from Fig 3–7, the smallest external watchman route for a triangle T comes from the wedge with the smallest angle. The shortest external watchman route W for any wedge in a triangle is either the edge opposite to that angle, or the perpendicular leg to the extension of one of the adjacent edges. In the figure, $W = ax$ for the wedge on c. In either way, since the smallest side is always opposite the smallest interior angle of a triangle, the conjecture works very well for the triangular case.

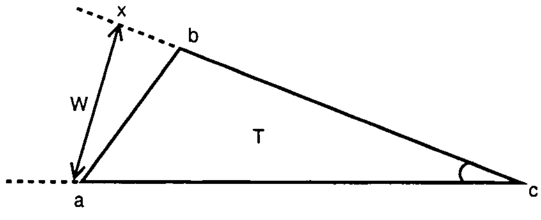


Figure 3–7: A triangle to prove the base case of Conjecture 1

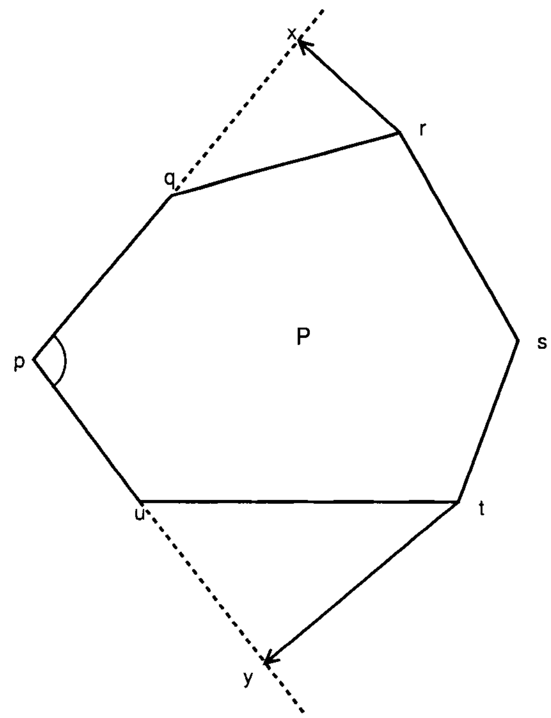


Figure 3–8: A convex obtuse polygon and one of its 2-leg routes

Another interesting question that arises from this conjecture is how it might affect the shortest external watchman route if all the interior angles of a polygon are relatively wide. A *Convex obtuse polygon* is defined as a convex polygon whose interior angles are all greater than 90°, i.e. a polygon that has no acute angles. An example is shown in Figure 3–8. As can be seen from this figure, the length of the path W for the wedge on vertex p looks like it is greater than $1/2CH(P)$ and this may be said for all its wedges. As was established previously, if none of the 2-leg route lengths are less than half the convex-hull length, than the convex-hull route is the shortest external watchman route for that polygon.

Therefore, now we make a new conjecture related to convex obtuse polygons.

Conjecture 2 : *All convex obtuse polygons have convex-hull routes as their shortest external watchman routes.*

This conjecture will also be looked into in Chapter 5.

3.1.2 External Watchman Routes on Convex Quadrilaterals

To study the questions posed in the previous section regarding external watchman routes on convex polygons of n vertices under the unlimited visibility model, we investigate the special case of convex polygons where $n = 4$. As was previously seen, *Conjecture 1* can be easily established for triangles ($n = 3$). If any of the conjectures can be proved or disproved in the case of arbitrary convex quadrilaterals, it would help further understand the problem for convex n-gons. Another motivation is that most conventional buildings are bounded by four walls, and if these outer walls have to be externally inspected, the study of convex quadrilaterals may prove to be useful.

Depending on the interior angles, convex quadrilaterals may be one of several shapes. The following are a few examples.

- All interior angles are 90°, as in a rectangle or square. In such cases, all routes, including the convex-hull route and two-leg routes on each of the four wedges of the rectangle, would have equal length (Figure 3–9).

- Another example is a parallelogram, with two acute angles, two greater than 90°, and opposite sides parallel. As can be seen in Figure 3–10, the shortest 2-leg routes W_a and W_c for the wedges with angles greater than 90° have path length equal to $(ad + ab)$ (or $(bc + cd)$). The routes W_d and W_b for the wedges with acute angles have path length equal to $(dx + dy)$, where dx is perpendicular to ab and dy is

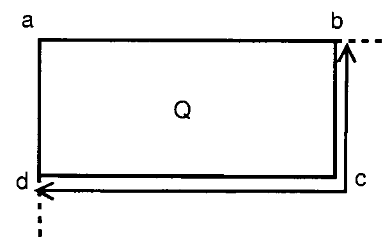


Figure 3–9: External watchman route on rectangles

perpendicular to bc. It can be seen that $(dx + dy) < (ad + ab)$, since $dx < ad$ (ad is the hypotenuse of $\triangle adx$) and $dy < ab$ ($ab = dc$ from the rectangle, and dc is the hypotenuse of $\triangle cdy$). Thus in the case of parallelograms, smaller angle wedges provide shorter routes.

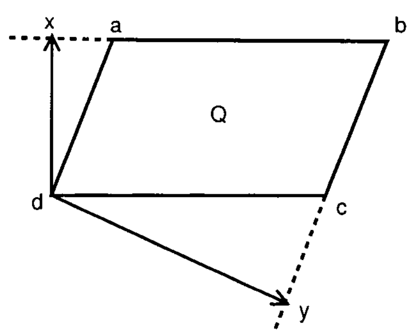


Figure 3–10: External watchman route on parallelograms

- Arbitrary quadrilaterals with non-parallel sides.

We study the structure of external watchman routes on different types of convex quadrilaterals to work our way through to either constructing a proof for *Conjecture 1* or finding a counterexample to the conjecture, and so we examined quadrilaterals with two equal acute interior angles. A question related to the conjecture is whether wedges of equal angles in a polygon always provide shortest 2-leg routes of equal length. If not, this may lead us to the construction of a counterexample.

A quadrilateral with two equal acute angles may have them placed either at adjacent vertices of the polygon, or at opposite vertices. We first look at the case where the two angles are at adjacent vertices of a quadrilateral.

Case 1: Quadrilateral with two equal acute angles at adjacent vertices

In quadrilateral ABC_0D (Figure 3–11), side AB is parallel to side C_0D, and $\angle A = \angle B = \theta$. This would mean $AD = BC_0$.

Now for the wedge B the route W_B is of length $AD + DY$ and the route W_A for wedge A is of length $BC_0 + C_0X_0$.

We can also say $\angle DC_0Y = \angle C_0DX_0 = \theta$ (corresponding angles) which means $DY = C_0X_0 = DC_0 \sin\theta$. Therefore, the route lengths for wedges A and B are equal, $W_A = W_B$, for the quadrilateral ABC_0D. This proves that for a quadrilateral with two parallel sides and two equal acute angle wedges on adjacent vertices, the watchman routes are of equal length.

Now let us consider the quadrilateral $ABCD$ shown in Figure 3–11, where side CD is *not* parallel to AB. The route for wedge A is now $W_A = (BC + CX)$ and the route for wedge B is $W_B = (AD + DY)$ (remains the same).

To compare the lengths of the two paths, it suffices to compare $(BC_0 + C_0X_0)$ with $(BC + CX)$, as it was shown that $(AD + DY) = (BC_0 + C_0X_0)$.

If we extend a perpendicular line from point C_0 to line CX we get the intersection point P. Now, $(BC + CX) = (BC + CP + PX)$ and $(BC_0 + C_0X_0) = (BC + CC_0 +$

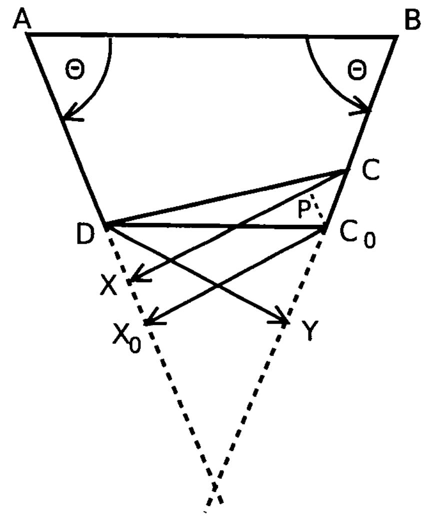


Figure 3–11: Convex quadrilaterals with adjacent equal smallest angles

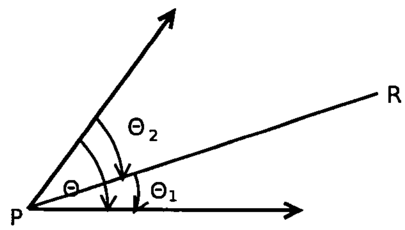
C_0X_0). Clearly, $PX = C_0X_0$ (sides of the rectangle XCC_0X_0). So, comparing CC_0 with CP, we can say $CC_0 > CP$ as CC_0 is the hypotenuse of $\triangle PCC_0$.

Therefore, routes $(BC_0 + C_0X_0) > (BC + CX)$, which equivalently means that the route for wedge A is not equal to the route for wedge B ($W_A \neq W_B$), even though $\angle A = \angle B$.

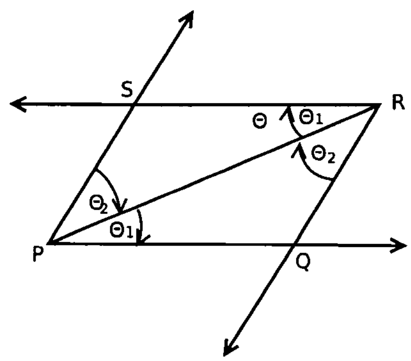
This shows that wedges of equal angles at adjacent vertices do not necessarily provide 2-leg routes of equal length. $\diamond$

Case 2: Quadrilateral with two equal acute angles at opposite vertices

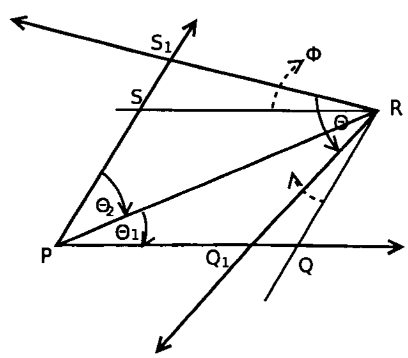
We now look at the case where the two equal acute angles of the convex quadrilateral are placed at opposite vertices. We propose a method that allows constructing a range of different convex quadrilaterals with equal acute angles opposite each other by changing the value of a variable ϕ.



(a) Wedge P



(b) Wedge P and R



(c) Rotating wedge R about PR

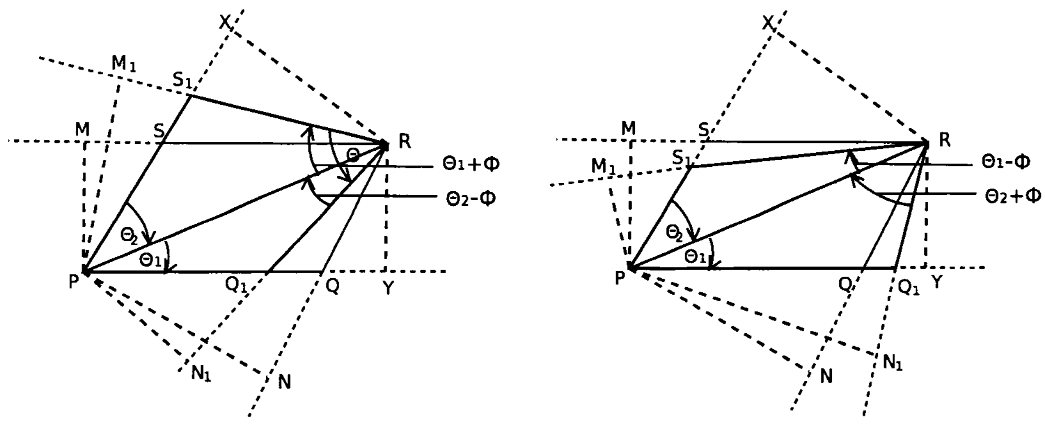
Figure 3–12: Generating convex quadrilaterals with two equal opposite acute angles

We take a wedge located at a point P and fix the interior angle at θ. We place a point R somewhere within this wedge, so that the line PR divides the angle θ into θ_1 and θ_2 as shown in Figure 3–12(a). Now we place a wedge of the same angle θ on point R. If the wedge is placed in such away that $\angle SRP = \angle RPQ = \theta_1$ and $\angle QRP = \angle SPR = \theta_2$, we get a parallelogram (shown in Figure 3–12(b)). However if we keep the wedge at P fixed, together with angles θ_1 and θ_2, by keeping point R fixed, and we only rotate the wedge R about line PR, we can achieve several convex quadrilaterals with acute angles θ opposite each other. The wedge R can be rotated anti-clockwise at angles within $0 \leq \phi < \theta_2$ and can be rotated clockwise at angles within $-\theta_1 < \phi \leq 0$, where ϕ is the amount by which the wedge is rotated, as can be seen in Figure 3–12(c).

Figure 3–13(a) shows an example of the transformation of the quadrilateral $PQRS$ to the new quadrilateral PQ_1RS_1 when the wedge R is rotated clockwise by an angle ϕ. The corresponding 2-leg routes for each wedge is also illustrated. The 2-leg route W_R for wedge R of quadrilateral $PQRS$ is $(PM + PN)$, and W_R for quadrilateral PQ_1RS_1 is $(PM_1 + PN_1)$. The 2-leg route W_P for wedge P remains $(RX + RY)$ even through rotation, as can be seen from the figure. Figure 3–13(b) similarly shows the formation of the quadrilateral PQ_1RS_1 by rotating the wedge R anti-clockwise by angle ϕ, and its corresponding 2-leg routes.

Now, we can define the path lengths for the wedges at P and R, with respect to the angles θ_1, θ_2 and ϕ. The path length for wedge P is

$$W_P = PR(\sin\theta_1 + \sin\theta_2)$$



(a) (b)

Figure 3–13: Illustration of rotating wedge R by angle ϕ

and the path length for wedge R is

$$W_R = PR(\sin(\theta_1 + \phi) + \sin(\theta_2 - \phi))$$

where $-\theta_1 < \phi < \theta_2$.

We now define a function of ϕ that computes the difference between the path lengths of wedges P and R:

$$\begin{aligned} f(\phi) &= W_R - W_P \\ &= PR(\sin(\theta_1 + \phi) + \sin(\theta_2 - \phi) - (\sin\theta_1 + \sin\theta_2)). \end{aligned}$$

Since the length of diagonal PR is constant for a given quadrilateral, we can take it out, so that the function now becomes

$$f(\phi) = (\sin(\theta_1 + \phi) + \sin(\theta_2 - \phi)) - (\sin\theta_1 + \sin\theta_2).$$

If $(\sin\theta_1 + \sin\theta_2) = x$ and $(\cos\theta_1 - \cos\theta_2) = y$, then $f(\phi)$ can also be written as

$$\begin{aligned} f(\phi) &= \cos\phi(\sin\theta_1 + \sin\theta_2) + \sin\phi(\cos\theta_1 - \cos\theta_2) - (\sin\theta_1 + \sin\theta_2) \\ &= x\cos\phi + y\sin\phi - x \end{aligned}$$

where x and y are constant values as the angles θ_1 and θ_2 are fixed.

The value of ϕ is restricted between $-\theta_1 < \phi < \theta_2$ so that the sides do not pass the diagonals during rotation. Also, if we do not want the sides to go beyond the perpendicular legs RX and RY of wedge P, the value of ϕ has to also be restricted within $-90 + (\theta_1 + \theta_2) < \phi < 90 - (\theta_1 + \theta_2)$.

Using different data sets, the graph of this function was plotted to see the form of the function within the restricted values of ϕ. A generalized form of the function $f(\phi)$ is shown in Fig 3–14. As it can be seen, the function in this form gives a relative maximum. This can be easily proved using the second derivative test. Since this is an elementary mathematical proof, it is presented in Appendix A for reference, if needed.

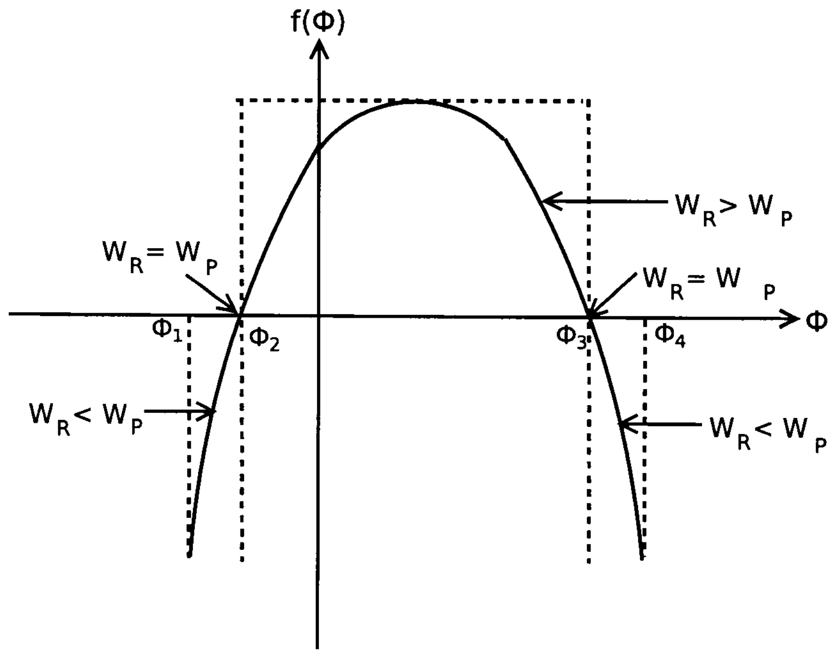


Figure 3–14: Form of the function $f(\phi)$

From Figure 3–14, we can say the following:

- When ϕ lies between ϕ_1 and ϕ_2 in Figure 3–14, or between ϕ_3 and ϕ_4, then the two-leg route for wedge R is shorter than that for wedge P.
- When ϕ lies between ϕ_2 and ϕ_3, then the two-leg route for wedge P is shorter than that for wedge R.
- When ϕ is equal to ϕ_2 or ϕ_3, only then the two-leg route lengths for wedge P and wedge R are equal.

Thus, it can be said that equal acute angle wedges that are opposite each other in convex quadrilaterals also do not always give shortest 2-leg routes that are equal in length. ◇

Now looking at Figure 3–14 we can see at certain points, such as the relative maximum point of $f(\phi)$, the difference between the route lengths is locally maximum. We used this information to create a counterexample for *Conjecture 1*.

Taking a data set where the wedges are of angle $\theta = 60°$, with $\theta_1 = 45°$ and $\theta_2 = 15°$, we found that the relative maximum occurs at $\phi = -15$. Thus at $\phi = -15$, the quadrilateral $PQRS$ is such that routes for wedges P and R are significantly unequal even though $\angle P = \angle R$. Now we reduce the angle of wedge R by a small amount, e.g. by reducing $(\theta_1 - \phi)$ by 2°, so that the angle at wedge R is now 58° instead of 60°. The route for wedge P remains the same length, but the route for wedge R consequently reduces in length.

For this particular example, shown in Figure 3–15, the values found for a quadrilateral with a diagonal $PR = 8.4$ cm are as follows.

When $\angle P = \angle R$ at $\phi = -15$

$$
\begin{aligned}
W_P &= RX + RY = 8.114 \\
W_R &= PM_1 + PN_1 = 8.4
\end{aligned}
$$

When $\angle P > \angle R$ at $\phi = -15$

$$
\begin{aligned}
W_P &= RX + RY = 8.114 \\
W_R &= PM' + PN_1 = 8.144
\end{aligned}
$$

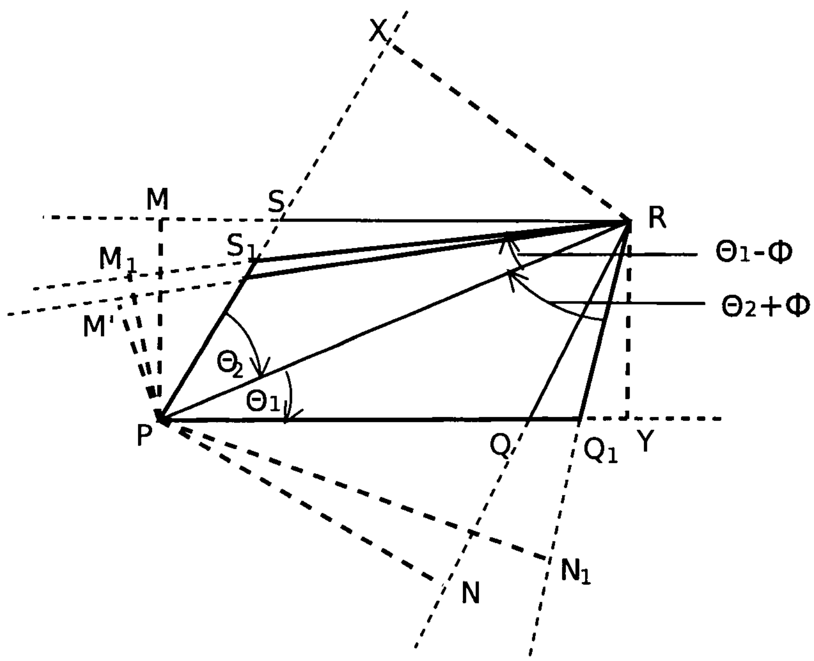


Figure 3–15: Counterexample of Conjecture 1

Thus even though wedge P is of a larger angle, it gives a shorter route than wedge R of smaller angle - a counterexample to *Conjecture 1*. Thus *Conjecture 1* has been disproved - smallest angle wedges do not always give the shortest routes. However the range of values for which a counterexample exists, and the amount by which the angle of a wedge can be reduced for the instance to remain a counterexample, is found to be quite narrow in most cases. In Chapter 5, we generate random convex polygons to see how likely it is to randomly produce convex quadrilaterals that satisfy these conditions to give rise to a counterexample to the conjecture. It was found that such conditions are usually rare in a set of random convex quadrilaterals.

We further study the unlimited visibility model on convex polygons in Chapter 5, using the random generation of convex polygons of different sizes. Hence, in the next section we move on to the discussion of inspecting simple polygons under the unlimited visibility model.

3.2 Simple Polygons

In this section, we present a brief discussion on the external watchman routes for simple, not necessarily convex, polygons. It is mostly an expository explanation of such routes using the unlimited visibility model; since it was not the focus of our work, no proofs or algorithms are given.

Simple polygons may be of two types - one where a watchman following the convex hull of the polygon would be able to see the entire exterior of the polygon, or one where it would not. Simple polygons of the former type are called *weakly externally visible polygons.*

Def: A polygon P is defined as being *weakly externally visible* if each point x on the boundary of P is visible from a circle at infinity.

Def: A *cave* is any region between the boundary of a polygon P and the convex hull of P, such that there is at least one point in it that is not visible from some point on the convex hull of P.

The problem of determining if a polygon is weakly externally visible was studied in [83]. From the above definitions it is apparent that a simple polygon that is *not* weakly externally visible has at least one or more caves. For such polygons, following the convex hull is not sufficient for external inspection. It is necessary for the watchman to enter the caves to see all the edges inside it.

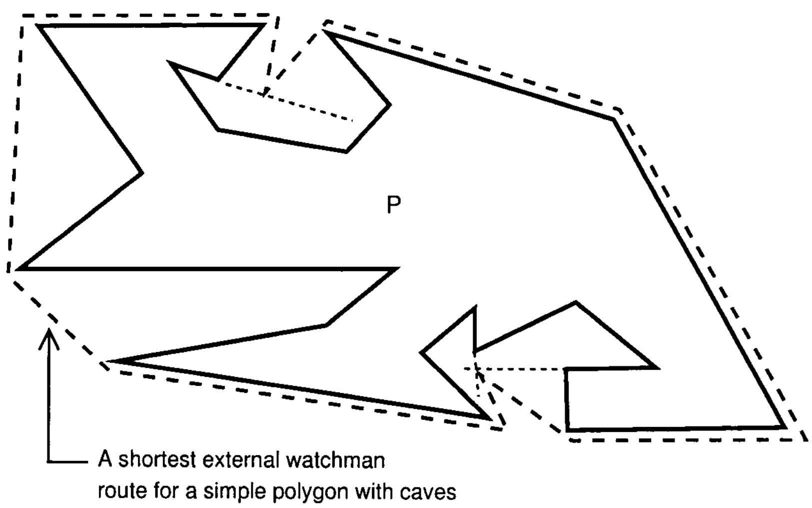


Figure 3–16: A shortest convex-hull route for a simple polygon

As mentioned in Section 3.1, an external watchman route would either have to touch or cross over the exterior extension of each edge, or enter the region from which the entire edge is visible, to ensure seeing the the whole polygon exterior. Again, the two types of routes can be either the convex-hull route, which is a closed curve enclosing P, or the nonconvex-hull route, which could be an open or closed curve that does not contain P. *Theorem 1* proved that the inner path W of a 2-leg route would be the same as the return path R for the shortest external watchman route on a convex polygon. However this is not necessarily true for simple polygons. The inner path W might have to enter some caves to see all of the exterior of P, but the return path R could just take the shortest path back, which would follow the convex hull of P in those areas.

In [64] an $O(n^4)$ time solution was given to this problem by converting it to a set of internal problems. An illustration of a solution instance is shown in Figure 3–16. In this example, it can be seen that there are two caves in the external boundary of the polygon, and the external watchman route has to enter each of these caves to be able to see the entire exterior. As shown in the figure, the part of the route that enters these caves contacts the extensions of the edges of the polygon that are not weakly externally visible.

CHAPTER 4

External Watchman Routes: The Limited Visibility Model

4.1 Convex polygons: Route Length as a Function of the Visibility Range

Although the internal watchman route problem under limited visibility has been studied ([63], [5]), no such study has been published to date for the external watchman route problem under limited visibility. In this section, we examine the effects of incorporating a limited visibility range constraint on the watchman for the problem of inspecting the exterior of a convex polygon.

To study this problem we observed what occurs when the visibility range d is reduced from infinity to zero. We can say:

- When $d = \infty$, the shortest watchman route for a convex polygon P can be either a 2-leg route or a convex-hull route (as seen in Chapter 3).
- When $d \to 0$ or is near 0, the shortest watchman route would have to be the convex-hull route, following the boundary very closely, to be able to see the entire boundary of P.

For the polygons where the shortest route is the convex-hull route when $d = \infty$, reducing the range d does not have any effect, since the shortest route for all values of d would be the same. However for those convex polygons where the shortest external route is a 2-leg route, a smaller d would affect the length of the route.

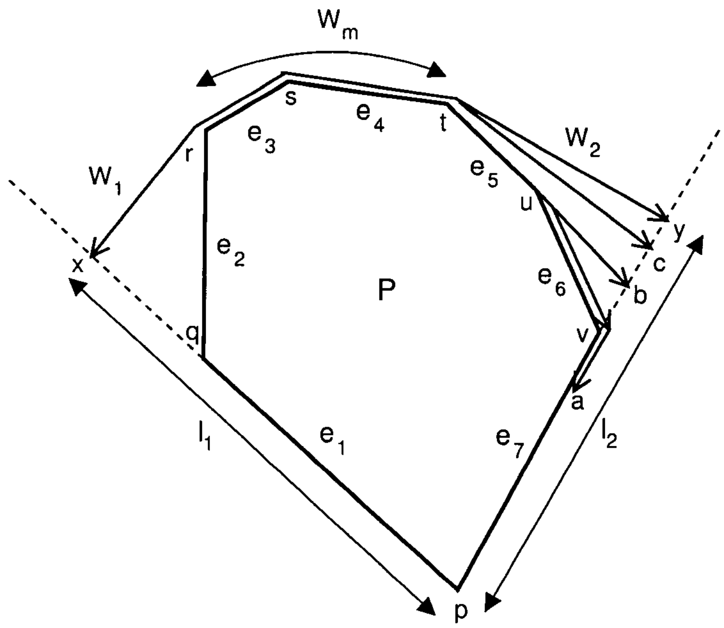


Figure 4–1: Reducing the visibility range on a shortest 2-leg route

From Figure 4–1 we can see the illustration of a convex polygon P with the shortest watchman route as a 2-leg route. We signify $W(d)$ as the length of the shortest watchman path (the inner path of the 2-leg route) under visibility range d.

$$W(\infty) = W_1 + W_m + W_2$$

where any of the components - the two legs W_1 and W_2, or the middle body W_m, may be empty for arbitrary convex polygons.

Let k be a point on the boundary of P, denoted as $\delta(P)$, and let $D(k)$ be the distance from point k to the closest point on route W that is visible to k. For all k, let $l = max\{D(k)\}$, where $k \in \delta(P)$. Thus, l is the maximum distance of a point k on the boundary of P to the closest d-visible point on W. As long as the range d is greater than the distance l, the shortest watchman route remains the same as the optimum route under the limited visibility model.

For instance, suppose this longest distance the watchman has to see from the route W to the boundary of polygon P, in the case of the polygon in Figure 4–1, is the length l_2 from point y to vertex p, where $l_2 > l_1$; this means that as long as the range $d \geq l_2$, the shortest external watchman route for P remains $W(\infty)$.

Now we look at an instance when d is reduced from l_2; the path $W(d)$ would have to change to adjust to the new visibility range. If d is reduced slightly so that $d < l_2$ and the longest distance covered by the visibility range from point p is to the point c in Figure 4–1, then the perpendicular leg W_2 would have to move down to point c, in order to be able to see until the vertex p. This means the leg is no longer normal to the extension of the edge e_7 and is longer than W_2. The more d is reduced, the longer the leg becomes, and at a certain value of d the leg absorbs edge e_5, so that $W_m = e_3 + e_4 + e_5$ and $W_2 = ub$. Similarly, at a certain value of d, edge e_6 is also absorbed. If d is reduced even further, the path would have to extend from vertex v to some point a on the edge e_7.

Simultaneous to the changes taking place at the right part of $W(d)$, as seen in Figure 4–1, the left part of the route would also be similarly altering and adjusting, depending on the value of d. At a particular value of d, the path length,

$$W(d) = 1/2CH(P)$$

where $CH(P)$ is the convex-hull length. Thus, at this point, the total length of the 2-leg route (which is $2W(d)$) and the convex-hull route become equal. We refer to the value of

d at this point as the critical value d^*. Reducing the value of d further would not change the length of $W(d)$ anymore, since the shortest route would be the convex-hull route for all $d < d^*$.

Length of the watchman route as a function of the visibility range

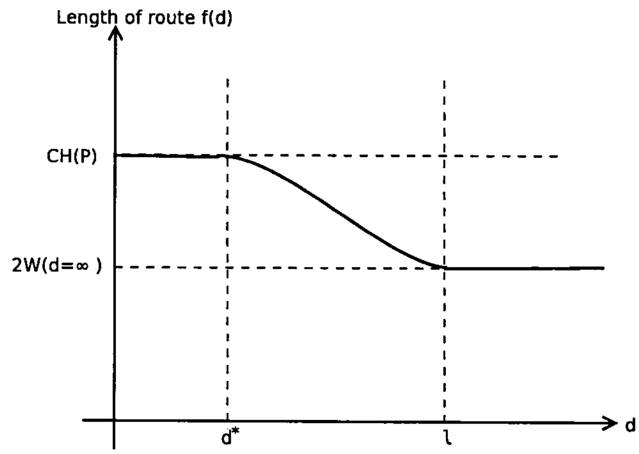
When the shortest external watchman route for a convex polygon P is a 2-leg route W under unlimited visibility, and l is the maximum distance that the watchman has to see from the route W to the boundary of P, or $l = max\{D(k)\}$, $k \in \delta(P)$, then we can summarize and say that under d-visibility, if

- $l \leq d < \infty$, the optimal route remains the same 2-leg route.
- $d^* < d < l$, the optimal route length would depend on a function of d, increasing as d decreases.
- $0 \leq d \leq d^*$, the optimal route length is equal to the convex-hull route.

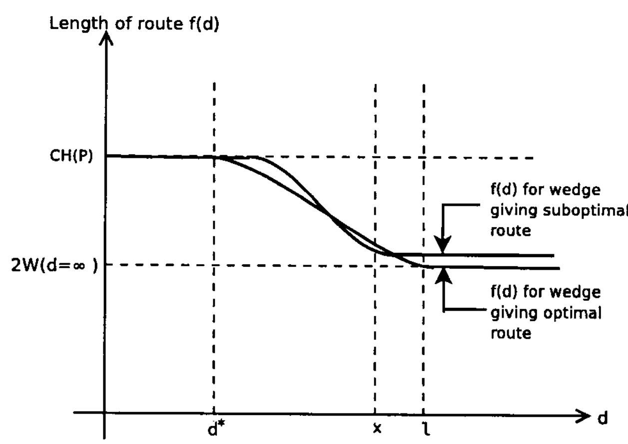
This concept has been illustrated in Figure 4–2(a). The shape of the function between $d^* < d < l$ is investigated in Chapter 5, in the experimental work.

One question that arises is whether *the wedge giving the optimal solution for a polygon P using the unlimited visibility model always provides the optimal solution for all values of d in the limited visibility model.*

To illustrate this, suppose the function $f(d)$ giving the length of each 2-leg route with different values of d is plotted for every wedge of a polygon P. If there is a crossover of the curve for a wedge that gives a suboptimal route in the unlimited visibility model with the curve for the optimal wedge, as shown in the Figure 4–2(b), there may be a value $d = x$ at which the suboptimal wedge gives a shorter route than the optimal wedge. This question is also investigated in the experimental work in Chapter 5.



(a) The length of an external route as a function of d



(b) Function of d for two wedges of polygon P

Figure 4–2: Graph of the route length as a function of the visibility range d

4.2 Simple polygons

It is known that the internal watchman route problem for simple polygons under limited visibility is NP-hard [63], and so it can be said that the external watchman route problem under limited visibility is also hard (the external problem is at least as hard as the internal one, as described in Chapter 3). Thus we propose an approximate solution for the external problem under a visibility range of d.

As mentioned in Chapter 3 (Section 3.4), externally inspecting simple polygons under unlimited visibility means categorizing them into polygons that are weakly externally visible or not. The shortest route for weakly externally visible polygons would either be the convex-hull route or a 2-leg route. The shortest inspection route for a polygon that is *not* weakly externally visible would be one where the route has to enter each of the caves of the polygon P to see all the edges of P.

Under limited visibility of range d, however, it is not important if the polygon P is weakly externally visible or not, because even if it is, that does not necessarily mean that following the convex hull of the polygon would be sufficient to see the entire exterior of P. If d is small, then the route might have to enter each concave region of P, even if it is a weakly externally visible polygon.

To construct an external watchman route under d-visibility would require placing circular sectors of radius d centered at each reflex vertex of P. The watchman route W would have to follow the convex hull between convex vertices and visit each d-radius disk at the reflex vertices in the concave regions of P so that the whole exterior is visible from W, as shown in Figure 4–3. The shortest route that does this would be the optimal external inspection route under d-visibility.

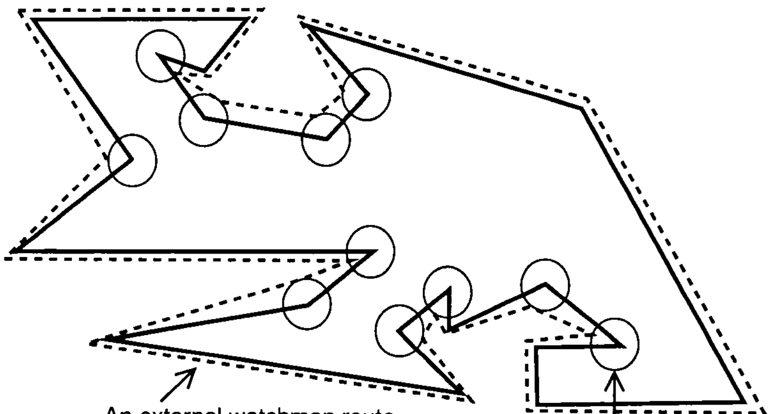


Figure 4–3: A shortest convex-hull route for a simple polygon under d-visibility

This concept is similar to the d-watchman route described in [63] (also refer to Section 2.2.2) for internal watchman routes under d-visibility. Using the method described there, the solution can be approximated using convex regular k-gons inscribed in the disks of radius d. Thus, the approximate solution we propose entails finding the shortest safari route that visits the set of regular k-gons at each reflex vertex of P, for each concave region.

Each cave or concave region of P can be considered a polygon itself. Hence the shortest external watchman route under limited visibility can be converted to a set of internal problems under limited visibility [63]. The shortest safari route that visits the set of k-gons centered at each reflex vertex can be found for each cave or concave region of P. The rest of the polygon is patrolled by simply following the convex-hull route (Figure 4–3).

The shortest safari route problem can be solved in $O(mn^2)$, where the route visits a set P' of m convex polygons attached to the boundary of the polygon P and n is the total number of vertices in P and P' [63]. Since we have $O(n)$ inscribed convex k-gons for the external watchman route problem under d-visibility, and the total number of vertices is kn, the complexity of the algorithm should be similar to the $O(k^2n^3)$ time given in [63] for internal d-watchman routes. Similarly, the solution should approach the optimal solution as k is increased, since increasing the number of sides of the inscribed polygons better approximates the circular sectors of radius d.

CHAPTER 5
Experimental Work

5.1 Generating Convex Polygons

Generating random geometric objects is often necessary for testing and evaluating various computational geometry and pattern recognition algorithms. The goal is to generate a wide and diverse collection of test data which would allow the algorithm to perform on all types of highly probable data.

For testing the external watchman route under both the unlimited and limited visibility models, we required the generation of random convex polygons. However it is difficult to find an accepted definition of a random polygon. There even exists no polynomial time solution for uniform generation of random simple polygons with n vertices.

Some very simple means to generate convex polygons would be to randomly generate a set of points and then to construct their convex hull, or to choose points randomly until their convex hull is an n-gon. However these methods are not very random unbiased procedures and do not produce a sufficiently wide range of polygonal data to work with.

The generation of random convex polygons has been investigated in several papers, such as [28] where random convex hulls are generated, [88] where a random convex polygon with vertices that are a subset of a given set of n points is generated, and many more. However, most of these do not allow specifying the number of sides of the generated polygon. An algorithm presented in [67] does allow this option, but since the algorithm works by randomly choosing an angle for each corner as it goes around the boundary, after the selection of the first few angles, the rest are forced to be very close to straight angles.

We use the algorithm presented in [58] where this feature of specifying n for generating a random convex n-gon is present. The approach used is to select a random topological triangulation of a polygon and to build a convex polygon whose Delaunay triangulation is homeomorphic to this. The implementation of this algorithm generates convex n-gons in $O(n)$ time [57].

Using this implementation of the algorithm [58] to generate random convex polygons of n sides allows us to experiment with different sizes of polygons and to test how the limited visibility and unlimited visibility models of the external watchman route algorithms perform on the generated polygons. We also tested the conjectures made in Section 3.2 using the random convex polygons generated. The description of these experiments as well as the results are presented in the following sections.

5.2 Evidence for Conjectures of Chapter 3

In Chapter 3 (Section 3.1.1), we presented a few conjectures on the shortest external watchman routes on convex polygons and their computation. In this section, we will perform experiments on random convex polygons of n vertices, with n starting at 4 (the $n = 3$ case is trivial) to support these conjectures.

In the previous section, the generation of random convex polygons was discussed. We implemented the method described in [58] to produce random convex polygons of a specified number of n vertices in time $O(n)$. We then implemented the method described in Chapter 3 (and [64]) to find the 2-leg route for each pair of adjacent edges of a convex polygon and computed its length. A Java application was written to implement both these procedures. Hence, at each run of the program, the following is performed for the unlimited visibility model:

- A random convex polygon with a specified number of n vertices is generated.
- The interior angle between each pair of adjacent edges of the generated polygon is computed.
- The length of the shortest 2-leg route for each of these wedges is computed.
- The length of the shortest convex-hull route is also computed.

The computation of the individual 2-leg route lengths for a convex polygon allows us to compare the lengths of the 2-leg routes for each wedge of a polygon, from a set of random convex polygons. Since the set is generated at random, we can see how likely it is that the narrowest wedge gives the shortest 2-leg route, thus supporting *Conjecture 1*, or how likely it is that the convex-hull route is the optimal route for convex polygons with no acute interior angles, thus supporting *Conjecture 2*.

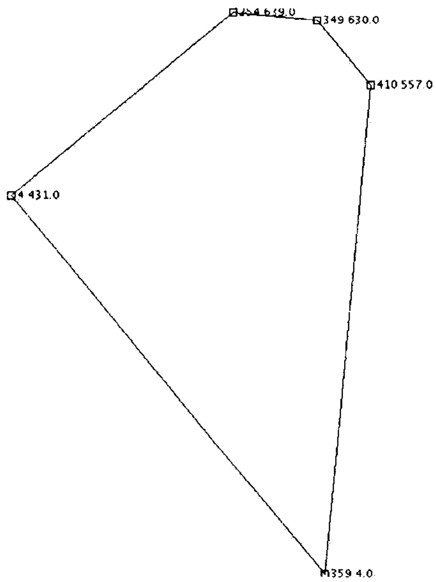
We performed the experiments on polygons of different sizes of n, ranging from $n = 4$ to $n = 20$. For lower values of n, such as $n = 4$, after fifteen or more runs, the structure of the convex polygons seem to become similar to previous runs, since the number of sides is few. And for higher values of n, such as $n = 9$ and higher, the convex polygons generated become more and more flat or wide, and the results also begin to appear similar to previous runs.

Table 5–1 shows a few samples of the data generated during experimental runs of the program. The output data for three sample random convex polygons are given. The first column defines the number of sides of the convex polygon for that particular run of the experiment and the second column shows the length of the shortest convex-hull route for that polygon. This length gives an idea of the size of the polygon, since it is basically the length of the perimeter of the polygon. The third and fourth columns give each interior angle of the polygon and the length of the shortest 2-leg route for the corresponding wedge. As can be seen, the smaller angles seem to give shorter routes, and the larger the angle, the longer the route becomes. The fifth column gives the ratio of the length of the 2-leg route for that wedge to the length of the shortest convex-hull route for that polygon. This gives an idea of the length of the 2-leg route associated with each polygon wedge relative to the convex-hull route for the polygon. Some of the figures in the table have been rounded: the angle size (given in degrees) to the nearest integer, and the lengths (given in arbitrary units of distance measure) to the nearest second decimal place, for clarity of illustration. The three polygons corresponding to the three convex polygons generated in Table 5–1 are shown in Figure 5–1.

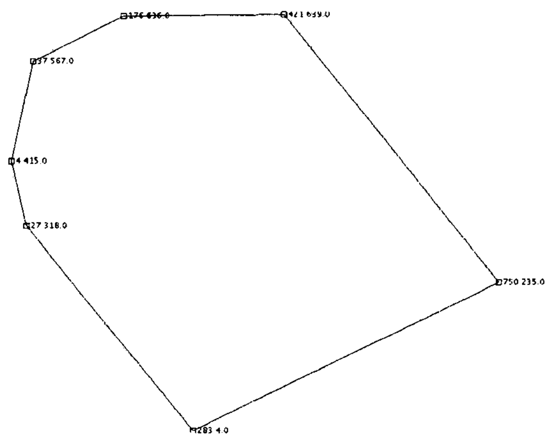
Table 5–2 gives some of the results found by running the program on convex polygons of different n, and comparing the lengths of the routes for each wedge for every polygon generated. The experiment was run 50 times for each value of n. The first column of the table gives the number of sides specified for the convex polygon during generation. The second column gives the average minimum angle over the fifty runs for that specific polygon size. This gives an idea whether the minimum angles of the polygons generated were acute or not. The third column states the percentage of times, out of the 50 times the experiment was run on each value of n, that the convex-hull route proved to be the shortest external inspection route for that polygon. And the fourth column specifies the percentage of experiments where the shortest route for a convex polygon with n vertices was achieved by the 2-leg route corresponding to the wedge with the smallest interior angle.

Table 5–1: Sample data generated for the unlimited visibility model

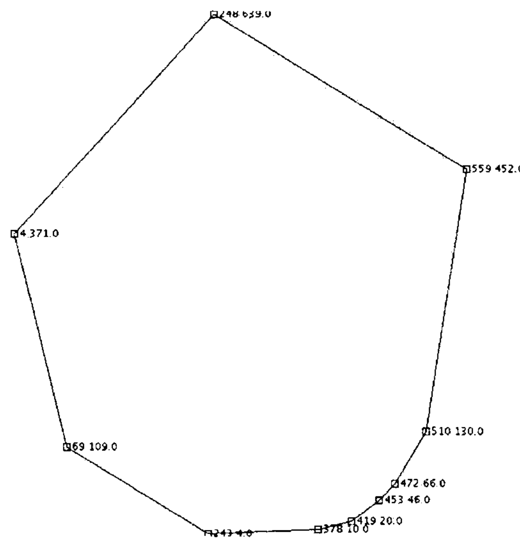
No. of sides	CH-route length	Angle sizes	2-leg-route length for wedge	Ratio of 2-leg to CH-route length
n=5	1678.31	45	1007.10	0.60
		90	1481.30	0.88
		135	2431.58	1.45
		135	2429.38	1.45
		135	2429.48	1.45
n=7	1719.60	77	1624.84	0.95
		103	1775.85	1.03
		128	2039.75	1.18
		129	2036.55	1.19
		154	2444.08	1.42
		154	2576.05	1.46
		155	2539.22	1.48
n=10	2159.51	104	2437.65	1.13
		114	2605.86	1.20
		123	2710.35	1.25
		133	2873.28	1.33
		142	3088.55	1.43
		151	3045.44	1.41
		160	3343.04	1.55
		168	3398.29	1.56
		172	3401.07	1.58
		173	3497.50	1.62



(a) Polygon with n=5



(b) Polygon with n=7



(c) Polygon with n=10

Figure 5–1: The three convex polygons from Table 5–1

Table 5–2: Experimental data for the unlimited visibility model

No. of sides	Average minimum angle	Percentage of runs CH-route is the shortest route	Percentage of runs the narrowest wedge gives the shortest 2-leg route
$n = 4$	36	0%	100%
$n = 5$	45	2%	98%
$n = 6$	67	4%	96%
$n = 7$	81	26%	74%
$n = 8$	87	58%	42%
$n = 9$	107	100%	0%
$n = 10$	113	100%	0%
$n = 15$	133	100%	0%
$n = 20$	145	100%	0%

Evidence for Conjecture 1

In *Conjecture 1* (refer to Section 3.1.1) it was proposed that the smallest interior angle of a convex polygon corresponds to the wedge that gives the optimal external watchman route for that polygon. However in Section 3.1.2, it was shown that it is possible to construct counterexamples to this conjecture. We use the experimental evidence found to evaluate how likely it is that the statement of the conjecture holds in a set of random convex polygons.

From the results displayed in Table 5–2, the likelihood of the convex-hull route being the optimal external route for convex polygons can be seen to be very low for smaller values of n, while becoming more and more likely as the value of n becomes larger. From Table 5–2 it can be noted that for $n = 9$ and above, the convex-hull route is very likely to be the optimal route.

From the column in Table 5–2 giving the percentage of experiments where the wedge with the smallest angle gave the optimal external route, it can be seen that 2-leg routes give the optimal route more frequently for smaller values of n. We observed that the smallest interior angle of the polygon always gave the shortest 2-leg route in these cases, and the only times it did *not* was when the smallest angle was greater than 90°. It was observed that whenever the smallest angle was greater than 90°, the convex-hull route gave the optimal route.

Hence for the convex polygons of n vertices with the smallest interior angle as an acute angle, the shortest external route for that polygon tended to be the 2-leg route for the wedge corresponding to that smallest angle. Thus even though it was possible to construct counterexamples for *Conjecture 1* in Section 3.1.1, these exceptions occur only under special circumstances, and only when the difference between the smaller angles of the polygon are only a few degrees. For most randomly generated convex polygons though, it is safe to say that the smallest acute angle wedge is likely to give a 2-leg route shorter than the convex-hull route and all other 2-leg routes for that polygon.

Evidence for Conjecture 2

The experiments showed that when an interior angle of a convex polygon is greater than 90°, the corresponding 2-leg route tends to be longer than the convex-hull route. In *Conjecture 2* (refer to Section 3.1.1), it was conjectured that all convex obtuse polygons have convex-hull routes as their shortest external watchman route. Convex obtuse polygons are those that have no acute interior angles.

This conjecture is strongly supported by the experimental evidence. Almost all the random convex polygons produced, with larger values of n, were convex obtuse polygons.

Thus these polygons had no acute angles and it was seen that non-acute angles did not tend to produce 2-leg routes shorter than the convex-hull route in the experimental data. This can also be seen in the data displayed in Table 5–2. The column giving the number of experiments where the convex-hull route was the shortest of all external routes shows that this number gets higher as the polygons tend to become more and more convex obtuse. And the shortest convex-hull route turned out to be the best route, shorter than all 2-leg routes, only when the smallest angle was greater than 90°.

Thus, according to the experimental evidence, it can be claimed that *Conjecture 2* tends to be true for randomly generated convex polygons: the convex-hull route is likely to be the optimal external inspection route for convex polygons with no acute interior angles.

5.3 Limited Visibility Experimental Work and Results

We performed the experiments described in the previous section using the limited visibility model on random convex polygons. Thus in this section, we used an additional parameter, the visibility range d, to observe how the length of the 2-leg route for each wedge changed with the value of d. The procedure described in Section 4.1 is used to do this.

In Section 4.1, it was mentioned that for the 2-leg route for each wedge under the unlimited visibility model, there was a maximum distance l that the watchman has to see from some point along the route (typically from one of the extreme points of the route), to be able to see the entire exterior of the polygon. As long as the visibility range d is greater or equal to this maximum distance l, the shortest route corresponding to that particular wedge would be its shortest 2-leg route (if the route was shorter than the convex-hull route). Therefore, starting from this value of $d = l$, and by entering smaller values of d at every step, it was possible to observe the increasing length of the 2-leg route of that wedge, until at a particular input value of d, the route length becomes equal to or greater than the convex-hull route length. In this way, it was possible to find the critical value of d, or d^* as previously defined in Chapter 4, where the 2-leg route is no longer shorter than the convex-hull route for visibility range capacities lower than this.

An example of this concept is illustrated for a basic quadrilateral that was generated by the program, and shown in Figure 5–2. The curve plotted in Figure 5–3 is for the route corresponding to the smallest wedge of the quadrilateral, which is 36°. The value of l (the maximum distance required to be seen from a point along the route) is computed to be 602.7 unit distance measures, and the critical value d^* is found to be 373.1 units, giving rise to the shape of the curve seen in Figure 5–3.

In Section 4.1 it was mentioned that this curve, or the graph of function $f(d)$, can be plotted for each wedge of a polygon P, to see if it is possible to get any intersection points or crossovers between the different curves for each wedge. If such crossover regions were found, it would mean that the wedge giving the optimal 2-leg route in the unlimited visibility model does not necessarily provide the optimal solution in the limited visibility model for certain values of d. This was previously illustrated in Figure 4–2(b).

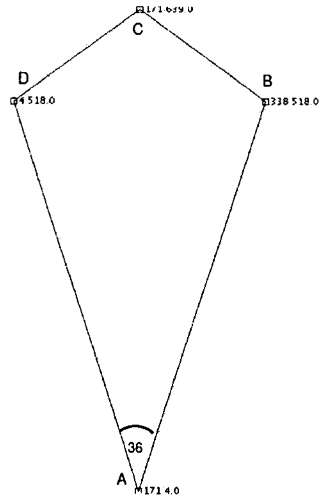


Figure 5–2: A convex quadrilateral

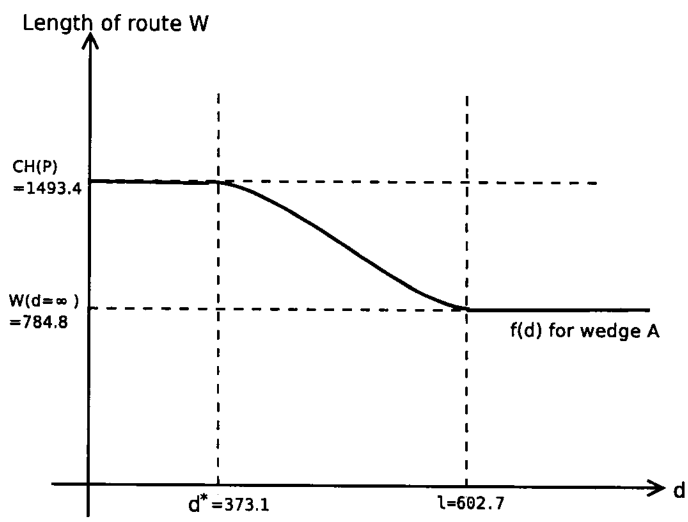


Figure 5–3: Function $f(d)$ for the convex quadrilateral

However, in Section 5.2, it was found that for most random convex polygons, the 2-leg route for a wedge is shorter than the convex-hull route when the interior angle of that wedge is less than 90°. Thus, in other words, the curves of function $f(d)$ need to be plotted only for the acute angle wedges of a polygon, in most cases. For wedges with interior angle greater than a right-angle, the 2-leg route length was usually found to be greater than the convex-hull route length, as can be easily seen in the experimental data given in Table 5–1 of Section 5.2. Hence the curve of function $f(d)$ for the non-acute wedges of the polygon would be the straight line corresponding to the route length $W = CH(P)$, where $CH(P)$ is the convex-hull length of the polygon P. This is because the convex-hull route would be the shortest route for all values of d for that wedge. Here W is used to denote the whole route, including both the inner and return path of a 2-leg route.

Now if we concentrate mainly on the function of the visibility range d for only the acute interior angles of a convex polygon, it is useful to note that the maximum number of acute angles possible for a convex n-gon is 3, for any $n \geq 3$. This can be easily shown using the following argument:

> It is a known mathematical fact that the sum of the exterior angles for any polygon of n vertices with $n \geq 3$ is always equal to 360°. For any acute interior angle x of a polygon, $x < 90°$, and hence the corresponding exterior angle $y = (180 - x) > 90°$. Thus, if there are more than three acute interior angles in a convex polygon, the sum of the exterior angles would be $> 360°$ (for four exterior angles greater than 90°), which is not possible. Therefore, any convex polygon may have at most three acute interior angles.

The experiments with different values of d for acute wedges of randomly generated polygons were performed for polygons of varying sizes. One sample convex polygon of n vertices with $n = 5$ is shown in Figure 5–4. Table 5–3 displays some of the experimental data for this polygon and each of its wedges, and Figure 5–5 shows the curves of the function for the wedges A and B, where $A = 45°$ and $B = 88°$, for different values of d. The other wedges with non-acute angles give 2-leg routes longer than the convex-hull route, as can be seen in Table 5–3, and so the convex-hull route is the shortest route for these wedges at all values of d. This is illustrated by the straight line $W = CH(P)$ in Figure 5–5.

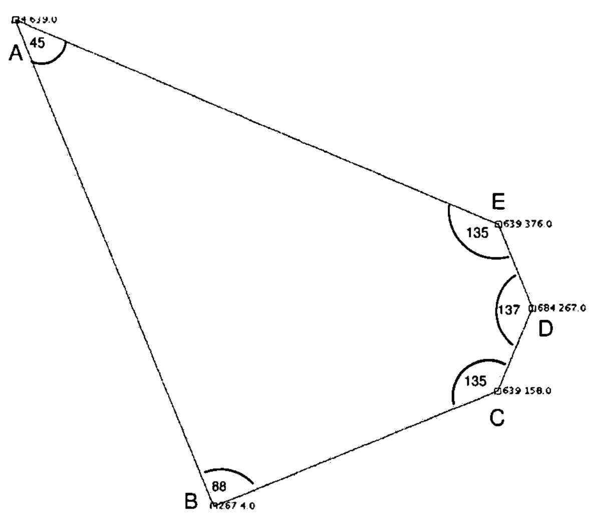


Figure 5–4: A convex polygon with n=5 vertices

Table 5–3: Route lengths for each wedge of polygon in Figure 5–4 if $d = \infty$

Wedge	Angle size	2-leg route length	CH-route length
A	45°	1208.05	
B	88°	1777.46	
C	135°	2346.22	2013.08
D	137°	2615.62	
E	135°	2348.96	

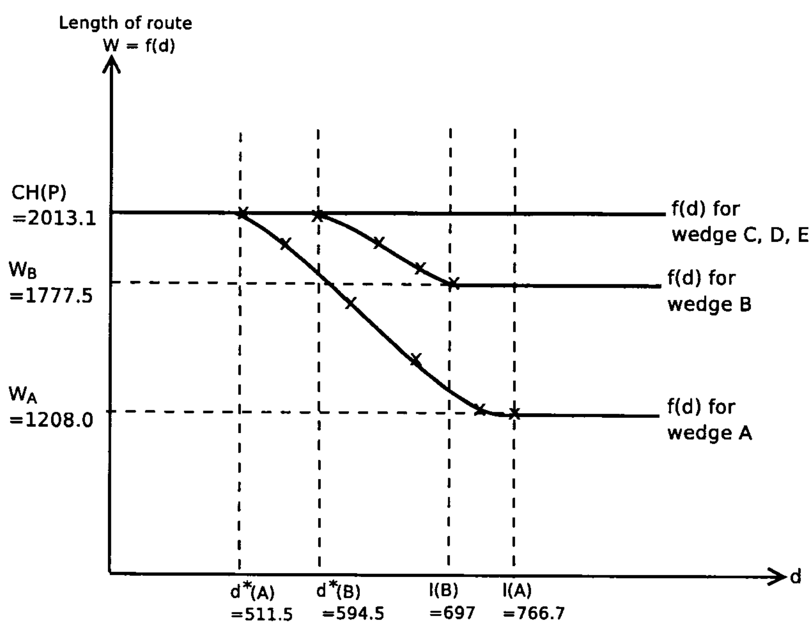


Figure 5–5: The route length as a function of the visibility range d for each wedge of the polygon in Figure 5–4

It can be seen from Figure 5–5 that there are no crossover points between the two curves for the wedges A and B. Suppose we are given a watchman or robot with a maximum visibility range of d, that has to externally patrol the polygon in Figure 5–4. If we take that value of d on the x-axis or the horizontal axis, and then move vertically up till we hit a point on one of the curves, that would give us the length of the shortest route for that polygon corresponding to that specific visibility range. The curve we hit also indicates the wedge which would provide the the optimal 2-leg route for that visibility range d, unless we hit the convex-hull route line, in which case the convex-hull route would be the shortest route.

As can be seen in Figure 5–5, in this case the wedge A, which is the narrowest wedge for the polygon, gives the optimal route for all values of $d \geq d^*(A)$ (the critical value of d for wedge A). Below this value of d, the convex-hull route provides the optimal path.

In all the runs of the experiment carried out on the limited visibility model, using different generated convex polygons of varying sizes, we found no crossover points as was illustrated in Figure 4–2(b). Most of the graphs were found to be similar to Figure 5–5. Hence, we can claim that it is most likely that the wedge that provides the shortest 2-leg route in the unlimited visibility model also provides the shortest inspection route in the limited visibility model, for all $d \geq d^*$.

CHAPTER 6
Discussions and Conclusions

6.1 Overview

In this thesis, we have studied the problem of inspecting the exterior of a polygonal region using a moving guard or robot. The main goal of the problem is to achieve the shortest route for the guard along which the entire external boundary of the polygon is visible. We have studied two different models for this problem: the unlimited visibility model, in which the guard is assumed to have an infinite visibility range, and the limited visibility model, in which the guard can see only as far as a fixed maximum viewing distance. Both models assume a panoramic field of view and a priori information on the polygon's shape.

Under the unlimited visibility model, we studied previous work in this area and elaborated on and extended the solution. We conjectured that the shortest inspection route for a convex polygon could be achieved using the wedge between the pair of adjacent edges that has the smallest interior angle. We were able to construct counterexamples to this theory. However our experimental work on convex polygons supports the conjecture, as long as the smallest angle is acute. We also conjectured that if all interior angles of a convex polygon are non-acute, then the shortest route is to simply follow the boundary of the polygon. This theory has also been strongly supported by the experimental evidence. We also briefly described the external inspection problem for simple polygons under unlimited visibility.

Under the limited visibility model, we studied how the length of the optimal route, computed under unlimited visibility, changes with the incorporation of a finite visibility range. We conjectured that the polygon wedge providing the optimal solution in the unlimited visibility model also provides the best solution in the limited visibility model. This theory is supported by the experimental work performed on random convex under a varying visibility range. We also studied the external inspection of a simple polygon under this model, and briefly described an approximate solution to the problem.

6.2 Conclusions and Future Work

The shortest external inspection route for a convex polygon may either be the route that follows the boundary of the polygon, or it may be a route that spans across the extensions of two adjacent edges of the polygon, as was previously described. We were able to disprove the conjecture that the pair of adjacent edges that have the narrowest acute interior angle in the polygon provides the shortest inspection of the latter type by the construction of a counterexample. However, our experimental results on random convex polygons show that this conjecture is often true, since the counterexample arises only under special circumstances. As such, this would suggest that given a convex polygon, a good solution may be found by simply computing the shortest 2-leg route for the narrowest wedge, as long as it is an acute interior angle, rather than computing all routes and comparing to determine the shortest. This may be useful if a short route must be found quickly for each of several convex polygonal structures. It may also be the best way to go in cases where adequate information of the polygon is not available to determine the optimal external route.

We do note, however, that no proper definition of random polygons is currently available. And the structure of a building or establishment that might require external inspection is not randomly constructed. Hence, the special case where the conjecture does not hold may arise in such structures, and although choosing the 2-leg route of the smallest wedge might give a reasonably good solution, it may in such cases not be the best solution.

We observed that the margin of values of the interior angles for which the counterexample for convex quadrilaterals comes up is very narrow. It was created by taking two equal interior angles opposite each other and rotating one wedge through an angle ϕ (see Section 3.1.2). Only within a certain range of values of ϕ can one of the wedges be reduced by a limited amount to create a quadrilateral where the narrowest wedge does not give the shortest 2-leg route. For future work, it may be interesting to find the exact range of ϕ and the range of values by which the wedge angle can be reduced for it to remain a counterexample. This would give a better idea of exactly when the conjecture can be expected not to stand.

One of the drawbacks of the random convex polygon generator used in the experiments was that the angles were not very diverse after a number of iterations, especially for polygons with very large number of sides. If the randomness of the interior angles could be implemented more effectively during generation, this may provide a set giving more diverse results.

Finally, we conclude with a discussion of a few more possible problems for future research related to this topic that we find especially interesting.

- *Different polygon classes:* This thesis concentrated to a considerable degree on convex polygons and hence the results are significant for structures such as buildings, the boundaries of establishments in urban or rural settings, and even ship hulls at a given depth, which are all convex in nature most of the time. However the simple polygon case could be further studied for ways of improving the approximation algorithm as well as its computational complexity. Optimal external routes for special cases such as orthogonal polygons can also be a subject for future research, since orthogonal structures are one of the most common in everyday life.

- *External inspection of multiple polygons:* The inspection route problem considered here has been for the exterior of a single polygon. The problem of externally inspecting more than one convex, simple or orthogonal polygon under limited visibility is an open problem and may be an interesting future research direction.
- *Multi-robot inspection:* The basic single-polygon external inspection problem using a single mobile robot with limited visibility can also be extended to consider multiple mobile robots. If each robot is equipped with sensors having different visibility ranges, it would be interesting to study the computation of an optimal way of utilizing these robots, to cover sub-routes, that guard the entire external boundary.
- *3D external inspection:* A good idea for future work would also be extending the problem from planar external inspection to 3-dimensional inspection of workspaces. This would especially be useful where ground-level inspection is not sufficient. An example would be inspections in space or underwater where it is clear 3D-inspection capability would be essential.

Appendix A

In Chapter 3 (Section 3.1.2), a function $f(\phi)$ was presented, and it was claimed that the function has a relative maximum. The proof of this, using the second derivative test, is given here.

Proof:

The function $f(\phi)$ has been previously defined as

$$\begin{aligned} f(\phi) &= \cos\phi(\sin\theta_1 + \sin\theta_2) + \sin\phi(\cos\theta_1 - \cos\theta_2) - (\sin\theta_1 + \sin\theta_2) \\ &= x\cos\phi + y\sin\phi - x \end{aligned}$$

where $x = (\sin\theta_1 + \sin\theta_2)$ and $y = (\cos\theta_1 - \cos\theta_2)$, and x and y are constant values as the angles θ_1 and θ_2 are fixed.

The first derivative: $f'(\phi) = -x\sin\phi + y\cos\phi$

The second derivative: $f''(\phi) = -x\cos\phi - y\sin\phi$

Now, if $f'(\phi) = 0$, then $\sin\phi/\cos\phi = y/x$ and $\phi = \tan^{-1}(y/x)$.

Therefore,

$$\begin{aligned} f''(\phi) &= -x\cos(\tan^{-1}(y/x)) - y\sin(\tan^{-1}(y/x)) \\ &= -x(1/\sqrt{1+(y/x)^2}) - y((y/x)/(1/\sqrt{1+(y/x)^2})) \\ &= (-x/\sqrt{1+y^2/x^2}) - (y^2/(x(1/\sqrt{1+y^2/x^2})) \\ &= (-x^2/\sqrt{x^2+y^2}) - (y^2/\sqrt{x^2+y^2}) \end{aligned}$$

$$
\begin{aligned}
&= (-x^2 - y^2)/\sqrt{x^2 + y^2}) \\
&= -\sqrt{x^2 + y^2}).
\end{aligned}
$$

This value of $f"(\phi)$ is always less than 0 for all values of x and y. When $f'(\phi) = 0$ and $f"(\phi) < 0$, then $f(\phi)$ is a function with a relative maximum at $\phi = \tan^{-1}(y/x)$, according to the second derivative test.

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Glossary
of Some Key Terms

Art Gallery Problem: The problem of finding the smallest number of guards necessary to cover a polygon with n vertices.

Cave: Any region between the boundary of a polygon P and the convex hull of P, such that there is at least one point in it that is not visible from some point on the convex hull of P.

Convex Hull: The convex hull for an object or a set of objects is the minimal convex set containing the given objects.

Convex Obtuse Polygon: A convex polygon with no acute angles.

Convex Polygon: A polygon such that if u and v are any two points inside or on the boundary of the polygon then the entire line segment uv lies inside or on the boundary of the polygon.

Delaunay Triangulation: The Delaunay triangulation for a set P of points in the plane is the triangulation $DT(P)$ of P such that no point in P is inside the circumcircle of any triangle in $DT(P)$.

Diagonal Guard: A guard that can patrol on the straight line connecting non-adjacent vertices of a polygon. This is sometimes also referred to as a mobile guard.

Edge Extension: The infinite line on which the edge of a polygon lies.

Edge Guard: A guard that may patrol one edge of a polygon.

Exterior Visibility: Two points in the exterior of an arrangement are considered visible if the line segment between them contains only points in the exterior of the arrangement.

Fixed Watchman Route: The watchman route that is forced though a specified starting point.

Floating Watchman Route: The watchman route that has no starting point specified.

Floodlight Illumination Problem: The floodlight illumination problem asks whether there exists a one-to-one placement of n floodlights illuminating infinite wedges of angles $\alpha_1, ..., \alpha_n$ at n locations $p_1, ..., p_n$ in a plane such that a given infinite wedge W of angle T located at point q is completely illuminated by the floodlights.

Kernel: The set of all points in a polygon that can see every point in the polygon.

Monotone Polygon: A polygon is monotone with respect to a line if it can be split into two polygonal chains such that each chain is monotone with respect to the line.

Orthogonal Polygon: A polygon in which each pair of adjacent edges meets orthogonally.

Point Guard: A guard that can stand at any given point of a polygon.

Polygon Cover: A collection of subsets of a polygon P if the union of these subsets is exactly P.

Polygon Decomposition: The problem of breaking up polygons into simpler pieces (for example, refer to *Triangulation*).

Polygon: An ordered sequence of at least three points, v_1, v_2, ..., v_n, in the plane, called vertices, and the n line segments v_1v_2, v_2v_3, ..., $v_{n-1}v_n$, v_nv_1, called edges.

Robot Exploration Problem: The problem in which a robot has to construct a complete map of an unknown environment using a path that is as short as possible.

Robot Localization Problem: The problem for a robot determining its location in an environment.

Robot Navigation Problem: Given a robot and specified initial and final locations, the task of finding a collision-free path with certain optimization criteria in a real environment, which could vary with time or be only partially known.

Safari Route: The watchman route that has to touch or enter a number of subpolygons inside an enclosing polygon.

Simple Polygon: A polygon with the constraint that non-consecutive edges do not intersect.

Spiral Polygon: A simple polygon of which the boundary can be subdivided into a chain of reflex vertices, and a chain of convex vertices.

Star-shaped Polygon: A polygon that contains a kernel from which all points in the polygon can be seen (refer to *kernel*).

Traveling Salesman Problem (TSP): The problem of finding the shortest closed path that visits every point in a given set.

Triangulation Graph: A graph that represents a triangulated polygon. Each node of the graph corresponds to a specific vertex of the polygon. Each edge of the graph corresponds to a specific edge of a triangle in the triangulated polygon. (Also refer to *Triangulation*).

Triangulation: A triangulation of a polygon is a decomposition or partitioning of the polygon into a set of triangles without adding vertices, done by chopping the polygon with diagonals between non-adjacent vertices.

Vertex Guard: A guard that can be stationed only at the vertices of a polygon.

Visibility Graph: A graph of inter-visible locations. Each node or vertex in the graph represents a point location, and each edge represents a visible connection between them (that is, if two locations can see each other, an edge is drawn between them).

Watchman Route Problem: The problem of finding the shortest route from which every point in a polygon can be seen.

Weakly Externally Visible Polygon: A polygon P such that each point x on the boundary of P is visible from a circle at infinity.

d-visible: When visibility is limited within a maximum range of distance d.