M. A. Economics

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Introduction to the Generalized Theory of Multi-System Macroeconomics

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PREFACE

This thesis is largely original. The first chapter was inspired by a desire to improve upon Ragnar Frisch's equation for the principle of acceleration. This inspiration was found in a paragraph in Bretherton, Burchardt & Rutherford's Public Investment and the Trade Cycle, on pp. 319-20. Prof. Tinbergen's article, An Acceleration Principle for Stocks, also influenced Chapter 1, although the substance of the chapter had been conceived before I read it.

The formulation of the multiplier in Chapter 2 (Equation 9) was inspired by Dr. Lange's article, The Theory of the Multiplier. The diagrammatic analysis by which I arrive at the same formulation as Dr. Lange's is, however, original.

Chapter 3 and the first three parts of Chapter 4 are considerably influenced by Prof. Machlup's International Trade and the National Income Multiplier; however the diagrammatic analysis, the incorporation of the investment and export functions in the multiplier, and the development of the excess-saving and trade-balance functions, are entirely original.

The remainder of the thesis - parts IV, V and VI of Chapter 4, and Chapters 5, 6 and 7 - is entirely original.

REMARKS

In accordance with Regulation 60(d) I suggest the title, "Generalized Theory of Multi-System Macroeconomics".

Since this thesis is original, and not a piece of research, I have included a preface, according to Regulation 60(f), although it is an M.A. and not a Ph.D. thesis, outlining my claims to originality. This Preface may or may not be included in the thesis according to the wishes of the Faculty.

With regard to Regulation 60(g), I have not mentioned in the Preface, nor in the text, the extent to which assistance has been received from "members of the staff, fellow-students, technicians and others", as no such assistance has been received.

1. The Generalized Principle of Acceleration

I

The acceleration principle is as yet an unprecise concept, unsuitable for rigorous analysis. The first cause of weakness is the neglect of the rôle of inventories, and a tendency to take account of only one kind of investment, namely investment in fixed capital plant and equipment. This leads to a fundamental weakness in the acceleration principle, owing to the fact that investment in inventories may be affected by consumption in an opposite way to the effect of consumption on investment in fixed capital. Increases in consumption deplete inventories, thus causing a certain amount of disinvestment.

Secondly, this weakness could be disregarded but for the fact that there is a certain positive period of production. If production were instantaneous, inventory depletions would be completely offset by induced inventory investment. But since there is a time lag between consumption and production, the depletion or apletion of inventories caused by an increase or decrease in the rate of consumption will only be made up if consumption changes at a constant rate. Thus the neglect of the period of production is the second and fundamental weakness of the acceleration principle as it has been conceived up to the present.

The unrealistic assumptions on which the formulations of the accelerator have been based have necessitated the qualifications that the principle is not wholly applicable at the turning-points of the business cycle, and have led economists to remark that the acceleration principle does not operate in periods of excess inventories, such as the trough of the business cycle.

In this paper I will attempt to formulate a statement of the acceleration principle which is applicable in all phases of the business cycle and is fully generalized.

II

Output can be considered as being composed of changes in inventories, and of consumption. This can be written:-

$$O = \Delta V + C$$

Inventories are held because of irregularities in consumption, and discontinuities between production and consumption. It will be assumed that entrepreneurs will wish to hold inventories at a constant proportion to consumption. Thus,

$$V = vC$$

A change in consumption will cause a change in inventories in the same direction, in the next period. The period of production, measured in units of time, we shall call θ.

It will be further assumed that entrepreneurs produce on the basis of present consumption, so that an increase in consumption of $C_1 - C_0$ will cause an increase in production of $C_1 - C_0$ after θ time units have elapsed.

The immediate effect of an increase in consumption from C_0 to C_1 will be the depletion of inventories to the amount of $C_1 - C_0$. A change in inventories at a given point of time is then composed of (1) a proportion v of the change in consumption in the previous period, (2) the change in consumption itself in the previous period, and (3) the change in inventories caused by a change in consumption in the present period. This can be written:-

$$V_t - V_{t-\theta} = v(C_{t-\theta} - C_{t-2\theta}) + (C_{t-\theta} - C_{t-2\theta}) - (C_t - C_{t-\theta}) \quad \ldots\ldots\ldots(1)$$

If the rate of consumption is constant, or changing at a constant rate, then $(C_{t-\theta} - C_{t-2\theta}) = (C_t - C_{t-\theta})$ and $\Delta V = v\Delta C$. Similarly, if consumption is changing at a declining rate (i.e., if $d^2C/dt^2 < 0$) then $\Delta V > v\Delta C$, as $(C_{t-\theta} - C_{t-2\theta}) - (C_t - C_{t-\theta}) > 0$. Likewise, if consumption is changing at an expanding rate (if $d^2C/dt^2 > 0$), then $\Delta V < v\Delta C$. Thus an accelerated increase in consumption will cause a decline in investment in inventories, and vice versa. This will tend to cushion the top of the downswing and also to dampen the recovery.

Equation (1) can be written (the subscript t referring to the period t-θ to t):

$$V'_t = (v+1)C'_{t-\theta} - C'_t \qquad \text{............(2)}$$

Output therefore consists of this expression plus C_t. The rate of change of output is written

$$O'_t = C'_t + (v+1)C''_{t-\theta} - C''_t \qquad \text{............(3)}$$

III

We must now consider investment in fixed capital. Under given technological conditions, assuming fixed technical coefficients, entrepreneurs will wish to produce capital (Q) in a fixed proportion (q) to output (O). Thus, after a time-lag (τ),

$$Q_t = qO_{t-\tau} \qquad \text{............(4)}$$

Investment in fixed capital will consist of replacement (R) and net investment (dQ/dt). Replacement is a function of the durability of capital. If capital instruments last γ years, and if replacement is not bunched, then $\frac{1}{\gamma}$ of these assets will be replaced annually. The ratio of replacement to capital we shall call r. Over a period of time τ,

$$r = \frac{\tau}{\gamma},$$

where γ is the durability of capital, and both γ and τ are measured in years.

We might express the replacement-capital ratio in period τ as being equal to τ times the annual replacement-capital ratio. Thus replacement is expressed as a function of capital:

$$R_t = rQ_t \qquad \text{...............(5)}$$

Substituting (4),

$$R_t = rqO_{t-\tau} .$$

Similarly, using the prime notation to express the derivative with respect to time,

$$Q'_t = qO'_{t-\tau}.$$

Gross investment in fixed capital (F) is therefore expressed as follows:

$$F = rqO_{t-\tau} + qO'_{t-\tau} .$$

When investment in inventories is added to this, we obtain the following equation for gross investment (I):

$$I_t = rqO_{t-\tau} + qO'_{t-\tau} + V'_t .$$

Substituting (2) and (3), this becomes:

$$I_t = rq\left\{C_{t-\tau} + (v+1)C'_{t-\tau-\theta} - C'_{t-\tau}\right\} + q\left\{C'_{t-\tau} + (v+1)C''_{t-\tau-\theta} - C''_{t-\tau}\right\}$$
$$+ (v+1)C'_{t-\theta} - C'_t .$$

This becomes:

$$I_t = rqC_{t-\tau} + \left\{(v+1)C'_{t-\theta} - C'_t\right\} + q\left\{r(v+1)C'_{t-\tau-\theta} - (r-1)C'_{t-\tau}\right\}$$
$$+ q\left\{(v+1)C''_{t-\tau-\theta} - C''_{t-\tau}\right\} \qquad \text{................(6)}$$

For this, let us write:-

$$I = A + \lambda_1 + \lambda_2 + \lambda_3 .$$

We will proceed to examine λ_1, λ_2, and λ_3 separately.

IV

<u>Conditions for λ_1</u>,

$$\lambda_1 = (v+1)C'_{t-\theta} - C'_t$$

$$\therefore \quad C'_t - C'_{t-\theta} = vC'_{t-\theta} - \lambda_1 .$$

When $C'_{t-\tau} - C'_{t-\theta} = 0, \quad \lambda_1 = vC'_{t-\theta}$

$$> 0, \quad < vC'_{t-\theta}$$

$$< 0, \quad > vC'_{t-\theta}.$$

Further, $\lambda_1 = 0$ when $C'_t - C'_{t-\theta} = vC'_{t-\theta}$

$$> 0 \quad < vC'_{t-\theta}$$

$$< 0 \quad > vC'_{t-\theta}.$$

Thus, as long as the increment in the rate of change of consumption is less than the planned change in inventories, $\partial\lambda_1/\partial C > 0$. However, as long as this increment is greater than the planned change in inventories, $\partial\lambda_1/\partial C < 0$.

Conditions for λ_2.

$$\lambda_2 = q\left\{r(v+1)C'_{t-\tau-\theta} - (r-1)C'_{t-\tau}\right\}.$$

It follows that

$$r(v+1)C'_{t-\tau-\theta} - (r-1)C'_{t-\tau} = \frac{\lambda_2}{q}$$

and

$$C'_{t-\tau} = \frac{r(v+1)}{r-1}C'_{t-\tau-\theta} - \frac{1}{q(r-1)}\lambda_2$$

$$\therefore C'_{t-\tau} - C'_{t-\tau-\theta} = \frac{r(v+1) - (r-1)}{r-1}C'_{t-\tau-\theta} - \frac{1}{q(r-1)}\lambda_2.$$

Thus,

$$C'_{t-\tau} - C'_{t-\tau-\theta} = \frac{1}{q(r-1)}\left\{q(rv+1)C'_{t-\tau-\theta} - \lambda_2\right\}$$

Now, with finite changes in the rate of change of consumption,

When $C'_{t-\tau} - C'_{t-\tau-\theta} = 0, \quad \lambda_2 = q(rv+1)C'_{t-\tau-\theta}$;

When $C'_{t-\tau} - C'_{t-\tau-\theta} > 0, \quad \lambda_2 > q(rv+1)C'_{t-\tau-\theta}$ if $r<1$,

$$= q(v+1)C'_{t-\tau-\theta} \text{ if } r=1,$$

$$< q(rv+1)C'_{t-\tau-\theta} \text{ if } r>1;$$

When $C'_{t-\tau} - C'_{t-\tau-\theta} < 0, \quad < q(rv+1)C'_{t-\tau-\theta}$ if $r<1$,

$$= q(v+1)C'_{t-\tau-\theta} \text{ if } r=1,$$

$$> q(rv+1)C'_{t-\tau-\theta} \text{ if } r>1.$$

Similarly,

$$\lambda_2 = 0 \quad \text{when } C'_{t-\tau} - C'_{t-\tau-\theta} = \frac{rv+1}{r-1} C'_{t-\tau-\theta} \;;$$

$$> 0 \qquad \lessgtr \frac{rv+1}{r-1} C'_{t-\tau-\theta}, \text{ with } r \gtrless 1;$$

and

$$< 0 \qquad \gtrless \frac{rv+1}{r-1} C'_{t-\tau-\theta}, \text{ with } r \gtrless 1.$$

Since $r = \frac{T}{\gamma}$ and the durability of capital (γ) in a highly developed community tends to be large and is likely to be much greater than the period of production, we can assume $r<1$ in an advanced community. In such a community, $\partial\lambda_2/\partial C < 0$ if, τ time-units ago, the increment in the rate of change of consumption over θ time-units was less than $\frac{rv+1}{r-1} C'_{t-\tau-\theta}$. However, if the increment was more than this quantity then $\partial\lambda_2/\partial C > 0$. This is the opposite sort of relation to that which holds with λ_1. We can therefore expect that λ_1 and λ_2 will offset each other to a certain extent, if the rate of acceleration or deceleration of consumption τ time-units ago (multiplied by the coefficient $\frac{rv+1}{r-1}$) is equivalent to the present rate of acceleration or deceleration (multiplied by the coefficient v).

In a primitive community r may equal or exceed unity. In the latter case, $\partial\lambda_2/\partial C$ may be negative.

<u>Conditions for λ_3.</u>

$$\lambda_3 = q\left\{(v+1)C''_{t-\tau-\theta} - C''_{t-\tau}\right\}.$$

$$\therefore C''_{t-\tau} - C''_{t-\tau-\theta} = vC''_{t-\tau-\theta} - \frac{1}{q}\lambda_3$$

When $C''_{t-\tau} - C''_{t-\tau-\theta} = 0, \quad \lambda_3 = qvC''_{t-\tau-\theta}$,

$$> 0, \qquad < qvC''_{t-\tau-\theta},$$

$$< 0, \qquad > qvC''_{t-\tau-\theta}.$$

And $\lambda_3 = 0$ if $C''_{t-\tau} - C''_{t-\tau-\theta} = vC''_{t-\tau-\theta}$;

$$> 0 \qquad < vC''_{t-\tau-\theta};$$

$$< 0 \qquad > vC''_{t-\tau-\theta}.$$

If there is no change in the acceleration of consumption, then λ_3 is positive. If the acceleration of consumption τ time-units ago was greater than the planned acceleration of inventory investment at that time, then λ_3 will be negative. If the acceleration of consumption was smaller, λ_3 will again be positive. If fluctuations in the acceleration of consumption were within the planned changes in inventory acceleration, $\partial\lambda_3/\partial C > 0$; otherwise $\partial\lambda_3/\partial C < 0$.

V

If the velocity of consumption over the last $\tau+\theta$ time-units is constant, then equation (6) becomes

$$I_t = q(rC_t + C'_t) + v(rq+1)C'_t \qquad \ldots\ldots\ldots\ldots\ldots(7)$$

If $v = 0$, then this reduces to Ragnar Frisch's equation.[1] The velocity of investment then becomes a positive function of both the velocity and acceleration of consumption. However, this is only a special case. There are many circumstances in which this will not be so.

Case 1.

In certain cases, λ_1, λ_2 and λ_3 will all be negative with rising consumption. $\lambda_1 < 0$ if $C'_t - C'_{t-\theta} > vC'_{t-\theta}$. $\lambda_2 < 0$ if (in an advanced community) $C'_{t-\tau} - C'_{t-\tau-\theta} < \frac{rv+1}{r-1} C'_{t-\tau-\theta}$. $\lambda_3 < 0$ if $C''_{t-\tau} - C''_{t-\tau-\theta} > vC''_{t-\tau-\theta}$.

If, τ units ago, the velocity of consumption was steadily falling, then $\lambda_2 < 0$. If it was steadily falling at a slackened pace, then $\lambda_3 < 0$ also. If it is now rising rapidly, $\lambda_1 < 0$. The level of consumption in this case has reached a minimum point, and it is likely that A will be low, especially in an economy with very durable capital. We may thus expect that $\lambda_1 + \lambda_2 + \lambda_3$ will offset A. $\partial I/\partial C$ in this situation will therefore be negative.

1. "The Interrelation Between Capital Production and Consumer Taking", Journal of Political Economy, 1931.

Since this situation characterizes the trough of the business cycle, it is clear that a mild public works program, or private investment, carried on at that stage of the cycle would be ineffective, and that government consumption expenditure might, for the moment, only deepen the depression. λ_2 may also be negative if, τ units ago, consumption was falling and then rising (over a period, of course, of θ). $\lambda_2 < 0$ as long as the net increase in velocity did not exceed $\frac{rv+1}{r-1} C'_{t-\tau-\theta}$.

Case 2.

Likewise, if consumption was rising at a slackened pace, or rising and then falling slightly, τ units ago, and if consumption is now falling more rapidly than the planned reduction in inventory accumulation, $\lambda_1 + \lambda_2 + \lambda_3$ will be positive. A will also be positive. Therefore $\partial I/\partial C$, at the peak of the business cycle, will be negative.

That/is not apparently in accordance with experience is obviously due to the other factors which influence investment. Investment is a function of the marginal efficiency of capital, and the rate of interest. If the downswing of consumption raises the marginal efficiency of capital, pessimistic expectations of the future will lower it. As Keynes pointed out,[2] the sudden fall in the marginal efficiency of capital will be followed by a sudden rise in liquidity preference, which raises the interest rate, and further lowers investment. Thus the unexpected inventory accumulation will only cushion the downswing. Likewise, a determined and large-scale public works program in a depression would permit recovery to set in.

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2. "The General Theory of Employment, Interest and Money", p. 316.

Case 3.

If consumption is rising rapidly, λ_1 and λ_3 are negative. This will be offset by λ_2 if consumption τ units ago was rising just as rapidly. λ_2 will also be negative, as we saw, if consumption $\tau+\theta$ units ago was falling. Somewhere in between, λ_2 and $\lambda_1+\lambda_3$ may just offset each other. This will occur when consumption is rising a certain amount more rapidly than it was τ units ago. The phase of the business cycle at which $A+\lambda_2 = \lambda_1+\lambda_3$ we shall call the inflexion-point.[3] At this phase, $\partial I/\partial C$ (usually called "the Relation") will be equal to zero. There will be four such inflexion points, between the peak and downswing, downswing and trough, trough and upswing and peak. Thus the relation will become successively positive and negative during the course of the business cycle.

In a primitive economy where replacement is very high, and where an increase in consumption may manifest itself as a decline in capital stock, the effect will be a considerable decline in investment, or disinvestment. Thus $\lambda_2 < 0$ if $C'_{t-\tau} - C'_{t-\tau-\theta} > \frac{rv+1}{r-1} C'_{t-\tau-\theta}$. Both τ and θ will be very short compared to the periods of production in an advanced community. Whether it is likely that $\lambda_2 < 0$ depends on the values of r and v. It is possible that $\partial\lambda_2/\partial C < 0$ when consumption accelerates. Now if r is large A will be an important factor, which will be a stabilizer, and the acceleration of consumption will have to be large for $\lambda_1+\lambda_2+\lambda_3$ to exceed A. If $\partial I/\partial C < 0$, consumption cannot accelerate for long, for the induced reduction of investment will sooner or later curtail consumption.

3. This need not correspond with the mathematical point of inflexion. This phase is, of course, an inflexion point only when λ_1 and λ_3 are negative.

We have been assuming that entrepreneurs will produce on the basis of the present rate of change of consumption. The only reason the relation is negative or zero in certain phases of the business cycle is that entrepreneurs' expectations about consumption are not fulfilled. If there were perfect foresight, equation (7) would be fully general. However this does not conform to the real world. If there should be a long run steady acceleration of consumption, then expectations might adjust themselves and we might expect equation (7) to hold. If however expectations are based on the acceleration of consumption, throughout the cycle, then the tendency for the relation to be negative would be much more marked than we have described.

2. The Generalized Theory of the Multiplier

We shall now attempt to synthesize the Relation with the Principle of the Multiplier, in order to obtain a generalized theory of the multiplier.

The most correct synthesis would be one between the Relation and the Dynamic Multiplier. The time-period appropriate to the latter is the income-propagation period. The latter is no doubt quite different from the period of production for inventories, and certainly quite distinct from the period of production for fixed capital. Until the length of these three periods can be determined, little sense can be obtained from an integration of the relation and multiplier in terms of time-lags. We shall therefore pursue a different approach, and attempt a synthesis with the static multiplier.

We may regard the Keynesian static multiplier in the following way. Investment is independent of income (being a function of the marginal efficiency of capital and the rate of interest) and is therefore perfectly elastic with respect to income. The savings function on the contrary, is an increasing function of income.

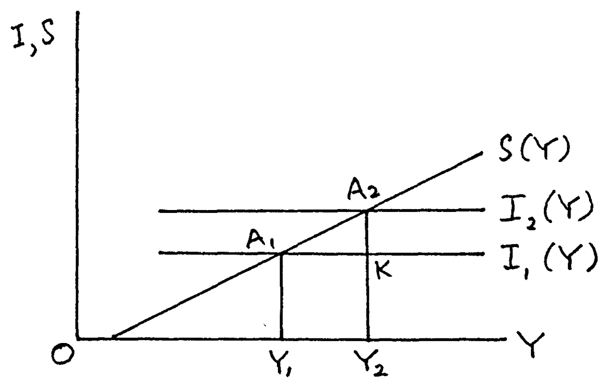

Fig. 1.

When investment increases from I_1 to I_2, income increases from Y_1 to Y_2.

The multiplier,

$$k = \frac{A_1 K}{A_2 K}$$

$$= \frac{1}{\frac{A_2 K}{A_1 K}}$$

$$= \frac{1}{dS/dY}$$

The multiplier, stated as the reciprocal of the marginal propensity to save, is static.[4] The word "multiplier" is, accordingly, misleading. It is quite analogous to the theory of the firm in perfect competition. Thus, as

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4. We can nevertheless distinguish between the short-run and long-run "static" multiplier.

We can assume that no income is spent on consumption instantaneously with the receipt of it. Thus the marginal propensity to consume in the very short-run is equal to zero. The short-run marginal propensity to save is therefore equal to one.

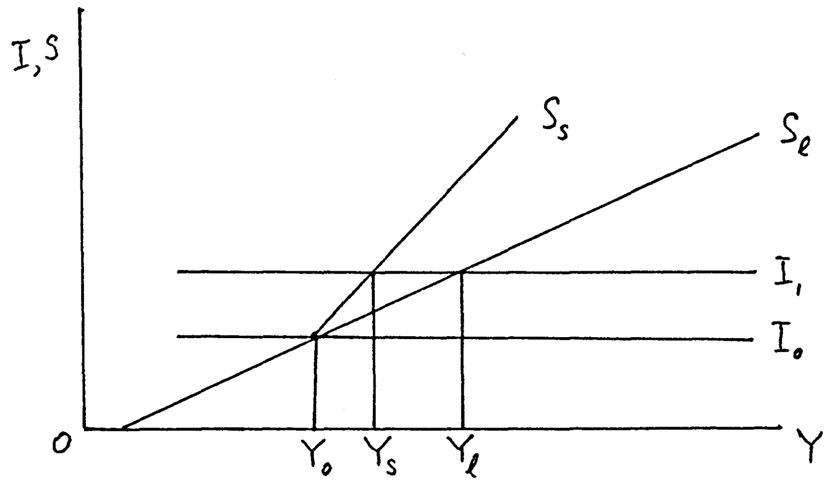

Fig. 2.

The short-run effect of an increase in investment from I_0 to I_1 is an equal increase in income of Y_0Y_s, determined by the short-run marginal propensity to save - the slope of S_s. In the long run, when the forces have fully worked themselves out (as Marshall would have said) income rises to OY_ℓ. The length of time it takes for income to reach this level is infinite, but it is practically reached in a finite period of time (a few years or less). For any finite period, the slope of the savings function is between that of S_s and S_ℓ.

In the very long run - to continue the Marshallian analogy - the effect of a permanent rise in the rate of investment will probably be to increase the marginal propensity to save, thus lowering the very-long-run multiplier.

shown in Fig. 3, an increase in

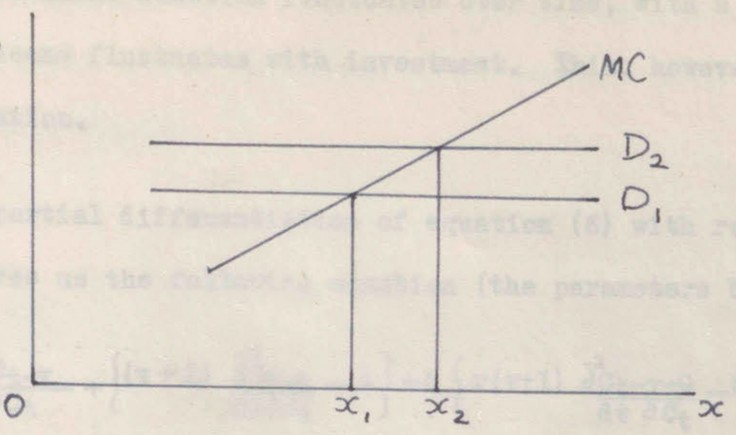

Fig. 3.

demand from D_1 to D_2 will increase output by x_1x_2. We might well say that the increase in output divided by the increase in price is a "multiplier", equal to "the reciprocal of the marginal propensity to cost".

It is assumed in the Keynesian multiplier that a rise in the savings function, though reducing income, would leave investment unchanged. It might lead, however, to a fall in investment, just as a rise in wages, though reducing the output, might raise demand.

A partial way of avoiding this difficulty is to show the movement of savings, investment and income in time.

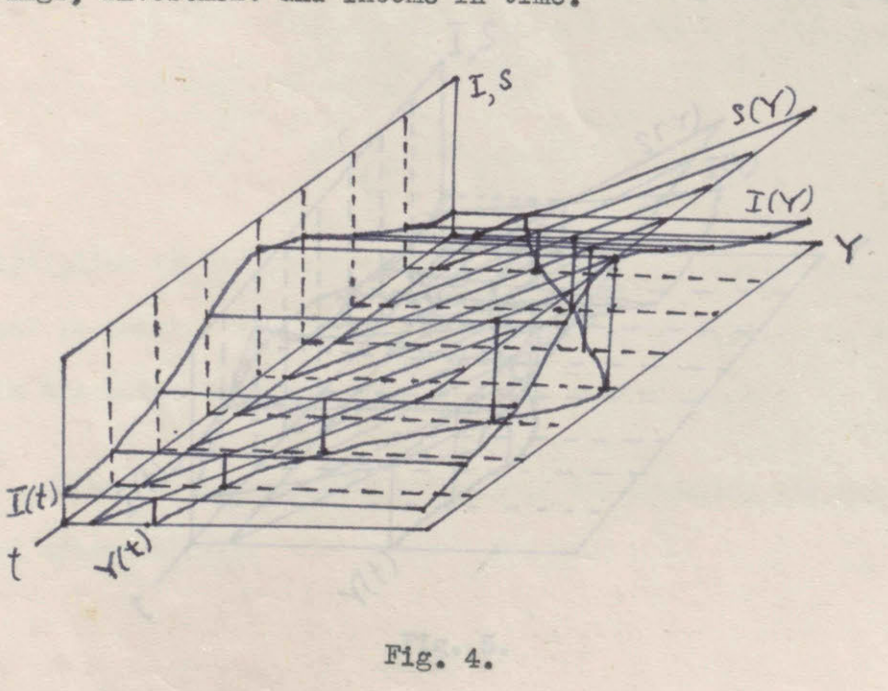

Fig. 4.

When the investment function fluctuates over time, with a constant savings function, income fluctuates with investment. This, however, is only a partial solution.

A partial differentiation of equation (6) with respect to consumption gives us the following equation (the parameters being kept constant):

$$\frac{\partial I_t}{\partial C_t} = rq\,\frac{\partial C_{t-\tau}}{\partial C_t} + \left\{(v+1)\,\frac{\partial^2 C_{t-\theta}}{dt\,\partial C_t} - 1\right\} + q\left\{r(v+1)\,\frac{\partial^2 C_{t-\tau-\theta}}{dt\,\partial C_t} - (r-1)\,\frac{\partial^2 C_{t-\tau}}{dt\,\partial C_t}\right\}$$
$$+ q\left\{(v+1)\,\frac{\partial^3 C_{t-\tau-\theta}}{d^2t\,\partial C_t} - \frac{\partial^3 C_{t-\tau}}{d^2t\,\partial C_t}\right\} \quad \text{.............(8)}$$

This equatioh is the Relation. We have already described this function as changing its shape over the business cycle. If we multiply the Relation by the consumption function, we obtain the investment function:-

$$\frac{\partial I_t}{\partial C_t} \cdot \frac{dC_t}{dY_t} = \frac{\partial I_t}{\partial Y_t}$$

Since the consumption function is assumed to be stable, the investment function will behave in the same manner as the relation. Therefore the investment function will not only shift in time (Fig. 4) but will vary in shape from positive to negative. This can be expressed diagrammatically:

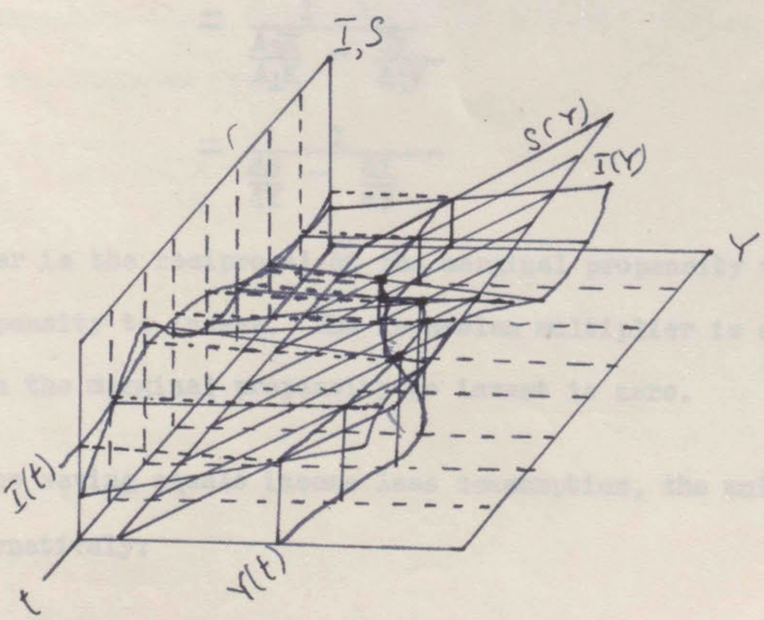

Fig. 5.

When the investment function is not perfectly elastic, the multiplier is no longer equal to the reciprocal of the marginal propensity to save. This is shown in Fig. 6:

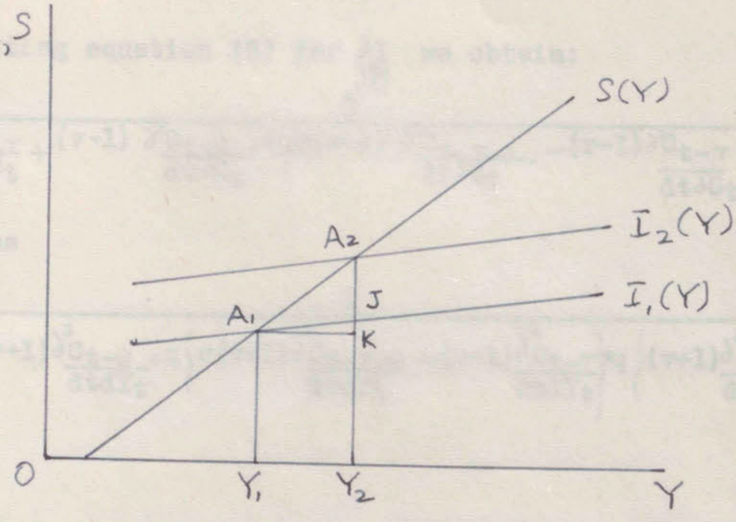

Fig. 6.

The multiplier is determined as follows:-

$$k = \frac{A_1K}{A_2J}$$

$$= \frac{A_1K}{A_2K - JK}$$

$$= \frac{1}{\frac{A_2K}{A_1K} - \frac{JK}{A_1K}}$$

$$= \frac{1}{\frac{dS}{dY} - \frac{dI}{dY}} \qquad \text{..............(9)}$$

The multiplier is the reciprocal of the marginal propensity to save minus the marginal propensity to invest. The Keynesian multiplier is simply a special case in which the marginal propensity to invest is zero.

Since saving equals income less consumption, the multiplier can be written alternatively:

$$k = \frac{1}{1 - \frac{dC}{dY} - \frac{dI}{dY}}$$

$$= \frac{1}{1 - \frac{dC}{dY}\left(1 + \frac{\partial I}{\partial C}\right)} \qquad \ldots\ldots\ldots\ldots\ldots(10)$$

By substituting equation (8) for $\frac{\partial I}{\partial C}$ we obtain:

$$k_t = \frac{1}{1 - \frac{dC_t}{dY_t}\left[rq\frac{\partial C_{t-\tau}}{\partial C_t} + (v+1)\frac{\partial^2 C_{t-\theta}}{dt\,\partial C_t} + q\left\{r(v+1)\frac{\partial^2 C_{t-\tau-\theta}}{dt\,\partial C_t} - (r-1)\frac{\partial^2 C_{t-\tau}}{dt\,\partial C_t}\right\} + q\left\{(v+1)\frac{\partial^3 C_{t-\tau-\theta}}{d^2t\,\partial C_t} - \frac{\partial^3 C_{t-\tau}}{d^2t\,\partial C_t}\right\}\right]}$$

which becomes

$$k_t = \frac{1}{1 - \left[rq\frac{\partial C_{t-\tau}}{dY_t} + (v+1)\frac{\partial^2 C_{t-\theta}}{dt\,dY_t} + q\left\{r(v+1)\frac{\partial^2 C_{t-\tau-\theta}}{dt\,dY_t} - (r-1)\frac{\partial^2 C_{t-\tau}}{dt\,dY_t}\right\} + q\left\{(v+1)\frac{\partial^3 C_{t-\tau-\theta}}{d^2t\,dY_t} - \frac{\partial^3 C_{t-\tau}}{d^2t\,dY_t}\right\}\right]}$$

$$\ldots\ldots\ldots\ldots\ldots(11)$$

A proviso must be added. The investment function can be more properly stated as being composed as follows:

$$\frac{\partial I}{\partial Y} = \frac{\partial I}{\delta I} \cdot \frac{\partial I}{\delta C} \cdot \frac{dC}{dY}$$

That is to say, investment is a function not only of consumption but also of itself. It is an inverse function of its own time-integral (the quantity of capital). This functional relationship is perhaps the most important in the analysis of the business cycle.

3. The Generalized Open-System Multiplier

Equilibrium in a closed system can be depicted in the following diagram:-

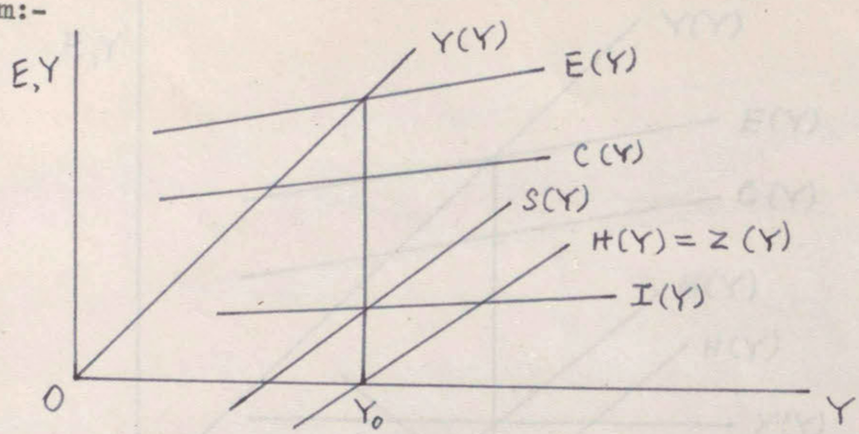

Fig. 7,

The level of income in equilibrium is OY_0, established at the point of intersection of the savings (S(Y)) and investment (I(Y)) functions. If we define excess savings (Z) as the difference between savings and investment (S-I), then Z = 0 in equilibrium.

Expenditure (E) is composed of consumption and investment, and equilibrium can alternatively be considered as being established where income is equal to expenditure. The difference between income and expenditure (Y-E) we shall define as hoarding (H). When E>Y (or I>S) income is rising, and vice-versa. Therefore stable equilibrium is only possible if $\frac{dI}{dY} < \frac{dS}{dY}$ or if $\frac{dE}{dY} < 1$. That is to say, the marginal propensity to hoard must be positive; but hoarding, in equilibrium, must be zero.

An open system can be analyzed in a similar way. If there is no savings or investment, expenditure consists of consumption and exports. Equilibrium is established when exports (X) equal imports (M). The marginal propensity to import can be assumed to be positive. We shall further assume the marginal propensity to export to be negative, since with rising income resources are likely to move into domestic industries. Exports minus imports equal the balance of trade (L). Hoarding (which in this case is imports minus exports) is equal to minus the balance of trade. The balance of trade function must therefore be negative. This is depicted in Fig. 8.

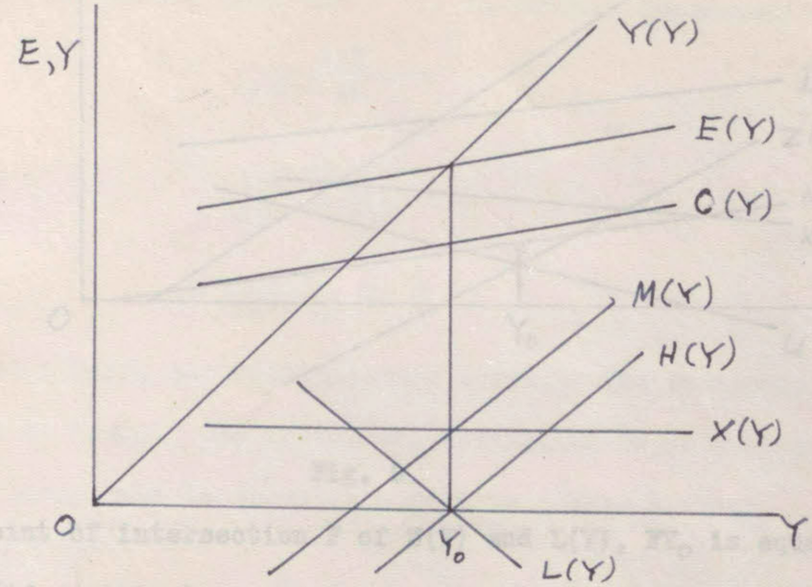

Fig. 8.

By the same method as that used on page 15 above (Fig. 6), the multiplier in this case is of the following nature:

$$k = \frac{1}{\frac{dM}{dY} - \frac{dX}{dY}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

For simplicity, this will be written:-

$$k = \frac{1}{M' - X'}$$

We shall now analyze an open-system model with saving and investment. Y in this case is composed of C+S+M, and E of C+I+X . The equilibrium condition is that Y - E = H = 0. Thus,

$$\begin{aligned} H &= Y - E \\ &= (S+M) - (I+X) \\ &= (S - I) - (X - M) \\ &= Z - L \end{aligned}$$

Equilibrium is then set at the intersection-point of the excess-saving and balance of trade functions (Fig. 9).

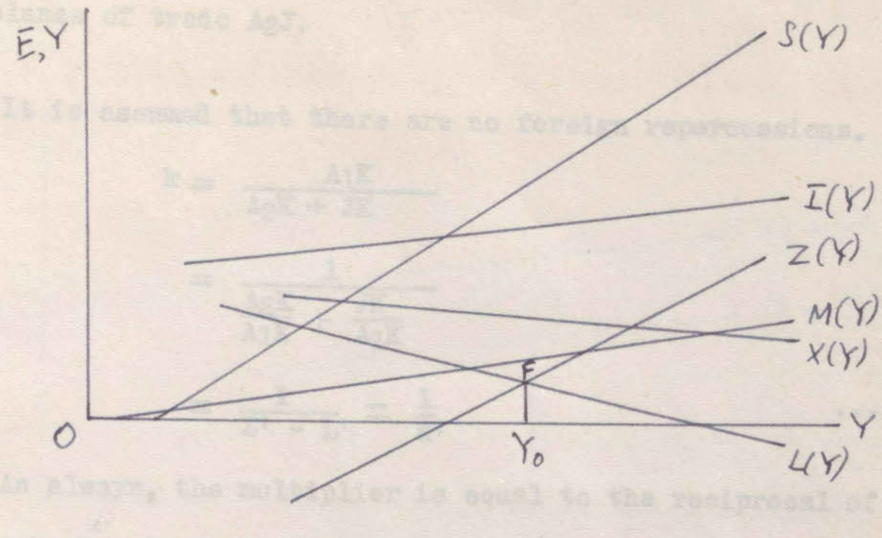

Fig. 9.

At the point of intersection F of Z(Y) and L(Y), FY_0 is equal to net foreign lending (if positive) or net foreign borrowing (if negative). OY_0 is the equilibrium national income which determines the volume of employment.

An increase in exports will be shown by an upward movement of L(Y) (Fig. 10).

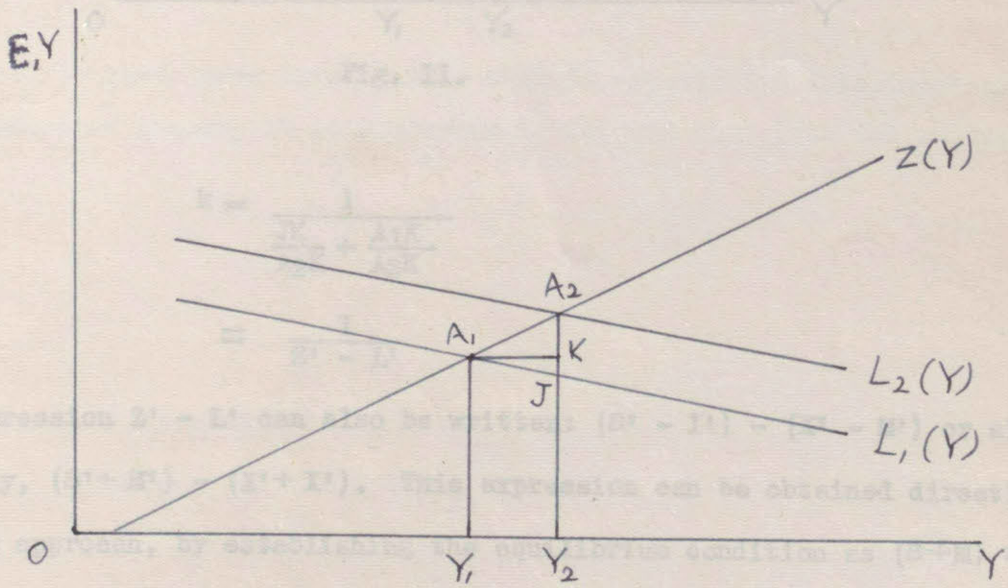

Fig. 10.

The Multiplier is the increase in income A_1K divided by the initial increase in the balance of trade A_2J.

It is assumed that there are no foreign repercussions. Thus,

$$k = \frac{A_1K}{A_2K + JK}$$

$$= \frac{1}{\frac{A_2K}{A_1K} + \frac{JK}{A_1K}}$$

$$= \frac{1}{Z' - L'} = \frac{1}{H'} \qquad \text{.............. (13)}$$

As always, the multiplier is equal to the reciprocal of the marginal propensity to hoard. The investment multiplier is of the same magnitude -- a rise in investment is shown as a fall in excess savings.

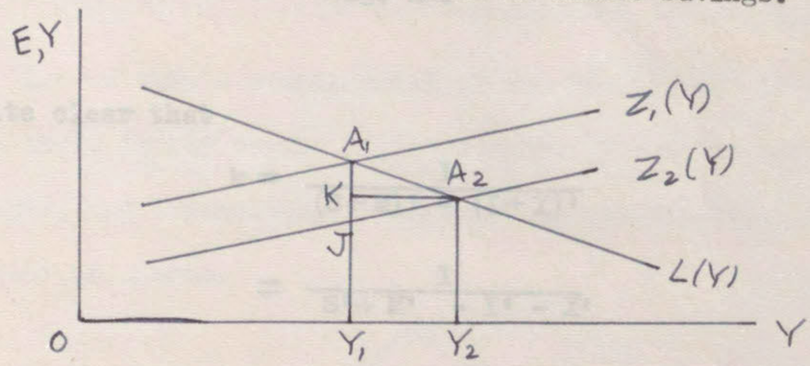

Fig. 11.

Again,

$$k = \frac{1}{\frac{JK}{A_2K} + \frac{A_1K}{A_2K}}$$

$$= \frac{1}{Z' - L'}$$

The expression $Z' - L'$ can also be written: $(S' - I') - (X' - M')$ or alternatively, $(S' + M') - (I' + X')$. This expression can be obtained directly by another approach, by establishing the equilibrium condition as $(S+M) = (I+X)$ which of course holds when $Y = E$.

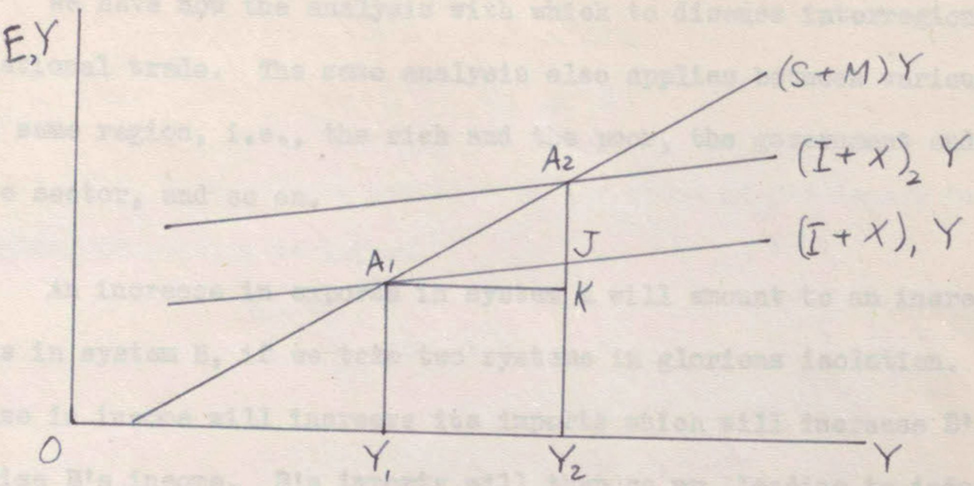

Fig. 12.

It is quite clear that

$$k = \frac{1}{(S+M)' - (I+X)'}$$

$$= \frac{1}{S' + M' - I' - X'}$$

It should be clear from this formula that if exports and investment are not independent of income, it is erroneous to impound them into the multiplicand.

4. The Generalized Inter-System Multiplier

I

We have now the analysis with which to discuss interregional and international trade. The same analysis also applies between various sectors of the same region, i.e., the rich and the poor, the government and the private sector, and so on.

An increase in exports in system A will amount to an increase in imports in system B, if we take two systems in glorious isolation. A's increase in income will increase its imports which will increase B's exports and raise B's income. B's imports will then go up, leading to induced exports in A.

Let us take a simple model of two "countries" in trade, in neither of which there is any saving or investment. Income is composed of consumption and imports, and expenditure of consumption and exports. Exports are independent of income.

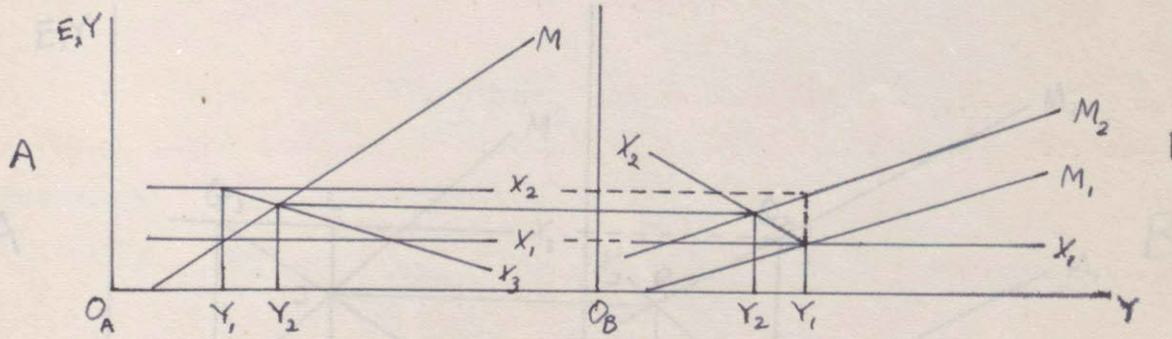

Fig. 13.

Using subscripts a, h and f for 'autonomous', 'home-induced' and 'foreign-induced', there are:-

$$\text{for A:}\left\{\begin{array}{l} {}_aX_A \\ {}_hM_A \\ -\ {}_fX_A = -\ {}_hM_B \end{array}\right.$$

$$\text{for B:}\left\{\begin{array}{l} {}_aM_B \\ {}_hM_B \\ {}_fX_B = {}_hM_A \end{array}\right.$$

For both countries, the equilibrium condition is that $X = M$.

For A,

$$_aX_A - {_fX_A} = {_hM_A}$$

$$_aX_A - {_hM_B} = {_hM_A}$$

$$X_3 = M$$

The slope of X_3 is equal and opposite to the slope of B's import function, or marginal propensity to import.

For B,

$$_aM_B - {_hM_B} = {_fX_B}$$

$$= {_hM_A}$$

$$M_2 = X_2$$

Again, the slope of X_2 is equal and opposite to A's marginal propensity to import.

The magnitude of the multiplier is shown as follows:

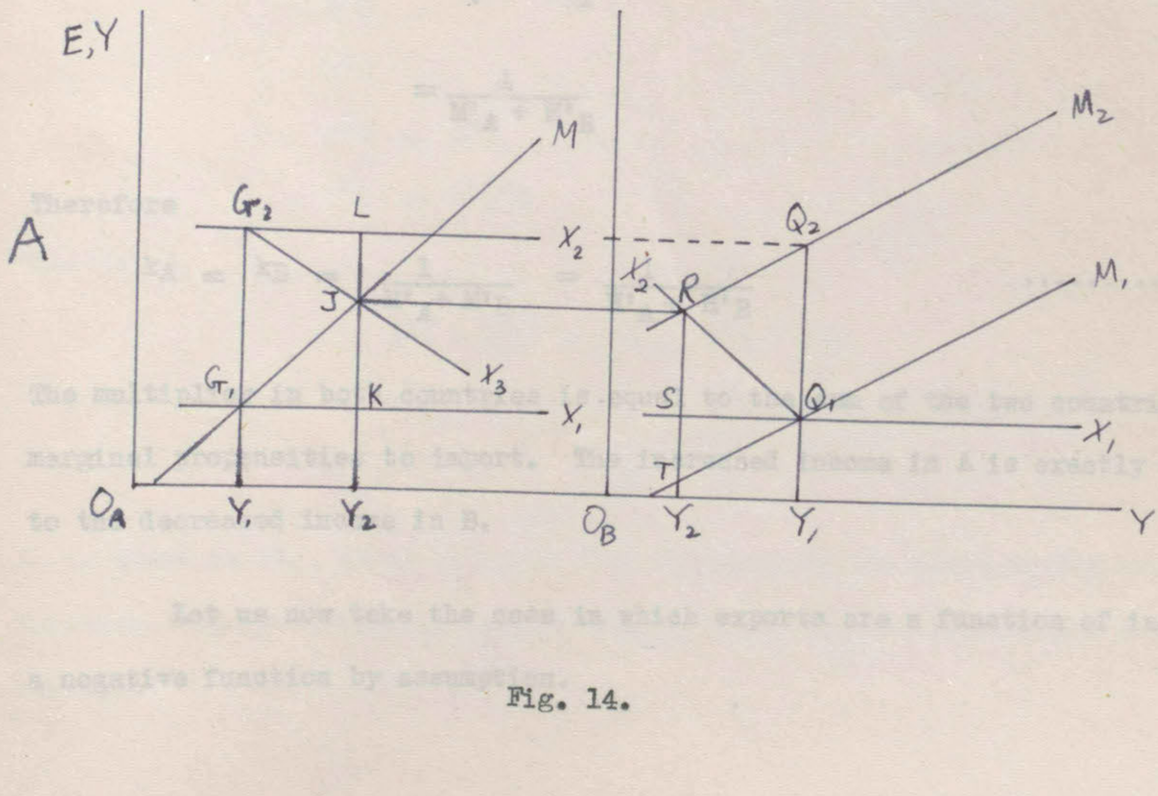

Fig. 14.

In A,

$$k_A = \frac{Y_1Y_2}{G_1G_2}$$

$$= \frac{G_1K}{LJ + JK}$$

$$= \frac{1}{\frac{LJ}{G_1K} + \frac{JK}{G_1K}}$$

$$= \frac{1}{\frac{LJ}{G_2L} + \frac{JK}{G_1K}}$$

$$= \frac{1}{M'_B + M'_A}$$

In B,

$$k_B = \frac{Q_1\ S}{Q_1Q_2}$$

$$= \frac{Q_1S}{RS + ST}$$

$$= \frac{1}{\frac{RS}{Q_1S} + \frac{ST}{Q_1S}}$$

$$= \frac{1}{M'_A + M'_B}$$

Therefore

$$k_A = k_B = \frac{1}{M'_A + M'_B} = \frac{1}{H'_A + H'_B} \qquad \text{............(14)}$$

The multiplier in both countries is equal to the sum of the two countries' marginal propensities to import. The increased income in A is exactly equal to the decreased income in B.

Let us now take the case in which exports are a function of income, a negative function by assumption.

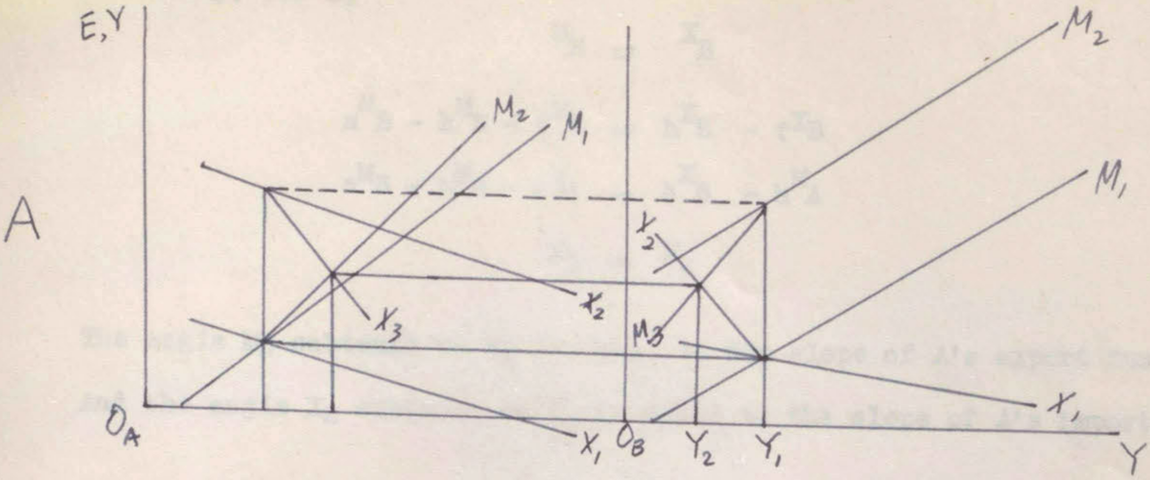

Fig. 15.

For A there are

$$\begin{cases} a^{X}_{A} \\ h^{M}_{A} \\ -\,h^{X}_{A} \\ -\,f^{X}_{A} = -\,h^{M}_{B} \\ f^{M}_{A} = h^{X}_{B} \end{cases}$$

For B there are

$$\begin{cases} a^{M}_{B} \\ -\,h^{M}_{B} \\ h^{X}_{B} \\ f^{X}_{B} = h^{M}_{A} \\ -\,f^{M}_{B} = -\,h^{X}_{A} \end{cases}$$

Thus for A,

$$X_A = M_A$$

$$a^{X}_{A} - h^{X}_{A} - f^{X}_{A} = h^{M}_{A} + f^{M}_{A}$$

$$a^{X}_{A} - h^{X}_{A} - h^{M}_{B} = h^{M}_{A} + h^{X}_{B}$$

$$X_3 = M_2$$

X_3 is drawn in Fig. 15 at an angle from X_2 equal to the slope of B's import function. M_2 is drawn at an angle from M_1 equal to the slope of B's export function.

Similarly, for B,

$$M_B = X_B$$

$$ {}_aM_B - {}_hM_B + {}_fM_B = {}_hX_B - {}_fX_B $$

$$ {}_aM_B - {}_hM_B - {}_hX_A = {}_hX_B - {}_hM_A $$

$$M_3 = X_2$$

The angle M_3 subtends on M_2 is equal to the slope of A's export function. And the angle X_2 subtends on X_1 is equal to the slope of A's import function.

The proof of the multiplier follows:-

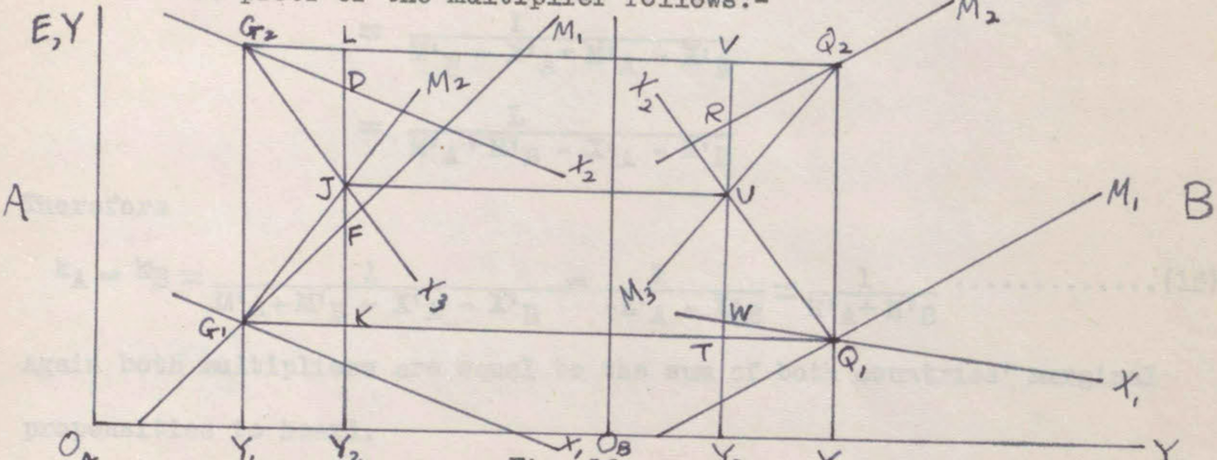

Fig. 16.

$$k_A = \frac{Y_1Y_2}{G_1G_2}$$

$$= \frac{G_1K}{LJ + JK}$$

$$= \frac{1}{\frac{LJ}{G_1K} + \frac{JK}{G_1K}}$$

$$= \frac{1}{\frac{LD}{G_2L} + \frac{DJ}{G_2L} + \frac{FK}{G_1K} + \frac{JF}{G_1K}}$$

$$= \frac{1}{-X'_A + M'_B + M'_A - X'_B}$$

$$= \frac{1}{M'_A + M'_B - X'_A - X'_B}$$

Similarly,

$$k_B = \frac{Y_1Y_2}{Q_1Q_2}$$

$$= \frac{Q_1T}{VU+UT}$$

$$= \frac{1}{\frac{VU}{Q_1T} + \frac{UT}{Q_1T}}$$

$$= \frac{1}{\frac{VR}{Q_2V} + \frac{RU}{Q_2V} + \frac{UW}{Q_1T} + \frac{WT}{Q_1T}}$$

$$= \frac{1}{M'_B - X'_A + M'_A - X'_B}$$

$$= \frac{L}{M'_A + M'_B - X'_A - X'_B}$$

Therefore

$$k_A = k_B = \frac{1}{M'_A + M'_B - X'_A - X'_B} = \frac{1}{-L'_A - L'_B} = \frac{1}{H'_A + H'_B} \quad \ldots\ldots\ldots\ldots\ldots (15)$$

Again both multipliers are equal to the sum of both countries' marginal propensities to hoard.

II

Now let us consider the case in which there is saving and investment in both countries. We will consider first the export multiplier.

In equilibrium,

$$E = Y$$

$$C + I + X = C + S + M$$

$$X - M = S - I$$

$$\Delta X - \Delta M = \Delta S - \Delta I$$

A's balance of trade must be equal and opposite to B's. In order to state the equilibrium condition in terms of the propensities, and since there are no autonomous changes in excess savings, we write:

$$\Delta Z_A = -\Delta Z_B$$

$$Z'_A \Delta Y_A = - Z'_B \Delta Y_B$$

$$\Delta Y_B = \frac{- Z'_A}{Z'_B} \Delta Y_A \qquad \ldots\ldots\ldots\ldots\ldots(16)$$

Now A's final increase in income is made up as follows:-

$$\Delta Y_A = a\Delta X_A + {}_h\Delta C_A + {}_h\Delta I_A + {}_h\Delta X_A - {}_f\Delta X_A + {}_f\Delta M_A$$

$$= \Delta X_A + C'_A \Delta Y_A + I'_A \Delta Y_A + X'_A \Delta Y_A - L'_B \Delta Y_B$$

Substituting (16),

$$\Delta Y_A = \Delta X_A + C'_A \Delta Y_A + I'_A \Delta Y_A + X'_A \Delta Y_A + L'_B \frac{Z'_A}{Z'_B} \Delta Y_A$$

$$\Delta X_A = \Delta Y_A \left(1 - C'_A - I'_A - X'_A - L'_B \frac{Z'_A}{Z'_B}\right)$$

$$= \Delta Y_A \left(Z'_A - L'_A - L'_A \frac{Z'_A}{Z'_B}\right).$$

Thus,

$$k_A = \frac{\Delta Y_A}{\Delta X_A} = \frac{1}{Z'_A - L'_A - L'_B \frac{Z'_A}{Z'_B}} \qquad \ldots\ldots\ldots\ldots\ldots(17)$$

Similarly it can be proved that

$$k_B = \frac{1}{Z'_B - L'_B - L'_A \frac{Z'_B}{Z'_A}}$$

In the special case in which I' and X' are both equal to zero, the multiplier is of the following magnitude:

$$k_A = \frac{1}{S'_A + M'_A + M'_B \frac{S'_A}{S'_B}}$$

This is Prof. Machlup's export multiplier.[5]

5. Fritz Machlup: "International Trade and the National Income Multiplier", Blakiston, Philadelphia, 1943, p. 78.

The export multiplier can be shown graphically:

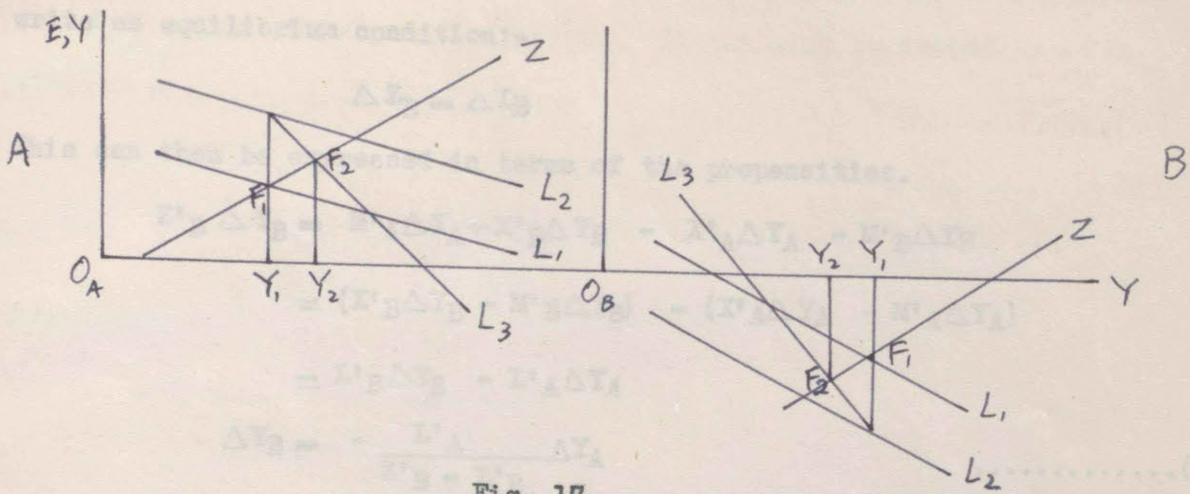

Fig. 17.

In Fig. 17, $L'_B \frac{Z'_A}{Z'_B}$ is the angle L_3 makes on L_2 for country A, and the angle L_3 makes on L_2 in country B is correspondingly $L'_A \frac{Z'_B}{Z'_A}$. Before the autonomous change in the trade balance, A's net foreign lending (F_1Y_1) was equal to B's net foreign borrowing. The change in trade has resulted in an expansion in A's income, a fall in B's income, and a rise in A's net foreign lending, to F_2Y_2, equal to the rise in B's net foreign borrowing.

Equation (17) can also be written:

$$k_A = \frac{1}{(1 - \frac{L'_B}{Z'_B})Z'_A - L'_A} \qquad \text{...........(18)}$$

III

Now let us consider the multiplier when investment (rather than exports) is the multiplicand. Again the equilibrium condition is that net foreign lending equals net foreign borrowing.

$$\Delta Z_B = -\Delta Z_A$$

However the expression Z_A includes the autonomous change in A's investment. In order to express the equilibrium condition in terms of induced changes, and since there are no autonomous changes in the trade balance, we shall write as equilibrium condition:-

$$\Delta Z_B = \Delta L_B$$

This can then be expressed in terms of the propensities.

$$Z'_B \Delta Y_B = M'_A \Delta Y_A + X'_B \Delta Y_B - X'_A \Delta Y_A - M'_B \Delta Y_B$$

$$= (X'_B \Delta Y_B - M'_B \Delta Y_B) - (X'_A \Delta Y_A - M'_A \Delta Y_A)$$

$$= L'_B \Delta Y_B - L'_A \Delta Y_A$$

$$\Delta Y_B = - \frac{L'_A}{Z'_B - L'_B} \Delta Y_A \qquad \ldots\ldots\ldots\ldots\ldots(19)$$

A's final increase in income is composed as follows:

$$\Delta Y_A = {}_a\Delta I_A + {}_h\Delta C_A + {}_h\Delta I_A + {}_h\Delta X_A - {}_f\Delta X_A + {}_f\Delta M_A$$

$$= \Delta I_A + C'_A \Delta Y_A + I'_A \Delta Y_A + X'_A \Delta Y_A - L'_B \Delta Y_B$$

$$\Delta Y_A (1 - C'_A - I'_A - X'_A) = \Delta I_A - L'_B \Delta Y_B$$

$$\Delta Y_A (Z'_A - L'_A) = \Delta I_A - L'_B \Delta Y_B$$

Substituting (19),

$$\Delta Y_A (Z'_A - L'_A) = \Delta I_A + \Delta Y_A \left(\frac{L'_A L'_B}{Z'_B - L'_B} \right)$$

$$\Delta Y_A \left[(Z'_A - L'_A) - \frac{L'_A L'_B}{Z'_B - L'_B} \right] = \Delta I_A$$

$$k_A = \frac{\Delta Y_A}{\Delta I_A} = \frac{Z'_B - L'_B}{(Z'_A - L'_A)(Z'_B - L'_B) - L'_A L'_B}$$

$$= \frac{Z'_B - L'_B}{Z'_A Z'_B - L'_A Z'_B - Z'_A L'_B}$$

$$= \frac{1 - \frac{L'_B}{Z'_B}}{Z'_A - L'_A - L'_B \frac{Z'_A}{Z'_B}} \qquad \ldots\ldots\ldots\ldots(20)$$

alternatively,

$$k_A = \frac{1 - \frac{L'_B}{Z'_B}}{(1 - \frac{L'_B}{Z'_B}) Z'_A - L'_A} \qquad \ldots\ldots\ldots\ldots(21)$$

It will be observed that the investment multiplier is equal to $(1 - \frac{L'_B}{Z'_B})$ times the export multiplier. It can also be stated in a simpler way:

$$k_A = \frac{1}{Z'_A - L'_A\left(\frac{1}{1 - \frac{L'_B}{Z'_B}}\right)} \qquad \text{.............(22)}$$

Similarly,

$$k_B = \frac{1}{Z'_B - L'_B\left(\frac{1}{1 - \frac{L'_A}{Z'_A}}\right)}$$

Or again, we may write:

$$k_A = \frac{1.}{Z'_A - L'_A - \frac{L'_A}{1 - \frac{L'_B}{Z'_B}} - L'_A}$$

$$= \frac{1}{Z'_A - L'_A - \frac{L'_A}{\frac{Z'_B}{L'_B} - 1}} \qquad \text{.............(23)}$$

Equation (23) can be shown diagrammatically:

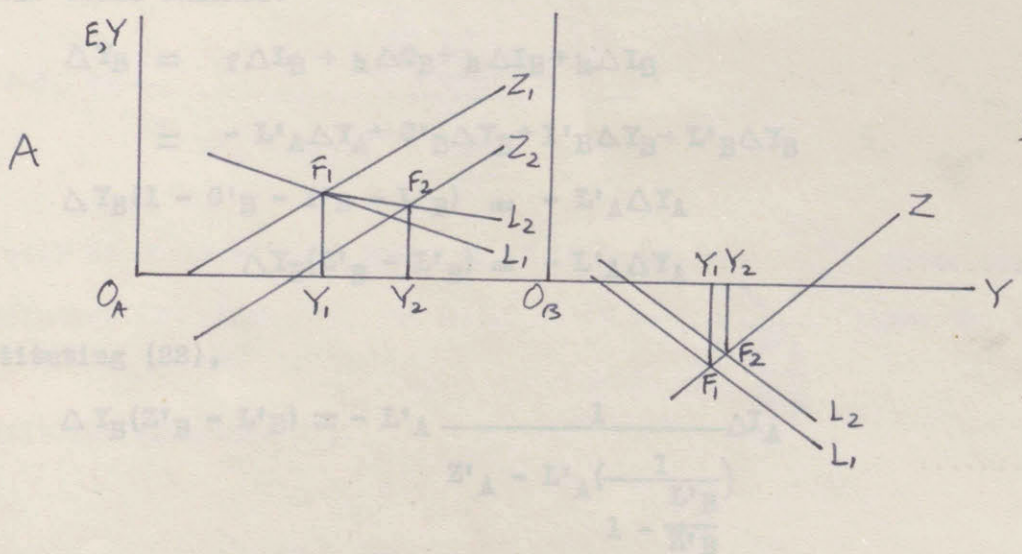

Fig. 18.

For A, the angle L_2 makes on L_1 is equal to $\dfrac{L'_A}{\dfrac{Z'_B}{L'_B} - 1}$. The induced fall in A's trade balance becomes, for B, an autonomous rise in its trade balance, which raises B's income according to the multiplier $\dfrac{1}{Z'_B - L'_B}$. The effect of the autonomous investment in A is to increase both systems' incomes, and to reduce the movement of capital between them.

IV

The rise in B's trade balance (Fig. 18), when considered as the multiplicand, raises B's income according to the open-system multiplier (Equation 13), since the multiplicand is induced from system A, so that there are no further foreign-induced changes in the trade balance.

We must now inquire, however, into the nature of the multiplier when A's autonomous investment is considered as the multiplicand.

The change in B's income is composed of the initial increment in the trade balance, plus the home-induced changes in consumption, investment, and the trade balance:

$$\Delta Y_B = f\Delta L_B + h\Delta C_B + h\Delta I_B + h\Delta L_B$$

$$= - L'_A\Delta Y_A + C'_B\Delta Y_B + I'_B\Delta Y_B + L'_B\Delta Y_B$$

$$\Delta Y_B(1 - C'_B - I'_B - L'_B) = - L'_A\Delta Y_A$$

$$\Delta Y_B(Z'_B - L'_B) = - L'_A\Delta Y_A$$

Substituting (22),

$$\Delta Y_B(Z'_B - L'_B) = - L'_A \frac{1}{Z'_A - L'_A\left(\dfrac{1}{1 - \dfrac{L'_B}{Z'_B}}\right)}\Delta I_A$$

where ΔI_A is the autonomous increment of investment in A.

This becomes

$$\Delta Y_B(Z'_B - L'_B) = - \frac{1}{\frac{Z'_A}{L'_A} - \frac{1}{1 - \frac{L'_B}{Z'_B}}} \Delta I_A$$

$$\therefore k_B = \frac{\Delta Y_B}{\Delta I_A} = - \frac{1}{(Z'_B - L'_B)\left(\frac{Z'_A}{L'_A} - \frac{Z'_B}{Z'_B - L'_B}\right)}$$

$$= \frac{1}{Z'_B - (Z'_B - L'_B)\frac{Z'_A}{L'_A}}$$

Similarly,

$$k_A = \frac{1}{Z'_A - (Z'_A - L'_A)\frac{Z'_B}{L'_B}} \qquad \text{..........(24)}$$

This can also be written:

$$k_A = \frac{-\frac{L'_B}{Z'_B}}{\left(1 - \frac{L'_B}{Z'_B}\right) Z'_A - L'_A} \qquad \text{..........(25)}$$

$$\text{and} \quad k_A = \frac{\frac{1}{1 - \frac{Z'_B}{L'_B}}}{Z'_A - L'_A\left(\frac{1}{1 - \frac{L'_B}{Z'_B}}\right)} \qquad \text{..........(26)}$$

It will be observed that Equation 25 is equal to $-\frac{L'_B}{Z'_B}$ times the export multiplier (Equation 18), and Equation 26 to $\frac{1}{1 - \frac{Z'_B}{L'_B}}$ times the investment multiplier (Equation 22).

V

The three inter-system multipliers we have formulated (Equations 18, 22 and 24) can be further generalized.

In the first place, it should be observed that consumption, as well as investment, can be taken as the multiplicand.[6] The generalized multiplicand is an autonomous increment in expenditure. In the same way, a fall in imports has the same effect as a rise in exports, and an increment in the balance of trade can be taken as a generalized multiplicand.

It should be further observed that an autonomous export from A to B, being a movement of money income from B to A, is identical (for our purposes) with an autonomous movement of capital from B to A, which again is identical for our purposes with an autonomous increase in expenditure

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6. In a closed system, an autonomous rise in consumption may be treated as a fall in the savings function (Fig. 19):

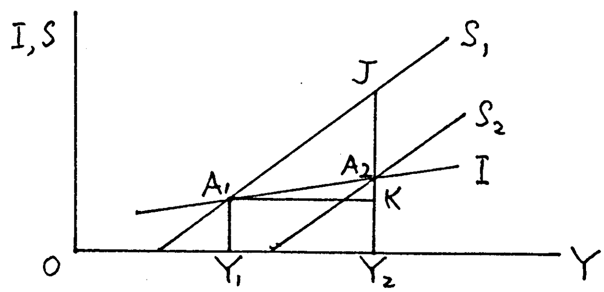

Fig. 19

The multiplier is equal to

$$\frac{A_1K}{A_2J}, = \frac{1}{\frac{JK}{A_1K} - \frac{A_2K}{A_1K}}$$

$$= \frac{1}{S' - I'} .$$

Similarly, the open-system and inter-system consumption and investment multipliers are identical.

in A, accompanied by a simultaneous autonomous decrease in expenditure in B. As far as our analysis is concerned, an "export" from A to B is the same thing as a simultaneous and equal creation of money in A and destruction of money in B. The assumptions under which this holds are no less realistic than those which hold in the usual multiplier analysis.

The "investment multiplier" (Equation 22) we shall now call the "inter-system home-expenditure multiplier", which should be understood to refer to the inter-system multiplier appropriate when the multiplicand consists of home expenditure (it being assumed, of course, that there are two systems under consideration). Equation 24 we shall then call the "inter-system foreign-expenditure multiplier", being the multiplier which is appropriate when the multiplicand is expenditure in the "foreign" system. For these two multipliers we shall use the symbols ${}_hk$ and ${}_fk$ respectively.

The trade-balance multiplier (${}_\ell k$) is then equal to the difference between the home-expenditure and foreign-expenditure multipliers:

$$\Delta E_A \cdot {}_hk_A = {}_h\Delta Y_A$$

$$-\Delta E_B \cdot {}_fk_A = -{}_f\Delta Y_A$$

Adding, $\Delta E_A \cdot {}_hk_A - \Delta E_B \cdot {}_fk_A = {}_h\Delta Y_A - {}_f\Delta Y_A$

Since ΔE_A is equal and opposite to ΔE_B,

$$\Delta E\,({}_hk_A - {}_fk_A) = {}_\ell\Delta Y_A$$

$$\therefore {}_\ell k_A = \frac{\Delta Y_A}{\Delta E} = {}_hk_A - {}_fk_A$$

which can be proved.[7]

VI

We may now formulate the generalized inter-system total-expenditure multiplier (${}_tk$).

The multiplicand will consist of an increment of domestic expenditure, or of foreign expenditure, or of a combination of both. The combination may consist of a positive domestic increment and an equal negative foreign increment (the trade-balance multiplier); or of unequal opposite changes; or of increments in the same direction. For example, if some of the materials bought in a public expenditure in A are bought in B, then there will be two autonomous, simultaneous and positive increments of expenditure in the same direction in both systems. The total increment of income in A (${}_t\Delta Y_A$) is determined as follows:

---

7. By equations (21) and (25)

$$ {}_hk_A - {}_fk_A = \frac{1 - \frac{L'_B}{Z'_B}}{(1 - \frac{L'_B}{Z'_B})Z'_A - L'_A} - \frac{-\frac{L'_B}{Z'_B}}{(1 - \frac{L'_B}{Z'_B})Z'_A - L'_A} $$

$$ = \frac{1}{(1 - \frac{L'_B}{Z'_B})\ Z'_A - L'_A} $$

$= {}_\ell k_A$ by Equation (18).

$$\Delta E_A \cdot {}_hk_A = {}_h\Delta Y_A$$
$$\Delta E_B \cdot {}_fk_B = {}_f\Delta Y_A$$

Adding, $\Delta E_A \cdot {}_hk_A + \Delta E_B \cdot {}_fk_A = {}_t\Delta Y_A$

Let $\Delta E_A = \alpha \Delta E$

$\Delta E_B = \beta \Delta E$

Then $\Delta E(\alpha\, {}_hk_A + \beta\, {}_fk_A) = \Delta Y_A$

$${}_tk_A = \frac{\Delta Y_A}{\Delta E} = \alpha\, {}_hk_A + \beta\, {}_fk_A$$

Substituting Equations (21) and (25),

$${}_tk_A = \frac{\alpha - (\alpha + \beta)\frac{L'_B}{Z'_B}}{(1 - \frac{L'_B}{Z'_B})Z'_A - L'_A} \qquad \ldots\ldots\ldots\ldots(27)$$

Likewise,

$${}_tk_B = \frac{\beta - (\beta + \alpha)\frac{L'_A}{Z'_A}}{(1 - \frac{L'_A}{Z'_A})Z'_B - L'_B}$$

The home-expenditure multiplier (Equation 22) is the special case in which $\alpha = 1$ and $\beta = 0$. The foreign-expenditure multiplier (Equation 24) is the special case in which $\alpha = 0$ and $\beta = 1$. The trade-balance multiplier (Equation 18) is the special case in which $\alpha = 1$ and $\beta = -1$. In the case of a public expenditure, some of which goes into the cost of imported materials, both α and β are positive fractions. Equation (27) is a fully-generalized multiplier, applicable to any multiplicand, holding in a bi-system economy.

5. The Generalized Intra-System Multiplier

If it should be desirable to consider systems A and B jointly, then the two total-expenditure multipliers may be combined. The product would then be the net or total change in income in the joint system, and the multiplicand would be the sum of the multiplicands of each system (which includes the transfer of expenditure from one part of the joint system to the other). The multiplier, then, would be the sum of the two total-expenditure inter-system multipliers:

$$_t k_{A+B} = \frac{\alpha - (\alpha+\beta)\frac{L'_B}{Z'_B}}{(1 - \frac{L'_B}{Z'_B})Z'_A - L'_A} + \frac{\beta - (\beta+\alpha)\frac{L'_A}{Z'_A}}{(1 - \frac{L'_A}{Z'_A})Z'_B - L'_B} \qquad \text{............(28)}$$

This becomes:

$$k_{A+B} = \frac{\alpha\left[\left\{(1-\frac{L'_A}{Z'_B})Z'_B - L'_B\right\} - \left\{(L'_A + L'_B)\ (1 - \frac{L'_A}{Z'_A} - \frac{L'_B}{Z'_B})\right\}\right] + \beta\left[\left\{(1-\frac{L'_B}{Z'_B})Z'_A - L'_A\right\} - \left\{(L'_A + L'_B)\ (1 - \frac{L'_A}{Z'_A} - \frac{L'_B}{Z'_B})\right\}\right]}{\left\{(1 - \frac{L'_B}{Z'_B})Z'_A - L'_A\right\}\ \left\{(1 - \frac{L'_A}{Z'_A})Z'_B - L'_B\right\}}$$

$$= \frac{\alpha\left\{(1-\frac{L'_A}{Z'_A})Z'_B - L'_B\right\}\ (1 - \frac{L'_A + L'_B}{Z'_B}) + \beta\left\{(1-\frac{L'_B}{Z'_B})Z'_A - L'_A\right\}\ (1 - \frac{L'_A + L'_B}{Z'_A})}{\left\{(1-\frac{L'_B}{Z'_B})\ Z'_A - L'_A\right\}\ \left\{(1 - \frac{L'_A}{Z'_A})\ Z'_B - L'_B\right\}}$$

$$= \frac{\alpha(1 - \frac{L'_A + L'_B}{Z'_B})}{(1 - \frac{L'_B}{Z'_B})Z'_A - L'_A} + \frac{\beta(1 - \frac{L'_A + L'_B}{Z'_A})}{(1 - \frac{L'_A}{Z'_A})Z'_B - L'_B} \qquad \text{..........(29)}$$

This can also be written:

$$_t k_{A+B} = \frac{\alpha(Z'_B - L'_A - L'_B) + \beta(Z'_A - L'_A - L'_B)}{Z'_A Z'_B - Z'_A L'_B - L'_A Z'_B} \qquad \text{.........(30)}$$

or

$$_t k_{A+B} = \frac{\alpha Z'_B + \beta Z'_A - (\alpha+\beta)\ (L'_A + L'_B)}{Z'_A Z'_B - Z'_A L'_B - L'_A Z'_B} \qquad \text{..........(31)}$$

This may be called the generalized intra-system total-expenditure multiplier.

As a special case, the transfer-expenditure intra-system multiplier may be written as follows:

$$ {}_{\ell}k_{A+B} = \pm \frac{Z'_B - Z'_A}{Z'_A Z'_B - Z'_A L'_B - L'_A Z'_B} \qquad \text{..........(32)} $$

$$ = \pm \frac{1 - \dfrac{Z'_A}{Z'_B}}{\left(1 - \dfrac{L'_B}{Z'_B}\right) Z'_A - L'_A} = \mp \frac{1 - \dfrac{Z'_B}{Z'_A}}{\left(1 - \dfrac{L'_A}{Z'_A}\right) Z'_B - L'_B} \qquad \text{......(33)} $$

As another special case, the partial-expenditure intra-system multiplier may be written as follows (for expenditure in A):

$$ {}_{A}k_{A+B} = \frac{Z'_B - L'_A - L'_B}{Z'_A Z'_B - Z'_A L'_B - L'_A Z'_B} \qquad \text{........(34)} $$

or:

$$ {}_{A}k_{A+B} = \frac{1 - \dfrac{L'_A + L'_B}{Z'_B}}{\left(1 - \dfrac{L'_B}{Z'_B}\right) Z'_A - L'_A} \qquad \text{.........(35)} $$

which can also be written:

$$ {}_{A}k_{A+B} = \frac{\left(1 - \dfrac{L'_B}{Z'_B}\right) \left(1 - \dfrac{\frac{L'_A}{Z'_B}}{1 - \frac{L'_B}{Z'_B}}\right)}{\left(1 - \dfrac{L'_B}{Z'_B}\right) \left(Z'_A - \dfrac{L'_A}{1 - \frac{L'_B}{Z'_B}}\right)} $$

$$ = \frac{1 - \dfrac{\frac{L'_A}{Z'_B}}{1 - \frac{L'_B}{Z'_B}}}{1 - \dfrac{\frac{L'_A}{Z'_B}}{1 - \frac{L'_B}{Z'_B}} - 1 + \dfrac{Z'_A}{Z'_B}} $$

$$= \cfrac{1}{1 - \cfrac{1 - \frac{Z'_A}{Z'_B}}{\cfrac{L'_A}{Z'_B - L'_B}}}$$

$$= \cfrac{1}{1 - \cfrac{(Z'_B - L'_B)\left(1 - \frac{Z'_A}{Z'_B}\right)}{L'_A}} \qquad \text{..............(36)}$$

$$= \cfrac{1}{1 - \cfrac{(Z'_B - Z'_A)\left(1 - \frac{L'_B}{Z'_B}\right)}{L'_A}} \qquad \text{..............(37)}$$

Similarly,

$$_B k_{A+B} = \cfrac{1}{1 - \cfrac{(Z'_A - Z'_B)\left(1 - \frac{L'_A}{Z'_A}\right)}{L'_B}}$$

6. The Class Struggle Multiplier

I

The generalized bi-system multiplier is applicable to any two related systems, whether the systems be related regions, or related sections of the same community. It is therefore applicable to the analysis of the two predominant classes - capitalists and workers. As a special case of the bi-system multiplier, the "class struggle multiplier", as it may be called, is significant enough to merit special attention.

The two systems under consideration may be called P (capitalists or profit-earners) and W (workers or wage-earners). The proportion of the multiplicand spent in P and W are respectively π and ω. The multipliers corresponding to each system are then respectively:

$$ {}_{t}k_{P} = \frac{\pi - (\pi+\omega)\dfrac{L'_W}{Z'_W}}{(1 - \dfrac{L'_W}{Z'_W})Z'_P - L'_P} \qquad \ldots\ldots\ldots\ldots\ldots(38) $$

and
$$ {}_{t}k_{W} = \frac{\omega - (\omega+\pi)\dfrac{L'_P}{Z'_P}}{(1 - \dfrac{L'_P}{Z'_P})Z'_W - L'_W} $$

It is not necessary to make a too rigid division between capitalists and workers. It is to be understood that many capitalists are in receipt of salaries, and that a number of workers are in receipt of dividends. It is also to be understood that there is a certain amount of investment (and saving) on the part of workers, and that there is, of course, a great deal of consumption on the part of capitalists.

It will therefore be understood that there is consumption, saving and investment in both groups, and trade of consumption and investment goods between them. It will also be understood that wage-payments to workers include whatever dividend payments to workers there may be; and that income accruing to capitalists includes salaries of capitalists as well as investment income.

Because of the nature of the two systems, the parametres will take on a new meaning, which will not be in accordance with the customary meanings of the terms. When a capitalist purchases a consumption-good, his purchase will find its way, some of it into profits, some of it into wages. The first we call "consumption", the second, "imports". When a capitalist invests, some of the funds find their way into profits, and some into wages. The first is "investment" and the second, again "imports". Similarly, "consumption" for workers refers only to that part of the workers' consumption disbursements going into wage-payments; and "Investment" refers only to that part of wage-earners' investment going into wage-payments. "Imports" refer to that part of wage-earners' consumption and investment expenditures finding their way into profits. Wage-earners' imports and exports are, of course, identical with profit-earners' exports and imports respectively.

As profits rise, capitalists' consumption will rise also, as will their imports of consumption goods. Capitalists' marginal propensity to consume and import consumption goods will not, however, be as high as that of workers. Again, as profits rise and fall, investment will rise and fall correspondingly. The impact of the change will be mostly on wages. There will then be, for capitalists, a positive marginal propensity to invest and to import investment goods. Wage-earners' marginal propensity to invest and to import investment goods may be expected to be quite low.

As profits rise, ceteris paribus, a greater proportion of increments in workers' expenditures may find their way into profits. Similarly with changes in wages. However these phenomena must be taken as reflecting the fact that greater proportions of expenditure-increments are accruing to profits or wages, as the case may be, and not causing the fact. Hence we shall consider the marginal propensity to export in both systems to be equal to zero.

II

The class struggle multipler may be applied in many ways. In the first place, it can describe the effect of an autonomous increase in profits (e.g., brought about by increased efficiency or a fall in profits taxes, etc.). In this case, $\pi = 1$ and $\omega = 0$. The result of the increase is plainly a large rise in profits and a small rise in wages. The result is of benefit to workers during times of unemployment, but harmful in inflationary periods.

Similarly an autonomous increase in wage-expenditure ($\pi = 0, \omega = 1$) will increase the incomes of wage-earners considerably, and that of capitalists to a lesser extent. Social security payments are an example of this kind of expenditure (assuming a progressive income tax structure).

A third example is an autonomous investment, in which $\pi + \omega = 1$, and both are positive fractions. The values of π and ω will vary according to the nature of the investment. If the project is a public investment whose object it is to increase national employment, it can be shown that the most effective public investment would be that in which a large proportion of the investment funds accrue to labour, since the wage-multiplier is higher than the profit-multiplier.

Let us assign the following values to the marginal propensities of capitalists:

$C'_P = .2$

$S'_P = .3$ $\qquad I'_P = .2$ $\qquad Z'_P = .1$

$M'_P = .5$ $\qquad X'_P = 0$ $\qquad L'_P = -.5$

and of workers:

$C'_W = .7$

$S'_W = .1$ $\qquad I'_W = .05$ $\qquad Z'_W = .05$

$M'_W = .2$ $\qquad X'_W = 0$ $\qquad L'_W = -.2$

The values of the two class struggle multipliers are then as follows:

$$ {}_tk_P = \frac{\pi - (\pi+\omega)\frac{-.2}{.05}}{(1+\frac{.2}{.05}).1 + .5} \qquad {}_tk_W = \frac{\omega - (\omega+\pi)\frac{-.5}{.1}}{(1+\frac{.5}{.1})\ .05 + .2} $$

$$ = \frac{\pi + 4(\pi+\omega)}{.5 + .5} \qquad = \frac{\omega + 5\ (\omega+\pi)}{.3 + .2} $$

$$ = \pi + 4\ (\pi+\omega) \qquad = 2\left[\omega + 5\ (\omega+\pi)\right] $$

Now, if $\pi = 1$, $\omega = 0$,

${}_tk_P = 5$ $\qquad {}_tk_W = 10$

and if $\pi = \frac{1}{2}$, $\omega = \frac{1}{2}$,

${}_tk_P = 4.5$ $\qquad {}_tk_W = 11$

and if $\pi = 0$, $\omega = 1$,

${}_tk_P = 4$ $\qquad {}_tk_W = 12$

Now, employment may be regarded as being a function of wage-income. And since, during periods of unemployment, the elasticity of wage-rates with respect to wage-income is low (i.e., the elasticity of the aggregate supply curve of labour is high), the wage-multiplier will reflect itself in employment. Since the wage-multiplier is at its highest when the proportion of the initial public expenditure paid out to labour is at its maximum, a wise public works programme should concentrate mainly on labour-employing projects for the swiftest and most effective recovery.

It is very unlikely that in recession periods capitalists' marginal propensity to invest would be high. However let us consider the case in which labour's marginal propensity to consume is low and capital's marginal propensity to invest and to import investment goods is very high. Let the values of the propensities be as follows: for capital,

$C'_P = .1$

$S'_P = .3$ $\quad I'_P = .25$ $\quad Z'_P = .05$

$M'_P = .6$ $\quad X'_P = 0$ $\quad L'_P = -.6$

and for labour,

$C'_W = .5$

$S'_W = .3$ $\quad I'_W = 0$ $\quad Z'_W = .3$

$M'_W = .2$ $\quad X'_W = 0$ $\quad L'_W = -.2$

The values of the class struggle multipliers then become:

$$t^{k_P} = \frac{\pi - (\pi+\omega)\frac{-.2}{.3}}{(1+\frac{.2}{.3}).05 + .6} \qquad t^{k_W} = \frac{\omega - (\omega+\pi)\frac{-.6}{.05}}{(1+\frac{.6}{.05}).3 + .2}$$

$$= \frac{\pi + \frac{2}{3}(\pi+\omega)}{.68} \qquad = \frac{\omega + 12(\omega+\pi)}{4.1}$$

$$= 1.47\left[\pi + \tfrac{2}{3}(\pi+\omega)\right] \qquad = .24\left[\omega + 12(\omega+\pi)\right]$$

Now if $\pi = 1, \quad \omega = 0,$

$_tk_P = 2.4$ $\qquad$ $_tk_W = 2.9$

and if $\pi = \frac{1}{2}$ $\omega = \frac{1}{2}$,

$_tk_P = 1.7$ $\qquad$ $_tk_W = 3.0$

and if $\pi = 0$, $\omega = 1$,

$_tk_P = 1.0$ $\qquad$ $_tk_W = 3.1$

Even in this extreme case, workers are better off when the initial expenditure accrues preponderantly to workers. However, the difference is slight, and the main effect of a large proportion of expenditure initially going to profits is the enlargement of profits.

This application of the multiplier is also useful for analyzing the special case of "economic imperialism", in which the capitalists reside in one country or region, and the workers in another.

III

A fourth application of the class struggle multiplier is of special interest, as it treats of the question, Can workers, by striking, increase their real wages? It is the case of an autonomous payment of wages out of profits. In this case, $\pi = -1$ and $\omega = +1$. The multipliers are accordingly as follows:

$$_tk_P = \frac{-1}{\left(1 - \frac{L'_W}{Z'_W}\right) Z'_P - L'_P}$$

and

$$_tk_W = \frac{1}{\left(1 - \frac{L'_P}{Z'_P}\right) Z'_W - L'_W}$$

This is of course a new form of the trade-balance multiplier, or export multiplier. An autonomous increase in wages paid by capitalists is equivalent to an autonomous "export" from labour to capital. Attempts by labour, then, to improve its position by striking is analagous to (and identical with, for our purposes) attempts by nations to improve their trade-balance by erecting tariff barriers.

In order to compare the "trade-balance" multiplier with the open-system multiplier ($\frac{1}{Z' - L'}$ -- Equation 13), let us rewrite it as follows, according to Equqtion (17):

$$t^{k}P = - \frac{1}{(Z'_P - L'_P) - L'_W \frac{Z'_P}{Z'_W}}$$

and

$$t^{k}W = \frac{1}{(Z'_W - L'_W) - L'_P \frac{Z'_W}{Z'_P}}$$

Using the same values as those used in the first numerical example (p. 44), the values of the two multipliers may be written:

$$t^{k}P = - \frac{1}{.6 + .4} \qquad t^{k}W = \frac{1}{.25 + .25}$$

$$= -1 \qquad = 2$$

There is a multiplier of + 1 in the national economy whose multiplicand is a shift of funds from one sector of the economy to the other. The transfer of income has raised national income, and has given labour a larger share of the national income. Whether the increase takes the form of greater employment or higher prices, labour's real income is higher.

The open-system multipliers are as follows:

$$k_P = -\frac{1}{.6} \qquad\qquad k_W = \frac{1}{.25}$$

$$= -1.7 \qquad\qquad = 4$$

This shows that profits are prevented from falling by 1.7 times the rise in the wage-bill, by the increased profits induced by the increased expenditure of labour on consumption goods. Similarly, wage-income is prevented from multiplying to four times the wage-bill increase by the falling off in investment and consumption expenditure of capitalists.

Though workers can improve their position in this way, it is not to be denied that a reaction to their increased wage-income, in the form of "autonomous" reductions in capitalists' payments to workers, might nullify the benefits they received; just as the benefits of tariff increases may be nullified by retaliatory tariff increases on the part of the foreign country.

7. The Regional Multiplier

I

The bi-system multiplier may be applied to the problem of regionalism. We may consider "regions" as being geographical areas whose propensities differ.

We may consider an "advanced" region (A) and a "backward" region (B). The advanced region, being mature, may be expected to have a high marginal propensity to save, and also a considerable marginal propensity to invest. The backward region will have a low marginal propensity to save and probably a very low marginal propensity to invest. The advanced region, being more self-sustaining, will have a low marginal propensity to import, whereas the backward region, depending upon the advanced region for finished goods, may be expected to have a high marginal propensity to import.

Let us then assign the following values to the marginal propensities:

for A,

$C'_A = .4$		
$S'_A = .4$	$I'_A = .15$	$Z'_A = .25$
$M'_A = .2$	$X'_A = 0$	$L'_A = -.2$

for B,

$C'_B = .4$		
$S'_B = .2$	$I'_B = .01$	$Z'_B = .19$
$M'_B = .4$	$X'_B = 0$	$L'_B = -.4$

The multipliers are then as follows (using Equation 27):

$$t^{k}{}_{A} = \frac{\alpha - (\alpha+\beta)\frac{-.4}{.19}}{(1+\frac{.4}{.19}).25 + .2} \qquad t^{k}{}_{B} = \frac{\beta - (\beta+\alpha)\frac{-.2}{.25}}{(1+\frac{.2}{.25}).19 + .4}$$

$$= \frac{\alpha + 2.1(\alpha+\beta)}{.98} \qquad = \frac{\beta + .8(\beta+\alpha)}{.74}$$

$$= \alpha + 2.1(\alpha+\beta) \qquad = 1.4\left[\beta + .8(\beta+\alpha)\right]$$

If $\alpha = 1, \beta = 0,$

$t^{k}{}_{A} = 3.1 \qquad t^{k}{}_{B} = 1.1$

If $\alpha = \frac{1}{2}, \beta = \frac{1}{2},$

$t^{k}{}_{A} = 2.6 \qquad t^{k}{}_{B} = 1.8$

If $\alpha = 0, \beta = 1,$

$t^{k}{}_{A} = 2.1 \qquad t^{k}{}_{B} = 2.5$

If $\alpha = .175, \beta = .825$

$t^{k}{}_{A} = 2.275 \qquad t^{k}{}_{B} = 2.275$

The fact arises that if there is a public expenditure in B, over 82½% of the onsight expenditure must be directly spent in B, if B is to benefit more than A. If over 17½% of the public expenditure goes into imported materials or profits accruing directly to A, then A benefits more than B.

II

Let us now suppose that the government of a bi-system economy intends to maximize total income in the thriftiest manner. The objective, then, is to find that multiplicand for which the total intra-system multiplier is the highest.

If a public expenditure is fully spent in the given bi-system economy, then $\alpha + \beta = 1$, and the total-expenditure intra-system multiplier (Equation 31) may be rewritten as follows:

$$t^{k}_{A\,B} = \frac{\alpha Z'_B + \beta Z'_A - L'_A - L'_B}{(Z'_B - L'_B)Z'_A - L'_A Z'_B} \qquad \text{.........(39)}$$

Using the same values for the propensities, this becomes:

$$t^{k}_{A+B} = \frac{.19\alpha + .25\beta + .6}{.19}$$

$$= \alpha + 1.4\beta + 3.2$$

The multiplier reaches its maximum value (neither α nor β, of course, exceeding unity) when $\alpha = 1$ and $\beta = 0$. It is at its highest, that is, when all of the public expenditure is made in the backward region. The reason for this, it must be noted, lies not in the marginal propensities to trade, but in the marginal propensities to excess-save; the reason is that the coefficient of β, Z'_A, exceeds the coefficient of α, Z'_B.

If A's marginal propensity to invest is so high that the amount by which it exceeds B's marginal propensity to invest is greater than the amount by which its marginal propensity to save exceeds B's marginal propensity to save, then the multiplier is higher if the expenditure is made in A. If, however, A's marginal propensity to invest is very low (as it is likely to be from time to time), then the multiplier will be higher when the expenditure is made in B. If the latter supposition is assumed to be characteristic of periods when public investment is needed, then we must conclude that a nation will recover best from a depression if the public expenditure is concentrated in the less developed regions.

III

National income may also be increased by a movement of capital from the advanced to the backward region. The relevant multiplier is Equation (33), which is written as follows for $\alpha = -1$ and $\beta = +1$:

$$k_{A\,B} = \frac{1 - \frac{Z'_B}{Z'_A}}{\left(1 - \frac{L'_A}{Z'_A}\right) Z'_B - L'_B}$$

Using the same values, this becomes:

$$k_{A\,B} = \frac{1 - \frac{.19}{.25}}{\left(1 + \frac{.2}{.25}\right) .19 + .4}$$

$$= .43$$

The multiplier is not large. However it should be noted that a development program in a planned economy involving the movement of capital to depressed areas has the added effect of raising money income. If there is full employment, the mere transfer will have to be offset by higher taxation if it is not to be inflationary.

The same effect will be produced by a steepening of the progressive tax structure. If the tax structure is changed so that its incidence shifts from the backward to the advanced region, without any change in the tax yield, then the effect of the change is employment-producing in slack periods, and inflationary during full-employment periods.

It will be noted that as Z'_A and Z'_B diverge from each other the multiplier becomes larger. If Z'_B should exceed Z'_A, a movement of capital into B would lower national income. As Z'_A and Z'_B approach equality the multiplier approaches zero. Thus when adjacent regions' marginal propensities to excess-save approach equality, they cease to have significance as regions as far as the intra-system multiplier is concerned.

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