ALMOST PERIODIC FUNCTIONS OF MANY COMPLEX VARIABLES.

by

Basil A. Rattray

Submitted in partial fulfillment of the requirements for the degree of Master of Science.

McGill University. April, [illegible]

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I am indebted to Professor Tornehave
for his advice and criticism.

CONTENTS

CHAPTER 1.

INTRODUCTION.

The theory of almost periodic functions was created by H.Bohr two decades ago.

I. A (complex-valued) function f(x) defined for all real x is almost periodic if

(a) f(x) is continuous for all x.

(b) Given $\varepsilon > 0$, there is a relatively dense set of translation numbers corresponding to ε, $\tau = \tau_f(\varepsilon)$, such that

$$|f(x+\tau) - f(x)| \leq \varepsilon \qquad \text{for every } x.$$

A set of points on the real axis is said to be relatively dense if we can find l such that every interval of length l contains a point of the set.

Any a.p. function is bounded and uniformly continuous in $[-\infty < x < \infty]$. The set of a.p. functions is closed under addition, and multiplication. If a sequence of a.p. functions converges, uniformly in $[-\infty < x < \infty]$, to f(x), then f(x) is a.p. If $\underset{-\infty < x < +\infty}{\text{l.b.}} |f(x)| > 0$ and f (x) is a.p. then $\frac{1}{f(x)}$ is a.p. If the derivative of an a.p. function is uniformly continuous, it is a.p. If an indefinite integral of an a.p. function is bounded, it is a.p.

If the trigonometric series

$$\sum_{n=1}^{\infty} a_n e^{i\lambda_n x}$$

(where the a's are complex and the λ's real) is uniformly convergent in $[-\infty < x < +\infty]$, its sum-function is almost periodic.

If f(x) is a.p. the mean value

$$\underset{t}{M}\{f(t)\} = \lim_{T\to\infty} \frac{1}{T} \int_{\gamma}^{\gamma+T} f(t)\, dt$$

exists uniformly in γ.

Moreover $a(\lambda) = M\{f(x)e^{-i\lambda x}\}$

differs from zero for an at most enumerable set of values

$\lambda_1, \lambda_2, \ldots$

If $a(\lambda_n) = a_n$, the formal infinite series

$$\sum_{n=1}^{\infty} a_n e^{i\lambda_n x}$$

is called the Fourier series of $f(x)$. We write

$$f(x) \sim \sum_{n=1}^{\infty} a_n e^{i\lambda_n x}$$

It is easily seen that if $f(x)$ is periodic this Fourier series is its ordinary Fourier series.

The sum-function of a uniformly convergent trigonometric series has that series as its Fourier series.

The Fourier series of a sum or a uniform limit is the formal sum or limit of the corresponding Fourier series.

Theorem of Identity: If two a.p. functions have the same Fourier series, they are identically equal.

Parseval's Theorem: If $f(x)$ is a.p. and

$$f(x) \sim \sum_{n=1}^{\infty} a_n e^{i\lambda_n x}$$

then $$M\{|f(x)|^2\} = \sum_{n=1}^{\infty} |a_n|^2$$

Multiplication Theorem: The Fourier series of two a.p. functions may be multiplied term by term, i.e., if

$$f(x) \sim \sum_{n=1}^{\infty} a_n e^{i\mu_n x} \qquad g(x) \sim \sum_{n=1}^{\infty} b_n e^{i\nu_n x}$$

then $$f(x)\, g(x) \sim \sum_{n=1}^{\infty} c_n e^{i\lambda_n x}$$

where $$c_n = \sum_{\mu_r + \nu_s = \lambda_n} a_r b_s$$

<u>Approximation Theorem</u>: If $f(x) \sim \sum_{n=1}^{\infty} a_n e^{i\lambda_n x}$

we can find a sequence of polynomials

$$s_N(x) = \sum_{n=1}^{\infty} k_n^{(N)} a_n e^{i\lambda_n x}.$$

(where $0 \leq k_n^{(N)} \leq 1$, for any fixed N only a finite number of the $k_n^{(N)}$ are different from zero, and $\lim_{N\to\infty} k_n^{(N)} = 1$ for a fixed value of n) such that

$s_n(x) \longrightarrow f(x)$, uniformly in $[-\infty < x < \infty]$, as $N \to \infty$.

In fact, we can construct these approximating sums by a generalization of the Féjer summation used for ordinary Fourier series. For a periodic function

$$g(x) \sim \sum_{n=-\infty}^{\infty} b_n e^{i\frac{2\pi}{p}nx}$$

we construct a series of approximating sums

$$s_N(x) = \underset{t}{M}\{g(x-t) K_N(\tfrac{2\pi}{p}t)\} = \sum_{n=-\infty}^{+\infty} k_n^{(N)} b_n e^{i\frac{2\pi}{p}nx}$$

by means of the Fejer kernel

$$K_N(t) = \sum_{\nu=-N}^{+N}\left(1 - \frac{|\nu|}{N}\right) e^{i\nu t}.$$

This sequence has the properties described above.

Let B_1, B_2, ... be a rational base of $\{\lambda_n\}$, i.e. a set of numbers such that $r_1 B_1 + .. + r_n B_n = 0$, $r_1, r_2 .., r_n$ rational, implies $r_1 = r_2 = .. = r_n = 0$, and any λ_n can be expressed in the form

$\lambda_n = r_1 B_1 + .. + r_k B_k$ where $r_1, r_2 .. r_k$ are rational.

We define

$$k_N(t) = K_{N!N}\left(\frac{B_1}{N} t\right) \ldots K_{N!N}\left(\frac{B_N}{N} t\right)$$

$$s_N(x) = \underset{t}{M}\{f(x-t) k_N(t)\} = \sum_{n=1}^{\infty} k_n^{(N)} a_n e^{i\lambda_n x}$$

Then $S_N(x)$ is the required sequence of approximating sums.

The coefficients $k_n^{(N)}$ depend only on $\{B_n\}$ and N, and not otherwise upon f(x).

In the particular cases when (a) the exponents $\{\lambda_n\}$ are linearly independent, or (b) the Fourier coefficients $\{a_n\}$ are positive, the Fourier series is absolutely convergent.

II. The set of points $s = \sigma + it$ with $\alpha < \sigma < \beta$ is called the strip (α, β). A closed strip is denoted by $\{\alpha, \beta\}$. If a property holds in every strip (α_1, β_1), where $\alpha < \alpha_1 < \beta_1 < \beta$, it is said to hold in $\langle \alpha, \beta \rangle$.

Let f(s) be a function analytic in a strip (α, β). A real number τ such that

$$|f(s + i\tau) - f(s)| \leq \varepsilon$$

At all points s of the strip is called a translation number of f(s) belonging to ε. If the set of these translation numbers is relatively dense for every $\varepsilon > 0$, the function f(s) is called almost periodic in (α, β). We may define functions a.p. in $\{\alpha, \beta\}$ or in $\langle \alpha, \beta \rangle$ similarly.

A function f(s) a.p. in (α, β) is an a.p. function of t on any line $s = \sigma_0 + it$ of the strip.

If f(s) is analytic in (α, β), bounded in $\langle \alpha, \beta \rangle$ and an a.p.function of t on the line σ_0 of (α, β) then it is a.p. in $\langle \alpha, \beta \rangle$.

If f(s) is a.p. in $\langle \alpha, \beta \rangle$, it is bounded in $\langle \alpha, \beta \rangle$. Hence it is uniformly continuous in $\langle \alpha, \beta \rangle$ together with all its derivatives.

The set of functions a.p. in a strip $\langle \alpha, \beta \rangle$ is closed under addition, multiplication, differentiation, and integration if the integral is bounded, and division by an a.p. function with no zeros in the strip. If a sequence $f_n(s)$ of a.p. functions in a strip (α, β) converges uniformly to $f(s)$, then $f(s)$ is a.p. in (α, β).

The sum of an exponential series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}$$

uniformly convergent in a strip (α, β), is a.p. in (α, β).

If $f(s)$ is a.p. in (α, β) then $f(s)$, considered as a function of t, has the Fourier series

$$f(\sigma + it) \sim \sum_{n=1}^{\infty} A_n e^{\lambda_n \sigma} e^{i\lambda_n t}$$

for each σ in (α, β). The series

$$\sum_{n=1}^{\infty} A_n e^{\lambda_n s}$$

is called the Dirichlet series of $f(s)$ in the strip (α, β).

The sum-function $f(s)$ of an exponential series uniformly convergent in some strip has this series as its Dirichlet series.

To any a.p. function corresponds a Dirichlet series, and if two functions, a.p. in the same strip, have the same Dirichlet series, then they are identical.

For any function

$$f(s) \sim \sum_{n=1}^{\infty} A_n e^{\lambda_n s}$$

a.p. in a strip (α, β) the Parseval equation

$$\underset{t}{M}\{|f(\sigma+it)|^2\} = \sum_{n=1}^{\infty} |A_n|^2 e^{2\lambda_n \sigma}$$

holds for all σ of the interval (α, β).

<u>Approximation Theorem:</u> To any function a.p. in a strip $\langle \alpha, \beta \rangle$, corresponds a sequence of exponential polynomials converging to

the function uniformly in $\langle \alpha, \beta \rangle$.

The class of functions bounded and a.p. in a half-plane $(\alpha, +\infty)$ and the class of functions with non-positive exponents are identical.

If a function

$$f(s) \sim \sum_{n=1}^{\infty} A_n e^{\lambda_n s}$$

is a.p. in $\langle \alpha, +\infty)$ and if

$$\lambda_n < \lambda < 0$$

for all n, then an indefinite integral of f(s) is a.p. in $\langle \alpha, +\infty)$.

If a function

$$f(s) \sim \sum_{n=1}^{\infty} A_n e^{\lambda_n s}$$

is a.p. in $\langle \alpha, \beta \rangle$ and if one of the following conditions is satisfied:

(a). The exponents $\{\lambda_n\}$ are linearly independent,

(b). The coefficients A_n are all positive,

(c). The series $\sum_{n=1}^{\infty} e^{-|\lambda_n|\delta}$ converges for any $\delta > 0$, then the series

$$\sum_{n=1}^{\infty} A_n e^{\lambda_n s}$$ converges absolutely in (α, β).

III. S.Bochner has developed the theory of almost periodic functions of any number of real variables.

Given a function $f(x_1, \ldots, x_n)$ and an $\varepsilon > 0$, a vector $\tau = (\tau_1, \ldots, \tau_n)$ is said to be a translation vector belonging to ε if

$$|f(x_1 + i\tau_1, \ldots, x_n + i\tau_n) - f(x_1, \ldots, x_n)| \leq \varepsilon$$

for all $(x_1, \ldots, x_n)$.

A set of points $(x_1, \ldots, x_n)$ is said to be relatively dense if there exists a number l such that any interval

$$a_i < x_i < a_i + l \qquad (i=1,\dots,n)$$

contains at least one point of the set.

A function $f(x_1,\dots, x_n)$ is called almost periodic if it is continuous for all $\bar{x}$, and the set of translation vectors belonging to ε is relatively dense for any $\varepsilon > 0$.

An almost periodic function is bounded and uniformly continuous.

A necessary and sufficient condition for a continuous function $f(x_1,\dots,x_n)$ to be a.p. is that the set of τ_i such that

$$|f(x_1, x_2,\dots, x_{i-1}, x_i + \tau_i, x_{i+1}, \dots, x_n) - f(x_1,\dots,x_n)| \le \varepsilon$$

for all $(x_1,\dots x_n)$ is relatively dense on the real axis for $i = 1,\dots, n$ and every $\varepsilon > 0$.

Hence, if r of the variables have some constant value, an a.p. function $f(x_1,\dots, x_n)$ is a.p. in the remaining $(n-r)$ variables.

The set of a.p. functions is closed under addition, multiplication and uniform convergence. If $f(x_1,\dots,x_n)$ is a.p. then

$$M_{x_1 x_2 \dots x_i} \{f(x_1,\dots,x_n)\}$$

is a.p. in $(x_{i+1}, \dots, x_n)$. The mean value of f with respect to i of the variables is independent of the order in which mean values with respect to the individual variables are taken.

If $f(x_1,\dots, x_n)$ is a.p. and $\frac{\partial}{\partial x_r} f(x_1,\dots,x_n)$ is uniformly continuous then $\frac{\partial f}{\partial x_r}$ is a.p. If $f(x_1,\dots,x_n)$ is bounded and has a.p. partial derivatives it is a.p.

The set of values of $\bar{\lambda} = (\lambda_1, \dots\dots, \lambda_n)$ for which

$$a(\bar{\lambda}) = M_{\bar{x}} \{ f(\bar{x})\, e^{-i\bar{\lambda}\cdot\bar{x}} \}$$

differs from zero is at most denumerable. If we allow the addition to this set of a denumerable number of values of $\bar{\lambda}$ for which $a(\bar{\lambda}) = 0$, we can obtain a set

$$\{\bar{\lambda}_{\bar{m}}\} = \{(\lambda_{1,m_1}, \ldots, \lambda_{n,m_n})\} \text{ where } m_i = 1, 2, \ldots \text{ for } i = 1, \ldots, n.$$

We call the formal multiple series

$$\sum_{\substack{1 \le m_i < \infty \\ i = 1,\ldots,n}} a_{\bar{m}} e^{i\bar{\lambda}_{\bar{m}} \cdot \bar{x}} = \sum_{\substack{1 \le m_i < \infty \\ i = 1,\ldots,n}} a_{m_1,\ldots,m_n} e^{i(\lambda_{1,m_1} x_1 + \ldots + \lambda_{n,m_n} x_n)}$$

the Fourier series of $f(\bar{x})$, and write

$$f(\bar{x}) \sim \sum a_{\bar{m}} e^{i\bar{\lambda}_{\bar{m}} \cdot \bar{x}}$$

If two functions have the same Fourier series they are identically equal.

For any a.p. function $f(\bar{x}) \sim \sum a_{\bar{m}} e^{i\bar{\lambda}_{\bar{m}} \cdot \bar{x}}$ the Parseval equation

$$M_{\bar{x}}\{|f(x)|^2\} = \sum_{\substack{1 \le m_i < \infty \\ i = 1,\ldots,n}} |a_{\bar{m}}|^2$$

Approximation Theorem:

If $f(\bar{x})$ is a.p. and

$$f(\bar{x}) \sim \sum a_{\bar{m}} e^{i\bar{\lambda}_{\bar{m}} \cdot \bar{x}}$$

we can find a sequence of polynomials

$$s_N(\bar{x}) = \sum k_{\bar{m}}^{(N)} a_{\bar{m}} e^{i\bar{\lambda}_{\bar{m}} \cdot \bar{x}}$$

(where $0 \le k_{\bar{m}}^{(N)} \le 1$, $k_{\bar{m}}^{(N)} \to 1$ as $N \to \infty$ for fixed $\bar{m}$, $k_{\bar{m}}^{N}$ for fixed N differs from zero for only a finite number of values of $\bar{m}$) which converges uniformly to $f(\bar{x})$.

In fact, we can take

$$s_N(\bar{x}) = M_{\bar{t}}\{f(x_1 - t_1, \ldots, x_n - t_n) K_N(t_1, \ldots, t_n)\}$$

where $K_N(\bar{x}) = k_N(x_1) \ldots k_N(x_n)$.

If a function $f(\bar{x})$ is a.p. and if one of the following conditions is satisfied:

(a). The exponents $\{\lambda_{i,m_i}\}$ are linearly independent for $i = 1,..,n$

(b). The coefficients $a_{\bar{m}}$ are all positive,

then the series $\sum a_{\bar{m}} e^{i\bar{\lambda}_{\bar{m}}\cdot\bar{x}}$ is absolutely convergent.

IV. N.Brazma has discussed almost periodic functions of several complex variables.

A function $f(z_1,..,z_n)$ analytic in the interval

I: $a_1 < x_1 < b_1, \ldots, a_n < x_n < b_n$, $(y_1, .. , y_n)$ unrestricted, is said to be almost periodic in I if for any $\varepsilon > 0$ there exists a relatively dense set of translation vectors $\bar{\tau} = (\tau_1, .., \tau_n)$ such that

$$|f(z_1 + i\tau_1, \ldots, z_n + i\tau_n) - f(z_1, .., z_n)| \leq \varepsilon$$

for all $(z_1,.., z_n)$ in I.

These functions have properties analogous to those of previous types. A Dirichlet series exists and is summable to the function. A function is almost periodic if it is bounded and has almost periodic partial derivatives.

In the following discussion, these functions are defined in more general domains, and their properties are derived independently of Brazma's results. This more general point of view makes it possible to obtain results on the analytic continuation of the functions.

Chapter 2.

STRUCTURAL PROPERTIES.

We shall consider functions of n complex variables $(z_1, .. , z_n)$ analytic in tubes. A tube is a set of all finite points $(z_1, .., z_n)$ such that $(x_1, .., x_n)$ lies in an open connected set in R_n or its closure.

We denote tubes by $\mathfrak{A}, \mathfrak{A}', \mathfrak{B}$, etc. and the corresponding open connected sets in R_n by A, A^1, B, etc.

If the closure of A^1 is contained in A, the tube $\mathfrak{A}'$ is said to be inside the tube $\mathfrak{A}$. If a property P of tubes holds for a tube $\mathfrak{A}$, it is said to hold in $\mathfrak{A}$. If it holds for every tube $\mathfrak{A}'$ inside $\mathfrak{A}$ it is said to hold inside $\mathfrak{A}$. The set of points $(z_1, .., z_n)$ such that $(x_1, .., x_n)$ lies in the closure, of A will be denoted by $\overline{\mathfrak{A}}$.

If $\mathfrak{A}'$ is inside $\mathfrak{A}$, A^1 is bounded. For otherwise at least one infinite point would lie in the closure of A^1, and hence in A.

Let $f(z_1, .., z_n)$ be analytic in a tube $\mathfrak{A}$. A vector $\overline{\tau} = (\tau_1, .., \tau_n)$ in R_n is called a translation vector of $f(z_1,..,z_n)$ belonging to ε if

$$|f(z_1+i\tau_1,\ldots,z_n+i\tau_n) - f(z_1,\ldots,z_n)| \leq \varepsilon$$

for all points $(z_1, .., z_n)$ of the tube $\mathfrak{A}$.

If the set of translation vectors of $f(z_1, .., z_n)$ belonging to ε is relatively dense in R_n for every $\varepsilon > 0$ then $f(z_1, .., z_n)$ is said to be almost periodic in $\mathfrak{A}$.

Then $f(z_1, .., z_n)$ is a.p. inside $\mathfrak{A}$ if it is a.p. in every tube inside $\mathfrak{A}$.

A function $f(z_1, .., z_n)$ a.p. in $\mathfrak{a}$ is an almost periodic function of the n real variables $(y_1, .., y_n)$ for any fixed $(x_1, .., x_n)$ in A.

<u>Theorem 1.</u> A function $f(z_1, .., z_n)$ almost periodic inside $\mathfrak{a}$ is bounded inside $\mathfrak{a}$.

Proof: Let l be an edge-length of an inclusion interval α corresponding to 1 of $f(z_1, .., z_n)$ in $\bar{\mathfrak{a}}'$ inside $\mathfrak{a}$. Then $f(z_1, .., z_n)$ is analytic in the domain comprising that part of $\bar{\mathfrak{a}}'$ for which

$$0 \le y_i \le l \qquad (i=1,..,n)$$

Hence $|f(z_1, .., z_n)| < K$ in this domain. Hence $|f(x_1, .., z_n)| < K+1$ in $\mathfrak{a}'$. That is, $f(z_1, .., z_n)$ is bounded inside $\mathfrak{a}$.

<u>Theorem 2.</u> If $f(z_1, .., z_n)$ is analytic in a tube $\mathfrak{a}$ and bounded inside $\mathfrak{a}$, and is an almost periodic function of $(y_1, .., y_n)$ at a point $(x_1^0, .., x_n^0)$ of A, then $f(z_1, .., z_n)$ is almost periodic inside $\mathfrak{a}$.

Proof: Putting $F(z_1,...,z_n) = f(z_1+i\tau_1,...,z_n+i\tau_n) - f(z_1,...,z_n)$ the theorem reduces to proving that if $F(z_1, .., z_n)$ is analytic in $\mathfrak{a}$ and

$$|F(z_1,...,z_n)| \le K$$

in $\bar{\mathfrak{a}}'$ inside $\mathfrak{a}$ and containing $(x_1^0+iy_1, ..., x_n^0+iy_n)$, then, given $\varepsilon > 0$, there exists a $\delta > 0$, such that if $|F(x_1^0+iy_1,...,x_n^0+iy_n)| \le \delta$ for all $(y_1, .., y_n)$ then $|F(z_1, .., z_n)| \le \varepsilon$ in $\mathfrak{a}^*$ inside $\mathfrak{a}'$. δ depends on $\mathfrak{a}$, $\mathfrak{a}'$, $\mathfrak{a}^*$ and upon ε and K, but not otherwise upon $F(z_1, .., z_n)$.

We prove first, by an induction on n, that if $F(z_1, .., z_n)$

is analytic and, $|F(z_1, .., z_n)| \le K$ in the interval

$a_i' \le x_i \le b_i' \quad (i=1,..,n)$ and $a_i' < a_i < x_i^0 < b_i < b_i'$

then, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that, if

$$|F(x_1^0 + iy_1, .., x_n^0 + iy_n)| \le \delta \text{ for all } (y_1, .., y_n)$$

then $|F(z_1,..,z_n)| \le \varepsilon$ in the interval

$$a_i \le x_i \le b_i \quad (i=1,..,n)$$

The number δ depends upon the a's, b's, a''s and b''s, and upon ε and K, but not otherwise upon $F(z_1, .., z_n)$.

This theorem is known to be true for one variable. It follows that given $\varepsilon > 0$, we can choose $\delta_1 > 0$, so that if

$$|F(z_1, .., z_n)| \le \delta_1$$

for $a_i'' \le x_i \le b_i''$ $(i = 2, ..., n)$, $x_1 = x_1^0$,

(where $a_i' < a_i'' < a_i < b_i < b_i'' < b_i'$) then,

$$|F(z_1, .., z_n)| \le \varepsilon \quad \text{for } a_i \le x_i \le b_i \quad (i = 1, .., n)$$

but by induction hypothesis, we find $\delta > 0$ so that if

$$|F(z_1, .., z_n)| \le \delta$$

for $x_i = x_i^0$ $(i = 1, .., n)$ then

$$|F(z_1, .., z_n)| \le \delta_1$$

for $a_i'' \le x_i \le b_i''$, $(i = 2,..,n)$, $x_1 = x_1^0$

Hence this theorem is proved for all n.

Now, given any point p of $\bar{A}$ and any interval I lying in A^1 and containing the poiht, we can for each $\varepsilon > 0$, find $\delta > 0$ such that if $|F(z_1, .., z_n)| \le \delta$ when $(x_1, .., x_n)$ is the point p, then $|F(z_1, .., z_n)| \le \varepsilon$ when $(x_1, .., x_n)$ lies in I.

By the Heine- Borel theorem, we find ~~that~~ a finite covering of $\bar{A}^*$ by the interiors of intervals lying in A^1. Any one I_ν of these intervals can be joined by a finite chain of intervals to the interval containing $(x_1^0, .., x_n^0)$. Hence, by a finite number of applications of the theorem proved above, we can for each $\varepsilon > 0$ find $\delta_\nu > 0$ such that if

$$|F(z_1, .., z_n)| \leq \delta_\nu$$

when $x_1 = x_1^0, .., x_n = x_n^0$, then

$$|F(z_1, .., z_n)| \leq \varepsilon$$

in I_ν. For each $\varepsilon > 0$, we take δ to be minimum of the finite number of δ_ν's.

Then $|F(z_1, .., z_n)| \leq \varepsilon$ in $\mathfrak{A}^*$ if $|F(z_1, .., z_n)| \leq \delta$ for $x_1 = x_1^0, .., x_n = x_n^0$.

It follows from this theorem that if $f(z_1, .., z_n)$ is analytic in a tube $\mathfrak{A}$ and a.p. as a function of $(y_1, .., y_n)$ for $x_1 = x_1^0, .., x_n = x_n^0$, where $(x_1^0, .., x_n^0)$ lies in A, there is a largest tube contained in $\mathfrak{A}$ and containing $(x_1^0, .., x_n^0)$ inside which $f(z_1, .., z_n)$ is a.p. This is also the largest tube contained in $\mathfrak{A}$ inside which $f(z_1, .., z_n)$ is bounded.

Theorem 3. The sum and product of two functions $f(z_1, .., z_n)$, $g(z_1, .., z_n)$, a.p. inside a tube $\mathfrak{A}$, are a.p. inside $\mathfrak{A}$.

Proof: $f(z_1, .., z_n)$ and $g(z_1, .., z_n)$ are bounded inside $\mathfrak{A}$. Hence $f + g$ and $f \times g$ are bounded inside $\mathfrak{A}$. Also, they are a.p. functions of $(y_1, .., y_n)$ for $x_1 = x_1^0, .., x_n = x_n^0$, if $(x_1^0, .., x_n^0)$ lies in A. Hence they are a.p. inside $\mathfrak{A}$.

Hence any exponential polynomial

$$\sum_{n=1}^{N} A_n e^{(\Lambda_1 z_1 + \dots + \Lambda_n z_n)}$$

is a.p. inside the tube

$$-\infty < x_1 < \infty, \dots, -\infty < x_n < \infty.$$

<u>Theorem 4.</u> If $\{f_m(z_1, .., z_n)\}$ converges to $f(z_1, .., z_n)$ uniformly in $\mathfrak{a}$, and $f_m(z_1, .., z_n)$ is a.p. in $\mathfrak{a}$ for every m, then $f(z_1, .., z_n)$ is a.p. in $\mathfrak{a}$.

Proof: Given $\varepsilon > 0$, we find m such that

$$|f(z_1, .., z_n) - f_m(z_1, .., z_n)| \leq \frac{\varepsilon}{3} \text{ in } \mathfrak{a}.$$

Let $\tau = \tau_{f_m}(\frac{\varepsilon}{3})$ be any one of the relatively dense set of translation vectors of $f_m(z_1, .., z_n)$ belonging to ε. Then

$$\begin{aligned}
&|f(z_1+i\tau_1,\dots,z_n+i\tau_n) - f(z_1,\dots,z_n)| \\
&\quad \leq |f(z_1+i\tau_1,\dots,z_n+i\tau_n) - f_m(z_1+i\tau_1,\dots,z_n+i\tau_n)| \\
&\quad + |f_m(z_1+i\tau_1,\dots,z_n+i\tau_n) - f_m(z_1,\dots,z_n)| \\
&\quad + |f_m(z_1, .., z_n) - f(z_1, .., z_n)|
\end{aligned}$$

Hence, for any $\varepsilon > 0$, there exists a relatively dense set of $\tau_f(\varepsilon)$'s. Hence $f(z_1, .., z_n)$ is a.p. in $\mathfrak{a}$.

It follows that the sum of a series

$$\sum_{\substack{1 \leq m_1 < \infty \\ \vdots \\ 1 \leq m_n < \infty}} A_{m_1,\dots,m_n} e^{(\Lambda_{1,m_1} z_1 + \dots + \Lambda_{n,m_n} z_n)}$$

uniformly convergent inside a tube $\mathfrak{a}$, is a.p. inside $\mathfrak{a}$.

<u>Theorem 5</u>. If $f(z_1, .., z_n)$ is a.p. in the tube $\mathfrak{a}$ and has no zeros in $\mathfrak{a}$, and $\mathfrak{a}^1$ is a tube inside $\mathfrak{a}$, then

$$\text{l.b.} |f(z_1, .., z_n)| > 0$$
$$(z_1, .., z_n) \text{ in } \mathfrak{a}^1.$$

Proof: By the Heine-Borel theorem, we can find a finite number of intervals I covering $\mathfrak{a}'$, and each inside another interval contained in $\mathfrak{a}$. Thus we need only prove that the lower bound of $|f(z_1, .., z_n)|$ is positive in any one of the intervals I.

We begin from a theorem on functions of one complex variable, which states: If (1) a set of functions f(z) is uniformly

bounded in the rectangle $a^1 < x < b^1$, $-k^1 < y < k^1$,
(2) each function is analytic and has no zeros in this rectangle,
(3) the set does not have zero as a limit function, i.e. there is no sequence of functions $\{f_m(z)\}$ of the set converging to zero uniformly in the rectangle, then if $a' < a < b < b'$, $0 < k < k'$, we can find $m > 0$ such that

$$\text{l.b.} |f(z)| > m$$

in the rectangle $a < x < b$, $-k < y < k$ for each function $f(z)$ in the set.

It follows that if a set of functions $f(z)$ is uniformly bounded and uniformly almost periodic in a strip, $a' < x < b^1$, no function of the set has zeros in the strip and there exist a point z_0 in the strip $a < x < b$ and a $k > 0$ such that $|f(z_0)| > k$ for every function $f(z)$ of the set, then if $a' < a < b < b'$ we can find $m > 0$ such that

$$\text{l.b.} |f(z)| > m$$

in the strip $a < x < b$ for each function $f(z)$ of the set. To prove this, consider the set of functions $f(z + iy^*)$ in the rectangle

$$a' < x < b' \quad , -\frac{l}{2} - 1 < y < \frac{l}{2} + 1$$

where l is an inclusion interval corresponding to $\frac{k}{2}$ of the set of functions $f(z)$ in $a' < x < b'$, and the parameter y^* assumes all real values for each function $f(z)$.
This set of functions obviously fulfills the first two conditions of the above theorem, and since, for any f and any y^*, we can find a value of z in the given rectangle for which $|f(z + iy^*)| > \frac{k}{2}$, the third condition is also fulfilled.

We now prove the main theorem by an induction on n. The function $f(z_1, \ldots, z_n)$ is analytic and has no zeros in

$$a_1' < x_1 < b_1', \ldots, a_n' < x_n < b_n'$$

and $|f(z_1, \ldots, z_n)| \leq K$ in this interval. We prove that $\text{l.b.}\ |f(z_1, \ldots, z_n)|$ is positive in

$$a_1 < x_1 < b_1, \ldots\ldots, a_n < x_n < b_n \quad \text{where}$$

$$a_1' < a_1 < b_1 < b_1', \ldots, a_n' < a_n < b_n < b_n'.$$

Take x_1^0 such that $a_1 < x_1^0 < b_1$. By the induction hypothesis, $\text{l.b.}\ |f(x_1^0, z_2, \ldots, z_n)|$ is positive in

$$a_2 < x_2 < b_2, \ldots, a_n < x_n < b_n$$

The set of functions $f(z_1, \ldots, z_n)$ of z_1 in the strip $a_1' < x_1 < b_1'$, depending upon (n-1) parameters $z_2, \ldots, z_n$, which run through the above interval, is uniformly bounded and uniformly almost periodic in the strip and no function of the set has zeros in the strip. Also $\text{l.b.}\ |f(x_1^0, z_2, \ldots, z_n)|$, where $z_2, \ldots, z_n$ runs through the above interval, is positive. Hence, by the previous theorem, there exists an $m > 0$ such that

$$|f(z_1, \ldots, z_n)| > m$$

when $a_1 < x_1 < b_1, \ldots, a_n < x_n < b_n$

Since the theorem is true for $n = 1$ it is true for every n.

<u>Theorem 6.</u> If $f(z_1, \ldots, z_n)$ is a.p. inside a tube $\mathfrak{A}$, and has no zeros in $\mathfrak{A}$, then $\dfrac{1}{f(z_1, \ldots, z_n)}$ is a.p. inside $\mathfrak{A}$.

Proof: Take $\mathfrak{A}'$ inside $\mathfrak{A}$, and $(x_1^0, \ldots, x_n^0)$ in A^1.

Then $g(z_1, \ldots, z_n) = \dfrac{1}{f(z_1, \ldots, z_n)}$ is bounded in $\mathfrak{A}'$ and is a.p. as a function of $(y_1, \ldots, y_n)$ for $x_1 = x_1^0, \ldots, x_n = x_n^0$. Hence $g(z_1, \ldots, z_n)$ is a.p. in $\mathfrak{A}'$.

<u>Theorem 7.</u> An analytic function $f(z_1, \ldots, z_n)$ bounded inside a tube $\mathfrak{A}$ is uniformly continuous inside $\mathfrak{A}$, as is each of its partial derivatives, of any order.

Proof: Since any tube inside $\mathfrak{A}$ can be covered by a finite number of intervals, we need only prove that if the interval

I^1: $a_1 - \delta < x_1 < b_1 + \delta, \ldots, a_n - \delta < x_n < b_n + \delta$

is inside $\mathfrak{A}$, then $f(z_1, \ldots, z_n)$ is uniformly continuous in

I: $a_1 < x_1 < b_1, \ldots, a_n < x_n < b_n$

together with its derivatives. Now

$$|f(z_1, \ldots, z_n)| \leq K$$

in I^1, whence

$$\left| \frac{\partial^{m_1} \ldots \partial^{m_n}}{\partial z_1^{m_1} \ldots \partial z_n^{m_n}} f(z_1, \ldots, z_n) \right| \leq \frac{m_1! \ldots m_n!}{\delta^{m_1 + \ldots + m_n + 1}} K$$

in I. If $(z_1^{(1)}, \ldots, z_n^{(1)})$ and $(z_1^{(2)}, \ldots, z_n^{(2)})$ are any two points of I,

$$\left| \frac{\partial^{m_1} \ldots \partial^{m_n}}{\partial z_1^{m_1} \ldots \partial z_n^{m_n}} f(z_1^{(1)}, \ldots, z_n^{(1)}) - \frac{\partial^{m_1} \ldots \partial^{m_n}}{\partial z_1^{m_1} \ldots \partial z_n^{m_n}} f(z_1^{(2)}, \ldots, z_n^{(2)}) \right|$$
$$\leq \sum_{i=1}^{n} \frac{(m_1+1)! \ldots (m_n+1)!}{\delta^{m_1 + \ldots + m_n + 1}} K |z_i^{(1)} - z_i^{(2)}|$$

Hence each derivative is uniformly continuous in I.

The partial derivatives are clearly bounded inside $\mathfrak{A}$.

<u>Theorem 8.</u> If $f(z_1, \ldots, z_n)$ is a.p. inside a tube $\mathfrak{A}$, so are its partial derivatives.

Proof: $f(z_1, \ldots, z_n)$ is bounded inside $\mathfrak{A}$. Hence $\frac{\partial}{\partial z_r} f(z_1, \ldots, z_n)$ is uniformly continuous inside $\mathfrak{A}$, and, in particular, is uniformly continuous as a function of $(y_1, \ldots, y_n)$ for $x_1 = x_1^0, \ldots, x_n = x_n^0$, where $(x_1^0, \ldots, x_n^0)$ lies in A.

Then $\frac{\partial}{\partial y_r} f(z_1, \ldots, z_n) = i \frac{\partial}{\partial z_r} f(z_1, \ldots, z_n)$ is uniformly continuous as a function of $(y_1, \ldots, y_n)$ for $x_1 = x_1^0, \ldots, x_n = x_n^0$ and $r = 1, \ldots, n$.

Then $\frac{\partial}{\partial y_r} f(z_1, \ldots, z_n)$ and hence $\frac{\partial}{\partial z_r} f(z_1, \ldots, z_n)$ is a.p. as a function of $(y_1, \ldots, y_n)$ for $x_1 = x_1^0, \ldots, x_n = x_n^0$. Also $\frac{\partial}{\partial z_r} f(z_1, \ldots, z_n)$ is bounded inside $\mathfrak{a}$. Hence it is a.p. inside $\mathfrak{a}$.

Theorem 9. If $f(z_1, \ldots, z_n)$ is analytic in a tube $\mathfrak{a}$, bounded inside $\mathfrak{a}$, and its partial derivatives are a.p. inside $\mathfrak{a}$, then $f(z_1, \ldots, z_n)$ is a.p. inside $\mathfrak{a}$.

Proof: For $x_1 = x_1^0, \ldots, x_n = x_n^0$, $f(z_1, \ldots, z_n)$ is an a.p. function of $(y_1, \ldots, y_n)$. ~~inside~~ . Hence it is a.p. inside $\mathfrak{a}$.

CHAPTER 3.

DIRICHLET SERIES AND ANALYTIC CONTINUATION.

Theorem 10. If $f(z_1, .., z_n)$ is a.p. in a tube $\mathfrak{a}$ then the Fourier series of $f(x_1 + iy_1, .., x_n + iy_n)$ considered as a function of $(y_1, .., y_n)$ has the same expression.

$$f(x_1 + iy_1, .., x_n + iy_n) \sim \sum_{\substack{1 \le m_1 < \infty \\ \cdots \\ 1 \le m_n < \infty}} A_{m_1, .., m_n} e^{\Lambda_{1,m_1} x_1 + \dots + \Lambda_{n,m_n} x_n} e^{i(\Lambda_{1,m_1} y_1 + \dots + \Lambda_{n,m_n} y_n)}$$

for all $(x_1, .., x_n)$ in A.

Proof: We have to prove that

$$M_{y_1, .., y_n}\left\{ f(x_1 + iy_1, \dots, x_n + iy_n)\, e^{-(\Lambda_1 z_1 + \dots + \Lambda_n z_n)} \right\}$$

is independent of $(x_1, .., x_n)$. To prove this we need only prove that it is independent of $(x_1, .., x_n)$ in any interval inside $\mathfrak{a}$. That is, we must prove that if the interval

$$a_1 < x_1 < b_1, \dots, a_n < x_n < b_n$$

is inside $\mathfrak{a}$ and the points $(x_1^{(1)}, .., x_n^{(1)})$, $(x_1^{(2)}, .., x_n^{(2)})$ lie in this interval then the mean value above is the same for these two points. This follows from the corresponding theorem for functions of one complex variable if we consider the chain of points $(x_1^{(2)}, .., x_n^{(2)})$, $(x_1^{(1)}, x_2^{(2)}, .., x_n^{(2)})$ ....., $(x_1^{(1)}, .., x_{n-1}^{(1)}, x_n^{(2)})$,

$(x_1^{(1)}, .., x_n^{(1)})$.

The mean value is unchanged as we go from any one of these to the next. Hence it is the same for

$(x_1^{(1)}, .., x_n^{(1)})$ and $(x_1^{(2)}, .., x_n^{(2)})$.

We associate with $f(z_1, .., z_n)$ the "Dirichlet series"

$$\sum_{\substack{1\le m_1<\infty \\ \cdots \\ 1\le m_n<\infty}} A_{m_1,\dots,m_n} e^{\Lambda_{1,m_1} z_1 + \dots + \Lambda_{n,m_n} z_n}$$

and write

$$f(z_1, .., z_n) \sim \sum_{\substack{1\le m_1<\infty \\ \cdots \\ 1\le m_n<\infty}} A_{m_1,\dots,m_n} e^{\Lambda_{1,m_1} z_1 + \dots + \Lambda_{n,m_n} z_n}$$

It is convenient to use vector notation and write

$$f(\bar{z}) \sim \sum A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}} \cdot \bar{z}}$$

The following theorems are immediate consequences of the corresponding theorems for a.p. functions of n real variables.

Theorem 11. The a.p. sum-function of a series $\sum A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}} \cdot \bar{z}}$ uniformly convergent in a tube $\mathfrak{a}$ has that series as its Dirichlet series in $\mathfrak{a}$.

Theorem 12. (Uniqueness Theorem). If two functions a.p. in a tube $\mathfrak{a}$ have the same Dirichlet series in $\mathfrak{a}$, they are identically equal.

Theorem 13. If $f(\bar{z})$ is a.p. in a tube $\mathfrak{a}$ and

$$f(\bar{z}) \sim \sum A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}} \cdot \bar{z}}$$

then the Parseval equation

$$M_{\bar{y}} \{ |f(\bar{x}+i\bar{y})|^2 \} = \sum |A_{\bar{m}}|^2 e^{2\bar{\Lambda}_{\bar{m}} \cdot \bar{x}}$$

holds for all $\bar{x}$ in A.

<u>Theorem 14.</u> (Approximation Theorem). If $f(\bar{z})$ is a.p. inside a tube $\mathfrak{A}$, its Dirichlet series

$$\sum A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}}\cdot\bar{z}}$$

is summable to $f(\bar{z})$ uniformly inside $\mathfrak{A}$, i.e. there exists a sequence of exponential polynomials

$$s_N(\bar{z}) = \sum k_{\bar{m}}^{(N)} A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}}\cdot\bar{z}} \qquad (N=1,\ 2,\ \ldots\ldots)$$

(for which $0 \leq k_{\bar{m}}^{(N)} \leq 1$, $k_{\bar{m}}^{(N)} \to 1$ as $N \to \infty$ for fixed $\bar{m}$, $k_{\bar{m}}^{N} = 0$, for any fixed N, for all but a finite number of $\bar{m}$) converging to $f(\bar{z})$ uniformly inside $\mathfrak{A}$.

Proof: The sequence of polynomials

$$s_N(\bar{z}) = M_{\bar{t}}\{f(\bar{x} + i(\bar{y} - \bar{t}))K_N(\bar{t})\}$$

(whose coefficients satisfy the above conditions) converges to $f(\bar{z})$ uniformly in $\bar{y}$ for each fixed $\bar{x}$ in A. The convergence is uniform in $\bar{x}$ in any interval contained in A (this follows from the generalised Phragmen - Lindelof theorem). Hence, by the Heine-Borel theorem, it is uniform in any tube inside $\mathfrak{A}$.

<u>Theorem 15.</u> If $f(\bar{z})$ is a.p. inside $\mathfrak{A}$,

$$f(\bar{z}) \sim \sum A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}}\cdot\bar{z}}$$

and one of these conditions is satisfied:

1. The exponents $\{\Lambda_{r,m_r}\}$ are linearly independent for each fixed r,
2. The coefficients $A_{\bar{m}}$ are all positive,
3. The series
$$\sum e^{-(|\Lambda_{1,m_1}| + \ldots + |\Lambda_{n,m_n}|)\delta}$$
converges for each $\delta > 0$,

then the series $\sum A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}}\cdot\bar{z}}$ converges absolutely in $\mathfrak{A}$, and uniformly inside $\mathfrak{A}$.

Proof: For the first two cases the absolute convergence follows from the corresponding theorem for Fourier series. The uniform convergence then follows by the Phragmen - Lindelof and Heine - Borel theorems.

In case 3, let $\bar{x}^0$ be any fixed point of A. Choose $\delta > 0$ such that $\bar{x}^0 + (\pm\delta, \ldots\ldots, \pm\delta)$ belong to A. The series

$$\sum A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}} \cdot (\bar{x}^0 + (\pm\delta, \ldots, \pm\delta))} e^{i\bar{\Lambda}_{\bar{m}} \cdot \bar{y}}$$

are Fourier series of a.p. functions of $\bar{y}$. Hence their coefficients are bounded

$$\left| A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}} \cdot (\bar{x}^0 + (\pm\delta, \ldots, \pm\delta))} \right| < K$$

Therefore $\left| A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}} \cdot \bar{x}^0} \right| < K e^{\bar{\Lambda}_{\bar{m}} (\pm\delta, \ldots, \pm\delta)}$

and $\left| A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}} \cdot \bar{z}} \right| < K e^{-(|\Lambda_{1,m_1}| + \ldots + |\Lambda_{n,m_n}|)\delta}$

so that $\sum A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}} \cdot \bar{z}}$ is absolutely convergent. The uniform convergence inside $\mathfrak{A}$ follows immediately.

<u>Theorem 16.</u> A function $f(\bar{z})$ a.p. inside a tube $\mathfrak{A}$ can be continued analytically into a convex tube, inside which it is a.p..

Proof: We prove first the following lemma:

Lemma. If $g(\bar{z})$ is analytic and bounded in a tube $\mathfrak{B}$, the points $\bar{x}^{(1)}$ and $\bar{x}^{(2)}$ lie in $\mathfrak{B}$, and

$$|g(\bar{z})| \leq \varepsilon \quad \text{for } \bar{x} = \bar{x}^{(1)} \text{ or } \bar{x}^{(2)} \text{ and all } \bar{y},$$

then $|g(\bar{z})| \leq \varepsilon$ for $\bar{x} = \tau_1 \bar{x}^{(1)} + \tau_2 \bar{x}^{(2)}$ $(\tau_1 \geq 0, \tau_2 \geq 0, \tau_1 + \tau_2 = 1)$ and all $\bar{y}$.

Proof: By a change of variable

$$z_i = \sum_{j=1}^{n} a_{ij} w_j + x_i^{(1)} \qquad (i = 1, \dots, k)$$

where the a_{ij} are real and det $a_{ij} = 1$,

$$g(\bar{z}) = g\left(\sum_{j=1}^{n} a_{1j} w_j + x_1^{(1)}, \dots, \sum_{j=1}^{n} a_{nj} w_j + x_n^{(1)}\right) = h(\bar{w})$$

becomes a function of $\bar{w}$ analytic in a tube in the $\bar{w}$ space containing $\bar{u}^{(1)}$ and $(0, \dots, 0)$, corresponding to $\bar{x}^{(2)}$ and $\bar{x}^{(1)}$ in the $\bar{z}$ space. We can choose the a_{ij} so that $\bar{u}^{(1)} = (u_1^1, 0, \dots, 0)$. Then

$$|h(\bar{w})| \leq \varepsilon$$

for $\bar{u} = (0, \dots, 0)$ or $(u_1^1, 0, \dots, 0)$ and all $\bar{V}$. For any fixed values of $V_2, \dots, V_n$, and $u_2 = \dots u_n = 0$, $h(\bar{w})$ becomes an analytic and bounded function of w_1, in the strip $0 \leq u_1 \leq u_1^{(1)}$ $(\ast)$, whose modulus is $\leq \varepsilon$ on the boundary of the strip. By the Phragmen-Lindelof theorem,

$$|h(\bar{w})| \leq \varepsilon$$

in the strip. Hence $|g(\bar{z})| \leq \varepsilon$ for $\bar{x} = \tau_1 \bar{x}^{(1)} + \tau_2 \bar{x}^{(2)}$ $(\tau_1 \geq 0, \tau_2 \geq 0, \tau_1 + \tau_2 = 1)$ and all $\bar{y}$.

The main theorem can now be proved as follows:

We can find a sequence $\{s_N(\bar{z})\}$ of exponential polynomials converging to $f(\bar{z})$ uniformly inside $\mathcal{a}$. If $\{s_N(\bar{z})\}$ converges uniformly in $\bar{y}$ for $\bar{x} = \bar{x}^{(0)}$ and $\bar{x}^{(1)}$, we can find N_0 such that

$$|s_N(\bar{z}) - s_M(\bar{z})| \leq \varepsilon$$

for $\bar{x} = \bar{x}^{(0)}$ or $\bar{x}^{(1)}$ and all $\bar{y}$, for $N, M > N_0$.

By the lemma, $|\rho_N(\bar z)-\rho_M(\bar z)| \leq \varepsilon$ for any point $\bar x$ on the line segment joining $\bar x^{(0)}$ and $\bar x^{(1)}$, all $\bar y$, and all N, M such that $N > N_0$, $M > N_0$. Hence the set of values of $\bar X$, for which $\{\rho_N(\bar z)\}$ converges uniformly for all $\bar y$, is convex. Its interior is an open convex set containing A. The sequence converges uniformly inside the corresponding tube, and hence represents a function a.p. inside this tube, and equal to $f(\bar z)$ in $\mathcal{O}\!\iota$.

If M is any set in R_n, the $+r$ extension of M consisting of the points $(x_1, .., x_n)$ such that a point $(x_1, .., x'_r, .., x_n)$ with $x'_r \leq x_r$ lies in M is denoted by M_{+r}. We define M_{-r} similarly. The $+s$ extension of M_{+r} is denoted by $M_{+r,+s}$, and we denote other extensions similarly. Clearly a multiple extension is independent of the order in which the extensions are taken, e.g. $M_{+r,+s} = M_{+s,+r}$. If M is open or convex, the extensions of M are open or convex.

<u>Theorem 17</u>. If $f(\bar z)$ is a.p. inside a tube $\mathcal{O}\!\iota$, the exponents $\{\Lambda_{r_1,m_{r_1}}\}, \ldots\ldots, \{\Lambda_{r_p,m_{r_p}}\}$ are non-positive, the exponents $\{\Lambda_{r_{p+1},m_{r_{p+1}}}\}, \ldots\ldots, \{\Lambda_{r_q,m_{r_q}}\}$ are non-negative, and $\mathcal{O}\!\iota'$ is a tube inside $\mathcal{O}\!\iota$, then $f(\bar z)$ can be continued analytically into a function a.p. and bounded in the tube corresponding to the extension $A^1_{+r_1,\ldots,+r_p,-r_{p+1},\ldots,-r_q}$ of A^1.

Proof: Since A^1 can be covered by a finite number of intervals I contained in A, we need only prove that $f(\bar z)$ is a.p. in $I_{+r_1,+r_2,\ldots,+r_p,-r_{p+1},\ldots,-r_q}$.

We can find a sequence $\{\Delta_N(\bar{z})\}$ of exponential polynomials converging to $f(\bar{z})$ uniformly in I. It is known that for an exponential polynomial $\Delta(\bar{z})$ in one variable $\underset{-\infty<y<\infty}{\text{u.b.}} |\Delta(\bar{z})|$ is a decreasing function of x if all the exponents of the polynomial are non-positive and an increasing function of x if all the exponents are non-negative. Applying this theorem q times, it follows that if $s(\bar{z})$ is an exponential polynomial with exponents from among those of $f(\bar{z})$ and $|s(\bar{z})| \leq \varepsilon$ in I, then $|s(\bar{z})| \leq \varepsilon$ in $I_{+r_1,\ldots,+r_p,-r_{p+1},\ldots,-r_q}$. Now given $\varepsilon > 0$ we can find N_0 such that $|s_N(\bar{z}) - s_M(\bar{z})| \leq \varepsilon$ in I for $N > N_0$, $M > N_0$. Hence $|s_N(\bar{z}) - s_M(\bar{z})| \leq \varepsilon$ in $I_{+r_1,\ldots,+r_p,-r_{p+1},\ldots,-r_q}$. Hence the sequence $\{s_N(\bar{z})\}$ converges uniformly in $I_{+r_1,\ldots,+r_p,-r_{p+1},\ldots,-r_q}$, so that its sum-function, which is an analytic continuation of $f(\bar{z})$, is a.p. and bounded in $I_{+r_1,\ldots,+r_p,-r_{p+1},\ldots,-r_q}$.

Theorem 18. If $f(\bar{z})$ is a.p. inside a tube , and the exponents $\{\Lambda_{r_1,m_{r_1}}\},\ldots,\{\Lambda_{r_p,m_{r_p}}\}$ and $\{\Lambda_{r_{p+1},m_{r_{p+1}}}\},\ldots,\{\Lambda_{r_q,m_{r_q}}\}$ are bounded above and below respectively, then $f(\bar{z})$ is a.p. inside the tube corresponding to $A_{+r_1,\ldots,+r_p,-r_{p+1},\ldots,-r_q}$.

Proof:

$$\Lambda_{r,m_r} < K \qquad \text{for } r = r_1, \ldots, r_p \text{ and all } m_r$$

$$\Lambda_{r,m_r} > k \qquad \text{for } r = r_{p+1}, \ldots, r_q \text{ and all } m_r$$

Hence if $\mathcal{O}'$ is inside $\mathcal{O}$, $g(\bar{z}) = f(\bar{z})e^{-K(z_{r_1}+\ldots+z_{r_p}) - k(z_{r_{p+1}}+\ldots+z_{r_q})}$ is a.p. in the tube corresponding to $A^1_{+r_1,\ldots,-r_q}$. Since any tube inside the extension of $\mathcal{O}$ can be contained in a tube of this type, $g(\bar{z})$ is a.p. inside this extension, the tube corresponding to $A_{+r_1,\ldots,-r_q}$. Hence $f(\bar{z})$ is a.p. inside this tube.

Theorem 19. If $f(\bar{z})$ is a.p. in some tube $\mathfrak{a}$ and has all its exponents bounded then $f(\bar{z})$ is a.p. inside the tube

$$-\infty < x_1 < \infty, \ldots, -\infty < x_n < \infty$$

Proof: This follows immediately from Theorem 18.

Theorem 20. If $f(\bar{z})$ is a.p. in a tube $\mathfrak{a}$ such that $A = A_{+r}$, and bounded in $\mathfrak{a}$, then it has no positive exponents $\bar{\Lambda}_{r,mr}$.

Proof: Suppose $f(\bar{z}) \sim \sum A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}} \cdot \bar{z}}$

and $|f(\bar{z})| \leq C$ in $\mathfrak{a}$.

Then $$\left|A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}} \cdot \bar{x}}\right| = \left|M_{\bar{y}}\{f(\bar{x}+i\bar{y})e^{-i\bar{\Lambda}_{\bar{m}} \cdot \bar{y}}\}\right| \leq C$$

for fixed values of $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n$ and arbitrarily large values of x_r. This is only possible if $\Lambda_{r,m_r} \leq 0$ for every m_r.

Combining Theorem 17 and Theorem 20 we obtain:

Theorem 21. The class of a.p. functions such that

$$\Lambda_{r,m_r} \leq 0 \qquad \text{for } r = r_1, \ldots, r_p, \quad 1 \leq m_r < \infty$$

$$\Lambda_{r,m_r} \geq 0 \qquad \text{for } r = r_{p+1}, \ldots, r_q, \quad 1 \leq m_r < \infty$$

is identical with the class of functions analytic in some tube $\mathfrak{a}$ such that $A_{+r_1,\ldots,+r_p,-r_{p+1},\ldots,-r_q} = A$, and bounded and a.p. in the $+r_1, \ldots, +r_p, -r_{p+1}, \ldots, -r_q$ extension of any tube inside $\mathfrak{a}$.

The $r_1, \ldots, r_q$ section of a tube $\mathfrak{a}$ is defined to be the set of points $(z'_{r_{q+1}}, \ldots, z'_{r_n})$ where $r_1, \ldots, r_n$ is a rearrangement of $1, \ldots, n$, for which some point $(z_1, \ldots, z_n)$, with

$z_{r_{q+1}} = z'_{r_{q+1}}$, .., $z_{r_n} = z'_{r_n}$, lies in $\mathfrak{a}$. Any section of a tube is also a tube, in a space of fewer dimensions. Clearly, if A is bounded, the r_1, .., r_q section of $\mathfrak{a}$ is actually the section of $\mathfrak{a}_{r_1, .., r_p, -r_{p+1}, .., -r_q}$ by a hyperplane $z_{r_1} =$ K, .., $z_{r_q} = -K$ for all sufficiently large values of K. That is, we can find K_1, such that if $K > K_1$, the set of points $(z'_{r_{q+1}}, \ldots, z'_{r_n})$ such that some point $(z_{r_1}, .., z_{r_q}, z'_{r_{q+1}}, .., z'_{r_n})$ lies in $\mathfrak{a}$ is identical with the set of points $(z'_{r_{q+1}}, .., z'_{r_n})$ such that the point $(z_1, .., z_n)$ with $z_{r_1} = .., = z_{r_p} = -z_{r_{p+1}} = .., = -z_{r_n} = K$, $z_{r_{q+1}} = z^1_{r_{q+1}}$, .., $z_{r_n} = z^1_{r_n}$, lies in $\mathfrak{a}$.

Theorem 21A. If $f(\bar{z})$ is any function of the class described in Theorem 21, then

$$\lim_{\substack{x_{r_i} \to \infty \\ x_{r_j} \to -\infty}} f(\bar{z}) \qquad (i = 1, .., p, \ j = p+1, .., q)$$

exists uniformly in y_{r_1}, .., y_{r_q} and uniformly in $\bar{Z} = (z_{r_{q+1}}, .., z_{r_n})$ (where r_1, .., r_n is a rearrangement of 1, .., n) in $\mathcal{E}$, the r_1, .., r_q section of $\mathfrak{a}$. Moreover, this limit is independent of y_{r_1}, .., y_{r_q} and is a function ~~of~~ $g(\bar{Z})$ of $\bar{Z}$. This function $g(\bar{Z})$ is a.p. inside $\mathcal{E}$, and

$$g(\bar{Z}) \sim \sum{}' A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}} \cdot \bar{Z}}$$

where $\sum'$ indicates a summation taken over all $\bar{m}$ for which

$$\Lambda_{r_i, m_{r_i}} = 0 \qquad (i = 1, .., q)$$

Proof: The Dirichlet series of $f(\bar{z})$ is summable to $f(\bar{z})$ uniformly in the r_1, .., r_p, $-r_{p+1}$, .., $-r_q$ extension of any tube $\mathfrak{a}'$ inside $\mathfrak{a}$. We can find $\{s_N(\bar{z})\}$ such that

$$\Delta_N(\bar{z}) = \sum d^{(N)}_{\bar{m}} A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}}\cdot\bar{z}} \longrightarrow f(\bar{z})$$

uniformly in this extension.

Given $\varepsilon > 0$, there exists N_0 such that

$|f(\bar{z}) - \Delta_N(\bar{z})| \leq \frac{\varepsilon}{4}$ for $N \geq N_0$, $\bar{z}$ in the extension of $\mathfrak{a}'$.

Now $\lim\limits_{\substack{x_{r_i} \to \infty \\ x_{r_j} \to -\infty}} \Delta_N(\bar{z}) = \sum{}' d^{(N)}_{\bar{m}} A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}}\cdot\bar{z}}$ $(i = 1, \ldots, p, j = p+1, \ldots, q)$

Hence we can find A_N such that

$$\left|\Delta_N(\bar{z}) - \sum{}' d^{(N)}_{\bar{m}} A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}}\cdot\bar{z}}\right| \leq \frac{\varepsilon}{4} \text{ for } x_{r_i} > A_N,\ x_{r_j} < -A_N,$$

$(i = 1, \ldots, p ; j = p+1, q)$

Then $\left|f(\bar{z}) - \sum{}' d^{(N)}_{\bar{m}} A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}}\cdot\bar{z}}\right| \leq \frac{\varepsilon}{2}$ for $x_{r_i} > A_N$, $x_{r_j} < -A_N$

$(i = 1, \ldots, p ; j = p+1, \ldots, q)$, $N \geq N_0$ and all $\bar{z}$ in $\mathcal{E}$.

Then $|f(\bar{z}) - f(\bar{z}')| \leq \varepsilon$ for $x_{r_i} > A_{N_0}$, $x'_{r_i} > A_{N_0}$, $x_{r_j} < -A_{N_0}$, $x'_{r_j} < -A_{N_0}$, $(i = 1, \ldots, p ; j = p+1, \ldots, q)$ and all $\bar{z}$ in $\mathcal{E}$.

Hence $\lim\limits_{\substack{x_{r_i} \to \infty \\ x_{r_j} \to -\infty}} f(\bar{z})$ $(i = 1, \ldots, p ; j = p+1, \ldots, q)$ exists uniformly in $y_{r_1}, \ldots, y_{r_q}$, uniformly in $\bar{z}$ in $\mathcal{E}$, and depends only upon $(\bar{z})$. Then, denoting this limit by $g(\bar{z})$

$$\left|g(\bar{z}) - \sum{}' d^{(N)}_{\bar{m}} A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}}\cdot\bar{z}}\right| \leq \frac{\varepsilon}{2} \text{ for } N \geq N_0 \text{ and all } \bar{z} \text{ in } \mathcal{E}.$$

Hence $g(\bar{z})$ is a.p. in $\mathcal{E}$ and

$$g(\bar{z}) \sim \sum{}' A_{\bar{m}} e^{\bar{\Lambda}_{\bar{m}} \cdot \bar{z}}$$

In the particular case when $q = n$, the theorem asserts

$$\lim_{\substack{x_{r_i} \to \infty \\ x_{r_j} \to -\infty}} f(\bar{z}) \qquad (i = 1, \ldots, p \,;\, j = p+1, \ldots, n)$$

is equal to the constant term of the Dirichlet series of $f(\bar{z})$.

<u>Theorem 22.</u> If $f(\bar{z})$ is analytic in a tube $\mathfrak{A}$ ~~and~~ such that $A = A_{+r_1, \ldots, +r_p, -r_{p+1}, \ldots, -r_n}$ where $r_1, \ldots, r_n$ is a rearrangement of $1, 2, \ldots, n$ and if the partial derivatives of $f(\bar{z})$ are a.p. in the $+r_1, \ldots, +r_p, -r_{p+1}, \ldots, -r_n$ extension of any tube inside $\mathfrak{A}$, and moreover

$$\Lambda_{r, m_r} < -\lambda < 0 \qquad \text{for } r = r_1, \ldots, r_p, \quad 1 \leq m_r < \infty$$

$$\Lambda_{r, m_r} > \lambda' > 0 \qquad \text{for } r = r_{p+1}, \ldots, r_n, \quad 1 \leq m_r < \infty$$

for the exponents of each of these derivatives, then $f(\bar{z})$ is a.p. in the $+r_1, \ldots, -r_n$ extension of any tube inside $\mathfrak{A}$.

Proof: By Theorem 17, we need only prove that $f(\bar{z})$ is a.p. inside $\mathfrak{A}$. By Theorem 9, we need only prove it to be bounded inside $\mathfrak{A}$, and, by the Heine-Borel theorem, need only prove it to be bounded in any interval

$$I: \quad a_1 \leq x_1 \leq b_1, \ldots, a_n \leq x_n \leq b_n,$$

inside $\mathfrak{A}$. We prove it to be bounded in J, the $+r_1, \ldots, +r_p, -r_{p+1}, \ldots, -r_n$ extension of I.

For $e^{-\lambda(z_{r_1}+\dots+z_{r_p})-\lambda'(z_{r_{p+1}}+\dots+z_{r_n})} \times \frac{\partial}{\partial z_r} f(\bar{z})$ is bounded in J, by Theorem 17.

Hence, in J, $\left|\frac{\partial}{\partial z_r} f(\bar{z})\right| < K e^{\lambda(x_{r_1}+\dots+x_{r_p})+\lambda'(x_{r_{p+1}}+\dots+x_{r_n})}$

for $r = 1, \dots, n$. It is then easily shown that for fixed values of $z_1, \dots, z_{r-1}, z_{r+1}, \dots, z_n$,

$$\left|\int_{z_r^{(1)}}^{z_r^{(2)}} \frac{\partial}{\partial z_r} f(\bar{z})\, dz_r\right| < 3\frac{K}{\lambda^*} e^{\lambda(a_{r_1}+\dots+a_{r_p})+\lambda'(b_{r_{p+1}}+\dots+b_{r_n})}$$

for $r=1,\dots,n$,

where $(z_1, \dots, z_r^{(1)}, \dots, z_n)$ and $(z_1, \dots, z_r^{(2)}, \dots, z_n)$ lie in J and $\lambda^* = \text{Min}(-\lambda, \lambda')$.

Hence

$$\left|f(\bar{z}) - f(\bar{z}^{(0)})\right| < 3\frac{Kn}{\lambda^*} e^{\lambda(a_{r_1}+\dots+a_{r_p})+\lambda'(b_{r_{p+1}}+\dots+b_{r_n})}$$

for any $\bar{z}$ in J' ($\bar{z}^{(0)}$ is some fixed point in J) so that $f(\bar{z})$ is bounded in J.

The definition of the extension of a set in R_n can be generalized to cover the notion of an extension in any direction, not simply in a direction parallel to one of the coordinate axes.

If M is any set in R_n, the $\bar{\mu}$ extension of M is defined as the set which contains all the line segments

$$x_i = x_i^0 + \mu_i t \quad (i = 1, \dots, n), \qquad t \geq 0$$

originating at points $(x_1^0, \dots, x_n^0)$ of M. This set is denoted by $M_{\bar{\mu}}$. The $\bar{\mu}^{(1)}, \bar{\mu}^{(2)}$ extension of M is defined as the $\bar{\mu}^{(2)}$ extension of the $\bar{\mu}^{(1)}$ extension of M. Extensions of higher order are defined similarly. Multiple extensions of M are independent of the order in which the extensions are taken, e.g. $M_{\bar{\mu}^{(1)},\bar{\mu}^{(2)}} = M_{\bar{\mu}^{(2)},\bar{\mu}^{(1)}}$. If M is open or convex the extensions of M are open or convex. The tube in $(z_1, .., z_n)$ space corresponding to the $\bar{\mu}$ extension of a set A is denoted by $\mathfrak{A}_{\bar{\mu}}$ and called the $\bar{\mu}$ extension of $\mathfrak{A}$.

Theorem 23. If $f(\bar{z})$ is a.p. inside a tube $\mathfrak{A}$, there exist q vectors $\bar{\mu}^{(1)}, .., \bar{\mu}^{(q)}$ such that

$$\bar{\mu}^{(i)}\bar{\Lambda}_{\bar{m}} \leq 0 \qquad i = 1, .., q$$

for all exponent vectors $\bar{\Lambda}_{\bar{m}}$, and $\mathfrak{A}'$ is a tube inside $\mathfrak{A}$, then $f(\bar{z})$ can be continued analytically into a function a.p. and bounded in the $\bar{\mu}^{(1)}, .., \bar{\mu}^{(q)}$ extension of $\mathfrak{A}'$.

Proof: We prove the theorem first in the important special case when $\bar{\mu}^{(1)}, .., \bar{\mu}^{(q)}$ are linearly independent. Then $q \leq n$, and we can find $\bar{\mu}^{(q+1)}, .., \bar{\mu}^{(n)}$ so that $\bar{\mu}^{(1)}, .., \bar{\mu}^{(n)}$ are linearly independent.

We make the change of variable

$$z_i = \sum_{j=1}^{n} \mu_i^{(j)} w_j \qquad (i = 1, .., n)$$

Then $g(\bar{w}) = f(\sum_{j=1}^{n}\mu_1^{(j)} w_j, \ldots\ldots\ldots, \sum_{j=1}^{n}\mu_n^{(j)} w_j)$ is a.p. inside the tube $\mathfrak{A}_1$ in $\bar{w}$ space into which $\mathfrak{A}$ is carried by the transformation. $\mathfrak{A}'$ is carried into $\mathfrak{A}_1'$ inside $\mathfrak{A}$, and the $\bar{\mu}^{(1)}, .., \bar{\mu}^{(q)}$ extension of $\mathfrak{A}'$ is carried into the $+1,.., +q$ extension of $\mathfrak{A}_1'$. By Theorem 17, $g(\bar{w})$ can be continued analytically into a function a.p. and bounded in this extension. Hence $f(\bar{z})$ can be continued analytically into a function a.p. and bounded in the $\bar{\mu}^{(1)}, .., \bar{\mu}^{(q)}$ extension of $\mathfrak{A}$.

To extend the proof to the general case, we note that the condition $\bar{\mu}\cdot\bar{\Lambda} \leq 0$

requires that the exponent vector $\bar{\Lambda}$ lie in the half-space on one side of an (n-1) space through the origin. If the q half-spaces of this kind have no common point except the origin, then the only possible exponent vector is (0,0,..,0) and $f(\bar{z})$ is a constant. Otherwise, some vector $-\bar{\alpha}$ lies in all these half-spaces, whence all the vectors $\bar{\mu}$ lie within an angle $\frac{\pi}{2}$ of $\bar{\alpha}$. We need therefore only prove the following lemma.

Lemma: If each of a finite set S of vectors in R_n makes an angle $\leq \frac{\pi}{2}$ with a vector $\bar{\alpha}$, the extension of any set M by means of the vectors of S is the sum of the $\bar{\mu}^{(1)}, .., \bar{\mu}^{(p)}$ extensions of M, where $\bar{\mu}^{(1)}, .., \bar{\mu}^{(p)}$ is any linearly independent set of vectors from S .

Proof: The theorem will be proved for any set M if it is proved for a single point, which we may take to be the origin. We transform the theorem into a theorem on point sets by considering the section of the S - extension of the origin by the hyperplane H: $\bar{\alpha}\cdot\bar{x} = 1$

The vectors of S go into points in H , some of which may be at infinity, a vector extension goes into the convex closure of the points corresponding to the vectors, the sets of linearly independent vectors go into sets of linearly independent points. Hence we need only prove:

The convex closure of any finite set S of points (some of which may be at infinity) in Euclidean n-space is the sum of the convex closures of all linearly independent sets of points in S.

We prove this by induction. If there are q points in S of which a is one, the convex closure of S is the convex closure of a and the set C which is the convex closure of the remaining (q-1) points of S . Now this convex closure C is a convex polytope of at most n dimensions, and its faces are convex polytopes of at most (n-1) dimensions, whose vertices are points of S . Any point p which lies on a line segment joining α to a point of C lies on a line segment joining α to a point of some face of C . By the induction hypothesis, this last point lies in the convex closure of a linearly independent set of points of S lying in this face. If α is linearly dependent upon these points, then p lies in an (n-1) space through the face, and, by the induction hypothesis, p lies in the convex closure of a linearly independent set of points of S . If α is linearly independent of these points we adjoin it to them, and p lies in the convex closure of the linearly independent set of points of S thus obtained.

It is clear from the proof of Theorem 2 , that if a vector lies inside the S - extension of the origin, it lies inside the $\bar{\mu}^{(1)}, \ldots, \bar{\mu}^{(p)}$ extension of the origin, where $\bar{\mu}^{(1)}, \ldots, \bar{\mu}^{(p)}$ is some maximum linearly independent set of vectors from S. The corresponding vector in $\bar{u}$ space lies inside the $+1, \ldots, +p$ extension of the origin. Hence, as $\bar{u} \to \infty$ along any line in the same direction as this vector, $u_1 \to \infty, \ldots\ldots, u_p \to \infty$. Hence, in the special case when S contains n linearly independent vectors, we can apply Theorem 21 A to the function $g(\bar{w})$ to obtain:

<u>Theorem 24.</u> If $f(\bar{z})$ is a.p. inside a tube $\mathfrak{a}$, there exist q vectors $\bar{\mu}^{(1)}, \ldots, \bar{\mu}^{(q)}$ (containing n linearly independent vectors) such that

$$\bar{\mu}^{(i)}\bar{\Lambda}_{\bar{m}} \leq 0 \qquad (i = 1, \ldots, q)$$

for all exponent vectors $\bar{\Lambda}_{\bar{m}}$, $\bar{a}$ is a point of A, and $\bar{\mu}$ lies in the $\bar{\mu}^{(1)}, \ldots, \bar{\mu}^{(q)}$ extension of the origin, then

$$f(\bar{a} + \bar{\mu}t + i\bar{y})$$

tends to the constant term of the Dirichlet series of $f(\bar{z})$ as $t \to \infty$, uniformly in $\bar{y}$, uniformly in $\bar{a}$ when $\bar{a}$ ranges over some bounded subset of A.

__Theorem 25.__ If $f(\bar{z})$ is a.p. inside a tube $\mathfrak{a}$, and

$$\bar{\mu}^{(i)}\bar{\Lambda}_{\bar{m}} < K \qquad (i = 1, \ldots, q)$$

for all exponent vectors $\bar{\Lambda}_{\bar{m}}$ then $f(\bar{z})$ is a.p. inside the $\bar{\mu}^{(1)}, \ldots, \bar{\mu}^{(q)}$ extensions of $\mathfrak{a}$.

Proof: Take $\bar{\alpha}$ such that $\bar{\mu}^{(i)} \cdot \bar{\alpha} > K$ for $i = 1, \ldots, q$. Then $g(\bar{z}) = f(\bar{z})\, e^{-\bar{\alpha}\cdot\bar{z}}$ is a.p. inside the $\bar{\mu}^{(1)}, \ldots, \bar{\mu}^{(q)}$ extension of $\mathfrak{a}$, by Theorem 22. Hence $f(\bar{z})$ is also a.p. inside this tube.

__Theorem 26.__ If $f(\bar{z})$ is a.p. in a tube $\mathfrak{a}$ which is identical with its $\bar{\mu}$ extension, and bounded in $\mathfrak{a}$, then

$$\bar{\mu} \cdot \bar{\Lambda}_{\bar{m}} \leq 0$$

for all the exponent vectors $\bar{\Lambda}_{\bar{m}}$ of $f(\bar{z})$.

Proof: As in Theorem 22, we transform $f(\bar{z})$ to a function $g(\bar{w})$. Applying Theorem 20 to this function, we obtain the theorem.

Combining Theorem 22 and Theorem 25, we obtain:

__Theorem 27.__ The class of a.p. functions such that

$$\bar{\mu}^{(i)} \cdot \bar{\Lambda}_{\bar{m}} \leq 0 \qquad (i = 1, \ldots, q)$$

is identical with the class of functions analytic in some tube $\mathfrak{a}$ which coincides with its $\bar{\mu}^{(1)}, \ldots, \bar{\mu}^{(q)}$ extension, and bounded and a.p. in the $\bar{\mu}^{(1)}, \ldots, \bar{\mu}^{(q)}$ extension of any tube $\mathfrak{a}'$ inside $\mathfrak{a}$.