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Doctoral Thesis

Distribution of Mass of Holomorphic Hecke Cusp Forms and Quantum Chaos

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Peter Zenz

Department of Mathematics and Statistics,

McGill University, Montréal,

Québec, Canada

Supervisors:

Maksym Radziwiłł and Dimitris Koukoulopoulos

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Abstract

Arithmetic Quantum Chaos is a central problem at the intersection of number theory and physics. One of the main goals in Arithmetic Quantum Chaos is to study the distribution of mass of automorphic forms on arithmetic hyperbolic surfaces. In this thesis we investigate the distribution of mass of holomorphic Hecke cusp forms in certain regions. As a first result, we show that the fourth moment of holomorphic Hecke cusp forms is bounded, assuming the Generalized Riemann Hypothesis. This work relies on the seminal work of Soundararajan and its extension by Harper on obtaining sharp bounds for the moments of the Riemann zeta function on the critical line. The second result is concerned about the mass distribution of holomorphic cusp forms restricted to a special one-dimensional subset of the fundamental domain. More precisely, we compute the so-called quantum variance of holomorphic Hecke cusp forms for smooth compactly-supported test functions on the vertical geodesic. We conclude the thesis by providing a short outlook for evaluating the sixth moment of cusp forms on average.

Abrégé

Le Chaos Quantique Arithmétique est un problème central à l'intersection de la théorie des nombres et de la physique. L'un de ses principaux objectifs est d'étudier la distribution de la masse des formes automorphes sur des surfaces hyperboliques arithmétiques. Dans cette thèse, nous étudions la distribution de masse des formes cuspidales de Hecke holomorphes dans certaines régions. Comme premier résultat, nous montrons que le quatrième moment des formes cuspidales de Hecke holomorphes est borné, en supposant l'hypothèse de Riemann généralisée. Ce travail s'appuie sur le résultat séminal de Soundararajan, et son extension par Harper, sur l'obtention de bornes optimales pour les moments de la fonction zêta de Riemann sur la ligne critique. Le deuxième résultat concerne la distribution de masse des formes cuspidales holomorphes restreintes à un sous-ensemble spécial unidimensionnel du domaine fondamental. Plus précisément, nous calculons la variance dite quantique des formes cuspidales de Hecke holomorphes pour les fonctions de test lisses à support compact sur la géodésique verticale. Nous concluons la thèse en donnant un bref aperçu de l'évaluation du sixième moment des formes cuspidiennes en moyenne.

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Contribution to knowledge

In Chapter 1 of this thesis we recall standard information about automorphic forms and L-functions, that will be required in the subsequent chapters of this thesis. We follow closely the notation and exposition of [IK04].

In Chapter 2 we recall basic ideas and problems in the world of Arithmetic Quantum Chaos. This information can also be found in the articles [Zel17], [Dya22], [Mar05], [Sar95], [Has11], which we used as a reference.

At the date of initial thesis submission, April 11, 2022, Chapter 3 was taken, up to the introduction and minor corrections, verbatim from our submitted article [Zen21b]. We are referring here to the version uploaded to the arXiv on August 31, 2021. The article was submitted to a journal and was at the date of initial thesis submission under review. The oral defense of this thesis was conducted on June 10, 2022. The final version of the article [Zen22] was accepted by the journal *International Mathematics Research Notices* on June 26, 2022 and published on July 28, 2022. The published version can be reached with the url `academic.oup.com/imrn/advance-article-abstract/doi/10.1093/imrn/rnac199/6650962`.

We note that the published article as well as the arXiv version differ slightly from the part used in the final version of this thesis, due to the timing of approval of the thesis, the approval of the final published article and the corresponding corrections. The article constitutes original work of the author of this thesis. We would like to acknowledge the continuous support and helpful discussions with Dimitris Koukoulopoulos and Maksym Radzwiłł, while

preparing the article. Furthermore we would also like to acknowledge helpful discussions with Andrei Shubin.

Chapter 4 is up to the introduction and minor corrections (like typographical errors and adjustment of notation) taken verbatim from the preprint article [Zen21a]. We are referring here to the version uploaded to the arXiv on November 8, 2021. This article constitutes original work of the author of this thesis. Again, we acknowledge the continuous support and helpful discussions with Maksym Radziwiłł and Dimitris Koukoulopoulos, while preparing the article.

The final chapter comprises ongoing original work on the sixth moment of holomorphic Hecke cusp forms.

Contents

Chapter 1

Introduction

1.1 Quantum Chaos

Quantum Chaos is concerned about the behaviour of Laplace eigenfunctions of dynamical quantum systems that are "chaotic" in nature. Very little can be proved in full generality but in certain number theoretic special cases, additional symmetries allow us to get a better understanding. One of the major goals of Quantum Chaos is to study the mass distribution of Laplace eigenfunctions on hyperbolic surfaces in the large eigenvalue limit. From a physical perspective this is analogous to asking for the likelihood of finding a quantum particle with large energy in a certain region of the surface. In the large energy limit the dynamics of the quantum particle should be reflected by the underlying nature of the classical dynamical system. If the underlying dynamical system is chaotic (essentially the trajectory of a particle is very sensitive to its initial conditions), then we expect that the probability of finding the particle in a specified region is proportional to the volume. In terms of eigenfunctions φ_j of the Laplace operator we then say that the L^2-mass of φ_j is *equidistributed* (see Chapter 2 for a more precise description). In this thesis we are interested in the distribution of mass of holomorphic Hecke cusp form on the fundamental domain $X := \Gamma\backslash\mathbb{H}$, where $\Gamma := \mathrm{SL}_2(\mathbb{Z})$

is the set of integer matrices with determinant 1 and $\mathbb{H}$ denotes the usual upper half-plane. The modular surface X is chaotic in nature and we expect similar equidistribution behaviour for our holomorphic objects of interest.

1.2 Thesis Outline

In the remaining part of the first chapter we introduce and define our mathematical objects that we use throughout the thesis.

In Chapter 2 we will elaborate on several important questions arising in the realm of Arithmetic Quantum Chaos, like the so-called *Quantum Unique Ergodicity* conjecture, the *Random Wave Conjecture* and the *Quantum Variance*. We will also state our main theorems and show how they fit into the above mentioned set of problems.

In Chapter 3 we prove our first main result: a sharp bound for the fourth moment of holomorphic Hecke cusp forms (see Theorem 2.2.3). This result was obtained in our work [Zen21b] and can be seen as progress toward the holomorphic Random Wave Conjecture (see Conjecture 2.2.2).

Chapter 4 is comprised of our second work [Zen21a]. The main theorem is a variance computation for holomorphic Hecke cusp forms on the vertical geodesic, i.e. the line connecting zero and infinity on the upper half-plane (see Theorem 2.3.2).

We conclude the thesis with Chapter 5, by discussing ongoing work on the sixth moment of holomorphic Hecke cusp forms on average.

1.3 General Notation

Throughout this thesis, we write $f(x) = O(g(x))$ or equivalently $f(x) \ll g(x)$ (or $g(x) \gg f(x)$) if there exists an absolute constant $C > 0$, such that $|f(x)| \leq C|g(x)|$ for all x sufficiently large. The asymptotic equivalence $f(x) \sim g(x)$ means that $g(x) \neq 0$ for sufficiently large x and $\lim_{x\to\infty} f(x)/g(x) = 1$. We write $f(x) \asymp g(x)$, if $f(x) \ll g(x)$ and $g(x) \gg f(x)$. The notation $f(x) \approx g(x)$ should only be interpreted informally and indicates that $f(x)$ and $g(x)$ are roughly the same (up to some technical factors). The indicator function 1_P will equal 1 if the statement P is true and 0 if it is false.

1.4 Holomorphic Cusp Forms

Let $\mathbb{H} := \{z = x + iy | x \in \mathbb{R}, y \in \mathbb{R}^+\}$ denote the usual upper half-plane and $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ denote the full modular group, i.e. the set of matrices

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

The modular group Γ acts on $\mathbb{H}$ via Möbius transformations:

$$g \cdot z := \frac{az + b}{cz + d} \quad \text{with} \quad g \in \Gamma, z \in \mathbb{H}.$$

For a function $f\colon \mathbb{H} \to \mathbb{C}$ and an integer $k \geq 1$ we have the action

$$(f|_k\ g)(z) = j(g, z)^{-k} f(gz), \quad \text{where } j(g, z) = cz + d \text{ with } g \in \Gamma.$$

We say that a function f on $\mathbb{H}$ is Γ-invariant if $(f|_k\ g)(z) = f(z)$ for $g \in \Gamma$. Note that $(f|_k\ \gamma)(z) = f(z)$ for $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ implies that $f(z + 1) = f(z)$.

A holomorphic modular form of weight k is a holomorphic function f on $\mathbb{H}$, such that

$$(f|_k\ g)(z) = f(z)$$

and f is holomorphic at ∞ (see [IK04, p.356] for a precise definition of this notion). A holomorphic modular form that vanishes at all cusps is called a *cusp form*. The space of holomorphic cusp forms, denoted by S_k, is a finite dimensional vector space and is endowed with a natural Hilbert space structure via the Petersson inner product. For $z = x + iy$ and $f, g \in S_k$ we define

$$\langle f, g \rangle := \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}.$$

For each $f \in S_k$ we have the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n)(4\pi n)^{(k-1)/2} e(nz),$$

where $e(x) := e^{2\pi i x}$. For the rest of the thesis we will be mostly interested in L^2-normalized cusp forms, i.e. cusp forms f of weight k such that $\langle f, f \rangle = 1$.

1.5 Maass Forms and Eisenstein series

We call a function $f : \mathbb{H} \to \mathbb{C}$ *automorphic with respect to* Γ if it is invariant under the group action, i.e.

$$f(g \cdot z) = f(z) \quad \text{for all } g \in \Gamma.$$

We denote the space of automorphic functions by $\mathcal{A}(\Gamma\backslash\mathbb{H})$. Let $\Delta := -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ denote the hyperbolic Laplace operator. Automorphic functions that are eigenfunctions of the hyperbolic Laplace operator are called automorphic forms, and the space of automorphic forms is denoted by

$$\mathcal{A}_s(\Gamma\backslash\mathbb{H}) = \{f \in \mathcal{A}(\Gamma\backslash\mathbb{H}) \mid \Delta f = \lambda f\},$$

with $\lambda = s(1-s)$.

Important examples for automorphic forms are given by Maass cusp forms. Maass cusp forms are L^2-integrable, i.e. $\int_{\Gamma\backslash\mathbb{H}} |f(z)|^2 \frac{dxdy}{y^2} < \infty$, satisfy an additional growth condition at infinity and their eigenvalue is given by $\lambda = 1/4 + R^2$, where $R > 0$ denotes the spectral parameter. Similarly to holomorphic cusp on $\Gamma\backslash\mathbb{H}$, Maass cusp forms admit a Fourier expansion, given by

$$f(z) = \sqrt{y}\sum_{n\neq 0} a_f(n) K_{iR}(2\pi|n|y)e(nx),$$

where $\Delta f = (1/4 + R^2)f$ and $K_s(y)$ denotes the K-Bessel function.

Eisenstein series (see [IK04, Chapter 15.4]) are also important examples of automorphic forms.

1.6 Hecke Operators

Establishing rigorous theorems in the world of quantum chaos is often very difficult. In number theoretic special cases, additional symmetries allow us to get a better understanding. These symmetries are given by the so-called Hecke operators. These linear operators act on the space of holomorphic modular forms (they can also be defined for Maass forms) and are explicitly defined by

$$T(n)f(z) = \frac{1}{n^{\frac{k+1}{2}}} \sum_{ad=n} a^k \sum_{b\,(\mathrm{mod}\,d)} f\Big(\frac{az+b}{d}\Big).$$

Hecke operators are commutative, i.e.

$$T(m)T(n) = \sum_{d|(m,n)} T(mn/d^2),$$

and self-adjoint with respect to the Petersson inner product (see [IK04, Prop. 14.9, Lem. 14.10]). Consequently, there exists an orthonormal basis of the space of cusp forms S_k, which consists of eigenfunctions of all the Hecke operators $T(n)$. We call such a basis a *Hecke basis.*

A Hecke cusp form, i.e. a cusp form that is an eigenfunction with respect to all Hecke operators, satisfies the relation $T(n)f = \lambda_f(n)f$ for all $n \geq 1$. The Fourier expansion of a

Hecke cusp form of weight k, is given by

$$f(z) = a_f(1) \sum_{n=1}^{\infty} \lambda_f(n)(4\pi n)^{(k-1)/2} e(nz),$$

with

$$|a_f(1)|^2 = \frac{2\pi^2}{\Gamma(k) L(1, \operatorname{sym}^2 f)}.$$

The constant $a_f(1)$ arises from the normalization

$$\int_X |f(z)|^2 y^k \frac{dxdy}{y^2} = 1.$$

The Fourier coefficients of a cusp form satisfy the following quasi-orthogonality relations (see [IK04, Proposition 14.5]):

Lemma 1.6.1 (Petersson Trace formula)**.** *Let B_k be a Hecke basis of weight k cusp forms. For any positive numbers n, m we have*

$$\frac{\zeta(2)}{(k-1)/12} \sum_{g \in B_k} \frac{\lambda_g(n)\overline{\lambda_g}(m)}{L(1, \operatorname{sym}^2 g)} = 1_{n=m} + 2\pi i^{-k} \sum_{c=1}^{\infty} \frac{S(m, n; c)}{c} J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big).$$

Here $L(1, \operatorname{sym}^2 f)$ is the symmetric square L-function defined in (1.3)*, $S(m, n; c)$ denotes the classical Kloosterman sum and $J_y(x)$ denotes the J-Bessel function of order y.*

In this thesis we will use a modified version of the Petersson Trace formula for a product of primes (see Lemma 3.2.2) in Chapter 3 and an averaged version due to Iwaniec, Luo and Sarnak (see Lemma 4.8) in Chapter 4.

1.7 L-functions

Next we recall some basic quantities in the theory of L-functions. We follow closely the exposition in [IK04]. We call $L(s, f)$ an L-function of degree $d \geq 1$ if we have an Euler product representation

$$L(s,f)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s}=\prod_p\left(1-\frac{\alpha_1(p)}{p^s}\right)^{-1}\cdots\left(1-\frac{\alpha_d(p)}{p^s}\right)^{-1},$$

with $\lambda_f(1)=1$, $\lambda_f(n)\in\mathbb{C}$ and $\alpha_i\in\mathbb{C}$. Moreover, the *gamma factor* is given by

$$L_\infty(s,f)=\prod_{j=1}^{d}\Gamma_{\mathbb{R}}(s+\kappa_j),$$

where $\Gamma_{\mathbb{R}}(s):=\pi^{-s/2}\Gamma(s/2)$ and $\kappa_j\in\mathbb{C}$. The complex numbers α_i for $1\le i\le d$, and κ_j for $1\le j\le d$ are called local parameters of $L(s,f)$ at the prime p and at infinity, respectively.

Together with the conductor $q(f)$ of $L(s,f)$, which is an integer $q(f)\ge 1$, such that $\alpha_i(p)\neq 0$ for $p\nmid q(f)$ and $1\le i\le d$, we can form the so-called *completed L-function* $\Lambda(s,f)$:

$$\Lambda(s,f):=q(f)^{s/2}L_\infty(s,f)L(s,f).$$

The completed L-functions satisfies the functional equation

$$\Lambda(s,f)=\varepsilon(f)\Lambda(1-s,\overline{f}),$$

where $\varepsilon(f)\in\mathbb{C}$, such that $|\varepsilon(f)|=1$ and $\overline{f}$ is the dual of f for which $\lambda_{\overline{f}}(n)=\overline{\lambda}_f(n)$, $L_\infty(s,\overline{f})=L_\infty(s,f)$ and $q(\overline{f})=q(f)$.

An important quantity that measures the complexity of L-functions is the *analytic conductor* of $L(s,f)$, which we define by

$$\mathcal{C}(s,f):=q(f)\prod_{j=1}^{d}(|s+\kappa_j|+3).$$

Remark 1.7.1. We say that the analytic conductor measures the complexity of L-functions because one can express an L-function as two partial sums of length roughly $\mathcal{C}(s,f)^{1/2}$

through the *approximate functional equation* (see [IK04, Theorem 5.3]).

The so-called *convexity bound* of L-functions says that

$$L(s,f) \ll \mathcal{C}(s,f)^{1/4+\varepsilon}, \tag{1.1}$$

for $\mathrm{Re}(s) = 1/2$. This can be easily shown by the Phragmén–Lindelöf principle or the approximate functional equation (see [IK04, Eq. 5.20]).

Remark 1.7.2. Heath-Brown showed in [HB09] that the ε in (1.1) is superfluous i.e. he proved $L(s,f) \ll \mathcal{C}(s,f)^{1/4}$ for $\mathrm{Re}(s) = 1/2$.

A major theme in analytic number theory is to "break" the convexity barrier (1.1) and to show that

$$L(s,f) \ll \mathcal{C}(s,f)^{1/4-\delta} \tag{1.2}$$

for some $\delta > 0$ and $\mathrm{Re}(s) = 1/2$. A bound of the form (1.2) is termed *subconvexity bound* and has often far reaching consequences in number theory. The *Arithmetic Quantum Unique Ergodicity Conjecture* for example, on which we will elaborate more in Chapter 2, would follow from a subconvexity bound for a special degree 8 L-function.

The *Generalized Lindelöf Hypothesis* (GLH) would show that $L(s,f) \ll \mathcal{C}(s,f)^{\varepsilon}$, for $\mathrm{Re}(s) = 1/2$.

1.7.1 Specific L-functions

We gather now more concrete information of several L-functions that arise in our study of the distribution of mass of holomorphic cusp forms. We recall standard information as also stated in our work [Zen21b, Section 3].

Let f be a Hecke cusp form for $\mathrm{SL}_2(\mathbb{Z})$ of weight k. Furthermore, let $\lambda_f(n)$ denote the

n-th Hecke eigenvalue of f. The associated L-function of degree 2 to f is given by

$$L(s,f)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s}=\prod_p\Big(1-\frac{\alpha_f(p)}{p^s}\Big)^{-1}\Big(1-\frac{\beta_f(p)}{p^s}\Big)^{-1},$$

where $\alpha_f(p),\beta_f(p)=\overline{\alpha_f(p)}$ are complex numbers of absolute value 1. Since $\lambda_f(p)=\alpha_f(p)+\beta_f(p)$, we have $|\lambda_f(p)|\leq 2$, to which we refer as the Deligne bound. The gamma factor is given by

$$L_\infty(s,f)=\Gamma_{\mathbb{R}}\Big(s+\frac{k-1}{2}\Big)\Gamma_{\mathbb{R}}\Big(s+1+\frac{k-1}{2}\Big).$$

The analytic conductor is consequently of size

$$\mathcal{C}(s,f)\asymp(|s|+k+3)^2.$$

We are mostly interested in the weight aspect of the analytic conductor (and s fixed) and thus we may also write $\mathcal{C}(f)\ll_s k^2$, where the implied constant depends on s.

For $\mathrm{Re}(s)>1$ we define the symmetric square L-function associated to our Hecke cusp form f by the following Euler product:

$$L(s,\mathrm{sym}^2 f)=\prod_p\Big(1-\frac{\alpha_f^2(p)}{p^s}\Big)^{-1}\Big(1-\frac{1}{p^s}\Big)^{-1}\Big(1-\frac{\beta_f^2(p)}{p^s}\Big)^{-1}. \tag{1.3}$$

The associated gamma factor is given by

$$L_\infty(s,\mathrm{sym}^2 f)=\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{R}}(s+k-1)\Gamma_{\mathbb{R}}(s+k).$$

It follows that the analytic conductor of the symmetric square L-function is of size

$$\mathcal{C}(s,\mathrm{sym}^2 f)\asymp(|s|+3)(|s|+k+3)^2.$$

$L(s,\mathrm{sym}^2 f)$ is entire, can be analytically continued to the entire complex plane and satisfies the functional equation

$$L_\infty(s,\mathrm{sym}^2 f)L(s,\mathrm{sym}^2 f) = L_\infty(1-s,\mathrm{sym}^2 f)L(1-s,\mathrm{sym}^2 f),$$

as proved in work of Shimura [Shi75].

Next we define a so-called triple product L-function. Let g be another Hecke cusp form of weight $2k$ with Hecke eigenvalues $\lambda_g(n)$. In terms of the local parameters we can express the Hecke eigenvalues at primes again as $\lambda_g(p) = \alpha_g(p) + \beta_g(p)$. Then

$$\begin{aligned} L(s, f\times f\times g) = \prod_p &\Big(1-\frac{\alpha_f(p)^2\alpha_g(p)}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_g(p)}{p^s}\Big)^{-2}\Big(1-\frac{\beta_f(p)^2\alpha_g(p)}{p^s}\Big)^{-1} \\ &\times\Big(1-\frac{\alpha_f(p)^2\beta_g(p)}{p^s}\Big)^{-1}\Big(1-\frac{\beta_g(p)}{p^s}\Big)^{-2}\Big(1-\frac{\beta_f(p)^2\beta_g(p)}{p^s}\Big)^{-1}, \end{aligned}$$

with gamma factor

$$\begin{aligned} L_\infty(s, f\times f\times g) =&\Gamma_{\mathbb{R}}(s+2k-3/2)\Gamma_{\mathbb{R}}(s+2k-1/2)\Gamma_{\mathbb{R}}(s+k-1/2)^2 \\ &\times\Gamma_{\mathbb{R}}(s+k+1/2)^2\Gamma_{\mathbb{R}}(s+1/2)\Gamma_{\mathbb{R}}(s+3/2), \end{aligned} \tag{1.4}$$

is the degree 8 triple product L-function of interest. Garrett [Gar87] showed that the completed L-function is entire, extends analytically to the entire complex plane and satisfies the functional equation

$$L_\infty(s, f\times f\times g)L(s, f\times f\times g) = L_\infty(1-s, f\times f\times g)L(1-s, f\times f\times g).$$

The analytic conductor is of size

$$\mathcal{C}(s, f\times f\times g) \asymp (|s|+3)^2(|s|+k+3)^6,$$

or, when s is fixed, $\mathcal{C}(f\times f\times g) \ll_s k^6$.

Chapter 2

Arithmetic Quantum Chaos

Quantum Chaos is concerned about the behaviour of Laplace eigenfunctions of dynamical quantum systems that are chaotic in nature. As mentioned in the introduction, one of the major goals is to study the mass distribution of automorphic forms. We will now describe several results and open questions in that regard.

2.1 Equidistribution Results

Let $\mathcal{M}$ be a smooth compact Riemannian manifold. Moreover, let Δ denote the Laplace–Beltrami operator on $\mathcal{M}$ and let φ_j be the corresponding L^2-normalized eigenfunctions with eigenvalue λ_j, i.e.

$$\Delta\varphi_j = \lambda_j\varphi_j \quad \text{with} \quad ||\varphi_j||_2^2 = 1.$$

We denote by $d\mu$ the volume measure on $\mathcal{M}$ and define the probability measures

$$d\mu_j := |\varphi_j|^2 d\mu.$$

We say that a subsequence φ_{j_k} equidistributes if the measures $d\mu_{j_k}$ converge weakly to $\frac{1}{\mathrm{Vol}(\mathcal{M})}d\mu$ as $k \to \infty$, i.e for all $\psi \in C^\infty(\mathcal{M})$

$$\int_{\mathcal{M}} \psi(z)|\varphi_{j_k}|^2 d\mu \to \frac{1}{\mathrm{Vol}(\mathcal{M})}\int_{\mathcal{M}} \psi(z) d\mu$$

as $k \to \infty$.

Remark 2.1.1. We will be especially interested in the modular surface $X = \Gamma\backslash\mathbb{H}$, which is non-compact but of finite volume. For a sequence of L^2-normalized Maass forms φ_j we then define the probability measures

$$d\mu_j := |\varphi_j|^2 \frac{dxdy}{y^2},$$

where $d\mu = \frac{dxdy}{y^2}$ is the volume measure on X.

A famous result due to Shnirelman [Š74], Zelditch [Zel87] and Colin de Verdiere [CdV85] shows that if the geodesic flow on a compact manifold is ergodic then there exists a density one subsequence of Laplace eigenfunctions that equidistributes. A result like this is known as "Quantum Ergodicity". Zelditch showed in [Zel92] the quantum ergodicity result for Maass forms on the modular surface X.

Having established a quantum ergodicity result, it is natural to ask whether *every* subsequence of Laplace eigenfunctions equidistributes. The so-called *Quantum Unique Ergodicity Conjecture*, introduced by Rudnick and Sarnak in [RS94], predicts that on negatively curved hyperbolic surfaces every sequence of eigenfunctions converges to the uniform distribution in the large eigenvalue limit.

Conjecture 2.1.1 (Quantum Unique Ergodicity, compact case)**.** *Let $\mathcal{M}$ be a compact manifold of negative curvature. Then the measures $d\mu_j$ converge weakly to $d\mu$ as $j \to \infty$.*

The Arithmetic Quantum Unique Ergodicity conjecture, i.e. the special case were additional structure from number theory is present, was solved by Lindenstrauss [Lin06] in the compact case and for the full fundamental domain X with an additional argument by

Soundararajan [Sou10]. Prior to that, Quantum Unique Ergodicty was also shown for Eisenstein series by work of Luo and Sarnak [LS95] and by Jakobson [Jak94].

It is natural to consider similar equidistribution questions for L^2-normalized holomorphic Hecke cusp forms f of weight k on X. To do so we define the probability measures

$$d\mu_f := |f(z)|^2 y^k \frac{dxdy}{y^2}.$$

Holowinsky and Soundararajan showed in [HS10] that the Quantum Unique Ergodicity conjecture holds for holomorphic Hecke cusp forms, i.e. $d\mu_f \to \frac{3}{\pi} d\mu$ as $k \to \infty$.

2.2 Random Wave Conjecture

Another important problem in the realm of AQC is to study L^p-norms of eigenfunctions, thus measuring additional aspects of the mass distribution of eigenfunctions. One of the major conjectures in this regard is the *Gaussian Moment Conjecture*, which is a particular instance of Berry's *Random Wave Conjecture* [Ber77]. The Random Wave Conjecture predicts that eigenfunctions in the large eigenvalue limit should behave as *random waves*. The notion of a random wave is not well defined but for the sake of exposition we will think of a random wave, as in work of Hejhal and Rackner [HR92], as a function on X given by

$$\Psi(x+iy) = \sum_{n=1}^{\infty} c_n \sqrt{y} K_{iR}(2\pi n y) \cos(2\pi n x), \tag{2.1}$$

where the coefficients c_n are chosen at random, with uniform distribution in $[-1, 1]$. The above notion is helpful from a conceptional point of view, as we can compare it with our deterministic objects, (even) Hecke Maass cusp forms f with spectral parameter R, whose

Fourier expansion is given by

$$f(x+iy) = C\sum_{n=1}^{\infty} \lambda_f(n)\sqrt{y}K_{iR}(2\pi ny)\cos(2\pi nx),$$

where $\lambda_f(n)$ denote the Hecke eigenvalues of f and C is a normalization constant.

Remark 2.2.1. An alternative definition of a random wave is given in work of Zelditch [Zel09]. For an orthornomal basis $\{\varphi_{\lambda_j}\}$ of Laplace eigenfunctions he defines Gaussian ensembles of random functions

$$f_\lambda = \sum_{j:\lambda_j\in\Lambda_\lambda} c_j\varphi_{\lambda_j}$$

where Λ_λ is either the interval $[\lambda, \lambda+1]$ or $[0, \lambda]$ and c_j are independent Gaussian random variables with mean 0 and properly normalized variance.

The *Gaussian Moment Conjecture* is a particular instance of the Random Wave Conjecture that predicts that the moments of a Hecke Maass cusp forms agree with the moments of a Gaussian random variable. More precisely, as in work of Humphries [Hum18]:

Conjecture 2.2.1 (Gaussian Moment Conjecture)**.** *Let B be any fixed compact set of $X = \Gamma\backslash\mathbb{H}$, so that the boundary of B has μ-measure zero, and let g be a Hecke-Maass eigenform with eigenvalue $\lambda = 1/4 + R^2$, normalized such that $\int_X |g(z)|^2 d\mu(z) = 1$. Then for every nonnegative integer n,*

$$\frac{1}{\mathrm{Var}_B(g)^{n/2}\,\mathrm{Vol}(B)}\int_B g(z)^n d\mu(z)$$

converges to

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}}dx = \begin{cases} \frac{2^{n/2}}{\sqrt{\pi}}\Gamma\left(\frac{n+1}{2}\right) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

as R (the spectral parameter) tends to infinity. Here

$$\mathrm{Var}_B(g) := \frac{1}{\mathrm{Vol}(B)}\int_B |g(z)|^2 d\mu(z).$$

Hejhal and Rackner [HR92] investigated this conjecture numerically and also provided heuristic arguments in support of this conjecture. To perform their heuristic arguments, Hejhal and Rackner use Theorem 3.1.1 of Salem and Zygmund [SZ54], which proves a central limit theorem for randomized Fourier series.

Similarly, we can formulate a conjecture for holomorphic Hecke cusp forms. Inspired by the heuristic argument of Hejhal and Rackner and the Fourier expansion of holomorphic cusp forms we consider randomized power series of the form

$$f(z) = \sum_{n=1}^{\infty} c_n \cdot a(n) e^{2\pi i n z}, \tag{2.2}$$

where c_n are again chosen randomly, say with uniform distribution in $[-1, 1]$ and suitable coefficients $a(n)$. A probabilistic model along these lines was also used by Gosh and Sarnak [GS12] to compute the expected number of real zeros of holomorphic cusp forms. Following Theorem 3.5.2 of Salem and Zygmund [SZ54], which proves a central limit theorem for randomized power series, holomorphic Hecke cusp forms should be modeled by complex Gaussian Random variables with mean 0 and variance $1/\operatorname{Vol}(X)$ as $k \to \infty$.

Conjecture 2.2.2 (Random Wave Conjecture for holomorphic Hecke cusp forms)**.** *Let B be any fixed compact set of $\Gamma\backslash\mathbb{H}$, and f a holomorphic Hecke cusp form of weight k, normalized such that $\int_X |f(z)|^2 y^k d\mu(z) = 1$. Then for every positive integer r,*

$$\frac{1}{\operatorname{Var}_B(f)^r \cdot \operatorname{Vol}(B)} \int_B |f(z)|^{2r} y^{rk} \frac{dxdy}{y^2}$$

converges to

$$\frac{1}{\pi} \int_{\mathbb{C}} |z|^{2r} e^{-\frac{|z|^2}{2}} dxdy = \Gamma(r+1),$$

as k tends to infinity. Here

$$\operatorname{Var}_B(f) := \frac{1}{\operatorname{Vol}(B)} \int_B |f(z)|^2 y^k \frac{dxdy}{y^2}.$$

2.2.1 Fourth Moment of Cusp Forms

Recently, there has been extensive work on the Random Wave Conjecture and in particular on the fourth moment of Hecke cusp forms. The fourth moment of Hecke cusp forms is of special interest to analytic number theorists, as there is a clear relation to moments of L-functions via Watson's formula [Wat08]. Buttcane and Khan [BK17] computed the fourth moment of Hecke Maass cusp forms, assuming the *Generalized Lindelöf Hypothesis* (GLH), and confirmed a special case of the Random Wave Conjecture. In [DK20], Djankovic and Khan evaluated the fourth moment of (suitably regularized) Eisenstein series, which provides the first unconditional result for a fourth moment of an automorphic form. Humphries and Khan [DK20] proved unconditionally the fourth moment instance of the Random Wave Conjecture for dihedral Maass forms.

The various fourth moment problems are, as mentioned, related to various moment problems of L-functions. Depending on the involved L-functions the corresponding problems vary in difficulty.

A useful heuristic to gauge the difficulty of an L-function moment problem is the logarithmic ratio of the size of the family of L-functions relative to the size of their analytic conductor. To exemplifiy this heuristic consider the fourth moment of the Riemann zeta function

$$\int_T^{2T} |\zeta(1/2+it)|^4 dt.$$

Here we average over a family of T L-functions, and the analytic conductor of the fourth power of zeta is of size T^4 (the analytic conductor of zeta is of size T). The logarithmic ratio of the size of the family relative to the analytic conductor of the L-function is consequently 4.

As we know how to compute an asymptotic formula for the fourth moment of zeta, we should also be able to compute L-function moment problems where the logarithmic ratio of the family versus the analytic conductor is of size 4.

Remark 2.2.2. This heuristic is of course a bit too naïve, as different aspect of L-functions come with a different set of difficulties.

To compute the fourth moment of Maass cusp forms we need to evaluate an L-function with analytic conductor of size T^8, averaged over a family of size T^2. The logarithmic ratio of these two quantities is again 4, which is why it is reasonable to expect that an asymptotic formula can be computed. Indeed, as mentioned above, Buttcane and Khan, solve this problem.

In this thesis we are interested in the distribution of mass of holomorphic Hecke cusp forms and in particular, we are also interested in the fourth moment of holomorphic cusp forms. Again, via Watson's formula the problem is related to an L-function moment problem (see Lemma 3.1.1). As seen later, we will need to average k L-functions, with analytic conductor of size k^6, and so the logarithmic ratio is 6. Based on the heuristic arguments above proving an asymptotic for the fourth moment of holomorphic cusp forms is thus expected to be at least as difficult as proving an asymptotic formula for the sixth moment of the Riemann zeta function. At the present moment no asymptotic formula for the sixth moment of the Riemann zeta function is known under any "reasonable " conjecture like the Riemann Hypothesis. Consequently, we also do not expect being able to show an asymptotic for the fourth moment of holomorphic cusp forms.

To obtain partial results in the direction of the Random Wave Conjecture for holomorphic cusp forms it is natural to increase the length of the family, i.e. by considering an additional averaging over the weight k. This was done by Khan in [Kha14], where he considered a family of size k^3 and the L-function's analytic conductor, as mentioned, is of size k^6. The logarithmic ratio of these quantities is thus reduced to 2, which is in the regime where concrete results can be obtained.

Without averaging the best unconditional result for the fourth moment of holomorphic Hecke cusp forms is obtained by Khan, Young and Blomer in [BKY13]. They show that the fourth moment is bounded by $k^{1/3+\varepsilon}$, while using GLH one can show immediately the bound $\int_{\Gamma\backslash\mathbb{H}} |f(z)|^4 y^{2k} \frac{dxdy}{y^2} \ll k^{\varepsilon}$. Conditionally on GRH we show in [Zen21b] that the fourth moment is bounded:

Theorem 2.2.3. *[Zen21b, Theorem 1.1] Let f be a holomorphic Hecke cusp form of even weight k, normalized so that $\langle f, f\rangle = 1$. Assuming the Riemann Hypothesis for $L(s, f\times f\times g)$ and $L(s, \mathrm{sym}^2 f)$, there exists a universal constant C such that*

$$\int_{\Gamma\backslash\mathbb{H}} |f(z)|^4 y^{2k}\frac{dxdy}{y^2} \leq C,$$

for k large enough.

In Chapter 3 we will restate verbatim the proof obtained in [Zen21b] as a main part of this thesis.

2.2.2 Higher Moments of Cusp Forms

We conclude this section, by mentioning that little is known regarding the Gaussian Moment Conjecture for $n > 4$ for any automorphic form. The best bounds for the sixth moment, for example, often stem from interpolation arguments between the L^4-norm and the L^∞-norm. One of the major reasons seems to be of course the lack of obvious relation to L-functions. Unlike the fourth moment of cusp forms, which is related to L-functions, via Parseval and Watson's formula, there is no quick relation for the sixth moment.

In the last chapter of this thesis we suggest a different approach to higher moments, namely directly via the Fourier expansion of cusp forms, as for example in Theorem 1.8 in [BKY13]. In some sense, we already encountered the two distinct approaches to mass equidistribution, namely one with L-functions and one directly working with Fourier coefficients, in the work of Soundararajan and Holowinsky on QUE for holomorphic Hecke cusp forms.

We should also remark that for higher moments of holomorphic Hecke cusp forms it is important to restrict our attention to compact subsets of the fundamental domain. As Xia showed in [Xia07] the L^∞-norm of a weight k cusp form is large as a function of k, with large values attained high up (depending on k) in the cusp. These large values get amplified by taking higher moments, and so higher moments of holomorphic cusp forms on the entire fundamental domain start to diverge. Indeed, Blomer, Khan and Young [BKY13] extend the argument of Xia and show that $||F||_p^p \gg k^{\frac{p}{4}-\frac{3}{2}-\varepsilon}$. In particular, the 8-th moment on the full fundamental domain will certainly diverge. This is not in contradiction to the Random Wave Conjecture, which should hold only for compact sets. For the fourth moment on the other hand, large values attained high in the cusp are still negligible, so that we can integrate over the full fundamental domain without issues.

In fact, we believe the lower bound of Blomer, Khan and Young should be the correct size for moments of holomorphic cusp forms on the full fundamental domain. In [Zen21b] we conjecture

Conjecture 2.2.4. *Let f be a holomorphic cusp form of even weight k, normalized so that $\langle f, f\rangle = 1$. Let r be an even number and $y_0 > 0$ then*

$$P_r(y_0) := \int_{y_0}^{\infty}\int_0^1 |f(x+iy)y^{k/2}|^r \frac{dxdy}{y^2} \ll k^{\frac{r}{4}-\frac{3}{2}+\epsilon} + \frac{1}{y_0}. \tag{2.3}$$

In [BKY13, Theorem 1.8] Blomer, Khan and Young consider the special case when $r = 4$ unconditionally. They relate $P_4(y_0)$ to a shifted convolution problem. Assuming square-root cancellation in this shifted convolution problem, which is out of reach unconditionally, we would obtain the upper bound suggested in equality (2.3). It is easy to generalize their computation to higher moments (see (5.3) in Chapter 5). If we then assume square-root cancellation in the resulting summation over the Hecke eigenvalues we are lead to Conjecture 2.2.4.

2.3 Quantum Variance

Once we have established the expected value of a quantity (like in a QE or QUE theorem), it is natural to ask how far the L^2-mass deviates from its mean value. This deviation is measured by the variance.

As before, we denote by $\mathcal{M}$ a smooth Riemannian manifold. Let K be a positive number. We denote by $\mathcal{B}_K$ an orthonormal basis of Laplace eigenfunctions φ_j on $\mathcal{M}$, with spectral parameter $\sqrt{\lambda_j}$ in a dyadic interval of size K, i.e. $\{\varphi_j : K \leq \sqrt{\lambda_j} \leq 2K\}$. For a smooth compactly supported function ψ we define

$$\mu_\varphi(\psi) := \int_{\mathcal{M}} \psi(z)|\varphi(z)|^2 d\mu(z) \quad \text{and} \quad \mathbb{E}(\psi) := \frac{1}{\mathrm{Vol}(\mathcal{M})} \int_{\mathcal{M}} \psi(z) d\mu(z).$$

We are then interested in the quantum variance, given by

$$V(\psi) := \frac{1}{|\mathcal{B}_K|} \sum_{\varphi \in \mathcal{B}_K} |\mu_\varphi(\psi) - \mathbb{E}(\psi)|^2,$$

as $K \to \infty$.

The quantum variance problem was first introduced by Zelditch in [Zel94]. Since then, many different aspects and variants of the original quantum variance problem for Laplace eigenfunctions were investigated. In [LS04] Luo and Sarnak computed an asymptotic formula for the quantum variance of holomorphic Hecke cusp forms on the full fundamendal domain. The case of Maass forms on $\Gamma\backslash\mathbb{H}$ was settled by Zhao in [Zha10] and Sarnak–Zhao in [SZ19] for more general observables on the modular surface. Moreover, Luo, Rudnick and Sarnak investigated the quantum variance for closed geodesics on the modular surface in [LRS09]. In the compact setting, Nelson used the theta correspondence to compute the variance for quaternion algebras [Nel16], [Nel17] and [Nel19]. Eisenstein series and dihedral Maass forms were treated by Huang [Hua21] and Huang–Lester [HL20] respectively. Recently, Nordentoft, Petridis and Risager computed in [NPR21] the variance in shrinking sets at infinity.

In our work [Zen21a] we compute for the first time the quantum variance restricted to the a one-dimensional set, namely the vertical geodesic connecting zero and infinity on the upper half-plane. Before stating the main theorem, which will be explored in Chapter 4, we digress shortly to equidistribution results restricted to certain hypersurfaces.

2.3.1 Restriction Theorems

We focus now on equidistribution questions for certain submanifolds. Rather than observing the behaviour of eigenfunctions on the entire manifold, we try to understand their behaviour restricted to various subregions. Questions like this are often referred to as *Quantum Ergodic Restriction Problems*. Quantum Ergodicity problems restricted to certain hypersurfaces were studied for Laplacian eigenfunction by Christianson–Toth–Zelditch in [CTZ13], Dyatlov–Zworski in [DZ13] and Toth–Zelditch in [TZ13].

Remark 2.3.1. One of the important applications of these restriction problems is the study of nodal domains of eigenfunctions. For a manifold $\mathcal{M}$ the nodal domains of φ are the connected components of $\mathcal{M}\backslash\{z \in \mathcal{M} : \varphi(z) = 0\}$. A fundamental problem in spectral geometry and quantum chaos is to count the number of nodal domains of φ (see for example the works [JZ16], [GRS13], [JJ18], [JY19]).

In Chapter 4 we analyze the distribution of holomorphic Hecke cusp forms on the vertical geodesic, meaning the line connecting zero and infinity on the upper half-plane. Young proposed the following Quantum Unique Ergodicity conjecture for the vertical geodesic:

Conjecture 2.3.1. *[You16, Conjecture 1.1] Suppose that $\psi\colon \mathbb{R}^+ \to \mathbb{R}$ is a smooth, compactly supported function. Then*

$$\lim_{k\to\infty} \int_0^\infty y^k |f(iy)|^2 \psi(y) \frac{dy}{y} = \frac{3}{\pi} \int_0^\infty \psi(y) \frac{dy}{y}, \tag{2.4}$$

where $f(z)$ runs over weight k holomorphic Hecke cusp forms that are L^2-normalized.

To provide evidence for this conjecture, Young relates the left-hand side of (2.4) to a (shifted) moment problem of L-functions (see Eq. 3.2 in [You16]), to which he applies the recipe of random matrix theory (see [CFK$^+$05]) to evaluate the main term.

Conjecture 2.3.1 is likely out of reach of current technology, which we motivate by the following standard computation that was done in [BKY13, Section 7]. First recall that the Mellin transform of holomorphic cusp forms is related to L-functions, i.e. for

$$f(z) = a_f(1)\sum_{n=1}^{\infty}\lambda_f(n)(4\pi n)^{(k-1)/2}e(nz), \quad |a_f(1)|^2 = \frac{2\pi^2}{L(1,\operatorname{sym}^2 f)\Gamma(k)},$$

we have

$$\begin{aligned}\int_0^\infty f(iy)y^{k/2}y^s\frac{dy}{y} &= a_f(1)\sum_{n=1}^{\infty}\lambda_f(n)(4\pi n)^{(k-1)/2}\int_0^\infty y^{k/2+s}e^{-2\pi n y}\frac{dy}{y} \\ &= a_f(1)\frac{2^{k/2}}{\sqrt{4\pi}}\frac{1}{(2\pi)^s}L(1/2+s,f)\Gamma(s+k/2).\end{aligned} \tag{2.5}$$

For simplicity we investigate the quantity

$$\mathcal{I} := \int_0^\infty |f(iy)|^2 y^k\frac{dy}{y},$$

rather than the left-hand side of (2.4), as the difficulty of those problems is likely comparable. From the Parseval theorem and the computation in (2.5) we get that

$$\mathcal{I} = \int_{-\infty}^{\infty} 2^{k-2}\frac{|\Gamma(k/2+it)|^2}{\Gamma(k)}\cdot\frac{|L(1/2+it,f)|^2}{L(1,\operatorname{sym}^2 f)}dt.$$

Applying Stirling's formula we get

$$\frac{|\Gamma(k/2+it)|^2}{\Gamma(k)} \sim \left(\frac{\pi}{2k}\right)^{1/2}2^{-(k-2)}e^{-2t^2/k}$$

and consequently

$$\mathcal{I} \sim \left(\frac{\pi}{2k}\right)^{1/2} \int_{-\infty}^{\infty} e^{-2t^2/k} \frac{|L(1/2+it,f)|^2}{L(1,\mathrm{sym}^2 f)} dt.$$

We notice now that an almost optimal bound $\mathcal{I} \ll k^{\varepsilon}$ would imply that $L(1/2,f) \ll k^{1/4+\varepsilon}$. As the analytic conductor $\mathcal{C}(f)$ is of size k^2, this would prove a subconvexity bound of the form $L(1/2,f) \ll \mathcal{C}(f)^{1/8+\varepsilon}$. We do not expect being able to obtain a bound of this strength, as it would go beyond what is even known for the simpler Riemann zeta function.

Rather than asking for equidistribution of all eigenforms (Quantum Unique Ergodicity), we settle for the easier question: whether equidistribution holds for almost all eigenforms (Quantum Ergodicity). In fact, we consider directly the more complicated problem of computing the quantum variance. This will not only show equidistribution for almost all Hecke cusp forms (in a large family), but it will also provide information about the deviation from its expected value. In Chapter 4 we review the work in [Zen21a] and one of the main theorems:

Theorem 2.3.2. *[Zen21a, Theorem 1.2] Let ψ_1, ψ_2 and h be smooth compactly supported functions on $\mathbb{R}^+$. Moreover, suppose that $\psi_i(y) = \psi_i(1/y)$ for $i = 1, 2$. Then*

$$\sum_{k \equiv 0 \,(\mathrm{mod}\, 2)} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k} L(1,\mathrm{sym}^2 f)\Big(\mu_f(\psi_1) - \mathbb{E}(\psi_1)\Big) \cdot \Big(\mu_f(\psi_2) - \mathbb{E}(\psi_2)\Big) = V(\psi_1,\psi_2) \quad (2.6)$$

with

$$\begin{aligned}V(\psi_1,\psi_2) =& K^{3/2}\log K\cdot\frac{\sqrt{2}\pi}{32}\widetilde{\psi_1}(0)\widetilde{\psi_2}(0)\cdot\int_0^\infty\frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}}du+\\&+K^{3/2}\frac{\sqrt{2}\pi}{64}\widetilde{\psi_1}(0)\widetilde{\psi_2}(0)\int_0^\infty\frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}}\log(u)du+\\&+K^{3/2}\int_0^\infty\frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}}du\cdot\left(\frac{\sqrt{2}\pi}{16}\left(\frac{3}{2}\gamma-\log(4\pi)\right)\widetilde{\psi_1}(0)\widetilde{\psi_2}(0)\right)+\\&+K^{3/2}\int_0^\infty\frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}}du\cdot\frac{\sqrt{2}\pi}{8}\frac{1}{2\pi i}\int_{(1)}\widetilde{\psi_1}(-s_2)\widetilde{\psi_2}(s_2)\zeta(1-s_2)\zeta(1+s_2)ds_2+\\&+O_{\psi_1,\psi_2}(K^{5/4+\varepsilon}),\end{aligned}$$

as $K\to\infty$. *Here* $\widetilde{\psi_i}(s) := \int_0^\infty \psi(1/y)y^{s-1}dy$ *for* $i=1,2$.

Chapter 3

Sharp Bound for the Fourth Moment of Holomorphic Hecke Cusp Forms

3.1 Background and Heuristics

We focus now on the proof of Theorem 2.2.3, which was established in our work [Zen21b]. We follow our exposition in [Zen21b] extremely closely.

First, we notice that the fourth moment of holomorphic Hecke cusp forms is related to a moment problem of L-functions, as seen for example seen in [Zen21b, Lemma 5.1].

Lemma 3.1.1. *Let f be a holomorphic Hecke cusp form of weight k, and let B_{2k} denote a Hecke basis for the space of holomorphic cusp forms of weight $2k$. Then*

$$\int_{\Gamma\backslash\mathbb{H}} |f(z)|^4 y^{2k} \frac{dxdy}{y^2} = \frac{\pi^3}{2(2k-1)} \sum_{g\in B_{2k}} \frac{L(1/2, f\times f\times g)}{L(1,\mathrm{sym}^2 f)^2 L(1,\mathrm{sym}^2 g)}. \tag{3.1}$$

Proof. Since f^2 is a cusp form of weight $2k$, we have the following decomposition in terms of Hecke eigenforms $g \in B_{2k}$:

$$\langle f^2, f^2 \rangle = \sum_{g \in B_{2k}} |\langle f^2, g \rangle|^2.$$

At this point we apply Watson's formula (see [Wat02, Theorem 3]) to the resulting inner product of three Hecke cusp forms (see also [BKY13, Eq. 2.7]) so that

$$\sum_{g \in B_{2k}} |\langle f^2, g \rangle|^2 = \frac{\pi^3}{2(2k-1)} \sum_{g \in B_{2k}} \frac{L(1/2, f \times \overline{f} \times g)}{L(1, \operatorname{sym}^2 f)^2 L(1, \operatorname{sym}^2 g)}.$$

Finally, we drop the complex conjugation bar of f, as the Fourier coefficients of f are real, and the lemma follows. □

Bounding moments of L-functions, like the right-hand side of (3.1), is a central problem in analytic number theory. Based on Random Matrix Theory Conrey et al. [CFK+05] provided heuristics to evaluate the main term of integral moments of various families of L-functions. Applying their recipe indicates that the right-hand side of (3.1), and hence the fourth moment of holomorphic Hecke cusp forms, converges indeed to the constant predicted by the Random Wave Conjecture (see Conjecture 2.2.2 for $r = 2$). This heuristic argument can be seen in [BKY13, Section 4].

It is natural to ask, whether one can evaluate moments of L-functions without invoking the Random Matrix Theory Conjectures, but say with the help of the Riemann Hypothesis. Indeed, in the breakthrough work [Sou09] Soundararajan obtained almost sharp bounds for the moments of the Riemann zeta function on the critical line:

Theorem 3.1.2 (Soundararajan [Sou09]). *Assume the Riemann Hypothesis. For every positive real number k, and every $\varepsilon > 0$ we have*

$$T(\log T)^{k^2} \ll_k \int_T^{2T} |\zeta(1/2 + it)|^{2k} dt \ll_{k,\varepsilon} T(\log T)^{k^2+\varepsilon}.$$

Harper built upon these techniques and improved them to achieve sharp bounds for the moments of the Riemann zeta function:

Theorem 3.1.3 (Harper [Har13]). *Assume the Riemann Hypothesis, and let $k \geq 0$ be fixed. Then for all large T we have*

$$\int_T^{2T} |\zeta(1/2+it)|^{2k} dt \ll_k T(\log T)^{k^2},$$

where the implicit constant depends on k only.

The approach of Soundararajan and Harper on bounding the moments of the Riemann zeta function is based on Selberg's Central Limit Theorem [Sel46], [Sel92], which shows that $\log|\zeta(1/2+it)|$ is approximately Gaussian with mean value 0 and variance $\frac{1}{2}\log\log T$ for $t \in [T, 2T]$ as $T \to \infty$.

For a Gaussian random variable X with mean μ and variance σ^2 we have the following standard computation

$$\begin{aligned}\mathbb{E}[e^{aX}] &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(at - \frac{(t-\mu)^2}{2\sigma^2}\right) dt \\ &= e^{a\mu + a^2\sigma^2/2} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu-a\sigma^2)^2}{2\sigma^2}\right) dt \\ &= e^{a\mu + a^2\sigma^2/2}.\end{aligned} \tag{3.2}$$

In the setting for the Riemann zeta function with X$=\log|\zeta(1/2+it)|$, $\mu = 0$ and $\sigma^2 = \frac{1}{2}\log\log T$ we have

$$\begin{aligned}\mathbb{E}(e^{2kX}) &= \frac{1}{T}\int_T^{2T} \exp(2k\log|\zeta(1/2+it)|) dt \\ &\asymp \exp\left((2k)^2 \frac{1}{4}\log\log T\right) \\ &= (\log T)^{k^2}.\end{aligned}$$

The probabilistic viewpoint and the techniques of Soundararajan and Harper are very robust and can also be applied to other families of L-functions. In particular, we adapt such a probabilistic viewpoint for $\log L(1/2, f \times f \times g)$. We expect that $\log L(1/2, f \times f \times g)$ has an approximately normal distribution with mean value

$$-\sum_{p \le k} \frac{\lambda_f(p)^4 + 4\lambda_f(p)^2 - 4}{2p}$$

and variance of size

$$\sum_{p \le k} \frac{\lambda_f(p)^4}{p}.$$

This can be seen from the Euler product represenation of our L-function (see Remark 3.2.1 for more details). Again, following the computation in (3.2) with X= $\log L(1/2, f \times f \times g)$ and mean and variance given above, we expect

$$\begin{aligned}\sum_{g \in B_{2k}}^h \exp(\log L(1/2, g \times f \times f)) &\approx \exp\left(-\sum_{p \le k} \frac{\lambda_f(p)^4 - 4\lambda_f(p)^2 + 4}{2p} + \frac{1}{2}\sum_{p \le k} \frac{\lambda_f(p)^4}{p}\right) \\ &= \exp\left(2\sum_{p \le k} \frac{\lambda_f(p)^2 - 1}{p}\right) \\ &\asymp L(1, \operatorname{sym}^2 f)^2.\end{aligned}$$

Inserting this bound into (3.1) demonstrates heuristically, why we expect the fourth moment of holomorphic Hecke cusp forms to be bounded.

Remark 3.1.1. In recent work Shubin [Shu21] used similar techniques as in the following sections to estimate the variance in Linnik's problem.

3.2 Proof of Theorem 2.2.3

We come now to the proof of Theorem 2.2.3 that we take verbatim from our work [Zen21b].

In view of the Petersson Trace Formula 3.2.2, it will be useful to introduce a normalized summation over a Hecke basis. Let B_k be a Hecke basis of weight k cusp forms. For any $\mathcal{S} \subseteq B_k$, we define

$$\sum_{g\in\mathcal{S}}^{h} \lambda_g(n) := \frac{2\pi^2}{k-1}\sum_{g\in\mathcal{S}} \frac{\lambda_g(n)}{L(1,\operatorname{sym}^2 g)}.$$

We also define the normalized measure of the set $\mathcal{S} \subset B_k$ by

$$\operatorname{meas}\{S\} := \sum_{g\in\mathcal{S}}^{h} 1 = \frac{2\pi^2}{k-1}\sum_{g\in\mathcal{S}} \frac{1}{L(1,\operatorname{sym}^2 g)}.$$

Now that we have reduced Theorem 4.12 to bounding an average of L-functions we will follow the approach of Soundararajan and Harper to control the right-hand side of (3.1). At first we need to approximate the logarithm of our L-function $L(1/2, f\times f\times g)$ with a short Dirichlet polynomial over primes. Working with this Dirichlet polynomial will enable us to detect the underlying Gaussian behaviour of $\log L(1/2, f\times f\times g)$. To accomplish this, we use an idea of Soundararjan [Sou09] as adapted by Chandee [Cha09] to our context.

Lemma 3.2.1. *Let f and g be Hecke cusp forms of even weight k and $2k$, respectively, for the full modular group. Assuming the Riemann Hypothesis for $L(1/2, f\times f\times g)$, we have for any $x \geq 2$*

$$\begin{aligned}\log L(1/2, f\times f\times g) \leq & \sum_{p\leq x} \frac{\lambda_f(p)^2\lambda_g(p)}{p^{1/2+1/\log x}}\frac{\log(x/p)}{\log x} \\ & + \sum_{p^2\leq x} \frac{(\lambda_f(p)^4-4\lambda_f(p)^2+4)(\lambda_g(p^2)-1)}{2p^{1+2/\log x}}\frac{\log(x/p^2)}{\log x} + \frac{\log k^6}{\log x} + O(1).\end{aligned}$$

Proof. We express the Hecke eigenvalues $\lambda(p)$ of f and g in terms of their Satake parameters $\alpha(p)$ and $\beta(p)$, more precisely $\lambda_f(p) = \alpha_f(p) + \beta_f(p)$ and $\lambda_g(p) = \alpha_g(p) + \beta_g(p)$. Now we can directly apply Theorem 2.1 in [Cha09] with $c = 1$ and get

$$\log L(1/2, f\times f\times g) \leq \sum_{\ell=1}^{\infty}\sum_{p^\ell\leq x} \frac{(\alpha_f(p)^{2\ell}+\beta_f(p)^{2\ell}+2)(\alpha_g(p)^\ell+\beta_g(p)^\ell)}{\ell p^{(\frac{1}{2}+\frac{1}{\log x})\ell}}\frac{\log(x/p^\ell)}{\log x} + \frac{\log k^6}{\log x} + O\left(\frac{1}{\log^2 x}\right).$$

Here we used that the analytic conductor of $L(1/2, f\times f\times g)$ is of size k^6, which can be seen from the gamma factor $L_\infty(s, f\times f\times g)$ in (1.4). By the Deligne bound $|\lambda_f(p)|\leq 2$ the contribution of the prime powers p^ℓ with $\ell\geq 3$ can be shown to be $O(1)$. Since $\alpha_f(p)^2+\beta_f(p)^2+2 = \lambda_f(p)^2$, $\alpha_f(p)^4+\beta_f(p)^4+2 = \lambda_f(p)^4-4\lambda_f(p)^2+4$ and $\alpha_g(p)^2+\beta_g(p)^2 = \lambda_g(p)^2-2 = \lambda_g(p^2)-1$ the lemma follows. □

Remark 3.2.1. Notice that on average over g the coefficients $\lambda_g(p)$ and $\lambda_g(p^2)$ are close to 0. Consequently, we expect the mean value of $\log L(1/2, f\times f\times g)$ to be essentially

$$\sum_{p^2\leq x}\frac{-(\lambda_f(p)^4-4\lambda_f(p)^2+4)}{2p^{1+2/\log x}}.$$

Since $\lambda_g(p)^2$ is close to 1 on average, the variance should be

$$\sum_{g\in B_{2k}}^{h}\left(\sum_{p\leq x}\frac{\lambda_f(p)^2\lambda_g(p)}{p^{1/2+\log x}}\right)^2 \sim \sum_{p\leq x}\frac{\lambda_f(p)^4}{p^{1+2/\log x}}.$$

3.2.1 Detecting Randomness

In his proof, Harper detected the randomness of the harmonics $\mathrm{Re}(p^{-it})$ with Proposition 2 in [Har13]. For our harmonics $\lambda_g(p)$, the role will be played by the following version of Petersson's Trace Formula:

Lemma 3.2.2 (Petersson Trace Formula). *Let k be large and let $n = p_1^{\alpha_1}\cdots p_r^{\alpha_r}\leq k^2/10^4$,* *where the p_i are distinct primes and $\alpha_i \in \mathbb{N}$ for all i. Then*

$$\sum_{g\in B_{2k}}^{h} \prod_{i=1}^{r} \lambda_g(p_i)^{\alpha_i} = h_1(n) + O(k^3 e^{-k}) \tag{3.3}$$

where

$$h_1(n) := \prod_{i=1}^{r} \frac{1_{2|\alpha_i} \cdot (\alpha_i)!}{((\alpha_i/2)!)^2(\alpha_i/2+1)},$$

in particular, $h_1(n) = 0$ if any of the exponents α_i is odd.

Moreover, if $n = p_1^{\beta_1} \cdots p_r^{\beta_r} \leq k/100$, with p_i distinct primes and $\beta_i \in \mathbb{N}$ for all i, then

$$\sum_{g\in B_{2k}}^{h} \prod_{i=1}^{r} \lambda_g(p_i^2)^{\beta_i} = h_2(n) + O(k^4 e^{-k}),$$

where

$$h_2(n) = \prod_{i=1}^{r} \sum_{\ell=0}^{\beta_i} \binom{\beta_i}{\ell} (-1)^{\ell} \frac{(2(\beta_i-\ell))!}{(\beta_i-\ell)!(\beta_i-\ell+1)!}$$

In particular, $h_2(n) = 0$ if $\beta_i = 1$ for some i, and $h_2(n) \leq \prod_{i=1}^{r} 3^{\beta_i}$ in general.

We also have the following combined result: Let $a = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $b = q_1^{\beta_1} \cdots q_s^{\beta_s}$, with $a \cdot b^2 \leq k^2/10^4$, p_i and q_j all distinct from each other. Then

$$\sum_{g\in B_{2k}}^{h} \prod_{i=1}^{r} \lambda_g(p_i)^{\alpha_i} \prod_{j=1}^{s} \lambda_g(q_j^2)^{\beta_i} = h_1(a)h_2(b) + O(k^5 e^{-k}). \tag{3.4}$$

Remark 3.2.2. Notice that h_1 is a multiplicative function supported on even numbers. This is reminiscent of the correlations of powers of independent Gaussian random variables. The multiplicativity of h_1 should be interpreted as quasi-independence and the support on even numbers reminds us that odd moments of Gaussian random variables vanish. We also highlight the condition that $n \leq k^2/10^4$. The total number of available harmonics is k, hence the length of the square of the Dirichlet polynomial should not exceed k^2, so that the only contribution comes from the main term. The bound $h_2(n) \leq \prod_{i=1}^{r} 3^{\beta_i}$ follows upon noting that $|\lambda_g(p^2)| \leq 3$ by the Deligne bound.

Proof. We want to use the Petersson Trace Formula in the form of Lemma 2.1 in [RS06] which says that

$$\sum_{g \in B_{2k}}^{h} \lambda_g(t)\lambda_g(u) = 1_{t=u} + O(e^{-k}), \tag{3.5}$$

if k is large and t and u are natural numbers with $tu \leq k^2/10^4$.

To do so we need to express $\lambda_g(p_i)^{\alpha_i}$ in terms of $\lambda_g(p^\ell)$ for $0 \leq \ell \leq \alpha_i$. This can be achieved via the Hecke relations of the Fourier coefficients. An easy computation, as done in Lemma 7.1 of [LL11], shows that

$$\lambda_f(p)^\alpha = \Big(A_\alpha + \sum_{\ell=1}^{\alpha/2} C_\alpha(\ell)\lambda_f(p^{2\ell})\Big)1_{2|\alpha} + \Big(B_\alpha\lambda_f(p) + \sum_{\ell=1}^{\alpha/2-1} D_\alpha(\ell)\lambda_f(p^{2\ell+1})\Big)1_{2|\alpha+1} \tag{3.6}$$

with

$$A_\alpha = \frac{(\alpha)!}{((\alpha/2)!)^2(\alpha/2+1)}, \qquad C_\alpha(\ell) = \frac{(\alpha)!(2\ell+1)}{(\alpha/2-r)!(\alpha/2+r+1)!},$$

(these coefficients only appear in the expression of $\lambda_f(p)^\alpha$ when α is even)

$$B_\alpha = \frac{2(\alpha)!}{((\alpha-1)/2)!((\alpha+3)/2)!} \quad \text{and} \quad D_\alpha(\ell) = \frac{(\alpha)!(2\ell+2)}{((\alpha-1)/2)-\ell)!((\alpha+3)/2+\ell)!}$$

(these coeffiecients only appear in the expression of $\lambda_f(p)^\alpha$ when α is odd). It follows that the left-hand side of equation (3.3) is given by

$$\sum_{g\in B_{2k}}^{h}\prod_{i=1}^{r}\left\{\Big(A_{\alpha_i} + \sum_{\ell_1=1}^{\alpha_i/2} C_{\alpha_i}(\ell_1)\lambda_f(p_i^{2\ell_1})\Big)1_{2|\alpha_i} + \Big(B_{\alpha_i}\lambda_f(p_i) + \sum_{\ell_2=1}^{\alpha_i/2-1} D_{\alpha_i}(\ell_2)\lambda_f(p_i^{2\ell+1})\Big)1_{2|(\alpha_i+1)}\right\}.$$

We apply identity (3.5) and get the main term

$$\prod_{i=1}^{r} A_{\alpha_i}1_{2|\alpha_i} = \prod_{i=1}^{r}\frac{(\alpha_i)!}{((\alpha_i/2)!)^2(\alpha_i/2+1)}1_{2|\alpha_i},$$

since the primes p_i are distinct for different $1 \leq i \leq r$. To bound the error term we first notice that $A_\alpha \leq 2^\alpha$, $\sum_{\ell=1}^{\alpha/2} C_\alpha(\ell) \leq 2^\alpha$, $B_\alpha \leq 2 \cdot 2^\alpha$ and $\sum_{\ell=1}^{(\alpha-1)/2} D_\alpha \leq 2 \cdot 2^\alpha$. Consequently, the error term is bounded by

$$O\Big(e^{-k} \prod_{i=1}^{r} 4 \cdot 2^{\alpha_i}\Big) = O\Big(e^{-k} k^3\Big).$$

Here we also used the crude bounds $4^r \ll k$ and $\prod_{i=1}^{r} 2^{\alpha_i} \leq k^2$. This shows the first part of the lemma.

The second part of the lemma follows similarly upon using $\lambda_g(p_i^2) = \lambda_g(p_i)^2 - 1$ and the binomial theorem. More precisely, we have

$$\begin{aligned}
\sum_{g \in B_{2k}}^{h} \prod_{i=1}^{r} \lambda_g(p_i^2)^{\beta_i} &= \sum_{g \in B_{2k}}^{h} \prod_{i=1}^{r} (\lambda_g(p_i)^2 - 1)^{\beta_i} \\
&= \sum_{g \in B_{2k}}^{h} \prod_{i=1}^{r} \sum_{\ell=0}^{\beta_i} \binom{\beta_i}{\ell} (-1)^\ell \lambda_g(p_i)^{2(\beta_i - \ell)} \\
&= \prod_{i=1}^{r} \sum_{\ell_i=0}^{\beta_i} \left(\binom{\beta_i}{\ell_i} (-1)^{\ell_i} \right) \sum_{g \in B_{2k}}^{h} \prod_{i=1}^{r} \lambda_g(p_i)^{2(\beta_i - \ell_i)}.
\end{aligned}$$

At this point we use relation (3.6) to rewrite $\lambda_g(p_i)^{2(\beta_i-\ell_i)}$ in terms of $\lambda_g(p_i^\ell)$ for $0 \leq \ell \leq 2(\beta_i - \ell_i)$. Again, an application of the Petersson Trace formula (see formula (3.5)), yields the main term

$$\prod_{i=1}^{r} \sum_{\ell=0}^{\beta_i} (-1)^\ell \frac{(2(\beta_i - \ell))!}{(\beta_i - \ell)!(\beta_i - \ell + 1)!}$$

as desired. The error term is given by

$$O\Big(e^{-k} \cdot \prod_{i=1}^{r} \sum_{\ell=0}^{\beta_i} \binom{\beta_i}{\ell} \cdot \Big(A_{2(\beta_i-\ell)} + \sum_{m=1}^{\beta_i - \ell} C_{2(\beta_i-\ell)}(m)\Big)\Big) = O\Big(e^{-k} 2^r \prod_{i=1}^{r} 5^{\beta_i}\Big),$$

as $A_{2(\beta_i-\ell)} \le 4^{\beta_i-\ell}$, $\sum_{m=1}^{\beta_i-\ell} C_{2(\beta_i-\ell)}(m) \le 4^{\beta_i-\ell}$ and

$$\sum_{\ell=0}^{\beta_i} \binom{\beta_i}{\ell} 4^{\beta_i-\ell} = 5^{\beta_i}.$$

So the contribution of the error term is given by

$$O\left(e^{-k} 2^r \prod_{i=1}^{r} 5^{\beta_i}\right) = O(e^{-k} k^4).$$

Finally, the last part of the lemma, namely equation (3.4), follows in a similar vein. The main term is, as desired, given by

$$\prod_{i=1}^{r} \frac{1_{2|\alpha_i} \cdot (\alpha_i)!}{((\alpha_i/2)!)^2(\alpha_i/2+1)} \cdot \prod_{j=1}^{s} \sum_{\ell=0}^{\beta_j} \binom{\beta_j}{\ell} (-1)^\ell \frac{(2(\beta_j-\ell))!}{(\beta_j-\ell)!(\beta_j-\ell+1)!}.$$

The error term is now given by

$$O\left(e^{-k} 2^{r+s} \prod_{i=1}^{r} 2^{\alpha_i} \cdot \prod_{j=1}^{s} 5^{\beta_i}\right) = O(e^{-k} k^5).$$

This concludes the proof of the entire lemma. □

Lemma 3.2.3. *Define the function h_1 as in Lemma 3.2.2 and let $u(p)$ be any real numbers. For any numbers $x_1, x_2 \ge 1$, we have*

$$\left| \sum_{x_1 < p_1, \ldots, p_n \le x_2} \frac{u(p_1) \cdots u(p_n)}{\sqrt{p_1 \cdots p_n}} h_1(p_1 \cdots p_n) \right| \le \frac{n!}{2^{n/2}(n/2)!} \left(\sum_{x_1 < p \le x_2} \frac{u(p)^2}{p} \right)^{\frac{n}{2}} \tag{3.7}$$

if n is even and 0 if n is odd.

Proof. Let U denote the sum on the left-hand side of the desired inequality (3.7). Recall that h_1 is supported only on squares. In particular, n has to be even and we write $n = 2\ell$, so that

$$U = \sum_{x_1 < p_1, \ldots, p_{2\ell} \le x_2} \frac{u(p_1) \cdots u(p_{2\ell})}{\sqrt{p_1 \cdots p_{2\ell}}} h_1(p_1 \cdots p_{2\ell}).$$

We now write $p_1 \cdots p_{2\ell} = q_1^{\alpha_1} \cdots q_r^{\alpha_r}$, where the primes q_i for $1 \le i \le r$ are distinct and $\alpha_i \ge 1$ for all $1 \le i \le r$. Then U equals

$$\sum_{1 \le r \le 2\ell} \sum_{\alpha_1 + \cdots + \alpha_r = 2\ell} \sum_{x_1 < q_1 < \ldots < q_r \le x_2} \binom{2\ell}{\alpha_1, \ldots, \alpha_r} \frac{u(q_1)^{\alpha_1} \cdots u(q_r)^{\alpha_r}}{\sqrt{q_1^{\alpha_1} \cdots q_r^{\alpha_r}}} h_1(q_1^{\alpha_1} \cdots q_r^{\alpha_r}),$$

where the multinomial coefficient counts the number of representations such that $p_1 \cdots p_{2\ell} = \prod_{1 \le i \le r} q_i^{\alpha_i}$. Since h_1 is supported only on squares we see that α_i for $1 \le i \le r$ is divisible by 2 and consequently $r \le \ell$. It follows that

$$\begin{aligned} U &= \sum_{1 \le r \le \ell} \sum_{\substack{\alpha_1 + \cdots + \alpha_r = 2\ell \\ 2 \mid \alpha_i}} \sum_{x_1 < q_1 < \ldots < q_r \le x_2} \binom{2\ell}{\alpha_1, \ldots, \alpha_r} \frac{u(q_1)^{\alpha_1} \cdots u(q_r)^{\alpha_r}}{\sqrt{q_1^{\alpha_1} \cdots q_r^{\alpha_r}}} h_1(q_1^{\alpha_1} \cdots q_r^{\alpha_r}) \\ &= \sum_{1 \le r \le \ell} \sum_{\substack{\beta_1 + \cdots + \beta_r = \ell \\ \beta_i \ge 1}} \sum_{x_1 < q_1 < \ldots < q_r \le x_2} \binom{2\ell}{2\beta_1, \ldots, 2\beta_r} \frac{u(q_1)^{2\beta_1} \cdots u(q_r)^{2\beta_r}}{q_1^{\beta_1} \cdots q_r^{\beta_r}} \prod_{1 \le i \le r} \frac{(2\beta_i)!}{\beta_i!(\beta_i + 1)!} \end{aligned}$$

We simplify and use the bound $(\beta_i + 1)! \ge 2^{\beta_i}$ so that

$$\begin{aligned} U &\le \frac{(2\ell)!}{\ell!} \sum_{1 \le r \le \ell} \sum_{\substack{\beta_1 + \cdots + \beta_r = \ell \\ \beta_i \ge 1}} \sum_{x_1 < q_1 < \ldots < q_r \le x_2} \binom{\ell}{\beta_1, \cdots, \beta_r} \frac{u(q_1)^{2\beta_1} \cdots u(q_r)^{2\beta_r}}{q_1^{\beta_1} \cdots q_r^{\beta_r}} \frac{1}{2^{\beta_1} \cdots 2^{\beta_r}} \\ &= \frac{(2\ell)!}{\ell!} \left(\sum_{x_1 < q \le x_2} \frac{u(q)^2}{2q} \right)^{\ell}. \end{aligned}$$

This conclues the proof of the lemma. □

Lemma 3.2.4. *Let $w(p)$ be any real numbers such that $|w(p)| \le C$ and define the function*

h_2 as in Lemma 3.2.2. Then

$$\Bigg| \sum_{2^m < p_1,\ldots,p_{2M} \le 2^{m+1}} \frac{w(p_1)\cdots w(p_{2M})}{p_1 \cdots p_{2M}} \cdot h_2(p_1 \cdots p_{2M}) \Bigg| \le \frac{(2M)!}{M!} \left(\frac{72C^2}{2^m} \right)^M . \tag{3.8}$$

Proof. The main difference to Lemma 3.2.3 is that the function h_2 is supported on integers that are divisible by squares, rather than integers that are squares. This leads to more difficult combinatorics. Let W denote the sum on the left-hand side of inequality (3.8). As in Lemma 3.2.3 we express $p_1 \cdots p_{2M}$ in terms of distinct primes, i.e. $p_1 \cdots p_{2M} = \prod_{i=1}^r q_i^{\alpha_i}$. Then

$$W = \sum_{1 \le r \le M} \sum_{\substack{\alpha_1 + \cdots + \alpha_r = 2M \\ \alpha_i \ge 2}} \sum_{2^m < q_1 < \ldots < q_r \le 2^{m+1}} \binom{2M}{\alpha_1, \ldots, \alpha_r} \frac{w(q_1)^{\alpha_1} \cdots w(q_r)^{\alpha_r}}{q_1^{\alpha_1} \cdots q_r^{\alpha_r}} h_2(q_1^{\alpha_1} \cdots q_r^{\alpha_r}).$$

Here we used that $h_2(q_1^{\alpha_1} \cdots q_r^{\alpha_r})$ is zero if $\alpha_i = 1$ for some i. Next, we apply the crude bound $h_2(q_1^{\alpha_1} \cdots q_r^{\alpha_r}) \le \prod_{1 \le i \le r} 3^{\alpha_i}$ and $|b(p_i)| \le C$ for $1 \le i \le r$. It follows that $|W|$ is bounded by

$$(3C)^{2M} \sum_{1 \le r \le M} \sum_{\substack{\alpha_1 + \cdots + \alpha_r = 2M \\ \alpha_i \ge 2}} \binom{2M}{\alpha_1, \ldots, \alpha_r} \sum_{2^m < q_1 < \ldots < q_r \le 2^{m+1}} \frac{1}{q_1^{\alpha_1} \cdots q_r^{\alpha_r}}$$

We omit the ordering of the primes and drop the condition that they are distinct so that

$$\begin{aligned} |W| \le & (3C)^{2M} \sum_{1 \le r \le M} \sum_{\substack{\alpha_1 + \cdots + \alpha_r = 2M \\ \alpha_i \ge 2}} \binom{2M}{\alpha_1, \ldots, \alpha_r} \frac{1}{r!} \prod_{1 \le i \le r} \left(\sum_{2^m < q_i \le 2^{m+1}} \frac{1}{q_i^{\alpha_i}} \right) \\ \le & (3C)^{2M} \sum_{1 \le r \le M} \sum_{\substack{\alpha_1 + \cdots + \alpha_r = 2M \\ \alpha_i \ge 2}} \frac{2^{mr}}{2^{m \cdot 2M}} \frac{(2M)!}{\alpha_1! \cdots \alpha_r!} \frac{1}{r!} \\ \le & (3C)^{2M} \frac{(2M)!}{2^{m \cdot 2M}} \sum_{1 \le r \le M} \sum_{\substack{\alpha_1 + \cdots + \alpha_r = 2M \\ \alpha_i \ge 2}} \frac{2^{mr}}{r!}. \end{aligned}$$

Comparing the ratios of consecutive terms of the sequence $2^{rm}/r!$ we see that the sequence is increasing and so its maximum is attained when $r = M$. Together with the trivial bound

$$\sum_{1\le r\le M} \sum_{\substack{\alpha_1+\cdots+\alpha_r=2M \\ \alpha_i\ge 2}} 1 \le M\cdot 2^{2M} \le 2^{3M}$$

we conclude that

$$|W| \le (72C^2)^M \frac{(2M)!}{M!}\frac{1}{2^{mM}},$$

which completes the proof of the lemma. □

We know that the expectation of a product of independent random variables is equal to the product of the expectations. The following lemma reminds us of this fact in our specialized setting.

Lemma 3.2.5. *Let $u(p), w(p)$ be any real numbers such that $|u(p)| \le p^{1/2}$ and $|w(p)| \le C \le p$, for a constant $C \ge 0$. Suppose k is large, fix the real numbers $1 \le y_{i-1} < y_i$ for $1 \le i \le I$ and let n_i, m, M be positive integers such that $2^{(m+1)\cdot 2M}\prod_{i=1}^{I} y_i^{n_i} \le k^2/10^4$. Moreover, let $M \le 2^m$ and $2^{m+1} \le y_0$ if $M \ne 0$, then*

$$\begin{aligned}
&\sum_{g\in B_{2k}}^{h} \prod_{1\le i\le I}\Bigg(\sum_{y_{i-1}<p\le y_i}\frac{u(p)\lambda_g(p)}{p^{1/2}}\Bigg)^{n_i}\cdot\Bigg(\sum_{2^m<q\le 2^{m+1}}\frac{w(q)\lambda_g(q^2)}{q}\Bigg)^{2M}\\
&\ll \prod_{1\le i\le I}\frac{1_{2|n_i}\cdot n_i!}{2^{n_i/2}(n_i/2)!}\Bigg(\sum_{y_{i-1}<p\le y_i}\frac{u(p)^2}{p}\Bigg)^{\frac{n_i}{2}}\cdot\frac{(2M)!}{M!}\Bigg(\frac{72C^2}{2^m}\Bigg)^{M} + k^7e^{-k}.
\end{aligned}$$

Proof. We want to apply the Petersson Trace Formula to detect the random behaviour of the coefficients $\lambda_g(p)$ and $\lambda_g(p^2)$. We start by expanding the n_i-th and $2M$-th powers.

$$\begin{aligned}
&\sum_{g\in B_{2k}}^{h} \prod_{1\le i\le I}\Bigg(\sum_{y_{i-1}<p\le y_i}\frac{u(p)\lambda_g(p)}{p^{1/2}}\Bigg)^{n_i}\cdot\Bigg(\sum_{2^m<q\le 2^{m+1}}\frac{w(q)\lambda_g(q^2)}{q}\Bigg)^{2M}\\
=&\sum_{g\in B_{2k}}^{h} \prod_{1\le i\le I}\Bigg(\sum_{y_{i-1}<p_1,\dots,p_{n_i}\le y_i}\prod_{1\le r\le n_i}\frac{u(p_r)\lambda_g(p_r)}{p_r^{1/2}}\Bigg)\cdot\Bigg(\sum_{2^m<q_1,\dots,q_{2M}\le 2^{m+1}}\prod_{1\le s\le 2M}\frac{w(q_s)\lambda_g(q_s^2)}{q_s}\Bigg).
\end{aligned}$$

Next we expand the product over i and interchange the order of summation. We get

$$\sum_{\widetilde{p}}\sum_{\widetilde{q}} C(\widetilde{p})D(\widetilde{q}) \cdot \sum_{g\in B_{2k}}^{h} \prod_{1\le i\le I}\prod_{1\le r\le n_i} \lambda_g(p_{i,r}) \prod_{1\le s\le 2M} \lambda_g(q_s^2), \tag{3.9}$$

where

$$C(\widetilde{p}) = \prod_{1\le i\le I}\left(\prod_{1\le r\le n_i} \frac{u(p_{i,r})}{p_{i,r}^{1/2}}\right) \quad \text{and} \quad D(\widetilde{q}) = \prod_{1\le s\le 2M} \frac{w(q_s)}{q_s}$$

with $\widetilde{p} = (p_{1,1}, p_{1,2}, \ldots p_{1,n_1}, p_{2,1}, \ldots p_{2,n_2}, \ldots p_{I,n_I})$ and $\widetilde{q} = (q_1, \ldots q_{2M})$. Each component of the vectors $\widetilde{p}$ and $\widetilde{q}$ is prime and they satisfy the conditions

$$y_{i-1} < p_{i,1}, \ldots, p_{i,I} \le y_i \quad \forall 1 \le i \le I \quad \text{and} \quad 2^m < q_1, \ldots q_{2M} \le 2^{m+1}.$$

By our assumption $\prod_{i=1}^{I} y_i^{n_i} \cdot 2^{(m+1)\cdot 2M} \le k^2/10^4$ and since $2^{m+1} \le y_0$ the primes $p_{i,r}$ are distinct from the primes q_s. Hence we can apply the Petersson Trace Formula, namely Lemma 3.2.2, and get

$$\sum_{g\in B_{2k}}^{h} \prod_{1\le i\le I}\prod_{1\le r\le n_i} \lambda_g(p_{i,r}) \prod_{1\le s\le 2M} \lambda_g(q_s^2) = h_1\Bigg(\prod_{1\le i\le I}\prod_{1\le r\le n_i} p_{i,r}\Bigg) \cdot h_2\Bigg(\prod_{1\le q\le 2s} q_s\Bigg) + O(k^5 e^{-k}).$$

It follows that expression (3.9) is equal to

$$\sum_{\widetilde{p}}\sum_{\widetilde{q}} C(\widetilde{p})D(\widetilde{q}) \cdot h_1\Bigg(\prod_{1\le i\le I}\prod_{1\le r\le j_i} p_{i,r}\Bigg) h_2\Bigg(\prod_{1\le s\le 2M} q_s\Bigg) + O\Bigg(e^{-k}k^5 \sum_{\widetilde{p}}\sum_{\widetilde{q}} |C(\widetilde{p})D(\widetilde{q})|\Bigg).$$

To bound the main term we notice that there is no dependency on the cusp forms g anymore and so we can analyze the sums over $\widetilde{p}$ and $\widetilde{q}$ separately. We begin with the summation over $\widetilde{p}$ and use the multiplicativity of $h_1(n)$ so that this part of the main term equals in absolute value

$$\begin{aligned}
&\Big|\sum_{\widetilde{p}} C(\widetilde{p}) \cdot h_1\bigg(\prod_{1\le i\le I}\prod_{1\le r\le j_i} p_{i,r}\bigg)\Big| \qquad (3.10)\\
=&\Big|\prod_{1\le i\le I}\bigg(\sum_{y_{i-1}<p_{i,1},\dots p_{i,n_i}\le y_i} \frac{u(p_{i,1})\cdots u(p_{i,n_i})}{\sqrt{p_{i,1}\cdots p_{i,n_i}}}\cdot h_1(p_{i,1}\cdots p_{i,n_i})\bigg)\Big|\\
\le&\prod_{1\le i\le I}\frac{1_{2|n_i}\cdot n_i!}{2^{n_i/2}(n_i/2)!}\bigg(\sum_{y_{i-1}<p\le y_i}\frac{u(p)^2}{p}\bigg)^{\frac{n_i}{2}}, \qquad (3.11)
\end{aligned}$$

by Lemma 3.2.3. Similarly, for the sum over $\widetilde{q}$ we get

$$\begin{aligned}
&\Big|\sum_{\widetilde{q}} D(\widetilde{q}) \cdot h_2\bigg(\prod_{1\le s\le 2M} q_s\bigg)\Big|\\
=&\Big|\sum_{2^m<q_1,\dots,q_{2M}\le 2^{m+1}}\frac{w(q_1)\cdots w(q_{2M})}{q_1\cdots q_{2M}}\cdot h_2(q_1\cdots q_{2M})\Big|\\
\le&\frac{(2M)!}{M!}\left(\frac{72C^2}{2^m}\right)^M, \qquad (3.12)
\end{aligned}$$

where we used Lemma 3.2.4. It remains to control the error term, which is given by

$$\begin{aligned}
&O\bigg(e^{-k}k^5\sum_{\widetilde{p}}\sum_{\widetilde{q}}|C(\widetilde{p})D(\widetilde{q})|\bigg)\\
=&O\Bigg(e^{-k}k^5\prod_{1\le i\le I}\bigg(\sum_{y_{i-1}<p\le y_i}\frac{|u(p)|}{p^{1/2}}\bigg)^{n_i}\cdot\bigg(\sum_{2^m<q\le 2^{m+1}}\frac{|w(q)|}{q}\bigg)^{2M}\Bigg)\\
=&O(e^{-k}k^7).
\end{aligned}$$

In the last line we used $|u(p)| \le p^{1/2}$, $|w(p)| \le p$ and the condition $\prod_{i=1}^I y_i^{n_i}\cdot 2^{(m+1)\cdot 2M}\le k^2/10^4$. Inserting (3.11) and (3.12) into (3.9) together with the error term calculation concludes the proof of Lemma 3.2.5. $\square$

3.2.2 Setup

Recall that Lemma 3.2.1 tells us essentially that

$$L(1/2, f \times f \times g) \ll \exp\Big(\sum_{p \le x} \frac{\lambda_f(p)^2 \lambda_g(p)}{p^{1/2}}\Big) \exp\Big(\sum_{p^2 \le x} \frac{(\lambda_f(p)^4 - 4\lambda_f(p)^2 + 4) \cdot \lambda_g(p^2)}{2p}\Big) \cdot \exp\Big(\sum_{p^2 \le x} \frac{-\lambda_f(p)^4 + 4\lambda_f(p)^2 - 4}{2p}\Big). \tag{3.13}$$

We want to average the right-hand side of expression (3.13) over g. Since the third exponential in (3.13) is independent of g, we can delay its treatment until the very end of our main proof (see Section 3.2.6). On the first two exponentials we will perform a Taylor expansion and so it is important to control the size of the corresponding Dirichlet polynomials. This is quite technical, and here it is how we do it precisely:

For k large enough, define the sequence $(\beta_i)_{i \ge 0}$ by

$$\beta_0 := 0; \quad \beta_i := \frac{20^{i-1}}{(\log\log k)^2} \quad \text{for all } i \ge 1, \tag{3.14}$$

and

$$I = I_k := 1 + \max\{i \colon \beta_i \le e^{-10000}\}.$$

To simplify notation write

$$x_j := k^{\beta_j} \quad \text{and} \quad u_{f,j}(p) := \frac{\lambda_f(p)^2}{p^{1/(\beta_j \log k)}} \frac{\log(x_j/p)}{\log x_j} \le \lambda_f(p)^2. \tag{3.15}$$

For each $1 \le i \le j \le I$ define

$$G_{(i,j)}(g) := \sum_{x_{i-1} < p \le x_i} \frac{\lambda_f(p)^2 \lambda_g(p)}{p^{1/2 + 1/(\beta_j \log k)}} \frac{\log(x_j/p)}{\log x_j} = \sum_{x_{i-1} < p \le x_i} \frac{u_{f,j}(p) \lambda_g(p)}{p^{1/2}}.$$

Let us now define the set of cusp forms for which a given Dirichlet polynomial is smaller than a suitable threshold by

$$\mathcal{G} = \mathcal{G}_k := \{g \in B_{2k} : |G_{(i,I)}(g)| \leq \beta_i^{-3/4} \text{ for all } i = 1, 2 \ldots I\}.$$

Finally, we define the exceptional sets where the given Dirichlet polynomials are large. These sets build the complement to $\mathcal{G}$ and the argument to handle these exceptional sets will be different. For $0 \leq j \leq I - 1$, we define

$$\begin{aligned}\mathcal{E}(j) = \mathcal{E}_k(j) :=\Big\{ & g \in B_{2k} \colon |G_{(i,\ell)}(g)| \leq \beta_i^{-3/4} \text{ for all } 1 \leq i \leq j, \text{ for all } i \leq \ell \leq I, \\ & \text{but } |G_{(j+1,\ell)}(g)| > \beta_{j+1}^{-3/4} \text{ for some } \ell \in \{j+1, \ldots, I\}\Big\}.\end{aligned}$$

Note that the variance of a Dirichlet polynomial of the form $\sum_{p \leq k} \frac{\lambda_g(p)}{p^{1/2}}$ is of size $\log\log k$. Hence it is a rare event that such a Dirichlet polynomial is larger than $(\log\log k)^{3/2}$, which is roughly $\beta_i^{-3/4}$. This motivates the choice of the parameters above.

The above definitions complete the required setting for the first Dirichlet polynomial on the right-hand side of expression (3.13). To handle the second Dirichlet polynomial of expression (3.13), where the summation ranges over the primes squared, it will be convenient to introduce the following notation:

$$w_{f,j}(p) = \frac{(\lambda_f(p)^4 - 4\lambda_f(p)^2 + 4)}{2p^{1/(\beta_j \log k)}} \frac{\log(x_j/p^2)}{\log x_j} \leq 2 \tag{3.16}$$

and

$$P_m(g) := \sum_{2^m < p \leq 2^{m+1}} \frac{w_{f,I}(p)\lambda_g(p^2)}{p}. \tag{3.17}$$

Furthermore, define for $m \geq 0$ the set

$$\mathcal{P}(m) := \{g \in B_{2k} : |P_m(g)| > 2^{-m/10},\ \text{but } |P_n(g)| \leq 2^{-n/10} \text{ for all } m+1 \leq n \leq \log k/\log 2\}. \tag{3.18}$$

In particular $\mathcal{P}(0)$ is the set of $g \in B_{2k}$ such that $P_n(g) < 2^{-n/10}$ for all n. The philosophy behind this definition is similar to the definition of the sets $\mathcal{E}(j)$. The variance of $P_m(g)$ is roughly of size 2^{-m}, hence it should happen rarely that this Dirichlet polynomial is larger than $2^{-m/10}$, say.

The following lemma, whose proof can be found in Section 3.2.5, will be used to show that Dirichlet polynomials of the form (3.17) are negligible.

Lemma 3.2.6. *Let k be large enough and define $\mathcal{P}(m)$ as in* (3.18). *Suppose* $(\log\log k)^2 < 2^{m+1} \leq x_I = k^{\beta_I}$*, then for any $1 \leq j \leq I$ we have*

$$\sum_{g \in \mathcal{P}(m)}^{h} \exp\left(2 \sum_{p \leq 2^{m+1}} \frac{w_{f,I}(p)\lambda_g(p^2)}{p}\right) \ll (\log k)^{-68}.$$

The next lemma allows us to replace the exponential series of a Dirichlet polynomial with a finite series. The truncation error is negligible, provided that the Dirichlet polynomial is small. In fact, this is the reason why we defined the set of Dirichlet polynomials $\mathcal{G}$.

Lemma 3.2.7. *Let $\mathcal{S} \subset B_{2k}$ be a set of cusp forms and let $u(p), w(p)$ be arbitrary real numbers. Let m, M be any non-negative integer and fix the real numbers $1 \leq y_{i-1} < y_i$ for $1 \leq i \leq I$. Furthermore, suppose that $2^{m+1} \leq y_0$ if $M \neq 0$ and*

$$\left| \sum_{x_{j-1} < p \leq x_j} \frac{u(p)\lambda_g(p)}{p^{1/2}} \right| \leq 2\beta_j^{-3/4} \tag{3.19}$$

for any $1 \le j \le I$ *and* $g \in \mathcal{S}$*. Then we have*

$$\begin{aligned}
&\sum_{g\in\mathcal{S}}^{h} \exp\left(\sum_{x_0<p\le x_j} \frac{u(p)\lambda_g(p)}{p^{1/2}}\right)\cdot\left(\sum_{2^m<p\le 2^{m+1}} \frac{w(p)\lambda_g(p^2)}{p}\right)^{2M} \\
&\ll \sum_{\tilde{n}} \prod_{1\le i\le j} \frac{1}{n_i!} \sum_{g\in B_{2k}}^{h} \prod_{1\le i\le j} \left(\sum_{x_{i-1}<p\le x_i} \frac{u(p)\lambda_g(p)}{p^{1/2}}\right)^{n_i}\cdot\left(\sum_{2^m<p\le 2^{m+1}} \frac{w(p)\lambda_g(p^2)}{p}\right)^{2M},
\end{aligned}$$

where $\tilde{n} = (n_1, \ldots, n_j)$ *and each component satisfies* $n_i \le 2\lceil 50\beta_i^{-3/4}\rceil$*.*

For the proof we refer the reader again to Section 3.2.5.

3.2.3 Main Contribution - Treating $\mathcal{G}$

We are now in the position to establish our main lemmas. The following lemma resembles the computation $\mathbb{E}[\exp(X)] = \exp(\mu + \sigma^2/2)$ for a Gaussian random variable with mean μ and variance σ^2. We can think of our Dirichlet polynomial $G_{(i,I)}$ as a random variable with mean $\mu = 0$ and variance $\sigma^2 = \sum_p \frac{\lambda_f(p)^4}{p}$. We do not know how to integrate exponentials, so we write them as finite sum using Taylor's theorem (see Lemma 3.2.7). Since our Dirichlet polynomials do not take large values, we only need a few terms in the Talyor expansion, so that the resulting Dirichlet polynomials have manageable length. Having done this, we can change the order of summation, which reminds us of the linearity of expectations in a probabilistic setting. The lemmas in the previous sections then allow us to deduce the desired random behaviour.

Lemma 3.2.8. *We follow the notation from Section 3.2.2. Let* $u(p)$ *be any real numbers such that* $|u(p)| \le p^{1/2}$ *and let* $\mathcal{S} \subset B_{2k}$ *such that*

$$\left|\sum_{x_{j-1}<p\le x_j} \frac{u(p)\lambda_g(p)}{p^{1/2}}\right| \le 2\beta_j^{-3/4} \tag{3.20}$$

for any $1 \le j \le I$ and $g \in \mathcal{S}$. Then we have for k large enough

$$\sum_{g\in\mathcal{S}}^{h} \exp\left(\sum_{p\le x_I} \frac{u(p)\lambda_g(p)}{p^{1/2}}\right) \ll \exp\left(\frac{1}{2}\sum_{p\le x_I}\frac{u(p)^2}{p}\right).$$

Proof. We abbreviate

$$U = \sum_{g\in\mathcal{S}}^{h} \exp\left(\sum_{p\le x_I} \frac{u(p)\lambda_g(p)}{p^{1/2}}\right).$$

Using our assumption (3.20) we can directly apply Lemma 3.2.7 with $y_i = x_i = k^{\beta_j}$, $M = 0$ and see that

$$U \ll \sum_{\widetilde{n}} \prod_{1\le i\le I} \frac{1}{n_i!} \sum_{g\in B_{2k}}^{h} \prod_{1\le i\le j} \left(\sum_{x_{i-1}<p\le x_i} \frac{u(p)\lambda_g(p)}{p^{1/2}}\right)^{n_i} \tag{3.21}$$

with $n_i \le 2\lceil 50\beta_i^{-3/4}\rceil$.

Note that

$$\prod_{1\le i\le I} x_i^{n_i} = \prod_{1\le i\le I} k^{\beta_i n_i} \le \prod_{1\le i\le I} k^{200\beta_i^{1/4}} \le k^{400\beta_I^{1/4}} \le k^2/10^4 \tag{3.22}$$

for k large enough. Here we used that $\beta_i^{1/4}$ form a geometric progression of ratio $20^{1/4} \ge 2$. We can now apply Lemma 3.2.5 to the right-hand side of (3.21) and see that U is bounded by

$$\sum_{\widetilde{n}} \prod_{1\le i\le I} \frac{1_{2|n_i}}{2^{n_i/2}(n_i/2)!}\left(\sum_{x_{i-1}<p\le x_i} \frac{u(p)^2}{p}\right)^{n_i/2} + O\left(k^7 e^{-k} \sum_{\widetilde{n}} \prod_{1\le i\le I} \frac{1}{n_i!}\right).$$

The error term is negligible since

$$k^7 e^{-k} \sum_{\widetilde{n}} \prod_{1\le i\le I} \frac{1}{n!} \le k^7 e^{-k} \prod_{1\le i\le I} \sum_{n_i\le 200\beta_i^{-3/4}} \frac{1}{n!} \le k^7 e^{-k} e^I \le k^8 e^{-k}. \tag{3.23}$$

Writing

$$\sum_{\widetilde{n}} \prod_{1\le i\le I} \frac{1_{2|n_i}}{2^{n_i/2}(n_i/2)!}\left(\sum_{x_{i-1}<p\le x_i} \frac{u(p)^2}{p}\right)^{n_i/2} = \exp\left(\frac{1}{2}\sum_{p\le x_I}\frac{u(p)^2}{p}\right)$$

completes the proof of the lemma. □

Now that we have considered the case for generic coefficients $u(p)$ let us focus on our Dirichlet polynomials of interest $G_{i,I}$, together with the Dirichlet polynomial that arises from summing over primes squared. The proof idea for the following lemma remains the same as for Lemma 3.2.8, albeit the proof being a bit more technical.

Lemma 3.2.9. *Let k be large enough and follow the notation from Section 3.2.2, then*

$$\sum_{g\in\mathcal{G}}^{h} \exp\left(\sum_{p\le x_I}\frac{u_{f,I}(p)\lambda_g(p)}{p^{1/2}}\right)\exp\left(\sum_{p^2\le x_I}\frac{w_{f,I}(p)\lambda_g(p^2)}{p}\right) \ll \exp\left(\frac{1}{2}\sum_{p\le x_I}\frac{u_{f,I}(p)^2}{p}\right). \tag{3.24}$$

Proof. Recall the definition of the set $\mathcal{P}(m)$ in (3.18). The left-hand side of (3.24) is bounded by

$$\sum_{0\le m\le\log k}\ \sum_{g\in\mathcal{G}\cap\mathcal{P}(m)}^{h} \exp\left(\sum_{p\le x_I}\frac{u_{f,I}(p)\lambda_g(p)}{p^{1/2}}\right)\exp\left(\sum_{p^2\le x_I}\frac{w_{f,I}(p)\lambda_g(p^2)}{2p}\right).$$

If $g\in\mathcal{P}(m)$ then clearly

$$\sum_{2^{m+1}<p\le\sqrt{x_I}}\frac{w_{f,I}(p)\lambda_g(p^2)}{p} = O(1).$$

In particular, if $g\in\mathcal{P}(0)$ then $\sum_{p^2\le x_I}\frac{w_{f,I}(p)\lambda_g(p^2)}{p} = O(1)$ and Lemma 3.2.9 follows directly from Lemma 3.2.8 after setting $u(p)=u_{f,I}(p)$. It thus suffices to bound the quantity

$$\sum_{1\le m\le\log k}\ \sum_{g\in\mathcal{G}\cap\mathcal{P}(m)}^{h} \exp\left(\sum_{p\le x_I}\frac{u_{f,I}(p)\lambda_g(p)}{p^{1/2}}\right)\exp\left(\sum_{p\le 2^{m+1}}\frac{w_{f,I}(p)\lambda_g(p^2)}{2p}\right).$$

Splitting into the sets $\mathcal{P}(m)$ enables us to show that the contribution from the primes squared part is negligible. We start by looking at the case when $m\le(2/\log 2)\log\log\log k$. Consider the following quantity for $g\in\mathcal{P}(m)$, which is a sum of the small primes of our Dirichlet polynomial over primes together with the primes squared Dirichlet polynomial:

$$\left|\sum_{p\le 2^{m+1}}\frac{u_{f,I}(p)\lambda_g(p)}{p^{1/2}} + \sum_{p\le 2^{m+1}}\frac{w_{f,I}(p)\lambda_g(p^2)}{2p}\right|. \tag{3.25}$$

We use the triangle inequality and the Deligne bound for the Fourier coefficients, i.e. $|u_{f,I}(p)\lambda_g(p)| \leq 8$ and $|w_{f,I}(p)\lambda_g(p^2)| \leq 6$, to see that (3.25) is bounded by

$$\sum_{p\leq 2^{m+1}} \frac{8}{\sqrt{p}} + \sum_{p\leq 2^{m+1}} \frac{6}{p} \leq 2^{m/2} + O(1).$$

This computation is useful so that $P_m(g)$ and the prime Dirichlet polynomial are running over disjoint primes, as we have an application of Lemma 3.2.5 in mind. We have

$$\begin{aligned} T(m) :=& \sum_{g\in\mathcal{G}\cap\mathcal{P}(m)}^h \exp\left(\sum_{p\leq x_I} \frac{u_{f,I}(p)\lambda_g(p)}{p^{1/2}}\right) \exp\left(\sum_{p\leq 2^{m+1}} \frac{w_{f,I}(p)\lambda_g(p^2)}{2p}\right) \\ \ll& e^{2^{m/2}} \sum_{g\in\mathcal{G}\cap\mathcal{P}(m)}^h \exp\left(\sum_{2^{m+1}<p\leq x_I} \frac{u_{f,I}(p)\lambda_g(p)}{p^{1/2}}\right) \\ \ll& e^{2^{m/2}} \sum_{g\in\mathcal{G}} \left(2^{m/10} P_m(g)\right)^{2M} \exp\left(\sum_{2^{m+1}<p\leq x_I} \frac{u_{f,I}(p)\lambda_g(p)}{p^{1/2}}\right), \end{aligned} \tag{3.26}$$

where M is any non-negative integer. We choose $M = \lfloor 2^{3m/4} \rfloor$ and this choice will become apparent in a calculation below.

Now we want to replace the exponential with a finite series. Since $g \in \mathcal{G}$ and $2^m \leq (\log\log k)^2$ we have that

$$\left| \sum_{2^{m+1}<p\leq x_I} \frac{u_{f,I}(p)\lambda_g(p)}{p^{1/2}} \right| \leq |G_{(i,I)}(g)| + \left| \sum_{p\leq 2^{m+1}} \frac{u_{f,I}(p)\lambda_g(p)}{p^{1/2}} \right| \leq 2\beta_i^{-3/4}$$

and the conditions of Lemma 3.2.7 are satisfied with $y_i := \max\{2^{m+1}, x_i\}$. Note that since $m \leq (2/\log 2)\log\log\log k$, we have that $y_i = x_i$ for $1 \leq i \leq I$ and $y_0 = 2^{m+1}$. An application of Lemma 3.2.7 to (3.26) shows that $T(m)$ is bounded by

$$e^{2^{m/2}} 2^{mM/5} \sum_{\widetilde{n}} \prod_{1\leq i\leq I} \frac{1}{n_i!} \sum_{g\in B_{2k}}^h \left\{ \prod_{1\leq i\leq I} \left(\sum_{x_{i-1}<p<x_i} \frac{u_{f,I}(p)\lambda_g(p)}{p^{1/2}} \right)^{n_i} \right\} \cdot \left(\sum_{2^m<p\leq 2^{m+1}} \frac{w_{f,I}(p)\lambda_g(p^2)}{p} \right)^{2M}. \tag{3.27}$$

The next step is to use the random behaviour of the coefficients $\lambda_g(p)$ and $\lambda_g(p^2)$ when averaged over $g \in B_{2k}$ as we did in Lemma 3.2.5. Note that $2^{(m+1)\cdot M} \ll (\log\log k)^{2\log\log k} = k^{o(1)}$ and so we have, as already seen for inequality (3.22),

$$2^{(m+1)M} \prod_{1\le i\le I} x_i^{n_i} \le k^{o(1)} \cdot k^{400\beta_I^{1/4}} \le k^2/10^4$$

for k large enough. We can therefore apply Lemma 3.2.5 with $u(p) = u_{f,I}(p)$ and $w(p) = w_{f,I}(p) \le 4$ so that $T(m)$ is bounded up to an error term by

$$e^{2^{m/2}} 2^{mM/5} \sum_{\widetilde{n}} \prod_{1\le i\le I} \frac{1_{2|n_i}}{2^{n_i/2}(n_i/2)!} \left(\sum_{x_{i-1}<p\le x_i} \frac{u_{f,I}(p)^2}{p} \right)^{n_i/2} \cdot \frac{(2M)!}{M!} \left(\frac{288}{2^m}\right)^M$$

with $\widetilde{n} = (n_1, \dots, n_I)$ and each component satisfies $n_i \le 2\lceil 50\beta_i^{-3/4}\rceil$. The mentioned error term is bounded as in inequality (3.23) by

$$k^7 e^{-k} \cdot e^{2^{m/2}} 2^{mM/5} \sum_{\widetilde{n}} \prod_{1\le i\le I} \frac{1}{n_i!} \ll k^9 e^{-k} \tag{3.28}$$

and is therefore negligible. Rearranging the Dirichlet polynomial over primes into an exponential and applying Stirling's formula, giving $(2M)!/M! \ll \left(\frac{2^{2M}M}{e}\right)^M$, we see that

$$T(m) \ll e^{2^{m/2}} \exp\left(\frac{1}{2} \sum_{2^{m+1}\le p\le x_I} \frac{u_{f,I}(p)^2}{p} \right) \cdot \left(\frac{2^{m/5}\cdot M \cdot 1152 \cdot 2^{-m}}{e} \right)^M. \tag{3.29}$$

By our choice of $M = \lfloor 2^{3m/4} \rfloor$ we have

$$\left(\frac{2^{m/5}\cdot 2^{3m/4} \cdot 1152 \cdot 2^{-m}}{e} \right)^{\lfloor 2^{3m/4}\rfloor} \ll e^{-2^{3m/4}}$$

and so

$$T(m) \ll e^{2^{m/2} - 2^{3m/4}} \cdot \exp\left(\frac{1}{2} \sum_{p\le x_I} \frac{u_{f,I}(p)^2}{p} \right).$$

Summing $T(m)$ over $m \leq (2/\log 2)\log\log\log k$ concludes the proof of the lemma in the given range of m.

In the remaining case, when $(\log\log k)^2 < 2^{m+1} \leq \log k$ an application of the Cauchy–Schwarz inequality will be enough to conclude the lemma. We have

$$\begin{aligned} T(m) &= \sum_{g\in\mathcal{G}\cap\mathcal{P}(m)}^{h} \exp\left(\sum_{p\leq x_I} \frac{u_{f,I}(p)\lambda_g(p)}{p^{1/2}}\right)\exp\left(\sum_{p\leq 2^{m+1}} \frac{w_{f,I}(p)\lambda_g(p^2)}{2p}\right) \\ &\leq \left(\sum_{g\in\mathcal{G}}^{h} \exp\left(2\sum_{p\leq x_I} \frac{u_{f,I}(p)\lambda_g(p)}{p^{1/2}}\right)\right)^{1/2} \cdot \left(\sum_{g\in\mathcal{P}(m)}^{h} \exp\left(2\sum_{p\leq 2^{m+1}} \frac{w_{f,I}(p)\lambda_g(p^2)}{p}\right)\right)^{1/2}. \end{aligned} \tag{3.30}$$

Using Lemma 3.2.8, the first factor of (3.30) is bounded by

$$\exp\left(\sum_{p\leq x_I} \frac{\lambda_f(p)^4}{p}\right) \ll (\log k)^{2^4}.$$

For the second part of (3.30) we apply Lemma 3.2.6. Combining these two bounds we see that $T(m) \ll (\log k)^{-18}$ for m such that $(\log\log k)^2 < 2^{m+1} \leq \log k$. Summing over m we have

$$\sum_{(\log\log k)^2<2^{m+1}\leq\log k} T(m) \ll (\log k)^{-17},$$

which is clearly negligible and so the claim of Lemma 3.2.9 follows.

□

3.2.4 New Exceptional Set Contribution - Treating $\mathcal{E}(j)$

In this section we treat the exceptional sets, i.e. those cusp forms where some (possibly all) parts of the Dirichlet polynomial are large. In this case we cannot apply our techniques from the last section. Although these large values cause some trouble, they are very rare. With a Markov inequality type argument, we can indeed show that the measure of these 'bad' sets is so small, that the entire contribution is negligible. Unsurprisingly, the argument will remind us of the treatment of the primes squared part in Lemma 3.2.9.

Recall that we are now interested in the set of cusp forms, where the corresponding Dirichlet polynomial might get large. For $0 \le j \le I-1$, we defined

$$\begin{aligned}\mathcal{E}(j) = \mathcal{E}_k(j) :=& \Big\{ g \in B_{2k} \colon |G_{(i,\ell)}(g)| \le \beta_i^{-3/4} \text{ for all } 1 \le i \le j, \text{ for all } i \le \ell \le I, \\ & \text{but } |G_{(j+1,\ell)}(g)| > \beta_{j+1}^{-3/4} \text{ for some } \ell \in \{j+1, \dots, I\} \Big\}.\end{aligned}$$

Lemma 3.2.10. *For k large enough and following the notation in Section 3.2.2, we have*

$$\operatorname{meas}\{\mathcal{E}(0)\} = \sum_{g \in \mathcal{E}(0)}^{h} 1 \ll e^{-(\log\log k)^2/C}$$

with $C = 2^5 \cdot 10/e$. Moreover, for any $1 \le j \le I-1$ we have that

$$\sum_{g \in \mathcal{E}(j)}^{h} \exp\left(\sum_{p \le x_j} \frac{u_{f,j}(p)\lambda_g(p)}{p^{1/2}} \right) \exp\left(\sum_{p^2 \le x_j} \frac{w_{f,j}(p)\lambda_g(p^2)}{p} \right) \ll \exp\left(\frac{1}{2} \sum_{p \le x_j} \frac{u_{f,j}(p)^2}{p} \right) e^{(4C\beta_{j+1})^{-1}\log\beta_{j+1}}.$$

Proof. We treat the primes squared part as in the proof of Lemma 3.2.9. By the exact same reduction as in Lemma 3.2.9 it suffices to control

$$\begin{aligned} S(m) :=& \sum_{g \in \mathcal{E}(j) \cap \mathcal{P}(m)}^{h} \exp\left(\sum_{p \le x_j} \frac{u_{f,j}(p)\lambda_g(p)}{p^{1/2}} \right) \exp\left(\sum_{p^2 \le x_j} \frac{w_{f,j}(p)\lambda_g(p^2)}{p} \right) \\ \ll& e^{2^{m/2}} \sum_{g \in \mathcal{E}(j)}^{h} \exp\left(\sum_{2^{m+1} < p \le x_j} \frac{u_{f,j}(p)\lambda_g(p)}{p^{1/2}} \right) \cdot (2^{m/10} P_m(g))^{2M} \end{aligned} \tag{3.31}$$

for $m \le (2/\log 2)\log\log\log k$. By the definition of the set $\mathcal{E}(j)$ and Markov's inequality

$S(m)$ is bounded by

$$e^{2^{m/2}} \sum_{\ell=j+1}^{I} \sideset{}{^h}\sum_{\substack{g\in B_{2k}:|G_{(i,j)}(g)|\le \beta_i^{-3/4}\forall 1\le i\le j,\\ |G_{j+1,\ell}(g)|>\beta_{j+1}^{-3/4}}} \exp\left(\sum_{2^{m+1}\le p\le x_j} \frac{u_{f,j}(p)\lambda_g(p)}{p^{1/2}}\right)\cdot (2^{m/10}P_m(g))^{2M} \quad (3.32)$$

$$\le e^{2^{m/2}} \sum_{\ell=j+1}^{I} \sideset{}{^h}\sum_{\substack{g\in B_{2k}:|G_{(i,j)}(g)|\le \beta_i^{-3/4}\\ \forall 1\le i\le j}} \exp\left(\sum_{2^{m+1}\le p\le x_j} \frac{u_{f,j}(p)\lambda_g(p)}{p^{1/2}}\right)\left(\beta_{j+1}^{3/4}G_{(j+1,\ell)}(g)\right)^{2L}\left(2^{m/10}P_m(g)\right)^{2M}, \quad (3.33)$$

where L is any non-negative integer, which we choose to be $L = \lfloor (C\beta_{j+1})^{-1}\rfloor$, with $C = 2^5\cdot 10/e$. Now we are again in the position to truncate the exponential and proceed as in Lemma 3.2.9, more precisely by Lemma 3.2.7 we get that $S(m)$ is bounded by

$$e^{2^{m/2}}2^{mM/5}\beta_{j+1}^{3L/2}\sum_{\ell=j+1}^{I}\sum_{\tilde{n}}\prod_{1\le i\le j}\frac{1}{n_i!}\sideset{}{^h}\sum_{g\in B_{2k}}\prod_{1\le i\le j}\left(\sum_{x_{i-1}<p\le x_i}\frac{u_{f,j}(p)\lambda_g(p)}{p^{1/2}}\right)\cdot G_{(j+1,\ell)}^{2L}\cdot P_m(g)^{2M}$$

with $\tilde{n} = (n_1,\dots n_j)$, and each component satisfies $n_i \le 2\lceil 50\beta_i^{-3/4}\rceil$. Again we use Lemma 3.2.5 to capture the random behaviour of the coefficients $\lambda_g(p)$ and $\lambda_g(p)^2$. This lemma is applicable since

$$2^{(m+1)2M}\cdot x_{j+1}^{2L}\prod_{1\le i\le j}x_i^{n_i} \le k^{o(1)}\cdot k^{2/C}\prod_{1\le i\le I}k^{100\beta_i{}^{1/4}} \le k^2/10^4.$$

Then the main term of $S(m)$ is bounded by

$$e^{2^{m/2}}2^{mM/5}\beta_{j+1}^{3L/2}\sum_{\ell=j+1}^{I}\sum_{\tilde{n}}\left\{\prod_{1\le i\le j}\frac{1_{2|n_i}}{2^{n_i/2}(n_i/2)!}\left(\sum_{x_{i-1}<p\le x_i}\frac{u_{f,j}(p)^2}{p}\right)^{n_i/2}\right\}\cdot$$
$$\cdot\frac{(2L)!}{L!}\left(\sum_{x_j<p\le x_{j+1}}\frac{u_{f,j+1}(p)^2}{p}\right)^{L}\cdot\frac{(2M)!}{M!}\left(\frac{288}{2^m}\right)^{M}.$$

As in Lemma 3.2.9 we write this in terms of an exponential and we use the bound

$u_{f,j+1}(p) \leq \lambda_f(p)^2$, so that $S(m)$ is controlled by

$$e^{2^{m/2}} 2^{mM/5} \beta_{j+1}^{3L/2} (I-j) \exp\left(\frac{1}{2} \sum_{p \leq x_j} \frac{u_{f,j}(p)^4}{p}\right) \cdot \frac{(2L)!}{2^L L!} \left(\sum_{x_j < p \leq x_{j+1}} \frac{\lambda_f(p)^4}{p} \right)^L \cdot \frac{(2M)!}{M!} \left(\frac{288}{2^m}\right)^M .$$

The error term arising from Lemma 3.2.5 is again negligible by the same computation as in (4.2). Together with a Stirling estimate this computation yields

$$S(m) \ll e^{2^{m/2}} (I-j) \exp\left(\frac{1}{2} \sum_{p \leq x_j} \frac{u_{f,j}(p)^2}{p}\right) \cdot \left(\frac{\beta_{j+1}^{3/2} \cdot 2L}{e} \sum_{x_j < p \leq x_{j+1}} \frac{\lambda_f(p)^4}{p} \right)^L \cdot \left(\frac{2^{m/5} \cdot M \cdot 1152 \cdot 2^{-m}}{e} \right)^M . \tag{3.34}$$

In the case $1 \leq j \leq I-1$ we have by the definition of β_j and I, that

$$I - j = \frac{\log(\beta_I/\beta_j)}{\log 20} \leq \frac{\log(1/\beta_j)}{\log 20}$$

and

$$\sum_{k^{\beta_j} < p \leq k^{\beta_{j+1}}} \frac{\lambda_f(p)^4}{p} \leq 2^4 (\log \beta_{j+1} - \log \beta_j + o(1)) = 2^4 (\log 20 + o(1)) \leq 2^4 \cdot 10.$$

Consequently,

$$(I-j) \cdot \left(\frac{\beta_{j+1}^{3/2} \cdot 2L}{e} \sum_{k^{\beta_j} < p \leq k^{\beta_{j+1}}} \frac{\lambda_f(p)^4}{p} \right)^L \leq \frac{\log(1/\beta_j)}{\log 20} (\beta_{j+1}^{1/2})^{1/(C \cdot \beta_{j+1})} \tag{3.35}$$

The right-hand side of inequality (3.35) is bounded by

$$e^{(4C \cdot \beta_{j+1})^{-1} \log \beta_{j+1}},$$

which is small since $\beta_{j+1} \leq \beta_I \leq 20 e^{-10^5}$. Summing over m as we did in Lemma 3.2.9 shows that (3.35) is bounded by

$$\exp\left(\frac{1}{2}\sum_{p\le x_j}\frac{u_{f,j}(p)^2}{p}\right)e^{(4C\beta_{j+1})^{-1}\log\beta_{j+1}}.$$

The remaining case when $m \geq (2/\log 2)\log\log\log k$ is negligible compared to the main term. As in the proof of Lemma 3.2.9 this can be seen by an application of the Cauchy–Schwarz inequality and Lemma 3.2.6. This finishes the proof of the lemma for the cases $1 \leq j \leq I-1$.

It remains to show the first assertion of the lemma, namely

$$\sideset{}{^h}\sum_{g\in\mathcal{E}(0)} 1 \ll e^{-\log\log k^2/C}.$$

Note that from the definition of I we see that $I \leq \log\log\log k$. Moreover,

$$\beta_0 = 0, \quad \beta_1 = \frac{1}{(\log\log k)^2}, \qquad \sum_{p\le k^{1/(\log\log k)^2}}\frac{\lambda_f(p)^4}{p} \leq 2^4\log\log k.$$

Following the argument from before for $1 \leq j \leq I$ without the exponential factors we see that

$$\begin{aligned}\sideset{}{^h}\sum_{g\in\mathcal{E}(0)} 1 &\ll I\cdot\left(\frac{\beta_1^{3/2}\cdot 2L}{e}\sum_{p\le k^{\beta_1}}\frac{\lambda_f(p)^4}{p}\right)\\ &\ll \log\log\log k\cdot\left(\frac{\beta_1^{3/2}\cdot 2L}{e}\cdot 2^4\log\log k\right)^L\\ &\ll e^{-(\log\log k)^2/C}\end{aligned}$$

by our choice of L and C. This finishes the proof of the entire lemma.

$\square$

3.2.5 Technical Lemmas

In this section we quickly prove certain technical statements that were used in the section before. We also gather some additional technical lemmas that are needed in the final proof of Theorem 4.12.

Proof of Lemma 3.2.6. Since $|\lambda_g(p^2)| \le 3$ and $|w_{f,I}(p)|^2 \le 2$ we have that

$$2 \sum_{p \le 2^{m+1}} \frac{b_{f,I}(p)\lambda_g(p^2)}{p} \ll 12 \log\log 2^{m+1}.$$

An application of Markov's inequality yields

$$\begin{aligned} B(m) := \sum_{g \in \mathcal{P}(m)}^h \exp\left(2 \sum_{p \le 2^{m+1}} \frac{w_{f,I}(p)\lambda_g(p^2)}{p}\right) &\ll (\log 2^{m+1})^{12} \sum_{g \in \mathcal{P}(m)}^h 1 \\ &\le (\log 2^{m+1})^{12} \sum_{g \in B_{2k}}^h (2^{m/10} P_m(g))^{2M} \end{aligned} \tag{3.36}$$

for any non-negative integer M. We apply Lemma 3.2.5 (with $n_i = 0$ for $1 \le i \le I$) to evaluate the above moment and get

$$B(m) \ll (\log 2^{m+1})^{12} \cdot \frac{(2M)!}{M!} \left(\frac{72C^2 \cdot 2^{m/5}}{2^m}\right)^M, \tag{3.37}$$

provided that $2^{(m+1)2M} \le k^2/10^4$. We first investigate the case when $\log k \le 2^{m+1} \le \sqrt{x_I}$. In this range we have

$$2^{(m+1)2M} \le k^{\beta_I M} \le k^{20e^{-10000}M}$$

and so $M = 100$ is certainly admissible. By our choice of M and taking into account the size of 2^m we see that

$$B(m) \ll (\log k)^{12} 2^{-400m/5} \ll (\log k)^{12} \cdot (\log k)^{-80} = (\log k)^{-68}.$$

Next we consider the case $(\log\log k)^2 \leq 2^{m+1} \leq \log k$. Since the primes $p \leq 2^{m+1}$ are smaller in size we can afford to take higher moments. We pick $M = \lfloor 2^{3m/4} \rfloor$ so that $2^{(m+1)2M} \leq (\log k)^{(\log k)^{3/4}} \ll k^{o(1)} \leq k^2/10^4$. Together with the Stirling bound $(2M)!/M! \ll (4M/e)^M$ we see that $B(m)$ is bounded by

$$
\begin{aligned}
(\log 2^{m+1})^{12} \left(\frac{M \cdot 4608 \cdot 2^{m/5}}{e \cdot 2^m} \right)^M &\ll (\log\log k)^{15} e^{-2^{3m/4}} \\
&\ll (\log\log k)^{12} \exp(-(\log\log k)^{3/2}) \\
&\ll (\log k)^{-68}.
\end{aligned}
$$

We used that $2^{-m/20} \cdot 4608 \leq 1$, if k is sufficiently large and therefore also m is sufficiently large. This completes the proof of the lemma. □

The following lemma, due to Radziwiłł and Soundararajan [RS15, Lemma 1], will be helpful in the process of replacing the exponential series with a finite sum.

Lemma 3.2.11. *Let ℓ be a non-negative even integer, and x a real number. Define*

$$
E_\ell(x) = \sum_{j=0}^{\ell} \frac{x^j}{j!}.
$$

Then $E_\ell(x)$ is positive and for any $x \leq 0$ we have $E_\ell(x) \geq e^x$. Moreover, if $x \leq \ell/e^2$, then we have

$$
\exp(x) \leq \exp\Big(O(e^{-\ell})\Big) E_\ell(x).
$$

Proof of Lemma 3.2.7. Our goal is to truncate the exponential series $\exp(x)$ and replace it with a finite series up to ℓ. During this process we incur a negligible error term, provided that x is smaller than ℓ (see for example Lemma 3.2.11). This is the case for our Dirichlet polynomials by assumption (3.19). With $\ell = 2\lceil 50\beta_i^{-3/4}\rceil$ we have

$$
\begin{aligned}
&\sum_{g\in\mathcal{S}}^{h} \exp\left(\sum_{x_0<p\le x_j} \frac{u(p)\lambda_g(p)}{p^{1/2}}\right)\cdot\left(\sum_{2^m<p\le 2^{m+1}} \frac{w(p)\lambda_g(p^2)}{p}\right)^{2M} \\
=&\sum_{g\in\mathcal{S}}^{h} \prod_{1\le i\le j} \exp\left(\sum_{x_{i-1}<p\le x_i} \frac{u(p)\lambda_g(p)}{p^{1/2}}\right)\cdot\left(\sum_{2^m<p\le 2^{m+1}} \frac{w(p)\lambda_g(p^2)}{p}\right)^{2M} \\
\le&\sum_{g\in\mathcal{S}}^{h} \prod_{1\le i\le j} \exp\left(O(e^{-100\beta_i^{-3/4}})\right)\sum_{0\le n\le\ell}\frac{1}{n!}\left(\sum_{x_{i-1}<p\le x_i} \frac{u(p)\lambda_g(p)}{p^{1/2}}\right)^{n}\left(\sum_{2^m<p\le 2^{m+1}} \frac{w(p)\lambda_g(p^2)}{p}\right)^{2M} \\
\ll&\sum_{g\in\mathcal{S}}^{h} \prod_{1\le i\le j}\sum_{0\le n\le\ell}\frac{1}{n!}\left(\sum_{x_{i-1}<p\le x_i} \frac{u(p)\lambda_g(p)}{p^{1/2}}\right)^{n}\left(\sum_{2^m<p\le 2^{m+1}} \frac{w(p)\lambda_g(p^2)}{p}\right)^{2M} \qquad (3.38)
\end{aligned}
$$

In the third equality we used assumption (3.19) and Lemma 3.2.11. Note that $\sum_{0\le n\le\ell}\frac{x^n}{n!}\ge 0$ for every x, as ℓ is even. Using this positivity, we replace the sum over the restricted set $\sum_{g\in\mathcal{S}}^{h}$ with the full sum $\sum_{g\in B_{2k}}^{h}$. Additionally, we expand the product over i and so (3.38) is equal to

$$
\sum_{\widetilde{n}}\prod_{1\le i\le j}\frac{1}{n_i!}\sum_{g\in B_{2k}}^{h}\left(\sum_{x_{i-1}<p\le x_i} \frac{u(p)\lambda_g(p)}{p^{1/2}}\right)^{n_i}\left(\sum_{2^m<p\le 2^{m+1}} \frac{w(p)\lambda_g(p^2)}{p}\right)^{2M}
$$

with $\widetilde{n} = (n_1,\dots,n_I)$ where each component satisfies $n_i\le\ell$. This concludes the proof. $\square$

For technical reason in the proof of Theorem 4.12 we will need the following lemma

Lemma 3.2.12. *For any $1\le i\le I$ and write $x_i = k^{\beta_i}$, then we have*

$$
\exp\left(\frac{1}{2}\sum_{p\le x_i}\frac{\lambda_f(p)^4}{p^{1+2/\log x_i}}\frac{\log^2(x_i/p)}{\log^2 x_i}\right)\cdot\exp\left(-\frac{1}{2}\sum_{p\le\sqrt{x_i}}\frac{\lambda_f(p)^4}{p^{1+2/\log x_i}}\frac{\log(x_i/p^2)}{\log x_i}\right) = O(1)
$$

Proof. At first we investigate the primes up to $\sqrt{x_i}$. We want to estimate

$$
\exp\left(\frac{1}{2}\sum_{p\le\sqrt{x_i}}\frac{\lambda_f(p)^4}{p^{1+2/\log x_i}}\frac{\log^2(x_i/p)}{\log^2 x_i}-\frac{1}{2}\sum_{p\le\sqrt{x_i}}\frac{\lambda_f(p)^4}{p^{1+2/\log x_i}}\frac{\log(x_i/p^2)}{\log x_i}\right)
$$

After expanding the smoothing of $\log(x_i/p)$ for both sums, we see that the only contribution that is left comes from

$$\exp\left(\frac{1}{2}\sum_{p\le\sqrt{x_i}}\frac{\lambda_f(p)^4}{p^{1+2/\log x_i}}\frac{(\log p)^2}{(\log x_i)^2}\right).$$

We bound $\frac{\lambda_f(p)^4}{p^{2/\log x_i}}\frac{\log p}{\log x_i}$ trivially by a constant (here we use the Deligne bound for the Fourier coefficients) and see that

$$\exp\left(\frac{1}{2}\sum_{p\le\sqrt{x_i}}\frac{\log p}{p}\frac{1}{\log x_i}\right) = O(1).$$

It remains to show that

$$\exp\left(\frac{1}{2}\sum_{\sqrt{x_i}<p\le x_i}\frac{\lambda_f(p)^4}{p^{1+2/\log x_i}}\frac{\log^2(x_i/p)}{\log^2 x_i}\right) \tag{3.39}$$

is bounded. Putting absolute values, and using again the Deligne bound, expression (3.39) is controlled by

$$\exp\left(\sum_{\sqrt{x_i}<p\le x_i}\frac{2^3}{p}\right) \ll \exp\left(\log\log x_i - \log\log\sqrt{x_i}\right) = O(1).$$

Hence, the lemma follows. ◻

Lemma 3.2.13. *Assume the Riemann Hypothesis for $L(s, \mathrm{sym}^2 f)$. For any $1 \le i \le I$ we have*

$$\frac{1}{L(1,\mathrm{sym}^2 f)^2}\cdot\exp\left(\sum_{p\le\sqrt{k^{\beta_i}}}\frac{2\lambda_f(p)^2-2}{p}\right) = O(1)$$

Proof. This is a small modification of Lemma 2 in [HS10]. Instead of the zero free region we use the Riemann Hypothesis for $L(s, \mathrm{sym}^2 f)$ to bound the contribution of the zeros. ◻

The next lemma is a crude bound for the second moment of our degree eight L-function. The ideas are from [Sou09] and adapted to our context.

Lemma 3.2.14. *Let f and g be Hecke cusp forms of even weight k and $2k$ respectively for the full modular group. Assuming the Riemann Hypothesis for $L(1/2, f \times f \times g)$*

$$\sum_{g\in B_{2k}}^{h} L(1/2, f \times f \times g)^2 \ll (\log k)^{10^{30}}.$$

Proof. Define $S(g,V) := \{g \in B_{2k} : \log L(1/2, f \times f \times g) \geq V\}$. Notice that

$$\sum_{g\in B_{2k}}^{h} L(1/2, f \times f \times g)^2 = \int_{-\infty}^{\infty} e^{2V} \operatorname{meas}(S(g,V))dV.$$

It suffices to investigate

$$\int_{10^{30}\log\log k}^{\infty} e^{2V} \operatorname{meas}(S(g,V))dV \tag{3.40}$$

as otherwise we trivially have the desired result.

From Lemma 3.2.1 we have for any $x \geq 2$ that

$$\begin{aligned}\log L(1/2, f \times f \times g) &\leq \sum_{p\leq x} \frac{\lambda_f(p)^2\lambda_g(p)}{p^{\frac{1}{2}+\frac{1}{\log x}}} \frac{\log(x/p)}{\log x} \\ &\quad + \sum_{p\leq\sqrt{x}} \frac{(\lambda_f(p)^4 - 4\lambda_f(p)^2 + 4)(\lambda_g(p^2) - 1)}{2p^{1+\frac{2}{\log x}}} \frac{\log(x/p^2)}{\log x} + \frac{\log k^6}{\log x} + O(1) \\ &\leq \sum_{p\leq x} \frac{\lambda_f(p)^2\lambda_g(p)}{p^{\frac{1}{2}+\frac{1}{\log x}}} \frac{\log(x/p)}{\log x} + 6\log\log x + \frac{6\log k}{\log x} + O(1).\end{aligned}$$

Here we used that $|\lambda_f(p)| \leq 2$ and $|\lambda_g(p)| \leq 3$. If we pick $x = k^{16/V}$, and notice that $6\log\log k \leq (6/10^{30})V$, then

$$\log L(1/2, f \times f \times g) \leq \sum_{p\leq x} \frac{\lambda_f(p)^2\lambda_g(p)}{p^{\frac{1}{2}+\frac{1}{\log x}}} \frac{\log(x/p)}{\log x} + \frac{3V}{4} + O(1).$$

Hence if $g \in S(g, V)$ then

$$\sum_{p \le x} \frac{\lambda_f(p)^2 \lambda_g(p)}{p^{\frac{1}{2}+\frac{1}{\log x}}} \frac{\log(x/p)}{\log x} \ge \frac{V}{4}.$$

By Markov's inequality, we have for any non-negative integer n

$$\mathrm{meas}(S(V,g)) \le \frac{4^{2n}}{V^{2n}} \sum_{g \in B_{2k}}^{h} \left(\sum_{p \le x} \frac{\lambda_f(p)^2 \lambda_g(p)}{p^{\frac{1}{2}+\frac{1}{\log x}}} \frac{\log(x/p)}{\log x} \right)^{2n}.$$

By Lemma 3.2.5 this is bounded by

$$\frac{4^{2n}}{V^{2n}} \cdot \frac{(2n)!}{2^{2n} n!} \left(\sum_{p \le x} \frac{\lambda_f(p)^4}{p} \right)^{n} \tag{3.41}$$

provided that $x^{2n} \le k^2/10^4$. From our choice of x we see that $n = \lfloor V/20 \rfloor$ is admissible. By Stirling and the Deligne bound quantity (3.41) is controlled by

$$\left(\frac{2^8 n \log\log k}{V^2 \cdot e} \right)^{n}.$$

This in turn is bounded by

$$\left(\frac{2^8}{20 \cdot 10^{30} e} \right)^{n} \ll e^{-3V}$$

by our choice of n and the lower bound $V \ge 10^{30} \log\log k$. We see that the contribution of the integral in (3.40) is negligible and consequently the result follows. □

3.2.6 Proof of Theorem 4.12

Proof of Theorem 4.12. Note that

$$\{g \in B_{2k}\} = \mathcal{G} \cup \bigcup_{j=0}^{I-1} \mathcal{E}(j),$$

hence our goal is to show that

$$\sum_{g\in\mathcal{G}}^{h} \frac{L(1/2, f\times f\times g)}{L(1,\operatorname{sym}^2 f)^2} + \sum_{j=0}^{I-1} \sum_{g\in\mathcal{E}(j)}^{h} \frac{L(1/2, f\times f\times g)}{L(1,\operatorname{sym}^2 f)^2} = O(1). \tag{3.42}$$

At first we approximate the L-functions with Dirichlet polynomials. Lemma 3.2.1 gives for $x = x_I = k^{\beta_I}$

$$\begin{aligned}\log L(1/2, f\times f\times g) \leq & \sum_{p\leq x_I} \frac{\lambda_f(p)^2\lambda_g(p)}{p^{1/2+1/\log x_I}} \frac{\log(x_I/p)}{\log x_I} \\ & + \sum_{p\leq\sqrt{x_I}} \frac{(\lambda_f(p)^4 - 4\lambda_f(p)^2+4)(\lambda_g(p^2)-1)}{2p^{1+2/\log x_I}} \frac{\log(x_I/p^2)}{\log x_I} + \frac{6}{\beta_I} + O(1).\end{aligned}$$

Consequently, the first sum in (3.42) is bounded by

$$\begin{aligned} e^{6/\beta_I} \sum_{g\in\mathcal{G}}^{h} & \exp\left(\sum_{p\leq x_I} \frac{\lambda_f(p)^2\lambda_g(p)}{p^{1/2+1/\log x_I}} \frac{\log(x_I/p)}{\log x_I}\right) \cdot \\ & \cdot \exp\left(\sum_{p\leq\sqrt{x_I}} \frac{(\lambda_f(p)^4 - 4\lambda_f(p)^2+4)\lambda_g(p^2)}{2p^{1+2/\log x_I}} \frac{\log(x_I/p^2)}{\log x_I}\right) \cdot \\ & \cdot \exp\left(-\sum_{p\leq\sqrt{x_I}} \frac{(\lambda_f(p)^4 - 4\lambda_f(p)^2+4)}{2p^{1+2/\log x_I}} \frac{\log(x_I/p^2)}{\log x_I}\right) \cdot \frac{1}{L(1,\operatorname{sym}^2 f)^2}. \end{aligned} \tag{3.43}$$

By Lemma 3.2.9 the contribution of the first two exponential sums is bounded by

$$\exp\left(\frac{1}{2}\sum_{p\leq x_I} \frac{\lambda_f(p)^4}{p^{1+2\log x_I}} \frac{\log^2(x_I/p)}{(\log x_I)^2}\right).$$

The last exponential factor of (3.43) can be written as

$$\exp\left(-\frac{1}{2}\sum_{p\leq\sqrt{x_I}} \frac{\lambda_f(p)^4}{p^{1+2/\log x_I}} \frac{\log(x_I/p^2)}{\log x_I}\right) \cdot \exp\left(\sum_{p\leq\sqrt{x_I}} \frac{(2\lambda_f(p)-2)}{p^{1+2/\log x_I}} \frac{\log(x_I/p^2)}{\log x_I}\right).$$

Therefore (3.43) can be bounded by

$$e^{6/\beta_I} \exp\left(\frac{1}{2}\sum_{p\le x_I} \frac{\lambda_f(p)^4}{p^{1+2\log x_I}} \frac{\log^2(x_I/p)}{(\log x_I)^2}\right) \cdot \exp\left(-\frac{1}{2}\sum_{p\le\sqrt{x_I}} \frac{\lambda_f(p)^4}{p^{1+2/\log x_I}} \frac{\log(x_I/p^2)}{\log x_I}\right)$$
$$\cdot \exp\left(\sum_{p\le\sqrt{x_I}} \frac{(2\lambda_f(p)-2)}{p^{1+2/\log x_I}} \frac{\log(x_I/p^2)}{\log x_I}\right) \cdot \frac{1}{L(1,\mathrm{sym}^2 f)^2}.$$

Since β_I is bounded, Lemma 3.2.12 and Lemma 3.2.13 show that (3.43) is of size $O(1)$.

We now treat the exceptional sets from the second term in (3.42). We begin, as before, by approximating the L-function with Dirichlet polynomials. Lemma 3.2.1 with $x = x_j = k^{\beta_j}$ shows that

$$\sum_{g\in\mathcal{E}(j)}^{h} \frac{L(1/2, f\times f\times g)}{L(1,\mathrm{sym}^2 f)^2}$$

is bounded by

$$e^{6/\beta_j} \cdot \sum_{g\in\mathcal{E}(j)}^{h} \exp\left(\sum_{p\le x_j} \frac{\lambda_f(p)^2\lambda_g(p)}{p^{1/2+1/\log x_j}} \frac{\log(x_j/p)}{\log x_j}\right) \cdot \tag{3.44}$$
$$\cdot \exp\left(\sum_{p\le\sqrt{x_j}} \frac{(\lambda_f(p)^4 - 4\lambda_f(p)^2+4)\lambda_g(p^2)}{2p^{1+2/\log x_j}} \frac{\log(x_j/p^2)}{\log x_j}\right) \cdot$$
$$\cdot \exp\left(-\sum_{p\le\sqrt{x_j}} \frac{\lambda_f(p)^4 - 4\lambda_f(p)^2+4}{2p^{1+2/\log x_j}} \frac{\log(x_j/p^2)}{\log x_j}\right) \cdot \frac{1}{L(1,\mathrm{sym}^2 f)^2}$$

for $1 \le j \le I-1$. By Lemma 3.2.10 the sum of the first two exponentials in (3.44) is bounded by

$$\exp\left(\frac{1}{2}\sum_{p\le x_j} \frac{\lambda_f(p)^4}{p^{1+2/\log x_j}} \frac{\log^2(x_j/p)}{(\log x_j)^2}\right) e^{(4C\beta_{j+1})^{-1}\log\beta_{j+1}}.$$

with $C = 2^5 \cdot 10/e$. Similarly as before, we use Lemma 3.2.12 and Lemma 3.2.13 to show that expression (3.44) is bounded by

$$e^{6/\beta_j} \cdot e^{(4C\beta_{j+1})^{-1}\log\beta_{j+1}} = e^{6/\beta_j + \log(\beta_{j+1})/(80C\beta_j)}.$$

Moreover, since $\beta_{j+1} \le \beta_I \le 20e^{-10^5}$ we have

$$e^{6/\beta_j + \log(\beta_{j+1})/(80C\beta_j)} \le e^{6/\beta_j - 10/\beta_j} = e^{-4/\beta_j}.$$

The sum over these values from $1 \le j \le I-1$ remains bounded and so we conclude the proof of the theorem for these exceptional sets.

The only case that is left is when $j = 0$. In that scenario, we win because the measure of $\mathcal{E}(0)$ is tiny. By Cauchy–Schwarz we have

$$\sum_{g \in \mathcal{E}(0)}^{h} \frac{L(1/2, f \times f \times g)}{L(1, \mathrm{sym}^2 f)^2} \le \left(\sum_{g \in \mathcal{E}(0)}^{h} 1 \right)^{1/2} \cdot \left(\sum_{g \in B_{2k}}^{h} \frac{L(1/2, f \times f \times g)^2}{L(1, \mathrm{sym}^2 f)^4} \right)^{1/2}. \tag{3.45}$$

Note that $L(1, \mathrm{sym}^2 f)^{-1} \ll \log k$ (see [HL94] and [GHL94]). Lemma 3.2.10 and Lemma 3.2.14 show that the right hand side of (3.45) is bounded by

$$e^{-(\log\log k)^2/(2C)} \cdot (\log k)^{(10^{30}+4)/2}.$$

For k large enough this is clearly bounded and therefore the theorem follows. $\square$

Chapter 4

Quantum Variance for Holomorphic Hecke Cusp Forms on the Vertical Geodesic

In the subsequent chapter we will discuss the proof of Theorem 2.3.2 that we obtained in our work [Zen21a]. Again, the introductory section 4.1 follows extremely closely our exposition in [Zen21a]. Section 4.2 is taken, up to minor notational changes and the correction of typographical errors, verbatim from [Zen21a].

4.1 High Level Sketch

The proof of Theorem 2.3.2 involves many technical details and so we begin by sketching the main ideas of the proof. First, we relate the quantum variance problem to a shifted convolution problem

$$\mathcal{M}(\psi) \approx \sum_{K<k\leq 2K} \sum_{f\in B_k} \left| \frac{1}{k} \sum_{0<|\ell|\leq k^{1/2+\varepsilon}} \sum_{n=1}^{\infty} \lambda_f(n)\lambda_f(n+\ell)\psi\left(\frac{k}{n}\right)\right|^2, \tag{4.1}$$

which follows quickly after using the Fourier expansion of $f(iy)$. If we show that $\mathcal{M}(\psi) = o(K^2)$ then we could deduce a Quantum Ergodicity result for the vertical geodesic, i.e. equidistribution on average. We could try to bound $\mathcal{M}(\psi)$ by detecting cancellation over the summation over n and forgoing any cancellation over the shifts ℓ, i.e.

$$\mathcal{M}(\psi) \ll \sum_{K<k\leq 2K} \sum_{f\in B_k} \Big|\frac{1}{k} \sum_{0<|\ell|\leq k^{1/2+\varepsilon}} \Big| \sum_{n=1}^{\infty} \lambda_f(n)\lambda_f(n+\ell)\psi\Big(\frac{k}{n}\Big)\Big|\Big|^2.$$

Obtaining square root cancellation over the summation of n, which is of length k, would then lead to the bound $\mathcal{M}(\psi) \ll K^{2+\varepsilon}$. Consequently, this crude estimate is not sufficient to obtain equidistribution on the vertical geodesic; we need to detect further cancellation over the shifts ℓ. It is also natural to expect square root cancellation over the shifts ℓ. In that case we would have $\mathcal{M}(\psi) \approx K^{3/2}$. We prove an asymptotic formula for $\mathcal{M}(\psi)$ confirming this heuristic.

To obtain an asymptotic formula for our averaged shifted convolution problem $\mathcal{M}(\psi)$, we open the square in (4.1). We then apply the Petersson Trace formula, which detects orthogonality relations between the Hecke eigenvalues. We are left with a diagonal contribution $\mathcal{D}$ that is easy to evaluate and an off-diagonal expression $\mathcal{OD}$, involving Kloosterman sums. The diagonal term is easily seen to be of size $K^{3/2}$. The off-diagonal term $\mathcal{OD}$ is roughly given by

$$\frac{1}{K} \sum_{0<|\ell_1|,|\ell_2|\leq K^{1/2+\varepsilon}} \sum_{K<n_1,n_2\leq 2K} \sum_{c\ll K^{\varepsilon}} \frac{S(n_1(n_1+\ell_1), n_2(n_2+\ell_2); c)}{\sqrt{c}} e_c(2\sqrt{n_1(n_1+\ell_1)n_2(n_2+\ell_2)}),$$

where $S(n,m;c)$ denotes the classical Kloosterman sum and $e_c(n) = e^{2\pi i n/c}$. Putting absolute values everywhere and using the Weil bound for Kloosterman sums we would get $\mathcal{OD} = O(K^{2+\varepsilon})$. To improve upon this bound, and consequently breaking the equidistribution barrier, we need to exploit further cancellation over the shifts ℓ_1, ℓ_2.

We notice that the summation range over the variable c in $\mathcal{OD}$ is very short. To reduce the complexity of our exposition consider therefore the special case when $c = 1$ (from a conceptional point of view not much is lost by this reduction). Then

$$\mathcal{OD} \approx \frac{1}{K} \sum_{0<|\ell_1|,|\ell_2|\leq K^{1/2+\varepsilon}} \sum_{K<n_1,n_2\leq 2K} e_1(2\sqrt{n_1(n_1+\ell_1)n_2(n_2+\ell_2)}).$$

When the shifts ℓ_1, ℓ_2 are of size $\sqrt{K}$, which is the critical range, the exponential is oscillating rapidly and we expect further cancellation. Indeed, subtracting the integer part of the phase function and using a Taylor expansion we get

$$\begin{aligned} f(n_1,n_2,\ell_1,\ell_2) :&= 2\sqrt{n_1(n_1+\ell_1)n_2(n_2+\ell_2)} - 2n_1n_2 - n_1\ell_2 - n_2\ell_1 \\ &= \frac{\ell_1\ell_2}{2} - \frac{1}{4}\frac{\ell_1^2 n_2}{n_1} - \frac{1}{4}\frac{\ell_2^2 n_1}{n_2} + \dots \end{aligned}$$

Since

$$\frac{\partial}{\partial n_1} f(n_1,n_2,\ell_1,\ell_2) \approx \frac{\ell_1^2 n_2}{2n_1^2} - \frac{\ell_2^2}{4n_2} \asymp 1 \quad \text{and} \quad \frac{\partial^2}{\partial n_1^2} f(n_1,n_2,\ell_1,\ell_2) \approx -\frac{3\ell_1^2 n_2}{2n_1^3} \asymp \frac{1}{K},$$

we expect to detect square root cancellation when summing over the variable n_1, by a classical Van der Corput estimate (see [IK04, Corollary 8.12]). We highlight here that the "stationary point" of the phase function $f(n_1,n_2,\ell_1,\ell_2)$ arises on the diagonal, i.e. when $n_1 = n_2$ and $\ell_1 = \ell_2$. A precise version of this argument will show that $\mathcal{OD} = O(K^{3/2+\varepsilon})$, as desired.

To compute an exact asymptotic formula for our quantum variance we keep track of the variable c. After an application of the Poisson summation formula and a stationary phase argument, which captures the square-root cancellation mentioned above, we need to evaluate

$$T(c) := \sum_{\substack{a_1 \,(\mathrm{mod}\, c) \\ a_2 \,(\mathrm{mod}\, c)}} \sum_{\substack{b_1 \,(\mathrm{mod}\, c) \\ b_2 \,(\mathrm{mod}\, c)}} S(a_1(a_1+b_1), a_2(a_2+b_2); c) e_c(2a_1a_2 + a_1b_2 + a_2b_1).$$

A quick computation shows that $T(c) = c^3\varphi(c)$, where $\varphi(c)$ denotes Euler's totient function. This simple expression allows us to evaluate the off-diagonal expression asymptotically. Putting the off-diagonal term and diagonal term together we arrive at our desired expression for the quantum variance.

4.2 Proof of Theorem 2.3.2

4.2.1 Setup

Recall from the introduction that ψ, h are smooth compactly-supported functions on $\mathbb{R}^+$ and B_k denotes a basis of Hecke cusp forms of weight k. We want to compute an asymptotic formula for

$$V(\psi_1,\psi_2) = \sum_{k\equiv 0 \,(\mathrm{mod}\, 2)} h\Big(\frac{k-1}{K}\Big) \sum_{f\in B_k} L(1,\mathrm{sym}^2 f)\Big(\mu_f(\psi_1)-\mathbb{E}(\psi_1)\Big)\cdot\Big(\mu_f(\psi_2)-\mathbb{E}(\psi_2)\Big)$$

with

$$\mu_f(\psi) = \int_0^\infty |f(iy)|^2 y^{k/2}\psi(y)dy \quad \text{and} \quad \mathbb{E}(\psi) = \frac{3}{\pi}\int_0^\infty \psi(y)\frac{dy}{y}.$$

Here the symmetric square L-function is given by

$$L(s,\mathrm{sym}^2 f) = \zeta(2s)\sum_{n=1}^\infty \frac{\lambda_f(n^2)}{n^s},$$

for $\mathrm{Re}(s)>1$. The Fourier expansion of a normalized Hecke cusp form is given by

$$f(z) = a_f(1)\sum_{n=1}^\infty \lambda_f(n)(4\pi n)^{(k-1)/2} e(nz),$$

with $|a_f(1)|^2 = \frac{2\pi^2}{\Gamma(k)L(1,\mathrm{sym}^2 f)}$ arising from the normalization $||f||_2^2 = 1$. Similar to Luo and Sarnak in [LS03, p. 877] we define the function $\widetilde{\psi}(s)$ by

$$\widetilde{\psi}(s) := \int_0^\infty \psi(y^{-1})y^{s-1}dy.$$

Then $\widetilde{\psi}$ is entire, and for any integer $j > 0$ and any vertical strip $a \le \mathrm{Re}(s) \le b$, it satisfies $\widetilde{\psi} \ll_{a,b,j} (|s|+1)^{-j}$. Mellin inversion yields

$$\psi(y) = \frac{1}{2\pi i}\int_{(\sigma)} \widetilde{\psi}(s)y^s ds, \quad \text{for } \sigma > 0, y > 0.$$

4.2.2 Reduction to a Shifted Convolution Problem

To evaluate the variance $V(\psi_1, \psi_2)$ we first use the Fourier expansion of $f(iy)$ and write

$$|f(iy)|^2 = \Big|a_f(1)\sum_n \lambda_f(n)(4\pi n)^{(k-1)/2}e^{-2\pi n y}\Big|^2.$$

We then expand the square, seperating the terms with $m = n$ from those with $m \neq n$. The terms with $m = n$ agree up to a small error term (which we will call $\mathcal{E}_\psi$ below) with the expected main term $\mathbb{E}(\psi)$. The terms with $m \neq n$ lead to a shifted convolution problem and this quantity will be denoted by S_ψ.

Lemma 4.2.1. *We have*

$$\begin{aligned}\mathcal{E}_\psi :=& \int_0^\infty \psi(y)y^k|a_f(1)|^2\sum_{n=1}^\infty \lambda_f(n)^2(4\pi n)^{k-1}e^{-4\pi n y}\frac{dy}{y} - \frac{3}{\pi}\int_0^\infty \psi(y)\frac{dy}{y} \\ =& \frac{2\pi^2}{L(1,\mathrm{sym}^2 f)} \cdot \frac{1}{2\pi i}\int_{(1/2)} \widetilde{\psi}(s-1)\frac{\zeta(s)L(s,\mathrm{sym}^2 f)}{(4\pi)^s\zeta(2s)}\frac{\Gamma(k+s-1)}{\Gamma(k)}ds.\end{aligned} \tag{4.2}$$

Proof.

$$
\begin{aligned}
&\int_0^\infty \psi(y)y^k|a_f(1)|^2\sum_{n=1}^\infty \lambda_f(n)^2(4\pi n)^{k-1}e^{-4\pi n y}\frac{dy}{y}\\
=&\frac{2\pi^2}{\Gamma(k)L(1,\mathrm{sym}^2 f)}\cdot\frac{1}{2\pi i}\int_{(2)}\widetilde{\psi}(s)\sum_{n=1}^\infty \lambda_f(n)^2(4\pi n)^{k-1}\int_0^\infty y^{k+s}e^{-4\pi n y}\frac{dy}{y}\\
=&\frac{2\pi^2}{L(1,\mathrm{sym}^2 f)}\cdot\frac{1}{2\pi i}\int_{(2)}\widetilde{\psi}(s)\frac{\zeta(1+s)L(1+s,\mathrm{sym}^2 f)}{(4\pi)^{1+s}\zeta(2(1+s))}\frac{\Gamma(k+s)}{\Gamma(k)}ds
\end{aligned}
$$

We used that we can write the Rankin–Selberg L-function in terms of the symmetric square as

$$
L(s,f\otimes f)=\frac{\zeta(s)L(s,\mathrm{sym}^2 f)}{\zeta(2s)}=\sum_{n=1}^\infty\frac{\lambda_f(n)^2}{n^s}.
$$

We then shift the contour from $\mathrm{Re}(s)=2$ to $\mathrm{Re}(s)=-1/2$ and pick up a pole at $s=0$ with residue

$$
\frac{2\pi}{L(1,\mathrm{sym}^2 f)}\cdot\widetilde{\psi}(0)\frac{L(1,\mathrm{sym}^2 f)}{4\pi\zeta(2)}=\frac{3}{\pi}\int_0^\infty\psi(y)\frac{dy}{y}.
$$

The lemma follows by making the change of variables $s\to s-1$ for the new line integral. □

The term $\mathcal{E}_\psi$ should be seen as an error term that is of size $K^{-1/2}$ (compare for example with [LS03, Section 5]). The off-diagonal term on the other hand is given by a shifted convolution (which we denote by S_ψ) of size $K^{-1/4}$ as the following lemma indicates:

Lemma 4.2.2. *We have*

$$
\begin{aligned}
S_\psi:=&\int_0^\infty\psi(y)y^k|a_f(1)|^2\sum_{n\neq m}\lambda_f(n)\lambda_f(m)(16\pi^2 nm)^{(k-1)/2}e^{-2\pi(n+m)y}\frac{dy}{y} \qquad (4.3)\\
=&\frac{\pi}{2L(1,\mathrm{sym}^2 f)}\sum_{\ell\neq 0}\sum_n\frac{\lambda_f(n)\lambda_f(n+\ell)}{\sqrt{n(n+\ell)}}\exp\Big(-\frac{k\ell^2}{2(2n+\ell)^2}\Big)\psi\Big(\frac{k}{2\pi(2n+\ell)}\Big)+O_\psi(k^{-1/2+\varepsilon}).
\end{aligned}
$$

Remark 4.2.1. Since ψ is smooth compactly-supported on $\mathbb{R}^+$, we see that $(2n+\ell)\asymp k$. The exponential factor limits the size of ℓ as otherwise we have rapid decay. It follows that $\ell\ll k^{1/2+\varepsilon}$ and consequently $n\asymp k$. These observations show that the off-diagonal term is related to a shifted convolution problem of the form

$$\frac{1}{k}\sum_{0<|\ell|\ll k^{1/2+\varepsilon}}\sum_{n\sim k}\lambda_f(n)\lambda_f(n+\ell).$$

Assuming square root cancellation in n and the shifts ℓ the expected size of S_ψ is $k^{-1/4+\varepsilon}$. The main part of the paper is attributed to showing this statement on average.

Proof of Lemma 4.2.2. First, we show by elementary means that only the terms satisfying $n+m \asymp k$ contribute to the main term of S_ψ. Since $\psi(y)$ is compactly-supported, there exist real numbers $0 < a < b$ such that $\psi(y)$ is supported in $[a,b]$. We write $L(y) = -2\pi(n+m)y + k\log y$ and set $y_0 = k/2\pi(n+m)$ so that $L(y)$ attains its maximum at y_0. Suppose that $y_0 \le a/2$. Then for all y in the support of ψ we have

$$L'(y) = -2\pi(n+m) + k/y \le -2\pi(n+m) + k/(2y_0) = -\pi(n+m).$$

Hence,

$$\begin{aligned}\int_0^\infty \psi(y)e^{L(y)}\frac{dy}{y} &\le \int_a^\infty |\psi(y)|e^{L(y_0)}e^{-(y-y_0)\pi(n+m)}\frac{dy}{y}\\ &\ll_\psi e^{L(y_0)}e^{-(a-y_0)\pi(m+n)}\\ &\ll_\psi e^{L(y_0)}e^{-a\pi(m+n)/2}.\end{aligned}$$

It follows that the contribution from n,m such that $k/(2\pi(n+m)) \le a/2$ to S_ψ is bounded by

$$\frac{2\pi^2}{L(1,\mathrm{sym}^2 f)}\sum_{\substack{n,m\\(n+m)\ge k/(a\pi)}}\frac{d(n)d(m)}{\sqrt{nm}}\cdot\left(\frac{2\sqrt{nm}}{n+m}\right)^k e^{-a\pi(n+m)/2}. \tag{4.4}$$

Here we used the Deligne bound $|\lambda_f(n)| \le d(n)$, where $d(n)$ is the divisor function. Expression (4.4) decays exponentially in k and is thus negligible. Similarly, we treat the case when $y_0 \ge 2b$. Note that $L''(y) = -k/y^2 \le -k/b^2$ for every y in the support of ψ.

Moreover, $(y-y_0)^2 \geq b^2$. We then have

$$\int_0^\infty \psi(y)e^{L(y)}\frac{dy}{y} \leq \int_0^\infty |\psi(y)|e^{L(y_0)-(y-y_0)^2\frac{k}{b^2}}\frac{dy}{y}$$
$$\ll_\psi e^{L(y_0)}e^{-k}.$$

The contribution from n,m such that $k/(2\pi(n+m)) \geq 2b$ to S_ψ is thus bounded by

$$\frac{2\pi^2}{L(1,\mathrm{sym}^2 f)} \sum_{\substack{n,m \\ (n+m)\leq k/(4\pi b)}} \frac{d(n)d(m)}{\sqrt{nm}} \cdot \left(\frac{2\sqrt{nm}}{n+m}\right)^k e^{-k}, \tag{4.5}$$

which decays exponentially in k.

Subsequently, we restrict our attention to the case $a/2 \leq y_0 \leq 2b$ and in particular, $n+m \asymp k$. We start by performing an inverse Mellin transform on ψ and evaluating the integral over y as a Gamma function:

$$\begin{aligned}
&\int_0^\infty \psi(y)y^k|a_f(1)|^2 \sum_{n\neq m} \lambda_f(n)\lambda_f(m)(16\pi^2 nm)^{(k-1)/2}e^{-2\pi(n+m)y}\frac{dy}{y} \\
=&\frac{2\pi^2}{\Gamma(k)L(1,\mathrm{sym}^2 f)} \sum_{n\neq m} \lambda_f(n)\lambda_f(m)(16\pi^2 nm)^{(k-1)/2}\frac{1}{2\pi i}\int_{(2)}\widetilde{\psi}(s)\int_0^\infty y^{k+s}e^{-2\pi(n+m)y}\frac{dy}{y}ds \\
=&\frac{2\pi^2}{L(1,\mathrm{sym}^2 f)} \sum_{n\neq m} \lambda_f(n)\lambda_f(m)(16\pi^2 nm)^{(k-1)/2}\frac{1}{2\pi i}\int_{(2)}\widetilde{\psi}(s)\frac{1}{\big(2\pi(n+m)\big)^{k+s}}\frac{\Gamma(k+s)}{\Gamma(k)}ds.
\end{aligned}$$

Similar to [LS03, Eq. 2.3] Stirlings formula yields that for any vertical strip $0 < a \leq \mathrm{Re}(s) \leq b$,

$$\frac{\Gamma(k+s)}{\Gamma(k)} = k^s \cdot (1 + O_{a,b}((1+|s|)^2 k^{-1})). \tag{4.6}$$

Using (4.6) we have

$$\begin{aligned} S_\psi =& \frac{\pi}{2L(1,\mathrm{sym}^2 f)} \sum_{n\neq m} \frac{\lambda_f(n)\lambda_f(m)}{\sqrt{nm}} \cdot \Big(\frac{2\sqrt{nm}}{n+m}\Big)^k \psi\Big(\frac{k}{2\pi(n+m)}\Big) \\ &+ O_\psi\Big(\frac{1}{k\cdot L(1,\mathrm{sym}^2 f)} \sum_{n\neq m} \frac{d(n)d(m)}{\sqrt{nm}} \cdot \Big(\frac{2\sqrt{nm}}{n+m}\Big)^k\Big). \end{aligned} \tag{4.7}$$

The factor $\big(\frac{2\sqrt{mn}}{n+m}\big)^k$ is forcing m and n to be close (roughly $|m-n| \ll k^{1/2+\varepsilon}$). More precisely, note that

$$\frac{m+n}{2\sqrt{mn}} = \sqrt{\frac{(m+n)^2}{4mn}} \geq \sqrt{1+\frac{(m-n)^2}{4(m+n)^2}}.$$

In particular,

$$\Big(\frac{2\sqrt{mn}}{n+m}\Big)^k \leq e^{-O(|m-n|^2/k)}$$

and the contribution from m,n with $|m-n| \geq k^{1/2+\varepsilon}$ is exponentially small in k. When $|m-n| \ll k^{1/2+\varepsilon}$ we have as in [BKY13, p. 9]

$$\begin{aligned} \Big(\frac{2\sqrt{mn}}{m+n}\Big)^k &= \Big(1-\frac{|m-n|^2}{2(m+n)^2} + O\Big(\frac{|m-n|^4}{(m+n)^4}\Big)\Big)^k \\ &= \exp\Big(k\log\Big(1-\frac{|m-n|^2}{2(m+n)^2} + O\Big(\frac{|m-n|^4}{(m+n)^4}\Big)\Big)\Big). \end{aligned}$$

By a Taylor expansion it follows that

$$S_\psi = \frac{\pi}{2L(1,\mathrm{sym}^2 f)} \sum_{m\neq n} \frac{\lambda_f(m)\lambda_f(n)}{\sqrt{mn}} \exp\Big(-\frac{k|m-n|^2}{2(m+n)^2}\Big)\psi\Big(\frac{k}{2\pi(n+m)}\Big) + O_\psi(k^{-1/2+\varepsilon}).$$

The lemma follows upon writing $m = n+\ell$. □

Subsequently, we only consider the case when $m > n$ and thus $\ell > 0$ as the case with $m < n$ is exactly the same upon relabelling the variables.

4.2.3 Cancellation in the Shifted Convolution Problem

Our goal now is to detect cancellation in the shifted convolution sum S_ψ (in an L^2 sense, when averaged over k and $f \in B_k$). To do this we will use an averaged Petersson trace formula:

Lemma 4.2.3. *[ILS00, Iwaniec, Luo, Sarnak, Lemma 10.1] For any positive numbers m, n we have*

$$\sum_{k\equiv 0 \,(\mathrm{mod}\, 2)} 2h\Big(\frac{k-1}{K}\Big)\frac{2\pi^2}{k-1}\sum_{f\in B_k}\frac{\lambda_f(m)\lambda_f(n)}{L(1,\mathrm{sym}^2 f)} = \tag{4.8}$$
$$=\hat{h}(0)K1_{m=n} - \pi^{1/2}(mn)^{-1/4}K\mathrm{Im}\Big(e^{-2\pi i/8}\sum_{c=1}^{\infty}\frac{S(m,n;c)}{\sqrt{c}}e_c(2\sqrt{mn})\hbar\Big(\frac{cK^2}{8\pi\sqrt{mn}}\Big)\Big)+$$
$$+O\Big(\frac{\sqrt{mn}}{K^4}\cdot\int_{-\infty}^{\infty}v^4|\hat{h}(v)|dv + 1_{m=n}\Big),$$

where $\hat{h}$ denotes the Fourier transform of h and $\hbar(v) = \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}}e^{iuv}du$.

Remark 4.2.2. We kept the dependency on h explicit in the error term, as our weight function will depend on n, m and K. Similar computations are also done by Khan in [Kha10] (see for example Lemma 2.6 and expression (2.29) therein).

Remark 4.2.3. Integrating by parts several times shows that $\hbar(v) \ll_A v^{-A}$ for any $A > 0$. In particular, the second term on the right hand side of (3.2.2) is absorbed in the error term if $cK^2/\sqrt{mn} > K^\varepsilon$. In our case mn will be of size K^4 and so this effectively restricts the range of c to $c \ll K^\varepsilon$.

4.2.4 Variance Computation

Now we compute the main term of the variance $V(\psi_1, \psi_2)$, which is given by the averaged shifted convolutioin problem

$$\mathcal{M}(\psi_1, \psi_2) = \sum_{k \equiv 0 \,(\mathrm{mod}\, 2)} h\Big(\frac{k-1}{K}\Big) \sum_{f \in B_k} L(1, \mathrm{sym}^2 f) S_{\psi_1} S_{\psi_2} \tag{4.9}$$

with

$$\begin{aligned}
S_\psi =& \frac{\pi}{2L(1,\mathrm{sym}^2 f)} \sum_{\ell \neq 0} \sum_{n} \frac{\lambda_f(n)\lambda_f(n+\ell)}{\sqrt{n(n+\ell)}} \exp\Big(\frac{-k\ell^2}{2(2n+\ell)}\Big) \psi\Big(\frac{k}{2\pi(2n+\ell)}\Big) + O_\psi(k^{-1/2+\epsilon}) \\
=& \frac{\pi}{2L(1,\mathrm{sym}^2 f)} \sum_{\substack{\ell, d \\ d|\ell}} \sum_{n} \frac{\lambda_f(n(n+\ell/d))}{\sqrt{d^2 n(n+\ell/d)}} \exp\Big(\frac{-k\ell^2}{2(2nd+\ell)^2}\Big) \psi\Big(\frac{k}{2\pi(2nd+\ell)}\Big) + O_\psi(k^{-1/2+\epsilon}) \\
=& \frac{\pi}{2L(1,\mathrm{sym}^2 f)} \sum_{d} \sum_{m} \sum_{n} \frac{\lambda_f(n(n+m))}{d\Big(n(n+m)\Big)^{1/2}} \exp\Big(-\frac{km^2}{2(2n+m)^2}\Big) \psi\Big(\frac{k}{2\pi d(n+m)}\Big) + O_\psi(k^{-1/2+\varepsilon}).
\end{aligned}$$

For the second equality we used the Hecke relations

$$\lambda_f(n)\lambda_f(n+\ell) = \sum_{d|n, d|\ell} \lambda_f\Big(\frac{n(n+\ell)}{d^2}\Big)$$

and replaced n by nd. For the third equality we wrote $\ell_i/d_i = m_i$. After expanding S_{ψ_1}, S_{ψ_2} we see that the main term of $\mathcal{M}(\psi_1, \psi_2)$ is equal to

$$\sum_{\substack{n_1, n_2 \\ d_1, d_2 \\ m_1, m_2}} \sum_{k \equiv 0 \,(\mathrm{mod}\, 2)} h^*\Big(\frac{k-1}{K}\Big) \frac{2\pi^2}{k-1} \sum_{f \in B_k} \frac{1}{L(1,\mathrm{sym}^2 f)} \frac{\lambda_f(n_1(n_1+m_1))\lambda_f(n_2(n_2+m_2))}{d_1 d_2 \Big(n_1(n_1+m_1)n_2(n_2+m_2)\Big)^{1/2}}$$

with

$$h^*(t) = h^*_{n_1,n_2,d_1,d_2,m_1,m_2,K}(t)$$
$$= h(t)\frac{tK}{8}\psi_1\Big(\frac{tK}{2\pi d_1(n_1+m_1)}\Big)\psi_2\Big(\frac{tK}{2\pi d_2(n_2+m_2)}\Big)\exp\Big(-\frac{tKm_1^2}{2(2n_1+m_1)^2}-\frac{tKm_2^2}{2(2n_2+m_2)^2}\Big).$$

We can now apply the averaged Petersson trace formula (Lemma 3.2.2) so that $\mathcal{M}(\psi_1,\psi_2) = \mathcal{D} + \mathcal{OD}$ with the diagonal

$$\mathcal{D} = K\sum_{\substack{n_1,n_2\\ d_1,d_2\\ m_1,m_2}}\frac{1_{n_1(n_1+m_1)=n_2(n_2+m_2)}}{d_1d_2\Big(n_1(n_1+m_1)n_2(n_2+m_2)\Big)^{1/2}}\cdot\widehat{h^*}(0) \tag{4.10}$$

and the off-diagonal

$$\mathcal{OD} = -\sqrt{\pi}K\mathrm{Im}\Bigg(e^{-2\pi i/8}\sum_{\substack{n_1,n_2\\ d_1,d_2\\ m_1,m_2}}\sum_{c=1}^{\infty}\frac{S(n_1(n_1+m_1),n_2(n_2+m_2);c)}{\sqrt{c}}e_c(2\sqrt{n_1(n_1+m_1)n_2(n_2+m_2)})\cdot \tag{4.11}$$
$$\times\frac{1}{d_1d_2\Big(n_1(n_1+m_1)n_2(n_2+m_2)\Big)^{3/4}}\cdot\hbar^*\Big(\frac{cK^2}{8\pi\sqrt{n_1(n_1+m_1)n_2(n_2+m_2)}}\Big)\Bigg),$$

where $\hbar^*(v) = \int_0^\infty\frac{h^*(\sqrt{u})}{\sqrt{2\pi u}}e^{iuv}du$. Now that we have established the formula

$$\mathcal{M}(\psi_1,\psi_2) = \mathcal{D} + \mathcal{OD}, \tag{4.12}$$

we start by evaluating the diagonal term.

4.2.5 Evaluating the Diagonal

Lemma 4.2.4. *If $\mathcal{D}$ is given by* (4.10) *then*

$$\begin{aligned}\mathcal{D} =& K^{3/2}\log K\cdot\frac{\sqrt{2}\pi}{32}\widetilde{\psi_1}(0)\widetilde{\psi_2}(0)\cdot\int_0^\infty\frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}}du+\\ &+K^{3/2}\frac{\sqrt{2}\pi}{64}\widetilde{\psi_1}(0)\widetilde{\psi_2}(0)\int_0^\infty\frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}}\log(u)du+\\ &+K^{3/2}\int_0^\infty\frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}}du\cdot\Big(\frac{\sqrt{2}\pi}{16}\Big(\frac{3}{2}\gamma-\log(4\pi)\Big)\widetilde{\psi_1}(0)\widetilde{\psi_2}(0)+\frac{\sqrt{2}\pi}{16}\widetilde{\psi_1}(0)\widetilde{\psi_2}'(0)\Big)+\\ &+K^{3/2}\int_0^\infty\frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}}du\cdot\frac{\sqrt{2}\pi}{16}\frac{1}{2\pi i}\int_{(1)}\widetilde{\psi_1}(-s_2)\widetilde{\psi_2}(s_2)\zeta(1-s_2)\zeta(1+s_2)ds_2+\\ &+O_{\psi_1,\psi_2}(K^{1+\varepsilon}),\end{aligned}$$

as $K\to\infty$.

Proof. To evaluate $\mathcal{D}$ (see (4.10)), we first show that most solutions to $n_1(n_1+m_1)=n_2(n_2+m_2)$ arise from the diagonal, i.e. $n_1=n_2$ and $m_1=m_2$. We assume that $n_1\neq n_2$ and $m_1\neq m_2$, as otherwise we are done. By completing the square the condition $n_1(n_1+m_1)=n_2(n_2+m_2)$ is equivalent to

$$(2n_1+m_1)^2-(2n_2+m_2)^2=m_1^2-m_2^2.$$

Since $m_i\ll K^{1/2+\varepsilon}$ there are at most $K^{1+2\varepsilon}$ choices for the integer $M=m_1^2-m_2^2$. Once M,m_1,m_2 are fixed, n_1 and n_2 are determined up to $K^{\varepsilon'}$. To see this abbreviate $A=(n_1+m_1+n_2+m_2)$ and $B=(n_1+m_1-n_2-m_2)$. Then $M=4AB$ and there are at most $K^{\varepsilon'}$ choices for A and B (as there are at most $K^{\varepsilon'}$ divisors of M). Now A,B,m_1,m_2 are determined and so are n_1,n_2. It follows that there are at most $K^{1+2\varepsilon+\varepsilon'}$ off-diagonal terms, whose contribution to $\mathcal{D}$ is bounded by

$$K \sum_{\substack{n_1,n_2,m_1,m_2 \\ n_1\neq n_2, m_1\neq m_2}} \sum_{d_1,d_2} \frac{1_{n_1(n_1+m_1)=n_2(n_2+m_2)}}{d_1 d_2\Big(n_1(n_1+m_1)n_2(n_2+m_2)\Big)^{1/2}} \cdot \widehat{h^*}(0) \ll_{\psi_1,\psi_2} K^{1+2\varepsilon+\varepsilon'}.$$

Hence,

$$\begin{aligned}\mathcal{D} =& \frac{\sqrt{2\pi}K^2}{16} \sum_{n_1,d_1,d_2,m_1} \frac{1}{d_1 d_2 n_1(n_1+m_1)} \cdot \\ & \times \int_0^\infty \frac{h(\sqrt{u})\sqrt{u}}{\sqrt{2\pi u}} \psi_1\Big(\frac{\sqrt{u}K}{2\pi d_1(n_1+m_1)}\Big)\psi_2\Big(\frac{\sqrt{u}K}{2\pi d_2(n_2+m_2)}\Big) \exp\Big(-\frac{\sqrt{u}Km_1^2}{(2n_1+m_1)^2}\Big)du \\ & + O_{\psi_1,\psi_2}(K^{1+\varepsilon}).\end{aligned}$$

Since $m_i d_i \ll K^{1/2+\varepsilon}$ and $m_i/n_i \ll K^{1/2+\varepsilon}$ for $i=1,2$, we can simplify the expression for the off-diagonal $\mathcal{D}$ by a Taylor expansion and get

$$\begin{aligned}\mathcal{D} =& \frac{\sqrt{2\pi}K^2}{16} \sum_{n_1,d_1,d_2,m_1} \frac{1}{d_1 d_2 n_1^2} \int_0^\infty \frac{h(\sqrt{u})\sqrt{u}}{\sqrt{2\pi u}} \psi_1\Big(\frac{\sqrt{u}K}{4\pi n_1 d_1}\Big)\psi_2\Big(\frac{\sqrt{u}K}{4\pi n_1 d_2}\Big) \exp\Big(-\frac{\sqrt{u}Km_1^2}{4n_1^2}\Big)du \\ & + O_{\psi_1,\psi_2}(K^{1+\varepsilon}).\end{aligned}$$

To evaluate the main term of $\mathcal{D}$ asymptotically we perform an inverse Mellin transform on the smooth compactly-supported functions ψ_1, ψ_2 and the exponential function. We then shift the contours and collect the residues. The main term of $\mathcal{D}$ is equal to

$$\begin{aligned}&\frac{\sqrt{2\pi}K^2}{16}\frac{1}{(2\pi i)^3}\int_{(1/2+\varepsilon)}\int_{(1)}\int_{(1)}\int_0^\infty \frac{h(\sqrt{u})\sqrt{u}}{\sqrt{2\pi u}} \widetilde{\psi}_1(s_1)\widetilde{\psi_2}(s_2)\Gamma(s_3) \cdot \\ &\qquad \cdot \sum_{n_1,d_1,d_2,m_1} \frac{1}{d_1 d_2 n_1^2}\Big(\frac{\sqrt{u}K}{4\pi n_1 d_1}\Big)^{s_1}\Big(\frac{\sqrt{u}K}{4\pi n_1 d_2}\Big)^{s_2}\Big(\frac{4n_1^2}{\sqrt{u}Km_1^2}\Big)^{s_3} du ds_1 ds_2 ds_3.\end{aligned}$$

This is turn can be rewritten as

$$\frac{\sqrt{2\pi}K^2}{16}\frac{1}{(2\pi i)^3}\int_{(1/2+\varepsilon)}\int_{(1)}\int_{(1)}\int_0^\infty \frac{h(\sqrt{u})\sqrt{u}}{\sqrt{2\pi u}}u^{(s_1+s_2-s_3)/2}K^{s_1+s_2-s_3}(4\pi)^{-s_1-s_2}4^{s_3}\cdot$$
$$\cdot\widetilde{\psi_1}(s_1)\widetilde{\psi_2}(s_2)\Gamma(s_3)\zeta(1+s_1)\zeta(1+s_2)\zeta(2+s_1+s_2-2s_3)\zeta(2s_3)du ds_1 ds_2 ds_3.$$

We start by shifting the contour from $\mathrm{Re}(s_3)=1/2+\varepsilon$ to $\mathrm{Re}(s_3)=100$. The integral on the new line $Re(s_3)=100$ is negligible by the rapid decay of $\widetilde{\psi_1}(s_1),\widetilde{\psi_2}(s_2)$ and the Gamma function (it contributes at most $O_{\psi_1,\psi_2}(K^{-96})$). The simpe pole at $s_3=1/2+s_1/2+s_2/2$ yields the residue

$$\frac{\sqrt{2\pi}K^2}{16}\frac{1}{(2\pi i)^2}\int_{(1)}\int_{(1)}\int_0^\infty \frac{h(\sqrt{u})\sqrt{u}}{\sqrt{2\pi u}}u^{(-1/2+s_1/2+s_2/2)/2}K^{-1/2+s_1/2+s_2/2}(4\pi)^{-s_1-s_2}4^{1/2+s_1/2+s_2/2}$$
$$\cdot\widetilde{\psi_1}(s_1)\widetilde{\psi_2}(s_2)\Gamma(1/2+s_1/2+s_2/2)\zeta(1+s_1)\zeta(1+s_2)\frac{1}{2}\zeta(1+s_1+s_2)du ds_1 ds_2. \tag{4.13}$$

Next we move the line $\mathrm{Re}(s_1)=1$ to $\mathrm{Re}(s)=-2+\varepsilon$ (stopping just before the pole of the Gamma function), picking up simple poles at $s_1=0$ and $s_1=-s_2$. We use again the rapid decay of $\widetilde{\psi_1}(s_1),\widetilde{\psi_2}(s_2)$ to show that the new line integral is bounded by $O_{\psi_1,\psi_2}(K^{1+\varepsilon})$. At $s_1=0$ the residue is

$$\frac{\sqrt{2\pi}K^2}{16}\frac{1}{2\pi i}\int_{(1)}\int_0^\infty \frac{h(\sqrt{u})\sqrt{u}}{\sqrt{2\pi u}}u^{(-1/2+s_2/2)/2}K^{-1/2+s_2/2}(4\pi)^{-s_2}4^{1/2+s_2/2} \tag{4.14}$$
$$\cdot\widetilde{\psi_1}(0)\widetilde{\psi_2}(s_2)\Gamma(1/2+s_2/2)\zeta(1+s_2)^2\frac{1}{2}du ds_2.$$

We follow up with the shift from $\mathrm{Re}(s_2)=1$ to $\mathrm{Re}(s_2)=-1+\varepsilon$ and pick up a double pole at $s_2=0$. The error term from the line at $\mathrm{Re}(s_2)=-1+\varepsilon$ is again $O_{\psi_1,\psi_2}(K^{1+\varepsilon})$. To compute the residue at the double pole we use the expansion $\zeta(1+s_2)^2=\frac{1}{s_2^2}+\frac{2\gamma}{s}+\cdots$ for s_2 close to 0, where γ is the Euler–Mascheroni constant. The residue is then given by

$$\frac{\sqrt{2\pi}K^{3/2}}{16}\cdot\int_0^\infty \frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}}\cdot\lim_{s_2\to 0}\frac{d}{ds}\left(s^2\cdot\left(\frac{K\sqrt{u}}{4\pi^2}\right)^{\frac{s_2}{2}}\widetilde{\psi_2}(s_2)\Gamma(1/2+s_1/2)\cdot\left(\frac{1}{s_2^2}+\frac{2\gamma}{s}+\cdots\right)\right)du.$$

The limit is equal to

$$\lim_{s_2\to 0}\left(\frac{K\sqrt{u}}{4\pi^2}\right)^{\frac{s_2}{2}}\cdot\left(\frac{1}{2}\log\left(\frac{K\sqrt{u}}{4\pi^2}\right)\widetilde{\psi_2}(s_2)\Gamma(1/2+s_2/2)+\right.$$
$$\left.+\widetilde{\psi_2}'(s_2)\Gamma(1/2+s_2/2)+\frac{1}{2}\widetilde{\psi_2}(s_2)\Gamma'(1/2+s_2/2)+2\gamma\widetilde{\psi_2}(s_2)\Gamma(1/2+s_2/2)\right).$$

We evaluate the limit, using $\Gamma'(1/2)=\sqrt{\pi}(-\gamma-\log 4)$, to

$$\frac{\sqrt{\pi}}{2}\widetilde{\psi_2}(0)\log K+\frac{\sqrt{\pi}}{4}\widetilde{\psi_2}(0)\log u+\sqrt{\pi}\widetilde{\psi_2}(0)\left(\frac{3}{2}\gamma-\log(4\pi)\right)+\sqrt{\pi}\widetilde{\psi_2}'(0).$$

Thus (4.14) is equal to

$$\begin{aligned}
&K^{3/2}\log K\cdot\frac{\sqrt{2}\pi}{32}\widetilde{\psi_1}(0)\widetilde{\psi_2}(0)\cdot\int_0^\infty\frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}}du\\
&+K^{3/2}\frac{\sqrt{2}\pi}{64}\widetilde{\psi_1}(0)\widetilde{\psi_2}(0)\int_0^\infty\frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}}\log(u)du+\\
&+K^{3/2}\int_0^\infty\frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}}du\cdot\left(\frac{\sqrt{2}\pi}{16}\left(\frac{3}{2}\gamma-\log(4\pi)\right)\widetilde{\psi_1}(0)\widetilde{\psi_2}(0)+\frac{\sqrt{2}\pi}{16}\widetilde{\psi_1}(0)\widetilde{\psi_2}'(0)\right)\\
&+O_{\psi_1,\psi_2}(K^{1+\varepsilon}).
\end{aligned}$$

This is our main term. There is another term of size $K^{3/2}$ coming from the residue of expression (4.13) at $s_1=-s_2$. This residue is given by

$$K^{3/2}\int_0^\infty\frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}}du\cdot\frac{\sqrt{2}\pi}{16}\frac{1}{2\pi i}\int_{(1)}\widetilde{\psi}_1(-s_2)\widetilde{\psi_2}(s_2)\zeta(1-s_2)\zeta(1+s_2)ds_2. \tag{4.15}$$

This completes the proof of the lemma. □

4.2.6 Auxiliary Lemmas

In the following section we record some lemmas that we use to compute the off-diagonal term asymptotically. We start with some observations regarding the function $\hbar(v)$, appearing in the off-diagonal term. For any complex number w define the function

$$\hbar_w^{\mathrm{Re}}(v) = \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}} u^{w/2} \cos(uv) du.$$

For $w = 0$ this is the real part of $\hbar(v)$. The Mellin transform of this function and its properties were evaluated by Khan [Kha10, Lemma 3.5] (and also Das-Khan [DK18, sec. 2.6]). Note that Khan and Das-Khan treat $\hbar_w(v)$ but the observations also go through for the real part that we consider. As in [DK18, sec. 2.6] we have by repeated integration by parts the bound

$$\frac{\partial^j}{\partial v^j} \hbar_w^{\mathrm{Re}}(v) \ll (1 + |w|)^A |v|^{-A} \tag{4.16}$$

for any non-negative integer j, A and the implied constant depending on $\mathrm{Re}(w), j, A$. We denote the Mellin transform of $\hbar_w^{\mathrm{Re}}$ by

$$\widetilde{\hbar}_w^{\mathrm{Re}}(s) = \int_0^\infty \hbar_w^{\mathrm{Re}}(v) v^s \frac{dv}{v}.$$

The bound (4.16) implies that the Mellin transform is absolutely convergent and holomorphic for $\mathrm{Re}(s) > 0$. Integrating by parts several times and using again the bound (4.16) shows that the Mellin transform deacys rapidly. More precisiely we have

$$\widetilde{\hbar}_w^{\mathrm{Re}}(s) \ll (1 + |w|)^{A + \mathrm{Re}(s) + 1} (1 + |s|)^{-A},$$

with the implied constants depending on $\mathrm{Re}(w)$ and A. By Mellin inversion we have for $c > 0$

$$\hbar_w^{\mathrm{Re}}(v) = \frac{1}{2\pi i} \int_{(c)} \widetilde{\hbar}_w^{\mathrm{Re}}(s) v^{-s} ds. \tag{4.17}$$

As in [Kha10, Lemma 3.5] we can explicitly evaluate the Mellin transform of $\hbar_w^{\mathrm{Re}}$ within the range $0 < \mathrm{Re}(s) < 1$. There we get

$$\widetilde{\hbar}_w^{\mathrm{Re}}(s) = \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}} u^{w/2} \Gamma(s) \cos(\pi s/2) du.$$

The next two lemmas will be useful to treat the exponential sum in the off-diagonal term.

Lemma 4.2.5 (Poisson summation)**.** *Let f be a rapidly decaying, smooth function, then*

$$\sum_{n \equiv a \,(\mathrm{mod}\, c)} f(n) = \frac{1}{c} \sum_n \hat{f}\Big(\frac{n}{c}\Big) e_c(an),$$

where $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e(-x\xi) dx$ denotes the Fourier transform of f.

Proof. This follows immediately from the classical Poisson summation formula and noting that $n \equiv a \,(\mathrm{mod}\, c)$ is a shifted lattice of $\mathbb{Z}$. □

We detect cancellation in the off-diagonal term with the stationary phase method. We use the following version of Blomer, Khan and Young, which is a special case of their Proposition 8.2 in [BKY13].

Lemma 4.2.6 (Stationary phase)**.** *Let $X, Y, V, V_1, Q > 0$ and $Z := Q + X + Y + V_1 + 1$, and assume that*

$$Y \geq Z^{3/20}, \quad V_1 \geq V \geq \frac{QZ^{1/40}}{Y^{1/2}}.$$

Suppose that h is a smooth function on $\mathbb{R}$ with support on an interval J of length V_1, satisfying

$$h^{(j)}(t) \ll_j XV^{-j}$$

for all $j \in \mathbb{N}_0$. Suppose f is a smooth function on J such that there exists a unique point

$t_0 \in J$ such that $f'(t_0) = 0$, and furthermore

$$f''(t) \gg YQ^{-2}, \quad f^{(j)}(t) \ll_j YQ^{-j}, \quad \textit{for } j \geq 1 \textit{ and } t \in J.$$

Then

$$\int_{-\infty}^{\infty} h(t)e^{2\pi i f(t)}dt = e^{\mathrm{sgn}(f''(t_0))\cdot \pi i/4}\frac{e^{2\pi i f(t_0)}}{\sqrt{|f''(t_0)|}}h(t_0) + O\Big(\frac{Q^{3/2}X}{Y^{3/2}} \cdot \Big(V^{-2} + (Y^{2/3}/Q^2)\Big)\Big).$$

In particular, we also have the trivial bound

$$\int_{-\infty}^{\infty} h(t)e^{2\pi i f(t)}dt \ll \frac{XQ}{\sqrt{Y}} + 1.$$

Proof. See Proposition 8.2 in [BKY13], with $\delta = 1/20$ and A sufficiently large. We bounded the contribution of the non-leading terms in the asymptotic expansion of [BKY13, Eq. 8.9]) trivially by $O\Big(\frac{Q^{3/2}X}{Y^{3/2}} \cdot \Big(V^{-2} + (Y^{2/3}/Q^2)\Big)\Big)$. □

The last lemma we need concerns the Kloosterman sum over arithmetic progressions.

Lemma 4.2.7. *Let $S(a, b; c)$ denote the classical Kloosterman sum, then*

$$\sum_{\substack{a_1 \,(\mathrm{mod}\, c),\\ a_2 \,(\mathrm{mod}\, c)}} \sum_{\substack{b_1 \,(\mathrm{mod}\, c),\\ b_2 \,(\mathrm{mod}\, c)}} S(a_1(a_1+b_1), a_2(a_2+b_2); c)e_c(2a_1a_2 + a_1b_2 + a_2b_1) = c^3\varphi(c),$$

where $\varphi(c)$ is Euler's totient function.

Proof. Let us call $\mathcal{T}$ the sum we must calculate. First we open the Kloosterman sum and get

$$\mathcal{T} = \sum_{\substack{a_1 \,(\mathrm{mod}\, c),\\ a_2 \,(\mathrm{mod}\, c)}} \sum_{\substack{b_1 \,(\mathrm{mod}\, c),\\ b_2 \,(\mathrm{mod}\, c)}} \sum_{\substack{x \,(\mathrm{mod}\, c)\\ (x,c)=1}} e_c\Big(a_1(a_1+b_1)\overline{x} + a_2(a_2+b_2)x + 2a_1a_2 + a_1b_2 + a_2b_1\Big).$$

Here $x\overline{x} \equiv 1 \pmod{c}$. Since $(x,c)=1$ we can substitute a_1 with a_1x and b_1 with b_1x. We get

$$\begin{aligned}
\mathcal{T} &= \sum_{\substack{a_1 \,(\mathrm{mod}\, c),\\ a_2 \,(\mathrm{mod}\, c)}} \sum_{\substack{b_1 \,(\mathrm{mod}\, c),\\ b_2 \,(\mathrm{mod}\, c)}} \sum_{\substack{x \,(\mathrm{mod}\, c)\\ (x,c)=1}} e_c\Big(a_1^2x + a_1b_1x + a_2^2x + a_2b_2x + 2a_1a_2x + a_1b_2x + a_2b_1x\Big) \\
&= \sum_{\substack{a_1 \,(\mathrm{mod}\, c),\\ a_2 \,(\mathrm{mod}\, c)}} \sum_{\substack{b_1 \,(\mathrm{mod}\, c),\\ b_2 \,(\mathrm{mod}\, c)}} \sum_{\substack{x \,(\mathrm{mod}\, c)\\ (x,c)=1}} e_c\Big((a_1+a_2)^2x + (a_1+a_2)(b_1+b_2)x\Big) \\
&= c^3\varphi(c).
\end{aligned}$$

To obtain the last equality we used orthogonality when summing over b_1 and b_2, i.e.

$$\sum_{b \,(\mathrm{mod}\, c)} e_c\Big(b(a_1+a_2)x\Big) = \begin{cases} c & \text{if } (a_1+a_2)x \equiv 0 \pmod{c} \\ 0 & \text{otherwise} \end{cases}$$

and so $a_1 \equiv -a_2 \pmod{c}$. □

Remark 4.2.4. We highlight here that the additional summation over b_1, b_2 (that originally stems from averaging over the shifts) is crucial here. In contrast to that, Luo and Sarnak [LS04, Appendix A.2] work for the full fundamental domain with fixed shifts. The corresponding summation over the Kloosterman sum then reduces "only" to an expression involving Salié sums. Lemma 4.2.7 might be seen as the reason why we can obtain a comparably clean formula for the quantum variance of the vertical geodesic.

4.2.7 Evaluating the Off-Diagonal

Our goal is to obtain an asymptotic formula for the off-diagonal $\mathcal{OD}$ given by

$$-\sqrt{\pi}\mathrm{Im}\Big(e^{-2\pi i/8}K\sum_{d_1,d_2}\sum_{n_1,n_2}\sum_{m_1,m_2}\sum_{c}\frac{S(n_1(n_1+m_1),n_2(n_2+m_2);c)}{\sqrt{c}}e_c(2n_1n_2+n_1m_2+n_2m_1)\cdot$$
$$\times\frac{e_c\Big(f(n_1,n_2,m_1,m_2)\Big)}{d_1d_2\Big(n_1(n_1+m_1)n_2(n_2+m_2)\Big)^{3/4}}\cdot\hbar^*_{d_1,d_2}\Big(\frac{cK^2}{8\pi\sqrt{n_1(n_1+m_1)n_2(n_2+m_2)}}\Big)\Big), \tag{4.18}$$

where

$$f(n_1,n_2,m_1,m_2)=2\sqrt{n_1(n_1+m_1)n_2(n_2+m_2)}-2n_1n_2-n_1m_2-n_2m_1$$

and

$$\begin{aligned}\hbar^*_{d_1,d_2}&\Big(\frac{cK^2}{8\pi\sqrt{n_1(n_1+m_1)n_2(n_2+m_2)}}\Big)= \\ &\frac{K}{8}\int_0^\infty\frac{h(\sqrt{u})\sqrt{u}}{\sqrt{2\pi u}}\psi_1\Big(\frac{\sqrt{u}K}{2\pi d_1(n_1+m_1)}\Big)\psi_2\Big(\frac{\sqrt{u}K}{2\pi d_2(n_2+m_2)}\Big)\cdot \\ &\quad\times\exp\Big(-\frac{\sqrt{u}Km_1^2}{(2n_1+m_1)^2}\Big)\exp\Big(-\frac{\sqrt{u}Km_2^2}{(2n_2+m_2)^2}\Big)e^{iu\frac{cK^2}{8\pi(n_1(n_1+m_1)n_2(n_2+m_2))^{1/2}}}du.\end{aligned} \tag{4.19}$$

This task requires several intermediate steps.

First we will detect square-root cancellation in the exponential sum

$$\sum_{n_1,n_2\asymp K}e_c\Big(f(n_1,n_2,m_1,m_2)\Big)$$

when m_1,m_2 are large. To see that this is possible, it is helpful to keep the critical ranges $m_i\asymp\sqrt{K}$ and $n_i\asymp K$ for $i=1,2$ in mind. We then have the following heuristic that guides our further analysis:

$$f(n_1,n_2,m_1,m_2)=-\frac{m_1^2n_2}{2n_1}-\frac{m_2^2n_1}{2n_2}+\dots$$

$$\frac{\partial}{\partial n_1} f(n_1,n_2,m_1,m_2) \approx \frac{m_1^2 n_2}{n_1^2} - \frac{m_2^2}{2n_2} \asymp O(1) \quad \text{and} \quad \frac{\partial^2}{\partial n_1^2} f(n_1,n_2,m_1,m_2) \approx -\frac{3m_1^2 n_2}{n_1^3} \asymp \frac{1}{K}.$$

From these bounds on the derivatives we can see that we expect square root cancellation in the summation over n_1 (see [IK04, Corollary 8.12]). To make this heuristic precise and to compute an asymptotic formula for the off-diagonal $\mathcal{OD}$, we will use the Poisson summation formula and the stationary phase method, which is the subject of the following lemmas.

Lemma 4.2.8. *Let $\mathcal{OD}$ be defined as in* (4.18). *Then*

$$\mathcal{OD} = -\sqrt{\pi} K \mathrm{Im}\Bigg(e^{-2\pi i/8} \sum_c \sum_{\substack{b_1 \,(\mathrm{mod}\, c),\\ b_2 \,(\mathrm{mod}\, c)}} \sum_{\substack{a_1 \,(\mathrm{mod}\, c),\\ a_2 \,(\mathrm{mod}\, c)}} \frac{S(a_1(a_1+b_1), a_2(a_2+b_2); c)}{c^{3/2}} e_c(2a_1a_2 + a_1b_2 + a_2b_1) \cdot$$
$$\times \sum_{d_1,d_2} \frac{1}{d_1d_2} \sum_{\substack{m_1 \equiv b_1 \,(\mathrm{mod}\, c)\\ m_2 \equiv b_2 \,(\mathrm{mod}\, c)}} \sum_{n_2 \equiv a_2 \,(\mathrm{mod}\, c)} \sum_{v \in \mathbb{Z}} \mathcal{I}_v(n_2, m_1, m_2, d_1, d_2, c) \Bigg)$$

with

$$\mathcal{I}_v(n_2,m_1,m_2,d_1,d_2,c) := \int_{-\infty}^{\infty} \frac{e_c\Big(f(x,n_2,m_1,m_2) - v(x+a_1)\Big)}{d_1d_2\Big(x(x+m_1)n_2(n_2+m_2)\Big)^{3/4}} \hbar^*_{d_1,d_2}\Bigg(\frac{cK^2}{8\pi\sqrt{x(x+m_1)n_2(n_2+m_2)}}\Bigg) dx. \tag{4.20}$$

Proof. We work with expression (4.18) and split the variables n_1, n_2, m_1, m_2 into residue classes modulo c. The off-diagonal OD is then given by

$$-\sqrt{\pi}K\mathrm{Im}\Bigg(e^{-2\pi i/8}\sum_{c}\sum_{\substack{b_1 \,(\mathrm{mod}\, c),\\ b_2 \,(\mathrm{mod}\, c)}}\sum_{\substack{a_1 \,(\mathrm{mod}\, c),\\ a_2 \,(\mathrm{mod}\, c)}}\frac{S(a_1(a_1+b_1),a_2(a_2+b_2);c)}{\sqrt{c}}e_c(2a_1a_2+a_1b_2+a_2b_1)\cdot \tag{4.21}$$

$$\times\sum_{d_1,d_2}\sum_{\substack{m_1\equiv b_1 \,(\mathrm{mod}\, c)\\ m_2\equiv b_2 \,(\mathrm{mod}\, c)}}\sum_{\substack{n_1\equiv a_1 \,(\mathrm{mod}\, c)\\ n_2\equiv a_2 \,(\mathrm{mod}\, c)}}\frac{e_c\Big(f(n_1,n_2,m_1,m_2)\Big)}{d_1d_2\Big(n_1(n_1+m_1)n_2(n_2+m_2)\Big)^{3/4}}. \tag{4.22}$$

$$\times\,\hbar^*_{d_1,d_2}\bigg(\frac{cK^2}{8\pi\sqrt{n_1(n_1+m_1)n_2(n_2+m_2)}}\bigg)\Bigg). \tag{4.23}$$

Here we used the fact that the Kloosterman sum only depends on the residue classes modulo c. The lemma follows now upon applying the Poisson summation formula (see Lemma 4.2.5) to the summation over n_1. □

Next we will analyze the quantity

$$\sum_{n_2\equiv a_2 \,(\mathrm{mod}\, c)}\sum_{v\in\mathbb{Z}}\mathcal{I}_v(n_2,m_1,m_2,d_1,d_2,c) \tag{4.24}$$

with the stationary phase method.

We record here some restrictions on the variables that will be useful for the further analysis. Since h,ψ_1,ψ_2 are compactly supported on $\mathbb{R}^+$ we have that $n_id_i \asymp K$ for $i = 1/2$. Additionally, $d_i \leq K^\delta$ for some $\delta > 0$, as noted in Remark 4.2.3. Indeed, if $d_i > K^\delta$ then $n_i \leq K^{1-\delta}$ and $cK^2/\sqrt{n_1(n_1+m_1)n_2(n_2+m_2)} \geq K^{2\delta}$ and the off-diagonal term can be absorbed in the error term. We will choose $\delta = 1/32$ for convenience. From the observations in Remark 4.2.3 we also have the condition $\frac{cK^2}{n_1n_2} \ll K^\epsilon$. The exponential functions ensure that $m_i \leq n_i/K^{1/2-\varepsilon} \leq K^{1/2+\varepsilon}$, as otherwise we have exponential decay. For technical purposes we also impose a lower bound on m_i. If both variables m_1, m_2 are small then the exponential $e_c(\sqrt{n_1(n_1+m_1)n_2(n_2+m_2)})$ is essentially smooth and the analysis is similar as in the case of full fundamental domain (see [LS04]). If $m_i \leq K^{1/8}$ for $i = 1, 2$, then we can bound OD trivially by $K^{5/4+\varepsilon}$, which is smaller than the expected main term of size $K^{3/2}$. In our stationary phase analysis we will only need one of the variables m_i to be large. Since OD is symmetric in m_i/n_i we can assume without loss of generality that $m_1 \geq K^{1/8}$. The most important case is of course when both m_1 and m_2 are of size $\sqrt{K}$ and so one should view this restriction only as a technical convenience. To summarize, we will work subsequently under the following conditions:

$$n_i d_i \asymp K \quad \text{and} \quad d_i \leq K^{1/32} \quad \text{for } i = 1, 2, \tag{4.25}$$

$$K^{1/8} \leq m_1 \leq K^{1/2+\varepsilon} \quad \text{and} \quad 1 \leq m_2 \leq K^{1/2+\varepsilon}, \tag{4.26}$$

$$\frac{cK^2}{n_1 n_2} \ll K^{\varepsilon} \quad \text{and in particular} \quad c \ll K^{\varepsilon}. \tag{4.27}$$

A stationary phase analysis leads to the following lemma:

Lemma 4.2.9. *Let $\mathcal{I}_v(n_2, m_1, m_2, d_1, d_2, c)$ be defined as in* (4.20)*. Then*

$$\begin{aligned}\mathcal{I}_v(n_2, m_1, m_2, d_1, d_2, c) = &\frac{e_c\Big(a_1 v + \frac{m_1}{2}\Big(v + m_2 - \sqrt{(v+m_2)^2 + 4vn_2}\Big)\Big) e^{-\pi i/4}\sqrt{c}}{\sqrt{|f''(x_v^*(n_2))|} \cdot x_v^*(n_2)^{3/2} \cdot n_2^{3/2}} \hbar^*_{d_1,d_2}\Big(\frac{cK^2}{8\pi n_2 x_v^*(n_2)}\Big) \\ &+ O(K^{-2+\varepsilon})\end{aligned}$$

with $x_v^*(n_2) = \frac{m_1}{2}\Big(-1 + \frac{v+m_2+2n_2}{\sqrt{(v+m_2)^2+4vn_2}}\Big)$.

Proof. We now need to check the conditions of the stationary phase Lemma 4.2.6 so that we can apply it to the quantity $\mathcal{I}_v(n_2, m_1, m_2, d_1, d_2, c)$. The stationary points $x_v^*(n_2)$ for fixed v are the solutions to the equation

$$\frac{\partial}{\partial x}\Big(f(x, n_2, m_1, m_2) - v(x - a_1)\Big) = 0$$

and are given by

$$x_v^*(n_2) = \frac{m_1\Big(m_2 + 2n_2 + v - \sqrt{(v+m_2)^2 + 4vn_2}\Big)}{2\sqrt{(v+m_2)^2+4vn_2}} = \frac{m_1 n_2}{\sqrt{(v+m_2)^2+4vn_2}} \cdot \Big(1 + O(K^{-1/2+1/32+\varepsilon})\Big). \tag{4.28}$$

From a Taylor expansion of $f(x, n_1, m_1, m_2)$ we see that

$$f(x, n_2, m_1, m_2) = -\frac{1}{4}\frac{n_2 m_1^2}{x} - \frac{1}{4}\frac{x m_2^2}{n_2} + \frac{m_1 m_2}{2} - \frac{1}{8}\frac{m_1^2 m_2}{x} - \frac{1}{8}\frac{m_2^2 m_1}{n_2} + \dots$$

and thus

$$\frac{\partial}{\partial x} f(x, n_1, m_1, m_2) = \frac{1}{4}\frac{n_2 m_1^2}{x^2} - \frac{1}{4}\frac{m_2^2}{n_2} + \frac{1}{8}\frac{m_1^2 m_2}{x^2} + \dots \ll K^{\varepsilon}.$$

From the bound on the first derivative of f it follows also that $v \ll K^{\varepsilon}$. The second derivate of the phase function f is given by

$$\frac{\partial^2}{\partial x^2} f(x, n_2, m_1, m_2) = -\frac{m_1^2}{2}\sqrt{\frac{n_2(n_2+m_2)}{(x(x+m_1))^3}}.$$

We note here that

$$\left|\frac{\partial^2}{\partial x^2} f(x, n_2, m_1, m_2)\right| \asymp \frac{m_1^2 n_2}{x^3} \quad \text{and} \quad \left|\frac{\partial^j}{\partial x^j} f(x, n_2, m_1, m_2)\right| \ll_j \frac{m_1^2 n_2}{x^{1+j}} \tag{4.29}$$

for any integer $j \geq 2$. If we evaluate the second derivative at the stationary point x_v^* we get

$$f''(x_v^*) = -\frac{((v+m_2)^2 + 4vn_2)^{3/2}}{2m_1 n_2 (m_2 + n_2)} = -\frac{((v+m_2)^2+4vn_2)^{3/2}}{2m_1 n_2^2} \cdot (1 + O(K^{-1/2+\epsilon})). \tag{4.30}$$

After a tedious computation we see that the exponential evaluated at the stationary point equals

$$e_c\Big(f(x_v^*, n_2, m_1, m_2) - v(x_v^* - a_1)\Big) = e_c\bigg(a_1 v + \frac{m_1}{2}\Big(v + m_2 - \sqrt{(v+m_2)^2 + 4vn_2}\Big)\bigg).$$

For the stationary phase analysis we also require bounds on the derivatives of the weight function $\hbar^*_{d_1,d_2}\Big(\frac{cK^2}{8\pi(x(x+m_1)n_2(n_2+m_2))^{1/2}}\Big)$, defined in (4.19). Conditions (4.25), (4.26) and (4.27) yield

$$\begin{aligned}&\frac{\partial^j}{\partial x^j}\frac{1}{\Big(x(x+m_1)n_2(n_2+m_2)\Big)^{3/4}}\hbar^*_{d_1,d_2}\Big(\frac{cK^2}{8\pi\sqrt{x(x+m_1)n_2(n_2+m_2)}}\Big)\\&\quad\ll\frac{1}{x^{3/2}n_2^{3/2}}K\Big(\frac{1}{x}+\frac{Km_1^2}{x^3}+\frac{cK^2}{n_2x^2}\Big)^j\\&\quad\ll\frac{1}{x^{3/2}n_2^{3/2}}K\Big(\frac{K^\epsilon}{x}\Big)^j\end{aligned}\tag{4.31}$$

for any fixed integer $j\geq 0$.

We are now in the position to determine the required parameters X,Y,V,V_1,Q and Z of Lemma 4.2.6. From the bounds on the derivatives (4.29) and the fact that $x\asymp K/d_1$ we see that $Y=\frac{m_1^2n_2d_1}{cK}$ and $Q=\frac{K}{d_1}$. From (4.31) we see that $X=d_1^{3/2}n_2^{-3/2}K^{-1/2}$, $V_1=V=\frac{K^{1-\varepsilon}}{d_1}$ and $Z\ll K^{1+\varepsilon+1/32}$. With these choices we also see that the condition $V_1\geq\frac{QZ^{1/40}}{Y^{1/2}}$ is satisfied, as long as $d_2\leq K^{1/32}$, say. Indeed, with the lower bound on $m_1\geq K^{1/8}$ we have

$$\frac{QZ^{1/40}}{Y^{1/2}}\ll\frac{K\sqrt{cK}Z^{1/40}}{d_1m_1\sqrt{n_2}}\ll\frac{K}{d_1}\cdot\frac{K^\varepsilon\sqrt{d_2}K^{1/30}}{m_1}\ll\frac{K}{d_1}\cdot K^{1/64+1/30+\varepsilon-1/8}\ll VK^{-1/16}.$$

We can now apply the stationary phase lemma (Lemma 4.2.6) and see that

$$\begin{aligned}\mathcal{I}_v(n_2,m_1,m_2,d_1,d_2,c)=&\frac{e_c\Big(f(x_v^*(n_2),n_2,m_1,m_2)-v(x_v^*-a_1)\Big)e^{-\pi i/4}\sqrt{c}}{\sqrt{|f''(x_v^*(n_2))|}\cdot\Big(x_v^*(n_2)(x_v^*(n_2)+m_1)n_2(n_2+m_2)\Big)^{3/4}}\\&\times\hbar^*\Big(\frac{cK^2}{8\pi\sqrt{x_v^*(n_2)(x_v^*(n_2)+m_1)n_2(n_2+m_2)}}\Big)+O(K^{-2}).\end{aligned}$$

Since $x_v^*(n_2)\asymp n_1$ we can simplify the above result slightly by using a Taylor expansion and conditions (4.25), (4.26). We get

$$\mathcal{I}_v(n_2,m_1,m_2,d_1,d_2,c) = \frac{e_c\Big(a_1 v + \frac{m_1}{2}\Big(v+m_2-\sqrt{(v+m_2)^2+4vn_2}\Big)\Big)e^{-\pi i/4}\sqrt{c}}{\sqrt{|f''(x_v^*(n_2))|}\cdot x_v^*(n_2)^{3/2}\cdot n_2^{3/2}}\hbar^*_{d_1,d_2}\Big(\frac{cK^2}{8\pi n_2 x_v^*(n_2)}\Big) + O(K^{-2+\varepsilon}).$$

This completes the proof of the lemma. □

We expect the main term of (4.24) to come from the 0-frequency, i.e. when $v=0$.

Lemma 4.2.10. *Let $\mathcal{I}_v(n_2,m_1,m_2,d_1,d_2,c)$ be defined as in* (4.20)*. Then*

$$\sum_{n_2\equiv a_2 \,(\mathrm{mod}\, c)} \mathcal{I}_0(n_1,m_1,m_2,d_1,d_2,c) = e^{-\pi i/4}\sqrt{2c}\sum_{n_2\equiv a_2\,(\mathrm{mod}\, c)}\frac{1}{m_1 n_2^2}\hbar^*\Big(\frac{cK^2 m_2}{8\pi n_2^2 m_1}\Big) + O(K^{-1+\varepsilon}).$$

Proof. This lemma is a direct consequence of Lemma 4.2.9, specialized to the case $v=0$. For $v=0$ the stationary point is given by $x_0^*(n_2) = \frac{m_1 n_2}{m_2}$. From (4.30) we see that

$$\sqrt{|f''\big(x_0^*(n_2)\big)|} = \frac{m_2^3}{2m_1 n_2(n_2+m_2)} = \frac{m_2^3}{2m_1 n_2^2}\cdot\Big(1+O(K^{-1/2+\varepsilon})\Big).$$

Moreover, a direct computation gives

$$e_c\Big(f(x_0^*(n_2),n_2,m_1,m_2)\Big) = e_c(0) = 1$$

and thus the claimed result.

□

On the other hand, the contribution of the non-zero frequencies, i.e. when $v\neq 0$, is negligible. To show this we will need to detect further cancellation in the summation over n_2.

Lemma 4.2.11. *Let $\mathcal{I}_v(n_2, m_1, m_2, d_1, d_2, c)$ be defined as in* (4.20)*. Then*

$$\sum_{n_2 \equiv a_2 \, (\mathrm{mod}\, c)} \sum_{v \neq 0} \mathcal{I}_v(n_2, m_1, m_2, d_1, d_2, c) \ll \frac{d_2^2}{\sqrt{c} K^{1-\varepsilon}}. \tag{4.32}$$

Proof. Let ET (for error term) denote the left-hand side of (4.32). First we use the Poisson summation formula for the summation over n_2. Together with Lemma 4.2.9 it follows that

$$ET = \frac{e^{-\pi i/4}}{c} \sum_{w} \sum_{v \neq 0} \int_{-\infty}^{\infty} e_c\Big(a_1 v + \frac{m_1}{2}\Big(v + m_2 - \sqrt{(v+m_2)^2 + 4vy}\Big) - w(y - a_2)\Big) \tag{4.33}$$

$$\times \frac{\sqrt{c}}{\sqrt{|f''\big(x_v^*(y)\big)|}} \frac{1}{x_v^*(y)^{3/2} y^{3/2}} \hbar^*\Big(\frac{cK^2}{8\pi x_v^*(y) y}\Big) dy + O(K^{-2+\varepsilon}) \tag{4.34}$$

with

$$x_v^*(y) = \frac{m_1\Big(m_2 + 2y + v - \sqrt{(v+m_2)^2 + 4vy}\Big)}{2\sqrt{(v+m_2)^2 + 4vy}} \asymp \frac{K}{d_1}. \tag{4.35}$$

As in Lemma (4.2.9) we perform a stationary phase analysis on the integral

$$\mathcal{J}_w(m_1, m_2, v) := \int_{-\infty}^{\infty} e_c\Big(g_{m_1,m_2,v}(y) - w(y - a_2)\Big) \frac{\sqrt{c}}{\sqrt{|f''\big(x_v^*(y)\big)|}} \frac{1}{x_v^*(y)^{3/2} y^{3/2}} \hbar^*\Big(\frac{cK^2}{8\pi x_v^*(y) y}\Big) dy$$

with

$$g_{m_1,m_2,v}(y) = a_1 v + \frac{m_1}{2}\Big(v + m_2 - \sqrt{(v+m_2)^2 + 4vy}\Big).$$

It is again useful to have the critical cases in mind when $y \sim K$ and $m_1, m_2 \sim \sqrt{K}$. The first derivative $g'_{m_1,m_2,v}(y)$ is then again roughly bounded, while $g''_{m_1,m_2,v}(y)$ is of size $1/K$. Consequently, we again expect square root cancellation in n_2 (see for example [IK04, Corollary 8.12]). We now perform the stationary phase method on the integral over y of expression ET. The stationary points are given by the solutions to the equation

$$\frac{\partial}{\partial y}\Big(g_{m_1,m_2,v}(y)-(w-a_2)\Big)=-\frac{m_1 v}{\sqrt{(v+m_2)^2+4vy}}-w=0.$$

Since $\frac{\partial}{\partial y}g_{m_1,m_2,v}(y)\ll K^{\varepsilon}$, we also see that $w\ll K^{\epsilon}$. Moreover, we have the following bounds on the higher derivatives:

$$\frac{\partial^2}{\partial y^2}g_{m_1,m_2,v}(y)=\frac{4m_1v^2}{\Big((v+m_2)^2+4vy\Big)^{3/2}}\quad\text{and}\quad\frac{\partial^j}{\partial y^j}g_{m_1,m_2,v}(y)\ll_j\frac{m_1v^j}{\Big((v+m_2)^2+4vy\Big)^{j-1/2}},\tag{4.36}$$

for $j\geq 3$. From the computations in the proof of Lemma 4.2.9, in particular equation (4.28) and (4.30) we have

$$\frac{\sqrt{c}}{\sqrt{|f''\big(x_v^*(y)\big)|}}\frac{1}{x_v^*(y)^{3/2}y^{3/2}}=\frac{\sqrt{2c}}{m_1y^2}\cdot\Big(1+O(K^{-1/2+1/32+\varepsilon})\Big).$$

Similarly to (4.31) we will need bounds on the derivatives of the involved weight function with respect to y. It is useful to first compute

$$\frac{\partial^j}{\partial y^j}\Big(x_v^*(y)\Big)\ll_j\frac{m_1v^{j-1}(m_2^2+vy)}{(4vy+(m_2+v)^2)^{1/2+j}}\ll_j\frac{K}{d_1}\Big(\frac{1}{y}\Big)^j.$$

Using the chain rule, a similar computation as in (4.31) yields

$$\frac{\partial^j}{\partial y^j}\frac{\sqrt{2c}}{m_1y^2}\hbar^*\Big(\frac{cK^2}{8\pi x_v^*(y)\cdot y}\Big)\ll_j\frac{\sqrt{c}}{m_1y^2}\cdot K\Big(\frac{K^\epsilon}{y}\Big)^j\ll_j\frac{\sqrt{c}}{m_1}\frac{d_2^2}{K}\cdot\Big(\frac{d_2}{K^{1-\varepsilon}}\Big)^j.\tag{4.37}$$

We can now again establish the various required quantities for Lemma 4.2.6. From the computation (4.37) we can see that $X=\sqrt{c}d_2^2/(m_1K)$ and $V=K^{1-\varepsilon}/d_2$. From the bounds (4.36) we find $Y=m_1\cdot\sqrt{(v+m_2)^2+4vK/d_2}$ and $Q=((v+m_2)^2+4vK/d_2)/v$. The trivial bound of Lemma 4.2.6 yields

$$\mathcal{J}_w(m_1, m_2, v) \ll \frac{XQ}{\sqrt{Y}} \ll \frac{\sqrt{c}d_2^2}{vK} \cdot \left(\frac{\sqrt{(v+m_2)^2 + 4vK/d_2}}{m_1}\right)^{3/2} \ll \frac{\sqrt{c}d_2^2}{vK}. \tag{4.38}$$

Here we used that $\frac{\sqrt{(v+m_2)^2+4vK/d_2}}{m_1} \asymp 1$, which can be deduced for example from the size of the stationary point (see (4.35)).

From the bounds $w \ll K^{\varepsilon}$, $v \ll K^{\varepsilon}$ and (4.38) it follows that

$$ET \ll \frac{d_2^2}{\sqrt{c}K^{1-\varepsilon}}.$$

This concludes the proof of the lemma. □

Using the previous lemmas we will obtain the following formula for the off-diagonal $\mathcal{OD}$:

Lemma 4.2.12. *Let $\mathcal{OD}$ be defined as in* (4.18)*. Then*

$$\begin{aligned}\mathcal{OD} =& \frac{\sqrt{2\pi}K^2}{8}\sum_c \frac{\varphi(c)}{c}\sum_{d_1,d_2}\sum_{n_2}\sum_{m_1,m_2}\frac{1}{d_1 d_2 m_1 n_2^2}\cdot \\ &\times \int_0^\infty \frac{h(\sqrt{u})\sqrt{u}}{\sqrt{2\pi u}}\psi_1\Big(\frac{\sqrt{u}Km_2}{4\pi m_1 d_1 n_2}\Big)\psi_2\Big(\frac{\sqrt{u}K}{4\pi d_2 n_2}\Big)\exp\Big(-\frac{\sqrt{u}Km_2^2}{4n_2^2}\Big)\cos\Big(u\frac{cK^2 m_2}{8\pi m_1 n_2^2}\Big)du \\ &+ O(K^{5/4}).\end{aligned}$$

Proof. Combining Lemma 4.2.8 and Lemma 4.2.10 we obtain the main term of the off-diagonal $\mathcal{OD}$ given by

$$\begin{aligned}-\sqrt{2\pi}K\mathrm{Im}\Bigg(e^{-\pi i/2}\sum_c \sum_{\substack{b_1 \,(\mathrm{mod}\, c),\\ b_2 \,(\mathrm{mod}\, c)}}\sum_{\substack{a_1 \,(\mathrm{mod}\, c),\\ a_2 \,(\mathrm{mod}\, c)}}\frac{S(a_1(a_1+b_1), n_2(n_2+b_2); c)}{c}e_c(2a_1a_2 + a_1b_2 + a_2b_1)\cdot \\ \times \sum_{d_1,d_2}\sum_{\substack{m_1\equiv b_1 \,(\mathrm{mod}\, c)\\ m_2\equiv b_2 \,(\mathrm{mod}\, c)}}\sum_{n_2\equiv a_2 \,(\mathrm{mod}\, c)}\frac{1}{d_1d_2m_1n_2^2}\hbar^*_{d_1,d_2}\Big(\frac{cK^2m_2}{8\pi m_1 n_2^2}\Big)\Bigg)\end{aligned}$$

with

$$\hbar^*_{d_1,d_2}\Big(\frac{cK^2m_2}{8\pi m_1 n_2^2}\Big) =$$
$$\frac{K}{8}\int_0^\infty \frac{h(\sqrt{u})\sqrt{u}}{\sqrt{2\pi u}}\psi_1\Big(\frac{\sqrt{u}Km_2}{2\pi m_1 d_1(2n_2+m_2)}\Big)\psi_2\Big(\frac{\sqrt{u}K}{2\pi d_2(2n_2+m_2)}\Big)\exp\Big(-\frac{\sqrt{u}Km_2^2}{(2n_2+m_2)^2}\Big)e^{iu\frac{cK^2m_2}{m_1n_2^2}}\,du.$$

On the other hand, by Lemma 4.2.8 and Lemma 4.2.11, the error term is bounded by

$$K^\varepsilon \sum_c \sum_{\substack{b_1 \,(\mathrm{mod}\, c),\\ b_2 \,(\mathrm{mod}\, c)}} \sum_{\substack{a_1 \,(\mathrm{mod}\, c),\\ a_2 \,(\mathrm{mod}\, c)}} \frac{|S(a_1(a_1+b_1), n_2(n_2+b_2); c)|}{c^2} \sum_{d_1,d_2} \sum_{\substack{m_1\equiv b_1 \,(\mathrm{mod}\, c)\\ m_2\equiv b_2 \,(\mathrm{mod}\, c)}} \frac{d_2}{d_1} \ll K^{5/4}.$$

We thus have

$$\mathcal{OD} = -\sqrt{2\pi}K\mathrm{Im}\Bigg(e^{-\pi i/2}\sum_c \sum_{\substack{b_1 \,(\mathrm{mod}\, c),\\ b_2 \,(\mathrm{mod}\, c)}} \sum_{\substack{a_1 \,(\mathrm{mod}\, c),\\ a_2 \,(\mathrm{mod}\, c)}} \frac{S(a_1(a_1+b_1), n_2(n_2+b_2); c)}{c} e_c(2a_1a_2+a_1b_2+a_2b_1)\cdot$$
$$\times \sum_{d_1,d_2} \sum_{\substack{m_1\equiv b_1 \,(\mathrm{mod}\, c)\\ m_2\equiv b_2 \,(\mathrm{mod}\, c)}} \sum_{n_2\equiv a_2 \,(\mathrm{mod}\, c)} \frac{1}{d_1d_2m_1n_2^2}\hbar^*_{d_1,d_2}\Big(\frac{cK^2m_2}{8\pi m_1n_2^2}\Big)\Bigg) + O(K^{5/4}).$$

Note that $\frac{cK^2m_2}{m_1n_2^2} \ll K^\varepsilon$ as otherwise we have rapid decay (using integration by parts). Additionally, we have the derivative bounds

$$\frac{\partial^j}{\partial n_2^j}\hbar^*\Big(\frac{cK^2m_2}{8\pi m_1n_2^2}\Big) \ll K\cdot\Big(\frac{K^\epsilon}{n_2}\Big)^j \ll K\cdot K^{-j/2}, \tag{4.39}$$

$$\frac{\partial^j}{\partial m_1^j}\hbar^*\Big(\frac{cK^2m_2}{8\pi m_1n_2^2}\Big) \ll K\cdot\Big(\frac{K^\epsilon}{m_1}\Big)^j \ll K\cdot K^{-j/16} \tag{4.40}$$

and

$$\frac{\partial^j}{\partial m_2^j}\hbar^*\Big(\frac{cK^2m_2}{8\pi m_1n_2^2}\Big) \ll K\cdot\Big(\frac{K}{m_1d_1n_2}+\frac{Km_2}{n_2^2}+\frac{cK^2}{m_1n_2^2}\Big)^j \ll K\cdot K^{-j/16}. \tag{4.41}$$

With the Poisson summation formula we see that

$$\sum_{n_2\equiv a_2\,(\mathrm{mod}\,c)} \frac{1}{d_1d_2m_1n_2^2}\hbar^*_{d_1,d_2}\Big(\frac{cK^2m_2}{8\pi m_1n_2^2}\Big) = \frac{1}{c}\sum_u \int_{-\infty}^{\infty} \frac{1}{d_1d_2m_1x^2}\hbar^*_{d_1,d_2}\Big(\frac{cK^2m_2}{8\pi m_1x^2}\Big)e_c\Big(-u(x-a_2)\Big)dx.$$

Repeated integration by parts and the bounds (4.39) show that the non-zero frequencies are bounded by $O_j(K^{-10})$. For the 0-frequency we have

$$\frac{1}{c}\int_{-\infty}^{\infty} \frac{1}{d_1d_2m_1x^2}\hbar^*_{d_1,d_2}\Big(\frac{cK^2m_2}{8\pi m_1x^2}\Big)dx = \frac{1}{c}\sum_{n_2} \frac{1}{d_1d_2m_1n_2^2}\hbar^*_{d_1,d_2}\Big(\frac{cK^2m_2}{8\pi m_1n^2}\Big) + O_j(K^{-10})$$

again by the Poisson summation formula. We proceed in the same way for the summation over m_1 and m_2, using the derivative bounds (4.40), (4.41) respectively, to bound the non-zero frequencies. It follows that the off-diagonal $\mathcal{OD}$ is up to a negligible error term given by

$$\begin{aligned}-\sqrt{2\pi}K\mathrm{Im}\Bigg(e^{-\pi i/2}\sum_c \sum_{\substack{a_1\,(\mathrm{mod}\,c),\\ a_2\,(\mathrm{mod}\,c)}} \sum_{\substack{b_1\,(\mathrm{mod}\,c),\\ b_2\,(\mathrm{mod}\,c)}} \frac{S(a_1(a_1+b_1),a_2(a_2+b_2);c)}{c^4}e_c(2a_1a_2+a_1b_2+a_2b_1)\cdot \\ \times \sum_{d_1,d_2}\sum_{n_2}\sum_{m_1,m_2} \frac{1}{d_1d_2m_1n_2^2}\hbar^*_{d_1,d_2}\Big(\frac{cK^2m_2}{8\pi m_1n_2^2}\Big)\Bigg).\end{aligned}$$

We now use Lemma 4.2.7 to simplify the summation of the Kloosterman sum over arithmetic progressions and see that

$$\mathcal{OD} = \frac{-\sqrt{2\pi}K^2}{8}\mathrm{Im}\Bigg(e^{-\pi i/2}\sum_c \frac{\varphi(c)}{c}\sum_{d_1,d_2}\sum_{n_2}\sum_{m_1,m_2} \frac{1}{d_1d_2m_1n_2^2}\hbar^*_{d_1,d_2}\Big(\frac{cK^2m_2}{8\pi m_1n_2^2}\Big)\Bigg) + O(K^{5/4}).$$

By a Taylor expansion, using (4.25) and (4.26), we see that

$$\begin{aligned}\hbar^*_{d_1,d_2}\Big(\frac{cK^2m_2}{8\pi m_1n_2^2}\Big) = &\int_0^\infty \frac{h(\sqrt{u})\sqrt{u}}{\sqrt{2\pi u}}\psi_1\Big(\frac{\sqrt{u}Km_2}{4\pi m_1d_1n_2}\Big)\psi_2\Big(\frac{\sqrt{u}K}{4\pi d_2n_2}\Big)\exp\Big(-\frac{\sqrt{u}Km_2^2}{4n_2^2}\Big)e^{iu\frac{cK^2m_2}{8\pi m_1n_2^2}}du \\ &+ O(K^{1/2+\varepsilon}),\end{aligned}$$

so that the lemma follows upon evaluating the imaginary part of $\mathcal{OD}$. $\square$

Finally, we are in the position to evaluate the off-diagonal $\mathcal{OD}$ asymptotically. To do so we will relate $\mathcal{OD}$ to a complex contour integral over several variables. We then evaluate this contour integral with the residue theorem.

Lemma 4.2.13. *Let $\mathcal{OD}$ be given by expression* (4.18)*. We have*

$$\mathcal{OD} = K^{3/2}\cdot \int_0^\infty \frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}}du \cdot \frac{\sqrt{2}\pi}{16}\frac{1}{2\pi i}\int_{(\varepsilon)}\widetilde{\psi_1}(s_4)\widetilde{\psi_2}(s_4)\zeta(1+s_4)\zeta(1-s_4)ds_4 + O(K^{5/4+\epsilon}).$$

Proof. From Lemma 4.2.12 we see that

$$\begin{aligned}\mathcal{OD} =& \frac{\sqrt{2\pi}K^2}{8}\sum_c \frac{\varphi(c)}{c}\sum_{d_1,d_2}\sum_{n_2}\sum_{m_1,m_2}\frac{1}{d_1d_2m_1n_2^2}\cdot \\ &\times \int_0^\infty \frac{h(\sqrt{u})\sqrt{u}}{\sqrt{2\pi u}}\psi_1\Big(\frac{\sqrt{u}Km_2}{4\pi m_1d_1n_2}\Big)\psi_2\Big(\frac{\sqrt{u}K}{4\pi d_2n_2}\Big)\exp\Big(-\frac{\sqrt{u}Km_2^2}{4n_2^2}\Big)\cos\Big(u\frac{cK^2m_2}{8\pi m_1n_2^2}\Big)du.\end{aligned}$$

To evaluate this expression asymptotically we perform an inverse Mellin transform on ψ_1, ψ_2 and the exponential function. Then $\mathcal{OD}$ is equal to

$$\begin{aligned}&\frac{\sqrt{2\pi}K^2}{8}\sum_c \frac{\varphi(c)}{c}\sum_{d_1,d_2}\sum_{n_2}\sum_{m_1,m_2}\frac{1}{d_1d_2m_1n_2^2}\cdot \\ &\quad\times \frac{1}{(2\pi i)^3}\int_{(1/2+\varepsilon)}\int_{(2)}\int_{(1+\varepsilon)}\widetilde{\psi_1}(s_1)\widetilde{\psi_2}(s_2)\Gamma(s_3)\Big(\frac{Km_2}{4\pi m_1d_1n_2}\Big)^{s_1}\Big(\frac{K}{4\pi d_2n_2}\Big)^{s_2}\Big(\frac{4n_2^2}{Km_2^2}\Big)^{s_3}\cdot \\ &\quad\times \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}}u^{(1+s_1+s_2-s_3)/2}\cos\Big(u\frac{cK^2m_2}{8\pi m_1n_2^2}\Big)du ds_1ds_2ds_3.\end{aligned} \tag{4.42}$$

Finally, we also perform an inverse Mellin transform on

$$\hbar^{\mathrm{Re}}_{1+s_1+s_2-s_3}(v) := \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}}u^{(1+s_1+s_2-s_3)/2}\cos(uv)du$$

as indicated in equation (4.17). We arrive at

$$\mathcal{OD} = \frac{\sqrt{2\pi}K^2}{8}\frac{1}{(2\pi i)^4}\int_{(1+\varepsilon)}\int_{(1/2+3\varepsilon)}\int_{(2)}\int_{(1+3\varepsilon)} \widetilde{\psi_1}(s_1)\widetilde{\psi_2}(s_2)\Gamma(s_3)\widetilde{\hbar}^{\mathrm{Re}}_{1+s_1+s_2-s_3}(s_4)\cdot \\ \times \sum_{d_1,d_2}\sum_{n_2}\sum_{m_1,m_2}\frac{1}{d_1d_2m_1n_2^2}\cdot\left(\frac{\sqrt{u}Km_2}{4\pi m_1d_1n_2}\right)^{s_1}\left(\frac{\sqrt{u}K}{4\pi d_2n_2}\right)^{s_2}\left(\frac{4n_2^2}{\sqrt{u}Km_2^2}\right)^{s_3} \\ \times \sum_c \frac{\varphi(c)}{c}\left(\frac{8\pi m_1n_2^2}{cK^2m_2}\right)^{s_4} du ds_1ds_2ds_3ds_4.$$

We were allowed to interchange the order of summation and integration by the absolute convergence of the integrand in the given ranges. We now rewrite the various summations in terms of zeta functions and get

$$\mathcal{OD} = \frac{\sqrt{2\pi}K^2}{8}\frac{1}{(2\pi i)^4}\int_{(1+\varepsilon)}\int_{(1/2+3\varepsilon)}\int_{(2)}\int_{(1+3\varepsilon)} \widetilde{\psi_1}(s_1)\widetilde{\psi_2}(s_2)\Gamma(s_3)\widetilde{\hbar}^{\mathrm{Re}}_{1+s_1+s_2-s_3}(s_4)\cdot \\ \times \zeta(1+s_1)\zeta(1+s_2)\zeta(2+s_1+s_2-2s_3-2s_4)\zeta(1+s_1-s_4)\zeta(-s_1+2s_3+s_4)\cdot \\ \times \zeta(s_4)/\zeta(1+s_4)(4\pi)^{-s_1-s_2}4^{s_3}(8\pi)^{s_4}K^{s_1+s_2-s_3-2s_4}ds_1d_2ds_3ds_4. \tag{4.43}$$

The zeta functions $\zeta(1+s_1)$ and $\zeta(1+s_2)$ arise from summing over d_1, d_2 respectively. The summation over n_2 yields $\zeta(2+s_1+s_2-2s_3-2s_4)$, while summing over m_1 gives rise to $\zeta(1+s_1-s_4)$. The m_2-variable leads to the factor $\zeta(-s_1+2s_2+s_4)$ and finally the summation over c gives $\zeta(s_4)/\zeta(1+s_4)$. Here we used that $\sum_c \frac{\varphi c}{c^s} = \frac{\zeta(s-1)}{\zeta(s)}$ for $\mathrm{Re}(s) > 2$. To evaluate expression $\mathcal{OD}$ asymptotically we will iteratively shift the contours and pick up poles.

We start to compute the contour integral (4.43) by shifting the line from $\mathrm{Re}(s_2) = 2$ to $\mathrm{Re}(s_2) = -100$. We pick up a simple pole at $s_2 = 0$ and $s_2 = -1 - s_1 + 2s_3 + 2s_4$. The new line integral is negligible by the rapid decay of $\widetilde{\psi_1}, \widetilde{\psi_2}, \widetilde{\hbar}$ and the Gamma function. The residue at $s_2 = 0$ is given by

$$\frac{\sqrt{2\pi}K^2}{8}\frac{1}{(2\pi i)^3}\int_{(1+\varepsilon)}\int_{(1/2+3\varepsilon)}\int_{(1+3\varepsilon)}\widetilde{\psi_1}(s_1)\widetilde{\psi_2}(0)\Gamma(s_3)\widetilde{h}^{\mathrm{Re}}_{1+s_1-s_3}(s_4)\cdot \tag{4.44}$$
$$\times\,\zeta(1+s_1)\zeta(2+s_1-2s_3-2s_4)\zeta(1+s_1-s_4)\zeta(-s_1+2s_3+s_4)\zeta(s_4)/\zeta(1+s_4)\cdot$$
$$\times\,(4\pi)^{-s_1}4^{s_3}(8\pi)^{s_4}K^{s_1-s_3-2s_4}ds_1ds_3ds_4.$$

Moving the line $\mathrm{Re}(s_1) = 1+2\varepsilon$ to $\mathrm{Re}(s_1) = -100$ yields poles at $s_1 = 0$ and $s_1 = s_4$. First, we consider the residue at $s_1 = 0$, which is given by

$$\frac{\sqrt{2\pi}K^2}{8}\frac{1}{(2\pi i)^2}\int_{(1+\varepsilon)}\int_{(1/2+2\varepsilon)}\widetilde{\psi_1}(0)\widetilde{\psi_2}(0)\Gamma(s_3)\widetilde{h}^{\mathrm{Re}}_{1-s_3}(s_4)\cdot \tag{4.45}$$
$$\times\,\zeta(2-2s_3-2s_4)\zeta(1-s_4)\zeta(2s_3+s_4)\zeta(s_4)/\zeta(1+s_4)4^{s_3}(8\pi)^{s_4}K^{-s_3-2s_4}ds_3ds_4.$$

Expression (4.45) is negligible upon shifting $\mathrm{Re}(s_3) = 1/2+3\varepsilon$ to $\mathrm{Re}(s_3) = 100$ and the rapid decay of $\widetilde{h}$ and the Gamma function. On the other hand the residue of the pole at $s_1 = s_4$ of (4.44) leads to

$$\frac{\sqrt{2\pi}K^2}{8}\frac{1}{(2\pi i)^2}\int_{(1+\varepsilon)}\int_{(1/2+2\varepsilon)}\widetilde{\psi_1}^{\mathrm{Re}}(s_4)\widetilde{\psi_2}(0)\Gamma(s_3)\widetilde{h}^{\mathrm{Re}}_{1+s_4-s_3}(s_4)\cdot \tag{4.46}$$
$$\times\,\zeta(2-2s_3-s_4)\zeta(2s_3)\zeta(s_4)(4\pi)^{-s_4}4^{s_3}(8\pi)^{s_4}K^{-s_3-s_4}ds_3ds_4.$$

This integral is again negligible after sending $\mathrm{Re}(s_3) = 1/2+3\varepsilon$ to $\mathrm{Re}(s_3) = 100$, as we pick up no poles and we can bound everything trivially. In total we found that the contribution from poles that arise after $s_2 = 0$ is negligible. We now evaluate the residue of (4.44) at the pole $s_2 = -1 - s_1 + 2s_3 + 2s_4$. We get

$$\frac{\sqrt{2\pi}K^2}{8}\frac{1}{(2\pi i)^3}\int_{(1+\varepsilon)}\int_{(1/2+3\varepsilon)}\int_{(1+3\varepsilon)}\widetilde{\psi_1}(s_1)\widetilde{\psi_2}(-1-s_1+2s_3+2s_4)\Gamma(s_3)\tilde{\hbar}^{\mathrm{Re}}_{s_3+2s_4}(s_4)\cdot \quad (4.47)$$
$$\times\,\zeta(1+s_1)\zeta(-s_1+2s_3+2s_4)\zeta(1+s_1-s_4)\zeta(-s_1+2s_3+s_4)\zeta(s_4)/\zeta(1+s_4)\cdot$$
$$\times\,(4\pi)^{1-2s_3-2s_4}4^{s_3}(8\pi)^{s_4}K^{-1+s_3}ds_1ds_3ds_4.$$

Next we move the line $\mathrm{Re}(s_3) = 1/2 + 3\varepsilon$ to $\mathrm{Re}(s_3) = \varepsilon$ (stopping before the pole of the Gamma function) and capture poles at $s_3 = 1/2 + s_1/2 - s_4$ and $s_3 = 1/2 + s_1/2 - s_4/2$. The new line integrals contribute at most $O(K^{1+\varepsilon})$. The residue at $s_3 = 1/2 + s_1/2 - s_4$ is given by

$$\frac{\sqrt{2\pi}K^2}{8}\frac{1}{(2\pi i)^2}\int_{(1+\varepsilon)}\int_{(1+3\varepsilon)}\widetilde{\psi_1}(s_1)\widetilde{\psi_2}(0)\Gamma(1/2+s_1/2-s_4)\tilde{\hbar}^{\mathrm{Re}}_{1/2+s_1/2+s_4}(s_4)\cdot \quad (4.48)$$
$$\times\,\zeta(1+s_1)\frac{1}{2}\zeta(1+s_1-s_4)\zeta(1-s_4)\zeta(s_4)/\zeta(1+s_4)\cdot$$
$$\times\,(4\pi)^{-s_1}4^{1/2+s_1/2-s_4}(8\pi)^{s_4}K^{-1/2+s_1/2-s_4}ds_1ds_4.$$

We then shift $\mathrm{Re}(s_1) = 1+3\varepsilon$ to $\mathrm{Re}(s_1) = -1+\varepsilon$ and pick up simple poles at $s_1 = 0$, $s_1 = s_4$ from the zeta functions and the simple pole $s_1 = -1 + 2s_4$ from the Gamma function. The new line integral is clearly negligible. The residue from the gamma function contributes at most $O(K)$. The residue at $s_1 = 0$, given by

$$\frac{\sqrt{2\pi}K^2}{8}\frac{1}{(2\pi i)^2}\int_{(1+\varepsilon)}\widetilde{\psi_1}(0)\widetilde{\psi_2}(0)\Gamma(1/2-s_4)\tilde{\hbar}^{\mathrm{Re}}_{1/2+s_4}(s_4)\cdot \quad (4.49)$$
$$\times\,\frac{1}{2}\zeta(1-s_4)^2\zeta(s_4)/\zeta(1+s_4)4^{1/2-s_4}(8\pi)^{s_4}K^{-1/2-s_4}ds_4,$$

is clearly also negligible. At the pole $s_1 = s_4$ the residue is given by

$$\begin{aligned}\frac{\sqrt{2\pi}K^2}{8}\frac{1}{(2\pi i)^2}\int_{(1+\varepsilon)} \widetilde{\psi_1}(s_4)\widetilde{\psi_2}(0)\Gamma(1/2-s_4/2)\widetilde{\hbar}^{\mathrm{Re}}_{1/2+3/2s_4}(s_4)\cdot \\ \times \frac{1}{2}\zeta(1-s_4)\zeta(s_4)(4\pi)^{-s_4}4^{1/2-s_4/2}(8\pi)^{s_4}K^{-1/2-s_4/2}ds_4,\end{aligned} \tag{4.50}$$

which again contributes only to the error term. It remains to compute the chain of residues of (4.47) starting with $s_3 = 1/2 + s_1/2 - s_4/2$. At this point we get

$$\begin{aligned}\frac{\sqrt{2\pi}K^2}{8}\frac{1}{(2\pi i)^2}\int_{(1+\varepsilon)}\int_{(1+3\varepsilon)} \widetilde{\psi_1}(s_1)\widetilde{\psi_2}(s_4)\Gamma(1/2+s_1/2-s_4/2)\widetilde{\hbar}^{\mathrm{Re}}_{1/2+s_1/2+3/2s_4}(s_4)\cdot \\ \times\zeta(1+s_1)\zeta(1+s_1-s_4)\frac{1}{2}\zeta(s_4)(4\pi)^{-s_1-s_4}4^{1/2+s_1/2-s_4/2}(8\pi)^{s_4}K^{-1/2+s_1/2-s_4/2}ds_1ds_4.\end{aligned} \tag{4.51}$$

We then shift $\mathrm{Re}(s_1) = 1 + 3\varepsilon$ to $\mathrm{Re}(s_1) = 2\varepsilon$ and pick up a simple pole $s_1 = s_4$. The new line integral is bounded by $O(K^{1+\epsilon})$ and is therefore negligible. Our expected main term, the residue of the pole $s_1 = s_4$, is given by

$$\frac{\sqrt{2\pi}K^{3/2}}{8}\frac{1}{2\pi i}\int_{(1+\varepsilon)} \widetilde{\psi_1}(s_4)\widetilde{\psi_2}(s_4)\Gamma(1/2)\widetilde{\hbar}^{\mathrm{Re}}_{1/2+2s_4}(s_4)\zeta(1+s_4)\zeta(s_4)(2\pi)^{-s_4}ds_4. \tag{4.52}$$

We now shift the line $\mathrm{Re}(s_4) = 1 + \varepsilon$ to $\mathrm{Re}(s_4) = \varepsilon$ to simplify this expression. Note that the residue of the pole at $s_4 = 1$ is 0, since $\widetilde{\hbar}^{\mathrm{Re}}_{3/2}(1) = 0$. On the new line $\mathrm{Re}(s_4) = \varepsilon$ we can explicitly evaluate $\widetilde{\hbar}^{\mathrm{Re}}_{1/2+2s_4}(s_4)$, leading to

$$K^{3/2}\cdot\int_0^\infty \frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}}du\cdot\frac{\sqrt{2}\pi}{8}\frac{1}{2\pi i}\int_{(\varepsilon)} \widetilde{\psi_1}(s_4)\widetilde{\psi_2}(s_4)\Gamma(s_4)\cos(\pi s/2)\zeta(1+s_4)\zeta(s_4)(2\pi)^{-s_4}ds_4. \tag{4.53}$$

Finally, we use the functional equation

$$\zeta(1-s) = 2(2\pi)^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s)$$

so that the off-diagonal is up to an error term of size $O(K^{5/4+\varepsilon})$ equal to

$$K^{3/2} \cdot \int_0^\infty \frac{h(\sqrt{u})u^{1/4}}{\sqrt{2\pi u}} du \cdot \frac{\sqrt{2}\pi}{16} \frac{1}{2\pi i} \int_{(\varepsilon)} \widetilde{\psi_1}(s_4)\widetilde{\psi_2}(s_4)\zeta(1+s_4)\zeta(1-s_4)ds_4.$$

This matches exactly with term (4.15) from the diagonal, if we suppose that $\psi_1(y)$ is even, i.e. $\psi_1(y) = \psi_1(1/y)$ and thus $\widetilde{\psi_1}(s) = \widetilde{\psi_1}(-s)$. □

4.2.8 Proof of the main theorem

Proof of Theorem 2.3.2. Recall the definition of $\mathcal{E}_\psi$ (se (4.2)) and S_ψ (see (4.3)). The work of Luo and Sarnak (see [LS03, Section 5]) shows that

$$\sum_{k\equiv 0 \,(\mathrm{mod}\, 2)} h\Big(\frac{k-1}{K}\Big) \sum_{f\in B_k} L(1, \mathrm{sym}^2 f)|E_\psi|^2 \ll K^{1+\varepsilon}. \tag{4.54}$$

Moreover, we have

$$\begin{aligned} V(\psi_1,\psi_2) &= \sum_{k\equiv 0 \,(\mathrm{mod}\, 2)} h\Big(\frac{k-1}{K}\Big) \sum_{f\in B_k} L(1, \mathrm{sym}^2 f)(S_{\psi_1}+E_{\psi_1})(S_{\psi_2}+E_{\psi_2}) \\ &= \sum_{k\equiv 0 \,(\mathrm{mod}\, 2)} h\Big(\frac{k-1}{K}\Big) \sum_{f\in B_k} L(1, \mathrm{sym}^2 f)(S_{\psi_1}S_{\psi_2}+S_{\psi_1}E_{\psi_2}+S_{\psi_2}E_{\psi_1}+E_{\psi_1}E_{\psi_2}). \end{aligned}$$

We evaluated the main term

$$\mathcal{M}(\psi_1,\psi_2) = \sum_{k\equiv 0 \,(\mathrm{mod}\, 2)} h\Big(\frac{k-1}{K}\Big) \sum_{f\in B_k} L(1, \mathrm{sym}^2 f)S_{\psi_1}S_{\psi_2} = \mathcal{D} + \mathcal{OD},$$

in Section 4.2.4 with Lemma 4.2.4 and Lemma 4.2.13. Theorem 2.3.2 follows then from the Cauchy–Schwarz ineqality and the bound (4.54). □

Chapter 5

Future Directions

Finally, we want to discuss further ongoing work regarding the Random Wave Conjecture for holomorphic Hecke cusp forms. To do so consider

$$P_{2r}(f) := \int_0^1 \int_0^\infty \psi(y)|f(z)|^{2r} y^{kr} \frac{dxdy}{y^2}, \tag{5.1}$$

where f denotes a holomorphic Hecke cusp form of weight k and $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a smooth compactly-supported weight function.

The quantity $P_{2r}(f)$ can be interpreted as a $2r$-moment of a Hecke cusp forms in a strip of the fundamental domain.

Remark 5.0.1. The definition of $P_{2r}(f)$ might seem a bit artificial but helps in concrete computations and should nonetheless lead to a better understanding of the Random Wave Conjecture. Alternatively, we could try to evaluate

$$\int_{\Gamma\backslash\mathbb{H}} \psi(z)|f(z)|^{2r} y^{kr} \frac{dxdy}{y^2}$$

with ψ a smooth compactly-supported test function on the upper half-plane. We would then expand $\psi(z)$ into so-called incomplete Eisenstein series and Poincaré series and proceed with a similar (although more difficult) analysis.

Remark 5.0.2. We are now considering also higher moments of holomorphic Hecke cusp forms, i.e. $r > 2$. It is therefore of increasing importance to restrict the integration range to a compact domain, in order to avoid large values of holomorphic cusp forms high in the cusp.

The following conjecture is a special case of Conjecture (2.2.2).

Conjecture 5.0.1. *Let f be a holomorphic Hecke cusp form of weight k, such that $\langle f, f\rangle = 1$ and let $\psi\colon \mathbb{R}^+ \to \mathbb{R}$ be a smooth compactly-supported test function. Then*

$$\frac{1}{\mathrm{Var}_f(P)^r \cdot \mathrm{Vol}(P)} \int_0^1 \int_0^\infty \psi(y)|f(z)|^{2r} y^{rk} \frac{dxdy}{y^2} \sim \Gamma(r+1),$$

where

$$\mathrm{Vol}(P) := \int_0^1 \int_0^\infty \psi(y) \frac{dy}{y^2} = \widetilde{\psi}(1),$$

$$\mathrm{Var}_f(P) := \frac{1}{\mathrm{Vol}(P)} \int_0^1 \int_0^\infty \psi(y)|f(z)|^2 y^k \frac{dxdy}{y^2} \sim \frac{3}{\pi},$$

as $k \to \infty$. In particular, we have

$$P_{2r}(f) \sim \Gamma(r+1) \cdot \left(\frac{3}{\pi}\right)^r \cdot \widetilde{\psi}(1),$$

as $k \to \infty$.

Remark 5.0.3. We highlight here that the computations in this chapter should mostly be regarded as heuristics. In particular, we ignore error terms for the sake of exposition, unless they are of conceptional importance.

5.1 The $2r$-th moment and shifted convolution sums

We begin our analysis by relating $P_{2r}(f)$ to shifted convolution sums. To do so, we use the Fourier expansion of f (similar to [BKY13, Section 3]).

Remark 5.1.1. For convenience of notation we will often suppress the summation range and write $\sum_n a(n)$ for the summation $\sum_{n=1}^{\infty} a(n)$.

$$\begin{aligned}
P_{2r}(f) = &\int_0^1 \int_0^\infty \psi(y)\Big|a_f(1)\sum_{n=1}^{\infty}\lambda_f(n)(4\pi n)^{(k-1)/2}e^{-2\pi n y}e(nx)\Big|^{2r} y^{rk}\frac{dxdy}{y^2}\\
=&|a_f(1)|^{2r}\sum_{n_1,\ldots,n_{2r}}\prod_{i=1}^{2r}\lambda_f(n_i)(4\pi n_i)^{(k-1)/2}\int_0^1 e\Big(\Big(\sum_{i=1}^{r}n_i-\sum_{j=r+1}^{2r}n_j\Big)x\Big)dx\\
&\times\int_0^\infty \psi(y)e^{-2\pi y\sum_{i=1}^{2r}n_i}y^{rk}\frac{dy}{y^2}.
\end{aligned}$$

Integrating over the variable x yields

$$P_{2r}(f) = |a_f(1)|^{2r}\sum_{\substack{n_1,\ldots,n_{2r}\\ n_1+\ldots+n_r=n_{r+1}+\ldots+n_{2r}}}\prod_{i=1}^{2r}\lambda_f(n_i)(4\pi n_i)^{(k-1)/2}\cdot\int_0^\infty \psi(y)e^{-2\pi y(n_1+\cdots+n_{2r})}y^{rk-1}\frac{dy}{y}$$

We then perform an inverse Mellin transform on ψ and interpret the integral over y as a Gamma function, so that $P_{2r}(f)$ is equal to

$$\begin{aligned}
&|a_f(1)|^{2r}\sum_{\substack{n_1,\ldots,n_{2r}\\ n_1+\ldots+n_r=n_{r+1}+\ldots+n_{2r}}}\prod_{i=1}^{2r}\Big(\lambda_f(n_i)(4\pi n_i)^{(k-1)/2}\Big)\frac{1}{2\pi i}\int_{(2)}\widetilde{\psi}(s)\int_0^\infty e^{-2\pi y(n_1+\cdots+n_{2r})}y^{rk-1+s}\frac{dy}{y}ds\\
=&\frac{|a_f(1)|^{2r}}{(4\pi)^{r-1}}\sum_{\substack{n_1,\ldots,n_{2r}\\ n_1+\ldots+n_r=n_{r+1}+\ldots+n_{2r}}}\lambda_f(n_1)\cdots\lambda_f(n_{2r})\frac{\Big((n_1\cdots n_r)^{1/r}\Big)^{r(k-1)/2}}{(n_1+\cdots n_r)^{(rk-1)/2}}\frac{\Big((n_{r+1}\cdots n_{2r})^{1/r}\Big)^{r(k-1)/2}}{(n_{r+1}+\cdots n_{2r})^{(rk-1)/2}}\\
&\times\frac{1}{2\pi i}\int_{(2)}\widetilde{\psi}(s)\frac{\Gamma(rk-1+s)}{\Big(4\pi(n_1+\cdots+n_r)\Big)^s}ds.
\end{aligned}$$

Note that from Stirling's formula, for fixed r, we have

$$\Gamma(k)^r \sim \Gamma(rk-1)\cdot(rk-1)(\sqrt{2\pi})^{r-1}r^{-rk+1/2}k^{(1-r)/2}$$

and (similarly as in [LS03, Eq. 2.3]) for any vertical strip $0 < a \leq \mathrm{Re}(s) \leq b$,

$$\frac{\Gamma(rk-1+s)}{\Gamma(rk-1)} = (rk-1)^s \cdot (1 + O_{a,b,r}((1+|s|)^2 k^{-1})). \tag{5.2}$$

Remark 5.1.2. The error term as stated in (5.2) is in general not good enough to be considered negligible. In practice this is not a problem, as we can easily compute an asymptotic expansion and evaluate the lower order terms.

Since $|a_1(f)|^2 = \frac{2\pi^2}{\Gamma(k)L(1,\mathrm{sym}^2 f)}$, we see that the main term of $P_{2r}(f)$ is equal to

$$\frac{\pi^{(r+3)/2}r^{r-1/2}}{2^{(3r-5)/2}}\frac{k^{(r-1)/2}}{rk-1}\frac{1}{L(1,\mathrm{sym}^2 f)^r}\sum_{\substack{n_1,\dots,n_{2r}\\ n_1+\dots+n_r=n_{r+1}+\dots+n_{2r}}}\frac{\lambda_f(n_1)\cdots\lambda_f(n_{2r})}{(n_1+\dots+n_r)^{r-1}}\times \tag{5.3}$$
$$\times\left(\frac{r(n_1\cdots n_r)^{1/r}}{n_1+\cdots+n_r}\right)^{r(k-1)/2}\left(\frac{r(n_{r+1}\cdots n_{2r})^{1/r}}{n_{r+1}+\cdots+n_{2r}}\right)^{r(k-1)/2}\cdot\psi\left(\frac{rk-1}{4\pi(n_1+\cdots+n_r)}\right).$$

In the following sections we use expression (5.3) for $r=2$ and $r=3$ to analyze the fourth and sixth moment of holomorphic Hecke cusp forms on average.

5.2 Fourth Moment Revisited

Aimed with the general computation for $P_{2r}(f)$ we sketch the evaluation of the fourth moment in vertical strips on average.

Remark 5.2.1. Khan [Kha14] computed the fourth moment of holomorphic Hecke cusp forms on average on the full fundamental domain. His approach is based on Watson's formula and the evaluation of an L-function moment problem on average. In Chapter 3 we treated this moment problem without averaging but under the assumption of the Generalized Riemann Hypothesis.

Before we begin with our analysis we notice the factor $L(1, \mathrm{sym}^2 f)^{-r}$ in our expression (5.3) for $P_{2r}(f)$. In view of the Petersson Trace formula (see Lemma 1.6.1) it will be helpful to analyze the average

$$\frac{2}{KW} \sum_{k \equiv 0 \pmod 2} w\left(\frac{k}{K}\right) \frac{\zeta(2) \cdot 12}{k} \sum_{f \in B_k} L(1, \mathrm{sym}^2 f)^{r-1} P_{2r}(f),$$

where $w \colon \mathbb{R}^+ \to \mathbb{R}^+$ is a smooth compactly supported function and $W = \int_0^\infty w(x)dx$. Inserting the factor $L(1, \mathrm{sym}^2 f)^{r-1}$ will affect our computation only by a constant. Indeed, as seen in the works [Luo99], [Roy01], [CM04] on the distribution of the symmetric square L-function at 1 we have for example for $r = 2$

$$\frac{2}{KW} \sum_{k \equiv 0 \pmod 2} w\left(\frac{k}{K}\right) \frac{\zeta(2) \cdot 12}{k} \sum_{f \in B_k} L(1, \mathrm{sym}^2 f) \sim \zeta(2)^3$$

and for $r = 3$

$$\frac{2}{KW} \sum_{k \equiv 0 \pmod 2} w\left(\frac{k}{K}\right) \frac{\zeta(2) \cdot 12}{k} \sum_{f \in B_k} L(1, \mathrm{sym}^2 f)^2 \sim \frac{\zeta(3)\zeta^6(2)}{\zeta(6)}, \tag{5.4}$$

as $K \to \infty$.

Conjecture 5.2.1. *Following the notation in the previous section, we have for $P_4(f)$, defined in equation* (5.1),

$$\frac{2}{KW} \sum_{k \equiv 0 \pmod 2} w\left(\frac{k}{K}\right) \frac{\zeta(2) \cdot 12}{k} \sum_{f \in B_k} L(1, \mathrm{sym}^2 f) P_4(f) \sim \zeta(2)^3 \cdot 2 \cdot \left(\frac{3}{\pi}\right)^2 \widetilde{\psi}(1), \tag{5.5}$$

as $K \to \infty$.

We know sketch a proof without providing details.

Proof sketch. From (5.3) with $r = 2$ we see that the main term of $P_4(f)$ is equal to

$$\frac{\pi^{5/2}}{\sqrt{k}L(1, \mathrm{sym}^2 f)^2} \sum_{\substack{n_1,\dots,n_4 \\ n_1+n_2=n_3+n_4}} \frac{\lambda_f(n_1)\cdots\lambda_f(n_4)}{n_1+n_2}\Big(\frac{2(n_1n_2)^{1/2}}{n_1+n_2}\Big)^{k-1}\Big(\frac{2(n_3n_4)^{1/2}}{n_3+n_4}\Big)^{k-1}\psi\Big(\frac{2k-1}{4\pi(n_1+n_2)}\Big).$$

Using the Hecke relations

$$\lambda_f(n_1)\lambda_f(n_2) = \sum_{d|(n_1,n_2)} \lambda_f\Big(\frac{n_1n_2}{d^2}\Big)$$

and relabelling the variables, we obtain (up to an error term)

$$\begin{aligned} P_4(f) =& \frac{\pi^{5/2}}{\sqrt{k}L(1, \mathrm{sym}^2 f)^2} \sum_{d_1,d_2} \sum_{\substack{n_1,\dots,n_4 \\ d_1(n_1+n_2)=d_2(n_3+n_4)}} \frac{\lambda_f(n_1n_2)\lambda_f(n_3n_4)}{d_1(n_1+n_2)} \times \\ &\times \Big(\frac{2(n_1n_2)^{1/2}}{n_1+n_2}\Big)^{k-1}\Big(\frac{2(n_3n_4)^{1/2}}{n_3+n_4}\Big)^{k-1}\psi\Big(\frac{2k-1}{4\pi d_1(n_1+n_2)}\Big). \end{aligned}$$

As a first step to evaluate the left-hand side of (5.5) we use the Petersson Trace formula (see Lemma 1.6.1), which leaves us with a diagonal term $\mathcal{D}$ and an off-diagonal term $\mathcal{OD}$. The diagonal term is given by

$$\mathcal{D} = \frac{\pi^{5/2}}{\sqrt{k}} \sum_{d_1=1}^{\infty}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \frac{1}{d_1(n_1+n_2)}\Big(\frac{2\sqrt{n_1n_2}}{n_1+n_2}\Big)^{2(k-1)}\psi\Big(\frac{2k}{4\pi d_1(n_1+n_2)}\Big).$$

Similarly, as in the computation for the quantum variance, the expression

$$\Big(\frac{2(n_1n_2)^{1/2}}{n_1+n_2}\Big)^{2(k-1)} \tag{5.6}$$

leads to exponential decay if $|n_1 - n_2| > k^{1/2+\varepsilon}$. We could then perform again a Taylor expansion of the quantity (5.6) and obtain a similar shifted convolution sum as in the quantum variance case (notice that we have here the additional constraint $d_1(n_1 + n_2) = d_2(n_3 + n_4)$). For the sake of exposition we choose here an alternative approach, that seems to be better suited for generalizations. Rather than using a Taylor expansion, we directly compute an inverse Mellin transform for (5.6).

For $\mathrm{Re}(b) > 0$, $0 < c < \mathrm{Re}(b)$, we have the integral representation (see [OLBC10, p. 143, Eq. 5.13.1])

$$\frac{1}{(n_1+n_2)^b} = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(\alpha)\Gamma(b-\alpha)}{\Gamma(b)} \frac{n_2^{\alpha-b}}{n_1} d\alpha. \tag{5.7}$$

First, we perform an inverse Mellin transform on ψ, and then we use formula (5.7) to see that

$$\begin{aligned}
\mathcal{D} =& \frac{\pi^{5/2}}{\sqrt{k}} \sum_{d_1} \sum_{n_1,n_2} \frac{1}{d_1(n_1+n_2)} \left(\frac{2\sqrt{n_1 n_2}}{n_1+n_2}\right)^{2(k-1)} \frac{1}{2\pi i}\int_{(3)} \widetilde{\psi}(s) \left(\frac{2k}{4\pi d_1(n_1+n_2)}\right)^s ds \\
=& \frac{\pi^{5/2}}{\sqrt{k}} \sum_{d_1} \sum_{n_1,n_2} \frac{1}{(2\pi i)^2} \int_{(3)} \int_{(k+2)} \widetilde{\psi}(s) \left(\frac{k}{2\pi}\right)^s \cdot \frac{2^{2(k-1)}}{d^{s+1}} (n_1 n_2)^{k-1} \cdot \frac{n_2^{\alpha-(1+s+2(k-1))}}{n_1^{\alpha}} \\
& \times \frac{\Gamma(\alpha)\Gamma(1+s+2(k-1)-\alpha)}{\Gamma(1+s+2(k-1))} d\alpha ds \\
=& \frac{\pi^{5/2}}{\sqrt{k}} \frac{1}{(2\pi i)^2} \int_{(k+2)} \int_{(3)} \widetilde{\psi}(s) \left(\frac{k}{2\pi}\right)^s 2^{2(k-1)} \zeta(1+s)\zeta(\alpha-(k-1))\zeta(1+s+(k-1)-\alpha) \\
& \times \frac{\Gamma(\alpha)\Gamma(1+s+2(k-1)-\alpha)}{\Gamma(1+s+2(k-1))} ds d\alpha.
\end{aligned}$$

Next, we evaluate the complex integral above by shifting the contour of s and α to the left and computing the residues of the poles that yield the main term. Since we only sketch a proof here, we ignore integrals that should be negligible compared to the main term.

We shift the line $\mathrm{Re}(s) = 3$ to $\mathrm{Re}(s) = \varepsilon$ and pick up a pole at $s = -(k-1)+\alpha$, whose residue is given by

$$\frac{1}{2\pi i}\int_{(k+2)} \widetilde{\psi}(-(k-1)+\alpha) \left(\frac{k}{2\pi}\right)^{-(k-1)+\alpha} 2^{2(k-1)} \zeta(2-k+\alpha) \frac{\Gamma(\alpha)\Gamma(k)}{\Gamma(k+\alpha)} \zeta(\alpha-(k-1)) d\alpha.$$

Shifting the line $\mathrm{Re}(\alpha) = k+2$ to $\mathrm{Re}(\alpha) = k-1+\varepsilon$ we pick up a pole at $\alpha = k$. The residue of this pole is given by

$$\widetilde{\psi}(1)\Big(\frac{k}{2\pi}\Big)2^{2(k-1)}\zeta(2)\frac{\Gamma^2(k)}{\Gamma(2k)} \sim \zeta(2)^3 \cdot \Big(\frac{3}{\pi}\Big)^2 \widetilde{\psi}(1), \tag{5.8}$$

as $K \to \infty$. There are two choices for n_1 to obtain a diagonal term (either $n_1 = n_3$ or $n_1 = n_4$). Once n_1 is determined, the other variables are also determined. Taking this into account we see that the result in (5.8) matches exactly the constant predicted by Conjecture 5.2.1.

Consider now the off-diagonal term $\mathcal{OD}$ that is roughly of the form

$$\frac{1}{K^{5/2}} \sum_{\ell_1,\ell_2 \le \sqrt{K}} \sum_{\substack{n_1,n_2 \asymp K \\ 2n_1+\ell_1=2n_2+\ell_2}} \sum_{c \ll K^{\varepsilon}} \frac{S(n_1(n_1+\ell_1), n_2(n_2+\ell_2); c)}{\sqrt{c}} e_c\Big(2\sqrt{n_1(n_1+\ell_1)n_2(n_2+\ell_2)}\Big).$$

Upon using the Weil bound for Kloosterman sums (see [IK04, Chapter 16])

$$|S(m,n;c)| \le (m,n,c)^{1/2} d(c) c^{1/2},$$

we see that the off-diagonal expression $\mathcal{OD}$ is bounded by $K^{-1/2+\varepsilon}$. Compared to the main term (5.8) that is of constant size, this is negligible, thus finishing our sketch of proof. □

Remark 5.2.2. Note that the diagonal and off-diagonal term for the fourth moment and the quantum variance, treated in Chapter 4.1, are very similar. The major difference is the additional condition $2n_1 + \ell_1 = 2n_2 + \ell_2$, arising in the computation of the fourth moment of holomorphic Hecke cusp forms. The variable n_2 can then be expressed in terms of n_1, ℓ_1, ℓ_2, thus reducing the complexity of the problem.

5.3 A glance at the sixth Moment

We will now continue with an investigation of the sixth moment, which is still work in progress. Unsurprisingly, the problem comes with a steep increase in difficulty. From a technical perspective the number of variables complicate computations considerably. The elevated "intrinsic" difficulty can be seen in the off-diagonal term, where it is necessary to detect significant cancellation.

In view of Conjecture 5.0.1 and equation (5.4) we propose the following averaged conjecture:

Conjecture 5.3.1. *Following the notation in the previous section, we have for $P_6(f)$, defined in equation* (5.1)*,*

$$\frac{2}{KW}\sum_{k\equiv 0\,(\mathrm{mod}\,2)} w\Big(\frac{k}{K}\Big)\frac{\zeta(2)\cdot 12}{k}\sum_{f\in B_k} L(1,\mathrm{sym}^2 f)^2 P_6(f) \sim \frac{\zeta(3)\zeta^6(2)}{\zeta(6)}\cdot 6\cdot\Big(\frac{3}{\pi}\Big)^3\cdot\widetilde{\psi}(1).$$

We conclude this thesis by sketching basic ideas for the treatment of Conjecture 5.3.1.

First, we use again (5.3) with $r=3$ to express $P_6(f)$ as a shifted convolution problem. We have

$$\begin{aligned}P_6(f) =&\frac{\pi^3\cdot 3^{5/2}}{4}\frac{k}{3k-1}\frac{1}{L(1,\mathrm{sym}^2 f)^3}\sum_{\substack{n_1,\dots,n_6\\ n_1+\ldots+n_3=n_4+\ldots+n_6}}\frac{\lambda_f(n_1)\cdots\lambda_f(n_6)}{(n_1+n_2+n_3)^2}\times\\ &\times\Big(\frac{3(n_1n_2n_3)^{1/3}}{n_1+n_2+n_3}\Big)^{3(k-1)/2}\Big(\frac{3(n_4n_5n_6)^{1/3}}{n_4+n_5+n_6}\Big)^{3(k-1)/2}\cdot\psi\Big(\frac{3k-1}{4\pi(n_1+n_2+n_3)}\Big).\end{aligned}$$

Note that

$$\Big(\frac{3(n_1n_2n_3)^{1/3}}{n_1+n_2+n_3}\Big)^{3(k-1)/2}\leq 1. \tag{5.9}$$

From

$$\begin{aligned}27n_1n_2n_3 =&(n_1+n_2+n_3)^3 - \frac{3}{2}(n_1+n_2+n_3)\Big((n_1-n_2)^2+(n_1-n_3)^2+(n_2-n_3)^2\Big)\\ &-(n_1+n_2-2n_3)(n_1-2n_2+n_3)(-2n_1+n_2+n_3)\end{aligned}$$

and a Taylor expansion, we can see with a bit more work that the left-hand side of (5.9) is decreasing exponentially in k, unless $|n_1-n_2| \leq K^{1/2+\varepsilon}$, $|n_1-n_3| \leq K^{1/2+\varepsilon}$ and $|n_2-n_3| \leq K^{1/2+\varepsilon}$. Rewriting $n_2 = n_1+\ell_1$, $n_3 = n_1+\ell_2$, $n_5 = n_4+\ell_3$ and $n_6 = n_4+\ell_4$ we see that $P_6(f)$ is "morally" equal to

$$\frac{1}{K^2}\sum_{\ell_1,\ell_2,\ell_3,\ell_4 \ll K^{1/2+\varepsilon}} \sum_{\substack{n_1,n_4 \asymp K\\ 3n_1+\ell_1+\ell_2=3n_4+\ell_3+\ell_4}} \lambda_f(n_1)\lambda_f(n_1+\ell_1)\lambda_f(n_1+\ell_2)\lambda_f(n_4)\lambda_f(n_4+\ell_3)\lambda_f(n_4+\ell_4).$$

Assuming for simplicity that the Hecke eigenvalues are completely multiplicative, we need to investigate

$$\frac{1}{K^2}\sum_{\ell_1,\ell_2,\ell_3,\ell_4 \ll K^{1/2+\varepsilon}} \sum_{\substack{n_1,n_4 \asymp K\\ 3n_1+\ell_1+\ell_2=3n_4+\ell_3+\ell_4}} \lambda_f(n_1(n_1+\ell_1)(n_1+\ell_2))\lambda_f(n_4(n_4+\ell_3)(n_4+\ell_4))$$

on average. As usual, we would then apply the Petersson Trace formula, leading to a diagonal term $\mathcal{D}$ and an off-diagonal term $\mathcal{OD}$.

It is reasonable to expect, though not trivial to show, that the diagonal term $\mathcal{D}$ arises when $n_1 = n_4$, $\ell_1 = \ell_3$ and $\ell_2 = \ell_4$. Under this assumption we see that the diagonal is essentially bounded:

$$\mathcal{D} \approx \frac{1}{K^2}\sum_{\ell_1,\ell_2 \ll K^{1/2+\varepsilon}} \sum_{n_1 \asymp K} 1 \ll K^{\varepsilon}.$$

Our next goal will be to show that the off-diagonal term $\mathcal{OD}$ is also bounded by K^{ε}. It will be comfortable to introduce the following polynomial in three variables:

$$\begin{aligned} p(r,s,t) :&= r(r+s)(r+t) \qquad (5.10)\\ &= \left(\frac{1}{3}(s+t)+r\right)^3 - \frac{1}{3}\left(s^2-st+t^2\right)\left(\frac{1}{3}(s+t)+r\right) + \frac{2s^3}{27} - \frac{1}{9}st^2 - \frac{1}{9}s^2t + \frac{2t^3}{27}\\ &= \left(\frac{1}{3}(s+t)+r\right)^3 - B(s,t)\left(\frac{1}{3}(s+t)+r\right) + C(s,t) \end{aligned}$$

with

$$B(s,t) := \frac{1}{3}\left(s^2-st+t^2\right) \quad \text{and} \quad C(s,t) := \frac{2s^3}{27} - \frac{1}{9}st^2 - \frac{1}{9}s^2t + \frac{2t^3}{27}. \qquad (5.11)$$

The off-diagonal expression is then roughly given by

$$\mathcal{OD} \approx \frac{1}{K^{7/2}} \sum_{\ell_1,\ell_2,\ell_3,\ell_4 \ll K^{1/2+\varepsilon}} \sum_{\substack{n_1,n_4\sim K\\ 3n_1+\ell_1+\ell_2=3n_4+\ell_3+\ell_4}} \sum_{c\ll K^{1+\varepsilon}} \frac{S(p(n_1,\ell_1,\ell_2),p(n_4,\ell_3,\ell_4);c)}{\sqrt{c}} \times e_c\Big(2\sqrt{p(n_1,\ell_1,\ell_2)\cdot p(n_2,\ell_3,\ell_4)}\Big).$$

Upon expressing n_4 in terms of $n_1, \ell_1, \ell_2, \ell_3, \ell_4$ and using the Weil bound for the Kloosterman sum, we see that a trivial bound for $\mathcal{OD}$ is given by

$$\mathcal{OD} \ll \frac{1}{K^{7/2}} \cdot (K^{1/2+\varepsilon})^4 \cdot K \cdot K^{1+\varepsilon} = K^{1/2+\varepsilon'}.$$

In particular, we need to save $K^{1/2}$ over the trivial bound, in order to show that the sixth moment on average is bounded by K^{ε}.

From

$$2\sqrt{ab} = \sqrt{(a+b)^2-(a-b)^2} = (a+b)\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}$$

and a Taylor expansion we see that $2\sqrt{ab} \sim (a+b)$, provided that a and b are sufficiently close. We want to apply this principle with $a = p(n_1,\ell_1,\ell_2)$ and $b = p(n_4,\ell_3,\ell_4)$. The condition $3n_1 + \ell_1 + \ell_2 = 3n_4 + \ell_3 + \ell_4$ and the polynomial representation (5.10) imply that $p(n_1, \ell_1, \ell_2)$ and $p(n_4, \ell_3, \ell_4)$ are indeed "close" in a precise quantitative sense. Ignoring error terms, it is thus sufficient to consider

$$\begin{aligned}\mathcal{OD} \approx \frac{1}{K^{7/2}} \sum_{\ell_1,\ell_2,\ell_3,\ell_4 \ll K^{1/2+\varepsilon}} \sum_{\substack{n_1,n_4 \sim K \\ 3n_1+\ell_1+\ell_2=3n_4+\ell_3+\ell_4}} \sum_{c \ll K^{1+\varepsilon}} &\frac{S(p(n_1,\ell_1,\ell_2), p(n_4,\ell_3,\ell_4); c)}{\sqrt{c}} \cdot \\ &\times e_c\Big(p(n_1,\ell_1,\ell_2) + p(n_4,\ell_3,\ell_4)\Big).\end{aligned}$$

Next we express the variable n_4 in terms of $n_1, \ell_1, \ell_2, \ell_3, \ell_4$. In particular, since $3n_1+\ell_1+\ell_2 = 3n_4 + \ell_3 + \ell_4$ we have

$$\begin{aligned}p(n_4,\ell_3,\ell_4) &= \left(\frac{1}{3}(\ell_3+\ell_4)+n_4\right)^3 - B(\ell_3,\ell_4)\left(\frac{1}{3}(\ell_3+\ell_4)+n_4\right) + C(\ell_3,\ell_4) \\ &= \left(\frac{1}{3}(\ell_1+\ell_2)+n_1\right)^3 - B(\ell_3,\ell_4)\left(\frac{1}{3}(\ell_1+\ell_2)+n_1\right) + C(\ell_3,\ell_4),\end{aligned}$$

where $B(s,t)$ and $C(s,t)$ are defined as in (5.11). Now that $\mathcal{OD}$ only depends on the variables n_1, ℓ_i with $1 \leq i \leq 4$ and c, we split n_1 into residue classes a_1 modulo c and obtain

$$\mathcal{OD} \approx \frac{1}{K^{7/2}} \sum_{\ell_1,\ell_2,\ell_3,\ell_4 \ll K^{1/2+\varepsilon}} \sum_{c \ll K^{1+\varepsilon}} \mathcal{T}(c),$$

with

$$\begin{aligned}\mathcal{T}(c) := \sum_{a_1 \,(\mathrm{mod}\, c)} &S(a_1^3 - B(\ell_1,\ell_2)\cdot a_1 + C(\ell_1,\ell_2), a_1^3 - B(\ell_3,\ell_4)\cdot C(\ell_3,\ell_4); c) \cdot \\ &\times e_c(a_1^3 - B(\ell_1,\ell_2)\cdot a_1 + C(\ell_1,\ell_2) + a_1^3 - B(\ell_3,\ell_4)\cdot C(\ell_3,\ell_4)).\end{aligned}$$

When c is a prime, which should be the most difficult case, $\mathcal{T}(c)$ can be interpreted with the formalism of trace functions over finite fields (see [FKMS19]). We will not elaborate more on trace functions at this point, as this would go beyond the scope of this thesis. We just remark that the Kloosterman sum as well as the exponential function $e_c(\cdot)$ can be interpreted as trace functions of the variable a_1 associated to special ℓ-adic *sheaves*.

A key feature of trace functions is that they satisfy quasi-orthogonality relations of the form

$$\Big| \sum_{a_1 \, (\mathrm{mod}\, p)} t_1(a_1)\overline{t_2(a_1)} \Big| \leq C_{t_1,t_2} \cdot \sqrt{p}, \tag{5.12}$$

where C_{t_1,t_2} is a constant that depends on the so-called conductor of the trace functions t_1, t_2 but is independent of p. This quasi-orthogonality feature is based on deep work of Deligne [Del80] on the Riemann Hypothesis for finite fields. For our purpose, it is best, to think of $t_1(a_1), t_2(a_1)$ as complex numbers of bounded size and so relation (5.12) amounts to square-root cancellation, unless there is an obvious obstruction.

The quasi-orthogonality relations applied to $\mathcal{T}(p)$ for a prime p, and trace functions

$$t_1(a_1) = \frac{1}{\sqrt{p}} S(f(a_1), g(a_1); p) \quad \text{and} \quad t_2(a_1) = e_p(f(a_1) + g(a_1)),$$

for some appropriate polynomials f, g would imply that

$$\mathcal{T}(p) \ll p.$$

Here the implied constant only depends on the degree of the polynomials f and g but not on their coefficients.

In the case when c is not a prime but a composite integer, usually more elementary methods suffice to prove good results (see [IK04, Chapter 12]). It is therefore reasonable to believe that one can show a result of the form $\mathcal{T}(c) \ll c^{1+\varepsilon}$ for primes as well as composite integers c. Under these assumptions we would have

$$\mathcal{OD} \approx \frac{1}{K^{7/2}} \sum_{\ell_1,\ell_2,\ell_3,\ell_4 \ll K^{1/2+\varepsilon}} \sum_{c \ll K^{1+\varepsilon}} \frac{\mathcal{T}(c)}{\sqrt{c}} \ll K^{\varepsilon},$$

leading to a sharp bound for the sixth moment of holomorphic Hecke cusp forms in a compact interval on average. In ongoing work we intend to prove such a sharp bound for the sixth moment on average rigorously. With more work it might be even possible to solve Conjecture 5.3.1.

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