# Asymptotics of Steklov eigenvalues for surfaces with finitely smooth boundary

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## Abstract

The Steklov eigenvalue problem has been one of the central topics in spectral geometry in the past decade. In particular, a lot of research has been focused on the asymptotic distribution of Steklov eigenvalues. In this manuscript, we investigate asymptotics for Steklov eigenvalues on surfaces with a boundary which is only smooth to finite order. In particular, we obtain remainder estimates in Weyl's law with a rate of decay depending on the order of smoothness, which improve upon results that were previously available in the literature. The proof uses pseudo-differential techniques for operators with non-smooth symbols inspired by the methods developed by G. Rozenblum.

## Résumé

Le problème aux valeurs propres de Steklov a été un sujet central en géométrie spectrale dans la dernière décennie. En particulier, plusieurs activités de recherche concernent la distribution asymptotique des valeurs propres de Steklov. Dans cette thèse, nous investigons le comportement asymptotique des valeurs propres de Steklov pour des surfaces dont la frontière a une régularité finie. En particulier, nous obtenons des estimés sur le reste dans la loi de Weyl avec un taux de décroissance qui dépend de l'ordre de la régularité, améliorant ainsi les résultats disponibles précédemment. La preuve utilise des méthodes pseudodifférentielles pour des opérateurs dont le symbole n'est pas lisse, inspirées des arguments développés par G. Rozenblum.

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*In memory of my father, Lesley Causley.*

# Contents

## Contributions

The main results of this thesis are as follows: Theorem 1.2 and 1.3 generalize a known result on asymptotics for Steklov eigenvalues on surfaces to $C^r$-smooth boundary, and Theorem 1.4 gives a quantitative estimate on the difference of eigenvalues after perturbing an operator, including $C^r$-smooth perturbations. The foundation for pseudo-differential operators with non-smooth symbols are given in the mapping properties of Proposition 2.2 and 2.3, and in the symbol decomposition of Lemma 2.4 and 2.5. Lemma 3.1 and 3.2 provide the operator inverse needed to invert the diagonalization procedure in Proposition 3.3, which yields the eigenvalue correspondence in Corollary 3.4.

These results are based on joint work with Jean Lagacé, who outlined the strategy for Theorem 1.2 and provided the computations for Theorem 1.4. The work in Proposition 2.2, Proposition 2.3, Lemma 2.4, Lemma 2.5, Lemma 3.1, Lemma 3.2, Proposition 3.3, and Corollary 3.4 was done solely by the author. In particular, the results Lemma 2.4, Lemma 2.5, Proposition 3.3 are previously known for the smooth case, but to best of the author's knowledge, are original contributions to knowledge for the non-smooth case.

CHAPTER 1

# Introduction and main results

Differential operators are useful because they represent movement and behaviour of natural objects in our physical world. When such an operator is elliptic, a door opens to find an inverse and hence solve many of the known equations that describe physical phenomena. The Laplacian is a natural operator, which, as the divergence of the gradient of a scalar function on Euclidean space, represents the gradient flow of a function. By using it in a heat equation, geometric properties of a shape can be recovered — in image processing, the heat kernel signature can perform edge detection and shape matching [**20**].

Pseudo-differential operators, which generalize differential operators using the framework of Fourier transforms, also play a key role in mathematical physics and geometric analysis. The Dirichlet-to-Neumann operator arises in scattering theory and is often used in medical imaging (to detect air and fluid flow) and geophysics (to detect the presence of stress fractures and mineral deposits). Using the same procedure of shape analysis as the Laplacian, one can capture the spatial embedding of a shape up to rigid motion [**22**].

In spectral geometry, one of the main goals is to relate the spectrum of an operator acting on a manifold to the global geometry of that manifold. A way to do this is to observe the asymptotic behaviour of the spectrum as it accumulates at infinity (or in some cases as it accumulates to zero). For example, we know that for elliptic, self-adjoint, bounded operators differing by a smoothing operator, eigenvalues converge to each other faster than any algebraic order [**6**, Lemma 2.1]. Of particular interest are the eigenvalues of the Dirichlet-to-Neumann operator, whose eigenvalues coincide with the eigenvalues from the Steklov problem. As demonstrated by [**15**] and [**6**], it is known that for any Steklov surface with smooth boundary, the eigenvalues will converge to those of a disjoint union of circles whose lengths are those of the boundary components, regardless of any local structure on the boundary (as well as the interior).

Spectral properties can be investigated using perturbation theory, which studies the behavior of operators undergoing a small change. When it comes to perturbing a Riemannian metric on a manifold or symbol of a pseudo-differential operator, it is often much simpler to work under the assumption that the setting is smooth. The present manuscript is concerned with eigenvalue asymptotics for a class of "rough" perturbations acting on elliptic, self-adjoint, bounded operators. This class includes $C^r$ perturbations, or more generally any operator $Q : H^\beta \to L^2$, where the error depends on $r$ or $\beta$ respectively.

Using this, we can tackle the asymptotic behaviour of Steklov eigenvalues of surfaces with $C^r$ boundary or with a $C^{r-1}$ weight function acting on the boundary. We generalize the known result that eigenvalues of a Steklov surface converge to those of a circle, and we capture the error term depending only on $r$.

One of the main ideas needed here is converting the asymptotics of Steklov eigenvalues on surfaces with $C^r$ boundary to weighted problems on unit disks by the use of conformal maps. See the work of ([**9**], [**10**], [**14**]) for details on the symbol of the Dirichlet-to-Neumann map, as well as the regularity of the coefficients using conformal maps.

## 1. Dirichlet-to-Neumann operator

Let $\Omega$ be a smooth surface with boundary $\partial\Omega$ and let $0 < \rho \in L^p(\partial\Omega)$, $p > 1$. The Steklov problem on $\Omega$ with density $\rho$ consists in finding the set of real numbers $\sigma$ satisfying the eigenvalue problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \partial_\nu u = \rho\sigma u & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The Dirichlet-to-Neumann map $\Lambda_\Omega : H^1(\partial\Omega) \to L^2(\partial\Omega)$ is defined as the map

$$f \mapsto \mathcal{H}f \mapsto \partial_\nu \mathcal{H}f, \tag{2}$$

where $\mathcal{H}f$ is the harmonic extension to the interior and $\partial_\nu$ is the exterior normal derivative. The eigenvalues of problem (1) are the eigenvalues of the operator $\Lambda_{\Omega,\rho} = M_{\rho^{-1}}\Lambda_\Omega$, where for any function $f$, $M_f$ is the operator of multiplication by $f$. From the work of Agranovich [**1**], it is known that if $\partial\Omega$ is $C^1$ except on a closed set of 0 measure, then problem (1) has a discrete set of eigenvalues

$$0 = \sigma_0(\Omega,\rho) < \sigma_1(\Omega,\rho) \leq \sigma_2(\Omega,\rho) \leq \ldots \nearrow \infty, \tag{3}$$

satisfying a Weyl's law

$$\sigma_k = C_\rho k(1 + o\,(1)), \tag{4}$$

as $k \to \infty$, where

$$C_\rho = \frac{2\pi}{\int_{\partial\Omega} \rho(x)dx}. \tag{5}$$

Our goal is to improve on the remainder estimate in (4) and to control it explicitly with dependence on the regularity of boundary $\partial\Omega$ and density $\rho$. In some sense, we will interpolate between this estimate and those obtained by Girouard, Parnovski, Polterovich and Sher [**6**] in the case $\rho \equiv 1$ and $\partial\Omega$ smooth with $J$ boundary components $\partial\Omega_j$, where they obtain that

$$\sigma_k = \left(\bigcup_{j=1}^{J} \operatorname{spec}(\Lambda_{a_j\mathbb{D}})\right)_k + O\left(k^{-\infty}\right), \tag{6}$$

with

$$a_j = \frac{|\partial\Omega_j|}{2\pi}, \tag{7}$$

and Rozenblum [**15**] who obtained the result (6) for simply connected domains, which was the first result of this kind.

Define $N(\sigma) = \#(\sigma_k < \sigma)$ as the eigenvalue counting function for Steklov eigenvalues. The author in [**8**] obtained the sharp Weyl's law for Steklov eigenvalues $\sigma_k$ when $\Omega$ has a *smooth* boundary $\partial\Omega$,

$$N(\sigma) = \#(\sigma_k < \sigma) = \frac{|\partial\Omega|}{\pi}\sigma + O(1), \tag{8}$$

which is equivalent to

$$\sigma_k = \frac{\pi k}{|\partial\Omega|} + O(1). \tag{9}$$

When the boundary is smooth, Rozenblum [15] and Edward [4] each showed that for a bounded simply connected planar domain,

$$\sigma_k = \frac{\pi k}{|\partial\Omega|} + O(k^{-\infty}). \tag{10}$$

In the non-smooth case, Sandgren [17] first obtained the result that for Euclidean domains with $C^2$ boundary,

$$N(\sigma) = \frac{|\partial\Omega|}{\pi}\sigma + o(\sigma), \tag{11}$$

which Agranovich [1] later proved will hold for bounded planar domains with piecewise $C^1$ boundary. Very recently, the authors in [11] showed that the same asymptotics will hold for bounded domains with Lipschitz boundary.

Improving the error term, the authors in [5] showed that for a $C^{2,\alpha}$ boundary, $\alpha > 0$,

$$N(\sigma) = \frac{|\partial\Omega|}{\pi}\sigma + O(1). \tag{12}$$

For a bounded planar domain $\Omega$, the authors [5] also showed that for a $C^{1,1}$ boundary,

$$\sigma_k = \frac{\pi k}{|\partial\Omega|} + O(1), \tag{13}$$

where the method only needs the normal vector of $\partial\Omega$ to be Lipschitz.

As far as we are aware, there are no intermediate results which capture the regularity needed to obtain a decaying error term as $k \to \infty$.

Before stating our next theorem, let us fix some notation for the regularity of maps.

Definition 1.1. For $n \in \mathbb{N}$, $t \in [0, 1)$ and $r = n + t$ we denote by $C^r$ the space of $n$ times differentiable functions whose $n$th derivative is Hölder of exponent $t$.

We obtain the following result.

Theorem 1.2. *Suppose that the boundary components of $\Omega$ are of class $C^r(\partial\Omega)$ for $r > 3$. Then, for any $\varepsilon > 0$, we have the estimate*

$$\sigma_k = \left(\bigcup_{j=1}^{J} \operatorname{spec}(\Lambda_{a_j\mathbb{D}})\right)_k + O\left(k^{\lceil \frac{3}{2} - \frac{r}{2} + \varepsilon \rceil}\right). \tag{14}$$

Note that a decaying error term first appears with $C^{5+t}(\partial\Omega)$ regularity on the boundary. In the case of a single boundary component, this can be written as

THEOREM 1.3. *Let $\Omega$ be a simply connected planar domain with $C^{5+t}(\partial\Omega)$ boundary for some $t > 0$. Then*

$$\sigma_{2k-1} = \sigma_{2k} = \frac{2\pi k}{|\partial\Omega|} + O\left(k^{-1}\right). \tag{15}$$

We are most interested in the case when density $\rho \equiv 1$ and the boundary of $\Omega$ is $C^r(\partial\Omega)$, however the same result will hold if the density function $\rho$ on the boundary is $C^{r-1}(\partial\Omega)$ (see section 3.1.1 for details about the conformal map on the boundary).

## 2. Eigenvalue problem

Let $M$ be a smooth $d$-dimensional closed manifold and consider the eigenvalue problem

$$A\varphi = \lambda\varphi, \tag{16}$$

where $A$ is a *self-adjoint, elliptic* and *semi-bounded below* pseudo-differential operator with rough symbol of order $m$. In other words, operator $A$ has symbol $a(x,\xi) \in C^r S^m_{1,0}$ or $C^r_* S^m_{1,0}$ for $r \geq 1$ in the notation of Taylor in [**21**]. Since the resolvent is compact, the eigenvalue problem (16) for $m > 0$ has a discrete set of eigenvalues

$$\lambda_1(A) \leq \lambda_2(A) \leq \ldots \nearrow \infty \tag{17}$$

accumulating only at infinity. Furthermore, they satisfy a Weyl's law,

$$\lambda_j(A) \asymp j^{\frac{m}{d}}. \tag{18}$$

Note that we say that an operator $A$ acting on $H^s$ with inner product $\langle\ ,\ \rangle_{H^s}$ is *self-adjoint* if and only if

$$\langle Au, v\rangle_{H^s} = \langle u, Av\rangle_{H^s} \tag{19}$$

for all $u, v \in H^s$. An operator $A = a(x, D)$ of order $m$ is called *elliptic* if, for some $c < \infty$,

$$|a(x,\xi)^{-1}| \leq C\langle\xi\rangle^{-m} \tag{20}$$

for $|\xi| \geq c$. If there is a constant $d > 0$ such that

$$\langle u, Au\rangle_{H^s} \geq d\|u\|^2 \tag{21}$$

for all $u \in H^s$, then operator $A$ is *semi-bounded from below*. We also note the case of a *smoothing* operator, which is a bounded operator of order $-\infty$.

### 2.1. Perturbations and eigenvalues.

Let $B := A + Q$ be a perturbation of $A$ by a relatively $A$-bounded operator $Q$. We are interested in the effect of the perturbation on the eigenvalue distribution. It has been proven in [**6**] (see also [**4**] and [**15**]) that if $A$ is a classical operator with smooth coefficients and $Q$ is a smoothing operator, then

$$|\lambda_j(A) - \lambda_j(B)| = O\left(j^{-\infty}\right). \tag{22}$$

We are interested in the case where $Q$ is of order $\beta > -\infty$ and in obtaining quantitative estimates on the size of the difference in (22), depending not only on $\beta$ but also on $m$ and $d$.

THEOREM 1.4. *Let $M$ be a compact manifold of dimension $d$. Let $A$ and $Q$ be operators acting on $M$, where $A$ is an elliptic, semi-bounded below, self-adjoint, pseudo-differential operator of order $m > 0$, and is $Q$ an operator of order $-\infty < \beta < m$ such that $Q : H^\beta \to L^2$. The eigenvalues of $A$ and $A + Q$ then satisfy*

$$|\lambda_k(A + Q) - \lambda_k(A)| = \mathcal{O}\left(k^{\frac{\beta}{d}}\right). \tag{23}$$

Note that we place no regularity restrictions on operator $Q$, and that it does not have to be pseudo-differential. Hence, we can have a "rough" perturbation of $A$. Estimates for operator $A$ with symbol $a(x,\xi) \in C^r S^m_{1,0}$ or $C^r_* S^m_{1,0}$ (and later for $a(x,\xi) \in {}^r\widetilde{\Psi}^m$, see section 2.2 for definition) when $r \geq 1$ will follow from [**21**] (see also [**3**] and [**19**]).

## 3. Organization of the thesis

The proof of Theorem 1.2 contains three main steps. The first step in section 3.2 follows a strategy similar to the one in [**6**] and isolates the boundary components, gluing a spherical cap on each. Since the symbol of the Dirichlet-to-Neumann map is determined up to isometries in the neighborhood of the boundary [**13**], the map acting on the union of the boundary components will be the same.

In the second step, we conformally map the problem to the unit disk in section 3.1.1, see the work of [9] and [11] for details of that mapping. This will make the Dirichlet-to-Neumann map into a pseudo-differential operator on the circle, with non-regular spatial coefficients. The regularity of the coefficients will be recovered from the boundary behaviour of conformal maps via the Kellogg-Warschawski theorem, see the monograph of Pommerenke [14, Chapter 3].

Finally, we obtain general results for eigenvalue asymptotics of perturbations of elliptic operators in Theorem 1.4, obtaining a quantitative version of [6, Lemma 2.1]. Combining this with Proposition 3.3 — a diagonalization procedure similar to [15], where we now keep track of dependence on regularity — we obtain precise eigenvalue asymptotics for the Dirichlet-to-Neumann operator on the circle.

The proof of Theorem 1.4 makes use of Weyl's law on relatively bounded operators. The procedure uses the variational characterization of eigenvalues and Min-Max theorem to arrive at a quantitative estimate on the difference of eigenvalues.

The overall outline of this manuscript is as follows: In Chapter 2 we cover the results needed to work with pseudo-differential operators with non-smooth symbols. Chapter 3 contains the strategy, similar to Rozenblum [15] and Edward [4], where we reduce an operator with rough symbol to be independent of the spatial variable. It also contains the main results of this manuscript, including the full proof of Theorem 1.2 and Theorem 1.4.

CHAPTER 2

# Pseudo-differential operators with non-smooth symbols

The purpose of this chapter is to develop the theory of non-smooth pseudo-differential operators ($\Psi$DOs). After a review of smooth $\Psi$DOs, we introduce the operator space $C^r_* S^m_{1,0}$ in the notation of Taylor in [21]. Although our operator space ${}^r\widetilde{\Psi}^m$ mostly resembles the symbols of $S^m_{1,0}$ which have a symbol decomposition, we introduce $C^r_* S^m_{1,0}$ as a way to recover mapping properties. In this section, we also detail two key lemmas which allow us to decompose and reconstruct non-smooth symbols in ${}^r\widetilde{\Psi}^m$.

## 1. Classic $\Psi$DOs

Consider the differential operator

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \tag{24}$$

which has the symbol

$$p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha, \tag{25}$$

and can be written using a Fourier transform as

$$Pu(x) = \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} p(x, \xi) \widehat{u}(\xi) d\xi. \tag{26}$$

The Fourier transform is beneficial in differential equations because it can reformulate problems in a format which is much easier to solve. Note that symbol $p(x, \xi)$ is a polynomial with respect to $\xi$. This theory can be generalized to operators defined by symbols which are not necessarily polynomials with respect to $\xi$.

Using the Fourier integral representation in (26), operators $p(x, D)$ are called *pseudo-differential operators*, provided the following bound for symbol $p(x, \xi)$ holds

$$|D^\alpha_x D^\beta_\xi p(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m - \rho|\beta| + \delta|\alpha|} \tag{27}$$

for all $\alpha$ and $\beta$ in $\mathbb{N}$. We say that $p(x,\xi) \in S^m_{\rho,\delta}$, and that $p(x,D) \in \Psi^m_{\rho,\delta}$.

Note that $\alpha$ can approach infinity in this definition, and hence the symbol $p(x,\xi)$ is smooth in $x$. We later introduce the case where $p(x,\xi)$ is only $C^r$ smooth in $x$, and hence we need to impose $\alpha \leq r$.

Given a symbol $a \in S^m_{\rho,\delta}$, one can use the formula (26) to define the map $\mathrm{Op} : S^m_{\rho,\delta} \to \Psi^m_{\rho,\delta}$, defined as

$$\mathrm{Op}(a) = a(x,D) = Au(x) = \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)d\xi. \tag{28}$$

Similarly, the symbol map $\sigma : \Psi^m_{\rho,\delta} \to S^m_{\rho,\delta}$ is defined as

$$\sigma(A) = \sigma\big(a(x,D)\big) = a_0(x,\xi), \tag{29}$$

where $a_0$ is called the *principal symbol* of $a(x,D) \in \Psi^m_{\rho,\delta}$.

If $\rho,\delta \in [0,1]$, then $p(x,D) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. Here, $\mathcal{S}$ is the Schwartz space of the set of smooth functions $\mathbb{R}^d \to \mathbb{C}$, such that for all $\alpha \in \mathbb{N}^d$ and all $N \geq 0$,

$$|\partial_x^\alpha u(x)| \leq C_{\alpha N}\langle x\rangle^{-N},\ x \in \mathbb{R}^d. \tag{30}$$

If we additionally restrict $\delta < 1$, then $p(x,D) : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. $\mathcal{S}'$ is the space of tempered distributions, the continuous dual space of Schwartz space $\mathcal{S}$.

An important subclass of symbols are $S^m_{1,0}$, where the bound

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|} \tag{31}$$

shows derivatives in $x$ have no impact on the exponent of $\langle\xi\rangle$.

A symbol in $S^m_{1,0}$ is said to be classical if there are smooth $p_{m-j}(x,\xi)$, homogeneous in $\xi$ of degree $m-j$ for $|\xi| \geq 1$ (that is, $p_{m-j}(x,s\xi) = s^{m-j}p_{m-j}(x,\xi)$ for $s$, $|\xi| \geq 1$), and if

$$p(x,\xi) \sim \sum_{j\geq 0} p_{m-j}(x,\xi) \tag{32}$$

in the sense that

$$p(x,\xi) - \sum_{j=0}^{N} p_{m-j}(x,\xi) \in S^{m-N-1}_{1,0}, \tag{33}$$

for all $N$. We write such classical symbols as $S^m$, and call $p_m(x,\xi)$ the *principal symbol* of $p(x,D)$.

We denote $S^{-\infty} = \cap_m S^m$, and say two operators differ by a *smoothing* operator if their difference represents a symbol in $S^{-\infty}$.

Given $a \in S^m$, the adjoint of $a$ is the symbol $a^* \in S^m$, where

$$a^*(x,\xi) \sim \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha \overline{a}(x,\xi). \tag{34}$$

Written as an operator, if $A = Op(a)$ is a pseudo-differential operator of order $m$, then $A^* = Op(a^*)$ is a pseudo-differential operator of order $m$.

Given two operators $a_1(x,D) \in \Psi^{m_1}_{\rho_1,\delta_1}$, $a_2(x,D) \in \Psi^{m_2}_{\rho_2,\delta_2}$, where $0 \leq \delta_2 < \rho \leq 1$ with $\rho = \min(\rho_1,\rho_2)$, their product is defined as

$$p_1(x,D)p_2(x,D) = q(x,D) \in \Psi^{m_1+m_2}_{\rho,\delta}, \tag{35}$$

with $\delta = \max(\delta_1,\delta_2)$, and

$$q(x,\xi) \sim \sum_{|\alpha|\geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p_1(x,\xi) D_x^\alpha p_2(x,\xi). \tag{36}$$

Note again that this product requires smoothness, particularly for the symbol $p_2(x,\xi)$.

The product in (36) captures the idea of Bony [2] who defined symbols of paradifferential operators where the regularity of each symbol term decreases provided the order also decreases. This leads into the next section where will define the operator space ${}^r\widetilde{\Psi}^m$, which represents non-smooth symbols behaving like $S^m_{1,0}$, but with a finite number of terms.

## 2. Notation and definitions

Recall that for $n \in \mathbb{N}$, $t \in [0,1)$ and $r = n+t$, $C^r$ is the space of $n$ times differentiable functions whose $n$th derivative is Hölder of exponent $t$.

Assuming that $r \in (0,\infty)$, $m \in \mathbb{R}$, we define ${}^rT^m_{1,0}(\Omega) = {}^rT^m(\Omega)$ to consist of functions $a(x,\xi) : \Omega \times \mathbb{R}^d \to \mathbb{C}$ which are $C^r(\Omega)$ in $x$ and $C^\infty(\mathbb{R}^d)$ in $\xi$, satisfying

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle\xi\rangle^{m-|\beta|} \tag{37}$$

for all $\beta$, and all $|\alpha| \leq r$, where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$.

We say that $a(x,\xi) \in {}^r\widetilde{S}^m(\Omega)$ if $a(x,\xi) = \sum_{j=0}^n a_j(x,\xi)$, where each $a_j(x,\xi) \in {}^{r-j}T^{m-j}(\Omega)$.

We now define the quotient space

$$ {}^r\widetilde{\Psi}^m(\Omega) = {}^r\Psi^m(\Omega)\big/_{\sim} \tag{38}$$

where

$$ {}^r\Psi^m(\Omega) = \{\mathrm{Op}\big(a(x,\xi)\big) + R \;\big|\; a(x,\xi) \in {}^r\widetilde{S}^m(\Omega),\ R : H^{m-r-\varepsilon}(\Omega) \to L^2(\Omega),\ \text{some } \varepsilon > 0\} \tag{39}$$

and $A$ is equivalent to $B$ if

$$ A - B : H^{s+m-r-\varepsilon}(\Omega) \to H^s(\Omega) \tag{40}$$

for some $\varepsilon > 0$ and $-r < s < r$.

As was the case with smooth pseudo-differential operators in $\Psi^m_{\rho,\delta}$, and symbols in $S^m_{\rho,\delta}$, we define the operator map $\mathrm{Op} : {}^r\widetilde{S}^m \to {}^r\widetilde{\Psi}^m$ using the Fourier transform as

$$ \mathrm{Op}(a)f(x) = \int a(x,\xi)\widehat{f}(\xi)e^{ix\xi}d\xi, \tag{41}$$

and define the symbol map $\sigma : {}^r\widetilde{\Psi}^m \to {}^r\widetilde{S}^m$ for any operator $A$,

$$ \sigma(A) = \sigma\big(a(x,D)\big) = a_0(x,\xi), \tag{42}$$

where $a_0(x,\xi)$ is now a principal symbol in ${}^rT^m$.

### 2.1. Besov and Zygmund spaces.

So far we only considered the space $C^r$ of Hölder-continuous functions $C^r$, however a slightly more general class of functions exists providing a framework useful to us. We will define this space to be $C^r_*$ below.

First, let us recall the $C^r$-norm for integer and non-integer values of $r$:

$$ \|f(x)\|_{C^r} = \|f(x)\|_{C^n} = \max_{|\alpha|\le n} \sup_{x\in\Omega} |D^\alpha f(x)|, \quad \text{if } r \in \mathbb{N}, \tag{43}$$

$$ \|f(x)\|_{C^r} = \|f(x)\|_{C^{n+t}} = \|f(x)\|_{C^n} + \max_{|\alpha|=n} \sup_{x\neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\|x-y\|^t}, \quad \text{if } r = n + t \notin \mathbb{N}. \tag{44}$$

Two common interpretations of $C^r_*$ (and hence the $C^r_*$-norm) is to view them as Zygmund spaces or $L^\infty$-based Besov spaces. Following Taylor [**21**] and Salo [**16**], we will first define the $C^r_*$-norm using a Littlewood-Paley partition of unity.

Define a smooth Littlewood-Paley partition of unity $1 = \sum_{j=0}^{\infty} \psi_j(\xi)$, where $\psi_0$ is supported in the annulus $|\xi| \leq 1$, $\psi_1$ is supported in $\frac{1}{2} \leq |\xi| \leq 2$, and for all $j > 1$, $\psi_j(\xi) = \psi_1(2^{1-j}\xi)$. We now define the $C^r_*$-norm as

$$\|f(x)\|_{C^r_*} = \sup_k 2^{kr} \|\mathrm{Op}\big(\psi_k(\xi)\big) f(x)\|_{L^\infty}. \tag{45}$$

Given $f(x) \in C^r$, it is known that $\|f(x)\|_{C^r_*} < \infty$. However, the converse breaks down if $r \in \mathbb{N}$. So we define the *$L^\infty$-based Besov space $C^r_*$* to consist of all functions $f(x)$ such the $C^r_*$-norm (45) is finite. In other words,

$$C^r = C^r_* \;\text{ if }\; r \in \mathbb{R}^+ \backslash \mathbb{N}, \quad C^r \subset C^r_* \;\text{ if }\; r \in \mathbb{N}. \tag{46}$$

Examples like the lacunary Fourier series $g(x) = \sum_{k=1}^{\infty} 2^{-k} e^{i2^k x}$ belong to $C^1_*$ but not $C^1$, so this inclusion is strict.

It is known that Zygmund spaces coincide with $L^\infty$-based Besov spaces (see [16]), so we define *Zygmund spaces* using an equivalent $C^r_*$-norm for integer and non-integer values of $r$:

$$\|f(x)\|_{C^r_*} = \|f(x)\|_{C^n_*} = \|f(x)\|_{C^{n-1}} + \sum_{|\alpha|=n-1} \sup_{x,h} \frac{|D^\alpha f(x+h) - 2D^\alpha f(x) + D^\alpha f(x-h)|}{|h|}, \tag{47}$$

$$\|f(x)\|_{C^r_*} = \|f(x)\|_{C^{n+t}_*} = \|f(x)\|_{C^{n+t}} = \|f(x)\|_{C^n} + \max_{|\alpha|=n} \sup_{x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\|x-y\|^t}. \tag{48}$$

We can now say that symbol $a(x,\xi) \in C^r_* S^m_{1,0}$, provided

$$|D^\beta_\xi a(x,\xi)| \leq C_\beta \langle \xi \rangle^{m-|\beta|}, \quad \text{and} \tag{49}$$

$$\|D^\beta_\xi a(x,\xi)\|_{C^r_*} \leq C_\beta \langle \xi \rangle^{m-|\beta|}. \tag{50}$$

We can also say that $a(x,\xi) \in C^r S^m_{1,0}$, provided we add the condition

$$\|D^\beta_\xi a(x,\xi)\|_{C^j} \leq C_\beta \langle \xi \rangle^{m-|\beta|}, \quad \text{where } \; 0 \leq j \leq r. \tag{51}$$

### 2.2. Mapping property.

We would like our operators in ${}^r\widetilde{\Psi}^m(\Omega)$ to have well behaved mapping properties. We begin by recalling a property known for $C^r_* S^m_{1,0}$.

PROPOSITION 2.1. (Taylor [21, Proposition 9.10]): If $a(x,\xi) \in C^r_* S^m_{1,0}$, then

$$a(x,D) : H^{s+m}(\mathbb{R}^d) \to H^s(\mathbb{R}^d), \quad \text{provided } -r < s < r. \tag{52}$$

It would be useful to take advantage of the mapping properties above — i.e. if our space ${}^r\widetilde{S}^m(\Omega)$ was a subset of $C^r_* S^m_{1,0}$ — hence, we will show the following inclusion.

PROPOSITION 2.2. For all $r \in (0,\infty)$, we have that ${}^r\widetilde{S}^m(\Omega) \subset C^r_* S^m_{1,0}$.

*Proof*:

Let $a(x,D) \in {}^r\widetilde{\Psi}^m(\Omega)$. In other words, let $a(x,\xi) = \sum_{j=0}^{n} a_j(x,\xi)$, where each $a_j(x,\xi) \in {}^{r-j}T^{m-j}(\Omega)$.

We satisfy the first condition (49),

$$|D^\beta_\xi a(x,\xi)| \le C_\beta \langle\xi\rangle^{m-|\beta|}, \tag{53}$$

of $C^r_* S^m_{1,0}$ directly by the definition of ${}^{r-j}T^{m-j}(\Omega)$ (by taking no derivatives in $x$).

To satisfy the second condition (50),

$$\|D^\beta_\xi a(x,\xi)\|_{C^r_*} \le C_\beta \langle\xi\rangle^{m-|\beta|}, \tag{54}$$

we observe the later two $C^r_*$-norms (47), (48) directly:

$$\begin{aligned}\|D^\beta_\xi a(x,\xi)\|_{C^r_*} =& \|D^\beta_\xi a(x,\xi)\|_{C^n_*}\\ =& \|D^\beta_\xi a(x,\xi)\|_{C^{n-1}} + \sum_{|\alpha|=n-1} \sup_{x,h} \frac{|D^\alpha_x D^\beta_\xi a(x+h,\xi) - 2D^\alpha_x D^\beta_\xi a(x,\xi) + D^\alpha_x D^\beta_\xi a(x-h,\xi)|}{|h|}\\ \le& C_{(n-1)\beta}\langle\xi\rangle^{m-|\beta|} + C_{n\beta}\langle\xi\rangle^{m-|\beta|},\end{aligned} \tag{55}$$

$$
\begin{aligned}
\|D_\xi^\beta a(x,\xi)\|_{C^r_*} =& \|D_\xi^\beta a(x,\xi)\|_{C^{n+t}_*} \\
=& \|D_\xi^\beta a(x,\xi)\|_{C^{n+t}} \\
=& \|D_\xi^\beta a(x,\xi)\|_{C^{n}} + \max_{|\alpha|=n}\sup_{x\neq y}\frac{|D_x^\alpha D_\xi^\beta a(x,\xi) - D_x^\alpha D_\xi^\beta a(y,\xi)|}{\|x-y\|^t} \\
\leq& C_{n\beta}\langle\xi\rangle^{m-|\beta|} + C_{nt\beta}\langle\xi\rangle^{m-|\beta|}.
\end{aligned} \tag{56}
$$

We now have the following mapping property for operators in ${}^r\widetilde{\Psi}^m(\Omega)$.

Proposition 2.3. If $a(x,D) \in {}^r\widetilde{\Psi}^m(\Omega)$, then

$$
a(x,D) : H^{s+m}(\Omega) \to H^s(\Omega), \quad \text{provided } -r < s < r. \tag{57}
$$

## 3. Symbol Calculus

In this section we present two lemmas which allow us to classify a symbol using a sum of derivatives, and allow a symbol to be constructed from a sum of individual terms.

Lemma 2.4. Let $A(x,y,\xi,\eta)$ in $\Omega\times\Omega\times\mathbb{R}^d\times\mathbb{R}^d$ be a function which is $C^{r-1}$ with respect to $(x,y)\in\Omega\times\Omega$ and $C^\infty$ with respect to $(\xi,\eta)\in\mathbb{R}^d\times\mathbb{R}^d$, and vanish if $y\in\Omega\backslash K$, where $K\subset\subset\Omega$. If

$$
|D_x^\alpha D_y^{\alpha'} D_\xi^\beta D_\eta^{\beta'} A(x,y,\xi,\eta)| \leq C_{\beta,\beta',\alpha,\alpha'}(1+|\xi|)^{m-|\beta|}(1+|\eta|)^{m'-|\beta'|} \tag{58}
$$

for all $\beta,\beta'$ and all $\alpha+\alpha'\leq r-1$, then we have that

$$
a(x,\xi) = \frac{1}{(2\pi)^d}\iint A(x,y,\xi,\eta)e^{i(x-y)(\eta-\xi)}dyd\eta \quad \in \; {}^{r-1}\widetilde{S}^{m+m'}(\Omega) \tag{59}
$$

and $a(x,\xi)-a_N(x,\xi)\in {}^{r-1-N}\widetilde{S}^{m+m'-N}(\Omega)$ for all $N<\frac{r-1+m'-d}{2}$, where

$$
a_n(x,\xi) = \sum_{|\alpha|<N}\frac{1}{\alpha!}D_y^\alpha\partial_\eta^\alpha A(x,y,\xi,\eta)\big|_{y=x,\eta=\xi}. \tag{60}
$$

*Proof*:

Assume that $N<\frac{r-1+m'-d}{2}$ and let

$$
a_N'(x,\xi) = \frac{1}{(2\pi)^d}\iint\sum_{|\alpha|<N}\frac{1}{\alpha!}(iD_\eta)^\alpha A(x,y,\xi,\eta)\big|_{\eta=\xi}(\eta-\xi)^\alpha e^{i(x-y)(\eta-\xi)}dyd\eta. \tag{61}
$$

By integrating by parts, we have that

$$
\begin{aligned}
a'_N(x,\xi) &= \frac{1}{(2\pi)^d}\iint \sum_{|\alpha|<N} \frac{1}{\alpha!}\partial_\eta^\alpha A(x,y,\xi,\eta)\big|_{\eta=\xi}(\eta-\xi)^\alpha e^{i(x-y)(\eta-\xi)}dyd\eta \\
&= \frac{1}{(2\pi)^d}\iint \sum_{|\alpha|<N} \frac{1}{\alpha!}\partial_\eta^\alpha A(x,y,\xi,\eta)\big|_{\eta=\xi}(-D_y)^\alpha e^{i(x-y)(\eta-\xi)}dyd\eta \\
&= \frac{1}{(2\pi)^d}\iint \sum_{|\alpha|<N} \frac{1}{\alpha!}D_y^\alpha\partial_\eta^\alpha A(x,y,\xi,\eta)\big|_{\eta=\xi} e^{i(x-y)(\eta-\xi)}dyd\eta \\
&= \frac{1}{(2\pi)^d}\int \left(\int \sum_{|\alpha|<N} \frac{1}{\alpha!}D_y^\alpha\partial_\eta^\alpha A(x,y,\xi,\eta)\big|_{\eta=\xi} e^{-iy(\eta-\xi)}dy\right) e^{ix(\eta-\xi)}d\eta.
\end{aligned}
\tag{62}
$$

Since the function $A(x,y,\xi,\xi)$, as well as it's derivatives up to $\alpha < N$, has compact support in $y$ for fixed $(x,\xi)$, the Fourier and inverse Fourier transform are well-defined. Therefore, we have that

$$
\begin{aligned}
a'_N(x,\xi) &= \frac{1}{(2\pi)^d}\int \sum_{|\alpha|<N} \frac{1}{\alpha!}(\eta-\xi)^\alpha \partial_\eta^\alpha \mathcal{F}_{y\to\eta-\xi}\big(A\big)(x,y,\xi,\eta)\big|_{\eta=\xi} e^{ix(\eta-\xi)}d\eta \\
&= \frac{1}{(2\pi)^d}\int \sum_{|\alpha|<N} \frac{1}{\alpha!}(\eta)^\alpha \partial_\eta^\alpha \mathcal{F}_{y\to\eta}\big(A\big)(x,\eta,\xi,\eta)\big|_{\eta=\xi} e^{ix\eta}d\eta \\
&= \sum_{|\alpha|<N} \frac{1}{\alpha!}D_y^\alpha\partial_\eta^\alpha A(x,y,\xi,\eta)\big|_{y=x,\eta=\xi} \\
&= a_N(x,\xi).
\end{aligned}
\tag{63}
$$

We know from the bound on the derivatives of $A(x,y,\xi,\eta)$ that $a_N \in {}^{r-1}\widetilde{S}^{m+m'}(\Omega)$. Using integration by parts on $a(x,\xi) - a'_N(x,\xi)$, we now want to show that

$$
a(x,\xi) - a_N(x,\xi) = \frac{1}{(2\pi)^d}\iint r_N(x,y,\xi,\eta)e^{i(x-y)(\eta-\xi)}dyd\eta, \tag{64}
$$

where

$$
r_N(x,y,\xi,\eta) = \sum_{|\alpha|=N} \frac{N}{\alpha!}(\eta-\xi)^\alpha \int_0^1 (1-t)^{N-1}\partial_\eta^\alpha A(x,y,\xi,\eta)\big|_{\eta=\xi+t(\eta-\xi)}dt \tag{65}
$$

also belongs to ${}^{r-1}\widetilde{S}^{m+m'}(\Omega)$.

Since

$$
(\eta-\xi)^\alpha e^{i(x-y)(\eta-\xi)} = (-D_y)^\alpha e^{i(x-y)(\eta-\xi)}, \tag{66}
$$

we can again integrate by parts to obtain

(67)
$$a(x,\xi) - a_N(x,\xi) = \frac{1}{(2\pi)^d} \sum_{|\alpha|=N} \frac{N}{\alpha!} \int d\eta \int_0^1 (1-t)^{N-1} dt \times \int D_y^\alpha \partial_\eta^\alpha A(x,y,\xi,\eta)\big|_{\eta=\xi+t(\eta-\xi)} e^{i(x-y)(\eta-\xi)} dy.$$

If $k$ is a natural number, we can use

$$e^{i(x-y)(\eta-\xi)} = (1-\Delta_y)^k e^{i(x-y)(\eta-\xi)} (1+|\eta-\xi|^2)^{-k} \tag{68}$$

to show that

$$a(x,\xi) - a_N(x,\xi) = \frac{1}{(2\pi)^d} \sum_{|\alpha|=N} \frac{N}{\alpha!} \int (1+|\eta-\xi|^2)^{-k} d\eta \int_0^1 (1-t)^{N-1} dt \times \int B_\alpha(t,x,y,\xi,\eta) e^{i(x-y)(\eta-\xi)} dy, \tag{69}$$

where

$$B_\alpha(t,x,y,\xi,\eta) = (1-\Delta_y)^k D_y^\alpha \partial_\eta^\alpha A(x,y,\xi,\eta)\big|_{\eta=\xi+t(\eta-\xi)}. \tag{70}$$

We are permitted to use this identity as long as $k \le \frac{r-1-N}{2}$.

Since the modulus of the integrand is not larger than

$$C(1+|\xi|)^m (1+|\xi+t(\eta-\xi)|)^{m'-N} (1+|\eta-\xi|^2)^{-k} h_1(y), \tag{71}$$

where $h_1 \in C_0^\infty(\mathbb{R}^d), h_1 \ge 0, h_1 = 1$ in a neighborhood of the compact set $K$, the following estimate

$$|a(x,\xi) - a_N(x,\xi)| \le C_1 \int \int_0^1 (1+|\xi|)^m (1+|\xi+t(\eta-\xi)|)^{m'-N} (1+|\eta-\xi|^2)^{-k} d\eta dt \tag{72}$$

is true, provided that $2k > m' - N + d$. Combining this result with $2k + N \le r - 1$ from above, this process holds as long as $d + m' < r - 1$.

We now divide the domain of integration into two:

$$\begin{aligned} \Omega_1 &= \{(t,\eta) : |t(\eta-\xi)| < |\frac{\xi}{2}|\}, \\ \Omega_2 &= \{(t,\eta) : |t(\eta-\xi)| \ge |\frac{\xi}{2}|\}. \end{aligned} \tag{73}$$

In the domain $\Omega_1$, the following inequality holds:

$$|\xi| \leq 2|\xi + t(\eta - \xi)| \leq 3|\xi|. \tag{74}$$

We then have that the integral over $\Omega_1$ does not exceed

$$C_2 \iint_{\Omega_1} (1+|\xi|)^{m+m'-N}(1+|\eta-\xi|^2)^{-k+\frac{|m'-N|}{2}} dt d\eta \leq C_3(1+|\xi|)^{m+m'-N}, \tag{75}$$

provided that we have $2k > d + |m' - N|$.

Note that the inequality $2k > d+|m'-N|$ additionally tells us that $d+2N-m' < r-1$. For a Dirichlet-to-Neumann operator acting on the circle, where $d = 1$ and $m' = 1$, this puts a cap on how big $N$ can be depending on the regularity $r$.

In the domain $\Omega_2$, the following inequalities hold:

$$\begin{aligned} |\xi| &\leq 2|\eta - \xi|, \\ 1 + |\xi + t(\eta - \xi)| &\leq 1 + 3|\xi - \eta|. \end{aligned} \tag{76}$$

We then have that the integral over $\Omega_2$ does not exceed

$$C_4 \iint_{\Omega_2} (1+|\xi|)^{m+m'-N}(1+|\eta-\xi|^2)^{-k+\frac{|m'-N|}{2}} dt d\eta \leq C_5(1+|\xi|)^{m+m'-N}, \tag{77}$$

provided that $2k > |m' - N| + d$.

To see this, when $m' \geq N$, we have that

$$\begin{aligned} (1+|\xi+t(\eta-\xi)|)^{m'-N} &\leq 3^{m'-N}(1+|\eta-\xi|)^{m'-N} \\ &\leq 3^{m'-N}(1+|\xi|)^{m'-N}(1+|\eta-\xi|)^{m'-N}. \end{aligned} \tag{78}$$

And when $m' < N$, we have that

$$(1+|\xi+t(\eta-\xi)|)^{m'-N} \leq 1 \leq 2^{N-m'}(1+|\xi|)^{m'-N}(1+|\eta-\xi|)^{N-m'}. \tag{79}$$

Therefore, we have that for all $2N < r - 1 + m' - d$,

$$|a(x,\xi) - a_n(x,\xi)| \leq C_6(1+|\xi|)^{m+m'-N}. \tag{80}$$

If we differentiate both sides of (65) with respect to $x$, then we obtain

$$|D_x^\alpha D_\xi^\beta [a(x,\xi) - a_N(x,\xi)]| \leq C'_{\alpha\beta}(1+|\xi|)^{m+m'-N-|\beta|}. \tag{81}$$

Observe that we cannot differentiate any more than $(r - 1 + m' - d - 2N)$ times in $x$. Using that ${}^r\widetilde{S}^m(\Omega)$ is a vector space, this concludes that $a \in {}^{r-1}\widetilde{S}^{m+m'}(\Omega)$.

LEMMA 2.5. Let $a_j(x,\xi) \in {}^{r-j}\widetilde{S}^{m-j}(\Omega)$ for all $0 \le j < r$. Then there exists a symbol $a \in {}^{r}\widetilde{S}^{m}(\Omega)$, such that for any $0 < N < r-1$,

$$a(x,\xi) - \sum_{j=0}^{N} a_j(x,\xi) \in {}^{r-N-1}\widetilde{S}^{m-N-1}(\Omega). \tag{82}$$

The symbol $a(x,\xi)$ is unique modulo an operator $R : H^{m-r-\varepsilon}(\Omega) \to L^2(\Omega)$, some $\varepsilon > 0$.

*Proof*:

Let $h \in C^\infty(\mathbb{R}^d)$ so that $h(\xi) = 0$ for $|\xi| \le 1$ and $h(\xi) = 1$ for $|\xi| \ge 2$. Let $\Omega'$ be an open set containing the closure of $\Omega$. We use $h$ to cut away the support near $\xi = 0$ as follows. Let $\{t_j\}$ be a decreasing sequence of positive numbers such that $t_j \to 0$ and define

$$a(x,\xi) = \sum_{j=0}^{\lfloor r \rfloor - 2} h(\xi t_j) a_j(x,\xi). \tag{83}$$

For any fixed $\xi$, $h(\xi t_j) = 0$ for all but a finite number of $j$, so this sum is well defined and continuous for $(x,\xi)$. For $0 < j \le r-2$ we have

$$|a_j(x,\xi)| \le C_j (1+|\xi|)^{m-j} = C_j \frac{(1+|\xi|)^m}{(1+|\xi|)^j}. \tag{84}$$

If $|\xi|$ is large enough, $\frac{C_j}{(1+|\xi|)^j}$ is as small as we like and therefore, by passing to a subsequence of $t_j$, we can assume that

$$\left|h(\xi t_j) a_j(x,\xi)\right| \le |a_j(x,\xi)| \le \frac{(1+|\xi|)^m}{2^j} \qquad \text{for } 0 < j \le r-2. \tag{85}$$

This implies that

$$|a(x,\xi)| \le \sum_{j=0}^{\lfloor r \rfloor - 2} \left|h(\xi t_j) a_j(x,\xi)\right| \le \sum_{j=0}^{\lfloor r \rfloor - 2} \frac{(1+|\xi|)^m}{2^j} \le C(1+|\xi|)^m. \tag{86}$$

We can use a similar argument for the derivatives (provided we stay within regularity for the finite number of terms), and use a diagonalization argument on the resulting subsequences to conclude that $a(x,\xi) \in {}^{r}\widetilde{S}^{m}(\Omega)$. For example, using that

$$a(x,\xi) = \sum_{j=0}^{\lfloor r \rfloor - 2} h(\xi t_j) a_j(x,\xi), \tag{87}$$

we have the derivative

$$D_x\big(a(x,\xi)\big) = \sum_{j=0}^{\lfloor r\rfloor-2} h(\xi t_j) D_x\big(a_j(x,\xi)\big). \tag{88}$$

Note that for the term with lowest regularity, $a_{\lfloor r\rfloor-2}(x,\xi)$ belongs to ${}^{2}\widetilde{S}^{m-r+2}(\Omega)$, so taking a derivative in $x$ still yields a well defined and continuous sum for $(x,\xi)$. For $0 < j \leq r-2$ we again have that

$$\big|D_x\big(a_j(x,\xi)\big)\big| \leq C_j(1+|\xi|)^{m-j} = C_j\frac{(1+|\xi|)^m}{(1+|\xi|)^j}. \tag{89}$$

Using the same strategy as above, we arrive at

$$\big|D_x\big(a(x,\xi)\big)\big| \leq \sum_{j=0}^{\lfloor r\rfloor-2} \big|h(\xi t_j) D_x\big(a_j(x,\xi)\big)\big| \leq \sum_{j=0}^{\lfloor r\rfloor-2} \frac{(1+|\xi|)^m}{2^j} \leq C(1+|\xi|)^m. \tag{90}$$

The supports of all $a_j(x,\xi)$ are contained compactly in $\Omega$, so the support of $a(x,\xi)$ is contained in $\overline{\Omega}$ which itself is contained in $\Omega'$.

Now we can apply exactly the same arguments to the sum $\sum_{j=1}^{N} a_j(x,\xi)$ to show that it belongs to ${}^{r-1}\widetilde{S}^{m-1}(\Omega)$. We continue in this fashion and use a diagonal argument on the resulting subsequences to conclude that

$$\sum_{j=N+1}^{\lfloor r\rfloor-1} h(\xi t_j) a_j(x,\xi) \in {}^{r-N-1}\widetilde{S}^{m-N-1}(\Omega). \tag{91}$$

Since

$$a_j(x,D) - h(Dt_j)a_j(x,D) : H^{m-j+s}(\Omega) \to H^s(\Omega), \tag{92}$$

provided $-(r-j) < s < r-j$, this implies that

$$a - \sum_{j=0}^{N} a_j \in {}^{r-N-1}\widetilde{S}^{m-N-1}(\Omega), \tag{93}$$

and concludes the proof.

CHAPTER 3

# Main results

## 1. Symbol of the Dirichlet-to-Neumann map

In this section we compute the symbol of the Dirichlet-to-Neumann map using the strategy of Rozenblum/Edward on "near-similar" operators with non-smooth symbols acting on the circle, and their relationships between eigenvalues.

### 1.1. Reduction to the disk.

Here we transform the Dirichlet-to-Neumann operator $\Lambda_\Omega(\partial\Omega)$ into $\Lambda_\delta(\mathbb{S}^1)$ using a conformal map between $\Omega$ and the unit disk.

In the proof of Theorem 1.2, we will have to isolate boundary components and glue a cap on each. Assume for now that $\Omega$ only has one boundary component. The strategy is to glue a disk to the collar neighbourhood of this boundary component, and discard the rest of the surface. Let $\Omega_C$ be this topological disk. Since the symbol of both $\Lambda_\Omega$ and $\Lambda_{\Omega_C}$ depend solely on data obtained from a neighbourhood of the boundary, which they share, they have the same symbol in ${}^r\widetilde{S}^1$. The conformal map in question is between the unit disk $\mathbb{D}$ and $\Omega_C$.

Let $\varphi$ be the conformal map (given by the Riemann mapping theorem) which maps $\mathbb{D}$ onto $\Omega$ and maps $\mathbb{S}^1$ to $\partial\Omega_C$. As shown in [11, Theorem 1.6], the following two Steklov problems are isospectral:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_C \\ \partial_\nu u = \sigma u & \text{on } \partial\Omega_C, \end{cases} \qquad\qquad \begin{cases} \Delta u = 0 & \text{in } \mathbb{D} \\ \partial_\nu u = \delta\sigma u & \text{on } \mathbb{S}^1. \end{cases} \tag{94}$$

Here, we have that $\delta = |\varphi'| > 0$ from our conformal map between $\mathbb{D}$ and $\Omega_C$.

Now, using ([14], Thm 3.6) and [9], we isolate each boundary component of $\Omega$ by cutting a thin neighbourhood of the boundary and gluing on a topological disk, and then conformally mapping each topological disk $\Omega_i$ to the unit disk $\mathbb{D}$. Given the initial $C^r$ regularity on the boundary, and the derivative required from the conformal map, each Dirichlet-to-Neumann operator $\Lambda_{\Omega_i} \in {}^r\widetilde{\Psi}^1(\partial\Omega_i)$ is transformed into $\Lambda_{\delta_i} \in {}^{r-1}\widetilde{\Psi}^1(\mathbb{S}^1)$. More precisely, let $\varphi_i$ be the conformal map from $\mathbb{D}$ to $\Omega_i$, and let $\delta_i = |\varphi_i'|$. We have that each Dirichlet-to-Neumann operator is given by $\Lambda_{\delta_i}$ with symbol $\frac{|\xi|}{\delta_i(x)}$ and corresponds to the weighted Steklov problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{D} \\ \partial_\nu u = \delta_i \sigma u & \text{on } \mathbb{S}^1. \end{cases} \tag{95}$$

### 1.2. Diagonalization of the symbol.

In this section, we diagonalize the symbol of operator $\Lambda_\delta(\mathbb{S}^1)$, which is motivated by [**12**]. The term *diagonalize* refers to the process of making the symbol independent of the spatial variable $x$ as far as regularity will allow.

The diagonalization is performed individually on each component, so let $\Omega_C$ be our topological disk. After our conformal map we have the following Steklov problem,

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{D} \\ \partial_\nu u = \delta \sigma u & \text{on } \mathbb{S}^1. \end{cases} \tag{96}$$

where $\delta = |\varphi'| > 0$ from the conformal map between $\mathbb{D}$ and $\Omega_C$.

Recalling the Dirichlet-to-Neumann map $\Lambda_\mathbb{D} : H^1(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$, the eigenvalues of problem (96) are the eigenvalues of the operator $\frac{1}{\delta(x)}\Lambda_\mathbb{D} = \Lambda_\delta$. The total symbol of the operator $\Lambda_\delta$ is given by $\frac{|\xi|}{\delta(x)}$, which is $C^{r-1}(\mathbb{S}^1)$ in $x$ (because $\delta(x)$ is $C^{r-1}$ from the conformal map using the Kellogg-Warschawski theorem [**14**]).

We now compare $\delta(x) \in C^{r-1}(\mathbb{S}^1)$ to Example 1 in [**15**], where $g(x) \in C^\infty(\mathbb{S}^1)$. We want to show, using a procedure of *near similarity*, that we can diagonalize the symbol of $\Lambda_\delta$ to depend only on $\xi$ (up to a perturbation $H^\beta \to L^2$ which depends on the regularity of $\delta(x)$).

We define

$$L = \frac{1}{2\pi}\int_0^{2\pi} \delta(x)dx \tag{97}$$

and

$$V(x,\eta) = \frac{\eta}{L}\int_0^x \delta(t)dt, \tag{98}$$

and since $V$ is a generating function for the canonical transformation $(y, \xi) = T(x, \eta)$ given by the relations $\xi = \frac{\partial V(x,\eta)}{\partial x}, y = \frac{\partial V(x,\eta)}{\partial \eta}$, we define the Fourier integral operator acting on $\mathbb{S}^1$,

$$\Phi u(x) = \int e^{iV(x,\xi)} \widehat{u}(\xi) d\xi. \tag{99}$$

We will later require that $\Phi$ is invertible. For our choice of $V(x, \xi)$ above, we can define $\Phi^{-1}$ explicitly using the following two lemmas.

LEMMA 3.1. The adjoint $\Phi^*$ of operator $\Phi$ is defined as

$$\Phi^* u(x) = \iint u(y) e^{-iV(y,\eta)} e^{ix\eta} dy d\eta.$$

*Proof*:

By using the inner product $\langle u, v \rangle_{L^2(\mathbb{S}^1)}$, we have

$$\begin{aligned}
&\big\langle \Phi^* u(x), v(x) \big\rangle \\
&= \big\langle u(x), \Phi v(x) \big\rangle \\
&= \left\langle u(x), \int e^{iV(x,\xi)} \widehat{v}(\xi) d\xi \right\rangle \\
&= \int u(x) \int e^{iV(x,\xi)} \widehat{v}(\xi) d\xi dx \\
&= \iiint u(x) v(y) e^{iy\xi} e^{iV(x,\xi)} dy d\xi dx \\
&= \left\langle \iint u(x) e^{iy\xi} e^{iV(x,\xi)} d\xi dx, v(y) \right\rangle.
\end{aligned} \tag{100}$$

LEMMA 3.2. The inverse $\Phi^{-1}$ of operator $\Phi$ is given by $\Phi^{-1} = \Phi^* M$, where $M$ is the operator of multiplication by $\frac{\delta(x)}{L}$.

*Proof*:

Given that $Mu(x) = \frac{\delta(x)}{L} u(x)$ and $\Phi v(x) = \int \widehat{v}(\xi) e^{iV(x,\xi)} d\xi$, we have

$$\begin{aligned}
M \Phi u(x) =& M \left( \int \widehat{u}(\xi) e^{iV(x,\xi)} d\xi \right) \\
=& \frac{\delta(x)}{L} \left( \int \widehat{u}(\xi) e^{iV(x,\xi)} d\xi \right) \\
=& \int \frac{\delta(x)}{L} \widehat{u}(\xi) e^{iV(x,\xi)} d\xi.
\end{aligned} \tag{101}$$

Now given $\Phi^* u(x) = \iint u(y) e^{-iV(y,\eta)} e^{ix\eta} dy d\eta$, we have

$$
\begin{aligned}
\Phi^* M \Phi u(x) = & \Phi^* \Big( \int \frac{\delta(x)}{L} \widehat{u}(\xi) e^{iV(x,\xi)} d\xi \Big) \\
= & \iint \Big( \int \frac{\delta(y)}{L} \widehat{u}(\xi) e^{iV(y,\xi)} d\xi \Big) e^{-iV(y,\eta)} e^{ix\eta} dy d\eta \\
= & \iiint \frac{\delta(y)}{L} \widehat{u}(\xi) e^{iV(y,\xi)} e^{-iV(y,\eta)} e^{ix\eta} d\xi dy d\eta.
\end{aligned}
\tag{102}
$$

After the change of variable $z = \frac{1}{L} \int_0^y \delta(t) dt$, we have

$$
\begin{aligned}
\Phi^* M \Phi u(x) = & \iiint \widehat{u}(\xi) e^{iz\xi} e^{-iz\eta} e^{ix\eta} d\xi dz d\eta \\
= & \iint u(z) e^{-iz\eta} e^{ix\eta} dz d\eta \\
= & \int \widehat{u}(\eta) e^{ix\eta} d\eta \\
= & u(x).
\end{aligned}
\tag{103}
$$

We now wish to diagonalize the symbol $\frac{|\xi|}{\delta(x)}$ of operator $\Lambda_\delta$ to depend only on $\xi$, as far as the regularity from $\delta(x) \in C^{r-1}(\mathbb{S}^1)$ will allow.

PROPOSITION 3.3. For all $N < \frac{r-1}{2}$, there exists an operator $B_N \in {}^{r-1}\widetilde{\Psi}^1(\mathbb{S}^1)$ such that the principal symbol (and following $N-1$ terms) of $B_N$ depends only on $\xi$, and where

$$
\Lambda_\delta \Phi - \Phi B_N \in {}^{r-1-N}\widetilde{\Psi}^{1-N}(\mathbb{S}^1).
\tag{104}
$$

*Proof*:

Our goal is to find

$$
b(x,\xi) = \sum_{j=0}^{N-1} b_{1-j}(\xi) + \widetilde{b}(x,\xi)
\tag{105}
$$

where each $b_j$ is homogeneous of order $j$ in $\xi$. To do this, we will decompose the symbol of both operators $\Lambda_\delta \Phi$ and $\Phi B_N$ acting in $L^2(\mathbb{S}^1)$ and relate terms of the same order.

We first observe the symbol of $\Lambda_\delta \Phi$:

$$
\begin{aligned}
\Lambda_\delta \Phi u(x) &= \Lambda_\delta\big( \int e^{iV(x,\xi)} \widehat{u}(\xi) d\xi \big) \\
&= \iiint \frac{|\eta|}{\delta(x)} e^{iV(y,\xi)} \widehat{u}(\xi) e^{-iy\eta} e^{ix\eta} d\xi dy d\eta \\
&= \iiint \frac{|\eta|}{\delta(x)} e^{i(x-y)\eta} e^{i\left(V(y,\xi)-V(x,\xi)\right)} e^{iV(x,\xi)} \widehat{u}(\xi) d\xi dy d\eta \\
&= \int a(x,\xi) e^{iV(x,\xi)} \widehat{u}(\xi) d\xi,
\end{aligned} \tag{106}
$$

where

$$
a(x,\xi) = \iint \frac{|\eta|}{\delta(x)} e^{i(x-y)\eta} e^{i\left(V(y,\xi)-V(x,\xi)\right)} dy d\eta. \tag{107}
$$

We localize our symbol $a(x,\xi)$ by introducing cutoff functions $h_1(x,y)$ and $h_2(\xi,\eta)$ to define $a'(x,\xi)$.

$$
a'(x,\xi) = \iint \frac{|\eta|}{\delta(x)} e^{i(x-y)\eta} e^{i\left(V(y,\xi)-V(x,\xi)\right)} h_1(x,y) h_2(\xi,\eta) dy d\eta \tag{108}
$$

Given the identity

$$
V(y,\xi) - V(x,\xi) = \frac{\xi}{L}(y-x) \cdot \int_0^1 \delta(x+t(y-x)) dt, \tag{109}
$$

we have that

$$
\begin{aligned}
a'(x,\xi) &= \iint \frac{|\eta|}{\delta(x)} e^{i(x-y)\eta - \frac{\xi}{L}\int_0^1 \delta(x+t(y-x))dt \cdot (y-x)} h_1(x,y) h_2(\xi,\eta) dy d\eta \\
&= \iint \frac{|\eta|}{\delta(x)} e^{i(x-y)(\eta - \frac{\xi}{L}\int_0^1 \delta(x+t(y-x))dt)} h_1(x,y) h_2(\xi,\eta) dy d\eta.
\end{aligned} \tag{110}
$$

Using the change of variable $\widetilde{\eta} = \eta - \frac{\xi}{L}\int_0^1 \delta(x+t(y-x))dt + \xi$,

$$
\begin{aligned}
a'(x,\xi) = \iint & \frac{|\widetilde{\eta} + \frac{\xi}{L}\int_0^1 \delta(x+t(y-x))dt - \xi|}{\delta(x)} e^{i(x-y)(\widetilde{\eta}-\xi)} h_1(x,y) \\
& \times h_2\left(\xi, \widetilde{\eta} + \frac{\xi}{L}\int_0^1 \delta(x+t(y-x))dt - \xi\right) dy d\widetilde{\eta} \\
= \iint & A(x,y,\xi,\widetilde{\eta}) e^{i(x-y)(\widetilde{\eta}-\xi)} dy d\widetilde{\eta},
\end{aligned} \tag{111}
$$

where

(112)
$$A(x,y,\xi,\widetilde{\eta}) = \frac{|\widetilde{\eta} + \frac{\xi}{L}\int_0^1 \delta(x+t(y-x))dt - \xi|}{\delta(x)} h_1(x,y) h_2\left(\xi, \widetilde{\eta} + \frac{\xi}{L}\int_0^1 \delta(x+t(y-x))dt - \xi\right).$$

Hence, using Lemma 2.4, we have that for all $N < \frac{r-1}{2}$,

$$a'(x,\xi) = \sum_{\alpha<N} \frac{1}{\alpha!} \partial_{\widetilde{\eta}}^{\alpha} D_y^{\alpha} A(x,y,\xi,\widetilde{\eta})|_{y=x,\ \widetilde{\eta}=\xi} + p(x,\xi), \tag{113}$$

where $\mathrm{Op}\big(p(x,\xi)\big) \in {}^{r-1-N}\widetilde{\Psi}^{1-N}(\mathbb{S}^1)$.

Since we want to compute the derivatives

$$D_y^{\alpha} \partial_{\widetilde{\eta}}^{\alpha} A(x,y,\xi,\widetilde{\eta}))|_{y=x,\ \widetilde{\eta}=\xi}, \tag{114}$$

we let $G'(x) = \delta(x)$ and show the following in advance:

$$\begin{aligned}
\int_0^1 \delta\big(x+t(y-x)\big)dt\Big|_{y=x} &= \lim_{y\to x} \int_0^1 \delta\big(x+t(y-x)\big)dt \\
&= \lim_{y\to x} \int_x^y \frac{\delta\big(s\big)}{y-x} ds \\
&= \lim_{y\to x} \frac{G(y)-G(x)}{y-x} \\
&= G'(x) \\
&= \delta(x).
\end{aligned} \tag{115}$$

We would like to make precise the cut-off function $h_2(\xi,\eta)$ so that $h_2\big(\xi, \frac{\xi}{L}\delta(x)\big) \equiv 1$ for any $x$. Since $\delta(x) > 0$ is bounded above and below, we use the following:

$$h_2(\xi,\eta) = \begin{cases} 1, & \frac{\min(\delta(x))}{L}\xi \le \eta \le \frac{\max(\delta(x))}{L}\xi \\ 0, & \eta \le \frac{\min(\delta(x))}{2L}\xi \\ 0, & \frac{2\max(\delta(x))}{L}\xi \le \eta. \end{cases} \tag{116}$$

*Case 1*, when $\alpha = 0$.

(117)
$$\begin{aligned}
&\frac{\left|\widetilde{\eta} + \frac{\xi}{L}\int_0^1 \delta(x+t(y-x))dt - \xi\right|}{\delta(x)} h_1(x,y) h_2\left(\xi, \widetilde{\eta} + \frac{\xi}{L}\int_0^1 \delta(x+t(y-x))dt - \xi\right)\Big|_{y=x,\ \widetilde{\eta}=\xi} \\
&= \frac{\left|\widetilde{\eta} + \frac{\xi}{L}\delta(x) - \xi\right|}{\delta(x)} h_2\left(\xi, \widetilde{\eta} + \frac{\xi}{L}\delta(x) - \xi\right)\Big|_{\widetilde{\eta}=\xi} \\
&= \frac{\left|\frac{\xi}{L}\delta(x)\right|}{\delta(x)} h_2\left(\xi, \frac{\xi}{L}\delta(x)\right) \\
&= \frac{\left|\frac{\xi}{L}\delta(x)\right|}{\delta(x)} \\
&= \frac{|\xi|}{L}.
\end{aligned}$$

*Case 2*, when $\alpha \geq 1$.

Looking at the expression

$$\frac{1}{\alpha!} D_y^\alpha \partial_{\widetilde{\eta}}^\alpha \left( \frac{\left|\widetilde{\eta} + \frac{\xi}{L}\int_0^1 \delta(x+t(y-x))dt - \xi\right|}{\delta(x)} h_1 h_2 \right)\Big|_{y=x,\ \widetilde{\eta}=\xi} \tag{118}$$

and using the fact that

$$\frac{\partial}{\partial \widetilde{\eta}} h_2\left(\xi, \widetilde{\eta} + \frac{\xi}{L}\int_0^1 \delta(x+t(y-x))dt - \xi\right)\Big|_{y=x,\ \widetilde{\eta}=\xi} = \frac{\partial}{\partial \widetilde{\eta}} h_2\left(\xi, \frac{\xi\delta(x)}{L}\right) = 0, \tag{119}$$

we can see that when $\alpha \geq 1$, any number of derivatives will result in zero (provided $\eta \neq 0$). Any derivative on $h_1$ and $h_2$ will become zero after restricting to $y = x$ and $\widetilde{\eta} = \xi$. And regardless of the number of derivatives in $y$ the term $\left|\widetilde{\eta} + \frac{\xi}{L}\int_0^1 \delta(x+t(y-x))dt - \xi\right|$ sees, adding a single derivative in $\widetilde{\eta}$ will make that zero.

In other words, we have that

$$a'(x,\xi) = \frac{|\xi|}{L} + 0 + \ldots + 0 + p(x,\xi), \tag{120}$$

where there are $N-1$ zeros, and where $\mathrm{Op}\big(p(x,\xi)\big) \in {}^{r-1-N}\widetilde{\Psi}^{1-N}(\mathbb{S}^1)$ for all $N < \frac{r-1}{2}$.

Now we want to observe the symbol of $\Phi B_N$:

$$
\begin{aligned}
\Phi B_N u(x) &= \Phi\left(\int b_N(x,\xi)\widehat{u}(\xi)e^{ix\xi}d\xi\right) \\
&= \iiint b_N(y,\xi)\widehat{u}(\xi)e^{iy(\xi-\eta)}e^{iV(x,\eta)}dyd\xi d\eta \\
&= \iiint b_N(y,\xi)e^{iy(\xi-\eta)}e^{i\left(V(x,\eta)-V(x,\xi)\right)}e^{iV(x,\xi)}\widehat{u}(\xi)d\xi dyd\eta \\
&= \int g(x,\xi)e^{iV(x,\xi)}\widehat{u}(\xi)d\xi,
\end{aligned}
\tag{121}
$$

where

$$
g(x,\xi) = \iint b_N(y,\xi)e^{iy(\xi-\eta)}e^{i\left(V(x,\eta)-V(x,\xi)\right)}dyd\eta. \tag{122}
$$

Following the same strategy as above, we look at $g(x,\xi)$ as a symbol in $\mathbb{S}^1$ and localize with cut-off functions $h_3(x,y)$ and $h_4(\xi,\eta)$ to define $g'(x,\xi)$.

$$
g'(x,\xi) = \iint b_N(y,\xi)e^{iy(\xi-\eta)}e^{i\left(V(x,\eta)-V(x,\xi)\right)}h_3(x,y)h_4(\xi,\eta)dyd\eta. \tag{123}
$$

Simplifying the expression $V(x,\eta)-V(x,\xi)$ gives

$$
g'(x,\xi) = \iint b_N(y,\xi)e^{i(\eta-\xi)(\frac{1}{L}\int_0^x\delta(t)dt-y)}h_3(x,y)h_4(\xi,\eta)dyd\eta. \tag{124}
$$

Using the change of variable $\widetilde{y} = y - \frac{1}{L}\int_0^x\delta(t)dt + x$, we have that

$$
\begin{aligned}
g'(x,\xi) &= \iint b_N\left(\frac{1}{L}\int_0^x\delta(t)dt+\widetilde{y}-x,\xi\right)h_3\left(x,\frac{1}{L}\int_0^x\delta(t)dt+\widetilde{y}-x\right)h_4(\xi,\eta)e^{i(\eta-\xi)(x-\widetilde{y})}d\widetilde{y}d\eta \\
&= \iint G(x,\widetilde{y},\xi,\eta)e^{i(\eta-\xi)(x-\widetilde{y})}d\widetilde{y}d\eta,
\end{aligned}
\tag{125}
$$

where

$$
G(x,\widetilde{y},\xi,\eta) = b_N\left(\frac{1}{L}\int_0^x\delta(t)dt+\widetilde{y}-x,\xi\right)h_3\left(x,\frac{1}{L}\int_0^x\delta(t)dt+\widetilde{y}-x\right)h_4(\xi,\eta). \tag{126}
$$

We can again use Lemma 2.4 to show that for all $N < \frac{r-1}{2}$,

$$
g'(x,\xi) = \sum_{\alpha<N}\frac{1}{\alpha!}D_{\widetilde{y}}^{\alpha}\partial_{\eta}^{\alpha}G(x,\widetilde{y},\xi,\eta)\big|_{\widetilde{y}=x,\ \eta=\xi} + q(x,\xi), \tag{127}
$$

where $\mathrm{Op}\big(q(x,\xi)\big) \in {}^{r-1-N}\widetilde{\Psi}^{1-N}(\mathbb{S}^1)$.

We again need to compute the derivatives $D^\alpha_{\widetilde{y}}\partial^\alpha_\eta G(x,\widetilde{y},\xi,\eta)\big|_{\widetilde{y}=x,\ \eta=\xi}$. This time, we would like to make precise the cut-off function $h_3(x,y)$ so that $h_3\big(x,\frac{1}{L}\int_0^x\delta(t)dt\big)\equiv 1$ for any $x$. We use the following:

$$h_3(x,y)=\begin{cases}1, & \frac{\min(\delta(x))}{L}x\le y\le\frac{\max(\delta(x))}{L}x\\ 0, & y\le\frac{\min(\delta(x))}{2L}x\\ 0, & \frac{2\max(\delta(x))}{L}x\le y.\end{cases}\tag{128}$$

*Case 1*, when $\alpha=0$.

$$\begin{aligned}&b_N\left(\frac{1}{L}\int_0^x\delta(t)dt+\widetilde{y}-x,\xi\right)h_3\left(x,\frac{1}{L}\int_0^x\delta(t)dt+\widetilde{y}-x\right)h_4(\xi,\eta)\Big|_{\widetilde{y}=x,\ \eta=\xi}\\&=b_N\left(\frac{1}{L}\int_0^x\delta(t)dt+\widetilde{y}-x,\xi\right)h_3\left(x,\frac{1}{L}\int_0^x\delta(t)dt+\widetilde{y}-x\right)\Big|_{\widetilde{y}=x}\\&=b_N\left(\frac{1}{L}\int_0^x\delta(t)dt,\xi\right)h_3\left(x,\frac{1}{L}\int_0^x\delta(t)dt\right)\\&=b_N\left(\frac{1}{L}\int_0^x\delta(t)dt,\xi\right).\end{aligned}\tag{129}$$

*Case 2*, when $\alpha\ge 1$.

Observe that no term in the product of $G(x,\widetilde{y},\xi,\eta)$ is a function of both $\widetilde{y}$ and $\eta$. We therefore have that

$$g'(x,\xi)=b_N\left(\frac{1}{L}\int_0^x\delta(t)dt,\xi\right)+0+...+0+q(x,\xi),\tag{130}$$

where there are $N-1$ zeros, and where $\mathrm{Op}\big(q(x,\xi)\big)\in{}^{r-1-N}\widetilde{\Psi}^{1-N}(\mathbb{S}^1)$ for all $N<\frac{r-1}{2}$.

By comparing terms of the same order from $a'(x,\xi)$ and $g'(x,\xi)$, we see that $b_N\big(\frac{1}{L}\int_0^x\delta(t)dt,\xi\big)=\frac{|\xi|}{L}$. The zeros also all match up to the error terms $p$ and $q$, which we note that both are order $1-N$. Lemma 2.5 guarantees the existence of operator $B_N\in{}^{r-1}\widetilde{\Psi}^1(\mathbb{S}^1)$.

Hence, we have that

$$\big(\Lambda_\delta\Phi-\Phi B_N\big)u(x)=\int\big(a(x,\xi)-g(x,\xi)\big)e^{iV(x,\xi)}\widehat{u}(\xi)d\xi,\tag{131}$$

where after setting $h_1=h_3$ and $h_2=h_4$, the localized symbol is

$$a'(x,\xi)-g'(x,\xi)\in{}^{r-1-N}\widetilde{S}^{1-N}(\mathbb{S}^1).\tag{132}$$

We conclude that

$$\Lambda_\delta \Phi - \Phi B_N \in {}^{r-1-N}\widetilde{\Psi}^{1-N}(\mathbb{S}^1). \tag{133}$$

Given the invertibility of $\Phi$ in Lemma 3.2, the following corollary immediately follows.

COROLLARY 3.4. The eigenvalues of $\Lambda_\delta$ and $B_N$ admit a one-to-one correspondence such that

$$|\sigma_k(\Lambda_\delta) - \sigma_k(B_N)| = O\left(k^{\lceil \frac{3}{2} - \frac{r}{2} + \varepsilon \rceil}\right). \tag{134}$$

*Proof*:

Proposition 3.3 tells us that there is an operator $R = \Lambda_\delta \Phi - \Phi B_N$ which belongs to ${}^{r-1-N}\widetilde{\Psi}^{1-N}(\mathbb{S}^1)$ for $N < \frac{r-1}{2}$.

If we let $N = \lfloor \frac{r-1}{2} - \varepsilon \rfloor$, we have that

$$\begin{aligned}(1-N) &= 1 - \lfloor \frac{r-1}{2} - \varepsilon \rfloor \\ &= 1 + \lceil -\frac{r-1}{2} + \varepsilon \rceil \\ &= \lceil 1 - \frac{r-1}{2} + \varepsilon \rceil \\ &= \lceil \frac{3}{2} - \frac{r}{2} + \varepsilon \rceil.\end{aligned} \tag{135}$$

Hence, the operator $R$ maps $H^{\lceil \frac{3}{2} - \frac{r}{2} + \varepsilon \rceil}(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$.

The inverse of $\Phi$ is computed exactly as an operator of order 0 in Lemma 3.2. The following computation,

$$\begin{aligned}&\Lambda_\delta \Phi - \Phi B_N = R \\ &\Phi^{-1}\Lambda_\delta \Phi - \Phi^{-1}\Phi B_N = \Phi^{-1} R \\ &\Phi^{-1}\Lambda_\delta \Phi - B_N = \Phi^{-1} R,\end{aligned} \tag{136}$$

demonstrates that $\Phi^{-1}R : H^{\lceil \frac{3}{2} - \frac{r}{2} + \varepsilon \rceil}(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$.

Combining this with the estimate in Theorem 1.4, we have

$$\left|\sigma_k(\Phi^{-1}\Lambda_\delta \Phi) - \sigma_k(B_N)\right| = O\left(k^{\lceil \frac{3}{2} - \frac{r}{2} + \varepsilon \rceil}\right). \tag{137}$$

It is known that for any invertable operator $\Phi$, $\lambda$ is an eigenvalue of $A$ if and only if it is an eigenvalue of $\Phi^{-1}A\Phi$. Therefore,

$$\left|\sigma_k(\Lambda_\delta) - \sigma_k(B_N)\right| = O\left(k^{\lceil \frac{3}{2} - \frac{r}{2} + \varepsilon \rceil}\right). \tag{138}$$

## 2. Proof of Theorem 1.2

We will first assume that there is only one boundary component.

Similar to [6], we glue a spherical cap to a collar neighbourhood of the boundary of $\Omega$ and discard the rest of the surface. Let $\Omega_C$ be this topological disk. Since $\Omega$ and $\Omega_C$ are isometric in a neighborhood of $\partial\Omega$, we have that the structure of the problem is unchanged because operators $\Lambda_\Omega$ and $\Lambda_{\Omega_C}$ have the same symbol in ${}^r\widetilde{S}^1$.

Note that when $\partial\Omega$ is smooth, Lee and Uhlmann [13] showed that operators $\Lambda_\Omega$ and $\Lambda_{\Omega_C}$ have the same *full* symbol. Our case is similar, except that symbols in ${}^r\widetilde{S}^1$ have a finite number of terms depending on the regularity. Taking into account the $C^r$ boundary $\partial\Omega$, we use the same recursive argument as [13] until we have to stop.

We have that $\Lambda_\Omega - \Lambda_{\Omega_C} : H^{1-r-\varepsilon}(\partial\Omega) \to L^2(\partial\Omega)$, for some $\varepsilon > 0$. Hence, Theorem 1.4 gives the estimate

$$\left|\sigma_k(\Lambda_\Omega) - \sigma_k(\Lambda_{\Omega_C})\right| = O(k^{1-r-\varepsilon}). \tag{139}$$

Now, using [14] and [9], we conformally map the topological disk $\Omega_C$ to the unit disk, which makes it a Dirichlet-to-Neumann map an operator in ${}^{r-1}\widetilde{\Psi}^m(\mathbb{S}^1)$. More precisely, let $\varphi$ be the conformal map from $\mathbb{D}$ to $\Omega_C$, and let $\delta = |\varphi'|$. As shown in [11, Theorem 1.6], the Dirichlet-to-Neumann map is given by $\frac{1}{\delta}\Lambda_{\mathbb{D}} = \Lambda_\delta$ and corresponds with the following two isospectral Steklov problems:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_C \\ \partial_\nu u = \sigma u & \text{on } \partial\Omega_C, \end{cases} \qquad\qquad \begin{cases} \Delta u = 0 & \text{in } \mathbb{D} \\ \partial_\nu u = \delta\sigma u & \text{on } \mathbb{S}^1. \end{cases} \tag{140}$$

Due to the conformal map, the operators $\Lambda_{\Omega_C}$ and $\Lambda_\delta$ have isospectral Steklov problems, and hence the spectrum of both coincide.

We now use a similar approach to Example 1 in [15], where instead of a smooth function, we have that $g(x) = \delta(x) \in C^{r-1}(\mathbb{S}^1)$. Using *near similarity* on operator $\Lambda_\delta$, we can diagonalize the symbol of $\Lambda_\delta$ to depend only on $\xi$ (up to a finite number of steps which depend only on the regularity of $\delta(x)$).

Proposition 3.3 and Corollary 3.4 tell us that for a given operator $\Lambda_\delta$, we can find an operator $B_N \in {}^{r-1}\widetilde{\Psi}^1(\mathbb{S}^1)$ such that

$$\Lambda_\delta \Phi - \Phi B_N \in {}^{r-1-N}\widetilde{\Psi}^{1-N}(\mathbb{S}^1). \tag{141}$$

for integer $N < \frac{r-1}{2}$, and that the eigenvalues of $\Lambda_\delta$ and $B_N$ admit a one-to-one correspondence

$$\left|\sigma_k(\Lambda_\delta) - \sigma_k(B_N)\right| = O\left(k^{\lceil \frac{3}{2} - \frac{r}{2} + \varepsilon \rceil}\right). \tag{142}$$

In other words, the eigenvalues of $\Lambda_\delta$ behave as

$$\sigma_k(\Lambda_\delta) = \sigma_k(B_N) + O\left(k^{\lceil \frac{3}{2} - \frac{r}{2} + \varepsilon \rceil}\right). \tag{143}$$

Recall that the symbol of $B_N$ is given explicitly as $b(x, \xi) = \frac{|\xi|}{L} + 0 + \ldots + 0 + r(x, \xi)$, and that the spectrum of an operator acting on $\mathbb{S}^1$ with symbol $\frac{|\xi|}{L}$ is well known [7] as

$$\operatorname{spec}(B_N) = \{0, \frac{2\pi}{L}, \frac{2\pi}{L}, \frac{4\pi}{L}, \frac{4\pi}{L}, \frac{6\pi}{L}, \frac{6\pi}{L}, \ldots\}. \tag{144}$$

Hence, after we combine that the symbols of $\Lambda_\Omega$ and $\Lambda_{\Omega_C}$ coincide to yield

$$\left|\sigma_k(\Lambda_\Omega) - \sigma_k(\Lambda_{\Omega_C})\right| = O(k^{1-r-\varepsilon}), \tag{145}$$

and that $\Lambda_{\Omega_C}$ and $\Lambda_\delta$ are isospectral to yield

$$\operatorname{spec}(\Lambda_{\Omega_C}) = \operatorname{spec}(\Lambda_\delta), \tag{146}$$

and that $\Lambda_\delta$ can be diagonalized as $B_N$ to yield

$$\left|\sigma_k(\Lambda_\delta) - \sigma_k(B_N)\right| = O\left(k^{\lceil \frac{3}{2} - \frac{r}{2} + \varepsilon \rceil}\right), \tag{147}$$

we conclude that

$$\sigma_k(\Lambda_\Omega) = \operatorname{spec}(B_N)_k + O\left(k^{\lceil \frac{3}{2} - \frac{r}{2} + \varepsilon \rceil}\right). \tag{148}$$

In other words, for one boundary component we have

$$\sigma_k(\Lambda_\Omega) = \frac{\pi k}{|\partial\Omega|} + O\left(k^{\lceil \frac{3}{2} - \frac{r}{2} + \varepsilon \rceil}\right). \tag{149}$$

Now let us consider multiple boundary components, where we will use that the Steklov spectrum of a disjoint union of domains is the union of the spectra.

Let $\Omega$ be a domain which has $J$ boundary components, and let $C^r$ be the lowest regularity of each of the boundary components $\partial\Omega_i$, $i = 1, ..., J$. For each boundary component, we build a topological disk $\Omega_i$ by keeping a neighborhood of the boundary $\partial\Omega_i$ and gluing on a spherical cap. Let $\Omega_\#$ be the disjoint union of the $J$ topological disks $\Omega_i$. Since $\Omega$ and $\Omega_\#$ are isometric in the neighborhoods of $\partial\Omega$, we again have that Dirichlet-to-Neumann operators $\Lambda_\Omega$ and $\Lambda_{\Omega_\#}$ have the same symbol.

Using Theorem 1.4, we again have the eigenvalue estimate for operators $\Lambda_\Omega$ and $\Lambda_{\Omega_\#}$,

$$\left|\sigma_k(\Lambda_\Omega) - \sigma_k(\Lambda_{\Omega_\#})\right| = O(k^{1-r-\varepsilon}). \tag{150}$$

It follows from [**14**] and [**9**] that each topological disk $\Omega_i$ is conformally equivalent to the unit disk $\mathbb{D}$. Let $\Lambda_{\Omega_i}$ be the Dirichlet-to-Neumann operator acting on boundary $\partial\Omega_i$, and let $\Lambda_{\delta,i}$ be the Dirichlet-to-Neumann operator acting on $\mathbb{S}^1$ after the conformal map. Finally, let $B_{i,N}$ be the diagonalized Dirichlet-to-Neumann operator of $\Lambda_{\delta,i}$ after applying Proposition 3.3.

For example, the spectrum of $B_{i,N}$ is given by

$$\mathrm{spec}(B_{i,N}) = \left\{0, \frac{2\pi}{L_i}, \frac{2\pi}{L_i}, \frac{4\pi}{L_i}, \frac{4\pi}{L_i}, \frac{6\pi}{L_i}, \frac{6\pi}{L_i}, ...\right\}, \tag{151}$$

where $L_i = |\partial\Omega_i|$ is the length of boundary component $\partial\Omega_i$.

By taking the union of all $J$ spectra, we therefore have

$$\sigma_k(\Lambda_\Omega) = \left(\bigcup_{j=1}^{J} \mathrm{spec}(B_{i,N})\right)_k + O\left(k^{\lceil \frac{3}{2} - \frac{r}{2} + \varepsilon \rceil}\right). \tag{152}$$

## 3. Proof of Theorem 1.4

It follows from the spectral theorem that $A$ has bounded below, discrete spectrum of eigenvalues $\{\lambda_j(A)\}$ accumulating at infinity, each with finite multiplicity, and there is a corresponding complete orthonormal basis $\{\varphi_j\}$ of $L^2(M)$. Because we will be taking asymptotics as $j \to \infty$, we can assume without loss of generality that the eigenvalues $\lambda_j$ are positive.

Let $E_k \in L^2(M)$ be the span of the first $k$ eigenfunctions $\varphi_1, ..., \varphi_k$, and denote $\|\cdot\|_2$ to be the $L^2$ norm $\|\cdot\|_{L^2(M)}$, and $\|\cdot\|$ to be the operator norm $\|\cdot\|_{L^2(M)\to L^2(M)}$.

*Case 1*, where $\beta \le 0$.

Following the variational characterization of eigenvalues we have

$$\lambda_{k+1}(A+Q) \;\ge\; \min_{\substack{f\perp E_k\\ \|f\|_2=1}} \left(\langle Af, f\rangle + \langle Qf, f\rangle\right) \;\ge\; \lambda_{k+1}(A) - \max_{\substack{f\perp E_k\\ \|f\|_2=1}} |\langle Qf, f\rangle|. \tag{153}$$

Using Cauchy-Schwarz and the fact that $\|f\|_2 = 1$, we have

$$\lambda_{k+1}(A) - \lambda_{k+1}(A+Q) \;\le\; \max_{\substack{f\perp E_k\\ \|f\|_2=1}} |\langle Qf, f\rangle| \;\le\; \max_{\substack{f\perp E_k\\ \|f\|_2=1}} \|Qf\|_2. \tag{154}$$

And also,

$$\max_{\substack{f\perp E_k\\ \|f\|_2=1}} \|Qf\|_2 = \max_{\substack{f\perp E_k\\ \|f\|_2=1}} \|QA^{\frac{-\beta}{m}}A^{\frac{\beta}{m}}f\|_2 \;\le\; \|QA^{\frac{-\beta}{m}}\| \max_{\substack{f\perp E_k\\ \|f\|_2=1}} \|A^{\frac{\beta}{m}}f\|_2. \tag{155}$$

Since $\beta \le 0$, and hence $\frac{\beta}{m} \le 0$, the Min-Max theorem gives us

$$\lambda_{k+1}(A) - \lambda_{k+1}(A+Q) \;\le\; C_{A,Q} \max_{\substack{f\perp E_k\\ \|f\|_2=1}} \|A^{\frac{\beta}{m}}f\|_2 \;=\; C_{A,Q}\,\big(\lambda_{k+1}(A)\big)^{\frac{\beta}{m}}. \tag{156}$$

Therefore, Weyl's law on $A$ gives us

$$\lambda_{k+1}(A) - \lambda_{k+1}(A+Q) \le C_{A,Q}(k+1)^{\frac{\beta}{d}}. \tag{157}$$

To show the reverse role, recall that $Q$ is relatively bounded with respect to $A$. Hence, we have that $A+Q$ behaves similarly to $A$ in that $A+Q$ has bounded below, discrete spectrum of eigenvalues which satisfy a Weyl's law $\lambda_j(A) \asymp j^{m/d}$. We repeat the same steps as above to show that

$$\lambda_{k+1}(A+Q) - \lambda_{k+1}(A) \le D_{A,Q}(k+1)^{\frac{\beta}{d}}. \tag{158}$$

Hence, when $\beta \le 0$, we conclude that

$$|\lambda_{k+1}(A) - \lambda_{k+1}(A+Q)| \asymp (k+1)^{\frac{\beta}{d}}. \tag{159}$$

*Case 2*, where $0 < \beta < m$.

Following the variational characterization of eigenvalues, we again have

$$\lambda_k(A+Q) \le \max_{\substack{f \in E_k \\ \|f\|_2 = 1}} \left(\langle Af, f\rangle + \langle Qf, f\rangle\right) \;\le\; \lambda_K(A) + \max_{\substack{f \in E_k \\ \|f\|_2 = 1}} |\langle Qf, f\rangle|. \tag{160}$$

In other words,

$$\lambda_k(A+Q) - \lambda_k(A) \;\le\; \max_{\substack{f \in E_k \\ \|f\|_2 = 1}} |\langle Qf, f\rangle| \;\le\; \max_{\substack{f \in E_k \\ \|f\|_2 = 1}} \|Qf\|_2. \tag{161}$$

And also,

$$\max_{\substack{f \in E_k \\ \|f\|_2 = 1}} \|Qf\|_2 \;=\; \max_{\substack{f \in E_k \\ \|f\|_2 = 1}} \|QA^{\frac{-\beta}{m}} A^{\frac{\beta}{m}} f\|_2 \;\le\; \|QA^{\frac{-\beta}{m}}\| \max_{\substack{f \in E_k \\ \|f\|_2 = 1}} \|A^{\frac{\beta}{m}} f\|_2. \tag{162}$$

Since $\beta > 0$, and hence $\frac{\beta}{m} > 0$, the Min-Max theorem now gives us

$$\lambda_k(A+Q) - \lambda_k(A) \;\le\; C_{A,Q} \max_{\substack{f \in E_k \\ \|f\|_2 = 1}} \|A^{\frac{\beta}{m}} f\|_2 \;=\; C_{A,Q} \left(\lambda_k(A)\right)^{\frac{\beta}{m}}. \tag{163}$$

Therefore, Weyl's law on $A$ gives

$$\lambda_k(A+Q) - \lambda_k(A) \le C_{A,Q} k^{\frac{\beta}{d}}. \tag{164}$$

For the reverse role, we use the same method of relative boundedness as in *Case 1* to realize that

$$\lambda_k(A) - \lambda_k(A+Q) \le D_{A,Q} k^{\frac{\beta}{d}}. \tag{165}$$

And so when $0 < \beta < m$, we conclude that

$$\left|\lambda_k(A+Q) - \lambda_k(A)\right| \asymp k^{\frac{\beta}{d}}, \tag{166}$$

this concludes the proof.

## 4. Conclusion

In conclusion, we proved that for a surface with finitely smooth boundary, the Steklov eigenvalues converge to those of a circle with a decay rate depending only on the order of smoothness. Since many of the results in this manuscript hold for any dimension (for example Theorem 1.4, Proposition 2.3, and Lemmas 2.4 and 2.5), further work in this direction may include extending the asymptotics of Steklov eigenvalues to manifolds with finitely smooth boundary of higher dimensions. It is the diagonalization theory of Rozenblum that appears to work only if the boundary is dimension 1 — i.e. there are no known global coordinates or simple canonical transformations in higher dimensions.

It is still an open question to determine the optimal regularity to ensure a decaying error term for Steklov surfaces. A result by Shamma in [**18**] required a $C^4$ boundary to obtain the error term $o(1)$. This makes our result of an $O(k^{-1})$ error term requiring a $C^{5+\alpha}$ boundary not so unreasonable. It was also recently shown in [**5**] that such an estimate will not hold for polygons.

One of the biggest impacts on our remainder term comes from the proof of Lemma 2.4, where our current method essentially cuts the regularity in half (i.e. $n < \frac{r-1}{2}$). None of the methods tried so far could remove this factor of two. It would be interesting to see if an alternative method exists where $n < r - 1$.

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