COMPLETION OF CATEGORIES UNDER CERTAIN LIMITS

by

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ABSTRACT

To complete a category is to embed it into a larger one which is closed under a given type of limits (or colimits). A fairly classical construction scheme, in which the objects of the new category are defined as "formal limits" of diagrams of the given one, is carried out in two specific examples.

In the case of the exact completion $\hat{C}$ of a finitely complete category C, we look at "formal coequalizers" of equivalence spans of C, which end up being the quotients (in $\hat{C}$) of the corresponding equivalence relation. The objects of $\hat{C}$ are hence objects of C together with an equivalence span, and its morphisms are equivalence classes of "compatible" morphisms of C. Thus constructed, $\hat{C}$ is an exact category whose structure is studied in detail in Chapter I, as well as the universal property it satisfies.

The second example is the category pro-C of pro-objects, or "formal cofiltered limits" of C. Its objects are cofiltered diagrams of C, but can be characterized in various ways. The known properties of pro-C are summarized, extended and used to lift to pro-C some regularity and exactness properties of C (Chapters II & III).

The category of finite sets is an interesting example: its category of pro-objects is equivalent to the category of Stone spaces which is not exact, but whose exact completion in the sense of [9] is the category of compact Hausdorff spaces.

RESUME

Compléter une catégorie, c'est la plonger dans une catégorie plus vaste qui est stable sous un type de limites (ou colimites) donné. Un mode de contruction assez classique par lequel les objets de la nouvelle catégorie sont définis comme "limites formelles" de diagrammes de l'ancienne, est développé ici dans deux cas précis.

Dans le cas de la complétion exacte $\hat{C}$ d'une catégorie cartésienne (à limites finies) C, il s'agit de "coégalisateurs formels" de pseudo-relations d'équivalence de C qui se révèlent être les quotients (dans $\hat{C}$) de la relation d'équivalence correspondante. Les objets de $\hat{C}$ sont donc des objets de C munis d'une pseudo-relation d'équivalence et ses morphismes, des classes d'équivalence de morphismes "compatibles" de C. Ainsi construite, $\hat{C}$ est une catégorie exacte dont la structure est étudiée en détail au Chapitre I, ainsi que la propriét universelle qu'elle satisfait.

Le deuxième exemple est la catégorie pro-C des pro-objets ou "limites cofiltrantes formelles" de C. Ses objets sont des diagrammes filtrants de C, mais peuvent être caractérisés de diverses façons; de même pour les morphismes. Les propriétés connues de pro-C sont rassemblées, étendues et utilisées pour étendre à pro-C certaines propriétés de régularité et d'exactitude de C (Chapitres II & III). La catégorie des ensembles finis est un example intéressant: la catégorie de ses pro-objects est equivalente à la catégorie des espaces de Stone qui n'est pas exacte, mais dont la complétion exacte dans le sens de [9] est la catégorie des espaces compacts.

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PREFACE

The object of this thesis is to present a detailed study of the essential regularity and exactness properties of two types of category-completions: the exact completion of a finitely complete (cartesian) category and the category of pro-objects.

In the first example (Chapter I), we construct a full extention $\hat{C}$ of a given cartesian category C and prove that $\hat{C}$ is exact and has the required universal property. The definition of $\hat{C}$-morphisms as equivalence classes of "compatible" C-morphisms made it rather difficult to grasp the structure of $\hat{C}$; so we were led to give a systematic description of limits, monos, regular epis etc. of $\hat{C}$ using the cartesian structure of C. For instance, we point out and use the fact that every commutative diagram in $\hat{C}$ gives rise to a diagram in C which commutes modulo some equivalence span. The results of this study confirm and clarify our original intuition that $\hat{C}$ is the category of "formal quotients" of C.

The basic definitions and properties of equivalence spans and compatible morphisms in C are given in section 1. They lead to the definition of objects and morphisms of $\hat{C}$ (section 2). In section 3, we give an "external" characterization of (finite) limits in $\hat{C}$, which suggests a way of "creating" finite limits in $\hat{C}$ using finite limits in C. Then, we use it to construct a terminal object and pull-backs in $\hat{C}$.

In a similar way, we can characterize monos and regular epis as "injections and surjections modulo some equivalence span" by analogy with the category of sets (section 4). The regular structure of $\hat{C}$ is a direct consequence of the fact that (as in sets) every morphism f of $\hat{C}$ can be made into a mono by choosing the appropriate equivalence span on the domain of f (i.e.: the largest equivalence span making f compatible).

Finally, we give a construction of a saturated representative of a given subobject in $\hat{C}$, less properly called saturated subobject (section 5). Then we can characterize equivalence relations in $\hat{C}$ and construct quotients in $\hat{C}$; this proves that $\hat{C}$ is exact (section 6).

The universal property of $\hat{C}$ follows from the "internal characterization of objects of $\hat{C}$" (proposition 7.1) using diagram-chasing techniques.

In the second example (Chapters II & III), we study some structural properties of the category of pro-objects. This category has been partially studied by various authors who use the properties of the category pro-C or its dual ind-C as tools for solving various problems in algebra, algebraic topology, topos theory and logic ([1], [3], [4], [10] and others). Since pro-objects have been previously studied and used, many of their elementary properties are fairly well-known, although it is usually difficult to find references for them. It is even more difficult to find detailed proofs of these properties, especially when C has no or very little additional structure.

The purpose of Chapter II, a published paper (reference [0]), is to give an overview of the various equivalent definitions of the category pro-C (sections 1 & 2), and to extend the "uniform approximation theorem" stated in [1] in two ways.

First, we show that if Δ is a finite loop-free category, there is an equivalence of categories between $(\text{pro-}C)^{\Delta}$ and $\text{pro-}C^{\Delta}$ (section 3).

Then, we prove that if Cart is the category of small finitely complete categories and if Δ is any finite category, then the endofunctors $(-)^{\Delta}$ & $(-)^{\Delta^{op}}$ on Cart are adjoint. The generalized version of the uniform approximation theorem, where the loop condition can be omitted provided C has finite limits follows as a corallary (section 4). Finally, a counterexample is given which shows that without finite limits, the generalized version does not hold (section 5).

In chapter III, we use the uniform approximation theorem to lift some exactness and regularity properties from C to pro-C. In section 1, we give the well-known result that finite completeness (cocompleteness) can be lifted from C to pro-C. This leads to the new (although quite obvious) "approximation result" that every finite limiting cone in pro-C can be approximated by a cofiltered family of limiting cones of the same type in C (and dually).

Then we look at unique-factorization systems M-E in C, and discover that they can be lifted to a unique-factorization system pro-M– pro-E in pro-C in a unique and natural way (section 2). In section 3 we prove that the stability under pulling-back can be lifted from E to pro-E, and we get as corollary, the fairly well-known result that if C is a regular category, so is pro-C (no reference is known for this result, but most elements of the proof are implicitly present in [4]).

We finally look at the lifting of exactness; this is shown to work in the case of a category in which reflexive and symmetric relations are automatically transitive, and to fail in the general case. The proof of the lifting of exactness (section 4) is a typical example of the use of the uniform approximation theorem in its generalized form: we prove that every reflexive and symmetric relation in pro-C can be uniformly approximated by a cofiltered family of reflexive and symmetric spans (hence relations) in C. The exactness of pro-C under these special conditions follows as an easy consequence.

The counterexample of finite sets is fairly well-known, although no detailed proof of the equivalence between pro-finite sets and Stone spaces is available in the literature. In section 5, we characterize these two categories as the Fix-subcategories of an adjunction between topological spaces and the opposite of the category of functors from finite sets to sets, which are known to be equivalent. So the category of pro-finite sets is not exact but can be completed using the construction given in [9]: we get the category of compact Hausdorff spaces. This fact which ends the thesis suggests that, provided the two constructions can be internalized in a topos, an internal concept of compact Hausdorff space could be defined given a concept of (internal) finiteness.

CHAPTER I

EXACT COMPLETION OF A FINITELY COMPLETE CATEGORY

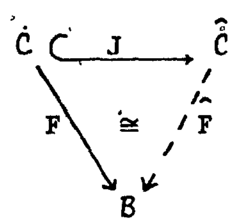
0) Introduction

The purpose of this chapter is to prove the following:

<u>Theorem</u>: The forgetful functor from the category Ex of exact categories to the category Cart of finitely complete categories (sometimes called cartesian categories) has a left pseudo-adjoint (i.e.: adjoint up to natural isomorphism) .

<u>Equivalently</u>: Given any finitely complete category C , we can construct its exact completion $\hat{C}$ i.e. a category such that:

1) $\hat{C}$ is an exact category .
2) There is an inclusion functor $J: C \hookrightarrow \hat{C}$ which preserves and reflects finite limits .
3) Given any left-exact functor $F: C \longrightarrow B$ where B is an exact category there exists a unique (up to natural isomorphism) exact functor $\hat{F}$ such that $F \cong \hat{F} \circ J$:



The proof of this theorem is constructive in the sense that the category $\hat{C}$ and its exact structure is constructed step by step using finite limits in C , and $\hat{F}$ is constructed from F in a similar way.

These constructions are done using a double "internal-external" approach, where:

"<u>internal</u>" means in terms of diagrams and finite limits in C (or $\hat{C}$) , and "external" means in terms of morphisms from an arbitraty object X of C into a given C , which can be viewed as "X-elements" of C .

So, the internal approach gives the formal construction of $\hat{C}$ and of its structure, and the external one, gives an intuitive insight into "what $\hat{C}$ looks like" which can otherwise be extremely messy.

For example, in section 2 we define $\hat{C}$ as the category of objects of C (or C-objects) equipped with an equivalence span, and equivalence classes of C-morphisms. Now, it results directly from the preliminaries of section 1 that "X-elements" of a $\hat{C}$-object $\hat{C}$ (i.e.: $\hat{C}$-morphisms $X \longrightarrow \hat{C}$) are equivalence classes of "X-elements" of C .

Hence every object of $\hat{C}$ can be viewed as the quotient of an object of C by an (external!) equivalence relation. Although this fact is "internalized" and used in section 7 only, it indicates intuitively that $\hat{C}$ is a natural exact completion of C , and constitutes a guideline to the whole construction.

So, this rather long and detailed construction serves the purpose of creating an appropriate machinery of definitions, lemmas and propositions of the form "T.F.A.E." (the following are equivalent), in order to be able to actually work in this category $\hat{C}$, and not merely show that "the construction works".

In particular, it seems that the notion of "canonical subobject " defined by J. Lambek in [5] could be lifted from C to $\hat{C}$, which would give a more precise version of the Theorem given above.

1) Equivalence spans and compatible morphisms in C:

a) Preliminaries:

Proposition 1.1: Given a parallel pair of morphisms of C: $A_1 \underset{a_1}{\overset{a_0}{\rightrightarrows}} A_0$

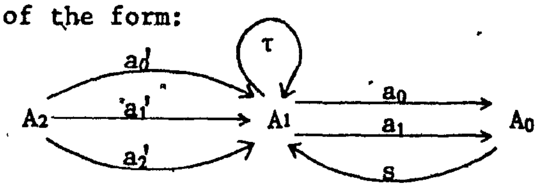
T.F.A.E.:

(i) For any object X of C, the relation defined in $C(X,A_0)$ by:

$$f \sim g \Leftrightarrow \exists h: X \longrightarrow A_1 \text{ s.t. } f = a_0 h \ \& \ g = a_1 h$$

is an equivalence relation.

(ii) The diagram $A_1 \underset{a_1}{\overset{a_0}{\rightrightarrows}} A_0$ can be completed into a simplicial diagram of the form:



where A_2 is the pull-back of a_0 & a_1, a_0' & a_2' being the projections, and which satisfies the relations:

(R) $a_0 s = a_1 s = 1_{A_0}$

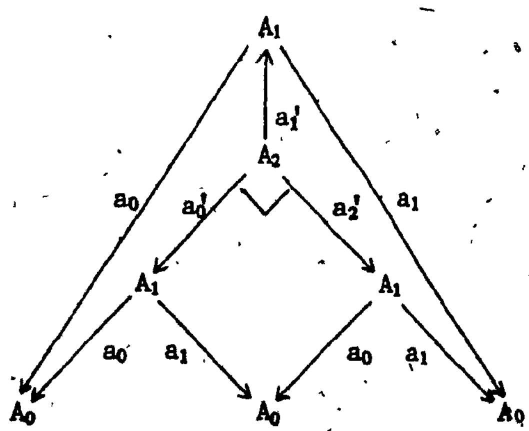
(S) $a_0 \tau = a_1$ & $a_1 \tau = a_0$

(T) $a_0 a_1' = a_0 a_0'$

$a_1 a_1' = a_1 a_2'$

$a_1 a_0' = a_0 a_2'$ (This square being a pull-back)

Note: (T) can be expressed by the following commutative diagram:



<u>Proof</u>: We are going to prove that

a) Reflexivity in (i) $\Leftrightarrow$ (R) in (ii)

b) Symmetry in (i) $\Leftrightarrow$ (S) in (ii)

c) Transitivity in (i) $\Leftrightarrow$ (T) in (ii)

a) <u>Reflexivity</u>:

$\Rightarrow$: If $\sim$ is reflexive, then for $X = A_0$, we must have $1_{A_0} \sim 1_{A_0}$

i.e.: $\exists\, s: A_0 \longrightarrow A_1$ s.t. $1_{A_0} = a_0 s = a_1 s$ (R).

$\Leftarrow$: Conversely if (R) holds, then for any X and any $f: X \longrightarrow A_0$,

we have: $f = 1_{A_0} f = a_0 sf = a_1 sf$ hence, $f \sim f$. Thus, $\sim$ is reflexive.

b) <u>Symmetry</u>:

$\Rightarrow$: If $\sim$ is symmetric, then for $X = A_1$, we have $a_0 \sim a_1$, hence we must have $a_1 \sim a_0$ i.e.: $\exists\, \tau: A_1 \longrightarrow A_1$ s.t. $a_1 = a_0 \tau$ & $a_0 = a_1 \tau$ (S)

$\Leftarrow$: Conversely if (S) holds and if f and g are morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} A_0$ such that $f \sim g$ i.e. $f = a_0 h$ & $g = a_1 h$ (for some h), then $g = a_0 \tau h$ & $f = a_1 \tau h$ hence $g \sim f$.

Thus, $\sim$ is symmetric.

c) <u>Transitivity</u>:

$\Rightarrow$: If $\sim$ is transitive, then for $X = A_2$ defined as the pull-back of a_0 and a_1, $f = a_0 a_0'$, $g = a_1 a_0' = a_0 a_2'$ and $h = a_1 a_2'$, we have $f \sim g$ and $g \sim h$ therefore $f \sim h$

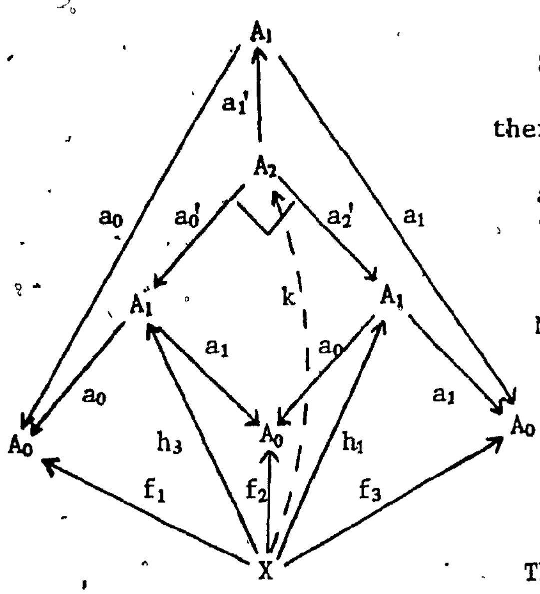
i.e.: $\exists\, a_1': A_2 \longrightarrow A_1$ such that $a_0 a_0' = a_0 a_1'$, & $a_1 a_2' = a_1 a_1'$ (T)

$\Leftarrow$: Conversely, if (T) holds and if f_1, f_2 and $f_3: X \longrightarrow A_0$ verify $f_1 \sim f_2$ and $f_2 \sim f_3$, i.e. for some h_3 & $h_1: X \rightrightarrows A_1$ we have

$$f_1 = a_0 h_3$$

$$f_2 = a_1 h_3 = a_0 h_1$$

$$f_3 = a_1 h_1$$



Since A_2 is the pull-back of a_0 & a_1, there exists a (unique) k s.t.:

$a_0'k = h_3$ & $a_2' k = h_1$

Now, taking $h_2 = a_1'k$ we have:

$f_1 = a_0 h_3 = a_0 a_0' k = a_0 a_1' k = a_0 h_2$

$f_3 = a_1 h_1 = a_1 a_2' k = a_1 a_1' k = a_1 h_2$

Thus $f_1 \sim f_3$, and $\sim$ is transitive.

Definition 1.2: Given an object A_0 of C, an equivalence span (A_1, a_0, a_1) on A_0 is such a pair of parallel maps: $A_1 \overset{a_0}{\underset{a_1}{\rightrightarrows}} A_0$.

We will write: $f \equiv g \pmod{A_1}$ or $f \equiv_{A_1} g$ instead of $f \sim g$ in the above sense.

Proposition 1.3: Given two objects A_0 & B_0, equipped with equivalence spans (A_1, a_0, a_1) & (B_1, b_0, b_1) respectively and a morphism $f: A_0 \longrightarrow B_0$, T.F.A.E.:

(i) For any pair of parallel morphisms: $X \overset{g_0}{\underset{g_1}{\rightrightarrows}} A_0$

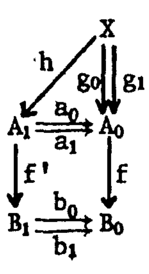
$g_0 \equiv g_1 \pmod{A_1} \Rightarrow fg_0 \equiv fg_1 \pmod{B_1}$

(ii) There exists a morphism $f': A_1 \longrightarrow B_1$

such that: $b_0 f' = fa_0$ & $b_1 f' = fa_1$

Proof (i) ⇒ (ii) : If we take $X = A_1$, $g_0 = a_0$ and $g_1 = a_1$, obviously $a_0 \equiv a_1 \pmod{A_1}$ therefore $fa_0 \equiv fa_1 \pmod{B_1}$

i.e.: $\exists f'$ s.t. $fa_0 = b_0 f'$ & $fa_1 = b_1 f'$.



(ii) ⇒ (i) : If $g_0 \equiv g_1 \pmod{A_1}$, there exists an $h: X \longrightarrow A_1$

such that: $g_0 = a_0 h$ & $g_1 = a_1 h$

therefore $fg_0 = fa_0 h = b_0 f'h$ & $fg_1 = fa_1 h = b_1 f'h$

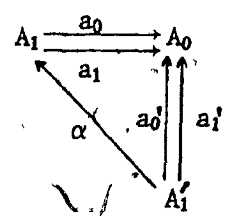
Thus: $fg_0 \equiv fg_1 \pmod{B_1}$

<u>Definition 1.4:</u> such a morphism is called compatible with A_1 and B_1 .

<u>Proposition 1.5</u>: Given A_0 and two equivalence spans (A_1, a_0, a_1) and (A_1', a_0', a_1') on A_0, T.F.A.E.:

(i) For any parallel pair of morphisms of $\mathcal{C}$, f & g: $X \rightrightarrows A_0$, $f \equiv g \pmod{A_1'} \Rightarrow f \equiv g \pmod{A_1}$.

(ii) There exists a morphism $\alpha: A_1' \longrightarrow A_1$ such that: $a_0' = a_0\alpha$ & $a_1' = a_1\alpha$



(ii') $a_0' \equiv a_1' \pmod{A_1}$

(ii") 1_{A_0} is compatible with A_1' & A_1 .

<u>Proof</u> <u>(i) ⇒ (ii)</u>: Assuming (i), if we take $X = A_1'$, $f = a_0'$ & $g = a_1'$, since $a_0' \equiv a_1' \pmod{A_1'}$ we have $a_0' \equiv a_1' \pmod{A_1}$ hence there exists an α verifying $a_0' = a_0\alpha$ & $a_1' = a_1\alpha$.

<u>(ii) ⇒ (i)</u>: Assuming (ii), $f \equiv g \pmod{A_1'} \Rightarrow$

$\Rightarrow \exists h : X \longrightarrow A_1'$ s.t. $f = a_0'h$ & $g = a_1'h$

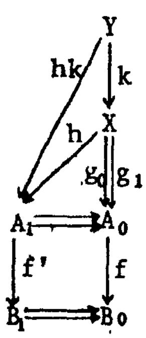
$\Rightarrow f = a_0\alpha h$ & $g = a_1\alpha h \Rightarrow f \equiv g \pmod{A_1}$.

<u>(ii) ⇔ (ii') ⇔ (ii")</u> is obvious by definition.

<u>Note</u>: The relation defined by (i) or (ii) is obviously a preorder relation between equivalence spans on A_0. We will then write $A_1' < A_1$.

b) <u>Properties of equivalence spans & compatible morphism</u>:

(1) The equivalence of morphisms is preserved by composition:



(i) on the right by any morphism k.

i.e.: $g_0 \equiv g_1 \pmod{A_1} \Rightarrow g_0k \equiv g_0k \pmod{A_1}$.

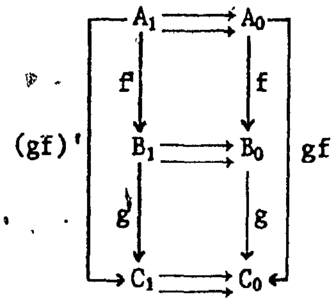
(ii) on the left by any compatible morphism f.

i.e.: [$g_0 \equiv g_1 \pmod{A_1}$ & f compatible with A_1 & B_1] $\Rightarrow$

$\Rightarrow fg_0 \equiv fg_1 \pmod{B_1}$.

(2) The identity morphism 1_{A_0} is compatible with any equivalence span A_1 and itself.

(3) Compatibility is closed under composition.



i.e.: if f is compatible with A_1 & B_1, and g is compatible with B_1 & C_1, then gf is compatible with A_1 & C_1.

(4) For any object A of C, $\Delta A = (A, 1_A, 1_A)$ and (AxA, π, π') where π & π' are the projections, are two trivial equivalence spans on A, and satisfy for any pair of parallel morphisms f & g: $X \longrightarrow A$:

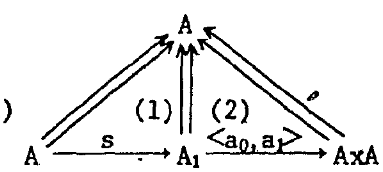
(i) $f \equiv g \pmod{\Delta A} \Leftrightarrow f = g$.

(ii) $f \equiv g \pmod{AxA}$.

Furthermore, if A_1 is any equivalence span on A, we have: $\Delta A \leq A_1 \leq AxA$ as shown in the following diagram where:

$a_0 s = 1_A = a_1 s$, so that (1) commutes

$\pi\langle a_0, a_1\rangle = a_0$ & $\pi'\langle a_0, a_1\rangle = a_1$, so that (2) commutes.



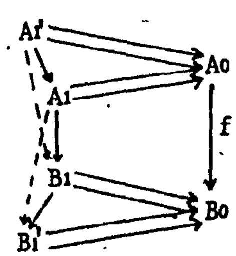
Hence ΔA is minimal & AxA maximal for this preorder.

(5) If A_1 & A_1' are two equivalence spans on A_0 satisfying $A_1 \leq A_1'$ & $A_1' \leq A_1$, then: $f \equiv g \pmod{A_1} \Leftrightarrow f \equiv g \pmod{A_1'}$. Yet A_1' & A_1 need not be isomorphic. We will then call them "equivalent".
But if A_1' & A_1 are relations (i.e.: if $\langle a_0, a_1\rangle$ & $\langle a_0', a_1'\rangle$ are monos), then they are isomorphic. Thus: "$\leq$" , restricted to relations, is an order relation.

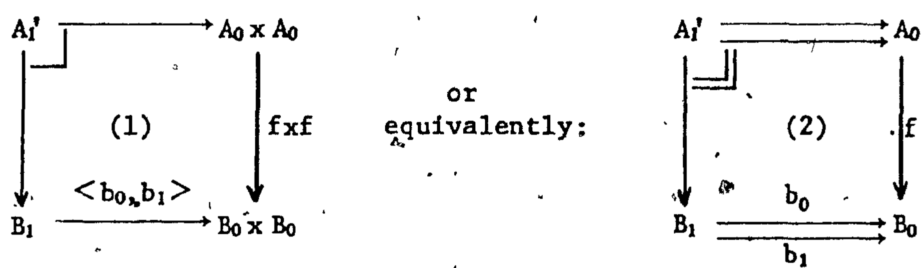
(6) For a moprhism f: $A_0 \longrightarrow B_0$, if f is compatible with A_1 & B_1 then:

(i) f is also compatible with any $A_1' \leq A_1$ & B_1

(ii) f is also compatible with A_1 and any $B_1' \geq B_1$.



(7) So, given $f: A_0 \longrightarrow B_0$, and B_1, the class of equivalence spans A_1^i, on A_0 such that f is compatible with A_1^i & B_1 is a lower class (property 6(i)) that has a minimal element, namely ΔA. It also has a maximal one, the "largest equivalence span that makes f compatible with B_1", which is the pull-back $A_1' = (f x f)^{-1}(B_1)$:



where (2) is a "short-hand" for (1) (" ⅃⅃ " means that the squares with same index commute and that we have the corresponding universal property. We will call it a "<u>joint pull-back</u>").

A_1' is an equivalence span because $(f x f)^{-1}(-)$ is a left-exact functor, and maximal by the pull-back property.

(8) This result can be generalized to a finite number of morphisms $f^i: A_0 \longrightarrow B_0^i$, $i = 1,\ldots,n$ where each B_0^i is equipped with a given equivalence span B_1^i. Then there is an A_1' which is the largest equivalence span that makes each f^i compatible with B_1^i.

For this we just have to take A_1' in the above diagram in which we replace:

$$B_0 \text{ by } B_0^1 \times B_0^2 \times \ldots\ldots \times B_0^n \quad , \quad b_0 \text{ by } b_0^1 \times b_0^2 \times \ldots\ldots \times b_0^n.$$

$$B_1 \text{ by } B_1^1 \times B_1^2 \times \ldots\ldots \times B_1^n \quad , \quad b_1 \text{ by } b_1^1 \times b_1^2 \times \ldots\ldots \times b_1^n$$

$$f \text{ by} < f_1, f_2, \ldots\ldots, f_n >$$

<u>Note</u>: Since $\leq$ is only a preorder, the maximal element defined in (7) & (8) is not unique as such, but A_1 is "canonical" in the sense that it is defined internally by a pull-back, and therefore is <u>unique</u> (up to isomorphism). Furthermore, any two such maximal elements are equivalent as in (5).

(9) If A_1 & B_1 are equivalence spans on A_0 & B_0 respectively, then $A_1 \times B_1$ is an equivalence span on $A_0 \times B_0$ satisfying:

(i) $\langle f,g\rangle \equiv \langle h,k\rangle (\mathrm{mod}\ A_1 \times B_1) \Leftrightarrow f \equiv h(\mathrm{mod}\ A_1)\ \&\ g \equiv k(\mathrm{mod}\ B_1)$

for any pairs: $f,h: X \rightrightarrows A_0$ & $g,k: X \rightrightarrows B_0$.

(ii) $A_1' \times B_1' \leqslant A_1 \times B_1 \Leftrightarrow A_1' \leqslant A_1$ & $B_1' \leqslant B_1$

for any other equivalence spans A_1' & B_1' on A_0 & B_0 respectively.

(10) An equivalence span on A_0 is a special type of span from A_0 to A_0 as defined in [6] and using the (abreviated) notations of [6] and our preorder relation (where $A_1 \leqslant B_1$ is equivalent to: "there exists a morphism of spans from A_1 to B_1"), we have for a span (A_1, a_0, a_1) on A_0:

(R) $\Leftrightarrow (A_0, 1_{A_0}, 1_{A_0}) \leqslant (A_1, a_0, a_1)$

(S) $\Leftrightarrow (A_1, a_1, a_0) \leqslant (A_1, a_0, a_1)$

(T) $\Leftrightarrow (A_2 = A_1 * A_1,\ a_0 a_0', a_1 a_2') \leqslant (A_1, a_0, a_1)$

It is also known ([6], proposition 3.3) that, if C is a regular category, the image A_1^+ of a span on A_0 is a relation and that $^+$ is a functor (in fact a reflector) from spans to relations on A_0, hence a closure operation for our preorder.

So, (R) & (S) are preserved by $^+$, and so is (T) because using lemma 3.5 of [6] we have:

$$A_1 * A_1 \leqslant A_1 \Rightarrow (A_1 * A_1)^+ \leqslant A_1^+ \Rightarrow A_1^+ * A_1^+ \leqslant (A_1^+ * A_1^+)^+ = (A_1 * A_1)^+ \leqslant A_1^+.$$

<u>Thus</u>: If A_1 is an equivalence span on A_0 then A_1^+ is a relation on A_0 satisfying (R), (S) & (T) hence an equivalence relation ([4], I (5.5)).

2) **Definition of the Category $\hat{C}$:**

a) Let us define a first category, called C_1, as follows:

(i) $\text{Obj}(C_1) = \{A \mid A = (A_0, A_1, a_0, a_1)$, where $A_0 \in \text{Obj}(C)$ & (A_1, a_0, a_1) is an equivalence span on $A_0\}$.

(ii) If $A = (A_0, A_1)$ & $B = (B_0, B_1)$ are objects of C, then

$C_1(A, B) = \{f \in C(A_0, B_0) \mid f$ is compatible with A_1 & $B_1\}$

C_1 is obviously a category with the same identities and composition as in C, because identities are compatible and compatibility is preserved by composition (properties 2 & 3).

Note: As above, we often write $A = (A_0, A_1)$, thus omitting to specify a_0 & a_1, or even just A when no confusion is possible.

b) Let us define the category $\hat{C}$ as follows:

(i) $\text{Obj}(\hat{C}) = \text{Obj}(C_1)$

(ii) $\hat{C}(A,B) = C_1(A,B)/_{\equiv_{B_1}}$ where $B = (B_0, B_1)$

i.e.: a morphism of $\hat{C}$ (or $\hat{C}$ – morphism) is an equivalence class modulo B_1 of morphisms of C (they will always be written as such).

Or : $\dot{f}: (A_0, A_1) \longrightarrow (B_0, B_1)$ is a $\hat{C}$ – morphism if and only if

$\dot{f} = \{g: A_0 \longrightarrow B_0 \mid g$ is compatible with A_1 & B_1, & $g \equiv f(\text{mod } B_1)\}$.

$\hat{C}$ is obviously a category because C_1 is one, and because equality of morphisms in $\hat{C}$ is preserved by composition (on either side) by a fixed morphism.

i.e.: if $\dot{f}$, $\dot{g}_0$, $\dot{g}_1$ & $\dot{h}$ are $\hat{C}$-morphisms:

$$A \xrightarrow{\dot{f}} B \underset{\dot{g}_1}{\overset{\dot{g}_0}{\rightrightarrows}} C \xrightarrow{\dot{h}} D,$$

then: $\dot{g}_0 = \dot{g}_1 \Rightarrow \dot{\widehat{g_0 f}} = \dot{\widehat{g_1 f}}$ & $\dot{\widehat{hg_0}} = \dot{\widehat{hg_1}}$ (property 1)

c) <u>$\hat{C}$ contains C as a full subcategory</u>:

(i) For each object A of C, $(A,\Delta A)$ is an object of $\hat{C}$.

(ii) For each C-morphism $f: A \longrightarrow B$,

(*) f is compatible with ΔA & ΔB

(**) $\dot{f} = \{f\}$ because $f \equiv g(\mathrm{mod}\Delta B) \Leftrightarrow f = g$

Therefore $\{f\}$ is a $\hat{C}$-morphism: $(A,\Delta A) \longrightarrow (B,\Delta B)$ and the functor J:

$$\begin{array}{rcl} C & \longrightarrow & \hat{C} \\ A & \longmapsto & (A,\Delta A) \\ f & \longmapsto & \{f\} \end{array}$$

is full and faithful.

d) <u>Conventions & Notations</u>:

(i) We will usually identify objects & morphisms of C with their images in $\hat{C}$.

(ii) We will denote $\hat{C}$-morphisms by $\dot{f}$ or $\dot{g}$ etc...., thus emphasizing that they are equivalence classes of C-morphisms. Then, given $\dot{f}$, f will automatically denote an arbitrary representative of $\dot{f}$.

3) <u>Limits in $\hat{C}$</u>:

a) <u>Preliminary observations</u>:

<u>Lemma 3.1</u>: If (C_0,C_1) and (A_0,A_1) are objects of $\hat{C}$ and $f: C_0 \longrightarrow A_0$ a C-morphism, then T.F.A.E.:

(i) f coequalizes c_0 & c_1 modulo A_1.

(ii) f is compatible with C_1 & A_1.

<u>Proof</u>: It obviously follows from the diagram:

$$\begin{array}{ccc} C_1 & \overset{c_0}{\underset{c_1}{\rightrightarrows}} & C_0 \\ {\scriptstyle h=f'}\downarrow & & \downarrow{\scriptstyle f} \\ A_1 & \rightrightarrows & A_0 \end{array}$$

<u>Corollary 3.2:</u> If (C_0, C_1) is an object of $\hat{C}$, and C_1' an equivalence span on C_0 s.t. $C_1' \leq C_1$, then (C_0, C_1) is the coequalizer of $\dot{c}_0$ & $\dot{c}_1$:

$$C_1 = (C_1, \Delta C_1) \underset{\dot{c}_1}{\overset{\dot{c}_0}{\rightrightarrows}} (C_0, C_1') \xrightarrow{\dot{1}_{C_0}} (C_0, C_1)$$

$$(C_0, C_1') \xrightarrow{\dot{f}} (A_0, A_1), \qquad (C_0, C_1) \overset{\ddot{f}}{\dashrightarrow} (A_0, A_1) \quad (1)$$

<u>Proof:</u> (i) Since $C_1' \leq C_1$, 1_{C_0} is compatible, therefore $\dot{1}_{C_0}$ is a $\hat{C}$-morphism .

(ii) Obviously $c_0 \equiv c_1 \pmod{C_1}$, hence $\dot{1}_{C_0}$ coequalizes $\dot{c}_0$ & $\dot{c}_1$.

(iii) If $\dot{f}$: $(C_0, C_1') \longrightarrow (A_0, A_1)$ coequalizes $\dot{c}_0$ & $\dot{c}_1$ (in $\hat{C}$) , then f: $C_0 \longrightarrow A_0$ coequalizes c_0 & $c_1 \pmod{A_1}$ (in C) .

Therefore f is compatible with C_1 & A_1 , hence defines a $\hat{C}$-morphism $\ddot{f}$: $(C_0, C_1) \dashrightarrow (A_0, A_1)$, which makes (1) obviously commutative.

<u>Note</u> that the two morphisms $\dot{f}$ and $\ddot{f}$ in (1) are not actually equal, but satisfy $\ddot{f} \subset \dot{f}$.

b) <u>External caracterisation of limits in $\hat{C}$:</u>

<u>Proposition 3,3:</u> If $(A^i)_{i \in I}$ is a Diagram of type I in $\hat{C}$ (where $A^i = (A_0^i, A_1^i)$ for each i) , and $L = (L_0, L_1)$ an object of $\hat{C}$, <u>then (i) ⇔ (ii) & (ii) ⇒ (iii)</u> where (i),(ii) & (iii) are defined as follows:

(i) $L = \lim\limits_{i \in I} A^i$ with projections $\dot{p}_i$: $L \longrightarrow A^i$

(ii) (a) $(L_0 \xrightarrow{p_i} A_0^i)_{i \in I}$ is an A_1-cone in C, or cone modulo $A_1 = (A_1^i)_{i \in I}$

i.e.: $\forall \alpha: i \longrightarrow j$ in I , $p_j \equiv A^\alpha p_i \pmod{A_1^j}$, so that:

$$\begin{array}{ccc} L_0 & \xrightarrow{p^i} & A_0^i \\ & \searrow^{p_j} \ \equiv_{A_1^j} & \downarrow A^\alpha \\ & & A_0^j \end{array}$$

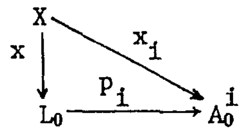
commutes modulo A_1^j .

(b) p_i is compatible with L_1 & A_1^i for each i, and for any

A_1-cone $(X \xrightarrow{x_i} A_0^i)_{i\in I}$ in C,

there is an L_1-unique morphism $x\colon X \longrightarrow L_0$

s.t.: $x_i \equiv p_i x \pmod{A_1^i}$ for each $i\in I$:



i.e.: if $x'\colon X \longrightarrow L_0$ also verifies $x_i \equiv p_i x' \pmod{A_1^i}\ \forall\, i\in I$,

then $x \equiv x' \pmod{L_1}$.

(iii)(a) L_0 is the vertex of an A_1-cone (as in (ii)(a)) .

(b) for any A_1-cone $(X \xrightarrow{x_i} A_0^i)_{i\in I}$, there is at least a morphism

$x\colon X \longrightarrow L_0$ s.t. $x_i \equiv p_i x \pmod{A_1^i}$ for each i (as in

(ii)(b) , except for L_1- uniqueness) .

(c) L_1 is a largest equivalence span on L_0 that makes the

projections p_i compatible.

<u>Furthermore</u>: if $(A^i)_{i\in I}$ is a <u>finite</u> diagram, then we <u>also</u>

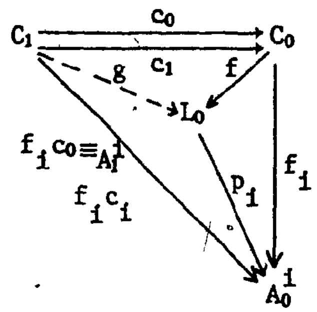
have <u>(iii) ⇒ (ii)</u> .

<u>Proof</u>: <u>(i) ⇒ (ii)</u> is obvious because (ii) is the expression of (i) in C, in the particular case of cones having their vertex in C.

<u>(ii) ⇒ (i)</u>: We want to prove that this restriction can in fact be done without loss of generality.

Assuming (ii), if $[\,(C_0,C_1) \xrightarrow{f_i} (A_0^i,A_1^i)\,]_{i\in I}$ is a cone in $\hat{C}$, then $[\,(C_0 \xrightarrow{f_i} A_0^i)\,]_{i\in I}$ is an A_1-cone in C. Therefore by (ii)(b), there is an L_1-unique $f\colon C_0 \longrightarrow L_0$ s.t. $f_i \equiv p_i f \pmod{A_1^i}$ for each i in I.

Now since f_i is compatible with C_1 & A_1^i , f_i coequalizes c_0 & c_1 modulo A_1^i .



So $(f_i c_0)_{i\in I}$ & $(f_i c_1)_{i\in I}$ are A_1-equivalent A_1-cones.

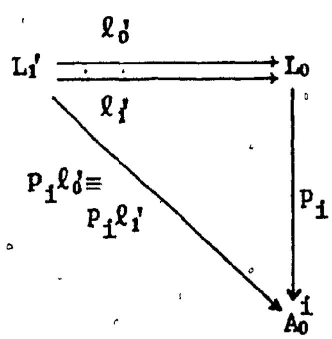
Hence, there exists an L_1-unique g, s.t.: $p_i g \equiv f_i c_0 \equiv f_i c_1 \pmod{A_1^i}$ but since $f_i \equiv p_i f \pmod{A_1^i}$, we have:

$$p_i g \equiv p_i f c_0 \equiv p_i f c_1 \pmod{A_1^i}.$$

- But g is L_1-unique. So $g \equiv fc_0 \equiv fc_1 \pmod{L_1}$.
- Therefore f coequalizes c_0 & c_1 modulo L_1.
- Therefore f is compatible with C_1 & L_1.

Thus f is a $\hat{C}$-morphism: $(C_0, C_1) \longrightarrow (L_0, L_1)$ such that $p_i f = f_i$ for each $i \in I$, and f is unique because f is L_1-unique.

(ii) ⇒ (iii): Assuming (ii), if (L_1', ℓ_0', ℓ_1') is another equivalence span on L_0 that makes each p_i compatible, then p_i coequalizes ℓ_0' & ℓ_1' modulo L_1 (Lemma 3.1):



Therefore as above, $(p_i \ell_0')_{i\in I}$ & $(p_i \ell_1')_{i\in I}$ are A_1-equivalent A_1-cones, hence there is an L_1-unique $h: L_1' \longrightarrow L_0$ s.t. $p_i \ell_0' \equiv p_i h \equiv p_i \ell_1' \pmod{A_1^i}$.

So by L_1-uniqueness we have $\ell_0' \equiv \ell_1' \pmod{L_1}$, which means that $L_1' \le L_1$ as desired.

Thus L_1 is a largest equivalence span on L_0 that makes the projections p_i compatible.

(iii) ⇒ (ii): Given any A_1-cone $(X \xrightarrow{x_i} A_0^i)_{i\in I}$, we know that there is at least one morphism $x: X \longrightarrow L_0$ s.t. $x_i \equiv p_i x \pmod{A_1^i}$ for each i.

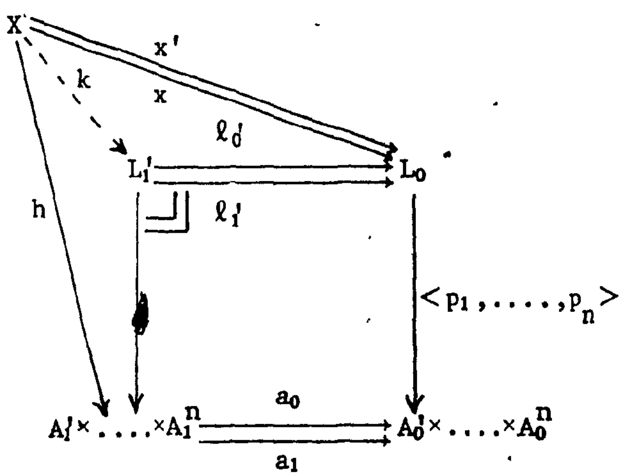
All we need to prove is that x is in fact L_1-unique.

But if I is finite $(Obj(I)=\{1,2,\ldots,n\})$, we can construct the canonical largest equivalence span L_1' making the projections compatible. So L_1' is equivalent to L_1 ($L_1' < L_1$ for the same reason as in (ii) $\Rightarrow$ (iii)). Hence, we only need to prove that x is L_1'-unique because for any $x': X \longrightarrow L_0$, $x \equiv x' \pmod{L_1} \Leftrightarrow x \equiv x' \pmod{L_1'}$.

So let x' verify $p_i x' \equiv x_i \equiv p_i x \pmod{A_1^i}$ for each i.

i.e.: $\forall i \in I, \exists h_i: X \longrightarrow A_1^i$, s.t. $p_i x' = a_0^i h_i$ & $p_i x = a_1^i h_i$

hence taking $h = \langle h_1, \ldots, h_n \rangle$, we have:



$\langle p_1, \ldots, p_n \rangle x' = a_0 h$

& $\langle p_1, \ldots, p_n \rangle x = a_1 h$

where $a_0 = a_0^1 \times \ldots \times a_0^n$

& $a_1 = a_1^1 \times \ldots \times a_1^n$

Hence there exists a (unique) $k: X \longrightarrow L_1$

s.t.: $x' = \ell_0' k$ & $x = \ell_1' k$

i.e.: $x \equiv x' \pmod{L_1}$.

<u>Hence</u> x is L_1'-unique therefore L_1-unique with the property that $p_i x \equiv x_i \pmod{A_1^i}$ for each i.

c) <u>Construction of finite limits in $\hat{C}$:</u>

(i) <u>Terminal object</u>: If we look at condition (iii) of proposition 3.3 (which is equivalent to (i) & (ii) in the finite case), our terminal object $T = (T_0, T_1)$ must satisfy:

(a) nothing particular.

(b) T_0 is non-empty, i.e.: $C(X, T_0) \neq \phi$ for every object X in C.

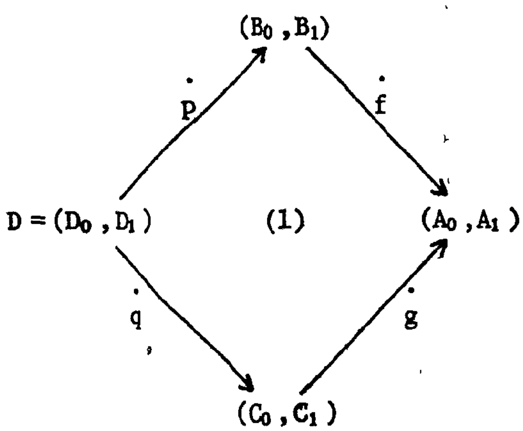
(c) T_1 is the largest equivalence span on T_0 i.e.: $T_1 = T_0 \times T_0$

This will be satisfied if for instance $T_0 = 1$ (terminal object of C) and $T_1 = 1 \times 1 \cong 1$.

<u>Hence</u>: $T = (1,1)$ is a terminal object in $\hat{C}$.

<u>Note</u>: T can be defined in many different ways (one for each non-empty object of C), which give rise to many different isomorphic copies of T in $\hat{C}$. But (1,1) is "canonical" because it is entirely defined in terms of limits in C.

(ii) <u>Pull-backs</u>: Given two $\hat{C}$-morphisms $\dot{f}$ & $\dot{g}$ with same codomains, their pull-back $D = (D_0, D_1)$ must satisfy (proposition 3.3 (iii)):



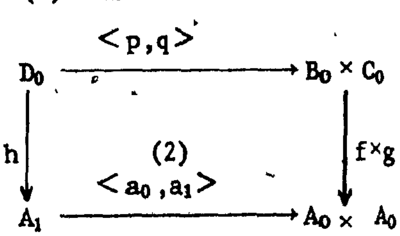
(a) $fp \equiv gq \pmod{A_1}$

i.e.: $fp = a_0 h$ & $gq = a_1 h$ for some $h: D_0 \longrightarrow A_1$

i.e.: $\langle fp, gq \rangle = \langle a_0, a_1 \rangle h$

i.e.: $(f \times g) \langle p, q \rangle = \langle a_0, a_1 \rangle h$

i.e.: (2) is commutative:



(b) For any pair of C-morphisms x & y: $X \xrightarrow{x} B_0$, $X \xrightarrow{y} C_0$

s.t.: $fx \equiv gy \pmod{A_1}$, i.e.: $(f \times g) \langle x, y \rangle = \langle a_0, a_1 \rangle k$ for some $k: X \longrightarrow A_1$, there exists a C-morphism $\ell: X \longrightarrow D_0$ s.t. $\langle p, q \rangle \ell \equiv \langle x, y \rangle \pmod{B_1 \times C_1}$

(a) & (b) will be satisfied if we choose D_0, s.t.:

(2) is a pull-back (then ℓ is unique satisfying $<p,q>\ell=<x,y>$)

(c) D_1 is a largest equivalence span making the projections compatible, for instance the canonical one:

$$D_1 = (<p,q>\times<p,q>)^{-1}(B_1\times C_1) .$$

Hence: (D_0,D_1) is the pull-back of f & g in $\hat{C}$.

Note: Here also there are possibly many different objects like D_0 satisfying (a) & (b) and equivalence spans like D_1 satisfying (c), giving rise to different isomorphic copies of the desired pull-back, but like the terminal object, (D_0,D_1) is "canonical" for similar reasons.

(iii) Exactness properties of the inclusion functor

So $\hat{C}$ is a category with finite limits.

It is obvious that if $(A^i)_{i\in I}$ is a diagram in C, then:

$$L = \lim_{i\in I} A^i \text{ in } C \Leftrightarrow (L,\Delta L) = \lim_{i\in I} (A^i,\Delta A^i) \text{ in } \hat{C}.$$

This is an immediate consequence of proposition 3.3 if we observe that:

(i) equivalence modulo ΔA^i in $\hat{C}$ coincides with equality in C,

(ii) ΔA-cones in $\hat{C}$ with vertex in C coincide with cones in C.

(iii) ΔL-uniqueness in $\hat{C}$ coincides with uniqueness in C.

Conclusion: $\hat{C}$ is finitely complete and the inclusion functor $C \hookrightarrow \hat{C}$ preserves & reflects all limits that exist in C.

4) The regular structure of $\hat{C}$:

a) Monos in $\hat{C}$:

Proposition 4.1 (Characterization of Monos): If $\dot{f}: (A_0, A_1) \longrightarrow (B_0, B_1)$ is a morphism of $\hat{C}$ then T.F.A.E.:

(i) $\dot{f}$ is a mono in $\hat{C}$.

(ii) for every pair of C-morphisms $x, y: X \longrightarrow A_0$,

$$fx \equiv fy \pmod{B_1} \Rightarrow x \equiv y \pmod{A_1}.$$

(iii) A_1 is a largest equivalence span making f compatible,

i.e.: A_1 is equivalent to A_1', the canonical largest equivalence span making f compatible (in the sense that $A_1 < A_1'$ & $A_1' < A_1$).

Note: in (ii) & (iii) f is any fixed, but arbitrarily chosen representative of $\dot{f}$, since if $f \equiv g \pmod{B_1}$, then obviously:

(ii) for f $\Leftrightarrow$ (ii) for g

Therefore in order to prove that $\dot{f}$ is a mono we will only need to prove (ii) or (iii) for any one representative f of $\dot{f}$.

Proof: (i) $\Rightarrow$ (ii): is trivial because (ii) is the "external" expression of (i) in the case of two morphisms x & y having their domains in C.

(ii) $\Rightarrow$ (i): is also obvious because in the definition of equality in $\hat{C}$, only the equivalence span of the codomain is taken into consideration:

for any pair of $\hat{C}$-morphisms $\dot{g}_1$ & $\dot{g}_2$:

$$(C_0, C_1) \underset{\dot{g}_2}{\overset{\dot{g}_1}{\rightrightarrows}} (A_0, A_1) \xrightarrow{\quad \dot{f} \quad} (B_0, B_1)$$

$$\ddot{fg_1} = \ddot{fg_2} \Rightarrow fg_1 \equiv fg_2 \pmod{B_1} \Rightarrow g_1 \equiv g_2 \pmod{A_1} \Rightarrow \dot{g}_1 = \dot{g}_2 .$$

(ii) $\Rightarrow$ (iii): (*) Since f is compatible with A_1 & B_1, we must have: $A_1 < A_1'$ (by definition of A_1').

(**) Since f is compatible with A_1' & B_1, f coequalizes a_0' & a_1' modulo B_1 (lemma 3.1):

$$A_1' \underset{a_1'}{\overset{a_0'}{\rightrightarrows}} A_0 \xrightarrow{\ f\ } B_0$$

hence by (ii) with $X = A_1'$, $x = a_0'$ & $y = a_1'$ we have:

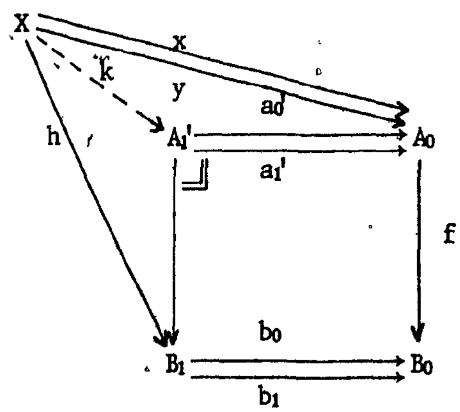
$$fa_0' \equiv fa_1' \pmod{B_1} \Rightarrow a_0' \equiv a_1' \pmod{A_1} \Rightarrow A_1' < A_1 .$$

(iii) ____ (ii): It suffices to prove that A_1' satisfies (ii) because, if A_1 is equivalent to A_1' and A_1' satisfies (ii) then:

$$fx \equiv fy \pmod{B_1} \Rightarrow x \equiv y \pmod{A_1'} \Rightarrow x \equiv y \pmod{A_1}$$

so we assume that $fx \equiv fy \pmod{B_1}$

i.e.: $\exists h$: $X \longrightarrow B_1$ s.t. $fx = b_0 h$ & $fy = b_1 h$ as in the following diagram:



then by the joint pull-back property, there exists a (unique) k s.t.: $x = a_0' k$ & $y = a_1' k$ i.e.: $x \equiv y \pmod{A_1'}$

<u>Corollary 4.2:</u> Given two monos:

$$(B_0, B_1) \xrightarrow{m} (A_0, A_1) \xleftarrow{n} (C_0, C_1),$$

$$\dot{m} < \dot{n} \Leftrightarrow \exists t: B_0 \longrightarrow C_0 \text{ s.t. } m \equiv nt \pmod{A_1}$$

<u>Proof:</u> " ⇒ " is trivial because:

$$\dot{m} < \dot{n} \Leftrightarrow \exists t: B_0 \longrightarrow C_0 \text{ s.t. } t \text{ is compatible with } B_1 \,\&\, C_1 \,\&\, m \equiv nt \pmod{A_1}$$

" ⇐ ": we only need to prove that t is compatible with B_1 & C_1.

So let x & y be $\mathcal{C}$-morphisms: $X \longrightarrow B_0$.

then: $x \equiv y \pmod{B_1}$ ⇒ $mx \equiv my \pmod{A_1}$ because m is compatible

⇒ $ntx \equiv nty \pmod{A_1}$ because $m \equiv nt \pmod{A_1}$

⇒ $tx \equiv ty \pmod{C_1}$ because n is a mono .

Hence t is compatible (proposition 1 - (ii)) .

<u>Consequence</u>: Given an object (B_0, B_1) of C, every C-morphism $f: A_0 \longrightarrow B_0$ can be made into a $\hat{C}$-mono $\dot{f}: (A_0, A_1) \rightarrowtail (B_0, B_1)$ by choosing as A_1, the canonical largest equivalence span making f compatible. Hence f defines a (unique) subobject $(A_0, A_1) \overset{\dot{f}}{\rightarrowtail} (B_0, B_1)$ in $\hat{C}$.

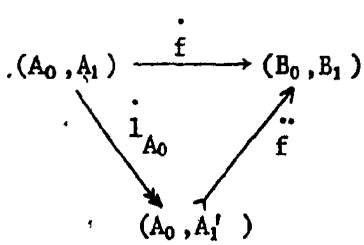
<u>Conversely</u>: Every subobject $(A_0, A_1) \overset{m}{\rightarrowtail} (B_0, B_1)$ in $\hat{C}$ determines a unique B_1-equivalence class of C-morphisms into B_0, where two morphisms are said to be B_1-equivalent if each one factors through the other modulo B_1.

<u>Furthermore</u>: This one-to-one correspondence preserves the order relation "factoring through".

b) <u>Factorization in $\hat{C}$</u>:

Let $\dot{f}: (A_0, A_1) \overset{f}{\longrightarrow} (B_0, B_1)$ be a $\hat{C}$-morphism.

If A_1' is the canonical largest equivalence span on A_0 that makes f compatible, then, we have a trivial factorization of $\dot{f}$:



where $\ddot{f}: (A_0, A_1') \longrightarrow (B_0, B_1)$ is a mono (proposition 5) and $\dot{1}_{A_0}: (A_0, A_1) \longrightarrow (A_0, A_1')$ a regular epi (corollary 3.2)

<u>Note</u>: If we choose another representative g of $\dot{f}$, then we get another equivalence span A_1'', but A_1' & A_1'' are equivalent because if $f \equiv g \pmod{B_1}$, then f is compatible with A_1'' & B_1 if and only if g is, therefore any largest equivalence span making g compatible is also a largest one making f compatible. In that case, (A_0, A_1') & (A_0, A_1'') are isomorphic.

So, this factorization does not depend (up to isomorphism) on the choice of representatives for $\dot{f}$.

We claim that this factorization is in fact unique but before we prove uniqueness, we shall see some equivalent characterization of regular epis.

c) <u>Characterization of regular epis in $\hat{C}$</u>

<u>Proposition 4.3</u>: for a $\hat{C}$-morphism $\dot{f}: (A_0, A_1) \longrightarrow (B_0, B_1)$

T.F.A.E.: (i) $\dot{f}$ is a regular epi

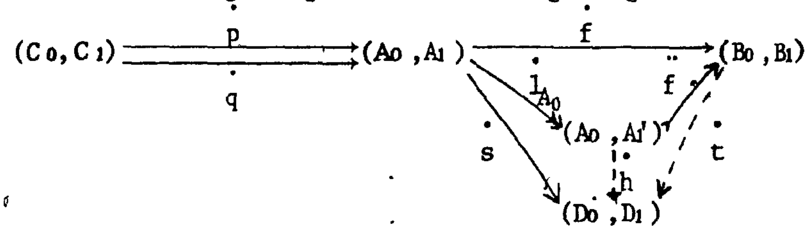
(ii) for any C-morphism $y: X \longrightarrow B_0$, there is an $x: X \longrightarrow A_0$ s.t. $fx \equiv y \pmod{B_1}$

(iii) there exist a C-morphism $g: B_0 \longrightarrow A_0$ s.t. $fg \equiv 1_{B_0} \pmod{B_1}$

(iv) $\ddot{f}: (A_0, A_1') \longrightarrow (B_0, B_1)$ is an isomorphism i.e.: $\dot{f}$ is equivalent to $\dot{1}_{A_0} : (A_0, A_1) \longrightarrow (A_0, A_1')$

<u>Proof</u>: <u>(i) ⇒ (iv)</u>: Let $\dot{f}$ be a regular epi, coequalizer of $\dot{p}$ & $\dot{q}$.

After factoring we get the following diagram:



We claim that (A_0, A_1') is also a coequalizer of $\dot{p}$ & $\dot{q}$, and therefore, $\ddot{f}: (A_0, A_1') \longrightarrow (B_0, B_1)$ is an isomorphism.

Indeed, since $\ddot{f}: (A_0, A_1') \longrightarrow (B_0, B_1)$ is a mono we have:

$\ddot{f}\dot{1}_{A_0}\dot{p} = \dot{f}\dot{p} = \dot{f}\dot{q} = \ddot{f}\dot{1}_{A_0}\dot{q} \Rightarrow \dot{1}_{A_0}\dot{p} = \dot{1}_{A_0}\dot{q}$. So $\dot{1}_{A_0}$ coequalizes $\dot{p}$ & $\dot{q}$. And if $\dot{s}: (A_0, A_1) \longrightarrow (D_0, D_1)$ coequalizes $\dot{p}$ & $\dot{q}$ there exists a unique $\dot{t}: (B_0, B_1) \longrightarrow (D_0, D_1)$ s.t. $\dot{t}\dot{f} = \dot{s}$ (outer triangle)

<u>Thus</u>: $\dot{h} = \dot{t}\ddot{f}: (A_0, A_1') \longrightarrow (D_0, D_1)$ is the unique morphism s.t. $\dot{s} = \dot{h}\dot{1}_{A_0}$

<u>(iv) ⇒ (i)</u> is obvious because we have proved that $\dot{1}_{A_0}: (A_0, A_1) \longrightarrow (A_0, A_1')$ is a regular epi. (Corollary 3.2)

(iv) $\Rightarrow$ (iii) is also obvious by definition of $\ddot{f}: (A_0, A_1') \longrightarrow (B_0, B_1)$ being an isomorphism.

(iii) $\Rightarrow$ (iv) assume that g satisfies $fg \equiv 1_{B_0} \pmod{B_1}$

Claim: g is compatible with B_1 & A_1'.

Indeed, if x & y are $\mathcal{C}$-morphisms: $A_0 \underset{f}{\overset{g}{\rightleftarrows}} B_0 \underset{y}{\overset{x}{\leftleftarrows}} X$

then $x \equiv y \pmod{B_1} \Rightarrow fgx \equiv fgy \pmod{B_1}$ (because $fg \equiv 1_{B_0} \pmod{B_1}$) $\Rightarrow$

$\Rightarrow$ $gx \equiv gy \pmod{A_1'}$ because $\ddot{f}$ is a mono: $(A_0, A_1') \longrightarrow (B_0, B_1)$

hence g is compatible with B_1 & A_1', by proposition 1.2

Thus: we have the following diagram in $\hat{\mathcal{C}}$:

$$(A_0, A_1') \underset{\ddot{f}}{\overset{\dot{g}}{\rightleftarrows}} (B_0, B_1) = B$$

with $\ddot{f}\dot{g} = 1_B$ and $\ddot{f}$ is a mono

so $\ddot{f}\dot{g}\ddot{f} = 1_B\ddot{f} = \ddot{f}1_{(A_0, A_1')}$

hence $\dot{g}\ddot{f} = 1_{(A_0, A_1')}$

Thus $\ddot{f}$ & $\dot{g}$ are inverse of each other, hence $\ddot{f}$ is an isomorphism as claimed.

Finally, (ii) $\Leftrightarrow$ (iii) is obvious because:

(iii) is a particular case of (ii) with $X = B_0$ & $y = 1_{B_0}$, and then g is the $x: B_0 \longrightarrow A_0$ of (ii).

and given $y: X \longrightarrow B_0$ then $x = gy$ (where g is given in (iii)) satisfies $fx = fgy \equiv 1_{B_0}y = y \pmod{B_1}$ in (ii).

Conclusion and remarks: We have successively proved that:

(iv) $\Leftrightarrow$ (i), (iv) $\Leftrightarrow$ (iii) and (iii) $\Leftrightarrow$ (ii), hence the equivalence of all these possible characterizations of regular epis.

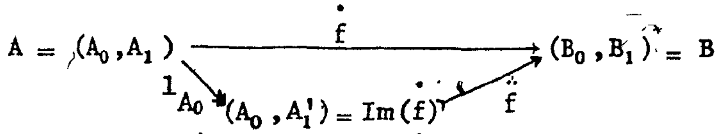
Note that (ii) expresses that every object of $\mathcal{C}$ is projective in $\hat{\mathcal{C}}$, or that regular epis are surjections with respect to $\mathcal{C}$-indexed elements.

Also, (iv) expresses that every regular epi is equivalent to one of the form $\dot{1}_{A_0}: (A_0, A_1) \longrightarrow (A_0, A_1')$.

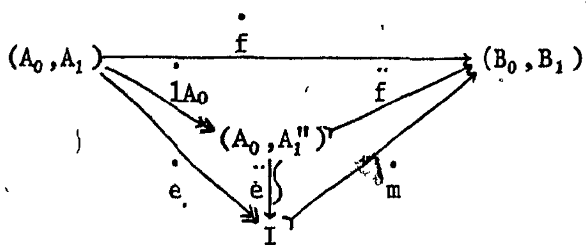
d) Unique factorization and stability:

Proposition 4.4: Every $\hat{C}$-morphism $\dot{f}: (A_0, A_1) \longrightarrow (B_0, B_1)$ has a unique (up to isomorphism) mono-regular-epi factorization.

Proof: As we have seen, $\dot{f}$ can be factored as:



Now, if $A \xrightarrow{\dot{e}} I \xrightarrow{\dot{m}} B$ is another mono-regular-epi factorization of $\dot{f}$, then by (iv) of proposition 4.3, $\dot{e}$ is equivalent to $\dot{1}_{A_0}$, in the sense that there is an isomorphism $(A_0, A_1'') \cong I$ (namely $\ddot{e}$) for some equivalence span A_1'', such that $\dot{e} = \ddot{e}\dot{1}_{A_0}$ as in the following diagram (lower-left triangle):



hence $\dot{m}\ddot{e}: (A_0, A_1'') \longrightarrow (B_0, B_1)$ is a mono ;

hence A_0'' is a largest equivalence span making me compatible .

But $\dot{f} = \dot{m}\dot{e} = \dot{m}\ddot{e}\dot{1}_{A_0} : A \longrightarrow (A_0, A_1'') \longrightarrow I \longrightarrow B$

So $f \equiv me \pmod{B_1}$.

Therefore A_0'' is a largest equivalence span making f compatible .

Hence A_1'' is equivalent to A_1' , i.e. $\dot{1}_{A_0}: (A_0, A_1') \longrightarrow (A_0, A_1'')$ is an isomorphism.

We thus have: $I \cong (A_0, A_1'') \cong (A_0, A_1')$ which shows uniqueness (up to isomorphism)

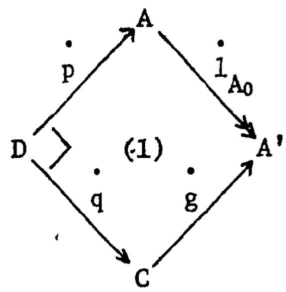
Proposition 4.5: Regular epis are stable.

Proof: Since by (iv) of proposition 4.3, regular epis are equivalent to one of the form $1_{A_0}: (A_0, A_1) \longrightarrow (A_0, A_1')$ where $A_1 \lessdot A_1'$, we only need to prove that the inverse images of those are regular epis.

So consider $\dot{1}_{A_0}: A = (A_0, A_1) \longrightarrow (A_0, A_1') = A'$ (where $A_1 \leq A_1'$)

and let $\dot{g}: C = (C_0, C_1) \longrightarrow (A_0, A_1')$ be an arbitrary $\hat{C}$-morphism.

Let us construct the pull-back $D = (D_0, D_1)$ of $\dot{1}_{A_0}$ & $\dot{g}$:

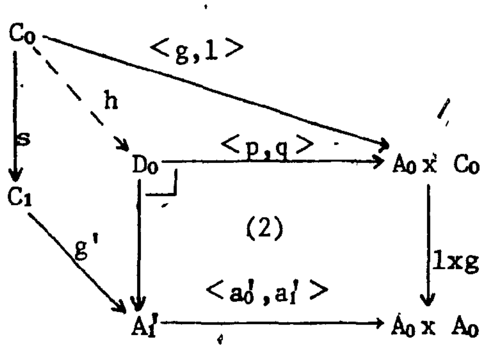


So we want to prove that:

$\dot{q} = \dot{g}^{-1}(\dot{1}_{A_0})$ is a regular epi.

From section 3-c we know that D_0 is constructed as the pull-back (2) below and q is the second component of the first projection: $q = \pi' \circ \langle p, q \rangle$

Now if we look at the two C-morphisms $\langle g, 1 \rangle$ and g', we have:



$c_1 s = c_0 s = 1_{C_0}$ (proposition 1.1) and

$a_0' g' = g c_0$ & $a_1' g' = g c_1$ (g is compatible)

Hence $\langle a_0', a_1' \rangle g' s = \langle a_0' g' s, a_1' g' s \rangle = \langle g c_0 s, g c_1 s \rangle$

$= \langle g, g \rangle = (1 \times g) \langle g, 1_{C_0} \rangle$

Hence there exists a (unique) h such that:

$\langle p, q \rangle h = \langle g, 1 \rangle$

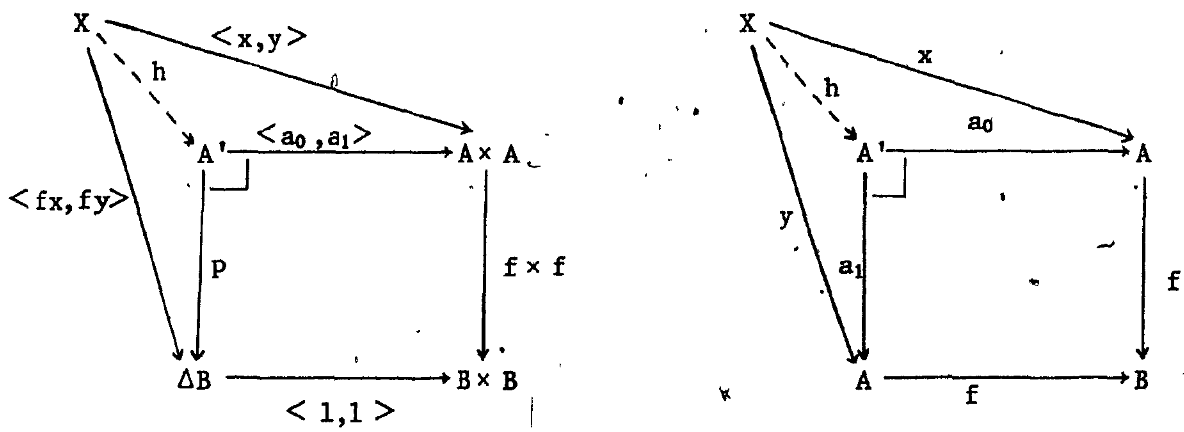
in particular we have $qh = 1_{C_0}$ hence $qh \equiv 1_{C_0}$ (mod C_1).

Therefore, by (iii) of proposition 4.3, $\dot{q}$ is a regular epimorphism.

<u>Conclusion:</u> $\hat{C}$ is a finitely complete category with unique factorizations and stable regular epis, <u>hence $\hat{C}$ is a regular category</u>.

e) <u>Canonical Images of $\mathcal{C}$-morphisms:</u>

If f is a $\mathcal{C}$-morphism: $A \longrightarrow B$, f has an image in $\hat{\mathcal{C}}$ of the form (A,A'), where A' can be chosen to be $A' = (f \times f)^{-1}(\Delta B)$ which is an equivalence relation on A, and in fact the equivalence relation defined by f (i.e. the kernel pair of f in $\mathcal{C}$), as shown in the following equivalent diagrams:



<u>Definition 4.6:</u> If $f: A \longrightarrow B$ is a $\mathcal{C}$-morphism, and A' its kernel pair, then (A,A') is called the <u>Canonical Image</u> of f (in $\hat{\mathcal{C}}$)

5) Saturated C-morphisms and subobjects in $\hat{C}$:

a) Saturated C-morphisms:

Proposition 5.1: Given (A_0, A_1) an object of $\hat{C}$ and a C-morphism $\mu : S \longrightarrow A_0$

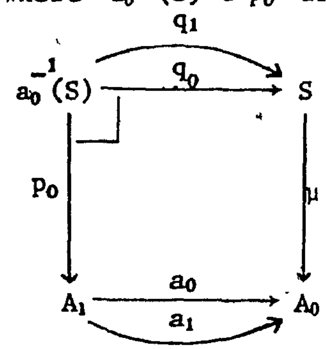
T.F.A.E.:

(i) For every pair of C-morphisms x & y: $X \rightrightarrows A_0$

$[\, x \equiv y (\mathrm{mod}\ A_1)$ & x factors through $S\,] \Rightarrow$ y factors through S.

(ii) $a_1 p_0 : a_0^{-1}(S) \longrightarrow A_1 \longrightarrow A_0$ factors through S,

where $a_0^{-1}(S)$ & p_0 are defined by the pull-back:



i.e.: if the "straight" square is defined to be a pull-back then there exists a q_1 such that the "rounded" square is commutative.

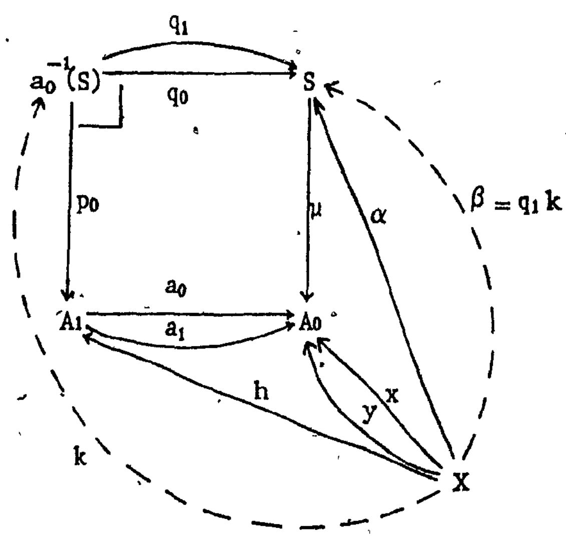
Proof (i) ⇒ (ii): Take $X = a_0^{-1}(S)$, $x = a_0 p_0$ & $y = a_1 p_0$, then $x \equiv y$ (mod A_1) by definition, x factors through S because $a_0 p_0 = \mu q_0$ (commutativity of the "straight square"). Therefore y should factor through S also:

i.e. $y = a_1 p_0 = \mu q_1$ for some $q_1 : a_0^{-1}(S) \longrightarrow S$.

(ii) ⇒ (i): Let x & y: $X \longrightarrow A_0$ satisfy $x \equiv y$ (mod A_1)

i.e.: $x = a_0 h$ & $y = a_1 h$ for some $h: X \longrightarrow A_1$, and x factor through S

i.e.: $x = \mu\alpha$ for some α as in the following diagram, in which all the "straight" and "rounded" squares and triangles commute:



So looking at the "straight" triangles, we have:

$x = \mu{\cdot}\alpha$ (upper one)

$x = a_0 h$ (lower one)

Hence there exists a (unique) $k: X \longrightarrow a_0^{-1}(S)$

s.t. $h = p_0 k$ (& $\alpha = q_0 k$).

So, if we now look at the "rounded" square and triangle we have:

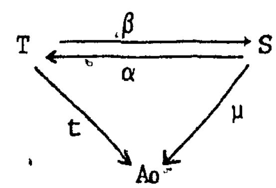
$y = a_1 h = a_1 p_0 k = \mu q_1 k$, hence taking $\beta = q_1 k$, we have $y = \mu\beta$

Hence y factors through S also.

Definition 5.2: Such a C-morphism (S,μ) will be called saturated for A_1 or A_1- saturated.

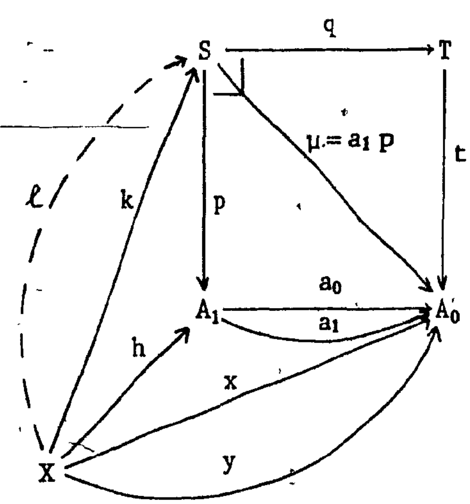
Proposition 5.3: Given (A_0, A_1) , every C-morphism $t: T \longrightarrow A_0$ is A_1-equivalent to an A_1-saturated one $\mu: S \longrightarrow A_0$,

i.e.: there exists a pair of morphisms α & β :



s.t.: $t\alpha \equiv \mu$ & $\mu\beta \equiv t$ (mod A_1)

Proof: Looking at (ii) of proposition 5.1, it is natural to take $S = a_0^{-1}(T)$ and $\mu = a_1 p$ where p is the projection $S \xrightarrow{p} A_1$ as in the diagram below:



(i) (S,μ) is saturated

Assume that x & y satisfy $x \equiv y$ (mod A_1) and x factors through S .

i.e.: $\exists k$ s.t. $x = \mu k = a_1 pk$.

We first claim that $x \equiv tqk$ (mod A_1).

Indeed , $x = a_1 pk$ & $tqk = a_0 pk$ because the square is a pull-back .

Therefore , by transitivity we have:

$x \equiv y$ & $x \equiv tqk$ (mod A_1) hence $y \equiv tqk$ (mod A_1)

i.e.: $\exists h: X \longrightarrow A_1$ s.t. $y = a_1 h$ & $tqk = a_0 h$

Thus: since qk & h satisfy $t(qk) = a_0 h$, there exists a (unique) $\ell : X \longrightarrow S$ s.t. $h = p\ell$ & $qk = q\ell$.

So $y = a_1 h = a_1 p \ell = \mu\ell$

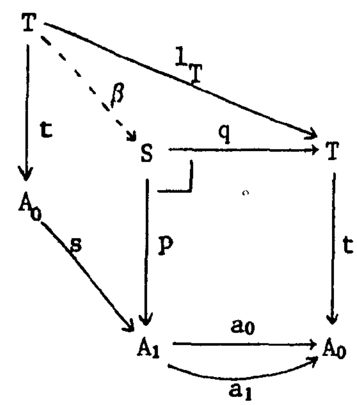
Hence y factors through S .

(S,μ) is thus saturated .

(ii) (S,μ) & (T,t) are A_1-equivalent

Taking $\alpha = q : S \longrightarrow T$ we have $\mu = a_1 p$ & $tq = a_0 p$ by definition of (S,μ) ; hence $\mu \equiv tq = t\alpha \pmod{A_1}$ as required .

For $\beta : T \longrightarrow S$, observe that if s is the reflexivity morphism of A_1 (i.e.: $a_0 s = a_1 s = 1_{A_0}$) , then we have:



$a_0 st = 1_{A_0} t = t1_T$

hence by the pull-back property,

there exists a (unique) β

s.t.: $1_T = q\beta$ & $st = p\beta$

So we have $t = a_0 st$ & $\mu\beta = a_1 p \beta = a_1 st$ $(= 1_{A_0} t = t)$

Hence $t \equiv \mu\beta \pmod{A_1}$

Conclusion: (S,μ) is A_1- saturated and A_1-equivalent to (T,t) as required.

Note: 1) We also have $\mu\beta = t$ hence t factors through S exactly (and not only modulo A_1) .

2) We have also proven that $q\beta = 1_T$ i.e. $\alpha\beta = 1_T$ hence β is a split mono & α a split epi in C.

Definition 5.4: We call (S,μ) the A_1-saturation of (T,t) .

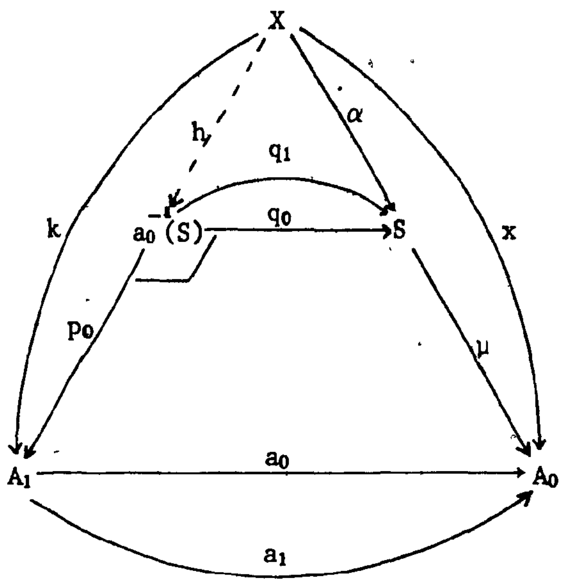
Lemma 5.5: Given (A_0, A_1) in $\hat{C}$ and two C-morphisms:

$x : X \longrightarrow A_0$ and $\mu : S \longrightarrow A_0$, if (S,μ) is saturated

then: (x factors through S modulo A_1) $\Rightarrow$ (x factors through S (exactly) in C)

Proof: Assume that x factors through S modulo A_1

i.e.: $\exists \alpha : X \longrightarrow S$ s.t. $x \equiv \mu\alpha \pmod{A_1}$, and construct the following diagram:



Where: (1) $\mu q_0 = a_0 p_0$ (pull-back)

(2) $\mu q_1 = a_1 p_0$ ((S, μ) is saturated)

(3) $a_1 k = x$ ($\mu\alpha \equiv x \pmod{A_1}$)

(4) $a_0 k = \mu\alpha$ ($\mu\alpha \equiv x \pmod{A_1}$)

but (4) $\Rightarrow \exists!$ h s.t. (5) & (6) where:

(5) $q_0 h = \alpha$

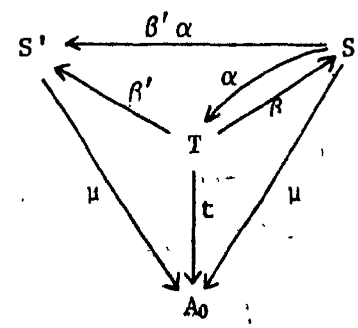
(6) $p_0 h = k$

So: (3) & (6) & (2) $\Rightarrow$ $a_1 p_0 h = x = \mu q_1 h$.

Thus x factors through (S,μ) in C as required .

Corollary 5.6 : The A_1-saturation (S,μ) of a given morphism (T,t) is the "smallest" A_1-saturated morphism through which t factors.

Proof: If (S',μ') is another saturated morphism through which T factors , then we have:



$\mu \equiv t\alpha \pmod{A_1}$ & $t = \mu'\beta$

so, $\mu \equiv \mu'\beta'\alpha \pmod{A_1}$

hence, since μ factors through S' modulo A_1, and S' is saturated, μ factors through S' exactly.

Note: Here also (like in section 2), since "factoring through" is only a preorder, the "smallest" A_1-saturated morphism through which t factors is not unique as such, but the A_1-saturation of t is "canonical" because it is defined as a pull-back in C.

Corollary 5.7: If two C-morphisms are A_1-equivalent, their respective A_1-saturations are equivalent (i.e. each one factors through the other exactly).

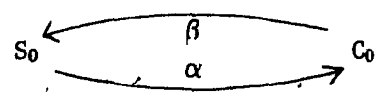
Proof: Obvious because A_1-equivalence between saturated morphisms implies (exact) equivalence.

b) Saturated subobjects in $\hat{C}$:

Let $\dot{f}: (C_0, C_1) \longrightarrow (A_0, A_1)$ be a $\hat{C}$-mono (we can assume that C_1 is the canonical largest equivalence span making f compatible), and let (S_0, φ) be the A_1-saturation of f (where f is a fixed but arbitrary representative of $\dot{f}$). φ can be made into a $\hat{C}$-mono $\dot{\varphi}: (S_0, S_1) \longrightarrow (A_0, A_1)$ by choosing the appropriate (canonical) S_1.

Proposition 5.8: $C = (C_0, C_1)$ and $S = (S_0, S_1)$ are isomorphic in $\hat{C}$. (hence define the same subobject of (A_0, A_1)).

Proof: We know (proposition 5.3) that there exists a pairs of C-morphisms α & β s.t. $f = \varphi\beta$ & $\varphi \equiv f\alpha \pmod{A_1}$



So first observe that α & β are compatible because $\dot{f}$ & $\dot{\varphi}$ are $\hat{C}$-monos:

(i) Given x & y: $X \rightrightarrows C_0$, we have:

$x \equiv y \pmod{C_1} \Rightarrow fx \equiv fy \pmod{A_1}$ (f is compatible)

$\Rightarrow \varphi\beta x \equiv \varphi\beta y \pmod{A_1}$ ($f = \varphi\beta$)

$\Rightarrow \beta x \equiv \beta y \pmod{S_1}$ ($\dot{\varphi}$ is a mono)

(ii) Given x & y: $X \rightrightarrows S_0$, we have:

$x \equiv y \pmod{S_1} \Rightarrow \varphi x \equiv \varphi y \pmod{A_1}$ (φ is compatible)

$\Rightarrow f\alpha x \equiv f\alpha y \pmod{A_1}$ ($\varphi \equiv f\alpha \pmod{A_1}$)

$\Rightarrow \alpha x \equiv \alpha y \pmod{C_1}$ ($\dot{f}$ is a mono)

Secondly we have: $\dot{f} = \dot{\varphi}\dot{\beta}$ & $\dot{\varphi} = \ddot{f\alpha}$ in $\hat{C}$ because $f = \varphi\beta$ & $\varphi \equiv f\alpha \pmod{A_1}$.
Therefore since $\dot{f}$ & $\dot{\varphi}$ are monos:

$$1_C = \ddot{\alpha\beta} \quad \& \quad 1_S = \ddot{\beta\alpha}$$

Thus $\dot{\alpha}$ & $\dot{\beta}$ are isomorphisms, so $C \cong S$ as required.

Definition 5.8:

a) Given an object (A_0, A_1) of $\hat{C}$, a saturated subobject $(S, \dot{\mu})$ of (A_0, A_1) is a $\hat{C}$-mono $\dot{\mu} : S = (S_0, S_1) \longrightarrow (A_0, A_1)$ such that:

(i) (S_0, μ) is an A_1-saturated C-morphism.

(ii) S_1 is the canonical largest equivalence span making μ compatible.

b) Given a $\hat{C}$-morphism $\dot{f}: (C_0, C_1) \longrightarrow (A_0, A_1)$ a saturated image I of $\dot{f}$ is a saturated subobject $(I_0, I_1) \longrightarrow (A_0, A_1)$ of (A_0, A_1) in which I_0 is the saturation of a representative of $\dot{f}$.

Conclusion: If we summerize the definition and properties of saturated C-morphisms and subobjects in $\hat{C}$ we get the following:

Proposition 5.10:

1) Every subobject in $\hat{C}$ can be represented by a saturated one (the saturation of a representative m, of a mono $\dot{m}$ that represents it with the appropriate equivalence span).

2) If (S, μ) & (S', μ') are saturated subobjects of the same object (of $\hat{C}$) then:

$S < S'$ (in $\hat{C}$) $\Leftrightarrow$ μ factors through S_0' (in C).

3) Equivalent (i.e. isomorphic) subobjects (in $\hat{C}$) are represented by equivalent (but not isomorphic!) saturated C-morphisms.

Thus the correspondance between subobjects of (A_0, A_1) (in $\hat{C}$) , and their saturated representations (viewed as C-morphisms) is a one-to-one order preserving correspondance between:

(i) subobjects of (A_0, A_1)

and (ii) equivalence classes of saturated C-morphisms into A_0 .

6) Quotients in $\hat{C}$:

a) Equivalence relations in $\hat{C}$:

An equivalence relation $(S,\dot{\mu}_0,\dot{\mu}_1)$ on $A = (A_0,A_1)$ in $\hat{C}$ can be defined as an equivalence span such that $<\dot{\mu}_0,\dot{\mu}_1>$ is a $\hat{C}$-mono: $S \rightarrowtail A\times A$ or equivalently, as a subobject of $A\times A$ such that for any object C of $\hat{C}$, $\hat{C}(C,S) \rightarrowtail \hat{C}(C,A) \times \hat{C}(C,A)$ is an equivalence relation (in sets) ([4], I (5.5)).

But, as we have seen previously every subobject can be represented by a saturated one (proposition 5.10) . We then have:

Proposition 6.1: If $<\dot{\mu}_0,\dot{\mu}_1> : S=(S_0,S_1) \longrightarrow A\times A$ is a saturated subobject, then T.F.A.E.:

(i) $(S,\dot{\mu}_0,\dot{\mu}_1)$ is an equivalence relation on A in $\hat{C}$.

(ii) (S_0,μ_0,μ_1) is an equivalence span on A_0 in $\hat{C}$ s.t. $A_1 \leq S_0$.

Proof (i) $\Rightarrow$ (ii): If S is an equivalence relation (in $\hat{C}$) , then for any object C of $\hat{C}$, $\hat{C}(C,S) \longrightarrow \hat{C}(C,A) \times \hat{C}(C,A)$ is an equivalence relation in sets which we will note "$\dot{\sim}$". But if we take $C = X$ an object of C, $\dot{\sim}$ defines a new equivalence relation $\sim$ on $C(X,A_0)$ as follows:

given x & y: $X \longrightarrow A_0$,

$$x \sim y \Leftrightarrow \dot{x} \dot{\sim} \dot{y} \Leftrightarrow <\dot{x},\dot{y}> \text{ factors through } S \text{ in } \hat{C} \qquad (1)$$

$$\Leftrightarrow <x,y> \text{ factors through } S_0 \text{ modulo } A_1 \times A_1 \qquad (2)$$

But since $<\mu_0,\mu_1>$ is a saturated C-morphism:

$$(2) \Leftrightarrow <x,y> \text{ factors through } S_0 \text{ exactly} \qquad (3)$$

Hence: $x \sim y \Leftrightarrow <x,y>$ factors through S_0

<u>So</u> by proposition 1.1 and by definition 1.2 ,

(S_0, μ_0, μ_1) is an equivalence span on A_0 (in C), and

$x \equiv y$ (modulo S_0) $\Leftrightarrow$ $x \sim y$.

<u>Now</u> , since $\dot{\sim}$ is reflexive, we have for any x & y: $X \longrightarrow A_0$:

$x \equiv y$ (modulo A_1) $\Rightarrow$ $\dot{x} = \dot{y}$ $\Rightarrow$ $\dot{x} \dot{\sim} \dot{y}$ $\Rightarrow$ $x \sim y$

$\Rightarrow$ $x \equiv y$ (modulo S_0)

<u>Hence</u> by proposition 1.5: $A_1 < S_0$.

<u>(ii) $\Rightarrow$ (i)</u> : If (S_0, μ_0, μ_1) is an equivalence span on A_0 s.t. $A_1 < S_0$, then for any object X of C, the relation $\dot{\sim}$ defined on $\hat{C}(X,A)$ by:

$\dot{x} \dot{\sim} \dot{y}$ $\Leftrightarrow$ $x \equiv y$ (mod S_0) is an equivalence relation

(Note that $\dot{\sim}$ is well defined and reflexive because $A_1 < S_0$) and we have for any pair of C-morphisms , x & y: $X \longrightarrow A_0$

$x \equiv y$ (mod S_0) $\Leftrightarrow$ (3) $\Leftrightarrow$ (2) $\Leftrightarrow$ (1) $\Leftrightarrow$ $\dot{y} \dot{\sim} \dot{y}$ as above .

<u>Hence</u> for any object X of C, $\hat{C}(X,S) \rightarrowtail \hat{C}(X,A) \times \hat{C}(X,A)$ is (the graph of) an equivalence relation in Sets.

<u>Now</u> if $C = (C_0, C_1)$ is an arbitrary object of $\hat{C}$, <u>we claim that</u>:

$\dot{f} \dot{\sim} \dot{g}$ $\Leftrightarrow$ $< f,g >$ factors through S in $\hat{C}$ (1')

$\Leftrightarrow$ $< f,g >$ factors through S_0 in C (3')

i.e.: $\dot{f} \dot{\sim} \dot{g}$ $\Leftrightarrow$ $f \equiv g$ (modulo S_0)

i.e.: we have the same situation as above ((1) $\Leftrightarrow$ (3)) but this time for <u>compatible</u> C-morphisms f & g , and therefore, that $\dot{\sim}$ is an equivalence relation on $\hat{C}(C,A) \subset \hat{C}(C_0,A)$ as above.

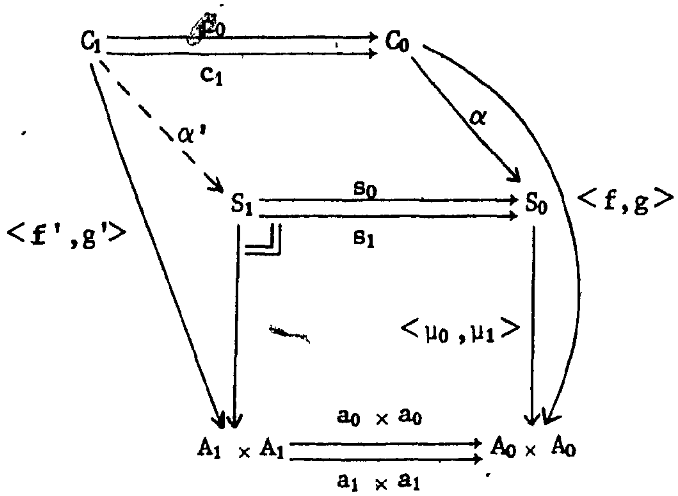
<u>Proof of claim</u>: (1') $\Rightarrow$ (3') is trivial .

<u>(3') $\Rightarrow$ (1')</u>: Suppose that $< f,g >$ factors through S_0

i.e.: $\exists \alpha$: $C_0 \longrightarrow S_0$ s.t. $< f,g > \doteq <\mu_0, \mu_1> \alpha$

All we have to prove is that α is compatible with C_1 & S_1 .

But since S is a saturated subobject, S_1 is the largest equivalence span on S_0 making $\langle\mu_0,\mu_1\rangle$ compatible. So we have the diagram below



$\langle f,g\rangle = \langle\mu_0,\mu_1\rangle\alpha$

$\langle f,g\rangle c_0 = (a_0\times a_0)\langle f',g'\rangle$

& $\langle f,g\rangle c_1 = (a_1\times a_1)\langle f',g'\rangle$

(because f&g are compatible)

So $\langle\mu_0,\mu_1\rangle(\alpha c_0) = (a_0\times a_0)\ \langle f',g'\rangle$

& $\langle\mu_0,\mu_1\rangle(\alpha c_1) = (a_1\times a_1)\ \langle f',g'\rangle$

<u>Hence</u> by the joint pull-back property, there exists a (unique) $\alpha': C_1 \longrightarrow S_1$ such that $\alpha c_0 = s_0\alpha'$ & $\alpha c_1 = s_1\alpha'$ (upper parallelogram commutes)

<u>So</u> α is compatible, and $\dot{\alpha}$ is a $\hat{C}$-morphism $C \longrightarrow S$ such that $\langle\dot{f},\dot{g}\rangle = \langle\dot{\mu}_0,\dot{\mu}_1\rangle\dot{\alpha}$ (1').

<u>Definition 6.2</u>: We will call such a subobject, a <u>saturated</u> equivalence relation (keeping in mind that every equivalence relation can be assumed to be saturated).

b) <u>Construction of Quotients in $\hat{C}$</u>:

Given a saturated equivalence relation $S = (S_0,S_1)$ on $A = (A_0,A_1)$, we know that S is an equivalence span on A_0 (in C) such that $A_1 \leq S_0$. Therefore (A_0, S_0) is an object of $\hat{C}$, and $\dot{1}_{A_0}: (A_0,A_1) \longrightarrow (A_0,S_0)$ a $\hat{C}$-morphism. We claim that (A_0,S_0) is the quotient of A by S in $\hat{C}$, and hence that S is effective:

<u>Proposition 6.3</u>: If $S = (S_0,S_1)$ is a saturated equivalence relation on $A = (A_0,A_1)$, then the following diagram is an exact sequence:

$$(S_0,S_1) \underset{\dot{\mu}_1}{\overset{\dot{\mu}_0}{\rightrightarrows}} (A_0,A_1) \xrightarrow{\dot{1}_{A_0}} (A_0,S_0)$$

i.e.: (i) $\dot{1}_{A_0}$ is the coequalizer of $\dot{\mu}_0$ & $\dot{\mu}_1$

(ii) (S_0, S_1) is the kernel pair of $\dot{1}_{A_0}$.

<u>Proof:</u> (i) is similar to Corollary 3.2:

That (A_0, S_0) is an object of $\hat{C}$, and $\dot{1}_{A_0}$ a $\hat{C}$-morphism results from Proposition 6.1 as previously remarked; and $\dot{1}_{A_0}\dot{\mu}_0 = \dot{1}_{A_0}\dot{\mu}_1$ i.e.: $\mu_0 \equiv \mu_1 \pmod{S_0}$ is trivial.

Now if $\dot{f}: (A_0, A_1) \longrightarrow (C_0, C_1)$ coequalizes $\dot{\mu}_0$ & $\dot{\mu}_1$, then f coequalizes μ_0 & μ_1 mod C_1 hence, by lemma 3.1 applied to (A_0, S_0) & (C_0, C_1), f is compatible with S_0 & C_1.

So, $\ddot{f}$ is a $\hat{C}$-morphism $(A_0, S_0) \longrightarrow (C_0, C_1)$ such that $\ddot{f}\dot{1}_{A_0} = \dot{f}$, and is obviously unique with that property.

(ii) Using the characterization of a kernel pair given in proposition 3.3 (iii), we have:

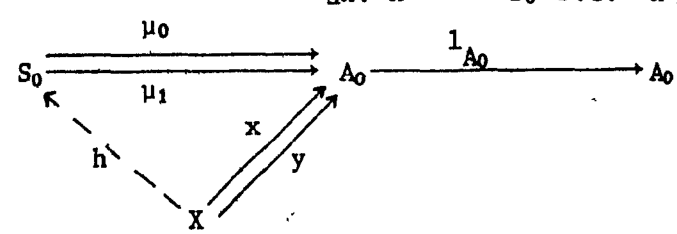
(a) $1_{A_0}\mu_0 \equiv 1_{A_0}\mu_1 \pmod{S_0}$ (as in (i))

(b) Given an object X of C and two C-morphisms

x & y: $X \longrightarrow A_0$,

$1_{A_0}x \equiv 1_{A_0}y \pmod{S_0}$ ⇒ $\exists h: X \longrightarrow S_0$ s.t. $x = \mu_0 h$ & $y = \mu_1 h$

⇒ $\exists h: X \longrightarrow S_0$ s.t. $x \equiv \mu_0 h$ & $y \equiv \mu_1 h \pmod{A_1}$



(c) Since S_1 is the canonical largest equivalence span on S_0 making $\langle\mu_0, \mu_1\rangle$ compatible (with $A_1 \times A_1$), and $\langle\mu_0, \mu_1\rangle$ compatible with $A_1 \times A_1$ ⇔ μ_0 & μ_1 compatible with A_1, S_1 is a largest equivalence span making the projections μ_0 & μ_1 compatible.

<u>Thus:</u> Conditions (iii) of proposition 3.3 is satisfied.

<u>Hence</u> $(S_0, \dot{\mu}_0, \dot{\mu}_1)$ is the kernel pair of $1_{A_0}: (A_0, A_1) \longrightarrow (A_0, S_0)$ which is the coequalizer of $\dot{\mu}_0$ & $\dot{\mu}_1$.

<u>So</u> (A_0, S_0) is the quotient of A by S .

<u>Conclusion:</u> Since every equivalence relation in $\hat{C}$ is equivalent to a saturated one S which always has a quotient $A/S = (A_0, S_0)$, we have:

<u>Corollary 6.4:</u> Every equivalence relation in $\hat{C}$ is effective.

i.e.: <u>$\hat{C}$ is an exact category</u>.

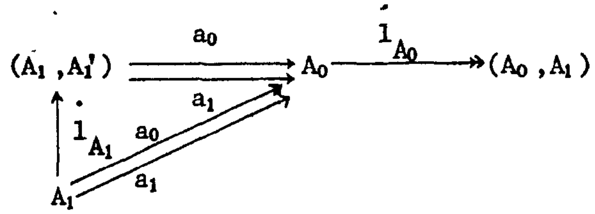
7) Universal property of the category $\hat{C}$:

Given a left-exact functor $F: C \longrightarrow B$, where B is an exact category, we want to prove that there exists a unique (up to isom orphism) exact functor $\hat{F} : \hat{C} \longrightarrow B$, such that: $F = \hat{F} \circ J$ where J is the (left-exact) inclusion functor: $J: C \longrightarrow \hat{C}$.

a) Internal characterization of objects of $\hat{C}$:

Lemma 7.1: Every object (A_0, A_1) of $\hat{C}$ is the quotient of A_0 by the canonical image (A_1, A_1') of $< a_0, a_1 >$ (Definition 4.6)

i.e.: The "horizontal" part of the following diagram (1) is an exact sequence.



Proof: From corallary 3.2, we know that (A_0, A_1) is the coequalizer of a_0 & $a_1 : A_1 \rightrightarrows A_0$, hence also of a_0 & $a_1 : (A_1, A_1') \rightrightarrows A_0$, because $\dot{1}_{A_1}$ is a (regular) epi.

So, we only need to prove that $(A_1, A_1') \overset{a_0}{\underset{a_1}{\rightrightarrows}} A_0$ is the kernel pair of $\dot{1}_{A_0} : A_0 \longrightarrow (A_0, A_1)$.

For that, let us verify the condition (iii) of proposition 3.3 :

(a) $\dot{1}_{A_0} a_0 = \dot{1}_{A_0} a_1 \Leftrightarrow a_0 \equiv a_1 \pmod{A_1}$ which is trivial .

(b) If x & $y : X \longrightarrow A_0$ are C-morphisms then:

$\dot{1}_{A_0} x = \dot{1}_{A_0} y \Rightarrow x \equiv y \pmod{A_1} \Rightarrow$

$\Rightarrow \exists h: X \longrightarrow A_1$ s.t. $x = a_0 h$ & $y = a_1 h$.

(c) A_1' is the (canonical) largest equivalence span making a_0 & a_1 compatible (section 4 -e).

So (A_1, A_1') is the kernel pair of $1_{A_0} : A_0 \longrightarrow (A_0, A_1)$ which is the coequalizer of a_0 & $a_1 : (A_1, A_1') \longrightarrow A_0$.

Therefore (A_0, A_1) is the quotient of A_0 by the (canonical) image of its equivalence span (A_1, a_0, a_1) .

b) Description and uniqueness of $\hat{F}$:

Given the left-exact functor $F: \mathcal{C} \longrightarrow \mathcal{B}$, we want $\hat{F}$ to be an exact extention of F to $\hat{\mathcal{C}}$ i.e.:

(i) $\hat{F}(A) = F(A)$ for every object A of $\mathcal{C}$

(ii) $\hat{F}(f) = F(f)$ for every $\mathcal{C}$-morphism f

(iii) $\hat{F}$ preserves all finite limits

(iv) $\hat{F}$ preserves monos and regular epis

Therefore:

(v) $\hat{F}$ preserves quotients i.e. exact sequences

So we have:

Lemma 7.2: If $\hat{F}$ exists, it is uniquely defined on objects by:

$\hat{F}(A_0, A_1) = FA_0/I_A$, where $I_A = \text{Im } F < a_0, a_1 >$

Proof: From lemma 7.1, we know that: $(A_0, A_1) = A_0/(A_1, A_1') = A_0/\text{Im} < a_0, a_1 >$

Hence (v) $\Rightarrow$ $\hat{F}(A_0, A_1) = \hat{F}A_0/\hat{F}(\text{Im} < a_0, a_1 >)$

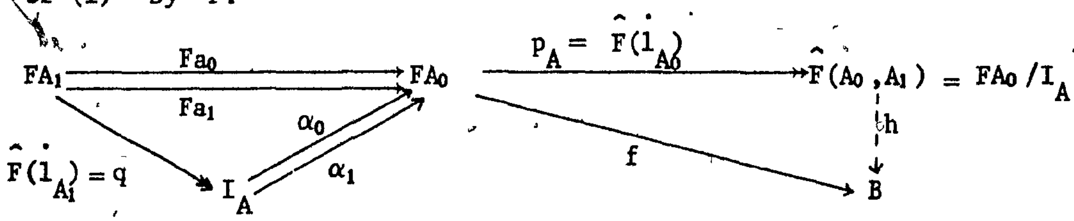
But (iv) $\Rightarrow$ $\hat{F}(\text{Im} < a_0, a_1 >) = \text{Im } \hat{F} < a_0, a_1 >$

And (i) $\Rightarrow$ $\hat{F}A_0 = FA_0$

(ii) $\Rightarrow$ $\hat{F} < a_0, a_1 > = F < a_0, a_1 >$

So: $\hat{F}(A_0, A_1) = \hat{F}A_0/\hat{F}(\text{Im} < a_0, a_1 >) = FA_0/\text{Im } F < a_0, a_1 > = FA_0/I_A$

<u>Corollary 7.3:</u> $\hat{F}(A_0, A_1)$ is (also) the coequalizer of Fa_0 & Fa_1, and $\hat{F}(\dot{1}_{A_0})$ is the projection p_A as in the diagram (2) below, image of (1) by $\hat{F}$:



(2)

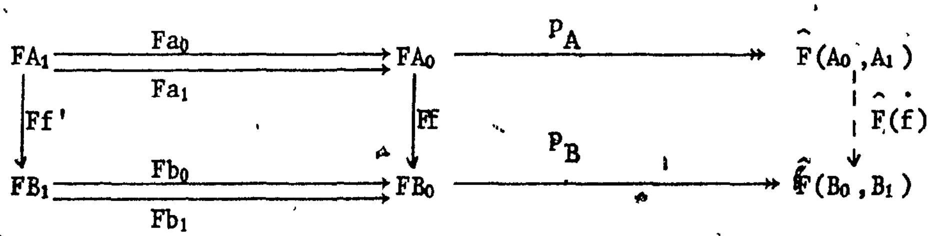
<u>Proof:</u> $\hat{F}(A_0, A_1)$ is the coequalizer of α_0 & α_1 (which equal $\hat{F}(a_0)$ & $\hat{F}(a_1)$ respectively), with p_A as projection.

So, if $f: FA_0 \longrightarrow B$ is a $\mathcal{B}$-morphism, then

$fFa_0 = fFa_1 \Rightarrow f\alpha_0 q = f\alpha_1 q \Rightarrow f\alpha_0 = f\alpha_1 \Rightarrow$

$\Rightarrow \exists h: \hat{F}(A_0, A_1) \longrightarrow B$ s.t. $f = hp_A$.

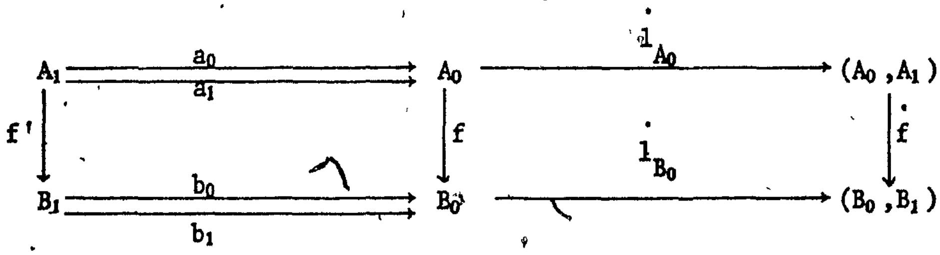
<u>Lemma 7.4</u>: If $\hat{F}$ exists and $\dot{f}: (A_0, A_1) \longrightarrow (B_0, B_1)$ is a $\hat{C}$-morphism, then $\hat{F}(\dot{f})$ is the unique morphism making the diagram (3) below commutative:



(3)

<u>Furthermore</u> $\hat{F}(\dot{f})$ does not depend on the choice of the representative f of $\dot{f}$.

<u>Proof:</u> (i) (3) is the image by $\hat{F}$ of the following diagram:



(ii) But we also know from corollary 7.3 that the two horizontal subdiagrams of (3) are coequalizers.

So: $p_B \circ Ff \circ Fa_0 = p_B \circ Fb_0 \circ Ff' = p_B \circ Fb_1 \circ Ff' = p_B \circ Ff \circ Fa_1$

Hence there exists a unique $\hat{F}(\dot{f})$ making the right square of (3) commutative.

Furthermore, if g is another representative of $\dot{f}$, then

$f = b_0 h$ & $g = b_1 h$ for some $h: A_0 \longrightarrow B_1$

hence: $p_B \circ Ff = p_B \circ Fb_0 \circ Fh = p_B \circ Fb_1 \circ Fh = p_B \circ Fg$

So f and g determine the same $\hat{F}(\dot{f})$.

Conclusion: $\hat{F}$ is uniquely determined on objects and morphisms, and therefore, using the characterizations given in Lemmas 7.2 & 7.4 as definition, $\hat{F}$ preserves identities and composition.

Thus: $\hat{F}$ is a functor: $\hat{C} \longrightarrow B$ s.t. $F = \hat{F} \circ J$.

c) Exactness properties of $\hat{F}$:

Lemma 7.5: $\hat{F}$ preserves products.

Proof: Given two objects (A_0, A_1) & (B_0, B_1) of $\hat{C}$, we have:

$(A_0, A_1) \times (B_0, B_1) = (A_0 \times B_0, A_1 \times B_1)$ and $\langle a_0 \times b_0, a_1 \times b_1 \rangle = \langle a_0, a_1 \rangle \times \langle b_0, b_1 \rangle$

So, since F preserves products, $F\langle a_0 \times b_0, a_1 \times b_1 \rangle = F\langle a_0, a_1 \rangle \times F\langle b_0, b_1 \rangle$

Hence: $\operatorname{Im} F\langle a_0 \times b_0, a_1 \times b_1 \rangle = \operatorname{Im} F\langle a_0, a_1 \rangle \times \operatorname{Im} F\langle b_0, b_1 \rangle = I_A \times I_B$

Thus: $\hat{F}[(A_0, A_1) \times (B_0, B_1)] = \hat{F}(A_0 \times B_0, A_1 \times B_1) = (FA_0 \times FB_0)/I_A \times I_B$

$$= FA_0/I_A \times FB_0/I_B$$

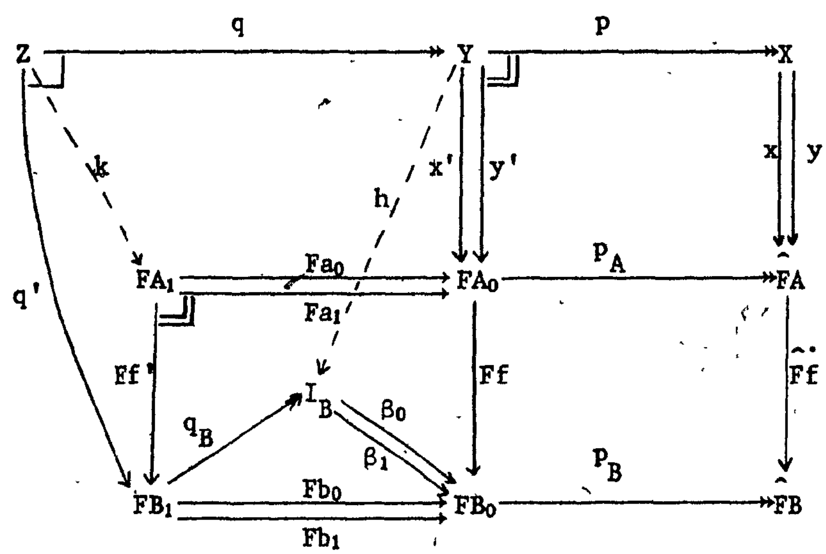
$$= \hat{F}(A_0, A_1) \times \hat{F}(B_0, B_1)$$

<u>Lemma 7.6:</u> $\hat{F}$ preserves monos:

<u>Proof:</u> Let $f\colon A = (A_0, A_1) \longrightarrow (B_0, B_1)$ be a mono (where A_1 is canonical, i.e. $A_1 = (f\times f)^{-1}<b_0, b_1>$) , and let x & $y\colon X \rightrightarrows \hat{F}(A)$ be morphisms s.t. $\hat{F}f\circ x = \hat{F}f\circ y$ (in $\mathcal{B}$) .

We want to prove that $x = y$.

For that, consider the following diagram:



In this diagram, Y is defined as the vertex of the pull-back $<x,y>\circ p = p_A \circ <x',y'>$ (drawn as a joint pull-back above). Therefore, we have:

(i) p is a regular epi because p_A is one and $\mathcal{B}$ is exact .

(ii) $p_B \circ Ff\circ x' = \hat{F}f\circ x\circ p = \hat{F}f\circ y\circ p = p_B\circ Ff\circ y'$

Hence, since I_B is the kernel pair of p_B, there exists a (unique) $h\colon Y \longrightarrow I_B$ such that $Ff\circ x' = \beta_0 h$ & $Ff\circ y' = \beta_1 h$.

If we now define Z as the vertex of the pull-back $hq = q_B q'$, we have:

(i) q is a regular epi because q_B is one and $\mathcal{B}$ is exact .

(ii) $Ff\circ x'\circ q = \beta_0 h\, q = \beta_0 q_B q' = Fb_0\circ q'$

& $Ff\circ y'\circ q = \beta_1 h\, q = \beta_1 q_B q' = Fb_1\circ q'$

But, since $\dot{f}$ is a "canonical" mono and F preserves pull-backs (hence joint pull-backs), FA_1 is the vertex of a joint pull-back and therefore, there exists a (unique) $k\colon Z \longrightarrow FA_1$ s.t. $Fa_0 \circ k = x'q$ & $Fa_1 \circ k = y'q$

<u>Thus</u>: $xpq = p_A x'q = p_A \circ Fa_0 \circ k = p_A \circ Fa_1 \circ k = p_A y'q = ypq$

but p & q are (regular) epis:

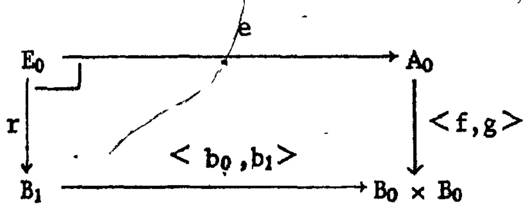
hence $xpq = ypq \Rightarrow x = y$

<u>Corollary 7.7</u>: $\hat{F}$ preserves equalizers.

<u>Proof</u>: Let $E = (E_0, E_1)$ be the equalizer of the two $\hat{C}$-morphisms $\dot{f}$ & $\dot{g}$: $A = (A_0, A_1) \longrightarrow (B_0, B_1) = B$

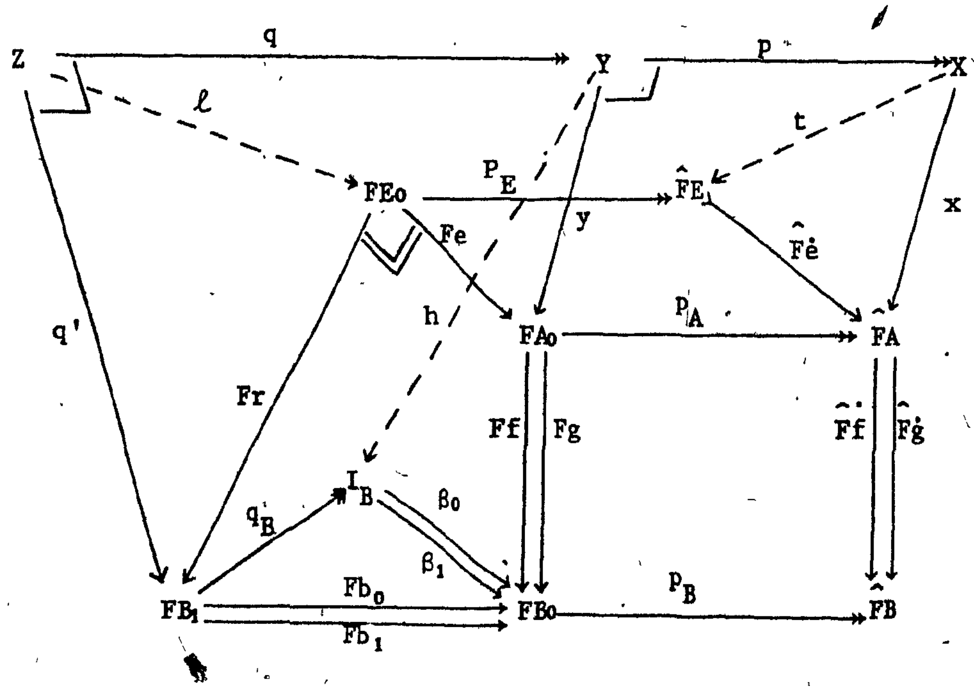
$$E \xrightarrow{\dot{e}} A \underset{\dot{g}}{\overset{\dot{f}}{\rightrightarrows}} B$$

It is easy to see that we can choose E_0 & e as in the pull-back:



and E_1 as in proposition 3.3 (iii).

Hence, since F preserves limits, FE_0 is the vertex of a joint pull-back in the diagram below:



And by proposition 7.6 above, $\hat{F}\dot{e}$ is a mono because $\dot{e}$ is one.

Now if $x: X \longrightarrow \hat{F}A$ satisfies $\hat{F}\dot{f} \circ x = \hat{F}\dot{g} \circ x$ and if we construct Y as the vertex of the pull-back $xp = p_A y$ we have:

(i) p is a regular epi

(ii) $p_B \circ Ff \circ y = \hat{F}\dot{f} \circ p_A \circ y = \hat{F}\dot{f} \circ x \circ p = \hat{F}\dot{g} \circ x \circ p = \hat{F}\dot{g} \circ p_A \circ y = p_B \circ Fg \circ y$.

Hence, since I_B is the kernel pair of p_B (Lemma 7.1), there exists a (unique) $h: Y \longrightarrow I_B$ s.t. $Ff \circ y = \beta_0 h$ and $Fg \circ y = \beta_1 h$.

If we now construct Z, as the vertex of the pull-back $hq = q_B q'$ we have:

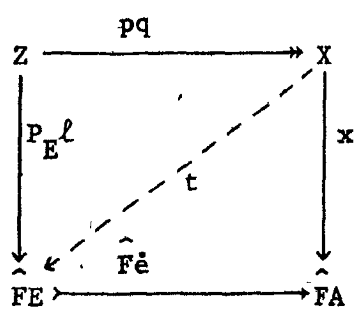
(i) q is a regular epi

(ii) $Ff \circ y \circ q = \beta_0 hq = \beta_0 q_B q' = Fb_0 \circ q'$

and $Fg \circ y \circ q = \beta_1 hq = \beta_1 q_B q' = Fb_1 \circ q'$.

Hence, since FE_0 is the vertex of a joint pull-back, there exists a (unique) $\ell : Z \longrightarrow FE_0$ s.t. $q' = Fr \circ \ell$ & $yq = Fe \circ \ell$.

<u>Hence</u> : $\hat{F}\dot{e} \circ p_E \circ \ell = p_A \circ Fe \circ \ell = p_A \circ y \circ q = xpq$ as the commutative square below:



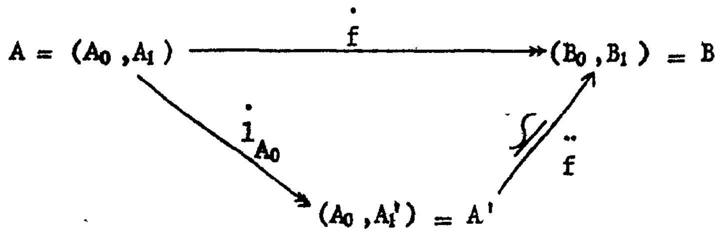
in which pq is a regular epi and $\hat{F}\dot{e}$ a mono.

Hence, by the diagonal property, there exists a unique t s.t. $x = \hat{F}\dot{e} \circ t$.

<u>Conclusion:</u> $\hat{F}E$ is the equalizer of $\hat{F}\dot{f}$ & $\hat{F}\dot{g}$.

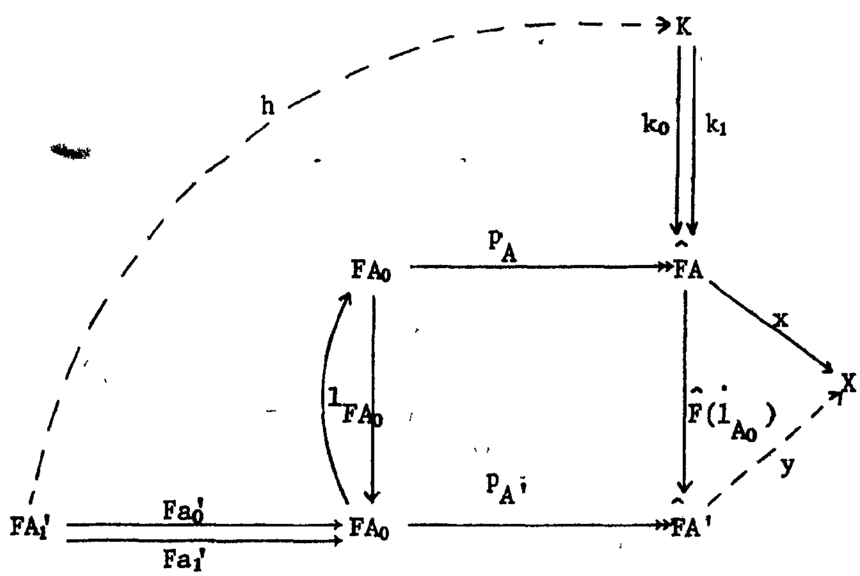
<u>Lemma 7.8:</u> $\hat{F}$ preserves regular epis.

From proposition 4.3, we know that every regular epi $\dot{f}$ is equivalent to one of the form $\dot{1}_{A_0}$ as in the diagram:



Therefore, since every functor preserves isomorphisms, it suffices to show that $\hat{F}(\dot{1}_{A_0})$ is a regular epi: $\hat{F}A \longrightarrow\!\!\!\!\rightarrow \hat{F}A'$ in $\mathcal{B}$, i.e. that $\hat{F}(\dot{1}_{A_0})$ is the coequalizer of its kernel pair.

So let (K, k_0, k_1) be the kernel pair of $\hat{F}(\dot{1}_{A_0})$, and consider the diagram below:



We have: $\quad p_{A'} = \hat{F}(\dot{1}_{A_0}) \circ p_A \circ 1_{FA_0}$

and $\quad p_{A'} \circ Fa_0' = p_{A'} \circ Fa_1'$

hence: $\quad \hat{F}(\dot{1}_{A_0}) \circ p_A \circ 1_{FA_0} \circ Fa_0' = \hat{F}(\dot{1}_{A_0}) \circ p_A \circ 1_{FA_0} \circ Fa_1'$

<u>Hence</u> since K is the kernel pair of $\hat{F}(\dot{1}_{A_0})$, there exists a (unique) $h\colon FA_1' \longrightarrow K$ s.t. $k_0 h = p_A \circ 1_{FA_0} \circ Fa_0'$ & $k_1 h = p_A \circ 1_{FA_0} \circ Fa_1'$

Thus, if $x\colon \hat{F}A \longrightarrow X$ is such that $xk_0 = xk_1$

<u>Then</u>: $x \circ p_A \circ 1_{FA_0} \circ Fa_0' = x \circ p_A \circ 1_{FA_0} \circ Fa_1'$

Hence, since $p_{A'}$ is the coequalizer of Fa_0' & Fa_1', there exists a unique $y: \hat{F}A' \longrightarrow X$ s.t. $y \circ p_{A'} = x \circ p_A \circ 1_{FA_0}$

i.e.: $y \circ \hat{F}(\dot{1}_{A_0}) \circ p_A = x \circ p_A$

<u>hence</u> $y \circ \hat{F}(\dot{1}_{A_0}) = x$ (since p_A is an epi)

<u>So</u> $\hat{F}(\dot{1}_{A_0})$ is the coequalizer of its kernel pair hence is a regular epi.

<u>Conclusion:</u> $\hat{\hat{F}}$ is a exact functor and the unique extension (up to isomorphism) of the left-exact functor $F: \mathcal{C} \longrightarrow \mathcal{B}$ to the exact category $\hat{\mathcal{C}}$, which terminates the proof of the theorem.

CHAPTER II

APPROXIMATION FILTRANTE DE DIAGRAMMES FINIS DANS Pro-C

0) Introduction.

Il est connu que tout foncteur F d'une catégorie C dans les ensembles est une colimite de foncteurs représentables: $F = \underset{\mathrm{Diagr}(F)}{\mathrm{colim}}\, C(C,-)$ où diagr(F) = $(C|F)$ est appelé le diagramme de F. Par conséquent, si on identifie C à une sous-catégorie pleine de $(\mathrm{Ens}^{C})^{op}$ par le foncteur d'Yonéda, F est une limite d'objets de C (à isomorphisme naturel près).

Nous nous proposons ici de nous concentrer sur le cas où F est un pro-objet de C, c'est a dire une limite cofiltrante d'objets de C. Dans ce cas, nous montrons que si Δ est une catégorie finie sans boucles (i.e. dans laquelle le domaine et le codomaine de tout morphisme qui n'est pas une identité sont distincts), nous avons une équivalence de catégories:

$$\text{Pro-}C^{\Delta} \cong (\text{Pro-}C)^{\Delta}$$

Ce théorème est une extention du "théorème d'approximation uniforme" cité dans [1], et dont une esquisse de démonstration est donnée. D'après ce théorème, tout diagramme de type Δ dans Pro-C peut être "approximé uniformément" par une famille de diagrammes de C.

Il est intéressant de remarquer que les catégories équivalentes $\text{Pro-}C^{\Delta}$ et $(\text{Pro-}C)^{\Delta}$ sont des sous-catégories pleines de $\mathrm{Ens}^{(C^{\Delta})}$ et $(\mathrm{Ens}^{C})^{\Delta} \cong \mathrm{Ens}^{C\times\Delta}$ respectivement, qui ne sont pas équivalentes en général.

Du point-de-vue pratique, ce résultat peut être utilisé pour étendre à Pro-C des propriétés de C comme l'existence des limites et colimites finies, ce qui est fait dans [1].

Dans le cas où C est une catégorie cartésienne (à limites finies), nous étendons ce résultat au cas où Δ est une catégorie finie quelquonque, en utilisant une démonstration complètement différente: nous montrons que les endo-foncteurs $(-)^{\Delta}$ et $(-)^{\Delta^{op}}$ de Cart, la grande catégorie des catégories cartésiennes sont adjoints (à droite et à gauche) l'un de l'autre, et nous en déduisons le résultat cherché en utilisant le fait qu'un pro-objet de C est un foncteur exact à gauche $C \longrightarrow Ens$, donc un élément de $Cart(C,Ens)$.

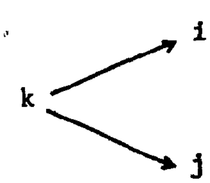
Finalement, nous donnons un contre-exemple dans le cas où C n'est pas cartésienne et où Δ a une boucle ($\Delta = \mathbb{Z}/_{2\mathbb{Z}}$).

L'auteur tient à remercier André JOYAL d'avoir supervisé la rédaction du présent article, et sans lequel sa publication aurait été impossible.

1) Concepts fondamentaux.

Définition: Une catégorie I est dite cofiltrante si et seulement si elle vérifie les deux conditions:

(i) Pour tout couple d'objets i et j de I, il existe un troisième objet k et deux morphismes:

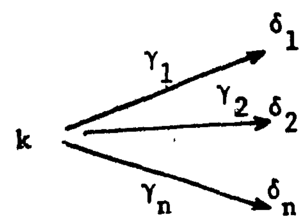


(ii) (unicité essentielle) pour tout couple de morphismes parallèles α et β: $i \rightrightarrows j$ il existe un objet k et un morphisme $k \longrightarrow i$ qui égalise α et β.

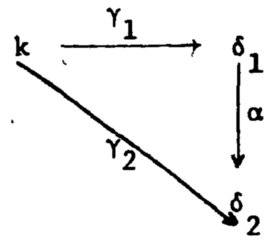
Propriété caractéristique: Une catégorie I est cofiltrante si et seulement si tout diagramme fini est la base d'un cône.

Démonstration: Remarquons d'abord que (i) et (ii) sont des cas particuliers de la propriété caractéristique.

Pour montrer la réciproque, considérons un diagramme fini Δ de I. Par récurence sur le nombre d'objets de Δ et en utilisant la propriété (i), nous pouvons dominer tous les objets $\delta_1, \delta_2, \ldots, \delta_n$ par un même objet k de I. On obtient un nouveau diagramme:

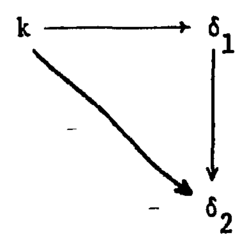


Maintenant, pour chaque morphisme α de Δ, $\alpha:\delta_1 \longrightarrow \delta_2$ on obtient un triangle:



qu'on peut rendre commutatif en appliquant l'unicité essentielle (ii) à γ_2 et $\alpha\circ\gamma_1$. En répètant (récurence) cette opération successivement pour tous les morphismes de Δ, on obtient un cône de base Δ.

<u>Remarque:</u> Cette construction permet d'ajouter un sommet "pseudo-initial" k au diagramme Δ, dans le sens qu'on peut construire k et des morphisms de k vers certains objets de Δ, mais pas nécessairement tous, et rendre certains mais pas nécessairement tous les triangles



commutatifs.

On peut ainsi complèter Δ en un diagramme fini de type quelquonque Δ_1 telque:

1) Δ est un sous-diagramme plein de Δ_1,

2) k est un sommet pseudo-initial (Δ_1 ne contient pas de morphisme de codomaine k),

3) $|\Delta_1| = |\Delta| \cup \{k\}$.

<u>Définition:</u> Un foncteur F d'une catégorie cofiltrante I dans une catégorie J est dit cofinal si et seulement s'il vérifie les deux conditions:

(i) pour tout objet j de J, il existe un objet i de I et un morphisme $F(i) \to j$.

(ii) Pour tout couple de morphismes parallèles α, β: $F(i) \rightrightarrows j$, il existe un objet k de I et un morphisme $\gamma: k \to i$ tel que $F(\gamma)$ égalise α et β: $F(k) \xrightarrow{F(\gamma)} F(i) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} j$

<u>Notes</u>: 1) Dans ces conditions, J est aussi cofiltrante.

2) Si I est une sous-catégorie pleine de J, alors, pour le foncteur d'inclusion, (ii) est une conséquence de (i).

<u>Propriété caractèristique</u>: Un foncteur $F: I \longrightarrow J$ est cofinal si et seulement si pour tout foncteur L de J dans une catégorie telle que $\lim LF$ existe, alors $\lim L$ existe aussi et le morphisme canonique $\lim L \longrightarrow \lim LF$ est un isomorphisme.

La démonstration de cette proposition, dont une variante est donnée dans [2, Ch IX, §3] et [1], est laissée au soin du lecteur.

<u>Note</u>: Les concepts de catégorie cofiltrante et de foncteur cofinal sont les concepts duals de ceux de catégorie filtrante et de foncteur final qui sont étudiés dans un cadre plus général dans [2].

2) La Catégorie Pro-$\mathcal{C}$:

Etant donnée une catégorie quelquonque $\mathcal{C}$, notre but est d'ajouter (de façon universelle) les limites cofiltrantes d'objets de $\mathcal{C}$ que nous appellerons pro-objets de $\mathcal{C}$.

a) Première caractérisation de la catégorie Pro-$\mathcal{C}$:

(i) Un objet de Pro-$\mathcal{C}$ (ou pro-objet de $\mathcal{C}$) est un diagramme cofiltrant de $\mathcal{C}$, c'est-à-dire un foncteur $X: I \longrightarrow \mathcal{C}$ où I est une (petite) catégorie cofiltrante.

Nous écrivons: $X = (X_i)_{i\in I}$

(ii) Si $X = (X_i)_{i\in I}$ et $Y = (Y_j)_{j\in J}$ posons:

$$(\text{Pro-}\mathcal{C})(X,Y) = \lim_{J^{op}} \operatorname{colim}_{I^{op}} \mathcal{C}(X_i,Y_j)$$

Au lieu de vérifier que Pro-$\mathcal{C}$ est une catégorie, nous allons la caractériser comme sous-catégorie pleine de $(\text{Ens}^{\mathcal{C}})^{op}$.

b) Deuxième caractérisation de la catégorie Pro-$\mathcal{C}$:

Proposition: Pro-$\mathcal{C}$ est équivalente à la sous-catégorie pleine de $(\text{Ens}^{\mathcal{C}})^{op}$ dont les objets sont des colimites filtrantes de foncteurs représentables dans $\text{Ens}^{\mathcal{C}}$ ou des limites cofiltrantes de foncteurs représentables dans $(\text{Ens}^{\mathcal{C}})^{op}$.

Démonstration: (i) $|\text{Pro-}\mathcal{C}| \hookrightarrow |\text{Ens}^{\mathcal{C}}| = |(\text{Ens}^{\mathcal{C}})^{op}|$

$$X = (X_i)_{i\in I} \longmapsto X(-) = \operatorname{colim}_{I^{op}} \mathcal{C}(X_i,-) \longmapsto X(-) = \lim_{I} \mathcal{C}(X_i,-)$$

(ii) Soit $X = (X_i)_{i\in I}$ et $Y = (Y_j)_{j\in I}$

alors: $(\text{Ens}^{\mathcal{C}})^{op}(X(-),Y(-)) \cong \text{Pro-}\mathcal{C}(X,Y)$ puisque:

$$f: \operatorname{colim}_{I^{op}} (X_i,-) \longleftarrow \operatorname{colim}_{J^{op}} (Y_j,-)$$

$$(f_j)_{j\in J^{op}} \text{ où } \operatorname{colim}_{I^{op}} (X_i,-) \xleftarrow{f_j} (Y_j,-)$$

--- (Yonéda)

$$(f_j^o)_{j\in J^{op}} \text{ où } f_j^o \in \operatorname{colim}_{I^{op}} (X_i,Y_j)$$

$$f^o \in \lim_{J^{op}} \operatorname{colim}_{I^{op}} (X_i,Y_j)$$

<u>Conclusion:</u> En identifiant les objets C de $\mathcal{C}$ aux foncteurs représentables $\mathcal{C}(C,-)$, nous obtenons la situation suivante: $\mathcal{C} \hookrightarrow \text{Pro-}\mathcal{C} \hookrightarrow (\text{Ens}^{\mathcal{C}})^{op}$ où tout pro-objet est une limite cofiltrante d'objets de $\mathcal{C}$.

c) <u>Troisième caractèrisation des pro-objets de $\mathcal{C}$:</u>

<u>Proposition:</u> Si $X(-)$ est un foncteur $\mathcal{C} \longrightarrow \text{Ens}$, $X(-)$ est un pro-objet si et seulement si son diagramme $\text{Diagr}(X)$ est filtrant où $\text{Diagr}(X) = (\mathcal{C}|X)$ est défini par: $|\text{Diagr}(X)| = \{(C,a)\colon C\in|\mathcal{C}|,\ a\in X(C)\}$, et un morphisme de $\text{Diagr}(X)$, $f\colon(C,a) \longrightarrow (C',a')$ est un morphisme de $\mathcal{C}$; $f\colon C' \longrightarrow C$

telque $X(f)(a') = a$

<u>Démonstration:</u> Puisqu'on a pour tout foncteur X:

$$X = \underset{\text{Diagr}(X)}{\text{colim}}\ \mathcal{C}(C,-) \text{ dans } \text{Ens}^{\mathcal{C}}$$

on a aussi:

$$X = \underset{(\text{Diagr}(X))^{op}}{\lim}\ \mathcal{C}(C,-) \text{ dans } (\text{Ens}^{\mathcal{C}})^{op}$$

donc: Si $\text{Diagr}(X)$ est filtrant, $(\text{Diagr}(X))^{op}$ est cofiltrant donc X est un pro-objet. Si X est un pro-objet: $X = \lim\limits_{I} X_i$ où $X_i\in\mathcal{C}$ est identifié à $\mathcal{C}(X_i,-)$ dans $(\text{Ens}^{\mathcal{C}})^{op}$, on a encore: $X = \underset{(\text{Diagr}(X))^{op}}{\lim}\ \mathcal{C}(C,-) = \underset{\{X\to C\}}{\lim}\ C$ où $(\text{Diagr}(X))^{op}$ est vu comme la catégorie des morphismes: $X \longrightarrow C$.

On a donc:

$$\lim_{i\in I} X_i = \lim_{\{X\to C\}} C$$

or

$$I \cong \{X \longrightarrow X_i\}_{i\in I} \longrightarrow (\text{Diagr}(X))^{op},$$

donc le foncteur d'inclusion est cofinal et par conséquent la catégorie $(\text{Diagr}(X))^{op}$ est cofiltrante.

d) <u>Quatrième caractèrisation dans le cas d'une catégorie cartésienne:</u>

<u>Proposition:</u> Si $\mathcal{C}$ est cartésienne et X est un foncteur: $\mathcal{C} \longrightarrow \text{Ens}$ alors, X est un pro-objet $\Leftrightarrow$ $X(-)$ est exact à gauche.

<u>Démonstration:</u> (i) Si X est un pro-objet, alors $X(-) = \underset{I^{op}}{\text{colim}}\ \mathcal{C}(X_i,-)$ dans $(\text{Ens}^{\mathcal{C}})$ où I est cofiltrante donc I^{op} est filtrante. Mais chaque

$C(X_i,-)$ préserve les limites finies et le foncteur "colimite filtrante": $Ens^{I^{op}} \longrightarrow Ens$ y commute (Voir [2], Ch. IX, § 2). Donc $X(-)$ est exact à gauche.

(ii) Si $X(-)$ est exact à gauche alors:

Si (C,a) et (C',a') sont des objets de $\left(\text{Diagr}(X)\right)^{op} = (C|X)^{op}$ alors les projections:

$$C \times C' \xrightarrow{\pi} C \qquad C \times C' \xrightarrow{\pi'} C'$$

sont préservés par $X(-)$ donc on a:

$$X(\pi)(a,a') = a$$
$$X(\pi')(a,a') = a'$$

π et π' sont donc des morphismes de $\left(\text{diagr}(X)\right)^{op}$. De même, si f et g $(C,a) \rightrightarrows (C',a')$ sont des morphismes de $\left(\text{Diagr}(X)\right)^{op}$, et si $E \xrightarrow{m} C$ est l'égalisateur de f et g dans C, alors $a \in X(E) \xrightarrow{X(m)} X(C)$ puisque $X(E)$ est l'égalisateur de $X(f)$ et $X(g)$.

Donc: $(E,a) \xrightarrow{m} (C,a) \underset{g}{\overset{f}{\rightrightarrows}} (C',a')$ est un diagramme de Diagr(X) où $f\,m = g\,m$. La catégorie $\left(\text{Diagr}(X)\right)^{op}$ est donc cofiltrante, et par conséquent Diagr(X) est filtrante.

3) Le Théorème d'approximation uniforme:

Remarque préliminaire: Si $X = (X_i)_{i \in I}$ et $Y = (Y_i)_{i \in I}$ sont des pro-objets de C indexés par la même catégorie I et si $f = (f_i)_{i \in I}$ est une famille compatible de morphismes: $f_i \in C(X_i, Y_i)$, alors f définit un morphisme de pro-objets: $f: X \longrightarrow Y$. En d'autres termes, on a une inclusion:

$$C^I(X,Y) \subset \text{Pro-}C(X,Y)$$

Dans ce cas bien particulier, on peut dire que f est une limite cofiltrante des morphismes f_i (de C) ou bien que f peut être approximée par des morphismes de C.

Dans le théorème d'approximation uniforme, nous montrons qu'étant donné un morphisme (ou un diagramme fini) de Pro-C $f: X \longrightarrow Y$ on peut ré-indexer X et Y par une même catégorie K de sorte f soit équivalent à une famille co-filtrante de morphismes (ou de diagrammes) de C, donc à un pro-objet de C^2:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \| \wr & \circlearrowright & \| \wr \\ (X_k)_{k \in K} & \xrightarrow[k \in K]{(f_k)} & (Y_k)_{k \in K} \end{array}$$

Plus précisément, K va être la catégorie des morphismes de C qui représentent f dans le sens suivant:

Définition: Si $X = (X_i)_{i \in I} \xrightarrow{f} Y = (Y_j)_{j \in J}$ est un morphisme de pro-objets, alors un morphisme de C,

$$\varphi: X_i \longrightarrow Y_j$$

représente f si et seulement si le diagramme suivant commute dans Pro-C:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \downarrow & \circlearrowright & \downarrow \\ X_i & \xrightarrow{\quad \varphi \quad} & Y_j \end{array}$$

En fait, nous montrons un résultat plus fort:

<u>Théorème d'approximation uniforme:</u> Si Δ est une catégorie finie ne comportant pas de boucles (i.e. chaque morphisme qui n'est pas une identité a un domaine et un codomaine distincts) alors le foncteur suivant définit une équivalence de catégories: F_Δ: Pro-$C^\Delta \rightleftarrows$ (Pro-C)$^\Delta$

$$F_\Delta: (D_i)_{i\in I} = ((X_i^\delta)^{\delta\in\Delta})_{i\in I} \longmapsto ((X_i^\delta)_{i\in I})^{\delta\in\Delta} = (X^\delta)^{\delta\in\Delta}$$

<u>Démonstration</u>: Nous allons montrer successivement que:

a) F_Δ est essentiellement surjectif.

b) F_Δ est pleinement fidèle.

a) Supposons comme hypothèse de récurence que pour toute catégorie finie sans boucles Δ' ayant n objets le foncteur $F_{\Delta'}$ est essentiellement surjectif.

Soit Δ une catégorie finie sans boucles ayant n+1 objets. Puisque Δ n'a pas de boucles, il est possible d'y trouver un objet pseudo-initial 0 dans le sens que Δ ne contienne pas de morphisme de codomaine 0. Soit alors Δ' la sous-catégorie pleine de Δ obtenue en supprimant l'objet 0. Δ est donc "décomposable" en 0, Δ' et quelques morphismes (un nombre fini) de 0 vers certains objets de Δ'. Symboliquement nous écrivons:

$$\Delta = \left(0 \dashrightarrow \Delta'\right)$$

Soit $D = (X^\delta)^{\delta\in\Delta}$ un diagramme de pro-objets que nous décomposons:

$$D = \left(X^0 \dashrightarrow (X^\delta)^{\delta\in\Delta'}\right)$$

Par hypothèse de récurence appliquée au diagramme $(X^\delta)^{\delta\in\Delta'} = D'$ de type Δ' (qui a n sommets), nous savons que D' est une limite cofiltrante de diagrammes de C: en d'autres termes il existe une catégorie cofiltrante J telle que $D' \cong (D_j')_{j\in J}$ dans (Pro-C)$^{\Delta'}$ et $D_j' = (X_j^\delta)^{\delta\in\Delta'} \in C^{\Delta'}\ \forall j\in J$. On a donc $(D_j')_{j\in J} \in$ Pro-$C^{\Delta'}$.

D'autre part X^0 est une limite cofiltrante d'objets de C: $X^0 = (X_i^0)_{i\in I}$ I cofiltrante.

Soit K la catégorie dont les objets sont les triplets (i,j,D_{ij}) où $i\epsilon I$, $j\epsilon J$, $D_{ij}\epsilon C^{\Delta}$ et D_{ij} est formé du sommet X_i^o, du diagramme D_j' et de morphismes $X_i^o \dashrightarrow D_j'$ à raison d'un morphisme φ: $X_i^o \longrightarrow X_j^{\delta}$ pour chaque morphisme $f: X^o \longrightarrow X^{\delta}$ de D, tel que φ représente f.

Notons que le diagramme D_{ij} représente D. Les morphismes de K sont définis de façon naturelle: $h: (i,j,D_{ij}) \longrightarrow (i',j',D_{i'j'})$ est la donnée de trois morphismes: $\alpha: i \longrightarrow i'$ de I

$\beta: j \longrightarrow j'$ de J

$\tau: D_{ij} \longrightarrow D_{i'j'}$ de C^{Δ}

tels que: $\tau^o = X_{\alpha}$: $X_i^o \longrightarrow X_{i'}^o$ dans C

et $\tau^{\Delta'} = D_{\beta}$: $D_j' \longrightarrow D_{j'}'$ dans $C^{\Delta'}$

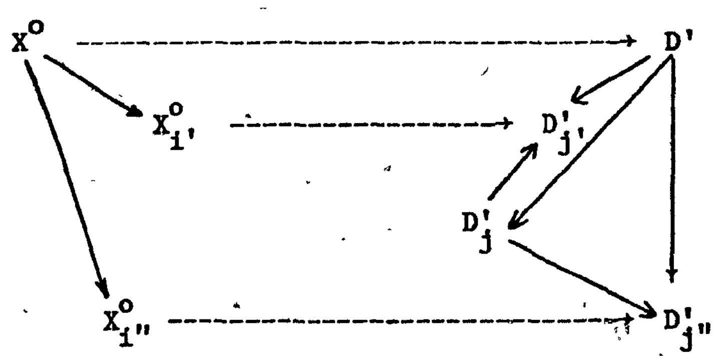
<u>Prop. 1: K est cofiltrante:</u>

(i) Soient $(i',j',D_{i'j'})$ et $(i'',j'',D_{i''j''})$ deux objets de K.

Puisque J est cofiltrante, il existe un j et deux morphismes:

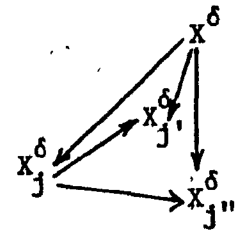
$$j \longrightarrow j', \quad j \longrightarrow j'' \quad \text{dans } J \text{ donc} \quad D_j' \longrightarrow D_{j'}', \quad D_j' \longrightarrow D_{j''}' \quad \text{dans } C^{\Delta'}$$

Dans la catégorie Pro-C nous avons le diagramme suivant:

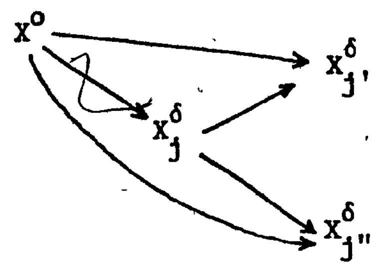


Dans la partie droite de ce diagramme, les sommets sont les objets de $C^{\Delta'}$ où de Pro-$C^{\Delta'} \hookrightarrow (\text{Pro-}C)^{\Delta'}$.

On a donc pour chaque $\delta\epsilon\Delta'$ un diagramme:



En composant les projections $X^\delta \longrightarrow X^\delta_j$ (par exemple) avec les morphismes (du diagramme donné D) $X^o \longrightarrow X^\delta$, on obtient:



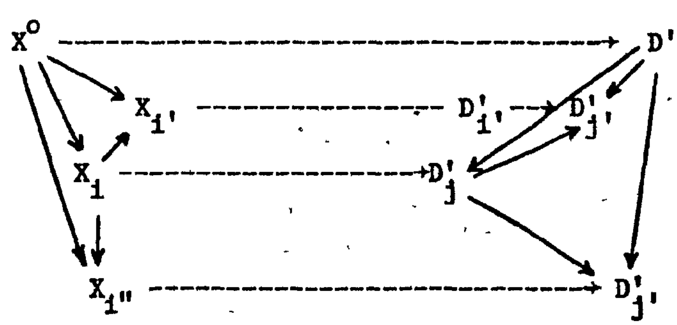
①

Maintenant, si on réunit en un seul diagramme (fini) tous les diagrammes du même type que ① pour tous les morphismes de D: $X^o \longrightarrow X^\delta$ $\delta\epsilon\Delta'$, ainsi que $X^o_{i'}$, et $X^o_{i''}$ avec leur projections, on obtient un diagramme de la catégorie $(X^o|C)$ qui est cofiltrante puisque X^o est un pro-objet et $(X^o|C) \cong (\mathrm{Diagr}(X^o))^{op}$

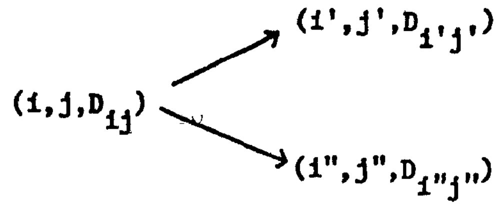
Par conséquent, ce grand diagramme est la base d'un cône dont on appelera le sommet X^o_i.

En fait, comme nous l'avons remarqué, nous n'avons pas besoin que X^o_i soit le sommet d'un cône. Nous pouvons nous limiter à construire un morphisme: $X^o_i \longrightarrow X^\delta_j$ pour chaque $X^o \longrightarrow X^\delta$ de D, de sorte que le diagramme $X^o_i \dashrightarrow D'_j$ soit de type Δ.

<u>Conclusion:</u> Nous obtenons finalement le diagramme:



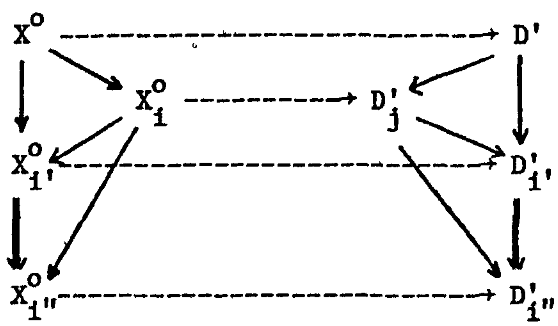
qui définit un objet (i,j,D_{ij}) de K et deux morphismes



(ii) Etant donnés deux morphismes parallèles:

$$(i',j',D_{i'j'}) \rightrightarrows (i'',j'',D_{i''j''}),$$

de façon analogue, nous obtenons d'abord un diagramme D'_j et un morphisme de $C^{\Delta'}: D'_j \longrightarrow D'_{j'}$, qui égalise les deux morphismes: $D'_{j'} \rightrightarrows D'_{j''}$, et ensuite un objet X^o_i et des morphismes tel qu'on ait le diagramme suivant:

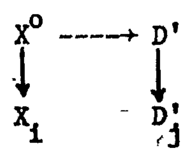


On a donc un morphisme de $K: (i,j,D_{ij}) \longrightarrow (i',j',D_{i'j'})$ qui égalise les deux morphismes donnés, ce qui prouve l'unicité essentielle et termine la démonstration de la proposition 1.

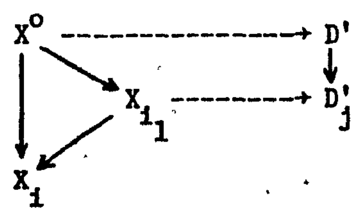
Prop. 2: Les foncteur "projection" $p_1: K \longrightarrow I$, $(i,j,D_{ij}) \longmapsto i$ et $p_2: K \longrightarrow J$, $(i,j,D_{ij}) \longmapsto j$ sont cofinals.

Démonstration: (i) Montrons qu'étant donnés $i' \in I$ et $j' \in J$ il existe un $k = (i_1,j_1,D_{i_1j_1})$ et deux morphismes: $i_1 \longrightarrow i$ et $j_1 \longrightarrow j$:

En utilisant le fait que la catégorie $(X^o \mid C)$ est cofiltrante, nous pouvons compléter le diagramme



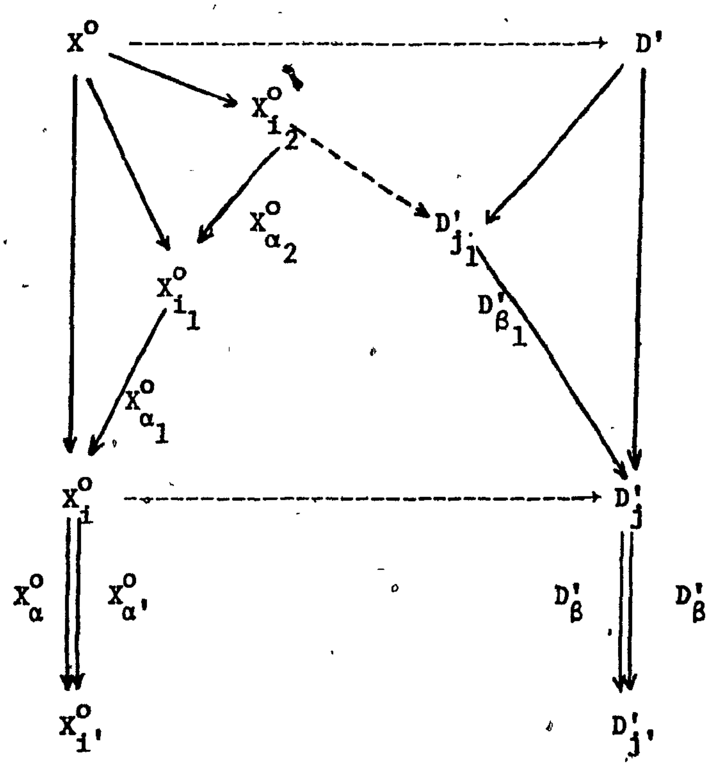
en:



ce qui détermine l'objet $k = (i_1, j, D_{i_1 j})$ avec $D_{i_1 j} = X_{i_1} \dashrightarrow D'_j$ (ici, $j_1 = j$).

Les morphismes cherchés sont $i_1 \longrightarrow i$, induit par $X_{i_1} \longrightarrow X_i$ et l'identité: $j \longrightarrow j$.

(ii) Etant donnés $i \in I$, $j' \in J$, $(i, j, D_{ij}) \in K$ et les morphismes: $\alpha, \alpha' : i \rightrightarrows i'$ et $\beta, \beta' : j \rightrightarrows j'$, nous pouvons égaliser séparément α et α' par $\alpha_1 : i_1 \longrightarrow i$ et β et β' par $\beta_1 : j_1 \longrightarrow j$ puis utiliser (i) pour obtenir le diagramme:



On a donc un objet $(i_2, j, D_{i_2 j_1})$ et un morphisme $(\alpha_1 \circ \alpha_2, \beta_1)$. $(i_2, j_1, D_{i_2, j_1}) \longrightarrow (i, j, D_{ij})$ de K tels que: $\alpha_1 \circ \alpha_2$ égalise α et α' et β_1 égalise β et β'.

<u>Conclusion</u>: Considèrons la famille (cofiltrante) $(D_k)_{k \in K}$ où pour chaque $k = (i, j, D_{ij})$, $D_k = D_{ij} = \left(D^o_i \dashrightarrow D'_j\right)$. La proposition 2 nous permet de conclure qu'on a des isomorphismes naturels de pro-C et pro-$C^{\Delta'} \hookrightarrow (\text{pro-}C)^{\Delta'}$:

$$\left(X^o_{p_1(k)}\right)_{k\in K} \cong \left(X_i\right)_{i\in I} = X$$

et

$$\left(D'_{p_2(k)}\right)_{k\in K} \cong \left(D'_j\right)_{j\in J} = D'$$

De plus, à cause de leur naturalité, ils forment des carrés commutatifs avec tous les morphismes que constituent les flêches pointillées.

On a donc un isomorphisme naturel de $(\text{pro-}C)^{\Delta}$

$$F_{\Delta}\left(\left(D_{p_1(k)p_2(k)}\right)_{k\in K}\right) \cong D$$

ce qui prouve que F_{Δ} est essentiellement surjectif.

b) <u>F_{Δ} est pleinement fidèle:</u>

Voyons comment F_{Δ} agit sur les morphismes:

Soit $f\colon D \longrightarrow E$, un morphisme de $\text{pro-}C^{\Delta}$. f est un objet de $(\text{Pro-}C^{\Delta})^{\mathbb{2}}$ où $\mathbb{2}$ est la catégorie: $\mathbb{2} = (1 \longrightarrow 2)$, qui n'a pas de boucle. Par conséquent, en appliquant la partie a) du théorème à $F_{\mathbb{2}}\colon \text{Pro-}\left((C^{\Delta})^{\mathbb{2}}\right) \longrightarrow (\text{Pro-}C^{\Delta})^{\mathbb{2}}$, on peut construire un pro-objet $C = (C_k)_{k\in K} \in \text{Pro-}\left((C^{\Delta})^{\mathbb{2}}\right)$ tel qu'on ait un isomorphisme de $(\text{Pro-}C^{\Delta})^{\mathbb{2}}$: $\Phi\colon F_{\mathbb{2}}(C) \cong (D \xrightarrow{f} E)$.

Bien entendu on a pour chaque $k\in K$: $C_k = \left(D_k \xrightarrow{f_k} E_k\right)$ dans C^{Δ} et on a le diagramme:

$$\begin{array}{ccc} D & \cong & (D_k)_{k\in K} \\ f\downarrow & & \downarrow (f_k)_{k\in K} \\ E & \cong & (E_k)_{k\in K} \end{array}$$

qui est commutatif puisque Φ est un isomorphisme naturel.

On a donc aussi: $D_k = (X_k^\delta)^{\delta\in\Delta}$ et $E_k = (Y_k^\delta)^{\delta\in\Delta}$

$\varphi: F_\Delta(D) \xrightarrow{\simeq} F_\Delta((D_k)_{k\in K}) = ((X_k^\delta)_{k\in K})^{\delta\in\Delta}$

et $\psi: F_\Delta(E) \xrightarrow{\simeq} F_\Delta((E_k)_{k\in K}) = ((Y_k^\delta)_{k\in K})^{\delta\in\Delta}$

et: $f_k = (f_k^\delta)^{\delta\in\Delta}$ puisque $f_k \in (C^\Delta)^{\mathbb{2}}$

par conséquent, il est naturel de prendre:

$$\boxed{F_\Delta(f) = \psi^{-1}\circ((f_k^\delta)_{k\in K})^{\delta\in\Delta}\circ\varphi}$$

$$\begin{array}{ccc} F_\Delta(D) & \xrightarrow[\simeq]{\varphi} & ((X_k^\delta)_{k\in K})^{\delta\in\Delta} \\ {\scriptstyle F_\Delta(f)}\downarrow & & \downarrow{\scriptstyle ((f_k^\delta)_{k\in K})^{\delta\in\Delta}} \\ F_\Delta(E) & \xrightarrow[\simeq]{\psi} & ((Y_k^\delta)_{k\in K})^{\delta\in\Delta} \end{array}$$

Avec cette définition, il est clair que si g est un morphisme $g: F_\Delta(D) \longrightarrow F_\Delta(E)$, après composition avec φ^{-1} et ψ on obtient une famille $(g_k^\delta)_{k\in K}^{\delta\in\Delta}$ définie de manière unique, donc un unique morphisme $g_1 = (g_k^\delta)_{k\in K}^{\delta\in\Delta}$

$$\begin{array}{cccc} g_1: & (D_k)_{k\in K} & \longrightarrow & (E_k)_{k\in K} \\ & \wr\| & & \wr\| \\ \text{unique } g_2: & D & \longrightarrow & E \end{array}$$

donc un unique g_2 tel que $F_\Delta(g_2) = g$.

<u>Conclusion:</u> F_Δ étant essentiellement surjectif et pleinement fidèle, c'est une équivalence de catégorie: $\text{Pro-}C^\Delta \simeq (\text{Pro-}C)^\Delta$.

4) Cas des Catégories à limites finies

Dans le cas où C est à limites finies, le résultat $(\text{Pro-}C)^{\Delta} \cong \text{Pro-}C^{\Delta}$ est valide pour toute catégorie finie Δ. De plus, il peut être démontré comme corollaire d'un théorème plus général:

Théorème: Si Cart est la catégorie des catégories à limites (à gauche) finies et si Δ est une catégorie finie, alors les endofoncteurs $(-)^{\Delta}$ et $(-)^{\Delta^{op}}$ sont adjoints (à droite et à gauche) l'un de l'autre.

Démonstration: Pour préserver la symétrie entre ces deux foncteurs nous allons utiliser la caractèrisation de l'adjonction $(-)^{\Delta} \dashv (-)^{\Delta^{op}}$ par unité et co-unité.

1) unité:

$$\eta : I \longrightarrow (-)^{\Delta\times\Delta^{op}}$$

$$\eta_A : A \longrightarrow A^{\Delta\times\Delta^{op}}$$

$$A \longmapsto A^{\Delta(-,-)}$$

2) co-unité:

$$\varepsilon : (-)^{\Delta^{op}\times\Delta} \longrightarrow I$$

$$\varepsilon_B : B^{\Delta^{op}\times\Delta} \longrightarrow B$$

$$F(-,-) \longmapsto \int F$$

où: $\int F$ est défini par:

$$\int F = \lim_{f\in\Delta} \left(\begin{array}{ccc} F(\sigma,\sigma) & \xrightarrow{F(\sigma,\ \sigma \xrightarrow{f} \delta)} & \\ & & F(\sigma,\delta) \\ F(\delta,\delta) & \xrightarrow{F(\sigma \xrightarrow{f} \delta,\ \delta)} & \end{array} \right)$$

3) Nous devons vérifier que les composés suivants sont des identités:

(i) $$A^{\Delta} \xrightarrow{(\eta_A)^{\Delta}} A^{\Delta\times\Delta^{op}\times\Delta} \xrightarrow{\varepsilon_{(A^{\Delta})}} A^{\Delta}$$

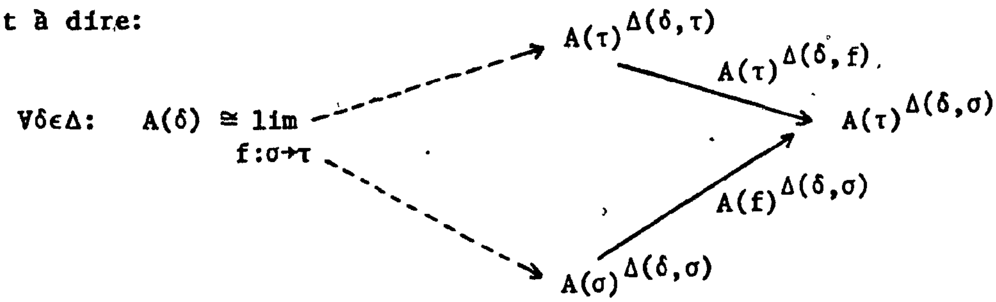
$$A(-_1) \longmapsto A(-_3)^{\Delta(-_1,-_2)} \longmapsto \int_{2,3} A(-_3)^{\Delta(-_1,-_2)} = A(-_1)$$

(ii) $$B^{\Delta^{op}} \xrightarrow{\eta_{(B^{\Delta^{op}})}} B^{\Delta^{op}\times\Delta\times\Delta^{op}} \xrightarrow{(\varepsilon_B)^{\Delta^{op}}} B^{\Delta^{op}}$$

$$B(-_1) \longmapsto B(-_1)^{\Delta(-_2,-_3)} \longmapsto \int_{1,2} B(-_1)^{\Delta(-_2,-_3)} = B(-_3)$$

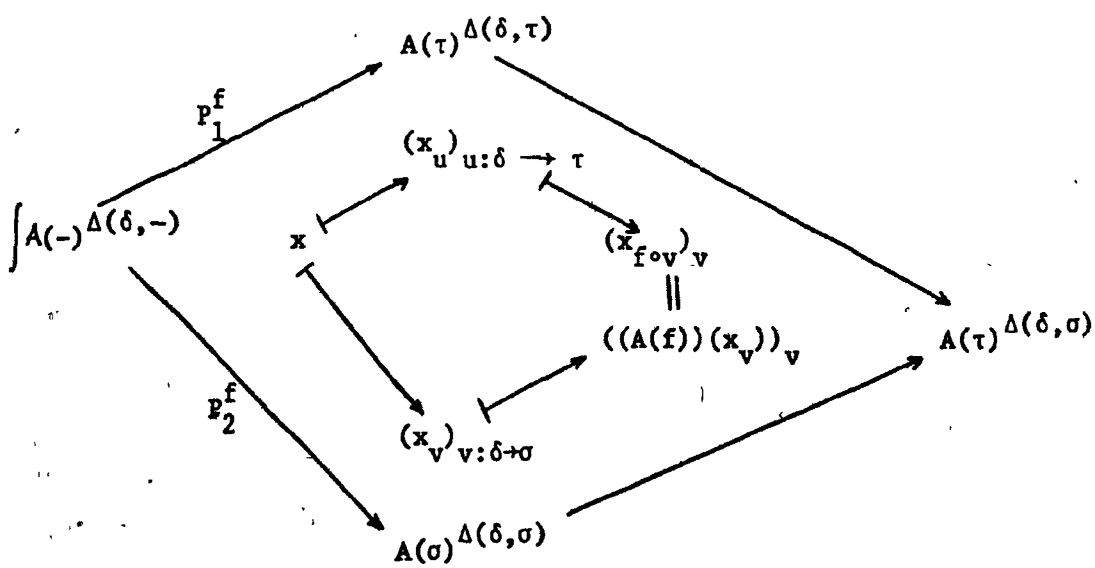
Nous voyons que (ii) est la transformée de (i) en changeant Δ en Δ^{op}, ce qui était prévisible puisqu'on obtient (ii) à partir de (i) en interchangeant les roles des 2 foncteurs.

Vu que Δ est une catégorie finie arbitraire, il suffit donc de montrer (i), c'est à dire:



Pour prouver cette égalité nous allons prouver que pour un $X \epsilon \mathcal{A}$ arbitraire, on a: $\mathrm{Hom}(X,A(\delta)) \cong \mathrm{Hom}\left(X,\int A(-)^{\Delta(\delta,-)}\right)$ ce qui revient à dire que $\mathcal{A}$ = Ens puisque $\mathrm{Hom}(X,\lim F) \cong \lim \mathrm{Hom}(X,F)$ et $\mathrm{Hom}\left(X,A(\tau)^{\Delta(\delta,\tau)}\right) \cong \mathrm{Hom}\left(X,A(\tau)\right)^{\Delta(\delta,\tau)}$.

Un élément x de $\int A(-)^{\Delta(\delta,-)}$ est donc une famille $(x_u)_{\substack{u:\delta\to\tau \\ \tau\epsilon|\Delta|}}$ satisfaisant $x_u \in A(\mathrm{codom}\ u) = A(\tau)$ et la relation de compatibilité: pour tout $f\colon \sigma \longrightarrow \tau$



par conséquent on a:

$$\underline{\forall f \in \Delta(\sigma,\tau)\ \forall v \in \Delta(\delta,\sigma)\ A(f)x_v = x_{f \circ v}}$$

En particulier, si on prend $\sigma = \delta$ et $v = 1_\delta$; on a:

$$\underline{\forall \tau \in |\Delta|,\ \forall f \in \Delta(\delta,\tau) \quad A(f)x_{1_\delta} = x_f}$$

On a donc une correspondance bi-univoque:

$$A(\delta) \longrightarrow \int A(-)^{\Delta(\delta,-)}$$

$$a \longmapsto (A(f)(a))_{f\in\Delta(\delta,\tau)} = (x_u)_{u\in\Delta(\delta,\tau)}$$

<u>Conclusion:</u> On a bien $\boxed{\int A(-)^{\Delta(\delta,-)} \cong A(\delta)}$ ce qui termine la démonstration du théorème.

<u>Corollaire:</u> Si C est une catégorie à limites à gauche finies et Δ est une catégorie finie, on a: $\underline{\text{Pro-}C^\Delta \cong (\text{Pro-}C)^\Delta}$.

<u>Démonstration:</u> A cause de l'adjonction précédemment démontrée on a:

$$(\text{Pro-}C^\Delta)^{op} \cong \widetilde{\text{Cart}}(C^\Delta,\text{Ens}) \cong \text{Cart}(C,\text{Ens}^{\Delta op}) \tag{1}$$

D'autre part, on a aussi une équivalence:

$$\bar{\text{Cart}}(C,\text{Ens})^{\Delta op} \cong \text{Cart}(C,\text{Ens}^{\Delta op}) \tag{2}$$

qui est la restriction aux foncteurs exacts à gauche de l'équivalence:

$$(\text{Ens}^C)^{\Delta op} \rightleftarrows \text{Ens}^{C\times\Delta op} \rightleftarrows (\text{Ens}^{\Delta op})^C$$

$$(\delta \mapsto (C \mapsto F^\delta(C))) \longmapsto ((C,\delta) \mapsto F^\delta(C)) \longmapsto (C \mapsto (\delta \mapsto F^\delta(C)))$$

Finalement, le lemme suivant nous fournit une troisième équivalence:

<u>Lemme:</u> Si $\mathcal{A}$ et $\mathcal{B}$ sont deux catégories, on a un foncteur involutif donc une équivalence:

$$\Phi: \mathcal{A}^{(\mathcal{B}^{op})} \rightrightarrows ((\mathcal{A}^{op})^{\mathcal{B}})^{op} .$$

En appliquant ce lemme à $\mathcal{A} = \mathrm{Cart}(C,\mathrm{Ens})$ et $\mathcal{B} = \Delta$ on obtient:

$$\mathrm{Cart}(C,\mathrm{Ens})^{\Delta op} \cong ((\mathrm{Cart}(C,\mathrm{Ens})^{op})^{\Delta})^{op} \cong ((\mathrm{Pro}\text{-}C)^{\Delta})^{op} \qquad (3)$$

<u>Conclusion</u>: En combinant les équivalences (1), (2) et (3), on obtient:

$$(\mathrm{Pro}\text{-}C^{\Delta})^{op} \overset{(1)}{\cong} \mathrm{Cart}(C,\mathrm{Ens}^{\Delta op}) \overset{(2)}{\cong} \mathrm{Cart}(C,\mathrm{Ens})^{\Delta op} \overset{(3)}{\cong} ((\mathrm{Pro}\text{-}C)^{\Delta})^{op}$$

donc:

$$\boxed{\mathrm{Pro}\text{-}(C^{\Delta}) \cong (\mathrm{Pro}\text{-}C)^{\Delta}}$$

<u>Démonstration du lemme:</u> Φ est défini de manière suivante:

(i) Pour $F \in \mathcal{A}^{(\mathcal{B}^{op})}$, $\Phi(F) = \tilde{F}$: $\begin{cases} B \longmapsto F(B) \\ f \longmapsto (F(f^{op}))^{op} \end{cases}$

Notons que si $f \in \mathcal{B}(B,B')$ alors $f^{op} \in \mathcal{B}^{op}(B',B)$, donc

$$F(f^{op}) \in \mathcal{A}(F(B'), F(B)) \text{ et } (F(f^{op}))^{op} \in \mathcal{A}^{op}(F(B), F(B'))$$

on a donc bien: $\tilde{F} = \Phi(F) \in |(\mathcal{A}^{op})^{\mathcal{B}}|$

(ii) Pour $\varphi \in \mathcal{A}^{(\mathcal{B}^{op})}(F,G)$, $\Phi(\varphi) = \tilde{\varphi}$: $\Phi(G) \longrightarrow \Phi(F)$

est définie par:

pour chaque $B \in \mathcal{B}$, $\tilde{\varphi}_B = \varphi_B^{op} \in \mathcal{A}^{op}(G(B), F(B))$

Vérifions que $\tilde{\varphi} = \Phi(\varphi)$ est naturelle:

Si $f \in \mathcal{B}(B,B')$, alors $f^{op} \in \mathcal{B}^{op}(B',B)$ donc on a:

$$\begin{array}{ccc} F(B') & \xrightarrow{\varphi_{B'}} & G(B') \\ {\scriptstyle F(f^{op})}\big\downarrow & \circlearrowright \ \text{dans } A & \big\downarrow{\scriptstyle G(f^{op})} \\ F(B) & \xrightarrow[\varphi_B]{} & G(B) \end{array}$$

donc:

$$\begin{array}{ccc} F(B') & \xleftarrow{\varphi_{B'}{}^{op}} & G(B') \\ {\scriptstyle F(f^{op})^{op}}\big\uparrow & \circlearrowright \ \text{dans } A^{op} & \big\uparrow{\scriptstyle G(f^{op})^{op}} \\ F(B) & \xleftarrow{\varphi_B{}^{op}} & G(B) \end{array}$$

Le deuxième diagramme pouvant être ré-écrit:

$$\begin{array}{ccc} \tilde{F}(B') & \xleftarrow{\tilde{\varphi}_{B'}} & \tilde{G}(B') \\ {\scriptstyle \tilde{F}(f)}\big\uparrow & \circlearrowright & \big\uparrow{\scriptstyle \tilde{G}(f)} \\ \tilde{F}(B) & \xleftarrow{\tilde{\varphi}_B} & \tilde{G}(B) \end{array}$$

il montre que $\tilde{\varphi} = \Phi(\varphi)$ est une transformation naturelle $\Phi(\varphi): \Phi(G) \longrightarrow \Phi(F)$.

<u>Conclusion</u>: Φ est bien un foncteur $A^{(B^{op})} \longrightarrow ((A^{op})^{B})^{op}$ qui est visiblement involutif; c'est donc une équivalence.

5) <u>Contre-exemple dans le cas où Δ a des boucles:</u>

Nous donnons ici un contre-exemple de la propriété duale: $\text{Ind-}(C^{\Delta}) \cong (\text{Ind-}C)^{\Delta}$ dans le cas où Δ est la catégorie $\mathbb{Z}_2$: $* \circlearrowleft \sigma$ où $\sigma^2 = 1_*$ et $\sigma \neq 1_*$.

Ici Ind-C est la catégorie des Ind-objets de C c'est à dire des colimites filtrantes de foncteur représentables dans $Ens^{C^{op}}$ qui contient C par le foncteur d'Yonéda.

Nous considérons la catégorie B dont les objets sont les boules unité de toute dimension $n \in \mathbb{N}$ et dont les morphismes sont les inclusions i_n et leurs opposés $s_n = -i_n$.

$$B_n = \{(x_1,\dots,x_n),\ \sum_1^n x_i^2 \leq 1\}$$

$$s_n, i_n : B_n \longrightarrow B_{n+1}$$

$$i_n : (x_1,\dots,x_n) \longmapsto (x_1,\dots,x_n, \sqrt{1 - x_1^2 \dots x_n^2})$$

$$s_n : (x_1,\dots,x_n) \longmapsto (-x_1,\dots,-x_n, -\sqrt{1 - x_1^2 \dots x_n^2})$$

Pour alléger les notations, nous écrirons i et s, au lieu de i_n et s_n.

<u>Notons que</u>: 1) $\text{Im}(i) \subset S_n$ & $\text{Im}(s) \subset S_n$

2) $i^2(x_1,\dots x_n) = (x_1,\dots,x_n, \sqrt{1 - x_1^2 \dots\dots x_n^2}, 0)$

$= s^2(x_1,\dots x_n) = (x_1,\dots,x_n, \sqrt{1 - x_1^2 \dots -x_n^2}, 0)$

3) $i \circ s(x_1,\dots,x_n) = s \circ i(x_1,\dots,x_n) = (-x_1,\dots,-x_n, -\sqrt{1 - x_1^2 \dots x_n^2}$

<u>On a donc les relations:</u>

$$i^2 = s^2 = -i \circ s = -s \circ i$$

et

$$-i = s$$

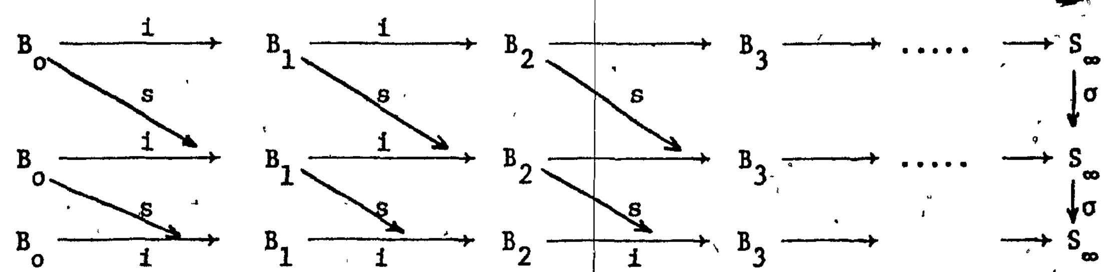
Il est évident que la catégorie B n'a pas de colimites finies puisqu'elle n'a ni coproduits finis, ni coégalisateurs. Par contre, on peut remarquer que les idempotents y sont trivialement scindés puisqu'il n'y en a pas d'autres que les identités.

Considèrons la sous-catégorie filtrante de $\mathcal{B}$ obtenue en supprimant les morphismes s:

$$B_o \xrightarrow{\ i\ } B_1 \xrightarrow{\ i\ } B_2 \cdots\cdots \longrightarrow B_n \longrightarrow \cdots\cdots$$

Soit $S_\infty = \operatorname{colim} B_n = \{(x_n)_{n\in\mathbb{N}}\ ,\ x_n = 0\ \forall_n \gg 0 \text{ et } \sum_0^\infty x_n^2 = 1\}$

Comme le montre le diagramme ci-dessous, S_∞ définit un ind-objet sur $\mathcal{B}$ et la famille des morphismes s définit un endomorphisme involutif $\sigma: S_\infty \longrightarrow S_\infty$, qui est en fait l'application antipode.



1) Le diagramme est commutatif puisque $i\circ s = s\circ i$ & $s\in\mathcal{B}(B_n,B_{n+1}) \subset \operatorname{colim}\mathcal{B}(B_n,B_p)$

donc $\sigma\in \text{Ind-}\mathcal{B}(S_\infty,S_\infty) = \lim_n \operatorname{colim}_p \mathcal{B}(B_n,B_p)$

2) $\sigma^2 = 1_{S_\infty}$ puisque $s\circ s = i\circ i$

3) $\sigma \neq 1_{S_\infty}$ puisque $s = -i$ et $s(x) = -i(x) \neq i(x)$.

<u>Conclusion:</u> σ est donc un endomorphisme de indobjets involutif mais qui n'est pas représentable par une famille d'involutifs puisqu'il n'y en a pas de non-triviaux.

CHAPTER III

REGULARITY AND EXACTNESS PROPERTIES OF THE CATEGORY OF PRO-OBJECTS

0) Introduction

In this chapter, we look more closely at some regularity and exactness properties that can be lifted from the category C to the category pro-C.

It is known for instance that finite limits and colimits that exist in C also exist in pro-C ([1], section 4). The proof of this lifting was the motivation for the "Uniform approximation theorem" discussed in the previous chapter.

After a brief overview of known lifting results (section 1), we look at the lifting of the unique-factorisation system M-E in C to pro-M— pro-E in pro-C, where pro-M and pro-E are defined in a natural way (section 2).

Then we look at the lifting of regularity properties, and prove that if E is stable under pulling-back in C, then so is pro-E in pro-C. Then, looking at the particular case where M & E are respectively the classes of all monos and all regular epimorphisms, we get as corollary the fact that if C is a regular category, then so is pro-C (section 3).

Finally, in sections 4 and 5, we look at the lifting of exactness, which works in all categories in which transitivity is redundant for equivalence relations. This proof is an application of the uniform approximation theorem in the case of a finite category with loops, which allows us to approximate any reflexive and symmetric relation in pro-C by a cofiltered family of reflexive and symmetric relations in C .

The counterexample given in section 5 is quite well-known: we give a direct proof of the equivalence between pro-finite sets and Stone spaces, which do not form an exact category. Then we show that the exact completion of the category of Stone spaces is the category of compact Hausdorf spaces. For that, we do not use the construction described in Chapter I because it does not preserve the regular structure of the category of Stone spaces; instead we use the construction given in [9], which extends a regular category C to an exact one $Q(C)$ in a universal way.

1) Pro-morphisms & pro-diagrams

Let I be a small cofiltered category and C be any category. There exists an obvious functor: $C^I \hookrightarrow$ pro-C which is an inclusion on objects, but is neither full, nor faithful .

For instance, the counterexample of the previous chapter provides us with a morphism σ in (Ind-B) (S_∞^-, S_∞) which cannot be defined as a filtered family of B-morphisms $\sigma_n: B_n \longrightarrow B_n$ because the only endomorphisms of B are the identities.

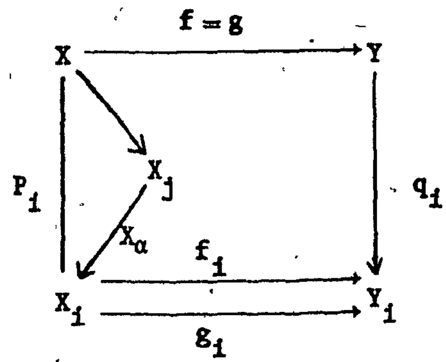
But the uniform approximation theorem gives us a "pseudo-fullness" condition as remarked in Section 3 of the previous chapter namely: every pro-morphism is equal (up to natural isomorphism) to the cofiltered family of morphisms that represent it.

Faithfulness also fails. In fact, we have the following:

Lemma 1.1: Two cofiltered families $(f_i)_{i\in I}$ & $(g_i)_{i\in I}$

$$\text{with} \qquad X_i \underset{g_i}{\overset{f_i}{\rightrightarrows}} Y_i$$

define the same pro-morphism (i.e.: $f = g$) if and only if for all i in I there is a j and an $\alpha: j \longrightarrow i$ in I such that $f_i \circ X\alpha = g_i \circ X\alpha$:



__Indeed,__

$$\begin{aligned} f = g \;&\Leftrightarrow\; q_i f = q_i g \text{ for all } i \in I \\ &\Leftrightarrow\; f_i p_i = g_i p_i \text{ for all } i \in I \\ &\Leftrightarrow\; f_i \circ X\alpha = g_i \circ X\alpha \text{ for some } \\ &\qquad \alpha: j \longrightarrow i \text{ in } I \end{aligned}$$

Note that in the last step, we use the fact that there is a cofinal functor $I \longrightarrow \text{Diagr}(X)^{op}$ the category of all pro-morphisms $X \longrightarrow C$ where C is any object of $\mathcal{C}$ (as in [1], proposition 2.7).

__Proposition 1.2:__ The above functor; $\mathcal{C}^I \longrightarrow$ pro-$\mathcal{C}$ commutes with all finite limits and colimits that exist in $\mathcal{C}^I$.

The proof is the same as the one given in [1] (proposition 4.1).

__Corollary 1.3:__ A cofiltered family of monos (epis) in $\mathcal{C}$ is a mono (an epi) in pro-$\mathcal{C}$.

__Proof:__ A cofiltered family $(f_i)_{i \in I}$ of monos (epis) in $\mathcal{C}$ is a mono (an epi) in $\mathcal{C}^I$ hence in pro-$\mathcal{C}$ since the functor: $\mathcal{C}^I \longrightarrow$ pro-$\mathcal{C}$ commutes with pull-backs (push-outs) that exist in $\mathcal{C}^I$, hence preserve monos (epis).

__Corollary 1.4:__ If $\mathcal{C}$ is closed under finite limits (colimits), then so is pro-$\mathcal{C}$.

__Proof:__ We only need to construct products and equalizers to have finite limits (and the dual for colimits).

For equalizers, for instance, given a pair of parallel pro-morphisms f & g, by the uniform approximation theorem, we can assume that they are represented by a cofiltered family of parallel morphisms: $f = (f_i)_{i \in I}$ & $g = (g_i)_{i \in I}$.

Now, by proposition 1.2, the family of equalizers E_i of f_i & g_i is the equalizer of f & g in pro-C .

$$E = (E_i)_{i \in I} \longrightarrow (X_i)_{i \in I} \cong X \overset{f}{\underset{g}{\rightrightarrows}} Y \cong (Y_i)_{i \in I}$$

$$(f_i)_{i \in I} : (X_i)_{i \in I} \to (Y_i)_{i \in I}, \qquad (g_i)_{i \in I} : (X_i)_{i \in I} \to (Y_i)_{i \in I}$$

In fact, we have the following:

<u>Proposition 1.5</u>: If C has finite limits, then given any limiting cone $L \xrightarrow{\cdot} D$ in pro-C where D is a finite pro-diagram, then there exists a cofiltered family of limiting cones $(L_i \xrightarrow{\cdot} D_i)_{i \in I}$ in C such that $D \cong (D_i)_{i \in I}$ & $L \cong (L_i)_{i \in I}$.

The dual results hold for finite colimits provided either D has no loops or C has finite limits.

<u>Proof:</u> By the uniform approximation theorem, there is a cofiltered family $(D_i)_{i \in I}$ such that $D \cong (D_i)_{i \in I}$ hence if $L_i = \lim D_i$, then by proposition 1.2, we have $(L_i)_{i \in I} = \lim (D_i)_{i \in I} \cong \lim D$ in pro-C .

<u>Hence:</u> $(L_i)_{i \in I} \cong L$.

<u>Note</u> that if D has no loops, I is in fact the category of diagrams representing D in C .

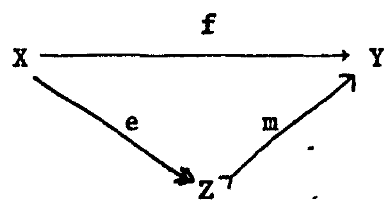
<u>Remark:</u> We have shown that limiting cones in pro-C can be approximated by limiting cones in C , but this result cannot be extended to an arbitrary finite diagram D such that one vertex is the limit of a subdiagram of D . In sections 4 & 5, we will see a typical counterexample.

2) Factorization in Pro-$\mathcal{C}$

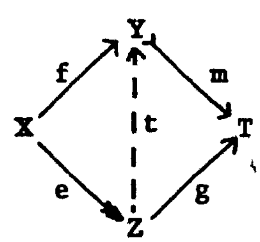
Let $\mathcal{C}$ be a finitely complete category.

Definition 2.1: A unique factorization system in $\mathcal{C}$ is given by a class $\mathcal{M}$ of monos and a class $\mathcal{E}$ of epis, both containing all isomorphisms such that:

(i) Every $\mathcal{C}$-morphism f is the composite $f = me$ where $m \in \mathcal{M}$ & $e \in \mathcal{E}$



(ii) (Diagonal property): If f, g, m, e are morphisms such that $m \in \mathcal{M}$, $e \in \mathcal{E}$ & $mf = ge$:



Then:

there exixts a (unique) $t: Z \longrightarrow Y$ such that $g = mt$ (hence $f = te$) .

Note: (ii) is equivalent to the uniqueness of the factorization in (i) (up to isomorphism).

The object Z (or the subobject of Y defined by m) is called the image of f: Z = Im(f) (or Im f)

$\mathcal{M}$ and $\mathcal{E}$ are just classes of morphisms, although they can be viewed as subcategories of $\mathcal{C}^{\mathbb{2}}$.

<u>Proposition 2.2</u>: If $\mathcal{C}$ has a unique-factorization system $\mathcal{M}$-$\mathcal{E}$ and if pro-$\mathcal{M}$ & pro-$\mathcal{E}$ are classes of pro-morphisms defined as follows:

$m \in$ pro-$\mathcal{M}$ ($e \in$ pro-$\mathcal{E}$) if and only if the category of morphisms representing m (e) has a cofinal subcategory whose objects are in $\mathcal{M}$ ($\mathcal{E}$),

<u>then</u>

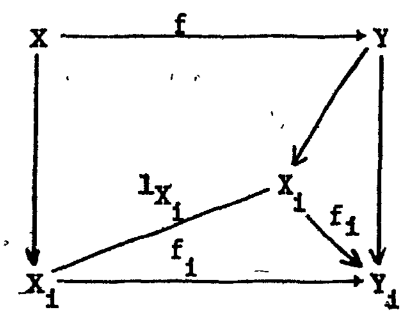
pro-$\mathcal{M}$ – pro-$\mathcal{E}$ is a unique-factorization system in pro-$\mathcal{C}$.

<u>Proof</u>: By definition, every pro-morphism of pro-$\mathcal{M}$ can be represented by a cofiltered family of morphisms of $\mathcal{M}$ which are monos, hence is a mono in pro-$\mathcal{C}$ (Corollary 1.3). Similarly, every pro-morphism of pro-$\mathcal{E}$ is an epi.

Also, if $f: X \longrightarrow Y$ is an isomorphism and if $X = (X_i)_{i \in I}$, then the family of isomorphisms $(1_{X_i})_{i \in I}$ is a cofinal subcategory of the category of morphisms f_i representing f:

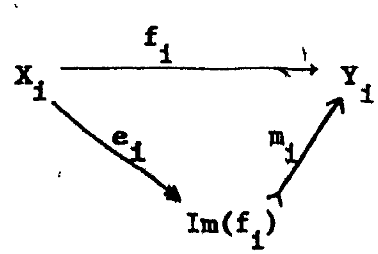
Hence, since $1_{X_i} \in \mathcal{M}$ ($\mathcal{E}$)

$f \in$ pro-$\mathcal{M}$ (pro-$\mathcal{E}$)

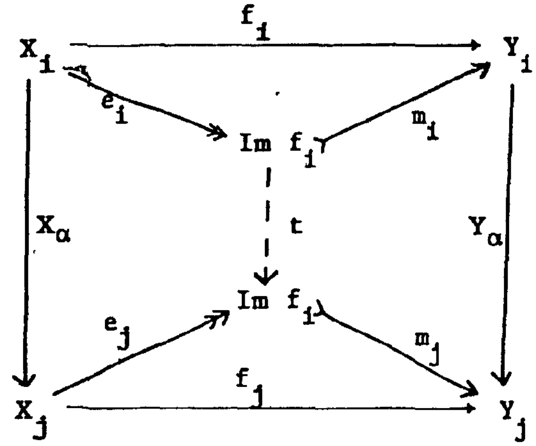


So, we only need to prove the properties (i) & (ii) of definition 2.1 :

(i) If f is a pro-morphism which we can assume to be a cofiltered family of morphisms: $f_i: X_i \longrightarrow Y_i$, then each f_i has a unique ($\mathcal{M}$-$\mathcal{E}$)-factorization :



Furthermore, if $\alpha: i \longrightarrow j$ is an I-morphism, then we have the following diagram (in C):



Since $m_j \circ (e_j \circ X_\alpha) = (Y_\alpha \circ m_i) \circ e_i$ there exists a unique $t: \mathrm{Im}(f_i) \longrightarrow \mathrm{Im}(f_j)$ s.t.: $te_i = e_j X_\alpha$ and

$m_j t = Y_\alpha m_i$.

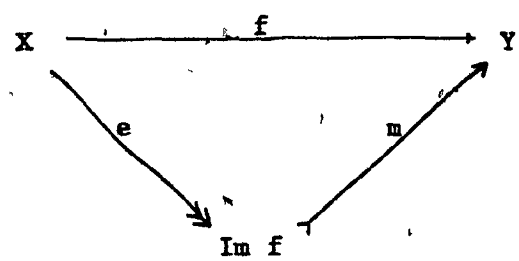
Hence if we set: $(\mathrm{Im}\ f)_i = \mathrm{Im}(f_i)$

and $(\mathrm{Im}\ f)_\alpha = t$ for $\alpha: i \longrightarrow j$ we define a functor $\mathrm{Im}(f): I \longrightarrow C$ and two natural transformations e & m

$$X \xrightarrow{\ e\ } \mathrm{Im}\ f \xrightarrow{\ m\ } Y$$

s.t.: $m \in M^I$ & $e \in E^I$.

Hence the respective images of Im f , e & m , by the functor: $C^I \longrightarrow$ pro-C, which sends M^I to pro-M and E^I to pro-E , gives a (pro-M — pro-E)-factorization of f in pro-C :



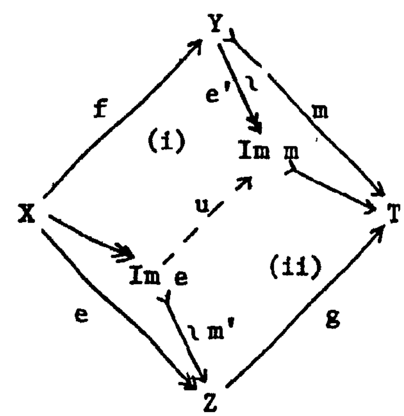
(we use the same letters to denote the objects & morphisms of C^I and pro-C) .

(ii) To prove the diagonal property, we first observe that every pro-morphism m in pro-$\mathcal{M}$ can be represented by a cofiltered family of morphisms of $\mathcal{M}$, hence that its image Im (m) is isomorphic to its domain.

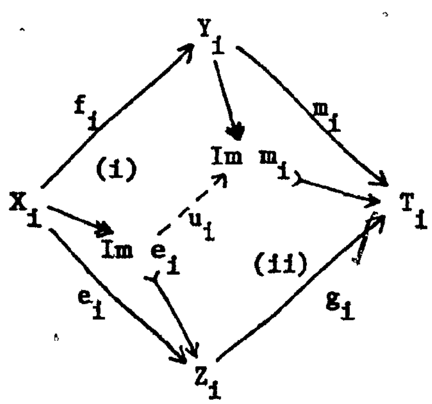
Similarly, if $e \in$ pro-$\mathcal{E}$, then Im (e) is isomorphic to its codomain. So, given the outer square D of the following pro-diagram (1)

where $m \in$ pro-$\mathcal{M}$

$e \in$ pro-$\mathcal{E}$



by the uniform approximation theorem, D can be assumed to be (up to isomorphism) a cofiltered family of diagrams of the same type: $D = (D_i)_{i \in I}$. Thus, for each i in I, we can construct the following diagram (2):



Where m_i & e_i do not need to be respectively in $\mathcal{M}$ & $\mathcal{E}$, but can be factored.

So, there exists a morphism u_i

Im e_i ⟶ Im m_i

making (i) & (ii) commutative.

Thus, since $m = (m_i)_{i \in I}$ & $e = (e_i)_{i \in I}$ we have (diagram (1)): $\operatorname{Im} m = (\operatorname{Im} m_i)_{i \in I}$ & $\operatorname{Im} e = (\operatorname{Im} e_i)_{i \in I}$, hence $u = (u_i)_{i \in I}$ defines a pro-morphism $u: \operatorname{Im} e \longrightarrow \operatorname{Im} m$ making (i) & (ii) commutative in the pro-diagram (1).

Thus, since $\operatorname{Im} m \cong Y$ (let e' be the isomorphism)

and $\operatorname{Im} e \cong Z$ (let m' be the isomorphism)

the pro-morphism $t = e'^{-1} \circ u \circ m'^{-1}$ is the required diagonal morphism such that: $m\,t = g$ and $t\,e = f$.

<u>Conclusion</u>: Pro-M — pro-E is a unique-factorization system in pro-E and satisfies for any cofiltered category I:

$|M^I| \subset$ pro-M & $|E^I| \subset$ pro-E (where $|M^I|$ ($|E^I|$) is the class of compatible I-indexed families of morphism of $M(E)$) .

In particular, if I is the one-point category, we get $M \subset$ pro-M & $E \subset$ pro-E .

<u>Hence</u>

pro-M — pro-E is a natural lifting of the unique-factorization system M-E .

~~In fact, it is unique in the following sense:~~

<u>Proposition 2.3</u>: If M'-E' is a unique-factorization system in pro-C such that for every (small) cofiltered category I, $|M^I| \subset M'$ & $|E^I| \subset E$, then M' = pro-M & E' = pro-E .

<u>Proof</u>: If $m \in$ pro-M , then m can be represented as a cofiltered family $(m_i)_{i \in I}$ where $m_i \in M$ for each i in I .

<u>hence</u> $m \in |M^I| \subset M'$ <u>so</u> pro-$M \subset M'$

<u>similarly</u> pro-$E \subset E'$

Conversely: If $m' \in M'$, then by the construction above, the (pro-M — pro-E)-factorization of m' is given by :

$$m' = (m'_i)_{i \in I} = (m_i e_i)_{i \in I} = (m_i)_{i \in I} \circ (e_i)_{i \in I} = m\, e$$

where $m = (m_i)_{i \in I} \in |M^I| \subset M'$

and $e = (e_i)_{i \in I} \in |E^I| \subset E'$

So, $m' = m\, e$ is also an $(M'$-$E')$— factorization, hence e is an isomorphism and thus: $m' \in$ pro-M .

(Note that in a unique- factorization system M-E the classes M & E are closed under composition.)

So, we have to prove that $M' \subset$ pro-M ,

similarly: $E' \subset$ pro-E .

Therefore M' = pro-M & E' = pro-E .

Conclusion and remarks: If M & E are viewed as categories then we still have: $M \subset$ pro-M & $E \subset$ pro-E but the functors $M^I \longrightarrow$ pro-M and $E^I \longrightarrow$ pro-E are not full and faithfull as remarked in section 1 .

However, since pro-M & pro-E , viewed as categories are exactly the categories of pro-objects of M & E respectively, every object of pro-M(pro-E) is a cofiltered limit of objects of $M(E)$. So, we have in fact:

Theorem 2.4: Every unique-factorization system M-E in a category C can be uniquely extended (up to isomorphism) to a unique-factorization system pro-M— pro-E of pro-C such that pro-M & pro-E contain M & E respectively and are closed under cofiltered limits.

3) Regularity properties of pro-$\mathcal{C}$

Definition 3.1: A category $\mathcal{C}$ is said to be $(\mathcal{M}\text{-}\mathcal{E})$-regular if, and only if it is closed under finite limits and has a unique-factorization system $\mathcal{M}\text{-}\mathcal{E}$ satisfying (as well as (i) & (ii) of 2.1) the following condition:

(iii) $\mathcal{E}$ is stable under pulling back.

Note: It results from 2.1 that $\mathcal{M}$ is automatically stable under pulling back.

Theorem 3.2: If $\mathcal{C}$ is $(\mathcal{M}\text{-}\mathcal{E})$-regular, then pro-$\mathcal{C}$ is (pro-$\mathcal{M}$-pro-$\mathcal{E}$)-regular.

Proof: We have to prove that if $\mathcal{E}$ is stable under pulling back in $\mathcal{C}$, then pro-$\mathcal{E}$ is stable under pulling back in pro-$\mathcal{C}$.

Let then D be the pro-diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\;e\;} & T \\ & & \uparrow f \\ & & Z \end{array}$$

such that $e \in$ pro-$\mathcal{E}$.

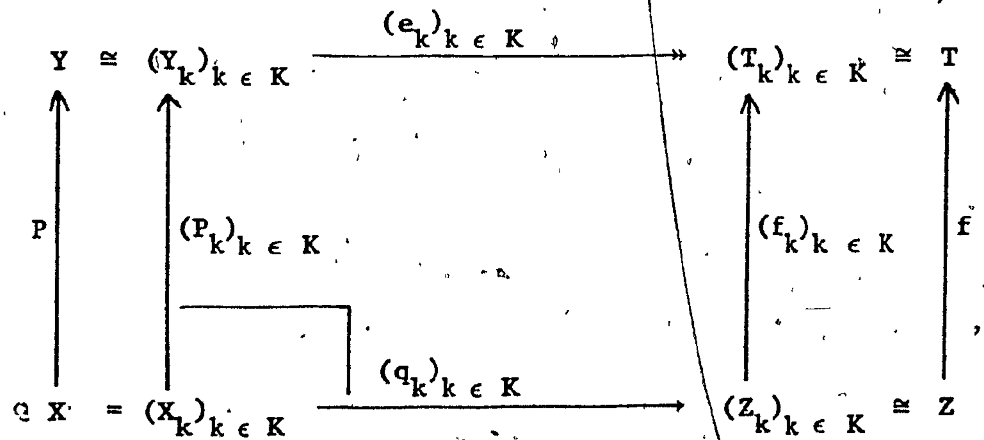
Since $e \in$ pro-$\mathcal{E}$, e can be assumed to be (up to isomorphism) a cofiltered family of morphisms of $\mathcal{E}$: $e = (e_j)_{j \in J}$.

So, if we reproduce the proof of the uniform approximation theorem for D using Z as "initial" vertex, we can prove that D is isomorphic to the cofiltered family $(D_k)_{k \in K}$ of diagrams representing D such that the morphism e_k representing e in D_k is in $\mathcal{E}$.

Note: We can also prove directly that the category of diagrams representing D such that the morphism representing e is in $\mathcal{E}$ is a cofinal subcategory of the category of all diagrams representing D.

Hence

by proposition 1.2, the pro-object $X = (X_k)_{k \in K}$ where X_k is the pull-back of D_k in C, is the pull-back of D in pro-C. And if $p = (p_k)_{k \in K}$ & $q = (q_k)_{k \in K}$ are the projections as shown below:



Then

$q_k \in E$ for each k since C is M-E-regular

Hence

$q = (q_k)_{k \in K} \in$ pro-E .

Thus

pro-C is (pro-M–pro-E)- regular .

Definition 3.3: A category is said to be regular if and only if is it (M-E)- regular where:

M is the class of all monos

E is the class of all regular epis .

Corollary 3.4: If $\mathcal{C}$ is regular then so is pro-$\mathcal{C}$.

Proof: If $\mathcal{C}$ is regular, then we know that pro-$\mathcal{E}$ is (pro-$\mathcal{M}$ — pro-$\mathcal{E}$)—regular where $\mathcal{M}$ ($\mathcal{E}$) is the class of all monos (regular epis) of $\mathcal{C}$.

So, all we have to prove is that:

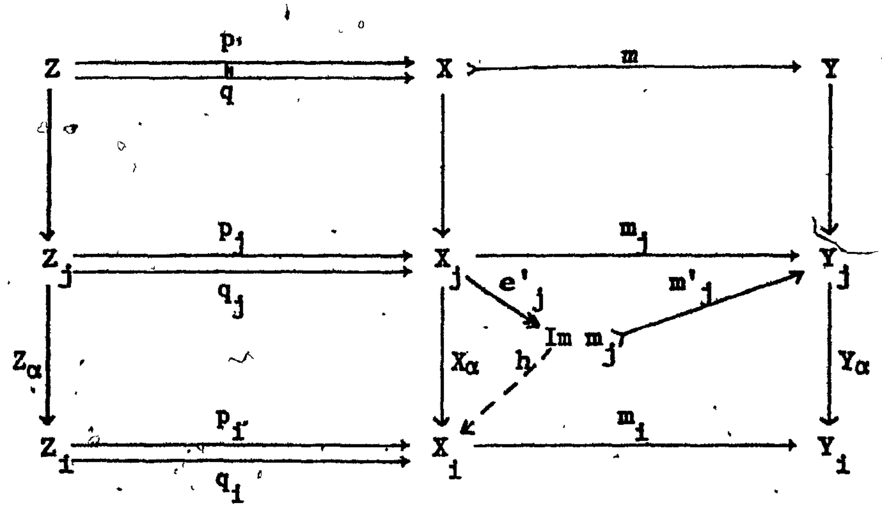
a) pro-$\mathcal{M}$ is the class of all monos of pro-$\mathcal{C}$

b) pro-$\mathcal{E}$ is the class of all regular epis of pro-$\mathcal{C}$

a) We already know that every element of pro-$\mathcal{M}$ is a mono in pro-$\mathcal{C}$.

Conversely, let $m = (m_i)_{i \in I}: X \rightarrowtail Y$ be a mono in pro-$\mathcal{C}$, and let $Z = (Z_i)_{i \in I}$ be its kernel pair, $p = (p_i)_{i \in I}$ & $q = (q_i)_{i \in I}$ being the projections (in $\mathcal{C}^I$ hence in pro-$\mathcal{C}$) .

Now, since m is a mono, the projections $p = (p_i)_{i \in I}$ & $q = (q_i)_{i \in I}$ are equal (and in fact isomorphisms). Hence by Lemma 1.1, for each i in I , there is a j in I and a morphism $\alpha: j \longrightarrow i$ such that: $p_i \circ Z_\alpha = q_i \circ Z_\alpha$ as in the diagram below:



So, if we take the Image of m_j we get a decomposition $m_j = m'_j \circ e'_j$ where

(i) m'_j is a mono

(ii) e'_j is a regular epi, hence the coequalizer of its kernel pair which is the same as the kernel pair of m_j namely Z_j .

Thus, looking at the left-lower square, we see that X_α coequalizes p_j & q_j:

$$X_\alpha \circ p_j = p_i \circ Z_\alpha = q_i \circ Z_\alpha = X_\alpha \circ q_j$$

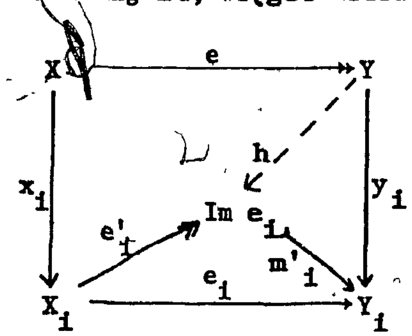
hence there exists a (unique) $h : \text{Im } m_j \longrightarrow X_i$

such that $X_\alpha = h\, e'_j$

So, $m'_j : \text{Im } m_j \rightarrowtail Y_j$ is a mono representing m that "dominates" m_i. Hence, the category of monos representing m is a cofinal subcategory of the category of all morphisms representing m, i.e.: $m \in$ pro-M.

b) For regular epis, it is obvious that a cofiltered family of regular epi $e = (e_i)_{i \in I}$ is a regular epi: since the functor $C^I \longrightarrow$ pro-C preserves both coequalizers and kernel pairs, e is the coequalizer of the family of kernel pairs of e_i, which is the kernel pairs of e.

Conversely, if e is a regular epi in pro-E and e_i a morphism representing it, we get after factoring e_i, the following diagram:



where e'_i is a regular epi in C hence in pro-C and m'_i is a mono in C hence in pro-C.

But in any category, the diagonal property holds for monos and regular epis. Hence there exists a (unique) $h: Y \longrightarrow \text{Im } e_i$ such that:

(i) $h\, e = e'_i x_i$ i.e. e'_i represents e

(ii) $m'_i h = y_i$ i.e. e'_i "dominates" e_i

So, the category of regular epis representing e is a cofinal subcategory of the category of all morphisms representing e.

i.e.: <u>$e \in$ pro-E</u>

<u>Conclusion:</u> Thus we have proved that pro-M & pro-E are respectively equal to the classes of monos and regular epis of pro-C.

<u>Thus</u>

<u>pro-C is regular</u>.

4) Exactness of pro-C under special conditions

It is known ([1] proposition 4.5) that abelianness (and additiveness) can be lifted from A to pro-A. This class of examples, in which exactness (which is part of the abelian structure) can be lifted, is special in the sense that in an abelian category, a relation only needs to be reflexive in order to be an equivalence relation.

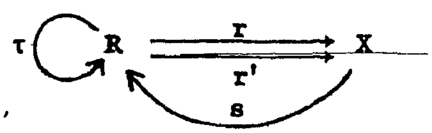
In this section, we prove that under the supplemental condition that every relation which is reflexive and symmetric is an equivalence relation, exactness can be lifted from C to pro-C.

This result is based on the following:

Lemma 4.1: If C is a regular category, every reflexive & symmetric relation in pro-C can be represented by a cofiltered family of reflexive and symmetric relations in C.

More precisely: Given a reflexive & symmetric relation R on X in pro-C, there exists a cofiltered category I and two pro-objects $(R'_i)_{i \in I} \cong R$ & $(X_i)_{i \in I} \cong X$ such that R'_i is a reflexive & symmetric relation on X_i for each i in I.

Proof: Let X be a pro-object and R a reflexive and symmetric relation on X, i.e. a subobject of $X \times X$ such that the two projections $R \underset{r'}{\overset{r}{\rightrightarrows}} X$ can be embedded into a diagram of the form:



satisfying the relations:

$$r s = r' s = 1_X$$

$$r \tau = r' \ \& \ r' \tau = r$$

We want to apply the uniform approximation theorem for finitely complete categories to the free category generated by this diagram, and for that, we have to prove that it is finite i.e. that all non-identity endomorphisms, namely τ , sr & sr' generate only a finite number of distinct morphisms. In fact we have:

(i) τ is an involution:

$$\langle r, r'\rangle\tau^2 = \langle r\tau^2, r'\tau^2\rangle = \langle r'\tau, r\tau\rangle = \langle r, r'\rangle$$

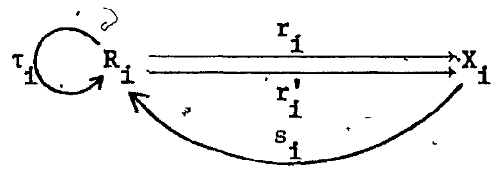
$\langle r, r'\rangle$ is a mono (i.e. R is a relation) ;

$$\tau^2 = 1_R$$

(ii) sr & sr' are idempotent:

$(sr)^2 = srsr = s1_X r = sr$ and similarly for sr' .

So this diagram is isomorphic (in the category of pro-diagrams of that type) to a cofiltered family of diagrams of the same type, i.e. of reflexive and symmetric spans (not necessarily relations):



where i is an arbitrary object of some cofiltered category I .

Now, $\langle r_i, r'_i\rangle$ need not to be a mono, but if we take its image $R'_i = \mathrm{Im}\langle r_i, r'_i\rangle$, we get a relation which is also reflexive and symmetric (Chapter I, section 1b (10). See also [6] proposition 3.3).

So, if we consider the family of relations $R' = (R'_i)_{i\in I}$ we get a pro-object which is the image of $\langle r, r'\rangle$, hence $R' \cong R$.

Conclusion: $R \cong (R'_i)_{i \in I}$

i.e.: R can be represented by a cofiltered family of reflexive and symmetric relations.

Corollary 4.2: If C is an exact category in which every reflexive and symmetric relation is also transitive, then pro-C is also exact (and has the same property as C)

Proof: Let X be a pro-object and R an equivalence relation on X (in fact R only needs to be reflexive and symmetric), then by Lemma 4.1, there exists a cofiltered family of reflexive and symmetric relations

$$\left(R'_i \overset{\rho_i}{\underset{\rho'_i}{\rightrightarrows}} X_i\right)_{i \in I} \quad \text{such that:}$$

$$(X_i)_{i \in I} \cong X, \quad (R'_i)_{i \in I} \cong R, \quad (\rho_i)_{i \in I} = r \quad \& \quad (\rho'_i)_{i \in I} = r'$$

So by assumption, R'_i is an equivalence relation on X_i for each i in I, and therefore, since C is exact, R'_i is the kernel pair of the coequalizer K_i of ρ_i & ρ'_i (which exists in a exact category). Thus, the following diagram is an exact sequence in C:

$$R'_i \overset{\rho_i}{\underset{\rho'_i}{\rightrightarrows}} X_i \xrightarrow{q_i} K_i$$

Hence if we set $K = (K_i)_{i \in I}$ & $q = (q_i)_{i \in I}$ the following pro-diagram is an exact sequence in C^I hence in pro-C (since by proposition 1.2 the functor $C^I \longrightarrow$ pro-C is exact):

$$R \overset{\rho}{\underset{\rho'}{\rightrightarrows}} X \xrightarrow{q} K .$$

Conclusion: R is effective (therefore is an equivalence relation even if we assumed it to be only reflexive and symmetric).

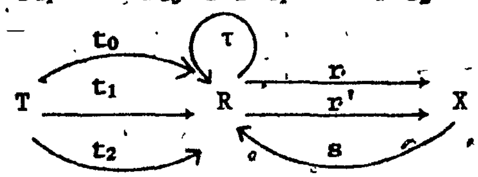
Thus: pro-C is exact .

Generalization attempt:

The key to the previous proof was that reflexivity & symmetry of a relation in pro-C can be expressed in terms of commutative diagrams (Chapter I, proposition 1.1), hence can be approximated by diagrams of the same type in C.

Now, if we attempt to approximate an equivalence relation which involves the diagram (1) below, where T is the pull-back of r & r' (t_0 & t_2 being the projections), and t_1 satisfies the relations:

$$rt_1 = rt_0 \;\&\; r't_1 = r't_2$$

(1) 

then:

a) The uniform approximation theorem will provide a cofiltered family of diagrams of the same type (with the commutation relations called (R), (S) & (T) in chapter I, 1.1) but T_i will not necessarily be the pull-back of r_i & r'_i.

b) Proposition 1.5 cannot be applied because of the morphism t_1 and the relations (T) it must satisfy (as remarked after proposition 1.5).

In fact, the next section gives an example of an exact category E_0 such that pro-E_0 is not exact, which actually proves that proposition 1.5 cannot be extended to arbitrary finite diagrams in which some vertex is a limit of a subdiagram of the original diagram.

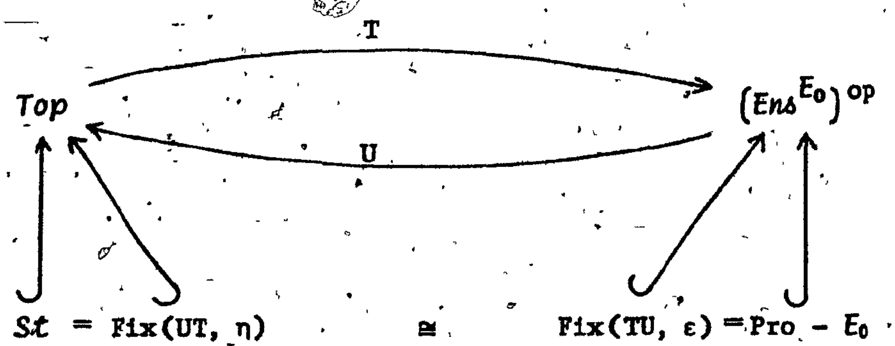
5) Counter-example of the general case: finite sets and Stone spaces

Let $\mathcal{St}$ be the category of Stone spaces (compact Hausdorff totally disconnected topological spaces) and let $\mathcal{E}_0$ be the category of finite sets (viewed as finite discrete spaces).

The purpose of this section is to show that $\mathcal{St} \cong$ pro $- \mathcal{E}_0$ (i.e.: every pro-finite set is isomorphic to some Stone space and conversely) and that $\mathcal{St}$ is not exact, but its exact completion is the category CH of compact Hausdorff spaces.

<u>Proposition 5.1</u> $\mathcal{St} \cong$ pro $- \mathcal{E}_0$

<u>Proof</u>: Consider the following diagram:



We are going to define an adjunction (T, U, η, ε) such that the full subcategory Fix(UT, η) and Fix(TU, ε) which are known to be equivalent, are respectively equal to $\mathcal{St}$ and Pro $- \mathcal{E}_0$.

<u>Claim 1</u>: The following data actually define an adjunction:

(i) For $P \in \mathcal{E}ns^{E_0}$, $\hat{U}(P) = \lim\limits_{(F,\ a) \in (\text{Diagr } P)^{op}} F$: where Diagr P is the comma category (E_0, P) of P - pointed finite sets (F, a) with $a \in PF$, and P-point preserving functions. Hence by Yoneda lemma, $(\text{Diagr } P)^{op}$ is equivalent to the category of $(\mathcal{E}ns^{E_0})^{op}$- morphisms from P to some representable functor $E_0(F, -)$ usually identified with the finite set F, which is here equiped with the discrete topology.

U is defined on morphisms the natural way:

If $\psi : P' \longrightarrow P$ is a morphism of $(\mathcal{E}ns^{E_0})^{op}$, then ψ is a natural transformation: $P(-) \longrightarrow P'(-)$

<u>hence</u> there is a unique $U(\psi): \lim\limits_{(\text{Diagr } P')^{op}} F \longrightarrow \lim\limits_{(\text{Diagr } P)^{op}} F$

such that: $p_a \circ U(\psi) = p_{\psi_F(a)}$

$$\begin{array}{ccc} \lim\limits_{(\text{Diagr } P')^{op}} F & \xrightarrow{U(\psi)} & \lim\limits_{(\text{Diagr } P)^{op}} F \\ & \searrow^{p_{\psi_F(a)}} & \downarrow p_a \\ & & F \end{array}$$

(ii) For $X \in \mathit{Top}$, $T(X) = \text{Cont}(X, -)$

<u>i.e.</u>: If F is a finite set, $TX(F)$ is the set of continuous functions from X into the discrete space F.

And if $f: X \longrightarrow Y$ is a continuous function,

$T(f): \text{Cont}(Y, -) \longrightarrow \text{Cont}(X, -)$ is defined as

$$Tf_F: \text{Cont}(Y, F) \longrightarrow \text{Cont}(X, F)$$
$$\alpha \longmapsto \alpha \circ f$$

(iii) For $X \in \mathit{Top}$, $\eta_X: X \longrightarrow UT(X) = \lim\limits_{a \in \text{Cont}(X, F)} F$

is defined by: $x \longmapsto (a(x))_{a \in \text{Cont}(X, F)}$

<u>Note that</u> $UT(X) = \lim\limits_{a \in \mathrm{Cont}(X,\, F)} F$

$= \{(x_a)_{a: X \to F} \mid x_a \in F \;\&\; x_{fa} = f(x_a) \;\; \forall f: F \to F' \}$ (with $a: X \to F$, $fa: X \to F'$)

(iv) For $P \in (\mathcal{E}ns^{E})^{op}$, ε_P is a natural transformation :

$$\varepsilon_P: P(-) \longrightarrow \mathrm{Cont}(\lim_{(F,\, a) \in (\mathrm{Diagr}\, P)^{op}} F,\; -)$$

$$\varepsilon_{PE}: P(E) \longrightarrow \mathrm{Cont}(\lim_{(F,\, a) \in (\mathrm{Diagr}\, P)^{op}} F,\; E)$$

$$a \longmapsto p_a \quad (\text{since } (E,\, a) \in (\mathrm{Diagr}\, P)^{op})$$

<u>Proof of Claim 1:</u>

We will check the equations:

$$\text{(a)} \quad \varepsilon_T * T\eta = 1_T$$

$$\text{(b)} \quad U\varepsilon * \eta_U = 1_U$$

(a) For $X \in \mathcal{T}op$ we want to prove that $\varepsilon_{TX} \circ T\eta_X = 1_{TX}$

$$TX \xrightarrow{T\eta_X} TUTX \xrightarrow{\varepsilon_{TX}} TX \quad \text{in } (\mathcal{E}ns^{E_0})^{op}$$

$$\mathrm{Cont}(X, -) \xrightarrow{\varepsilon_{TX}} \mathrm{Cont}(\lim_{a: X \to F} F,\, -) \xrightarrow{T\eta_X} \mathrm{Cont}(X, -) \quad \text{in } \mathcal{E}ns^{E_0}$$

$$\mathrm{Cont}(X, E) \longrightarrow \mathrm{Cont}(\lim_{a: X \to F'} F,\, E) \longrightarrow \mathrm{Cont}(X, E) \quad \text{in } \mathcal{E}ns$$

$$a \longmapsto p_a \longmapsto p_a \circ \eta_X$$

<u>But:</u> $p_a \circ \eta_X = a: \; X \longrightarrow \lim\limits_{a: X \to F} F \xrightarrow{p_a} E$

$$x \longmapsto (a(x))_{x: X \to F} \longmapsto a(x)$$

<u>So:</u> $\varepsilon_{TX} \circ T\eta_X = 1_{TX}$

(b) For $P \in Ens^{E_0}$ we want to prove that $U\varepsilon_P \circ \eta_{UP} = 1_{UP}$

$$\begin{array}{ccccccc}
UP & \xrightarrow{\eta_{UP}} & UTUP & \xrightarrow{U\varepsilon_P} & UP & & \\
\| & & \| & & \| & & \\
\lim\limits_{a \in PF} F & \longrightarrow & \lim\limits_{f \in Cont(\lim\limits_{a \in PF} F,\ E)} E & \longrightarrow & \lim\limits_{a \in PF} F & \xrightarrow{p_a} & F
\end{array}$$

$$(x_a)_a \overset{\eta_{UP}}{\longmapsto} (f((x_a)_a))_f \longmapsto x_a$$

Since, by definition of U on morphisms, we have for $a \in PF$:

$p_a \circ U\varepsilon_P = p_{\varepsilon_{PF}(a)} = p_{p_a}$, we also have:

$$\begin{aligned}
(p_a \circ U\varepsilon_P \circ \eta_{UP})\ ((x_a)_a) &= p_{p_a}\ ((f((x_a)_a))_f) \\
&= p_a\ ((x_a)_a) \\
&= x_a
\end{aligned}$$

(Note that $(x_a)_a$ stands for $(x_a)_{(F,\ a)\ \in\ (\text{Diagr } P)^{op}}$)

<u>Hence</u> $(U\varepsilon_P \circ \eta_{UP})\ ((x_a)_a) = (x_a)_a$

<u>i.e.</u>: $U\varepsilon_P \circ \eta_{UP} = 1_{UP}$

<u>Claim 2</u> : $\quad Fix(UT, \eta) \subset \mathcal{St}$

<u>Proof</u>: Observe that $Im\ (U) \subset \mathcal{St}$.

Indeed, for any P in $(Ens^{E_0})^{op}$, $U(P)$ is a limit of finite discrete spaces which are totally disconnected. Thus, since $\mathcal{St}$ is closed under limits, $U(P) \in \mathcal{St}$ for any functor P .

Hence, if $X \in Fix(UT, \eta)$, i.e.: if η_X is an isomorphism, then X is isomorphic to $UT(X)$ which is in $\mathcal{St}$, hence $X \in \mathcal{St}$.

Claim 3: Fix(TU, ε) ⊂ Pro − E_0

Proof: As above we have Im(T) ⊂ Pro − E_0 .
Indeed, for any X in *Top* , T(X) = Cont(X, −) preserves finite limits, hence is a pro-finite set.

Hence if P ∈ Fix(TU, ε) , i.e.: if ε_P is an isomorphism, then P is isomorphic to TU(P) which is a pro-finite set, hence P ∈ Pro − E_0 .

Claim 4: Fix(UT, η) ⊃ *St*

Proof:

(a) First observe that if X is totally disconnected and Hausdorf, then η_X is injective.

Indeed, if x ≠ y in X , then there exists a clopen partition A $\dot{\cup}$ B = X such that x ∈ A & y ∈ B , hence a continuous function X ⟶ {x, y} into the two-point set, namely the collapsing function s (i.e.: such that s(a) = x & s(b) = y for all a∈A & b∈B) .

Hence, since : s(x) = x ≠ y = s(y) and s = $p_s \circ \eta_X$:

$$X \xrightarrow{\eta_X} \lim_{a:X\to F} F \xrightarrow{p_s} \{x,y\}$$

we have then : $\eta_X(x) \neq \eta_X(y)$

i.e.: The s-components of $\eta_X(x)$ & $\eta_X(y)$ are different so $\eta_X(x)$ & $\eta_X(y)$ are themselves different.

(b) Then observe that if X is compact, then η_X is surjective; i.e.: if $x \in \lim_{a:X\to F} F$ then $\eta_x^{-1}\{x\} \neq \phi$.

For that, let $x = (x_a)_{a:X\to F}$ be an element of $\lim_{a:X\to F} F$.

Then:

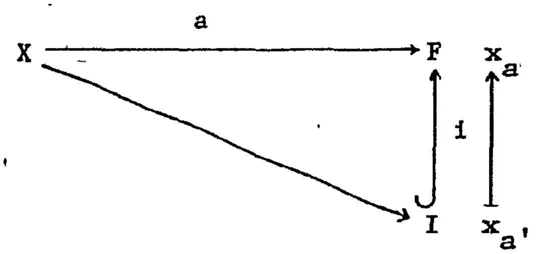
$$\eta_X^{-1}\{x\} = \{\xi \in X \mid a(\xi) = x_a \text{ for all } a: X \longrightarrow F\}$$

$$= \bigcap_{a:X\to F} a^{-1}\{x_a\} \qquad \left(\text{i.e.: } \bigcap_{(F,\, a) \in (\mathrm{Diagr}\ X)^{op}} a^{-1}\{x_a\}\right)$$

So we want to prove that $\bigcap_{a:X\to F} a^{-1}\{x_a\} \neq \phi$.

For that, we first observe that for each $a: X \longrightarrow F$, $a^{-1}\{x_a\} \neq \phi$.

Indeed, if $I = \mathrm{Im}(a)$ we have the diagram:



where $a = ia'$ so $x_a = i(x_{a'}) = x_{a'}$ since i is the inclusion. So we have $a^{-1}\{x_a\} = a'^{-1}\{x_{a'}\} \neq \phi$ because a' is surjective.

Secondly, we prove that $a^{-1}\{x_a\} \cap b^{-1}\{x_b\} \neq \phi$ for any two continuous functions:

$$a: X \longrightarrow F$$

$$b: X \longrightarrow F'$$

Indeed, since $\mathrm{Cont}(X, -)$ preserves finite limits, $(\mathrm{Diagr}\ X)^{op}$ is cofiltered and:

$< a, b > : X \longrightarrow F\times F'$ which is in $(\mathrm{Diagr}X)^{op}$ verifies:

$x_{<a,\, b>} = (x_a, x_b)$ in $F\times F'$ and therefore:

$$a^{-1}\{x_a\} \cap b^{-1}\{x_b\} = < a, b >^{-1}\{(x_a, x_b)\} = < a,b >^{-1}\{x_{< a,b >}\} \neq \phi .$$

In a similar way, we prove that for every finite subfamily of $(\mathrm{Diagr}X)^{op}$ $(a_i)_{i = 1,\ldots,n}$ where $a_i: X \longrightarrow F_i$, we have:

$$\bigcap_{i=1}^{n} a_i^{-1}\{x_i\} = \langle a_1,..,a_n \rangle^{-1}\{(x_{a_1},..,x_{a_n})\} = \langle a_1,..,a_n \rangle^{-1}\{x_{\langle a_1,..a_n \rangle}\} \neq \phi$$

Finally, since X is compact and $a^{-1}\{x_a\}$ is closed for each continuous a, we can conclude that : $\bigcap_{a:X\to F} a^{-1}\{x_a\} \neq \phi$.

Thus :

η_X is surjective.

<u>Conclusion</u>: If X is a Stone space, then η_X is both injective and surjective hence it is an isomorphism: $X \cong UT(X)$.

i.e.: <u>$Fix(UT, \eta) \supset St$</u>

<u>Claim 5</u> : $Fix(TU, \varepsilon) \supset$ pro-$\mathcal{E}_0$

<u>Proof:</u> Let $P = (F_i)_{i \in I}$ be a pro-finite set and let $p_i : P \longrightarrow F_i$ be the projections. Then the functor: $i \longmapsto (F_i, p_i)$ is a cofinal functor: $I \longrightarrow (\text{Diagr } P)^{op}$, so that we in fact have:

$$U(T) = \lim_{(F,a)\in(\text{Diagr } P)^{op}} F \cong \lim_{i\in I} F_i = L .$$

So, if E is an arbitrary finite set, we want to prove that ε_{PE} is a bijection, where :

$$\varepsilon_{PE} : P(E) \cong (\text{Pro-}\mathcal{E}_0)(P, E) \longrightarrow \text{Cont}(U(P), E)$$

$$(a: P \longrightarrow E) \longmapsto (p_a : \lim_{(F,a)} F \longrightarrow E) .$$

For that, we prove the following steps:

a) $P = (F_i)_{i \in I} \cong (E_i)_{i \in I}$ where the projections $q_i : P \longrightarrow E_i$ are regular epis in pro-$\mathcal{E}_0$ and ;

b) the transition map $E_\alpha : E_i \longrightarrow E_j$ for each $\alpha : i \longrightarrow j$ is surjective.

Consequently, the category I can be replaced by a partially ordered set.

c) The corresponding projection maps $p_k : U(P) \cong L = \lim_{i \in I} E_i \longrightarrow E_k$ are surjective.

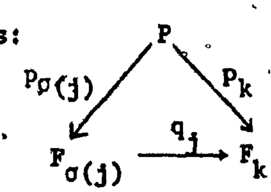
d) For every finite set E, $\varepsilon_{PE} : P(E) \longrightarrow \mathrm{Cont}(L, E)$ is injective.

e) For every finite set E, ε_{PE} is surjective.

<u>Proof of step a)</u>: We prove that for every (fixed) k in I, the image of p_k (in pro-E_0) is in fact a finite set E_k, and therefore that $(E_k)_{k \in I}$ is a pro-object isomorphic to P.

Indeed, by definition, $\mathrm{Im}\ p_k = (\mathrm{Im}\ q_j)_{j \in J}$ where J is the category of morphisms representing p_k:

i.e. For each j in J, there is an object $\sigma(j)$ in I and a map $q_j : F_{\sigma(j)} \longrightarrow F_k$ such that the following diagram commutes:



a morphism of J is then a morphism $\alpha : \sigma(j) \longrightarrow \sigma(j')$ of I such that $q_{j'} \circ \alpha = q_j$.

Note that σ, which is in fact the "projection functor" (Chapter 2, section 3) is cofinal so that:

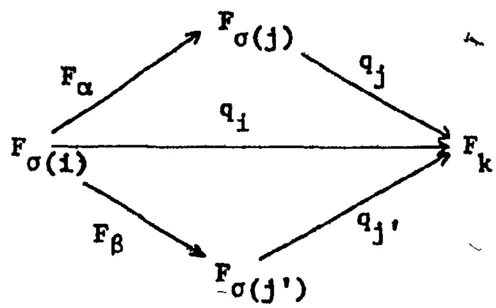
$$(F_i)_{i \in I} \cong (F_{\sigma(j)})_{j \in J}$$

Now consider the family $\mathcal{I} = (\text{Im } q_j)_{j \in J}$ of subsets of F_k .

Given the two objects j & j' in J , there is an object i and two morphisms:

$\alpha: i \longrightarrow j$ & $\beta: i \longrightarrow j'$ in J such that:

$q_j \circ F_\alpha = q_{j'} \circ F_\beta = q_i$

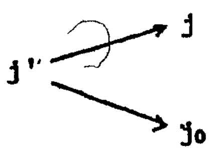


<u>Hence</u>: $\text{Im } q_i \subset \text{Im } q_j \cap \text{Im } q_{j'}$.

So, $\mathcal{I}$ is the base of a filter an F_k which is stationnary because F_k is a finite set.

Thus, there exists an object j_0 in J such that for every $\alpha: j \longrightarrow j_0$ in J , $\text{Im } q_j = \text{Im } q_{j_0} = E_k$.

Hence, given any j in J , there exists a j' and two morphisms α & β :

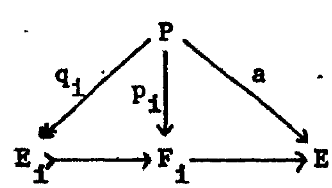


in J . <u>Hence</u>: $\text{Im } q_{j'} = E_k$ which means that the full subcategory J' of J whose objects j' satisfy $\text{Im } q_{j'} = E_k$ is cofinal.

<u>So</u>: $\text{Im } p_k = (\text{Im } q_j)_{j \in J} \cong (\text{Im } q_j)_{j \in J'} \cong E_k$.

<u>Consequently</u>: If we consider the pro-object $(E_i)_{i \in I}$, we know that the category of couples (F_i, p_i) is a cofinal subcategory of $(\text{Diagr } P)^{op}$, hence so is the category of couples (E_i, q_i) i.e.:

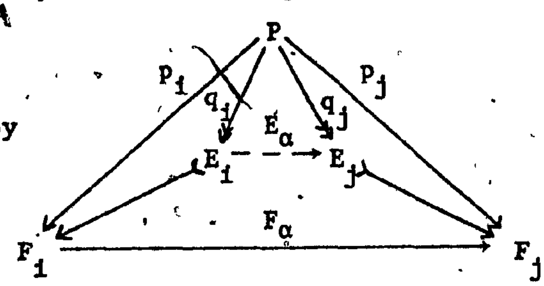
for any $q: P \longrightarrow E$ in $(\text{Diagr } P)^{op}$, the diagram (1) commutes for some i in I .



(1)

<u>So</u> $P = (F_i)_{i \in I} \cong (E_i)_{i \in I}$.

<u>Proof of step b)</u> Given any $\alpha : i \longrightarrow j$ in I we have the commutative diagram:



where E_α is uniquely defined by the diagonal property and is an epi (hence a surjection) because q_j is an epi .

<u>So</u> if E_α & E_β both satisfy: $E_\alpha \circ q_i = q_j = E_\beta \circ q_i$ then $E_\alpha = E_\beta$ because q_i is an epi.

So, if we replace I by the category having same objects as I , but where all morphisms with same domain and codomain are identified, we get the same pro-object P , but the indexing category is a partially ordered set.

<u>Thus</u> we can assume that $P = (E_i)_{i \in I}$ where :

(i) I is a directed partially ordered set ;

(ii) the transition maps are surjective : $E_i \longrightarrow E_j$ as well as the projections $P \longrightarrow F_i$;

(iii) the sets F_i are mutually disjoint ;

(iv) the category of couples (E_i, q_i) is a cofinal subcategory of $(\text{Diagr } P)^{op}$.

Therefore: $U(P) = \lim\limits_{(F,a) \in (\text{Diagr } P)^{op}} F \cong \lim\limits_{i \in I} E_i = L$

We will denote by p_i the projection: $L \longrightarrow E_i$ rather than p_{q_i} .

<u>Proof of step c)</u> Let c be a fixed element of E_k (where $k \in I$ is a fixed index) , and consider the set $E = \bigcup\limits_{i \in I} E_i$, ordered by $<$ defined as follows:

Given $x_i \in E_i \subset E$ & $x_j \in E_j \subset E$

$x_i < x_j \Leftrightarrow \exists \alpha : i \longrightarrow j$ such that $E_\alpha(x_i) = x_j$ (i.e.: Given x & y in E then $x = x_i$ & $y = x_j$ for some i & j in I, so $x < y \Leftrightarrow x_i < x_j$)

Now let $C = \{ x \in E \mid x < c \}$ and let F be the filter generated by C and $\{ F_i \}_{i \in I}$ where $F_i = \bigcup_{j<i} E_j$.

Notice that $\{ F_i \}_{i \in I}$ is itself the base of a filter and given i (and k) there is a j such that $j < i$ & $j < k$; so if α is the morphism $j \longrightarrow k$ then, since E_α is surjective, there is an $x_j \in E_j \subset F_i$ s.t.: $E_\alpha(x_j) = c$ i.e.: $x_j < c$ hence $x_j \in C \cap F_i$, which shows that for every i in I, $C \cap F_i \neq \phi$. So F exists and therefore there is an ultrafilter U finer than F i.e.: such that $C \in U$ and $F_i \in U$ for each i in I.

Now, for every i in I let:

$$E_i = \{ e_1 , \ldots , e_n \}$$

$$A_p = \{ x \in E \mid x < e_p \} \quad \text{for } p = 1,2,\ldots,n$$

<u>Then:</u> $$F_i = A_1 \dot{\cup} A_2 \dot{\cup} \ldots \dot{\cup} A_n \in U .$$

Hence, there is exactly one index p such that $A_p \in U$, which determines a unique element $x_i = e_p$ in each E_i.

But for $i = k$ we know that $C \in U$. Hence $x_k = c = e_p$ i.e.: $A_p = C$.

Also, if $j < i$ assuming that:

$$E_j = \{ d_1 , \ldots , d_m \}$$

$$B_q = \{ x \in E \mid x < d_q \} \quad \text{for } q = 1, \ldots, m$$

we have for any p & q:

$$d_q < e_p \Leftrightarrow d_q \in A_p \Leftrightarrow B_q \subset A_p$$

$$\text{and} \quad d_q \notin A_p \Leftrightarrow B_q \cap A_p = \phi .$$

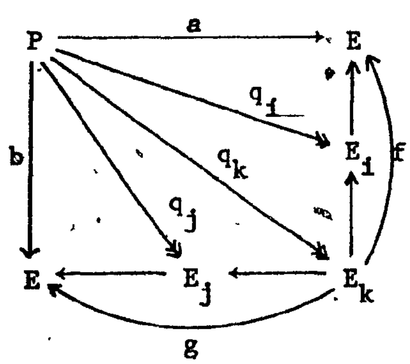
Hence if $A_p \in \mathcal{U}$ and $B_q \in \mathcal{U}$, then $x_j = d_q < e_p = x_i$.

Thus, $\mathcal{U}$ defines a (unique) element $(x_i)_{i \in I}$ of L such that $p_k(x) = x_k = c$ as desired.

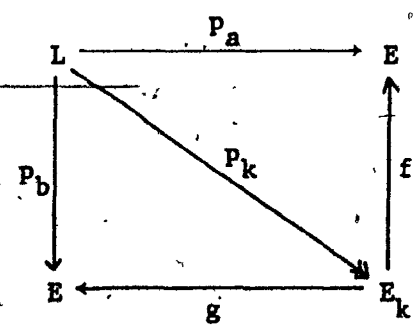
Note that there may be many such elements x (one for each $\mathcal{U}$ that contains $\mathcal{F}$), but our construction, which essentially depends on the axiom of choice, proves the existence of one of them.

<u>Proof of step d)</u> Let a & b be two pro-morphisms: $P \longrightarrow E$ (where E is a given finite set) such that $p_a = p_b : L \longrightarrow E$.

Since the category of couples (E_i, p_i) is a cofinal subcategory of $(\text{Diagr } P)^{op}$ and cofiltered, there are objects i, j & k of I and maps s.t.: the diagram (1) commutes:



(1)



(2)

So, that (2) is also commutative. (Note that f denotes the map: $E_k \rightarrow E_i \rightarrow E$ in both diagrams, and similarly for g: $E_k \rightarrow E_j \rightarrow E_k$).

So, $f \circ p_k = p_a = p_b = g \circ p_k$.

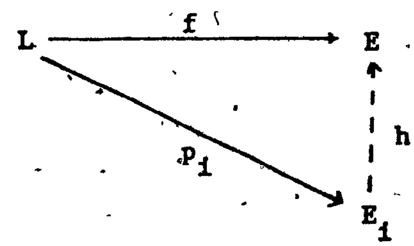
Hence $f = g$ since p_k is surjective (i.e.: an epi).

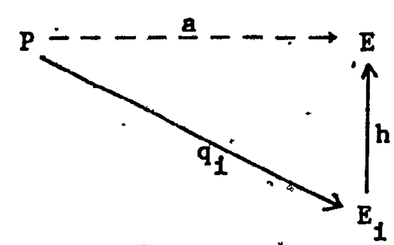
Therefore, in diagram (1) we have as required:

$$a = f \circ q_k = g \circ q_k = b$$

<u>Proof of step e)</u> Let $f: L \longrightarrow E$ be any continuous fonction into an arbitrary finite set $E = \{x_1, \ldots, x_n\}$. It is sufficient to show that f factors through E_i for some i in I.

Indeed, if $f = h \circ p_i$ as below, then taking $a = h \circ q_i$ we get:





So (E, a) is an object and h a morphism of $(\text{Diagr } P)^{op}$ $h: (E_i, q_i) \longrightarrow (E, a)$.
Hence $p_a = h \circ p_i = f$ as required.

Now, in order to prove that f can actually be factored, we look at the clopen partition $L = f^{-1}\{x_1\} \dot{\cup} \ldots\ldots \dot{\cup} f^{-1}\{x_n\}$ defined by f.

Since f is continuous and E is discrete, then $f^{-1}\{x_i\}$ is actually a clopen subset of L.

All we need to show is that given any finite clopen partition $L = A \dot{\cup} .. \dot{\cup} A_n$, there is a partition $B_1 \dot{\cup} \ldots \dot{\cup} B_n = E_i$ of some E_i s.t.: $A_p = p_i^{-1}(B_p)$ for every $p = 1, 2, \ldots, n$.

Indeed, if $f^{-1}\{x_p\} = p_i^{-1}(B_p)$ for the above partition of E_i, then the map $h: E_i \longrightarrow E$ defined by:

$$h(e) = x_p \iff e \in B_p \text{ for } p = 1, \ldots, n \qquad \text{satisfies also:}$$

$$\begin{aligned} h(p_i(x)) = x_p &\iff p_i(x) \in B_p \iff x \in p_i^{-1}(B_p) = f^{-1}\{x_p\} \\ &\iff f(x) = x_p \end{aligned}$$

Hence $f = h \circ p_i$ as desired.

Now let us look at clopen sets in L. For that, we recall that every open set A in L (which is indexed on a directed set) is the form:

$A = p_i^{-1}(B)$ for some open $B \subset E_i$ ([7] § 4.4 proposition 9).

But since E_i is discrete, B can be any subset of E_i and is automatically clopen hence A is also clopen.

Now if C is an arbitrary clopen subset of L then C open $\Rightarrow C = \bigcup_{k \in K} A_k$ where A_k is a basic open set and K is any set; and C closed $\Rightarrow$ C is compact.

<u>Hence</u>: $C = \bigcup_{j=1}^{p} A_{k_j}$ for some $k_j \in K$, $j = 1,\dots,p$.

But finite unions of basic open (hence clopen) sets are also basic open sets.

<u>Hence</u>: $C = p_i^{-1}(B)$ for some $B \subset E_i$.

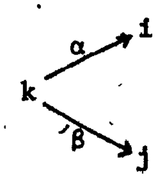
<u>Finally</u>, we proceed by induction on n. For $n = 1$ we have a trivial partition $L = L$.

Assuming the result for n, if $L = A_1 \dot{\cup} \dots \dot{\cup} A_n \dot{\cup} A_{n+1}$ is a clopen partition of L, then there exists an i in I and a partition $E_i = B_1 \dot{\cup} \dots \dot{\cup} B_n$ of E_i such that:

$$A_q = p_i^{-1}(B_q) \text{ for } q = 1,\dots,n-1 \text{ and } A_n \dot{\cup} A_{n+1} = p_i^{-1}(B_n)$$

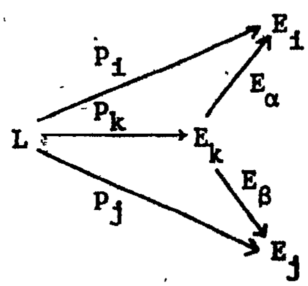
but since A_{n+1} is clopen, there is a j in I and a $B \subset E_j$ such that if $B' = E_j - B$ then $A_{n+1} = p_j^{-1}(B)$ and hence

$A_1 \dot{\cup} \dots \dot{\cup} A_n = p_j^{-1}(B')$.

Now, by cofilteredness, there is a k and two morphisms: 

such that : $E_\alpha \circ p_k = p_i$,

and $E_\beta \circ p_k = p_j$



Hence for $q = 1,\ldots,n$, we have:

$$A_q = p_i^{-1}(B_q) \cap p_j^{-1}(B') = p_k^{-1}(E_\alpha^{-1}(B_q)) \cap p_k^{-1}(E_\beta^{-1}(B'))$$
$$= p_k^{-1}(E_\alpha^{-1}(B_q) \cap E_\beta^{-1}(B'))$$

and for $q = n+1$

$$A_{n+1} = p_i^{-1}(B_n) \cap p_j^{-1}(B) = p_k^{-1}(E_\alpha^{-1}(B_n) \cap E_\beta^{-1}(B)) .$$

<u>Thus</u>: If we set : $C_q = E_\alpha^{-1}(B_q) \cap E_\beta^{-1}(B')$ for any $q = 1,\ldots,n$,

and $C_{n+1} = E_\alpha^{-1}(B_n) \cap E_\beta^{-1}(B)$.

Then, given $x_k \in E_k$, $x_k = p_k(x)$ for some x in L (p_k is surjective)

<u>hence</u> $E_\alpha(x_k) = p_i(x) \in B_q$ for exactly one q

<u>and</u> $E_\beta(x_k) = p_j(x) \in B \dot{\cup} B' = E_j$

<u>and</u> $E_\alpha(x_k) \in B \Rightarrow E_\beta(x_k) \in B_n$.

So, $x_k \in C_q$ for exactly one $q \in \{1,\ldots,n+1\}$.

<u>Conclusion</u>: $E_k = C_1 \dot{\cup} \ldots\ldots \dot{\cup} C_n \dot{\cup} C_{n+1}$ is a partition of E_k such that $A_q = p_k^{-1}(C_q)$ for each $q \in \{1,\ldots,n+1\}$ which ends the proof of proposition 5.1 .

<u>Consequently</u>: We have an example of an exact category $\mathcal{E}_0$ such that pro-$\mathcal{E}_0 \cong \mathcal{St}$ is not exact.

We could complete it using the construction described in Chapter I, but we know (corollary 3.4) that $\mathcal{St}$ is a regular category, and it can be seen that its regular structure is not preserved by the construction.

Instead, we use the construction given by R. Succi ([9], section 3), which extends every regular category C into an exact category $Q(C)$ in a universal way.

But in fact, we have the following:

Proposition 5.2: $Q(\text{pro-}E_0) \cong Q(St) \cong CH$, where CH denotes the category of compact Hausdorf topological spaces.

Proof:

Claim 1: CH is an exact category.

Indeed, given an object X of CH and an equivalence relation R on X in CH, then R is a compact subspace of X×X hence closed. Hence by the Alexandroff theorem ([8] ; 3.2.11). , the quotient space X/R is compact Hausdorf, i.e.: X/R is the quotient of X by R in the category CH .

Claim 2: Every compact Hausdorf space X is the quotient of some Stone space Y by an equivalence relation S (which is also a Stone space) .

The proof is a consequence of the following:

Theorem ([8] ; 3.2.2.): Every compact Hausdorf space of weight $m \geq \aleph_0$ is a continuous image of a closed subspace of the Cantor cube D^m .

Hence if the weight m of X is infinite, i.e.: if X has a basis of open sets having infinite cardinality m , then there exists a closed hence compact subspace C of the Stone space D^m , and a continuous surjection $f: C \xrightarrow{f} X$.

So, if S is the equivalence relation defined by f , then by the Alexandroff theorem mentioned in Claim 1, S is closed hence compact.

So S is a Stone subspace of C×C, and $X \cong C/S$ as required.

Now if X has finite weight, then it is finite hence discrete ([8] Theorem 1.5.1). So X is itself a Stone space.

<u>Claim 3:</u> $\mathcal{CH}$ satisfies the universal property of $\mathcal{Q}(\mathcal{St})$: The inclusion functor $\mathcal{St} \hookrightarrow \mathcal{Q}(\mathcal{St})$ is exact, and every exact functor G: $\mathcal{St} \longrightarrow \mathcal{B}$ (where $\mathcal{B}$ is any exact category) can be uniquely extended (up to isomorphism) to an exact functor $\mathcal{Q}(G)$: $\mathcal{CH} \longrightarrow \mathcal{B}$.

Indeed, by claim 2 we know that every compact Hausdorf space X is of the form $X \cong C/S$, where C and S are Stone spaces.

So, given G, the functor $\mathcal{Q}(G)$ defined on objects as:

$\mathcal{Q}(G)\ (X) \cong \mathcal{Q}(G)\ (C/S) = G(C)\ /\ G(S)$

is an obvious extention of G to the category $\mathcal{CH}$.

<u>Conclusion:</u> $\mathcal{Q}(\text{pro-}E_0) \cong \mathcal{Q}(ST) \cong \mathcal{CH}$. $\square$

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