ON THE RECONFIGURATION AND REACHABILITY OF CHAINS

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A THESIS

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Dedicated to my father

Abstract

A *chain* is a sequence of rigid rods or links consecutively connected at their endjoints, about which they may rotate freely. A *planar chain* is a chain whose links lie in the plane, with links allowed to cross over one another. For a chain Γ constrained to lie in a confining region P, the *reachability problem* for Γ is to determine, given a point $p \in P$ and an initial configuration of Γ inside P, whether Γ can be moved within P so that the endjoint of Γ reaches p, and if so, how this can be done.

This thesis solves the reachability problem of a planar chain Γ confined within a *convex obtuse polygon* P, a convex polygon whose interior angles each measure $\pi/2$ or more. In particular, we propose a uniform approach in which the geometry of Γ and its confining region P are studied together. We use this to obtain a family of pairs (Γ, P), which is largest possible in some sense, so that the reachability problem for each pair in the family can be solved quickly. We also examine the properties of the reachable region of Γ in such a pair.

This thesis also presents reconfiguration results for an n-link planar chain Γ

inside a circle. We show that if each link of Γ is less than the radius of its confining circle, then Γ can be moved between any of its configurations inside the circle in $O(n^2)$ time.

Our results demonstrate how to design short link chains within a given confining environment in order to ensure fast reconfiguration.

Résumé

Une *chaîne* est une suite de tiges rigides ou d'arètes consécutivement attachées à leurs extrémités. autour desquelles elles sont libres de se mouvoir. Une *chaîne planaire* en est une dont les arètes sont dans le plan. les croisements d'arètes étant permis. Pour une chaîne Γ contenue dans une région P. le *problème d'accessibilité* pour Γ est de déterminer. étant donné un point $p \in P$ et une configuration initiale de Γ à l'intérieur de P. si Γ peut être bougée à l'intérieur de P de telle sorte que l'extrémité de Γ coincide avec P. et si oui. de quelle façon on peut s'y prendre.

Cette thèse résouds le problème d'accessibilité pour une chaine planaire Γ circonscrite dans un *polygone convexe obtus* P. un polygone convexe dont les angles internes mesurent $\pi/2$ ou plus. En particulier. nous proposons une approche uniforme dans laquelle la géométrie de Γ et de la région P sont étudiées ensemble. Nous utilisons ceci pour obtenir une famille de paires $(\Gamma. P)$. maximale dans certain cas. pour lesquelles le problème d'accessibilité peut se résoudre rapidement. Nous examinons aussi les propriétés de la région accessible de Γ pour de telles paires.

Cette thèse présente aussi des résultats de reconfiguration pour une chaîne planaire Γ à n arètes à l'intérieur d'un cercle. Nous montrons que si chaque arète de Γ est plus petite que le rayon du cercle circonscrit, alors Γ peut être déplacée entre deux configurations quelconques dans le cercle en temps $O(n^2)$.

Nos résultats démontrent comment construire des chaînes avec de petites arètes à l'intérieur d'un environnement donné pour assurer des reconfigurations rapides.

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Although the effect on this thesis is indirect, I would like to mention my *Alma Mater*, Jilin University, where I obtained my B.S. and M.S. degrees and laid my foundations in mathematics and computer science. All the university faculty members who ever taught me deserve my gratitude; among them, my master's thesis supervisor, Prof. Xianghao Wang, contributed most to my studies there.

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Chapter 1

Introduction

This thesis concerns algorithmic motion planning from a geometric point of view. In this chapter, we first review previous work pertinent to the thesis and then describe our objectives.

1.1 Previous Work

With the advent of industrial automation and robotics, designing efficient algorithms for moving objects in 2- or 3- dimensional space subject to certain constraints has become increasingly important. The *mover's problem* is the following: given the initial and desired final configurations of an object in 2- or 3-dimensional space, and given a detailed description of obstacles in the space, determine if there is an obstacle-avoiding continuous motion of the object from the initial to the final configuration: find such a motion if it exists.

This problem has been studied by many researchers. See [AY90a], [AY90b], [Kor85], [Lat91], [SY87], [SSH87], [Whi85] for surveys. Schwartz and Sharir [SS83] and Canny [Can88] provided very general exact methods to solve the mover's problem. However, the methods do not provide polynomial time algorithms, much less linear time algorithms.

In general, the mover's problems in which the object has an unbounded number n of degrees of freedom, i.e., problems in which n is part of the input for a problem instance, are computationally intractable in the sense that such problems are often NP-hard or PSPACE hard. Even when the objects are very simple, n degree of freedom problems may remain NP- or PSPACE hard. See [SY87], [HJW85], [WZ89], [Rei79] and [JP85] for examples. If the number n of degrees of freedom is bounded, then in many cases the general methods of [SS83] and [Can88] provide polynomial time algorithms, although these may be far from practical. These results suggest that the feature that often makes motion planning intractable is the unboundedness of the number of degrees of freedom.

Therefore, to find fast reconfiguration algorithms it is essential to understand what relationships between moving objects and their environments enable some problems to be solved quickly in spite of having arbitrarily many degrees of freedom.

Several examples of fast reconfiguration have been provided for variants of the mover's problem, in which a *linkage*, and particularly, a *chain* or an *arm*, is considered as the object. See [Whi92] for a survey. A *linkage* is a collection of rigid rods or links connected together at their endjoints. A *planar* linkage has its links confined to the plane: links may cross over one another and the locations of certain joints may be required to remain fixed to the plane. A *chain* is a sequence of links consecutively connected at their endjoints. A *closed chain* is a chain such that the two endjoints are connected together. An *arm* is a chain in which a fixed location is associated with one endjoint of the chain.

For a linkage Γ constrained to lie in a confining region P, the *reachability problem* for Γ is to determine, given a point $p \in P$ and an initial configuration of Γ, whether Γ can be moved within P so that the endjoint of Γ reaches p; and if so, how this can be done.

Hopcroft, Joseph and Whitesides first studied the reconfiguration and reachability problems for n-link linkages. In [HJW84], they proved that the reachability problem for a planar linkage with no confining region is PSPACE hard. Joseph and Plantinga [JP85] proved that the reachability problem for a chain moving within a certain non-convex constraining environment is PSPACE hard. In [HJW85], Hopcroft, Joseph and Whitesides showed that the reachability problem for a planar arm constrained by arbitrary polygonal walls is NP-hard. However, when the constraint is an enclosing circle, they gave an algorithm to solve the reachability problem in $O(n^2)$ time. They also gave an algorithm to move the arm to any reachable configuration in $O(n^3)$ time. In [KK86], Kantabutra and Kosaraju improved the running time by reducing $O(n^3)$ to $O(n)$.

In [LW91], [LW92] and [LW95], Lenhart and Whitesides investigated the reconfiguration of closed chains and presented a linear time algorithm for reconfiguring closed chains in d-dimensional space. Also, Kantabutra ([Kan92] and [Kan95]) presented linear time algorithms for reconfiguring certain arms and chains inside squares.

Confining environments containing acute angles present special difficulties due to the fact that links may become stuck when they point into a corner. This phenomenon was studied by van Kreveld, Snoeyink and Whitesides [KSW95]. Here the problem of folding an l-ruler—an n-link chain whose links all have equal length l—onto one link inside an equilateral triangle of unit side was considered. In spite of the simplified situation, an unusual phenomenon occurs. For very small link lengths, the chain can always be folded. Of course for link lengths close to 1, the chain cannot always be folded. However, this property alternates not once but *twice* as link length increases from 0 to 1.

So far, algorithms for fast reconfiguration have been given for special situations that only involve very simple confining regions: circles, squares, equilateral triangles, or no confining region at all. Recently Whitesides and Pei [WP96] greatly extended

authors	*problem*	*linkage*	*confining region*	*bound on link length*	*time complexity*	
					decide	*move*
Hopcroft, Joseph, Whitesides	reconfiguration	arm	circle	diameter	$O(n^2)$	$O(n^3)$
Kantabutra, Kosaraju	reconfiguration	arm	circle	diameter		$O(n)$
Lenhart, Whitesides	reconfiguration	closed chain	none	none	$O(n)$	$O(n)$
Kantabutra	reconfiguration	arm	square	half of the side length	$O(n)$	
Kreveld, Snoeyink, Whitesides	folding	l-ruler	equilateral triangle of unit side	$l \leq c \approx 0.483$, $1/2 < l \leq \sqrt{3}/2$	$O(1)$	$O(n)$
Kantabutra	reachability	chain	square	side length	$O(n)$	
Whitesides, Pei	reachability	chain	convex obtuse polygon	length of the shortest side	$O(n)$	

Table 1.1: Fast algorithms for reconfiguring n-link linkages

the previous results by providing a polynomial algorithm to solve the reachability problem of certain n-linked planar chains confined within an m-sided *convex obtuse polygon*, a convex polygon whose interior angles all measure $\pi/2$ or more.

We summarize several fast reconfiguration results for various linkages in Table 1.1.

This thesis makes the point that by designing short link chains within a given confining environment, one can obtain fast reconfiguration algorithms. We propose a uniform approach in which the geometry of a chain and its confining region are studied in a coordinated way. Our results significantly contribute to understanding what relationships between objects and their confining environments ensure that typically hard reconfiguration problems become easy.

1.2 Objectives

1.2.1 Notation

Before proceeding further, we introduce terminology and notation, illustrated in Figure 1.1. For an n-link chain Γ with consecutive joints $A_0, \ldots, A_n$, the initial and final joints A_0 and A_n are called *endjoints* and the others are called *intermediate joints*. The link between A_{i-1} and A_i $(1 \leq i \leq n)$ is denoted by L_i, and the length of L_i is denoted by l_i. The angle at intermediate joint A_i, denoted by α_i, is that determined by rotating L_i about A_i counterclockwise to bring L_i to L_{i+1}. An intermediate joint A_i is called a *straight joint* if $\alpha_i = \pi$ and is called a *bending joint* otherwise. In particular, A_i is called a *closed joint* if $\alpha_i = 0$. Γ is said to be *folded* if its each intermediate joint is either straight or closed. The subchain of Γ with joints $A_i, A_{i+1}, \ldots, A_j (i < j)$ is denoted by $\Gamma(i, j)$. Subchain $\Gamma(i, j)$ is said to be *straight*, denoted by $[A_i, \ldots, A_j]$, if its links form a straight line segment with all interior joints straight. Also, we use $[A, B]$ to denote a single link chain having joints A and B.

We denote $\max_{1 \leq i \leq n}\{l_i\}$ by l_{max} and say that Γ is *bounded* by b, denoted by

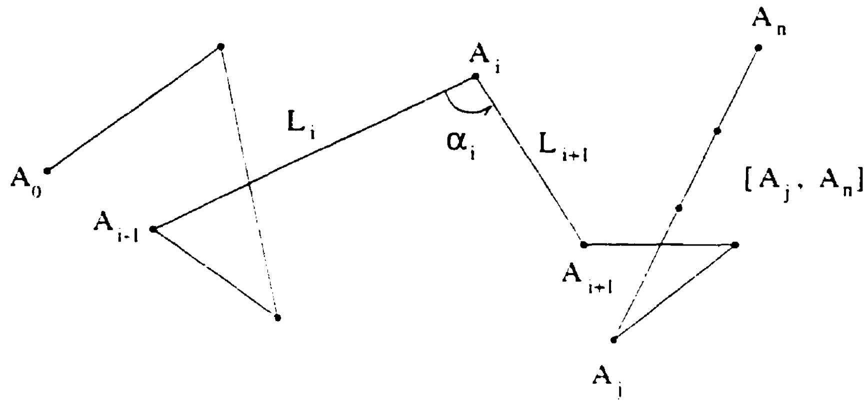


Figure 1.1: Notation for chains.

$\Gamma \prec b$, if $l_{max} < b$.

We denote the distance between two points x, y by $d(x, y)$; the line they determine by $l(x, y)$; the line segment they determine by xy; and the length of this segment by $|xy| = d(x, y)$. We denote the distance between two parallel lines l_1 and l_2 by $d(l_1, l_2)$.

We use P to denote a simple polygon and use V to denote the set of its vertices. We regard polygons as 2-dimensional closed sets and denote the boundary of P by ∂P. For $x, y \in \partial P$, we denote the counterclockwise polygonal chain from x to y by $Ch(x, y)$. For $p \in \partial P$, we denote a side of P containing p by $s(p)$. We denote the length of the shortest side of P by s_{min}. We define the *width* of P as the minimum possible distance between two parallel lines of support of P and denote it by w.

We denote a circle centered at o with radius r by $C(o, r)$, or simply by C if o, r are clear from the context. For a circle C and two points $x, y \in C$, we use $\widehat{xy}$ to denote the counterclockwise arc from x to y, and we use $d(\widehat{xy})$ to denote the measure of $\widehat{xy}$ in degrees.

For a closed region R in 2 dimensions, $v_{max}(p)$ denotes a point of R farthest from p, and $d_{max}(p)$ denotes $d(p, v_{max}(p))$. Obviously, if R is a polygon, $v_{max}(p)$ is a vertex of R farthest from p.

For an n-link chain Γ confined by a closed region R, the set of points of R that are reachable by A_n from each possible initial configuration of Γ, i.e., no matter where Γ initially lies, is called the *reachable region* of A_n and is denoted by $R_\Gamma(A_n)$. The complement of $R_\Gamma(A_n)$ with respect to R, denoted by $\overline{R_\Gamma(A_n)}$, is called the *unreachable region* of A_n. A point $p \in R$ is said to be an *l-reachable* point if $p \in R_\Gamma(A_n)$ for every $\Gamma \prec l$ no matter where Γ initially lies. The set of l-reachable points in R, denoted by R_l, is called the *l-reachable region* of R, and if $R_l = R$, R is said to be *l-reachable*.

1.2.2 Motivation

Let Γ be an n-link chain confined by a 2-dimensional closed region R, not necessarily convex, and let $p \in R$. We are particularly interested in the following condition:

For each i with $1 \le i \le n$,

$$l_i - \sum_{j=i+1}^{n} l_j \le d_{max}(p). \qquad (*)$$

This condition enjoys some nice properties related to the reachability of A_n, as described in the following.

Property 1 *(*) is independent of the initial configuration of* Γ.

Property 2 *(*) can be tested in $O(n)$ time whenever $d_{max}(p)$ is given. Furthermore, if R is a polygon, then (*) can be tested in $O(n)$ time.*

Property 3 *(*) is a necessary condition for p to be reachable by A_n.*

To see the necessity of (*), note that if this condition is not satisfied for some i, then A_i cannot lie within R when A_n lies at p.

The property below shows that the study of reachable regions would be greatly simplified if (*) were also sufficient for A_n to reach p.

Property 4 *If for every $p \in R$, (*) is a sufficient condition for A_n to reach p, then there exists a single link chain $\Gamma' = [A'_0, A'_1]$ such that $R_\Gamma(A_n) = R_{\Gamma'}(A'_1)$.*

Proof: Take $l' = \max_{1 \le i \le n}\{l_i - \sum_{j=i+1}^{n} l_j\}$ as the length of $[A'_0, A'_1]$. From Property 3, we have

$$p \in R_\Gamma(A_n) \iff \forall i (1 \le i \le n), l_i - \sum_{j=i+1}^{n} l_j \le d_{max}(p) \iff$$

$$l' = \max_{1 \le i \le n}\{l_i - \sum_{j=i+1}^{n} l_j\} \le d_{max}(p) \iff p \in R_{\Gamma'}(A'_1). \qquad \square$$

These nice properties motivate our investigation of the closed regions and confined chains for which (*) serves as a *sufficient* as well as a necessary condition for A_n to reach p.

In general, however, (*) is far from being sufficient to test the reachability of A_n, which usually depends on the initial configuration of Γ. Indeed, (*) is so mild that it cannot even guarantee the existence of a configuration of Γ inside R in which A_n touches p, as Figure 1.2 shows.

This figure shows an equilateral triangle $\triangle$ with unit side, which confines a folded 2-link chain Γ having joints A, B, C. Links of Γ have lengths 1 and 1/2, respectively. For endjoint C and any vertex v of $\triangle$, clearly condition (*) holds. But it is impossible to place C at v.

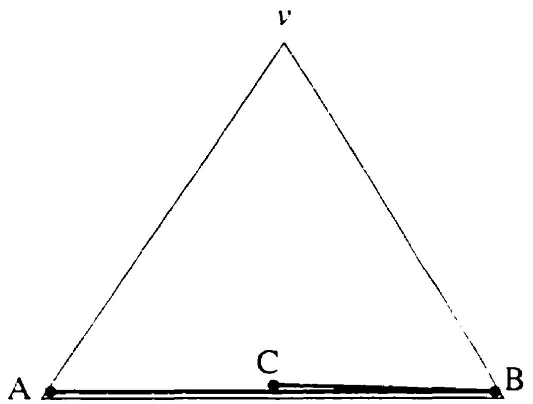


Figure 1.2: It is impossible to place C at v.

Intuitively, it is conceivable that if the link lengths of Γ are all sufficiently small in comparison to some measure of its confining region, Γ could be reconfigured within R so that A_n reaches p. Hence in such situations, the reachability of Γ would be *independent* of its initial configuration and $(*)$ would give a simple test for reachability.

Therefore, it is of interest to investigate, given a confining environment, how short the links of Γ are *required* to be in order to ensure the validity of $(*)$ to test the reachability of A_n, when neither the geometry of Γ nor its initial configuration are specified.

Kantabutra [Kan95] proved that if Γ is confined to a *square* and if Γ is bounded by the *side length* of the square, then $(*)$ is sufficient for A_n to reach p. This inspired our further investigation of more general cases in which $(*)$ serve as a sufficient condition as well.

As mentioned previously, confining regions involving acute angles, at which a link can jam, present special difficulties for reconfiguring chains. We thus consider *convex obtuse polygons*, a notion of our invention, as the confining regions. A convex obtuse polygon is a convex polygon whose interior angles are all at least $\pi/2$. In [WP96], we generalized Kantabutra's result from squares to arbitrary convex obtuse polygons.

In this thesis, we investigate how to *compute* a bound on Γ that depends on its given confining region so that the reachability of Γ can be easily determined. In particular, we consider a convex obtuse polygon as the confining region and ask how small a bound on Γ is *needed* so that $(*)$ is a sufficient and necessary condition for A_n to reach p.

We adopt the following philosophy common, for example, in engineering: by understanding where difficulties lie, we can plan how to avoid them. Hence, our study of reachability is novel in that we consider, given a confining environment, how to *design* chains in order to satisfy certain desirable properties; in particular we show how to design chains so that reachability problems become easier. This in turn suggests a *uniform* approach to reconfiguration that enables us to go beyond the discovery of individual special cases for which hard reachability problems become easy.

1.2.3 Results and Organization

Given a simple polygon P, we propose the following conditions, which will turn out to be interdependent, to bound the link lengths of Γ inside P.

Condition (S): $(*)$ is sufficient for A_n to reach p.

Based on **Condition (S)**, we define an "S" bound b^S as follows:

$b^S = \sup\{b \mid \text{If } \Gamma \prec b, \Gamma \text{ satisfies } \textbf{(S)}\ \}$.

Condition (F): Let $[A, B]$ be a single link chain with both A and B on ∂P. Then A can be moved completely around the entire ∂P in either direction and with no backtracking while B remains on ∂P.

Based on **Condition (F)**, we define an "F" bound b^F as follows:

$b^F = \sup\{b \mid \text{If } [A, B] \prec b, [A, B] \text{ satisfies } \textbf{(F)} \}$.

Condition (C): For any $o \in \partial P$, $C(o, r)$ intersects ∂P at exactly two points.

Based on **Condition (C)**, we define a "C" bound b^C as follows:

$b^C = \sup\{r \mid \text{If } r' < r, C(o, r') \text{ satisfies } \textbf{(C)} \}$

While **(S)** is intended to find a bound of Γ so that $(*)$ tests reachablity, we use **(F)** and **(C)** as the premise for **(S)**. In particular, we will show that **(C)** implies **(F)** and that **(F)** implies **(S)**. Also we will show that **(F)** characterizes the convex obtuse property of a simple polygon and that **(C)** characterizes the obtuse property of a convex polygon.

One of the main results in this thesis is the following, when the confining region P is a convex obtuse polygon.

$$s_{min} \leq b^C = b^F \leq b^S \leq w.$$

The rest of this thesis is organized as follows.

In Chapter 2 and Chapter 3, we consider convex obtuse polygons as the confining regions. Chapter 2 characterizes b^C and uses this to show that $s_{min} \leq b^C = b^F$. It proves that $b^C = w$ for any regular $2k$-gon. It also shows that **(F)** characterizes the convex obtuse property of a simple polygon and that **(C)** characterizes the obtuse property of a convex polygon.

Chapter 3 shows that $b^C \leq b^S$ by giving a polynomial time algorithm to bring A_n to any of its reachable points.

Chapter 4 examines the properties of the reachable region of Γ. Here we consider general convex polygons as the confining regions and prove that $b^S \leq w$. We use this to describe the shapes of reachable regions. We also characterize l-reachable convex polygons for $l \leq b^S$.

Chapter 5 handles reconfiguration of chains within circles. In particular, we show that if Γ is bounded by the radius r of the circle, then Γ can be moved between any of its configurations inside the circle in $O(n^2)$ time. Consequently, we are able to prove that any $\Gamma \prec r$ inside the circle can be folded and that any joint of $\Gamma \prec r$ can reach any point inside the circle. We treat circles as the extreme case of nice confining environments as circles have no corners. We believe that our results shed light on how the combination of short link chains and nice confining environments ensures fast reconfiguration.

Chapter 6 concludes with a summary of the results in this thesis and presents problems for future research.

Chapter 2

Chain Geometry

This chapter studies the geometry of chains and their confining convex obtuse polygons in a uniform, coordinated way so that condition $(*)$ tests the reachability. Section 2.1 characterizes b^C. Section 2.2 proves that $s_{min} \leq b^C$ and that $b^C = w$ for regular $2k$-gons. Section 2.3 shows that $b^C = b^F$ and that both **(F)** and **(C)** characterize convex obtuse polygons.

Assumptions: Throughout this and the next chapter, we assume, unless otherwise stated, that Γ denotes an n-link chain and P denotes an m-sided convex obtuse polygon. Γ is confined by P and joints of Γ may lie on ∂P.

2.1 Characterizing b^C

A key observation about b^C is the following.

Lemma 2.1 *Let $x, y \in \partial P$. If $y \notin V$ and $xy \perp s(y)$, then $b^C \leq |xy|$.*

Proof: Let uv be the side containing y. Without loss of generality, assume that x, u, v is a left turn, as Figure 2.1 shows.

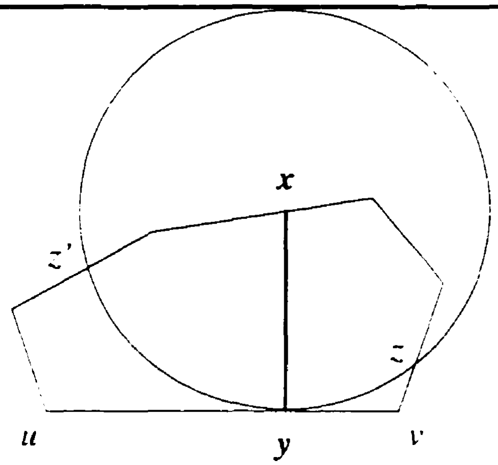


Figure 2.1: $C(x, |xy|)$ intersects ∂P at three points.

Then $C(x, |xy|)$ intersects ∂P at y. It also intersects $Ch(v, x)$, $Ch(x, u)$ at some points z, z', respectively. Since v, y, u are distinct, so are z, y, z'. Thus $C(x, |xy|)$ intersects ∂P at more than two points. Hence $b^C \leq |xy|$. $\square$

We will characterize b^C by proving that b^C is indeed the greatest lower bound of all such xy's. To this end, we need some preliminaries.

Given $p \in \partial P$ and $s(p)$, we define $l_p^{\perp}$ as the line through p perpendicular to $s(p)$. In these terms, we have

Lemma 2.2 *Let $p \in \partial P$ and $s(p)$ be given. If $p \notin V$, $l_p^{\perp}$ intersects ∂P at exactly two points p and p^0. If $p \in V$, $l_p^{\perp}$ intersects the sides of P that are non-adjacent to $s(p)$ at exactly one point. Finally,*

$$\min_{v \in V} d(v, v^0) = \inf_{p \notin V, p \in \partial P} d(p, p^0).$$

Proof: Since P is convex, if $p \notin V$, $l_p^{\perp} \cap \partial P$ contains exactly two points. Since P is obtuse, if $p \in V$, $l_p^{\perp}$ intersects the sides of P that are non-adjacent to $s(p)$ at exactly one point.

Let (u, u^0) be the pair achieving the minimum of $\{d(v, v^0) | v \in V\}$. Consider $p \notin V$ and the side $v_1 v_2$ containing p. Then clearly,

$$\min\{d(v_1, v_1^0), d(v_2, v_2^0)\} \leq d(p, p^0).$$

Hence $d(u, u^0) \leq d(p, p^0)$. Therefore,

$$d(u, u^0) \leq \inf_{p \notin V, p \in \partial P} d(p, p^0).$$

Note that

$$\lim_{p \to u, p \in \partial P} d(p, p^0) = d(u, u^0).$$

The result hence follows. $\square$

The width of an n point set was first introduced and studied by Houle and Toussaint. In [HT88], they proved the following.

Fact 2.1 *Let P be a simple polygon. Then the width of P is the minimum distance between two parallel lines of support of P, of which one passes through a vertex and the other passes through a side.*

By applying this, we have

Corollary 2.1 $\min_{v \in V} d(v, v^0) \leq w$.

Proof: By Fact 2.1, there exists two parallel lines of support of P, one passes through a vertex and the other passes through a side, achieving the width w. Without loss of generality, assume that l_1 passes through vertex u and l_2 passes through side $v_1 v_2$, and that u, v_1, v_2 is a left turn, as Figure 2.2 shows.

Let l be the perpendicular bisector of $v_1 v_2$ and let $y = l \cap v_1 v_2$. Then l intersects $Ch(v_2, v_1)$ at some point x. Since l_1, l_2 are support lines of P achieving the width, $|xy| \leq w$. By Lemma 2.2, $\min_{v \in V} d(v, v^0) \leq |xy| \leq w$. □

We are now ready to characterize b^C.

Theorem 2.1 $b^C = \min_{v \in V} d(v, v^0)$.

Proof: Let $\overline{b^C} = \min_{v \in V} d(v, v^0)$. For any $p \in \partial P$ and $p \notin V$, by Lemma 2.1, $b^C \leq d(p, p^0)$. Hence $b^C \leq \inf_{p \notin V, p \in \partial P} d(p, p^0)$. By Lemma 2.2, $b^C \leq \overline{b^C}$.

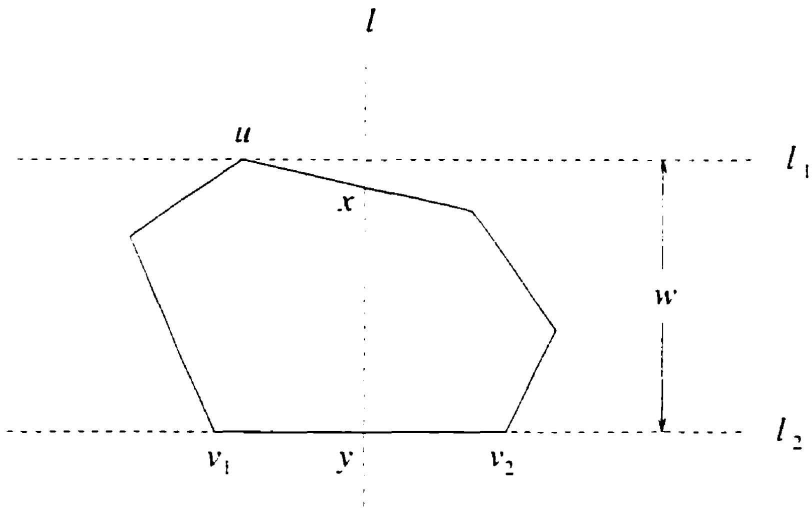


Figure 2.2: $\min_{v \in V} d(v, v^0) \leq w$.

We now verify that $\overline{b^C} \leq b^C$, i.e., $\forall r < \overline{b^C}$, $\forall o \in \partial P$, $C(o, r) \cap \partial P$ has exactly two points.

Assume otherwise. We first claim that $C(o, r) \cap \partial P$ has at least two points. If not, P would fall into a half circle determined by $C(o, r)$ and the line through $s(p)$. Then $r \geq w$. Hence $\overline{b^C} > w$. This contradicts Corollary 2.1.

Now suppose $C(o, r) \cap \partial P$ has at least three points, say, x, y, z. Without loss of generality, assume that x, o, z is a right turn and that x, y, z are consecutive intersection points on $C(o, r)$, i.e., $\widehat{xz}$ intersects ∂P at no points other than x, y, z. Then each of $Ch(x, y), Ch(y, z)$ is either completely outside or inside $C(o, r)$. We prove the result by these two cases.

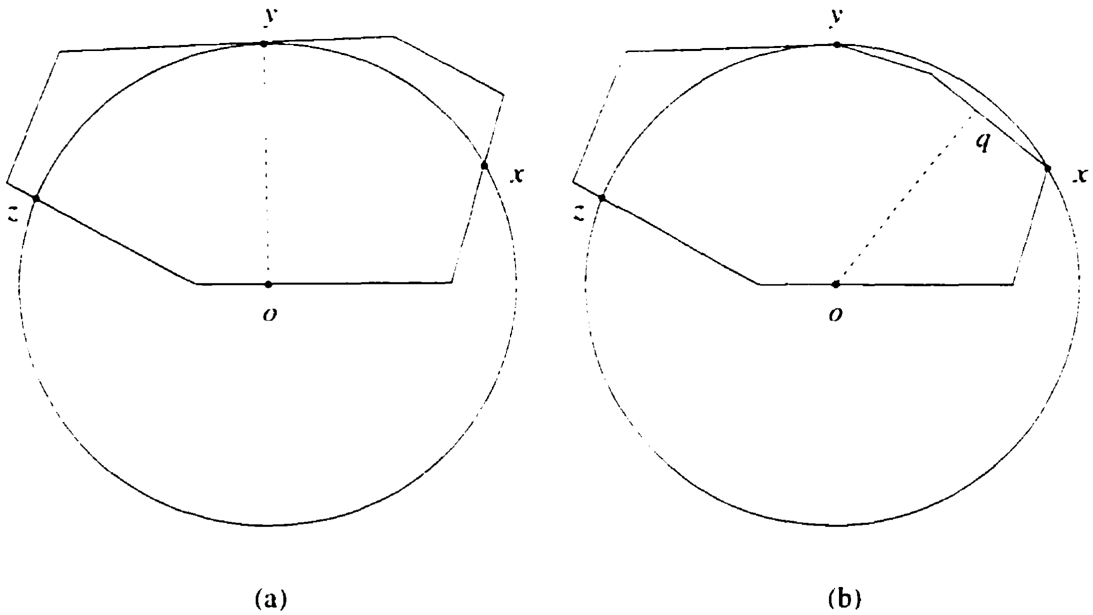


Figure 2.3: Two cases: $Ch(x, y)$ and $Ch(y, z)$ are both completely outside $C(o, r)$ or one of them is completely inside $C(o, r)$.

Case 1: $Ch(x, y)$ and $Ch(y, z)$ are both completely outside $C(o, r)$, as shown in (a) of Figure 2.3.

Then $y \notin V$ and hence $s(y)$ is tangent to $C(o, r)$. So $r = |oy|$. By Lemma 2.2, $|oy| \geq \overline{b^C}$. Hence $r \geq \overline{b^C}$. This contradicts the assumption.

Case 2: $Ch(x, y)$ or $Ch(y, z)$ is completely inside $C(o, r)$.

Without loss of generality, assume that $Ch(x, y)$ is completely inside $C(o, r)$, as shown in (b) of Figure 2.3. Then there exists a point $q \in Ch(x, y)$ such that $l(o, q)$ is perpendicular to $s(q)$. By Lemma 2.2, $|oq| \geq \overline{b^C}$. Note that $r \geq |oq|$, hence $r > \overline{b^C}$. Contradiction. $\square$

Corollary 2.2 $b^C \leq w$.

Proof: By Theorem 2.1 and Corollary 2.1. □

2.2 Bounds on b^C

Before proceeding to show that $b^C = b^F$, we present two essential results by applying Theorem 2.1. The first shows that, for any convex obtuse polygon, we can use s_{min} as the bound of Γ to satisfy **(C)** and the the second shows that, for any regular $2k$-gon, this bound can be pushed to w. These two results demonstrate that s_{min} and w are tight bounds on b^C.

We first present a crucial property of convex obtuse polygons.

Lemma 2.3 *If x and y lie on non-adjacent sides of P, then $|xy| \geq s_{min}$. Furthermore, if each interior angle of P measures $> \pi/2$, then $|xy| > s_{min}$.*

Proof: First, we show that $|xy| \geq s_{min}$. Let s_1, s_2 be the sides of P containing x, y, respectively. Let l_1, l_2 be the lines determined by s_1, s_2, respectively. Without loss of generality, assume that $x, l_1 \cap l_2, y$ is a right turn, as Figure 2.4 shows.

Let Q be $Ch(y, x)$. Let v_{max} be the vertex of Q farthest from xy and let l be the line through v_{max} parallel to xy. Let u, v be the vertices adjacent to v_{max}. Since s_1 and s_2 are non-adjacent sides, at least one of u, v is on Q. Without loss of generality, assume that $u \in Q$, as shown in Figure 2.4. We consider two cases.

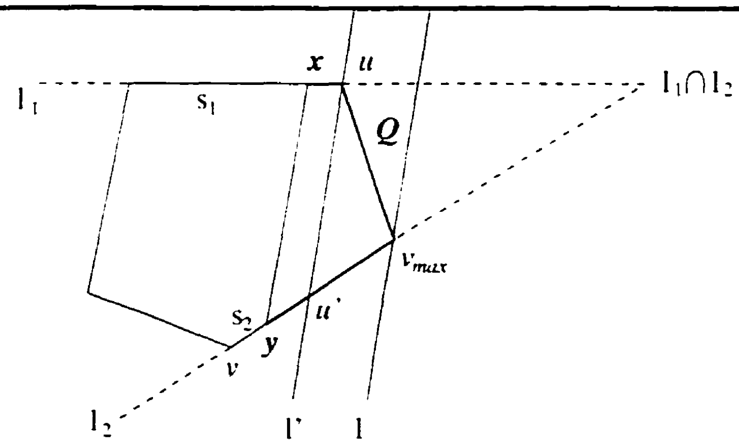


Figure 2.4: A crucial property of convex obtuse polygons

Case 1: $d(u, l) = 0$. Then $u \in l$. From $v_{max} \in l$ and $xy \parallel l$, we get $xy \parallel uv_{max}$. By the definition of Q, we have

$$|xy| \geq |uv_{max}| \geq s_{min}.$$

Case 2: $d(u, l) > 0$. Let l' be the line through u parallel to xy and let $l' \cap Q = \{u, u'\}$. Then $|xy| \geq |uu'|$.

Note that u, v_{max}, u' form a triangle. Since P is convex obtuse, $\angle uv_{max}u' \geq \pi/2$. Thus this angle is the largest interior angle of $\triangle uv_{max}u'$ and hence uu' is the longest side of $\triangle uv_{max}u'$. So we have $|uu'| > |uv_{max}|$. Therefore,

$$|uu'| > |uv_{max}| \geq s_{min}.$$

Hence $|xy| > s_{min}$.

Next, we show that $|xy| > s_{min}$ if each interior angle of P measures $> \pi/2$.

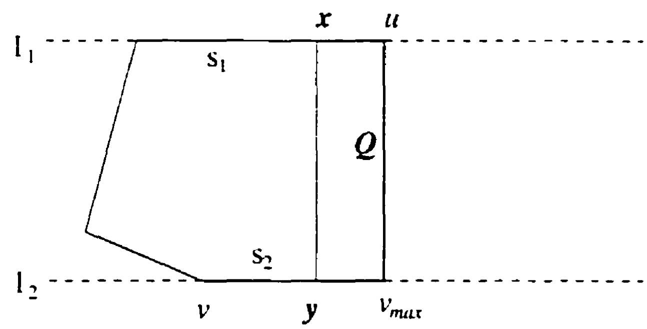


Figure 2.5: $|xy| > s_{min}$ if each interior angle of P measures $> \pi/2$.

Assume otherwise. Then $|xy| = s_{min}$ for some x, y on non-adjacent sides. From the above proof, this may hold only in Case 1, in which $u \in l$ and $|xy| \geq |uv_{max}| \geq s_{min}$. Hence $|xy| = |uv_{max}|$. So $l_1 \parallel l_2$.

Since P is obtuse, $\angle xuv_{max}$ and $\angle yv_{max}u$ are all $\geq \pi/2$. Since $l_1 \parallel l_2$, $\angle xuv_{max} + \angle yv_{max}u = \pi$. Thus $\angle xuv_{max} = \angle yv_{max}u = \pi/2$. Refer to Figure 2.5. This contradicts the assumption. □

Theorem 2.2 $s_{min} \leq b^C$. *Furthermore, if each interior angle of P measures* $> \pi/2$, $s_{min} < b^C$.

Proof: By Theorem 2.1, $b^C = \min_{v \in V} d(v, v^0)$. Let (u, u^0) be the pair achieving the minimum of $\{d(v, v^0) | v \in V\}$. By Lemma 2.2, u and u^0 lie on non-adjacent sides. By Lemma 2.3, $s_{min} \leq d(u, u^0) = b^C$, and if each interior angle of P measures $> \pi/2$, $s_{min} < d(u, u^0) = b^C$. The result hence follows. □

The above theorem shows that s_{min} can always be used as the bound of Γ to satisfy (C) and in case each interior angle of P measures $> \pi/2$. this bound can be improved.

It is of interest to find convex obtuse polygons in which $b^C = w$. As we will show that $b^C \leq b^S \leq w$. b^C would achieve its maximum and all gaps among b^C, b^F, w would be closed in such polygons. The theorem below provides examples of such convex obtuse polygons.

Theorem 2.3 *Let P be a regular $2k$-gon. Then $b^C = w$.*

Proof: Since P is regular, $d(v, v^0)$ is the same for all $v \in V$. Since P has $2k$ sides, each side is parallel to its opposite side. Hence $d(v, v^0) = w$. By Theorem 2.1, $b^C = w$. □

We remark that the above theorem suggests similar results for reconfiguring chains inside circles, which can be regarded as the limits of regular $2k$-gons. We will discuss the reconfiguration of chains within circles in more detail in Chapter 5.

2.3 $b^C = b^F$

This section presents the main result in this chapter: $b^C = b^F$. We first prove the following lemma.

Lemma 2.4 *Let $\Gamma = [A, B]$. Suppose that A, B lie at $p, q \in \partial P$, respectively, and that v, v' are vertices adjacent to q (q may be a vertex or not). Then if either $\angle pqv$*

or $\angle pqv'$ is $> \pi/2$. A can be moved away from p along ∂P in either direction while B remains on ∂P.

Proof: Let u be the vertex that we want A to move toward. Without loss of generality, assume that both u, p, q and p, q, v are both left turns. See Figure 2.6. Since either $\angle pqv$ or $\angle pqv'$ is $> \pi/2$, neither of them is $\pi/2$. Let $\alpha = \angle upq$, let $\beta = \angle pqv$ and let $\beta' = \angle pqv'$. We consider two cases.

Case 1: $\alpha \geq \pi/2$. We further consider two subcases.

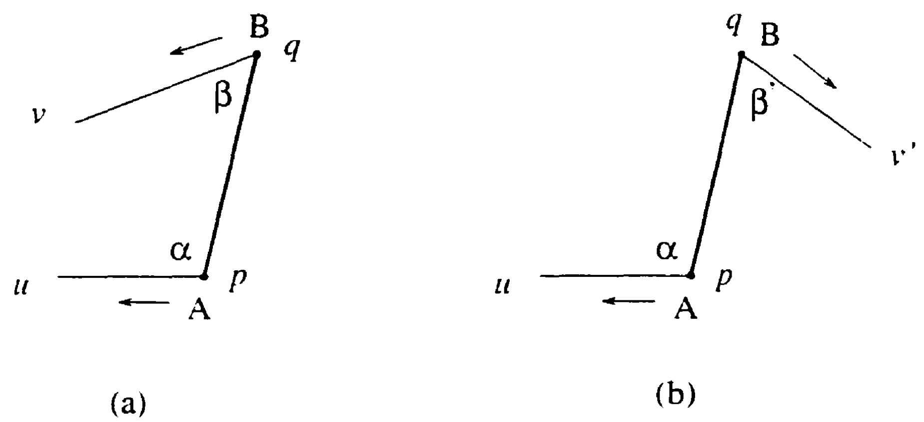
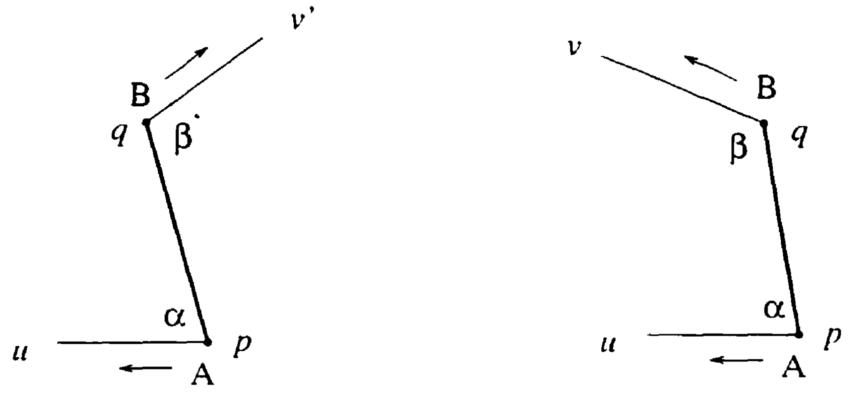


Figure 2.6: Case 1: $\alpha \geq \pi/2$.

Case 1a: $\beta < \pi/2$. Then while keeping B on vq, A can be moved away from p along ∂P in such a way that α increases and β decreases. See (a) of Figure 2.6.

Case 1b: $\beta > \pi/2$. Thus $\beta' < \pi/2$. Then while keeping B on qv', A can be moved away from p along ∂P in such a way that both α and β increase. See (b) of Figure 2.6.

Case 2: $\alpha < \pi/2$. We further consider two subcases.



(a) (b)

Figure 2.7: Case 2: $\alpha < \pi/2$.

Case 2a: $\beta < \pi/2$. Thus $\beta' > \pi/2$. Then while keeping B on qv', A can be moved away from p along ∂P in such a way that both α and β increase. See (a) of Figure 2.7.

Case 2b: $\beta > \pi/2$. Then while keeping B on vq, A can be moved away from p along ∂P in such a way that α increases and β decreases. See (b) of Figure 2.7.

This completes the proof. $\square$

Theorem 2.4 $b^C = b^F$.

Proof: First, we show that $b^C \leq b^F$, i.e., any single link chain $\Gamma = [A, B]$ having length $l < b^C$ satisfies **(F)**.

Suppose that A, B initially lie at $p, q \in \partial P$, respectively. Let v, v' be the vertices adjacent to q. We consider two cases.

Case 1: $q \notin V$. In this case, we show that until B reaches some vertex, A can be moved along ∂P in either direction while keeping B on ∂P.

Since $l < b^C$, by Lemma 2.1, $pq \not\perp vv'$. Hence either $\angle pqv$ or $\angle pqv'$ is $> \pi/2$. By Lemma 2.4, A can be moved away from p in either direction along ∂P while keeping B on ∂P. Also by Lemma 2.1, Γ remains not perpendicular to vv' until B reaches v or v'. So by Lemma 2.4, A can be moved along ∂P in any direction while keeping B on ∂P until B reaches some vertex.

Case 2: $q \in V$. In this case, we show that A can be moved away from p along ∂P in either direction while keeping B on ∂P.

To see this, we claim that either $\angle pqv$ or $\angle pqv'$ is $> \pi/2$. Assume otherwise. Then $\angle pqv$ and $\angle pqv'$ are all $\leq \pi/2$. If one of them, say, $\angle pqv = \pi/2$, then by Theorem 2.1 and Lemma 2.2, $b^C \leq |pq| = l$. Contradiction. Hence $\angle pqv$ and $\angle pqv'$ are both $< \pi/2$.

Also note that $q \in C(p, |pq|) \cap \partial P$; thus P has to lie completely inside $C(p, |pq|)$ for otherwise $C(p, |pq|) \cap \partial P$ has at least three points. Therefore, P falls into a half circle determined by $C(p, |pq|)$ and the line through $s(p)$, as Figure 2.8 shows.

Let l' be the tangent line of the half circle parallel to $s(p)$. Then l' and $s(p)$ are parallel lines of support of P. Since $d(l', s(p)) = |pq|$, $|pq| \geq w$. Thus $b^C > |pq| \geq w$.

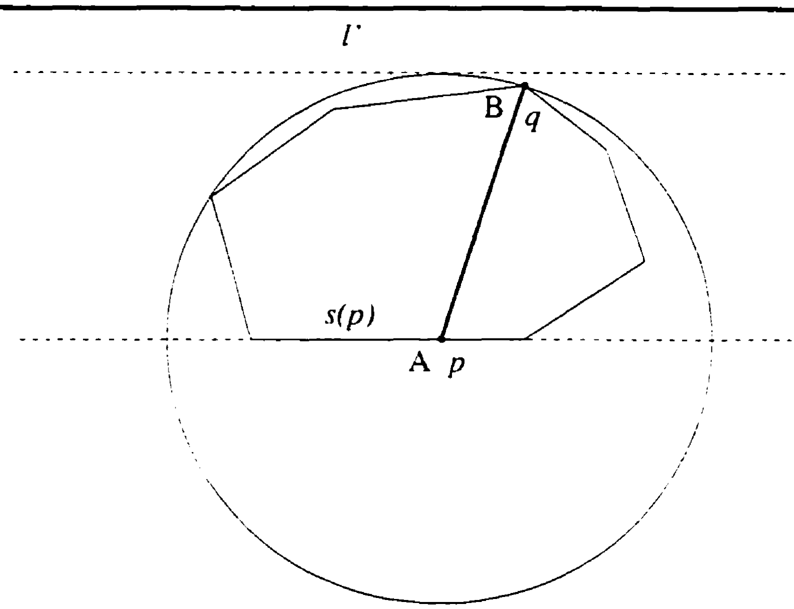


Figure 2.8: P lies completely inside a half circle.

This contradicts Corollary 2.2. Hence the claim.

From the claim and Lemma 2.1. A can be moved away from p along ∂P in either direction while keeping B on ∂P.

Note that in Case 2. A leaving p implies B leaving q. Hence by the above two cases. A can be moved completely around ∂P in either direction while keeping B on ∂P. So $b^C \leq b^F$.

Next. we show that $b^F \leq b^C$. If not. then $b^C < b^F$ for some P and we now consider such a P. Then for any l with

$$b^C < l < b^F$$

and any single link chain $\Gamma = [A, B]$ having length l. Γ satisfies **(F)**. We show that this is impossible.

To see this, note that by Theorem 2.1, there exists $u \in V$ such that $b^C = d(u, u^0)$. Suppose that v, v' are vertices adjacent to u and that w, w' are vertices adjacent to u^0. Without loss of generality, assume that $u^0u \perp uv'$ and that u, u^0, w' is a right turn, as Figure 2.9 shows.

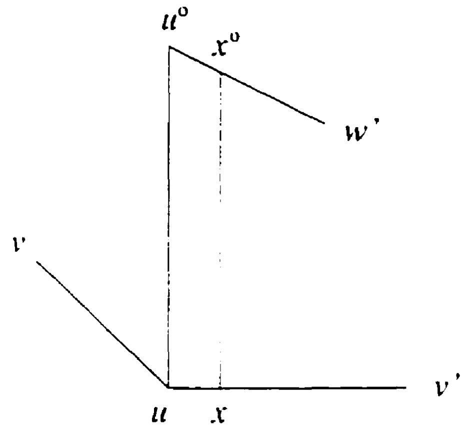


Figure 2.9: $|uu^0| > |xx^0|$

We claim that $\angle uu^0w' \geq \pi/2$. Otherwise $\angle uu^0w' < \pi/2$, as shown in Figure 2.9. Then for $x \in uv'$ sufficiently close to u, $x^0 \in u^0w'$. Hence $|uu^0| > |xx^0|$. Therefore $b^C > |xx^0|$. This contradicts Theorem 2.1 or Lemma 2.2. Hence the claim.

Note that $\angle vuu^0 < \pi/2$. Thus for $u' \in u^0w'$ sufficiently close to u^0, $\angle vuu' < \pi/2$, as Figure 2.10 shows. From the claim, $\angle uu'u^0 < \pi/2$, and clearly $\angle u'uv' < \pi/2$.

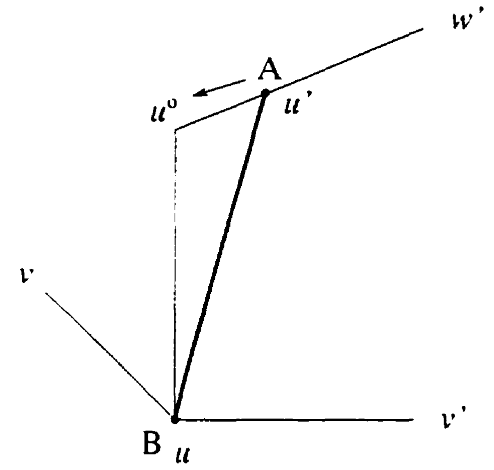


Figure 2.10: A cannot be moved towards u^0.

Consider a single link chain Γ whose joints A, B lie at u', u, respectively. Then,

$$b^C = |uu^0| < |uu'| < b^F.$$

But A cannot be moved towards u^0 any more along ∂P. □

Our next result shows that the family of convex obtuse polygons is the largest family of simple polygons for which condition **(F)** holds non-trivially, and is the largest family of convex polygons for which condition **(C)** holds non-trivially.

Theorem 2.5 *(1) Let P be a simple polygon. Then $b^F > 0$ if and only if P is convex obtuse.*

(2) Let P be a convex polygon. Then $b^C > 0$ if and only if P is convex obtuse.

Proof: (1) If P is convex obtuse, by Theorem 2.4, $b^F = b^C$. By Theorem 2.2, $b^C \geq s_{min} > 0$. Hence $b^F > 0$.

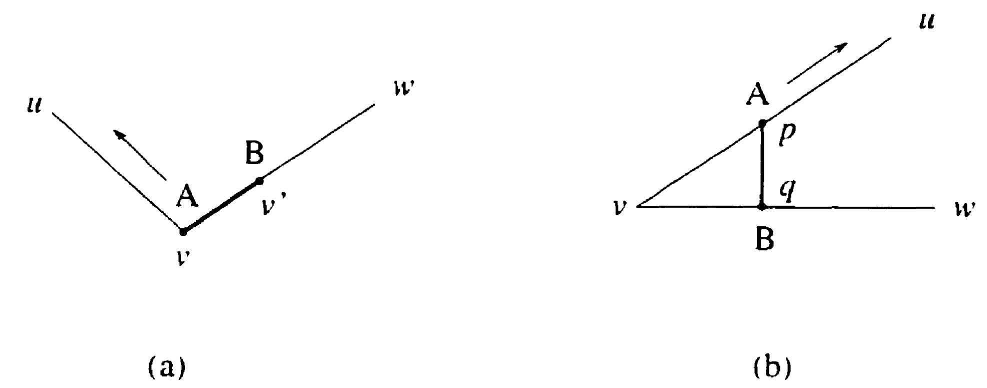


Figure 2.11: $b^F = 0$ if P is neither convex nor obtuse.

If P is not convex, then there exists an interior angle, say, $\angle uvw > \pi$. See (a) of Figure 2.11. For $l > 0$ small enough, let Γ be the single link chain having length of l and consisting of joints A, B which lie at $v, v' \in vw$. Then A cannot be moved towards u while keeping B on ∂P. Hence $b^F = 0$.

If P is not obtuse, then there exists an interior angle, say, $\angle uvw < \pi/2$. See (b) of Figure 2.11. For $l > 0$ sufficiently small, consider a single link chain $\Gamma = [A, B]$ having length l as follows: A, B lie at $p \in vu, q \in vw$, respectively and $pq \perp vw$. Then A cannot be moved towards u while keeping B on ∂P. Hence $b^F = 0$.

(2) If P is convex obtuse, by Theorem 2.2, $b^C \geq s_{min} > 0$.

If P is not obtuse, then there exists an interior angle, say, $\angle uvw < \pi/2$. See Figure 2.12. For $l > 0$ sufficiently small, let $o \in vu, q \in vw$ with $oq \perp vw$ and $|oq| = l$. Then $C(o, l)$ intersects vu at two points and intersects vw at one point. Thus $b^C = 0$. $\square$

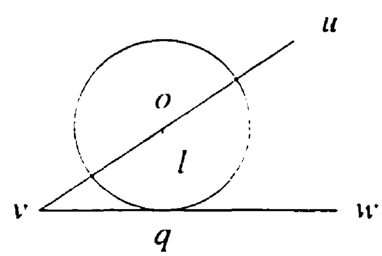


Figure 2.12: $b^C = 0$ if P is not obtuse.

Chapter 3

Reachability Algorithm

This chapter presents the reachability algorithm bringing A_n to any of its reachable points. This gives an algorithmic proof for $b^C \leq b^S$.

Section 3.1 defines normal forms and simple motions. Section 3.2 shows that any $\Gamma \prec b^F$ can be brought to RNF (to be defined). Section 3.3 shows that any $\Gamma \prec b^C$ which is already in RNF can be brought to TNF-i_0 (to be defined) for some i_0. Section 3.4 presents the reachability algorithm for $\Gamma \prec b^C$ which is already in TNF-i_0.

We recall that the **Assumptions** from Chapter 2 hold throughout this chapter.

3.1 Preliminaries

3.1.1 Normal Forms

We define three special configurations for a n-link chain Γ as follows, refer to Figure 3.1.

1. *Rim Normal Form* (denoted RNF): Γ is in RNF if all its joints lie on ∂P.

2. *k-Bending Rim Normal Form* (denoted k-BRNF): Γ is in k-BRNF if there exists k joints $A_{i_1}, \ldots, A_{i_k}$ such that

   (1) $A_{i_1}, \ldots, A_{i_k}$ lie on ∂P;

   (2) for any j with $0 < j < n$ and $j \neq i_1, \ldots, i_k$, $\alpha_j = \pi$.

3. *Tail Normal Form with index i* (denoted TNF-i): Γ is in TNF-i if there exists i with $i < n$ such that

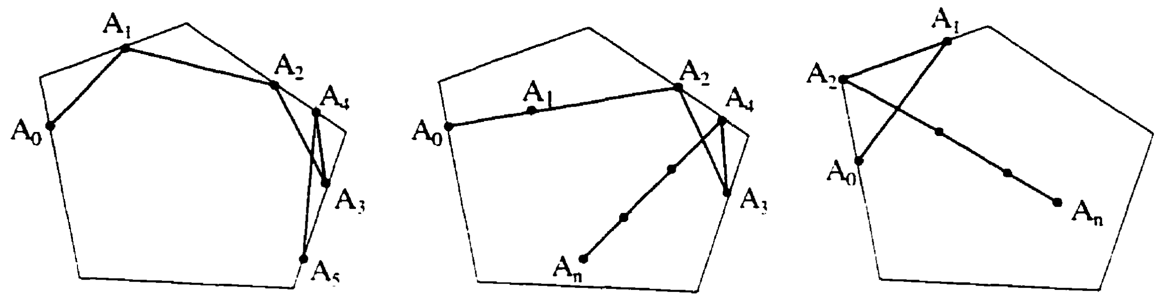
   (1) $A_0, \ldots, A_i$ lie on ∂P and A_i lies at a vertex of P;

   (2) $\Gamma(i, n)$ is straight.

From the above definitions, we have:

Observation 3.1 *Γ is in k-BRNF for some k if and only if all intermediate joints of Γ are either straight or on ∂P.*

Observation 3.2 *Γ is in n-BRNF if and only if Γ is in RNF.*



(a) RNF (b) 4-BRNF (c) TNF-2

Figure 3.1: Examples of normal forms.

Observation 3.3 *Γ is in 0-BRNF if and only if Γ is a straight chain with no joints on ∂P.*

Bringing a chain bounded by b^C to normal forms plays a crucial role in our reachability algorithm, described in Section 3.4. The next two sections elaborate on moving such a chain to normal forms. We first give an equivalent of **Condition (F)**, which will be referred to repeatedly later.

Lemma 3.1 *The following condition* **(F')** *is equivalent to* **(F)**.

Condition (F'): *Let Γ be an n-link chain in RNF. Then any joint A_i of Γ can be moved along any path on ∂P while keeping Γ in RNF.*

Proof: **(F)** follows trivially from **(F')**. We now verify that **(F)** implies **(F')**.

First, we consider the endjoints. Since the joint labels can be reversed, it suffices to consider A_n. We proceed by induction on n.

If $n = 1$, then Γ has only one link and the result is immediate from (**F**). Suppose that the result is true for any $(n-1)$-link chain, and now consider an n-link chain Γ. View Γ as two subchains $\Gamma(n-1, n)$ and $\Gamma(0, n-1)$ connecting at A_{n-1}. Given any path τ on ∂P, by the induction base, A_n of $\Gamma(n-1, n)$ can travel along τ while $\Gamma(n-1, n)$ remains in RNF. This yields a path τ' on ∂P that A_{n-1} travels. By the induction hypothesis, A_{n-1} of $\Gamma(0, n-1)$ can travel along τ' while $\Gamma(0, n-1)$ remains in RNF. So A_n can travel along τ while Γ remains in RNF and the induction is complete.

Next, we consider an intermediate joint A_i $(0 < i < n)$. Given any path τ on ∂P, view Γ as two subchains $\Gamma(0, i)$ and $\Gamma(i, n)$ connecting at A_i. Then both A_i of $\Gamma(0, i)$ and A_i of $\Gamma(i, n)$ can travel along τ while $\Gamma(0, i)$ and $\Gamma(i, n)$ remain in RNF. Thus A_i of Γ can travel along τ while Γ remains in RNF. This completes the proof. $\square$

3.1.2 Simple Motions

To analyze the time complexity of our motion planning algorithms, it is essential to define one or more kinds of simple motions so that complicated motions can be decomposed into a sequence of simple ones. Such a decomposition also gives us something to count, and hence some measure of the complexity of physical movement itself. Here is list of criteria, based on [HJW85], for "good" simple motions of a linkage.

Criteria:

1. The description of the motion should uniquely determine the geometric movement of all parts of the linkage.

2. The motion should be the one whose description can be computed.

3. If a joint angle changes, it should change monotonically. In other words, a motion in which an angle changes non-monotonically should be regarded as a combination of simple motions. This eliminates oscillating motion as candidates for simple motions.

With these criteria in mind, we define a simple motion of a chain as follows.

Definition 3.1 *A simple motion of a chain is a continuous motion during which at most four angles change.*

This type of simple motion was also used in [HJW85]. However, the simple motions chosen should not be limiting: it should be possible to carry out any reconfiguration in terms of the simple motions available to the algorithm. Indeed, other types of simple motions were also used. Refer to [LW92] and [KSW95].

Figure conventions: In some multi-part figures, the parts are intended to show possibilities for configurations, but the chain depicted may not be the same in all parts of the figure. Also, an unfilled circle o at a joint indicates that the joint is to be kept fixed during some motion of Γ.

3.2 Bringing a Chain to RNF

The key idea of the algorithm for bringing a chain $\Gamma \prec b^F$ to RNF is to use k-BRNF as a bridge. More specifically, we will show that if Γ takes the form $[A_0, \ldots, A_n]$ with A_0 and A_n on ∂P (a special 2-BRNF), then Γ can be brought to 3-BRNF while keeping A_0 fixed. By applying this manoeuvre to various subchains of Γ, it is possible to bring Γ to 4-BRNF, to 5-BRNF, ..., and finally, to n-BRNF, which is just RNF.

This algorithm consists of three main phases, which we describe in the next three lemmas.

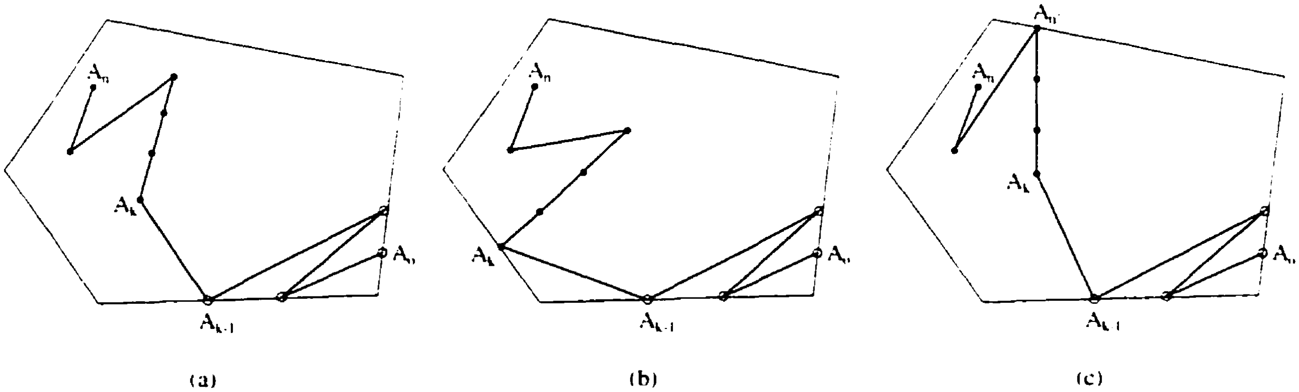


Figure 3.2: An initial configuration (a) and two possible final configurations (b) and (c) for Lemma 3.2

Lemma 3.2 *For an n-link chain Γ, suppose that $A_0, \ldots, A_{k-1}$ lie on ∂P. Then while $A_0, \ldots, A_{k-1}$ remain fixed, Γ can be moved to a configuration in which either A_k lies on ∂P or for some $n' > k$, $A_{n'}$ lies on ∂P and $\Gamma(k, n')$ is straight. Furthermore, this can be done with $O(n)$ simple motions. See Figure 3.2.*

Proof: If initially Γ is already in the expected configuration, then we are done. Otherwise we proceed as follows.

Let h be the highest index such that $\Gamma(k, h)$ is strsight. Then $k + 1 \leq h \leq n$. We show the result by induction on $h' = n - h$.

For $h' = 0$, i.e., for $h = n$, the configuration of Γ is as shown in (a) of Figure 3.3. Fix $A_0, \ldots, A_k$ and rotate $[A_k, \ldots, A_n]$ about A_k counterclockwise until A_n hits ∂P or α_k straightens to π. If A_k hits ∂P first, then we are done. If α_k straightens to π first, then $\Gamma(k-1, n)$ becomes straight. In this case, fix $A_0, \ldots, A_{k-1}$ and rotate $[A_{k-1}, \ldots, A_n]$ about A_{k-1} counterclockwise until A_n hits ∂P and we then establish the induction base with $n' = n$. This requires $O(1)$ simple motions.

Suppose that the result holds for any value less than $h' > 0$: we now show the result holds for h', i.e., for $h = n - h' < n$. Fix $A_0, \ldots, A_{k-1}$ and $A_{h+1}, \ldots, A_n$, and rotate A_k about A_{k-1} so that A_k moves away from A_{h-1} until A_k hits ∂P, or A_h hits ∂P, or α_h straightens to π. Only $O(1)$ simple motions are needed. See (b) of Figure 3.3.

If A_k hits ∂P first, then we are done. If A_h hits ∂P first, then the result holds

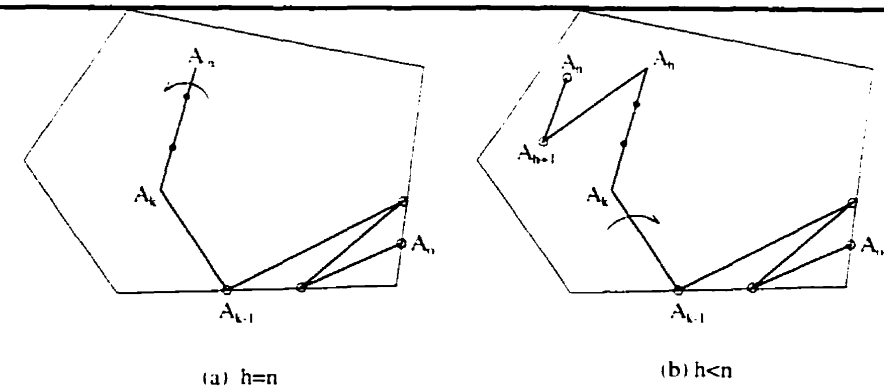


Figure 3.3: Bring $A_h(h > k)$ to ∂P with $[A_k, \ldots, A_h]$ straight

with $n' = h$. If α_h straightens to π first, then $\Gamma(k, h+1)$ becomes straight. In this case, by the induction hypothesis, Γ can be moved to a configuration in which either A_k lies on ∂P or for some $n' > k$, $A_{n'}$ lies on ∂P and $\Gamma(k, n')$ is straight. All together, $O(n)$ simple motion suffice. This completes the induction. $\square$

Lemma 3.3 *For an n-link chain $\Gamma \prec b^F$, suppose that $A_0, \ldots, A_l$ and A_n lie on ∂P and that $\Gamma(l, k)$ and $\Gamma(k, n)$ are straight, for some $l < k < n$.*

Then while A_n remains fixed and $\Gamma(0, l)$ remains in RNF, and while $[A_l, \ldots, A_k]$ and $[A_k, \ldots, A_n]$ remain straight, Γ can be moved to a configuration in which A_k either lies on ∂P or has $\alpha_k = \pi$. Furthermore, this can be done with $O(mn)$ simple motions. See Figure 3.4.

Proof: If initially Γ is already in the expected configuration, then we are done. Otherwise we proceed as follows.

Keeping A_n fixed, and keeping $[A_l, \ldots, A_k]$ and $[A_k, \ldots, A_n]$ straight, move A_l

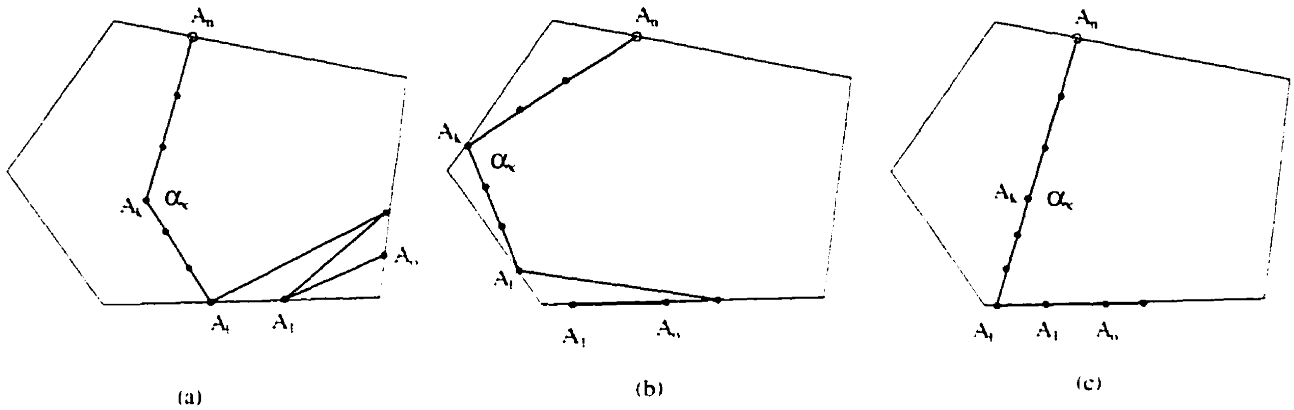


Figure 3.4: An initial configuration (a) and two possible final configurations (b) and (c) for Lemma 3.3. In (b) and (c), some joints are folded and some links overlap.

towards A_n along ∂P. By Lemma 3.1, this can be done while $A_0, \ldots, A_l$ remain on ∂P. Continue this process until A_k hits ∂P, or α_k straightens to π, or A_l coincides with A_n. Since P has m sides and $l \leq n$ and $\Gamma(0, l)$ undergoes no backtracking, $O(mn)$ simple motions suffice.

If A_k hits ∂P or α_k straightens to π first, then we are done. If A_l coincides with A_n first, then $[A_l, \ldots, A_k]$ and $[A_k, \ldots, A_n]$ coincide. In this case, fix A_l and A_n, and rotate $[A_k, \ldots, A_n]$ and $[A_l, \ldots, A_k]$ counterclockwise about A_n until A_k hits ∂P. All together, at most $O(mn)$ simple motions are needed. □

Lemma 3.4 *For an n-link chain $\Gamma \prec b^F$, suppose that $A_0, \ldots, A_l$ and A_n lie on ∂P and that $\Gamma(l, n)$ is straight, for some $l < n$.*

Then for any k with $l < k < n$, Γ can be moved to a configuration in which $A_0, \ldots, A'_l$ and A_k lie on ∂P and $\Gamma(l', k)$ is straight, for some $l' \leq l$. Moreover, during this *reconfiguration. A_n can be kept fixed, $[A_k, \dots, A_n]$ can be kept straight, and $\Gamma(0, l')$ can be kept in RNF. Furthermore, this can be done with $O(mn^2)$ simple motions. See Figure 3.5.*

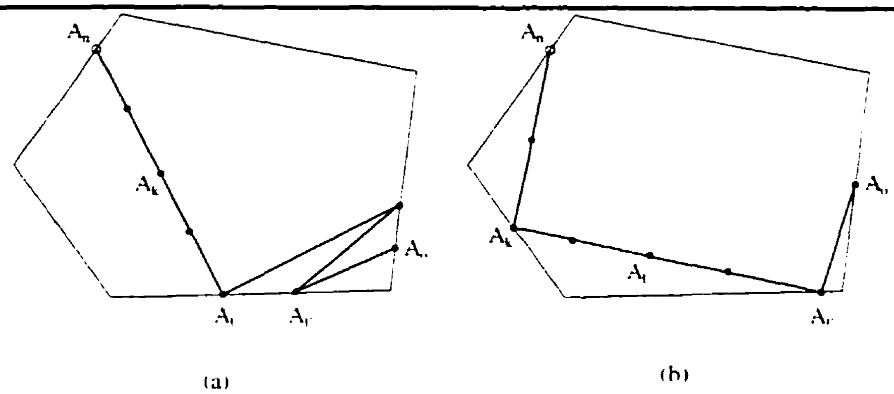


Figure 3.5: The initial (a) and final (b) configurations for Lemma 3.4

Proof: If initially Γ is already in the expected configuration, then we are done. Otherwise we proceed as follows.

Throughout the proof, A_n will remain fixed, and $[A_l, \dots, A_k]$ and $[A_k, \dots, A_n]$ will remain straight. We consider two cases.

Case 1: Initially A_l and A_n lie on adjacent sides. Let s_l and s_n be the sides that A_l and A_n lie on, respectively. Let v be the vertex where s_l and s_n meet. Without loss of generality, assume that A_l, v, A_n initially form a right turn. See Figure 3.6.

Keeping A_l, A_k, A_n a right turn, move A_l along s_l towards v. By Lemma 3.1, this can be done while $A_0, \dots, A_l$ remain on ∂P. Since P is obtuse, the interior angle at v is $\geq \pi/2$. Thus the distance between A_l and A_n is decreasing. Hence $\angle\alpha_k$ is

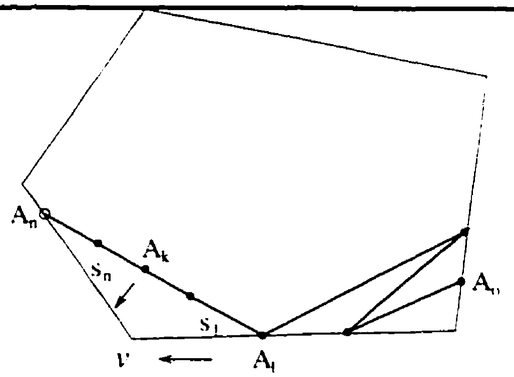


Figure 3.6: Moving A_l towards A_n along ∂P.

decreasing and A_k is moving towards ∂P. Continue this process until A_k hits ∂P and choose $l' = l$.

In this case, only $O(mn)$ simple motions are needed.

Case 2: Initially A_l and A_n lie on non-adjacent sides. We show the result by induction on l.

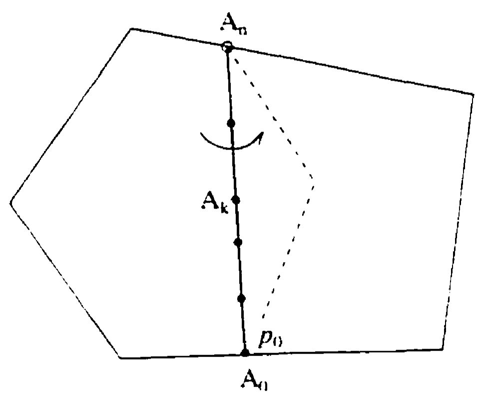


Figure 3.7: Rotating $[A_0, \ldots, A_k]$ about A_k.

For $l = 0$, let p_0 be the point where A_0 initially lies. Keeping p_0, A_0, A_k collinear, rotate $[A_k, \ldots, A_n]$ about A_n counterclockwise, as Figure 3.7 shows. Since the distance between A_k and p_0 is increasing, the above rotation is possible. Continue this process until A_k hits ∂P. Then fix $A_k, \ldots, A_n$, and rotate $[A_0, \ldots, A_k]$ about A_k counterclockwise until A_0 hits ∂P. We then establish the induction base with $l' = l = 0$ and only $O(1)$ simple motions are required.

Suppose that the result holds for all indices less than $l > 0$; we now prove the result for l. Fix $A_0, \ldots, A_{l-1}$, and rotate $[A_k, \ldots, A_n]$ about A_n so that A_k moves away from A_{l-1}, as Figure 3.8 shows.

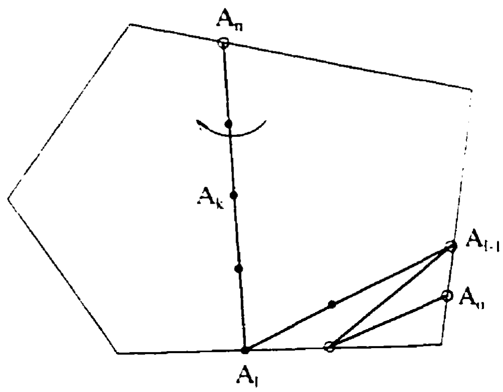


Figure 3.8: Rotating $[A_k, \ldots, A_n]$ about A_n so that A_k moves away from A_{l-1}.

We claim the following, which we will establish at the end of the proof.

Claim: During this rotation, A_l will *not* hit ∂P before A_k hits ∂P or α_l straightens to π.

From the claim, the above process terminates when A_k hits ∂P or α_l straightens to π, as shown in (a), (b) of Figure 3.9, respectively.

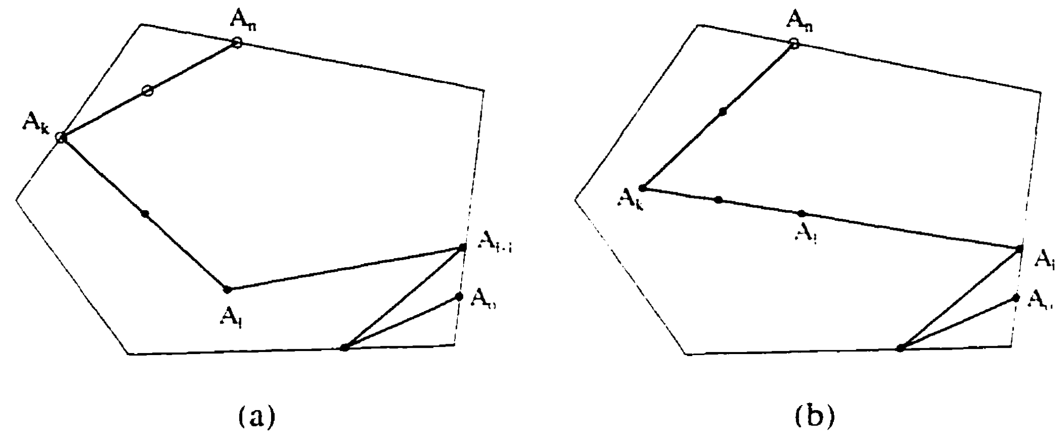


Figure 3.9: A_k hits ∂P first in (a): α_l straightens to π first in (b).

If A_k hits ∂P first, fix $A_k, \ldots, A_n$ and move $\Gamma(0, k)$ in accordance with Lemma 3.3. Then either A_l hits ∂P or α_l straightens to π, and we choose $l' = l$ or $l' = l - 1$, respectively. This requires $O(mn)$ simple motions.

If α_l straightens to π first, fix A_n and move Γ in accordance with Lemma 3.3. Then either A_k hits ∂P or α_k straightens to π. If A_k hits ∂P first, we choose $l' = l - 1$ and $O(mn)$ simple motions are involved. If α_k straightens to π first, then $\Gamma(l-1, n)$ becomes straight. In this case, if A_{l-1} and A_n lie on adjacent sides, then by Case 1, we conclude with $l' = l - 1$; if A_{l-1} and A_n lie on non-adjacent sides, then by the induction hypothesis, we conclude with some $l' \leq l - 1 < l$.

In this case, at most $mn + m(n-1) + \cdots, +m \in O(mn^2)$ simple motions are need.

Proof of the Claim: Let p_l, p_k be the points where A_l, A_k initially lie, respectively.

Let $C_l(o_l, r_l), C_k(o_k, r_k)$ be the circles that A_l, A_k travel along, respectively. Let t_l, t_k be the intersection points of $C_l(o_l, r_l)$ and ∂P, $C_k(o_k, r_k)$ and ∂P that A_l, A_k move towards, respectively. See Figure 3.10.

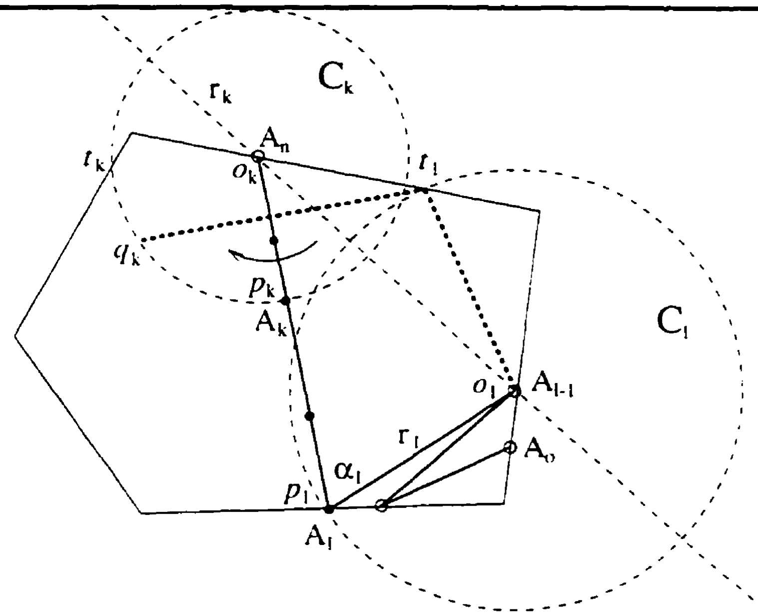


Figure 3.10: Case 1: o_k is outside C_l.

Then o_l, o_k are the points where A_{l-1}, A_n initially lie and remain fixed, respectively. Also r_l, r_k are the lengths of $[A_{l-1}, A_l]$, $[A_k, \ldots, A_n]$, respectively. By assumption, o_k and p_l lie on non-adjacent sides. We prove the result by two cases.

Case 1: o_k is outside C_l, as Figure 3.10 shows.

Without loss of generality, we assume that o_l, p_l, o_k form a right turn, i.e., initially $\alpha_l < \pi$. We show the result by contradiction.

Suppose A_l hits ∂P first, i.e., A_l reaches t_l while A_k lies at some $q_k \in \overset{\frown}{p_k t_k}$, as shown in the dotted line in Figure 3.10. Since o_k is outside C_l, o_k and p_l are on the same side of $l(t_l, o_l)$. Hence o_k, p_k, p_l are all on the same side of $l(t_l, o_l)$ and so are o_k, q_k, p_l. Note that o_l, p_l, o_k form a right turn, hence o_l, t_l, q_k form a left turn, i.e., $\alpha_l \geq \pi$ in this configuration.

Since α_l changes continuously, there exists some intermediate configuration in which $\alpha_l = \pi$. This means that α_l straightens to π before A_l hits ∂P. Contradiction.

Case 2: o_k is inside or on C_l, as Figure 3.11 shows.

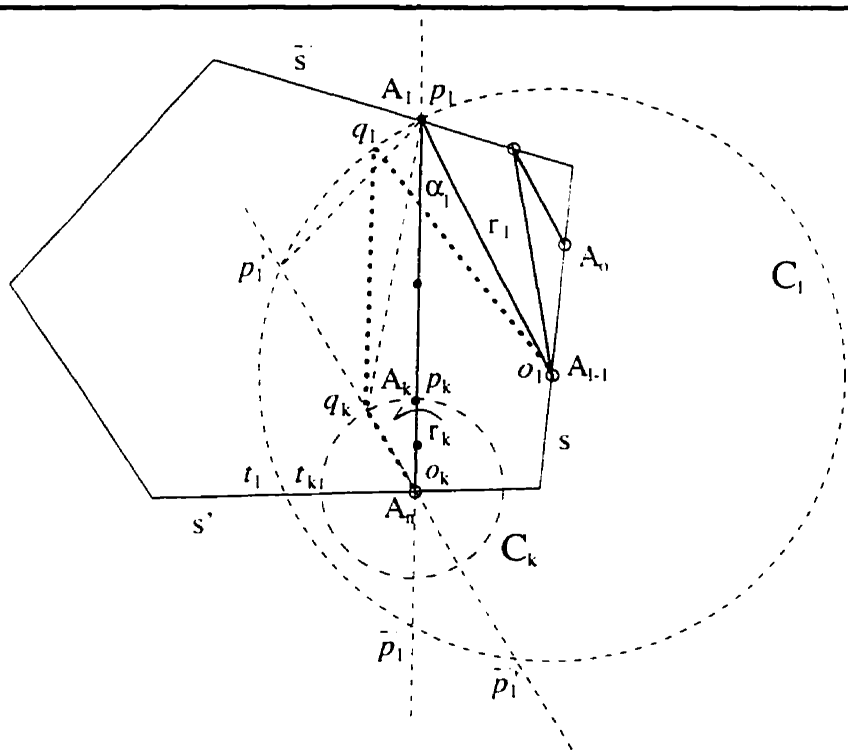


Figure 3.11: Case 2: o_k is inside or on C_l.

Without loss of generality, we assume that o_l, p_l, o_k form a left turn. We show that A_k hits ∂P first.

Suppose that $l(o_k, p_l) \cap C_l = \{p_l, \overline{p}_l\}$ and $l(o_k, q_k) \cap C_k = \{p'_l, \overline{p'}_l\}(p'_l \in \widehat{p_l\overline{p}_l})$. Then $o_l, p_l, \overline{p}_l$ is a left turn. So $d(\widehat{p_l\overline{p}_l}) < \pi$. Hence $d(\widehat{\overline{p}_l p_l}) = 2\pi - \widehat{p_l\overline{p}_l} > \pi$.

Since $p'_l \in \widehat{p_l\overline{p}_l}$, $o_l, p'_l, \overline{p'}_l$ is also a left turn. So $d(\widehat{p'_l\overline{p'}_l}) < \pi$. Therefore,

$$\widehat{\overline{p}_l p_l} > \widehat{p'_l\overline{p'}_l}$$

i.e. $\widehat{\overline{p}_l\overline{p'}_l} + \widehat{\overline{p'}_l p_l} > \widehat{p'_l\overline{p}_l} + \widehat{\overline{p}_l\overline{p'}_l}$. Thus $\widehat{\overline{p'}_l p_l} > \widehat{p'_l\overline{p}_l}$.

In $\triangle o_k p_l p'_l$, we have $\angle p_l p'_l o_k = \angle p_l p'_l \overline{p'}_l > \angle p'_l p_l \overline{p}_l = \angle p'_l p_l o_k$. Hence $|o_k p_l| > |o_k p'_l|$, i.e., $|o_k p_k| + |p_k p_l| > |o_k q_k| + |q_k p'_l|$. Since $|o_k p_k| = |o_k q_k| = r_k$, we get

$$|p_k p_l| > |q_k p'_l|. \tag{3.1}$$

In $\triangle o_k p_l q_k$, we have $|o_k p_k| + |p_k p_l| = |o_k p_l| < |o_k q_k| + |q_k p_l|$. Since $|o_k p_k| = |o_k q_k| = r_k$, we get

$$|p_k p_l| < |q_k p_l|. \tag{3.2}$$

From (3.1) and (3.2), we conclude that when A_k lies at q_k, A_l lies at some q_l inside $\widehat{p_l p'_l}$, as shown in the dotted line in Figure 3.11. Hence when A_k reaches t_k, A_l lies at some point inside $\widehat{p_l t_l}$. Therefore A_k hits ∂P first. $\square$

Corollary 3.1 *For an n-link chain $\Gamma \prec b^F$, suppose, as in Lemma 3.4, that $A_0, \ldots, A_l$ and A_n lie on ∂P and that $\Gamma(l, n)$ is straight, for some $l < n$. Then while A_n remains* *fixed.* Γ *can be moved to RNF with* $O(mn^3)$ *simple motions.*

Proof: By induction on n. If $n = 2$, the result is trivially true. Suppose that the result holds for $n < n'$: we now show that result holds for $n = n'$.

First, we move $A_{n'-1}$ to ∂P while keeping $A_{n'}$ fixed as follows. If $A_{n'-1}$ is already on ∂P, then we are done. Otherwise, fix $A_{n'}$ and move Γ in accordance with Lemma 3.4. Then Γ can be moved to a configuration in which $A_0, \ldots, A_{l'}$ and $A_{n'-1}$ lie on ∂P and $\Gamma(l', n'-1)$ is straight, for some $l' \leq n'-1$. This requires $O(mn^2)$ simple motions. See Figure 3.12.

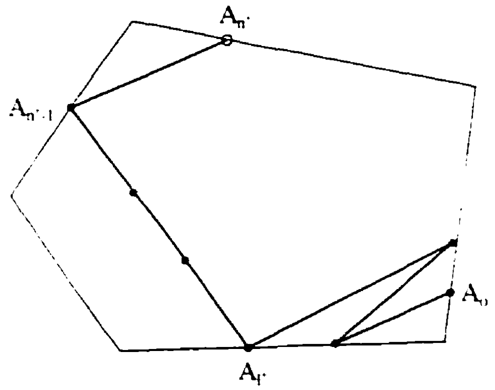


Figure 3.12: Bringing Γ to RNF.

Next, by the induction hypothesis, $\Gamma(0, n'-1)$ can be moved to RNF while $A_{n'-1}$ remains fixed. Hence Γ can be moved to RNF while A'_n remains fixed. All together, $mn^2 + m(n-1)^2 + \cdots + m \in O(mn^3)$ simple motion suffice.

This completes the induction proof. $\square$

Corollary 3.2 *Any chain $\Gamma \prec b^F$ in l-BRNF can be moved to RNF with $O(mn^4)$ simple motions.*

Proof: If $l = n$, then Γ is already in RNF. If $l < n$, suppose $A_{i_1}, \ldots, A_{i_l} (0 < i_1 < \ldots < i_l < n)$ are all intermediate joints on ∂P. If A_0 is not on ∂P, then fix $A_{i_1}, A_{i_1+1}, \ldots, A_n$ and rotate $[A_0, \ldots, A_{i_1}]$ about A_{i_1} counterclockwise until A_0 hits ∂P. Similarly A_n can be moved to ∂P. This requires $O(1)$ simple motions.

Now fix $A_{i_1}, A_{i_1+1}, \ldots, A_n$ and move $\Gamma(0, i_1)$ in accordance with Corollary 3.1. Then $\Gamma(0, i_1)$ can be moved to RNF with $O(mi_1^3)$ simple motions. Next, fix $A_{i_2}, A_{i_2+1}, \ldots, A_n$ and move $\Gamma(0, i_2)$ in accordance with Corollary 3.1. Then $\Gamma(0, i_2)$ can be moved to RNF with $O(mi_2^3)$ simple motion . Repeat this process until Γ is in RNF. All together, $mi_1^3 + mi_2^3 + \cdots + mi_l^3 \in O(mn^4)$ simple motions suffice. □

Now we are ready to present a crucial result in the following.

Theorem 3.1 *Any chain $\Gamma \prec b^F$ can be moved to RNF with $O(mn^4)$ simple motions.*

Proof: We give an algorithmic proof. The algorithm consists of an initial step, in which A_0 is brought to ∂P, followed by a main step, in which the lowest indexed joint not on ∂P is brought to ∂P. The main step is repeated until all joints lie on ∂P.

initial step: We bring A_0 to ∂P as follows. For $k = 1, 2, \ldots$, fix A_k and rotate

$[A_0, \ldots, A_k]$ about A_k. Repeat this process until either A_0 hits ∂P or the entire chain Γ becomes straight. If Γ straightens before A_0 hits ∂P, slide the straightened Γ along the line it determines towards ∂P until A_0 hits ∂P. This requires $O(n)$ simple motions.

main step: For A_k not on ∂P and $\Gamma(0, k-1)$ in RNF, bring A_k to ∂P as follows. Fix $A_0, \ldots, A_{k-1}$ and move Γ in accordance with Lemma 3.2. Then Γ can be moved to a configuration in which either A_k lies on ∂P or for some $n' > k$, A'_n lies on ∂P and $\Gamma(k, n')$ is straight, and this requires $O(n)$ simple motions.

In the latter case, fix $A'_n, \ldots, A_n$ and move $\Gamma(0, n')$ in accordance with Lemma 3.3. Then with $O(mn)$ simple motions, either A_k hits ∂P or α_k straightens to π. If α_k straightens to π first, fix $A'_n, \ldots, A_n$ and move $\Gamma(0, n')$ in accordance with Corollary 3.1. This puts A_k on ∂P with $O(mn^3)$ simple motions.

iteration steps: Once $A_0, \ldots, A_{k-1}, A_k$ lie on ∂P, repeat the main step to bring the $A_{k+1}, \ldots, A_n$ in turn to ∂P.

All together, at most $mn^3 + m(n-1)^3 + \cdots + m \in O(mn^4)$ simple motions are needed. $\square$

3.3 Bringing a Chain to TNF-i_0

This subsection shows that any chain bounded by b^C can be moved to TNF-i_0 for some i_0. First, we have

Lemma 3.5 *Let $\Gamma \prec b^C$ be a single link chain in RNF. Suppose that Γ has joints A, B which lie at x, y, respectively. If there exists a point $p \in P$ with $|xp| \geq |xy|$, then it is possible to move Γ onto xp by fixing x and rotating Γ about x.*

Proof: It suffices to show that B will not hit ∂P in the desired rotation.

Assume otherwise. Suppose that $l(x, p) \cap \partial P = \{x, q\}$ and that q, x, y is a right turn, as Figure 3.13 shows.

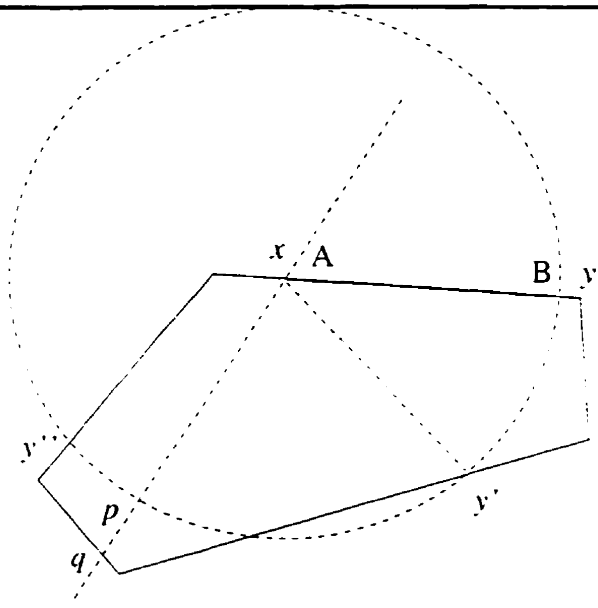


Figure 3.13: Γ can be moved onto xp by fixing x and rotating Γ about x.

Then there exists some $y' \in Ch(q, y)$ with $|xy'| = |xy|$. Note that since $|xq| \geq |xp| \geq |xy|$, there exists some $y'' \in Ch(x, q)$ with $|xy''| = |xy|$. Since $y, y'y''$ are distinct, $C(x, |xy|) \cap \partial P$ has at least three points. This contradicts $\Gamma \prec b^C$. $\square$

Theorem 3.2 *Let $\Gamma \prec b^C$ be an n-link chain. Suppose there exists a point $p \in P$, a vertex v of P and an index $i_0 < n$ such that*

$$d(p, v) \geq l_{i_0+1} + l_{i_0+2} + \cdots + l_n.$$

Then with $O(mn^4)$ simple motions, Γ can be moved to TNF-i_0 in which A_{i_0} lies at v.

Proof: Bring Γ to RNF in accordance with Theorem 3.1. Then, keeping Γ in RNF, move A_{i_0} around ∂P to v in accordance with Lemma 3.1. This requires $O(mn^4)$ simple motions.

Let $p_{i_0}, p_{i_0+1}, \ldots, p_n$ be the points that $A_{i_0}, A_{i_0+1}, \ldots, A_n$ now occupy, respectively, as Figure 3.14 shows. Then $p_{i_0} = v$. We claim that, for any k with $i_0 \leq k < n$,

$$d(p, p_k) > l_{k+1} + \cdots + l_n.$$

To see this, note that in $\triangle pvp_k$, we have

$$d(p, p_k) > d(p, v) - d(v, p_k).$$

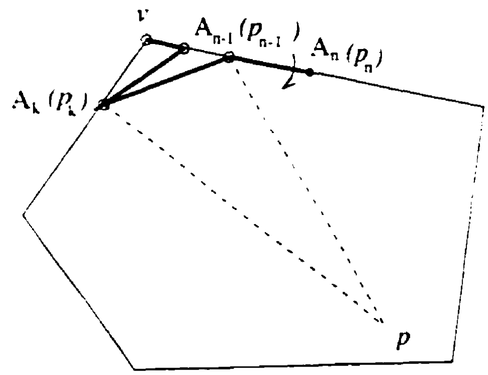


Figure 3.14: Bringing L_n onto $p_{n-1}p$. Point p_{i_0} and joint A_{i_0} are at v.

Note that A_{i_0}, A_k lie at v, p_k, respectively. Hence $d(v, p_k) \le l_{i_0-1} + l_{i_0-2} + \cdots + l_k$. Also $d(p, v) \ge l_{i_0+1} + l_{i_0+2} + \cdots + l_n$. Therefore,

$$d(p, p_k) > (l_{i_0+1} + l_{i_0+2} + \cdots + l_n) - (l_{i_0+1} + l_{i_0+2} + \cdots + l_k) = l_{k+1} + \cdots + l_n.$$

Hence the claim.

We now reconfigure Γ as follows. To begin, bring $[A_{n-1}, A_n]$ onto the line segment $p_{n-1}p$. To do this, fix $A_0, \ldots, A_{n-1}$ and rotate L_n about A_{n-1} until $[A_{n-1}, A_n]$ and p become collinear. Only $O(1)$ simple motions are needed. See Figure 3.14.

From the claim, $d(p, p_{n-1}) > l_n$. Also $\Gamma \prec b^C$. By Lemma 3.5, A_n will not hit ∂P during this rotation. Also from $d(p, p_{n-1}) > l_n$, A_n lies on pp_{n-1} when $[A_{n-1}, A_n]$ and p become collinear.

Next, straighten $\Gamma(n-2, n)$ and bring it onto $p_{n-2}p$ as follows. Fixing $A_0, \ldots, A_{n-2}$ and keeping $[A_{n-1}, A_n], p$ collinear, rotate L_{n-1} about A_{n-2} until $\Gamma(n-2, n)$ becomes straight. Again, only $O(1)$ simple motions are needed. See Figure 3.15.

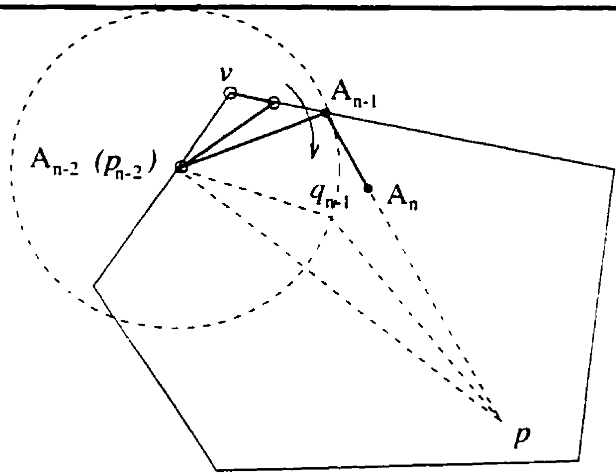


Figure 3.15: Straightening $\Gamma(n-2, n)$ onto $p_{n-2}p$.

We show that neither A_{n-1} nor A_n will hit ∂P during this reconfiguration. For A_{n-1}, this follows from $d(p_{n-2}, p) > l_{n-1} + l_n > l_{n-1}$ and Lemma 3.5. We now consider A_n.

Let q_{n-1} be the point where A_{n-1} lies at a stage of the reconfiguration, as shown in Figure 3.15. It suffices to show that A_n remains on $q_{n-1}p$, i.e., $d(q_{n-1}, p) > l_n$.

To see this, note that in $\triangle q_{n-1}p_{n-2}p$, $d(q_{n-1}, p) > d(p_{n-2}, p) - d(q_{n-1}, p_{n-2})$. By the claim, $d(p_{n-2}, p) > l_{n-1} + l_n$. Also $d(q_{n-1}, p_{n-2}) = l_{n-1}$. Therefore, $d(q_{n-1}, p) > (l_{n-1} + l_n) - l_{n-1} = l_n$.

At the end of the reconfiguration, $\Gamma(n-2, n)$ becomes straight and collinear with p. Also by $d(p_{n-2}, p) > l_{n-1} + l_n$, A_n lies on pp_{n-2}.

Repeat this process until $\Gamma(i_0, n)$ is straightened onto vp. This puts Γ in TNF-i_0 with A_{i_0} at v. All together, $O(mn^4)$ simple motions are needed. □

3.4 Reachable Points

We now give our main result in this chapter.

Theorem 3.3 *$b^C \leq b^S$, i.e., if $\Gamma \prec b^C$, then $(*)$ is a sufficient and necessary condition for A_n to reach p.*

Furthermore, if p is reachable by A_n, then A_n can be moved to p with $O(mn^4)$ simple motions.

Proof: The necessity of $(*)$ follows trivially from Property 3. Now we show the sufficiency by giving an algorithm to bring A_n to p with $O(mn^4)$ simple motions.

Let i_0 be the least index such that

$$\sum_{j=i_0+1}^{n} l_j \leq d_{max}(p).$$

Note that taking $i = n$ in $(*)$ gives $l_n \leq d_{max}(p)$, so $i_0 < n$. In accordance with Theorem 3.2, move Γ to TNF-i_0 with A_{i_0} at $v_{max}(p) = v$. This requires $O(mn^4)$ simple motions.

If $i_0 = 0$, then Γ is straight. Slide Γ along $l(v, p)$ towards p until A_n reaches p.

If $i_0 > 0$, move A_{i_0-1} along ∂P towards v while keeping $[A_{i_0}, \ldots, A_n]$ straight and collinear with p. By Lemma 3.1, this can be done while $\Gamma(0, i_0)$ remains in RNF. Continue this process until A_n reaches p or A_{i_0-1} reaches v. This requires $O(mn)$ simple motions. See (a) of Figure 3.16.

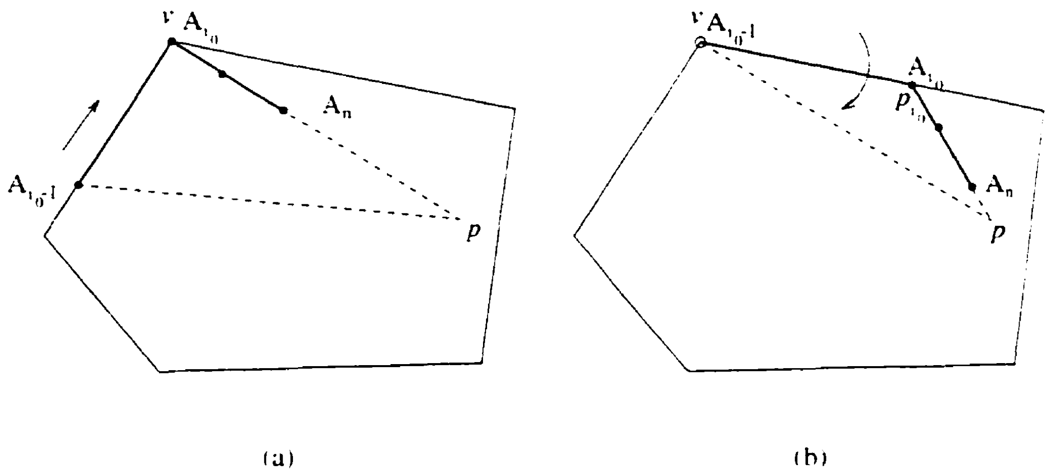


Figure 3.16: Moving A_{i_0} to v and then bringing L_{i_0} collinear with vp.

If A_n reaches p first, then we are done. If A_{i_0} reaches v first, let p_{i_0} be the point where A_{i_0} lies. Then,

$$l_{i_0-1} + \cdots + l_n < d(p_{i_0}, p). \tag{3.1}$$

Fixing $A_0, \ldots, A_{i_0-1}$ and keeping $[A_{i_0}, \ldots, A_n]$ straight and collinear with p, rotate L_{i_0} about A_{i_0-1}. This requires $O(1)$ simple motions. See (b) of Figure 3.16.

We claim that, before L_{i_0} is collinear with vp, A_{i_0} will not hit ∂P and A_n will reach p.

To see this, we consider two cases.

Case 1: $l_{i_0} \leq d_{max}(p)$. Since $\Gamma \prec b^C$, by Lemma 3.5, A_{i_0} will not hit ∂P before L_{i_0} is collinear with vp. We now show that A_n will reach p first.

Assume otherwise, i.e., that L_{i_0} becomes collinear with vp first. Then $\Gamma(i_0 - 1, n)$ is straight and collinear with vp, as shown in (a) of Figure 3.17. By definition of i_0,

$$l_{i_0} + l_{i_0+1} + \cdots + l_n > d_{max}(p) = d(v, p).$$

Or,

$$l_{i_0-1} + \cdots + l_n > d(v, p) - l_{i_0}.$$

Let q_{i_0} be the point where A_{i_0} lies. Then $d(v, p) - l_{i_0} = d(q_{i_0}, p)$. Therefore,

$$l_{i_0-1} + \cdots + l_n > d(q_{i_0}, p). \tag{3.2}$$

From (3.1) and (3.2), there exists some intermediate configuration in which A_{i_0} lies at t_{i_0} and $[A_{i_0}, \ldots, A_n]$ is straight and collinear with p, such that

$$l_{i_0+1} + \cdots + l_n = d(t_{i_0}, p).$$

This implies that A_n lies at p in this intermediate configuration. Contradiction.

Case 2: $l_{i_0} > d_{max}(p)$. Let $l(v, p) \cap \partial P = \{v, v'\}$, as shown in (b) of Figure 3.17. Clearly v is the vertex furthest from v'. Hence $d_{max}(v') = |vv'|$. By Lemma 4.1, $w \leq d_{max}(v')$. By Corollary 2.2, $b^C \leq w$. Hence $\Gamma \prec |vv'|$ and $l_{i_0} < |vv'|$. By Lemma 3.5, A_{i_0} will not hit ∂P before L_{i_0} is collinear with vp. We now show that A_n will reach p first.

Assume otherwise, i.e., that L_{i_0} becomes collinear with vp first. Then $[A_{i_0-1}, A_{i_0}]$, $[A_{i_0}, \ldots, A_n]$ are collinear with vp and A_{i_0} is closed, as shown in (b) of Figure 3.17.

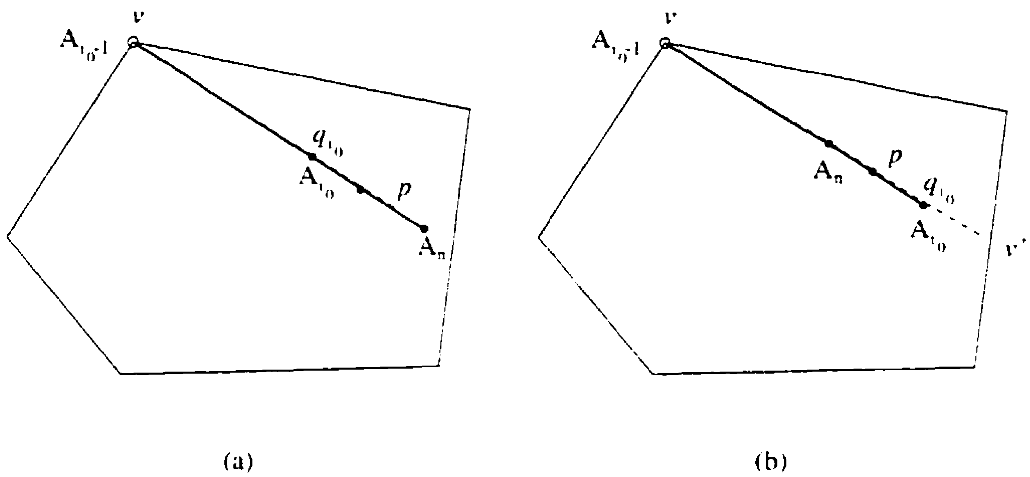


Figure 3.17: $l_{i_0} \leq d_{max}(p)$ in (a) and $l_{i_0} > d_{max}(p)$ in (b).

By (*),

$$l_{i_0} - (l_{i_0+1} + \cdots + l_n) \leq d_{max}(p) = d(v, p).$$

Or,

$$l_{i_0+1} + \cdots + l_n \geq l_{i_0} - d(v, p).$$

Let q_{i_0} be the point where A_{i_0} lies. Then $l_{i_0} - d(v, p) = d(q_{i_0}, p)$. Therefore,

$$l_{i_0+1} + \cdots + l_n \geq d(q_{i_0}, p). \tag{3.3}$$

From (3.1) and (3.3), there exists some intermediate configuration in which A_{i_0} lies at some t_{i_0} and $[A_{i_0}, \ldots, A_n]$ is collinear with p, such that

$$l_{i_0+1} + \cdots + l_n = d(t_{i_0}, p).$$

This implies that A_n lies at p in this intermediate configuration. Contradiction.

All together. $O(mn^4)$ simple motion suffice. This completes the proof. $\square$

Chapter 4

Reachable Regions

This chapter investigates the properties of reachable regions of chains. We consider general convex polygons as the confining regions throughout this chapter.

Section 4.1 proves that $b^S \leq w$ and that if P is l-reachable, then $l \leq w$. Section 4.2 describes the shapes of reachable regions and shows that they are linearly ordered by set inclusion. Section 4.3 recalls some basic facts about the minimal spanning circle of a convex polygon. This provides background for Section 4.4, which characterizes l-reachable convex polygons for $l \leq b^S$ and illustrates the applications of this result. Section 4.4 also characterizes the center of the minimal spanning circle of a convex polygon from a reachability point of view.

Assumptions: Throughout this Chapter, we assume, unless otherwise stated, that Γ denotes an n-link chain and P denotes an m-sided convex polygon. Γ is confined by P and joints of Γ may lie on ∂P.

4.1 $b^S \leq w$

To show $b^S \leq w$, we need the following.

Lemma 4.1 *If $p \in \partial P$, then $w \leq d_{max}(p)$. Furthermore, if $v \in V$, then $w < d_{max}(v)$.*

Proof: For any $p \in \partial P$, P lies completely inside $C(p, d_{max}(p))$. Hence P falls into a half circle determined by $C(p, d_{max}(p))$ and the line l through $s(p)$. So P lies between l and a tangent line of C parallel to l, whose distance is $d_{max}(p)$. Thus $w \leq d_{max}(p)$.

Let $v \in V$ and let u, w be the vertices adjacent to v. Suppose that l is the line through vw and that l' is the line through v perpendicular to l. Let v' be a vertex furthest from l and let u', w' be the vertices adjacent to v'. Without loss of generality, assume that u, v, w and w', u', v' are both left turns. See Figure 4.1.

We consider two cases.

Case 1: $v' \notin l'$. Let l'' be the line through v' parallel to l, as shown in Figure 4.1. By definition of v', points u' and w' are each on or below l''. Hence l'' and l are support lines of P. Since $v' \notin l'$, $d(l'', l)$ is strictly less than $d_{max}(v)$, the radius of

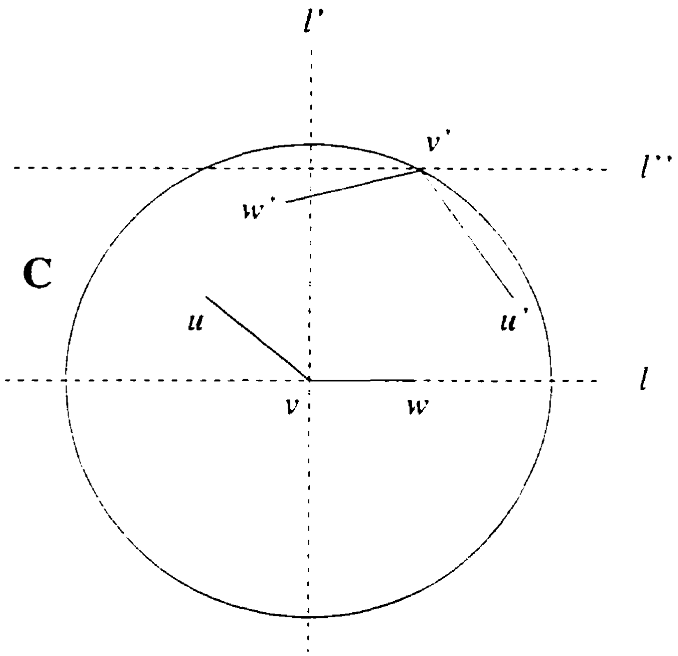


Figure 4.1: Case 1: $v' \notin l'$

C. Therefore,

$$w \leq d(l'', l) < d_{max}(v).$$

Case 2: $v' \in l'$. Let $s(uv), s(u'v')$ be the angles of $uv, u'v'$ with respect to l, respectively. We further consider two subcases.

Case 2a: $s(uv) \geq s(u'v')$, as shown in (a) of Figure 4.2.

Let l_1 be the line through uv and let l_2 be the line through v' parallel to l_1. Since $s(uv) \geq s(u'v')$, l_1 and l_2 are parallel lines of support of P. Note that l_2 is not parallel to l. Hence l_2 is not tangent to C. So $d(l_1, l_2)$ is strictly less than $d_{max}(v)$.

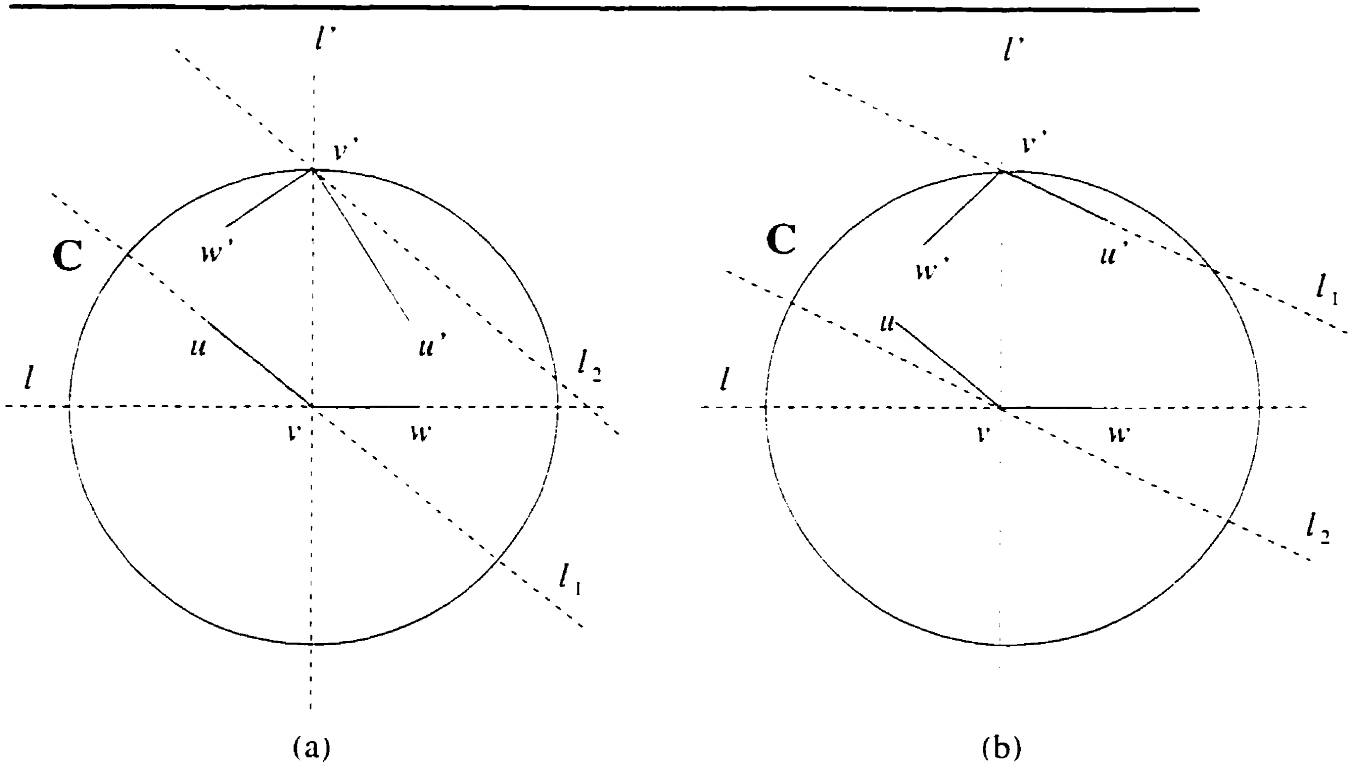


Figure 4.2: Case 2: $v' \in l'$.

the radius of C. Therefore,

$$w \leq d(l_1, l_2) < d_{max}(v).$$

Case 2b: $s(uv) < s(u'v')$, as shown in (b) of Figure 4.2.

Let l_1 be the line through $v'u'$ and let l_2 be the line through v parallel to l_1. Since $s(uv) < s(u'v')$, l_1 and l_2 are parallel lines of support of P. Note that l_1 is not parallel to l. Hence l_1 is not tangent to C. So $d(l_1, l_2)$ is strictly less than $d_{max}(v)$, the radius of C. Therefore,

$$w \leq d(l_1, l_2) < d_{max}(v).$$

This completes the proof. $\square$

Note that $w = d_{max}(p)$ may hold if $p \notin V$. To see this, let p be the midpoint of a side in an equilateral triangle.

Theorem 4.1 $b^S \leq w$.

Proof: Assume otherwise. Then $w < b^S$ for some P and we now consider such a P. Then for any l with

$$w < l < b^S$$

and any single link chain $\Gamma = [A, B]$ with length l, Γ satisfies **(S)**. We show that this is impossible.

By Fact 2.1, there exists two parallel lines l_1, l_2 of support of P, one passing through a vertex and the other passing through a side, achieving the width w. Suppose that l_1 passes through vertex u and l_2 passes through side v_1v_2. Let l be the line through v_2 perpendicular to v_1v_2. Without loss of generality, assume that l_1 is above l_2. See Figure 4.3.

Let v_2' be the vertex furthest from v_2. By Lemma 4.1, $w < d_{max}(v_2)$. Hence $v_2' \notin l$. First, assume that v_2' is on the left of l, as shown in Figure 4.3.

Let v be the rightmost vertex of P, with l_2 regarded as horizontal. Also by Lemma 4.1, $w < d_{max}(v)$. Take $\epsilon > 0$ sufficiently small such that

$$w + \epsilon < \min\{d_{max}(v_2), d_{max}(v)\}.$$

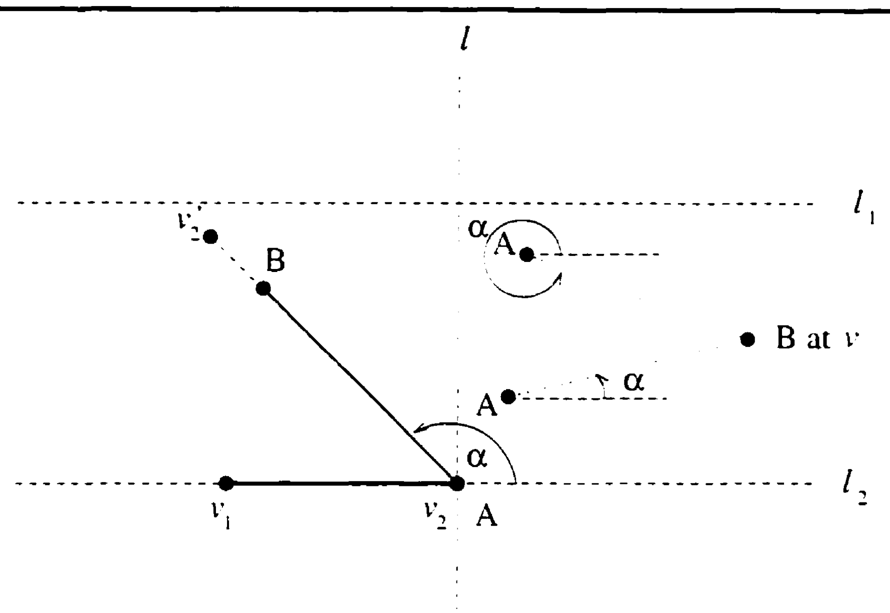


Figure 4.3: B cannot reach v.

Let $\Gamma = [A, B]$ have length of $w + \epsilon$. By the convexity of P, line segment v_2v_2' lies inside P. Since $w + \epsilon < d_{max}(v_2) = d(v_2, v_2')$, it is possible to place Γ on v_2v_2' with A at v_2, as Figure 4.3 shows. Consider this as the initial configuration of Γ. We claim that $v \notin R_\Gamma(B)$, i.e., that B cannot reach v.

To see this, view Γ as a vector $\vec{AB}$ and define α as the angle formed by rotating l to $\vec{AB}$ counterclockwise, as shown in Figure 4.3. Since v_2' is on the left of l, initially $\pi/2 < \alpha < \pi$.

If $v \in R_\Gamma(B)$, since v is the rightmost vertex, when B reaches v, either $2\pi \geq \alpha > 3\pi/2$ or $0 < \alpha < \pi/2$, as shown in the upper, lower dotted lines in Figure 4.3, respectively.

Since α changes continuously, there exists some intermediate configuration of Γ in which either $\alpha = 3\pi/2$ or $\alpha = \pi/2$, i.e., Γ is perpendicular to l_2. This is not possible. Hence the claim.

Note that $d_{max}(v) > w + \epsilon$. Hence Γ does not satisfy **(S)**. Contradiction.

For the case that v_2' is on the right of l, the proof is similar. □

We remark that b^S may be considerably less than w, as illustrated in Figure 4.4. In this figure, the polygon P is constructed by cutting off three congruent tiny right triangles of an equilateral triangle $\triangle$ with unit side. Thus $w = \sqrt{3}/2 - \epsilon$ for some small ϵ. We claim that $b^S < w$ in P.

To see this, consider a 3-link chain Γ having joints A, B, C, D whose initial configuration is as follows: B, C are at the midpoints of two sides of $\triangle$, respectively; A, D are at two right corners of $\triangle$, respectively. Then $\Gamma \prec 1/2$. If $b^S = w$, then the top vertex of $\triangle$ is reachable by D. But this is impossible as Γ is completely stuck.

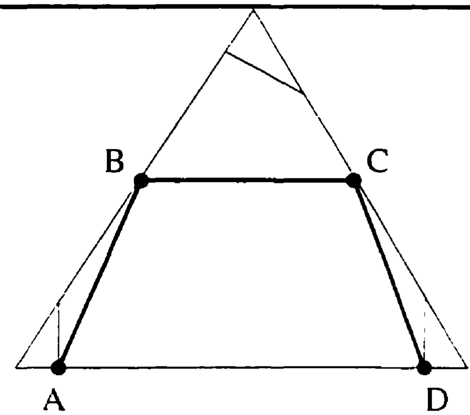


Figure 4.4: $b^S < w$.

One consequence of the above theorem is the following, which shows that w-reachability is the most one could hope for.

Corollary 4.1 *If P is l-reachable, then $l \leq w$.*

Proof: If not, then $l > w$. For any l' with $w < l' < l$, since P is l-reachable, P is l'-reachable. Then for any $\Gamma \prec l'$, Γ can reach each point of P. Therefore $l' \leq b^S$. By Theorem 4.1, $b^S \leq w$. So $l' \leq w$. Contradiction. $\square$

4.2 Properties of Reachable Regions

Property 4 from Chapter 1 shows the equivalence of the reachability between an n-link chain and a single link chain. This suggests the following.

Definition 4.1 *Let $\Gamma \prec b^S$ be an n-link chain and let $l_1, \ldots, l_n$ be the link lengths of Γ. We call the single link chain $\Gamma^e = [A, B]$ of length $l^e = \max_{1 \leq i \leq n}\{l_i - \sum_{j=i+1}^{n} l_j\}$ the* equivalent *chain of* Γ.

In terms of the above definition, Property 4 can be restated as follows: $R_\Gamma(A_n) = R_{\Gamma^e}(B)$.

Since $l^e \geq l_n > 0$ and $l^e \leq l_i$ for each i, we have

Observation 4.1 $0 < l^e \leq l_{max}$.

From the above observation, we have

Observation 4.2 *If* $\Gamma \prec b$, *then* $\Gamma^e \prec b$.

The following lemma shows that each point on the boundary of P is b^S-reachable.

Lemma 4.2 $\partial P \subseteq P_{b^S}$.

Proof: Let $\Gamma \prec b^S$ be an n-link chain and let $\Gamma^e = [A, B]$ be its equivalent chain. Then $R_\Gamma(A_n) = R_{\Gamma^e}(B)$.

By Observation 4.2, $\Gamma^e \prec b^S$. By Theorem 4.1, $b^S \leq w$. Hence $\Gamma^e \prec w$, i.e., $l^e < w$. For any $p \in \partial P$, by Lemma 4.1, $w \leq d_{max}(p)$. Thus $l^e \leq d_{max}(p)$. Since $\Gamma \prec b^S$, $(*)$ is satisfied. Hence $p \in R_{\Gamma^e}(B) = R_\Gamma(A_n)$. So p is b^S-reachable. Thus $\partial P \subseteq P_{b^S}$. □

Theorem 4.2 *Let* $\Gamma \prec b^S$. *Then* $\overline{R_\Gamma(A_n)}$ *is either empty or has boundary composed of at most* m *circular arcs centered at certain vertices of* P *and all having radius* l^e.

Proof: Let $\Gamma^e = [A, B]$ be the equivalent chain of Γ. Since $R_\Gamma(A_n) = R_{\Gamma^e}(B)$, $\overline{R_\Gamma(A_n)} = \overline{R_{\Gamma^e}(B)}$. By Observation 4.2, $\Gamma^e \prec b^S$. Therefore,

$$\overline{R_{\Gamma^e}(B)} = \{p | p \in P, d_{max}(p) < l^e\} = \bigcap_{i=1}^{m} \{p | p \in P, d(p, v_i) < l^e\}.$$

Note that for each i, $\{p | p \in P, d(p, v_i) < l^e\}$ is the subset of P inside the circle centered at v_i with radius l^e. By Lemma 4.2, each point of ∂P is b^S-reachable and hence is l^e-reachable. Thus for each i, $\{p | p \in P, d(p, v_i) < l^e\} \bigcap \partial P = \emptyset$. The result hence follows. □

We remark that the number of circular arcs bounding the unreachable region for different chains sharing the same P may change, as shown in Figure 4.5. This figure shows a convex 5-gon that is nearly regular. The solid and dashed arcs show the construction curves for the boundary of the unreachable region of a longer and a shorter single link chain, respectively. Note that the boundary of the unreachable region of the longer link chain is composed of five circular arcs, whereas the boundary of the unreachable region of the shorter link chain is composed of three circular arcs.

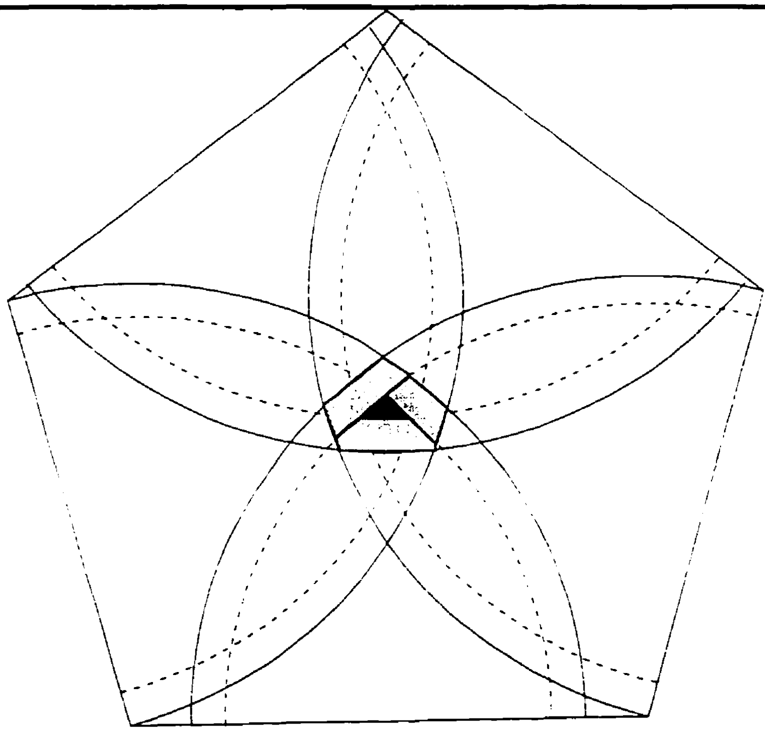


Figure 4.5: The circular arc number of unreachable regions may change

Theorem 4.3 *All $R_\Gamma(A_n)$ with $\Gamma \prec b^S$ are linearly ordered by set inclusion.*

Proof: Let $\Gamma, \tilde{\Gamma}$ be $n, \tilde{n}$ chains, respectively and both bounded by b^S. It suffices to show that one of $R_\Gamma(A_n), R_{\tilde{\Gamma}}(A_{\tilde{n}})$ contains the other.

Let $\Gamma^e = [A, B], \tilde{\Gamma}^e = [\tilde{A}, \tilde{B}]$ be the equivalent chains of $\Gamma, \tilde{\Gamma}$, respectively. Suppose Γ^e and $\tilde{\Gamma}^e$ have lengths of l^e and $\tilde{l}^e$, respectively. Then

$$R_\Gamma(A_n) = R_{\Gamma^e}(B), R_{\tilde{\Gamma}}(A_{\tilde{n}}) = R_{\tilde{\Gamma}^e}(\tilde{B}).$$

Suppose $\tilde{l}^e \leq l^e$ and consider $p \in R_{\Gamma^e}(B)$. By Observation 4.2, $\Gamma^e \prec b^S$. Hence $l^e \leq d_{max}(p)$. So $\tilde{l}^e \leq d_{max}(p)$. Again by Observation 4.2, $\tilde{\Gamma}^e \prec b^S$. So $p \in R_{\tilde{\Gamma}^e}(\tilde{B})$. Therefore $R_{\Gamma^e}(B) \subseteq R_{\tilde{\Gamma}^e}(\tilde{B})$.

Similarly $R_{\tilde{\Gamma}^e}(\tilde{B}) \subseteq R_{\Gamma^e}(B)$ if $l^e \leq \tilde{l}^e$. This completes the proof. $\square$

Definition 4.2 *Let $\Gamma \prec b^S$ be an n-link chain confined within P. We say that Γ is covering for P or covers P, denoted by $\Gamma \vdash P$, if $R_\Gamma(A_n) = P$.*

By the previous lemma, the unreachable regions of the noncovering chains are also linearly ordered by set inclusion. The supremum of these regions, $\bigcup_{\Gamma \prec b^S, \Gamma \not\vdash P} \overline{R_\Gamma(A_n)}$, is clearly the complement of P_{b^S}. In Section 4.4, we will show that the infimum of these regions, $\bigcap_{\Gamma \prec b^S, \Gamma \not\vdash P} \overline{R_\Gamma(A_n)}$, is a unique point that is "hardest reachable" (to be defined), which coincides with the center of the minimal spanning circle of P.

4.3 Minimal Spanning Circles

In this section, we review some well-known properties of minimal spanning circles of convex polygons. This provides needed background for the next section. Refer to [RT57] for details.

Let P be a convex polygon. A *spanning circle* of P is a circle C such that each vertex of P lies either inside or on C. A *minimal spanning circle* of P is a spanning circle of P having minimum radius.

Fact 4.1 *Every convex polygon P has a minimal spanning circle, which is unique.*

The notion of the minimal spanning circle can be generalized to a set of n points, and the corresponding results still hold. See [RT57] for references. [PS85] and [Meg83] show that the minimal spanning circle of an n-point set can be constructed in $\Theta(n)$ time.

Fact 4.2 *A spanning circle C of a convex polygon P is the minimal spanning circle of P if and only if C passes through two diametrically opposite vertices (i.e. the line segment between the two vertices defines a diameter of C) or through three vertices that define an acute triangle.*

Corollary 4.2 *Let P be a convex polygon and let $C(o, r)$ be its minimal spanning circle. Then $o \in P$ and $r = d_{max}(o)$.*

Corollary 4.3 *Let P be a convex polygon. Then the center of its minimal spanning circle is the unique point $o \in P$ at which $d_{max}(o)$ achieves its minimum.*

Corollary 4.3 shows that the problem of determining the minimal spanning circle of a convex polygon P is equivalent to that of seeking a point whose maximal distance to the vertices of P is minimal. Suppose vertices denote the locations of users, then the point to be sought is a kind of optimal position to place a public facility. This is the *minimax* problem in Operational Research, a classical problem with wide applications. See [NC71] and [TSRB71] for references.

4.4 l-Reachability

We now give our main result in this chapter.

Theorem 4.4 *Let P be a convex polygon and let $C(o, r)$ be its minimal spanning circle. Then for any $l \leq b^S$, the following are equivalent.*

(1) P is l-reachable;

(2) o is l-reachable;

(3) $l \leq r$.

Proof: (1) $\Longrightarrow$ (2) By Corollary 4.2, $o \in P$. Since P is l-reachable, o is l-reachable.

(2) $\Longrightarrow$ (3) By contradiction. If $l > r$, then there exists l' with $l > l' > r$.

Consider a single link chain $\Gamma = [A, B]$ having length l'. By Corollary 4.2, $r = d_{max}(o)$. Hence $l' > d_{max}(o)$. By Property 3 from Chapter 1, $o \notin R_\Gamma(B)$. Thus o is not l'-reachable. So o is not l-reachable.

(3) $\Longrightarrow$ (1) By Corollary 4.2, $r = d_{max}(o)$. Thus $l \leq d_{max}(o)$. For any $p \in P$, by Corollary 4.3, $d_{max}(o) \leq d_{max}(p)$. Hence $l \leq d_{max}(p)$.

Let $\Gamma \prec l$ be an n-link chain and let $\Gamma^e = [A, B]$ be the equivalent chain of Γ. By Observation 4.2, $\Gamma^e \prec l$. So $l^e < l \leq d_{max}(p)$. Since $\Gamma \prec b^S$, $(*)$ is satisfied. Therefore,

$$p \in R_{\Gamma^e}(B) = R_\Gamma(A_n).$$

So p is l-reachable. Hence P is l-reachable. □

Corollary 4.4 *Let P be a convex polygon and let r be the radius of its minimal spanning circle. If $w \leq r$, then P is b^S-reachable.*

Furthermore, if P is convex obtuse with $b^C = w$, then P is w-reachable if and only if $w \leq r$.

Proof: By Theorem 4.1, $b^S \leq w$. Hence $b^S \leq r$. By Theorem 4.4, P is b^S-reachable.

If P is convex obtuse and $b^C = w$, then $b^S = w$. By Theorem 4.4, P is w-reachable if and only if $w \leq r$. □

We now illustrate applications of the above in the next three theorems. Our results suggest that the shape of P determines its l-reachability.

Theorem 4.5 *Let $\triangle$ be a triangle with an interior angle $\geq \pi/2$. Then $\triangle$ is b^S-reachable.*

Proof: Note that w of $\triangle$ is the minimum height and that the radius r of the minimal spanning circle of $\triangle$ is half of the longest side. Hence $w \leq r$. By Corollary 4.4, $\triangle$ is b^S-reachable. $\square$

Observation 4.3 *Convex obtuse quadrilaterals are exactly rectangles.*

Theorem 4.6 *Let P be a rectangle having sides a, b with $a \geq b$. Then P is w-reachable if and only if $a/b \geq \sqrt{3}$.*

Proof: Let $C(o,r)$ be the minimal spanning circle of P. $C(o,r)$ is clearly the circumscribed circle of P. Hence $r = \sqrt{a^2+b^2}/2$. Note that in P, $b^C = w = b$. By Corollary 4.4,

$$P \text{ is } l - \text{reachable} \iff w \leq r \iff b \leq \sqrt{a^2+b^2}/2 \iff a/b \geq \sqrt{3}. \qquad \square$$

The above theorem shows that a rectangle is w-reachable if and only if it is "slim" enough.

Theorem 4.7 *Let P be a convex obtuse m-gon. If $m > 5$, then P is s_{min}-reachable.*

Proof: Suppose $v_1, v_2, \ldots, v_m$ are all the vertices of P and $C(o,r)$ is the minimal spanning circle of P. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the angles formed between ov_1 and ov_2, ov_2 and $ov_3, \ldots, ov_m$ and ov_1, respectively. See Figure 4.6.

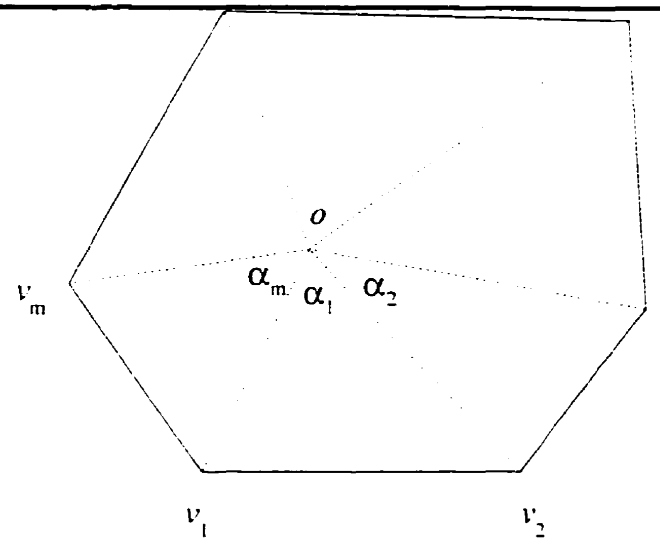


Figure 4.6: P is s_{min}-reachable if it has more than 5 sides.

Since $\sum_{i=1}^{m} \alpha_i = 2\pi$, there exists i_o with $\alpha_{i_0} \leq 2\pi/m$. Since $m \geq 6$,

$$\alpha_{i_0} \leq 2\pi/m \leq 2\pi/6 = \pi/3.$$

Without loss of generality, assume that $i_0 = 1$. Then in $\triangle ov_1v_2$,

$$\angle v_1ov_2 = \alpha_1 \leq \pi/3.$$

Note that $\angle v_1ov_2 + \angle ov_1v_2 + \angle ov_2v_1 = \pi$, hence $\angle ov_1v_2 + \angle ov_2v_1 \geq \pi - \pi/3 = 2\pi/3$. Therefore, one of $\angle ov_1v_2$, $\angle ov_2v_1$ is at least $\pi/3$. Without loss of generality, assume that $\angle ov_1v_2 \geq \pi/3$. We then have

$$\angle v_1ov_2 \leq \angle ov_1v_2.$$

Therefore

$$|v_1v_2| \leq |ov_2|.$$

By Corollary 4.2, $d_{max}(o) = r$. Hence,

$$s_{min} \leq |v_1 v_2| \leq |ov_2| = d(o, v_2) \leq d_{max}(o) = r.$$

Since P is convex obtuse, $s_{min} \leq b^S$. By Theorem4.4, P is s_{min}-reachable. $\square$

Next we characterize the center of the minimal spanning circle as the hardest reachable point of a convex polygon, defined in the following.

Definition 4.3 *Suppose P is not b^S-reachable. A point $o \in P$ is the hardest reachable point if for any n-link chain Γ, $o \in R_\Gamma(A_n)$ implies that $\Gamma \vdash P$.*

Theorem 4.8 *Let P be a convex polygon that is not b^S-reachable and let C be its minimal spanning circle with radius r. Then the following are equivalent.*

(1) o is the center of C;

(2) o is the hardest reachable point;

(3) o is the infimum of non-empty unreachable regions, i.e., $o = \bigcap_{\Gamma \prec b^S, \Gamma \not\vdash P} \overline{R_\Gamma(A_n)}$.

Proof: First, we show the equivalence of (1) and (2).

(1) $\Longrightarrow$ (2) For any $p \in P$, by Corollary 4.3, $d_{max}(o) \leq d_{max}(p)$.

For an n-link chain $\Gamma \prec b^S$, if $o \in R_\Gamma(A_n)$, then for any i with $1 \leq i \leq n$,

$$l_i - \sum_{j=i+1}^{n} l_j \leq d_{max}(o) \leq d_{max}(p).$$

Thus $p \in R_\Gamma(A_n)$. So o is the hardest reachable point.

(2) $\Longrightarrow$ (1) Let o^* be the center of C. If $o \neq o^*$, by Corollary 4.3, $d_{max}(o^*) < d_{max}(o)$.

Since P is not b^S-reachable, by Theorem 4.4, $r < b^S$. By Corollary 4.2, $r = d_{max}(o^*)$. Thus $d_{max}(o^*) < b^S$.

Take l with

$$d_{max}(o^*) < l < \min\{d_{max}(o), b^S\}.$$

Let $\Gamma = [A, B]$ be a single link chain of length l. Then $o^* \notin R_\Gamma(B)$ but $o \in R_\Gamma(B)$. So o is not the hardest reachable point. Contradiction.

Next, we show the equivalence of (3) and (1). To this end, we first claim that, if p is not the center of C, then

$$p \notin \bigcap_{\Gamma \prec b^S, \Gamma \not\vdash P} \overline{R_\Gamma(A_n)}.$$

To see this, note that from the proof of (2) $\Longrightarrow$ (1), we get the following. Let o be the center of C and let $p \neq o$. Then for some $\Gamma = [A, B]$, $o \notin R_\Gamma(B)$, $p \in R_\Gamma(B)$. Hence $\Gamma \not\vdash P$, $p \notin \overline{R_\Gamma(B)}$. Thus

$$p \notin \bigcap_{\Gamma \prec b^S, \Gamma \not\vdash P} \overline{R_\Gamma(A_n)}.$$

Hence the claim.

(3) $\Longrightarrow$ (1) Immediate from the claim.

(1) $\Longrightarrow$ (3) By the claim, it suffices to show that

$$o \in \bigcap_{\Gamma \prec b^S, \Gamma \not\vdash P} \overline{R_\Gamma(A_n)}.$$

For any chain Γ with $\Gamma \prec b^S$ and $\Gamma \not\vdash P$, there exists $p \notin R_\Gamma(A_n)$. By the equivalence of (1) and (2), $o \notin R_\Gamma(A_n)$, i.e., $o \in \overline{R_\Gamma(A_n)}$. Therefore,

$$o \in \bigcap_{\Gamma \prec b^S, \Gamma \not\vdash P} \overline{R_\Gamma(A_n)}.$$

This completes the proof. □

Corollary 4.5 *Suppose that P is not b^S-reachable. Then P has a unique hardest reachable point.*

Proof: By Theorem 4.8 and Fact 4.1. □

Chapter 5

Reconfiguring Chains inside Circles

This chapter handles reconfiguration of chains within circles. We treat circles as the extreme case of nice confining environments and believe that our results provide insights on how to design short link chains within a given confining region in order to ensure fast reconfiguration.

Section 5.1 shows that *any* n-link chain Γ confined within $C(0, r)$, whose links maybe as long as the diameter of C, can be brought to Rim Normal Form (RNF) with $O(n)$ simple motions. Except for the running time and the bound on Γ, the result is similar to that of Section 3.2. This enables us to present similar *reachability* results for chains inside circles. Section 5.2 shows that $\Gamma \prec r$ already in RNF can be brought to "right oriented Rim Normal Form (r-RNF)" (to be defined) with $O(n^2)$ simple motions. This yields the main result of this chapter in Section 5.3: $\Gamma \prec r$ implies Γ can be moved between any of its configurations inside C with $O(n^2)$ simple motions. Section 5.3 also illustrates two applications of this result that demonstrate that the bound on Γ to ensure only one equivalence class of configurations of Γ is best in some sense.

Assumptions: Throughout this chapter, Γ denotes an n-link chain confined inside circle $C(o, r)$ unless otherwise stated. Joints of Γ may lie on C.

5.1 Bringing a Chain to RNF

We define Rim Normal Form for a chain inside a circle in a similar way to what we did in Section 3.1 for a chain inside a convex obtuse polygon.

Definition 5.1 *A chain inside a circle is in* Rim Normal Form, *denoted* RNF, *if all its joints lie on C.*

We can give an algorithm for bringing Γ to RNF inside a circle similar to our reachability algorithm for Γ inside a convex obtuse polygon. The fact that circles are "corner free" enables us to give an algorithm that produces $O(n)$ simple motions in $O(n)$ computation time. Also note that for this RNF result, we need not restrict the lengths of Γ to be any less than the diameter of the confining circle.

We outline the main steps of the algorithm bringing Γ to RNF in the following.

Observation 5.1 *Let Γ be an n-link chain in RNF. Then any joint of Γ can be moved along any path on C with $O(1)$ simple motions while keeping Γ in RNF.*

Lemma 5.1 *Let Γ be an n-link chain. Then A_0 can be brought to C with $O(n)$ simple motions.*

Lemma 5.2 *For an n-link chain Γ, suppose that $A_0, \ldots, A_{k-1}$ lie on C. Then while $A_0, \ldots, A_{k-1}$ remain fixed, Γ can be moved to a configuration in which either A_k lies on C or for some $m > k$, A_m lies on C and $\Gamma(k, m)$ is straight. Furthermore, this can done with $O(n)$ simple motions. See Figure 5.1.*

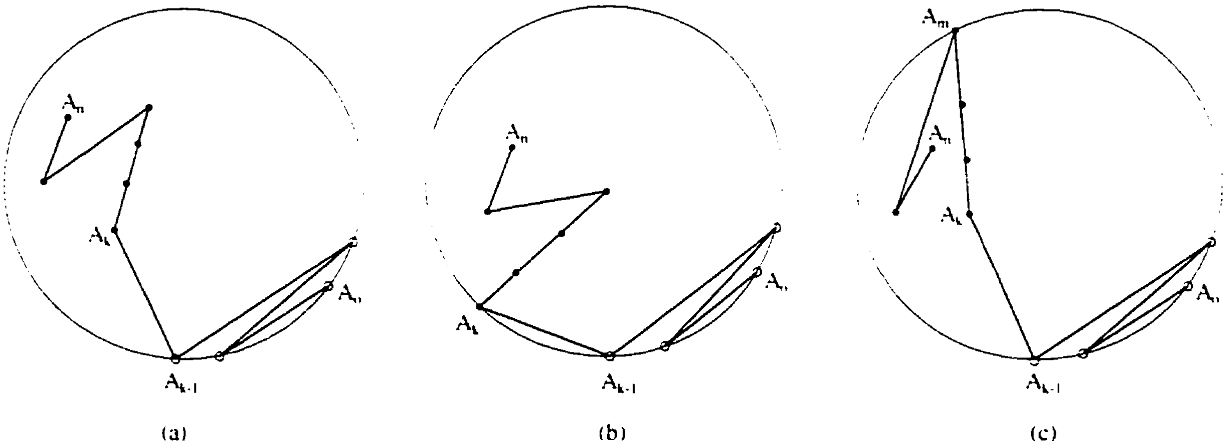


Figure 5.1: An initial configuration (a) and two possible final configurations (b) and (c) for Lemma 5.2

Proof: Similar to Lemma 3.2. □

Lemma 5.3 *For an n-link chain $\Gamma \prec b^F$. suppose that $A_0, \ldots, A_l$ and A_n lie on C and that $\Gamma(l,k)$ and $\Gamma(k,n)$ are straight. for some $l < k < n$.*

Then while A_n remains fixed and $\Gamma(0,l)$ remains in RNF. and while $[A_l, \ldots, A_k]$ and $[A_k, \ldots, A_n]$ remain straight. A_k can be brought to C. Furthermore. this can be done with $O(1)$ simple motions. See Figure 5.2.

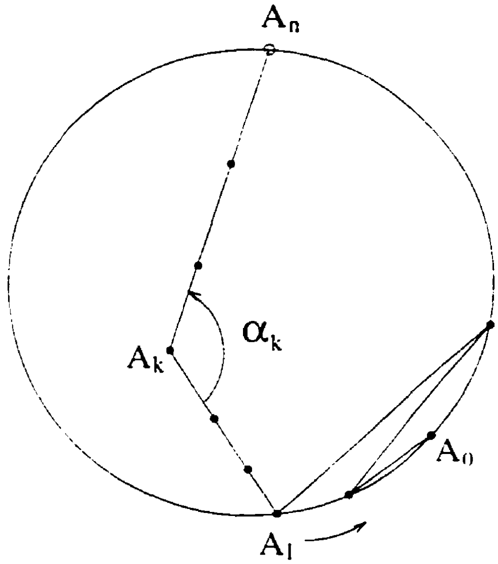


Figure 5.2: Bring A_k to C.

Proof: If initially A_k is already on C. then we are done. Otherwise we proceed as follows.

Keeping A_n fixed. and keeping $[A_l, \ldots, A_k]$ and $[A_k, \ldots, A_n]$ straight. move A_l towards A_n along C until A_k hits C. or α_k straightens to π. or A_l coincides with A_n. By Observation 5.1. this can be done with $O(1)$ simple motions while $A_0, \ldots, A_l$

remain on C.

If A_k hits C first, then we are done. If α_k straightens to π first, then keeping A_n fixed, keeping A_l on C, and keeping $[A_l, \ldots, A_k]$ and $[A_k, \ldots, A_n]$ straight, move A_k away from o until A_k hits C. By Observation 5.1, this can be done with $O(1)$ simple motions. If A_l coincides with A_n first, then $[A_l, \ldots, A_k]$ and $[A_k, \ldots, A_n]$ coincide. In this case, fix A_l and A_n, and rotate $[A_k, \ldots, A_n]$ and $[A_l, \ldots, A_k]$ counterclockwise about A_n until A_k hits C. All together, $O(1)$ simple motions suffice. $\square$

The above three lemmas yield the following, for which the proof is similar to Theorem 3.1 and is thus omitted.

Theorem 5.1 *Any n-link chain inside $C(o, r)$ can be moved to RNF with $O(n)$ simple motions.*

From the above theorem, the reachability algorithm for Γ inside a convex obtuse polygon can be extended in a straightforward way to handle Γ inside a circle. We mention the following, for which we omit details of the proof.

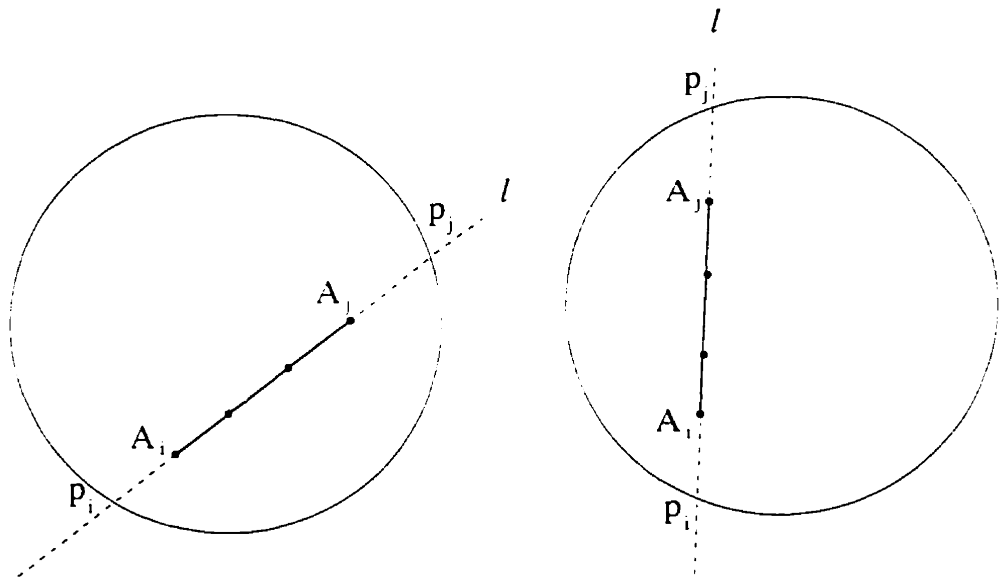
Theorem 5.2 *Let Γ be an n-link chain inside $C(o, r)$. For any p inside C, condition $(*)$ is a sufficient and necessary condition for A_n to reach p.*

Furthermore, A_n can be brought to any of its reachable points with $O(n)$ simple motions.

5.2 Bringing a Chain to r-RNF

For any given configuration of a chain inside a circle, we extend the definition of orientation of a single link in [HJW85] to a definition of orientation of a straight line segment composed of one or more links.

Definition 5.2 *Let Γ be a chain inside C. Suppose that $\Gamma(i,j)$ is straight and that l is the line containing $\Gamma(i,j)$. Assume that l intersects C at p_i and p_j, with A_i closer to p_i. Then $\Gamma(i,j)$ is said to have* right orientation *if $\widehat{p_i p_j} \leq \widehat{p_j p_i}$.* Left orientation *can be defined similarly. Refer to Figure 5.3.*



(a) right orientation (b) left orientation

Figure 5.3: Orientations

Clearly we have the following.

Observation 5.2 *$[A_i, A_j]$ lying on a diameter has both orientations. Moreover, while staying straight, $[A_i, A_j]$ can change its orientation only by moving to a configuration in which it lies along some diameter.*

Based on the above definition, we define right oriented RNF (left oriented RNF) as follows.

Definition 5.3 *A chain in RNF is in* right oriented RNF (left oriented RNF), *denoted* r-RNF (l-RNF), *if its links all have right (left) orientations.*

The following is immediate from Observation 5.1.

Observation 5.3 *Let Γ be an n-link chain in r-RNF. Then any joint of Γ can be moved along any path on C with $O(1)$ simple motions while keeping Γ in r-RNF.*

The key idea of the algorithm moving $\Gamma \prec r$ between any of its configurations inside $C(o, r)$ is to take r-RNF as a bridge. The rest of this section elaborates on moving such a Γ to r-RNF. First, we have

Lemma 5.4 *Suppose that $\Gamma = [A_0, A_n]$ has right orientation and that both A_0 and A_n initially lie on C. Then while A_n remains fixed and while A_0 remains on C, Γ can be moved to r-RNF with $O(n)$ simple motions.*

Proof: By induction on n. The result is trivially true when $n = 1$. Suppose that the result holds for any $n < m$: we now show the result holds for m.

Keeping A_m fixed, keeping A_0 on C, and keeping $[A_0, A_{m-1}]$ straight, rotate L_m about A_m counterclockwise until A_{m-1} hits C, as Figure 5.4 shows.

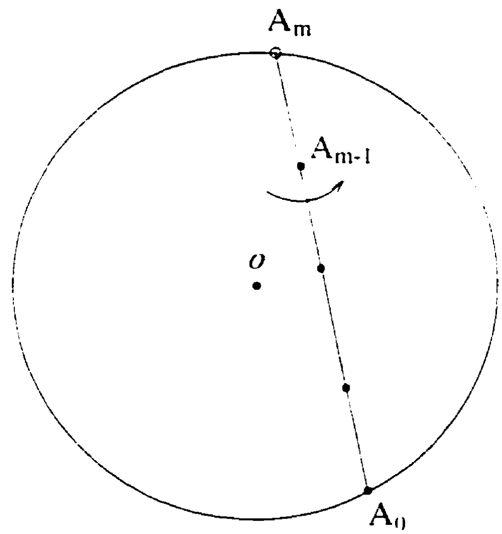


Figure 5.4: Move A_{m-1} to C.

Clearly both L_m and $[A_0, A_{m-1}]$ retain their orientations. Also this is done with $O(1)$ simple motions. Then fixing A_{m-1}, A_m and keeping A_0 on C, by the induction hypothesis, $[A_0, A_{m-1}]$ can be moved to r-RNF with $O(m)$ simple motions. This completes the induction proof. □

The following is the key step for bringing $\Gamma \prec r$ to r-RNF.

Lemma 5.5 *Let Γ be an n-link chain in RNF. Suppose that $\Gamma(0, n-1)$ is in r-RNF and that L_n has left orientation. Then while A_n remains fixed, Γ can be moved to* *r-RNF with $O(n)$ simple motions.*

Proof: We show the result by considering two cases.

Case 1: $l_n \leq l_{n-1}$, as Figure 5.5 shows.

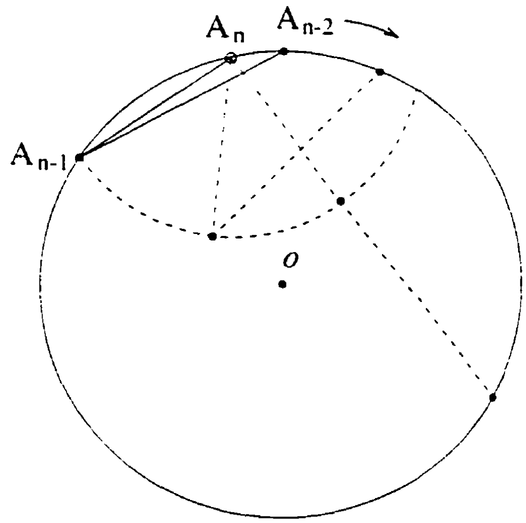


Figure 5.5: Case 1: $l_n \leq l_{n-1}$.

Fixing A_n, move A_{n-2} along C clockwise until α_{n-1} straightens to π. Since $l_n < r$ and $l_{n-1} < r$, $l_n + l_{n-1} < 2r$. Hence A_{n-2} cannot reach the point diametrically opposite to A_n. Since the distance between A_{n-2} and A_n increases, α_{n-1} increases. Therefore, the above reconfiguration is possible. By Observation 5.3, this can be done with $O(1)$ simple motions while $\Gamma(0, n-2)$ remains in r-RNF.

Note that A_n remains fixed, A_{n-2} remains on C and $l_n \leq l_{n-1}$. So L_{n-1} cannot lie on a diameter in the above process. By Observation 5.2, L_{n-1} retains right orientation in the above process. Hence $[A_{n-2}, A_n]$ has right orientation at the end of this process. Next, in accordance with Lemma 5.4, move $[A_{n-2}, A_n]$ to r-RNF while A_n remains fixed and A_{n-2} remains on C. By Observation 5.3, this can be done with $O(1)$ simple motions while $\Gamma(0, n-2)$ remains in r-RNF.

Thus in this case, Γ can be moved to r-RNF with $O(1)$ simple motions.

Case 2: $l_n > l_{n-1}$, as Figure 5.6 shows.

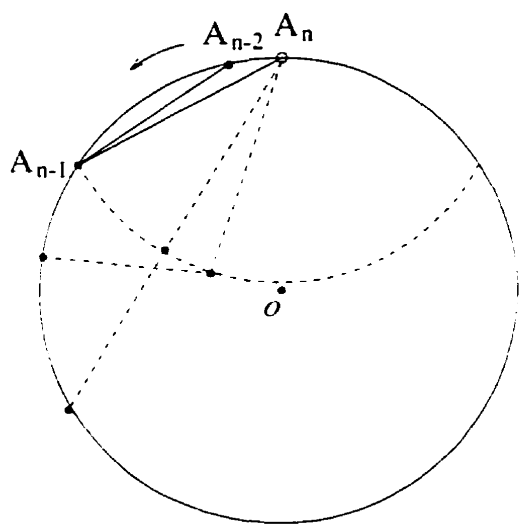


Figure 5.6: Case 2: $l_n > l_{n-1}$.

Fixing A_n, move A_{n-2} along C counterclockwise until α_{n-1} straightens to π. This is possible for the same reason as in Case 1. Also by Observation 5.3, this can be done with $O(1)$ simple motions while $\Gamma(0, n-2)$ remains in r-RNF.

Note that A_n remains fixed, A_{n-2} remains on C and $l_n > l_{n-1}$. So L_n cannot lie on a diameter in the above process. By Observation 5.2, L_n retains left orientation in the above process. Hence $[A_{n-2}, A_n]$ has left orientation at the end of this process.

If $l_{n-1} + l_n \leq l_{n-2}$, since $l_{n-2} < r$, by Case 1, Γ can be moved to r-RNF with $O(1)$ simple motions.

If $l_{n-1} + l_n > l_{n-2}$, we further consider two subcases.

Case 2a: $l_{n-2} + (l_{n-1} + l_n) > 2r$. Fixing A_n and keeping $[A_{n-2}, A_n]$ straight, move A_{n-3} along C counterclockwise until A_{n-2} hits C, as (a) of Figure 5.7 shows. This is possible since $l_{n-2} + (l_{n-1} + l_n) > 2r$. By Observation 5.3, this can be done with $O(1)$ simple motions while $\Gamma(0, n-3)$ remains in r-RNF.

Since A_{n-2} moves along C counterclockwise, $[A_{n-2}, A_n]$ rotates about A_n counterclockwise. Since $l_{n-1} + l_n \leq 2r$, $[A_{n-2}, A_n]$ lies on a diameter at some moment during the above process and passes from left orientation to right orientation. Next, in accordance with Lemma 5.4, move $[A_{n-2}, A_n]$ to r-RNF while A_n remains fixed and A_{n-2} remains on C. By Observation 5.3, this can be done with $O(1)$ simple motions while $\Gamma(0, n-2)$ remains in r-RNF.

Thus in this subcase, Γ can be moved, while A_n remains fixed, to r-RNF with $O(1)$ simple motions.

Case 2b: $l_{n-2} + (l_{n-1} + l_n) \leq 2r$. Fixing A_n and keeping $[A_{n-2}, A_n]$ straight, move A_{n-3} along C counterclockwise until α_{n-2} straightens to π, as (b) of Figure 5.7 shows. For the same reason as above, this is possible and can be done with $O(1)$ simple motions while $\Gamma(0, n-3)$ remains in r-RNF. Also $[A_{n-3}, A_n]$ has left orientation at the end of this process.

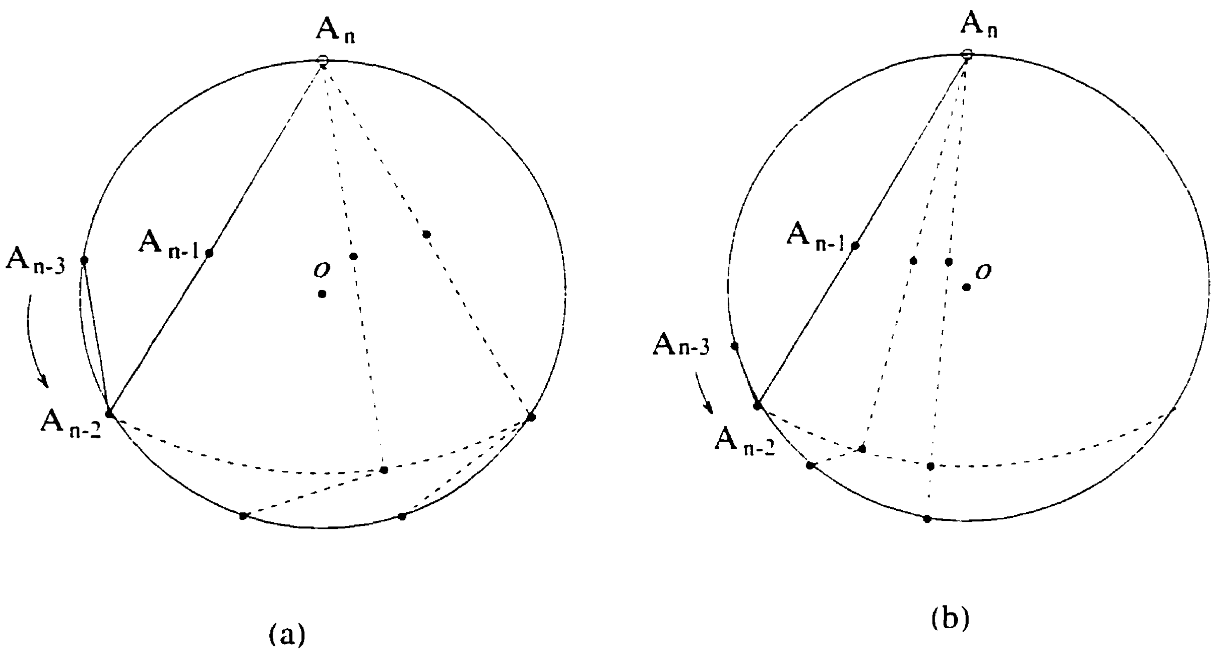


Figure 5.7: Two subcases of Case 2

Repeat this process until Γ is in r-RNF or the entire Γ becomes straight. In the latter case, fixing A_n and keeping $[A_0, A_n]$ straight, rotate $[A_0, A_n]$ about A_0 counterclockwise until A_0 hits C. Then $[A_0, A_n]$ has right orientation. By Lemma 5.4, $[A_0, A_n]$ can be moved to r-RNF while A_n remains fixed and A_0 remains on C.

Thus in this case, Γ can be moved to r-RNF with $O(n)$ simple motions. $\square$

Now we present a result that gives rise to the main result in this chapter.

Theorem 5.3 *Let $\Gamma \prec r$ be an n-link chain. Then Γ can be moved to r-RNF with $O(n^2)$ simple motions.*

Proof: First, move Γ ro RNF in accordance with Theorem 5.1. This requires $O(n)$ simple motions.

If L_1 has left orientation, then fix $A_1, A_2, \ldots, A_n$ and rotate L_1 about A_1 counterclockwise until A_0 hits C. This puts L_1 in right orientation.

Suppose that $L_1, L_2, \ldots, L_{k-1}$ all have right orientations and that L_k has left orientation. Fix $A_k, A_{k+1}, \ldots, A_n$ and move $\Gamma(0, k)$ in accordance with Lemma 5.5 to put $\Gamma(0, k)$ in r-RNF. This requires $O(k)$ simple motions.

Repeat the above process until Γ is in r-RNF. All together, at most $1+2+\cdots+n \in O(n^2)$ simple motions are needed. □

5.3 Main Result

Definition 5.4 *Two configurations of a chain are* equivalent *if one can be continuously moved to the other.*

The following is obvious.

Observation 5.4 *Definition 5.4 defines an equivalence relation on the set of configurations of a chain Γ. Furthermore, if Γ can be moved from an initial configuration to a final configuration, then Γ can be moved from the final configuration to the initial configuration with same number of simple motions.*

Now we present the main result in this chapter.

Theorem 5.4 *Let $\Gamma \prec r$ be an n-link chain. Then Γ has only one equivalence class of configurations. Furthermore, Γ can be moved between any of its configurations inside C with $O(n^2)$ simple motions.*

Proof: By Theorem 5.3, Observation 5.4 and Observation 5.3. □

Next we use the preceding theorem to obtain two results by examining possible configurations of $\Gamma \prec r$ within $C(o, r)$. Our results suggest that the bound r on Γ to ensure only one equivalence class of configurations of Γ is best in some sense.

Of all the configurations of a chain, the folded one in which all interior joint angles are either π or 0 is of particular interest. In [HJW85], Hopcroft, Joseph and Whitesides proved the following.

Fact 5.1 ([HJW85]) *Any n-link chain Γ can be folded into length $\leq 2l_{max}$ with $O(n)$ simple motions if there is no confining region. Furthermore, for any length less than $\leq 2l_{max}$, there exists a Γ having l_{max} that cannot be folded into that length.*

Consequently, we have the following.

Theorem 5.5 *Let $\Gamma \prec r$ be an n-link chain inside $C(o, r)$. Then Γ can always be folded within C with $O(n^2)$ simple motions.*

Furthermore, for any $r' > r$, there exists a $\Gamma \prec r'$ such that Γ cannot be folded within C.

Proof: Since any folded configuration of length $\leq 2r$ fits in C, by Fact 5.1 and $\Gamma \prec r$, a folded configuration of Γ within C is possible. By Theorem 5.4, Γ can be moved to this configuration with $O(n^2)$ simple motions.

The second part of theorem follows trivially from Fact 5.1. □

Note that the time complexity for folding Γ inside C is $O(n^2)$.

The results of [KSW95] for folding chains of links onto a single link are for chains whose links are of equal length. Note that, by Theorem 5.2, this situation is trivial for circles as confining regions, as any chains of equal length links at most the diameter can always be folded onto a single link.

We conclude by presenting a reachability result for $\Gamma \prec r$.

Theorem 5.6 *Let Γ be an n-link chain inside $C(o, r)$. Then no matter where Γ initially lies within C, any arbitrary joint of Γ can reach any arbitrary point inside C if and only if $\Gamma \prec r$.*

Furthermore, any joint of $\Gamma \prec r$ can be brought to any point inside C with $O(n^2)$ simple motions.

Proof: First, we show the "if" part. Note that for any p inside C, $r \leq d_{max}(p)$. By Theorem 5.2, the result holds for the endjoints.

For an intermediate joint A_i $(0 < i < n)$, view Γ as two subchains $\Gamma(0, i)$ and $\Gamma(i, n)$ connected at A_i. Since $\Gamma \prec r$, we have $\Gamma(0, i) \prec r$ and $\Gamma(i, n) \prec r$. Hence both $\Gamma(0, i)$ and $\Gamma(i, n)$ satisfy condition $(*)$. By Theorem 5.2, p is reachable by both A_i of $\Gamma(0, i)$ and A_i of $\Gamma(i, n)$.

Therefore, there is a configuration of Γ in which A_i lies at p. By Theorem 5.4, no matter where Γ initially lies, Γ can be moved to this configuration with $O(n^2)$ simple motions.

To show the "only if" part, note that if $\Gamma \prec r$ does not hold, then there exists some link, say, L_i, with length $l_i > r$. Clearly A_i cannot touch o, the center of C. $\square$

Chapter 6

Conclusion

6.1 Summary of the Results

We now briefly summarize the results in this thesis.

We have studied the reconfiguration and reachability problem for planar chains within two types of confining regions: convex obtuse polygons and circles. We have investigated how to design short link chains within a given environment so that an obviously necessary condition of reachability is sufficient as well. We have demonstrated that this approach enabled us to go beyond previous studies of anomalous special cases and provided insight into general reconfiguration problem.

In Chapter 1, we defined, given a simple polygon P as the confining region, three interdependent bounds on the maximum link length of Γ inside P, namely, b^S, b^F, and b^C.

Chapter 2 and 3 considered convex obtuse polygons as the confining regions. In Chapter 2, we characterized b^C and proved that $s_{min} \leq b^C = b^F$, where s_{min} is the length of the shortest side of P. We also proved that b^C achieves its maximum w, the width of P, if P is a regular $2k$-gon.

In Chapter 3, we presented a polynomial time algorithm bringing the endjoints of Γ to any of their reachable points. This gave an algorithmic proof for $b^C \leq b^S$.

In Chapter 4, we considered general convex polygons as the confining regions and examined the properties of reachable regions. We proved that $b^S \leq w$. We described the shapes of reachable regions and showed that they are linearly ordered by set inclusion. We developed the notion of l-reachable convex polygons and characterized such polygons for $l \leq b^S$.

In Chapter 5, we considered circles as the confining regions. We proved that if each link of Γ has length less than the radius of the confining circle, then Γ has only one equivalence class of configurations.

6.2 Problems for Future Research

Below we give a list of problems for future research.

(1) Determine the bound on Γ, which is confined within a convex polygon P, so

that Γ has only one equivalence class of configurations. We even don't know what is the bound of Γ to avoid completely stuck configurations of Γ inside P.

(2) Generalize the results we have obtained in this thesis to 3 dimensions. This is of particular interest in practice.

(3) Investigate the reachability problem for planar arms inside convex obtuse polygons.

(4) Consider the reconfiguration problem for planar chains if links are not allowed to cross over one another. There has been little known for the reconfiguration problem under this more restrictive type of motions.

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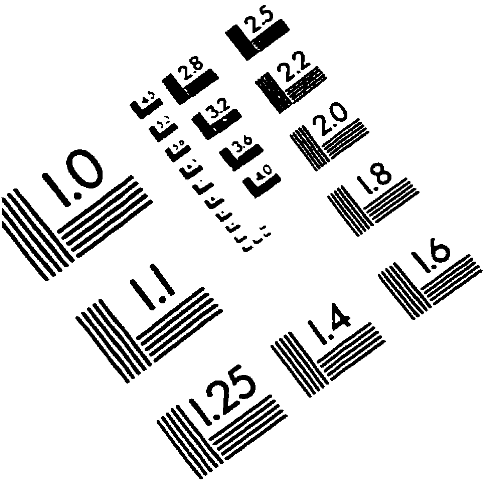
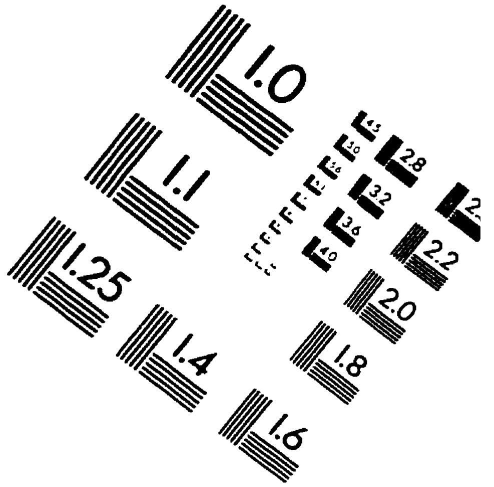
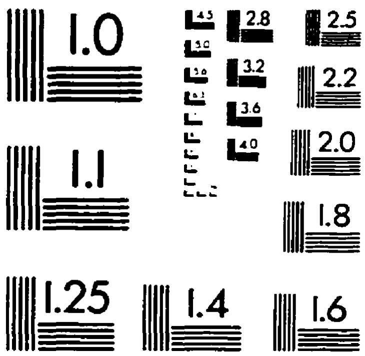
IMAGE EVALUATION TEST TARGET (QA-3)

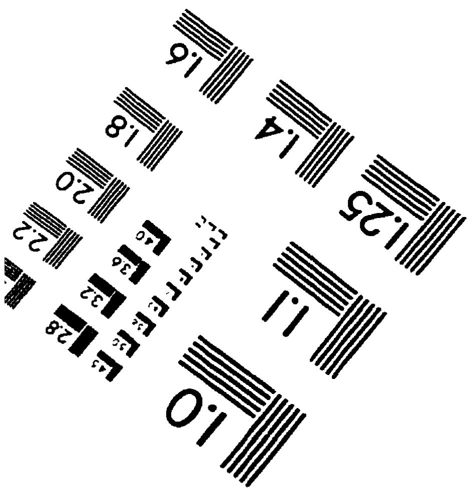


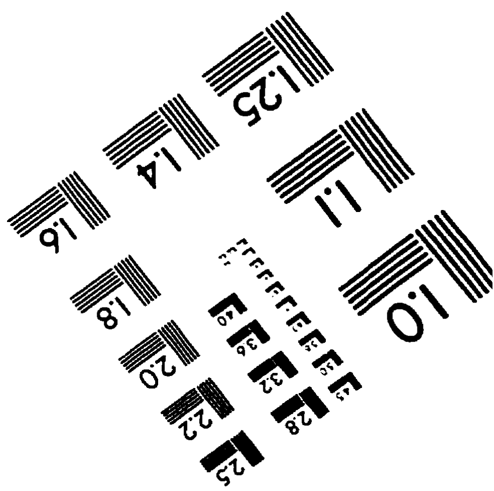




150mm

6"





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