EQUIDISTRIBUTION
AND
L-FUNCTIONS
IN
NUMBER THEORY

EQUIDISTRIBUTION AND L-FUNCTIONS IN NUMBER THEORY

by

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par

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RESUME

L'objet de cette thèse est d'obtenir à l'aide de la notion de suite équirépartie d'importants résultats classiques de théorie des nombres (théorème de Tchebotareff, "théorème des nombres premiers", théorème de Dirichlet). Les techniques utilisées font appel à l'analyse harmonique dans les groupes compacts (théorème de Peter-Weyl), à la théorie des variables complexes (théorème de Wiener-Ikehara), à la théorie des nombres algébriques.

ACKNOWLEDGEMENT

Professor John Labute introduced me to equidistribution after I had a glance at the subject during a seminar held at the Université de Montréal. This subject gains importance along with the renewal of interest in analytic number theory in the past few years: that is one more reason why I wish to express my gratitude to Professor Labute.

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INTRODUCTION

This thesis aims mainly at proving important density results from number theory. What we mean by density, we will clarify in our chapter 3 but in order to get the flavour of it let us state right now a typical result due to Dirichlet:

Dirichlet's Theorem: if m is a positive integer, if m and a are relatively prime, then there exists infinitely many primes p such that $p \equiv a \pmod{m}$.

This result is often referred to as Dirichlet's theorem on primes in an arithmetic progression.

To prove this we will need Čebotarev's theorem (see chapter 3), which is a density theorem following from the "equidistribution" of the Frobenius elements of a finite Galois group.

This explains why our first chapter has to be about "equidistribution": we define the notion, give criteria and consider the important cases when the elements of the sequences to be "equidistributed" belong to the space of conjugacy classes of a compact group (and this leads us to harmonic analysis on a compact group: see [5] or [12]) or to the two dimensional torus. Some of the facts about equidistribution we do not prove since they are not needed in the rest of the thesis but we do give references. Let us just mention that historically the notion of equidistribution arises from diophantine analysis in connection with the "distribution" in $[0, 1]$ of the

fractional parts of the elements of a sequence.

Then comes chapter 2: the technical core of the thesis. The so-called L-functions, generalization of the well-known zeta-functions, give us a sufficient condition for equidistribution. We use this condition to get the equidistribution of the Frobenius elements of a finite Galois group: then we are all set up to get in chapter 3 the required density results.

On our way, not in the main stream of the exposure though, we get the prime number theorem.

Our topic needs methods from various parts of mathematics: from the theory of the complex variable we will assume the Wiener-Ikehara theorem and the Abel summation trick. From the theory of the representations of a compact group we will assume the Peter-Weyl theorem: the reader can refer to the classical book Topological Groups by Pontryagin or to [12]. From algebraic number theory we assume the definitions and results from Théorie Algébrique des Nombres by Pierre Samuel (see [10]). The main reference though will be Abelian ℓ-adic Representations and Elliptic Curves by J.P. Serre (see [11], pages 18-29).

CHAPTER I

GENERAL

FACTS

ABOUT

EQUIDISTRIBUTION

1.1 Basic terminology.

Let X be a compact topological space.

Let C(X) be the Banach space of continuous, complex-valued functions on X with its usual norm $||f||_\infty = \sup_{x \in X} |f(x)|$.

For each $x \in X$ let δ_x be the Dirac measure associated to x: for $f \in C(X)$ we have $\delta_x(f) = f(x)$.

Let $(x_n)_{n \geq 1}$ be a sequence of elements of X.

For $n \geq 1$ we define $\mu_n = \dfrac{\delta_{x_1} + \ldots + \delta_{x_n}}{n}$

Let μ be a Radon measure in X (cf [12]).

Definition 1.1.1 The sequence (x_n) is said to be μ-equidistributed or μ-uniformly distributed or "équirépartie par rapport à μ" (in French) if $\mu_n \to \mu$ weakly as $n \to \infty$ (that is to say: if $\mu_n(f) \to \mu(f)$ as $n \to \infty$ for any $f \in C(X)$).

We get the following obvious results:

Note 1.1.2 $(x_n)_{n \geq 1}$ can be equidistributed relatively to at most one measure μ.

Assume $(x_n)_{n \geq 1}$ is "μ-and-ν-equidistributed"

then $\mu(f) = \nu(f)$ for all $f \in C(X)$

thus $\mu = \nu$

Note 1.1.3 if $(x_n)_{n \geq 1}$ is μ-equidistributed, μ is positive and of total mass 1.

For $f \geq 0$ $\quad \mu_n(f) = \dfrac{f(x_1) + \ldots + f(x_n)}{n} \geq 0.$

Hence $\mu(f) \geq 0$.

Take $f = \chi_X$ the characteristic function X.

$$\mu_n(\chi_X) = \frac{n}{n} = 1$$

Hence $\mu(\chi_X) = 1$.

1.2 General criteria.

We give two conditions for equidistribution: the first is sufficient; the second, necessary. Our first condition in fact amounts to noticing that we need check the weak convergence of the μ_n's only on a set of generators of a dense vector subspace of C(X).

lemma 1.2.1 let (ϕ_a) be a family of continuous functions on X with the property that their linear combinations are dense in C(X). Assume that for all a, $(\mu_n(\phi_a))_{n \geq 1}$ has a limit. Then $(x_n)_{n \geq 1}$ is equidistributed with respect to some measure μ: it is the unique measure such that $\mu(\phi_a) = \lim_{n \to \infty} \mu_n(\phi_a)$

Proof: for all a call $\mu(\phi_a)$ the limit of $(\mu_n(\phi_a))_{n \geq 1}$.

Let $V = \langle \phi_a \rangle$

i.e. V is the subspace generated by the ϕ_a's.

Let $f \in V$,

$f = \sum_{i=1}^{m} \lambda_i \phi_{a_i}$ for certain scalars λ_i.

$\mu_n(f) = \sum_{i=1}^{m} \lambda_i \mu_n(\phi_{a_i})$ since the μ_n's are clearly linear.

Hence $\mu_n(f) \to \sum_{i=1}^{m} \lambda_i \mu(\phi_{a_i})$ by the hypothesis

Let us call $\mu(f)$ the quantity $\sum_{i=1}^{m} \lambda_i \mu(\phi_{a_i})$.

Let $g \in C(X)$, $g \notin V$, $\varepsilon > 0$.

There exists $f \in V$ such that $||f - g||_\infty \leq \varepsilon$

and $\mu_n(f) \to \mu(f)$.

There exists N such that $n \geq N$ imply

$$|\mu_{n+p}(f) - \mu_n(f)| \leq \varepsilon$$

$$|\mu_{n+p}(g) - \mu_n(g)| \leq \left| \frac{(f-g)(x_1) + \ldots + (f-g)(x_{n+p})}{n+p} - \frac{(f-g)(x_1) + \ldots + (f-g)(x_n)}{n} \right| + \left| \frac{f(x_1) + \ldots + f(x_{n+p})}{n+p} - \frac{f(x_1) + \ldots + f(x_n)}{n} \right|$$

$$|\mu_{n+p}(g) - \mu_n(g)| \leq \frac{(n+p)\varepsilon}{(n+p)} + \frac{n\varepsilon}{n} + \varepsilon = 3\varepsilon$$

Hence $(\mu_n(g))$ is a Cauchy sequence and converges to a limit we will call $\mu(g)$.

This way we have defined a function:

μ: $C(X) \to \mathbb{C}$, the set of complex numbers.

$$g \to \mu(g) = \lim_{n \to \infty} \mu_n(g)$$

μ is linear since $\mu_n(f + g) = \mu_n(f) + \mu_n(g)$

$$\mu_n(\lambda f) = \lambda \mu_n(f)$$

for any $f, g \in C(X)$, λ complex.

Fix $\varepsilon > 0$.

Assume $||f - g||_\infty \leq \delta = \varepsilon$.

$$|\mu_n(f) - \mu_n(g)| = \frac{1}{n}|(f - g)(x_1) + \ldots (f - g)(x_n)| \leq \frac{n\delta}{n} = \varepsilon$$

Thus $|\mu(f) - \mu(g)| \leq \varepsilon$

and μ is continuous and a Radon measure.

Q.E.D.

We now give a necessary condition for equidistribution.

Theorem 1.2.2

Assume that $(x_n)_{n \geq 1}$ is μ-equidistributed.

Let $U \subset X$ have a boundary with μ-measure zero.

For all n let n_U be the number of $m \leq n$

such that $x_m \in U$.

Then $\lim_{n \to \infty} (n_U / n) = \mu(U)$

where $\mu(U) = \mu(U^\circ)$ and U° is the interior of U.

Proof: let $\varepsilon > 0$.

There exists $\phi \in C(X)$, $0 \leq \phi \leq 1$, $\phi = 0$ on $X - U^\circ$

such that $\mu(\phi) \geq \mu(U) - \varepsilon$

$$\mu_n(\phi) = \frac{\phi(x_1) + \ldots + \phi(x_n)}{n} \leq n_U / n$$

since $\phi(x_i) = 0$ if $x_i \notin U$

$0 \leq \phi(x_i) \leq 1$ if $x_i \in U$.

This result combined with the hypothesis $\mu_n(\phi) \to \mu(\phi)$

yields: $\liminf_{n \to \infty} n_U/n \geq \mu(\phi) \geq \mu(U) - \varepsilon$ for all ε.

Thus $\liminf_{n \to \infty} n_U/n \geq \mu(U)$.

Since the boundary of X - U has μ-measure zero, the same process for X - U yields:

$$\liminf_{n \to \infty} (n_{X - U}) / n = \liminf_{n \to \infty} (n - n_U) / n \geq \mu(X - U)$$

$$1 - \limsup_{n \to \infty} n_U / n \geq \mu(X - U) = 1 - \mu(U)$$

$$\liminf_{n \to \infty} n_U/n \geq \mu(U) \geq \limsup_{n \to \infty} n_U/n$$

Q.E.D.

1.3 The case of a compact group.

We now consider the case when the elements to be equidistributed belong to the space of conjugacy classes of a compact group; we will use the powerful methods of harmonic analysis on a compact group (mainly the Peter-Weyl theorem about the irreducible characters of a compact group): this will give us a first criterion of equidistribution. As compact groups are provided with a unique normalized Haar measure, we will find a second criterion in this particular case. For references, see [5] or [12].

Let G be a compact group and X, its space of conjugacy classes.

Let μ be a Radon measure on G.

Take $g \in C(X)$.

Let p be the canonical projection from G to X.

Define $\mu^*(g) = \mu(gp)$

Lemma 1.3.1. μ^* is a Radon measure on X. We will denote it by μ since there is no possible confusion.

μ^* is clearly linear.

Given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$||g_1 - g_2||_\infty < \delta \text{ imply } |\mu(g_1 p) - \mu(g_2 p)| = |\mu^*(g_1) - \mu^*(g_2)| < \varepsilon$$

since μ and p are continuous.

Q.E.D.

Under the above hypothesis we get:

Theorem 1.3.2

The sequence $(x_n)_{n \geq 1}$ is μ-equidistributed if and only if for any irreducible characters χ of G we have:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi(x_i) = \mu(\chi)$$

Proof: consider $C(X) \to C(G)$

$$g \to gp .$$

This is an isomorphism onto the space of central functions on G: by the Peter-Weyl theorem, the irreducible characters of G generate a dense subspace of C(X) (when viewed as functions $\chi^* \in C(X)$ where $\chi = \chi^* p$ but of course we will denote χ^* by χ).

Assume that $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi(x_i) = \mu(\chi)$.

Then by lemma 1.2.1 $(x_n)_{n \geq 1}$ is μ-equidistributed.

The converse is obvious since $\chi^* \in C(X)$.

Q.E.D.

In the case when μ is the Haar measure of G with $\mu(G) = 1$, we get the important corollary:

Theorem 1.3.3 (x_n) is μ-equidistributed in X if and only if for any irreducible character χ of G, $\chi \neq 1$, we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi(x_i) = 0$$

Proof: if $\chi \neq 1$, there exists y such that $\chi(y) \neq 1$.

Denote $\chi_y(x) = \chi(xy)$.

where the operation in G is denoted multiplicatively.

Then $\mu(\chi) = \mu(\chi_y)$ since the Haar measure is invariant by translation.

But $\chi_y(x) = \chi(xy) = \chi(x)\ \chi(y)$.

Therefore $\mu(\chi) = \mu(\chi_y\chi) = \chi_y\ \mu(\chi)$

$(1 - \chi(y))\ \mu(\chi) = 0$

Since $\chi(y) \neq 1$, $\mu(\chi) = 0$.

If $\chi = 1$, $\mu(\chi) = 1 = \mu(G)$.

The last two results and the theorem 1.3.2 yield the result.

Q.E.D.

We finish this section by quoting a result from [5] (without proof since we will not need it):

Theorem 1.3.4

Let G be a compact group,

μ its Haar measure,

$(x_n)_{n \geq 1}$ a sequence of elements of G.

Then the following statements are equivalent:

a) for all closed subset U of G the boundary of which has measure zero, we have:

$$\lim_{n \to \infty} n_U/n = \mu(U)$$

(with the notation of 1.2).

b) for all $f \in C(G)$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \mu(f)$$

(our definition of equidistribution).

c) for all not trivial unitary irreducible representation M of G, one has:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} M(x_i) = 0.$$

Note that here there is no question of the space of conjugacy classes. Compare b) with theorem 1.2.2: b) is stronger since it is an equivalence and theorem 1.2.2 is only a necessary condition of equidistribution. Compare c) and theorem 1.3.3: in the first case we use characters, in the second representations.

1.4 The case of R/Z.

This section is mainly a particular case of the preceeding one except that at the end we consider the classical case of equidistribution in $[0, 1]$ without using the compact group structure, turning instead to "visual methods".

Let $G = R/Z$

where R is the set of real numbers and Z is the set of integers. G is $[0, 1[$ with the addition modulo one.

G can also be called the two-dimensional torus because of an obvious identification mapping.

Let μ be the normalized Haar measure on G, i.e. $\mu(G) = 1$. Note that the space X of the conjugacy classes of G can be considered to be G itself since G is abelian. In these conditions we have:

Theorem 1.4.1 $(x_n)_{n \geq 1}$ is μ-equidistributed if and only if for any integer $m \neq 0$ we have

$$\frac{1}{N} \sum_{n \leq N} e^{2\pi i m x_n} \to 0 \quad \text{where } i^2 = -1.$$

The above result is known as Weyl's criterion. For classical proofs of it, see [4], page 89 or The American Mathematical Monthly, Volume 76, no. 6, page 654.

Proof: we will need:

Lemma 1.4.2. The irreducible characters χ of R/Z are the mappings:

$$r \to e^{2\pi i m r} \quad (m \in Z).$$

Proof of the lemma: (outline)

From [12], page 99 we accept the following powerful result: a bounded (or equivalently continuous) irreducible representation of an abelian group is nothing but a continuous solution $\chi(x)$ of $\chi(x + y) = \chi(x)\chi(y)$, $|\chi(x)| = 1$ and is undistinguishable from its character.

Since R/Z is compact, it has continuous characters: since it is abelian, its irreducible characters are as described above.

Now if ξ is an irrational in $[0,1[$, the set $\{n\xi\}_{n \geq 1}$ is dense in $[0,1[$.

The value $\chi(\xi)$ determines $\chi(n\xi)$ since $\chi(n\xi) = (\chi(\xi))^n$. Since χ is continuous, its values on a dense set determine it completely.

Put $\chi(\xi) = e^{2\pi mi\Theta}$. There exists a subsequence $(n_k\xi)_{k \geq 1}$ such that $|\chi(n_k\xi) - 1| < \varepsilon$ for any $\varepsilon > 0$ and this can be shown to force $\Theta = \xi$.

Q.E.D.

Back to theorem 1.4.1: we need only to put together theorem 1.3.3 and lemma 1.4.1.

Q.E.D.

Example 1.4.3 We prove the equidistribution in R/Z relatively to the Haar measure (the uniform distribution modulo one according to the classical terminology) of the fractional parts of the sequence $\{n\,\xi\}$ where ξ is irrational. Here we see the roots of the classical theory of equidistribution, i.e. the link with diophantine analysis.

Consider $x_n = \{n\,\xi\} = n\xi - [n\xi]$

where ξ is irrational,

$\{r\}$ denotes the fractional part of the real number r,

$[r]$ denotes the biggest integer not exceeding r.

Then $\frac{1}{n} \sum_{n=1}^{N} e^{2\pi i m (n\xi - [n\xi])} = \frac{1}{n} \sum_{n=1}^{N} e^{2\pi i m \xi n}$

Now $\left| \sum_{n=1}^{N} e^{2\pi i n \xi m} \right| = \left| \frac{e^{2\pi i (N+1) \xi m} - e^{2\pi i m \xi}}{e^{2\pi i m \xi} - 1} \right|$

$$< \frac{2}{\left| e^{2\pi i m \xi} - 1 \right|} = \frac{1}{|\sin \pi\, m\, \xi|} \quad \text{for } m \neq 0.$$

Thus $\frac{1}{n} \sum_{n=1}^{N} e^{2\pi i m (n\xi - [n\xi])} \to 0$. Conclude with 1.4.1.

Q.E.D.

For more examples, see [9], a book using ergodic theory.

A few remarks to conclude.

If one drops the group structure of $R/Z = [0, 1[$ and consider the compact topological space $[0, 1]$ (usual topology) instead, using the Lebesque measure, one gets:

Theorem 1.4.4 (cf [4] for a proof and look at 1.2.2)

(x_n) is μ-equidistributed if and only if for each interval $[a, b]$ of length $d > 0$ in $[0, 1]$ $\lim_{n \to \infty} (n_{[a, b]} / n) = d$.

Theorem 1.4.4 provides a very visual criterion and for an even better one in this respect see [8], where the notion of "almost-arithmetic progression" is defined.

Proof of theorem 1.4.4 (outline).

The necessity part is exactly theorem 1.2.2.

Let us prove the sufficiency part.

Let $h = \chi_{[a, b]}$ be the characteristic function of [a, h].

Then since $\sum_{i=1}^{n} h(x_i) = n_{[a, b]}$ we have:

$$\lim_{n \to \infty} \sum_{i=1}^{n} h(x_i)/n = d = \int_0^1 h(x)dx,$$

By linearity this extends to any step function g. Let f be any continuous function: for any $\varepsilon > 0$, there exists step functions g_1, g_2 such that $g_1 \leq f \leq g_2$, $\int_0^1 (g_2(x) - g_1(x))dx < \varepsilon$, so that the above equation still holds for f.

Q.E.D.

CHAPTER 2

L-FUNCTIONS

APPLIED

TO

EQUIDISTRIBUTION

2.1 L-functions associated to a compact group.

Let G be a compact group, and X its space of conjugacy classes.

Σ is a denumerable set, $\mathbb{N}$ is the set of natural numbers including zero.

$\{x_v\}_{v \in \Sigma}$ is a family of elements of X.

Consider the function N:

$$\Sigma \to \mathbb{N} - \{0, 1\}$$

$$v \to Nv$$

$\mathbb{C}$ is the set of complex numbers.

R(s) denotes the real part of s for $s \in \mathbb{C}$.

"det" stands for the determinant of a matrix.

Assumption 2.1.1

$\prod_{v \in \Sigma} \frac{1}{1 - (Nv)^{-s}}$ converges for every $s \in \mathbb{C}$ with $R(s) > 1$ and extends to a meromorphic function on $R(s) \geq 1$ having neither zero nor pole except for a simple pole at $s = 1$.

Assumption 2.1.2

Let ρ be an irreducible representation of G with character χ.

Put $L(s, \rho) = \prod_{v \in \Sigma} \frac{1}{\det (1 - \rho(x_v) (Nv)^{-s})}$.

Then $L(s, \rho)$ converges for $R(s) > 1$ and extends to a meromorphic function on $R(s) \geq 1$ having neither zero nor pole except possibly for $s = 1$.

The order of $L(s, \rho)$ at $s = 1$ will be denoted by $- c_\chi$.

Of course the functions just defined above will be referred to as L-functions.

We introduce a notation we will need all through this section. Given two sequences $u = (u_n)$ and $(v_n) = v$ we say that $u = o(v)$ if $\lim_{n \to \infty} \frac{u_n}{v_n} = 0$

Under the above assumptions, we get:

Theorem 2.1.3

a) The number of $v \in \Sigma$ with $Nv \leq n$ is asymptotic $n / \log n$ as $n \to \infty$.

b) If χ is irreducible,

then $\sum_{Nv \leq n} \chi(x_v) = c_\chi \, n / \log n + o(n / \log n)$

Proof:

Step 1: computation of L'/L (the logarithmic derivative of L).

Let d be the degree of ρ and take s, $R(s) \geq 1$, $s \neq 1$.

$$L(s, \rho) = \prod_{v \in \Sigma} \prod_{i=1}^{d} \frac{1}{1 - \lambda_v^{(i)} (Nv)^{-s}}$$

where $\lambda_v^{(i)}$ are the eigenvalues of $\rho(x_v)$.

$$\log L(s, \rho) = - \sum_{v \in \Sigma} \sum_{i=1}^{d} \log (1 - \lambda_v^{(i)} (Nv)^{-s})$$

The log is defined since $L(s, \rho) \neq 0$ by assumption 2.1.2.

$$\frac{d}{ds} \log L(s, \rho) = - \sum_{v \in \Sigma} \sum_{i=1}^{d} \frac{\lambda_v^{(i)} (Nv)^{-s} \log Nv}{1 - \lambda_v^{(i)} (Nv)^{-s}}$$

$$L'/L = \frac{d}{ds} \log L(s, \rho) = - \sum_{v \in \Sigma} \log Nv \sum_{i=1}^{d} \frac{\lambda_v^{(i)} (Nv)^{-s}}{1 - \lambda_v^{(i)} (Nv)^{-s}}$$

Now $\left| \lambda_v^{(i)} (Nv)^{-s} \right| = \dfrac{\left| \lambda_v^{(i)} \right|}{\left| Nv \right|^{R(s)}} \leq \frac{1}{2} < 1$,

since by the theory of compact groups $\left| \lambda_v^{(i)} \right| = 1$

and by assumptions $Nv \geq 2$, $R(s) \geq 1$.

Hence we can develop using geometric series:

$$L'/L = - \sum_{v \in \Sigma} \log Nv \sum_{i=1}^{d} \lambda_v^{(i)} (Nv)^{-s} \sum_{m=0}^{\infty} \lambda_v^{(i)m} (Nv)^{-sm}$$

$$L'/L = - \sum_{v \in \Sigma} \log Nv \sum_{m=1}^{\infty} \sum_{i=1}^{d} \lambda_v^{(i)m} (Nv)^{-sm}$$

by absolute convergence of $\sum_{i=1}^{d} \sum_{m=1}^{\infty} \lambda_v^{(i)m} (Nv)^{-sm}$.

$$L'/L = - \sum_{v \in \Sigma} \log Nv \sum_{m=1}^{\infty} (Nv)^{-sm} \chi(x_v^m)$$

$$L'/L = - \sum_{v \in \Sigma} \sum_{m=1}^{\infty} \frac{(\log Nv)\chi(x_v^m)}{(Nv)^{ms}}$$

Step 2: we prove:

Lemma 2.1.4 $\sum_{v \in \Sigma} \sum_{m \geq 2} \frac{\log Nv}{Nv^{ms}}$ converges

for $R(s) > 1/2$.

We will need:

Lemma 2.1.5 $\sum_{v \in \Sigma} \frac{\log Nv}{(Nv)^{\tau}} < \infty$ for $\tau > 1$.

Choose $\varepsilon > 0$ such that $\tau > \tau - \varepsilon > 1$.

Then $\frac{\log x}{x^{\varepsilon}} \to 0$ as $x \to \infty$,

$$\frac{\log Nv}{(Nv)^{\tau}} = \frac{\log Nv}{(Nv)^{\varepsilon}} \frac{1}{(Nv)^{\tau - \varepsilon}}$$

and there exists a constant C such that $\frac{\log Nv}{(Nv)^{\varepsilon}} < C$ for any v.

Thus $\sum_{v \in \Sigma} \frac{\log Nv}{(Nv)^{\tau}} \leq C \sum_{v \in \Sigma} \frac{1}{(Nv)^{\tau - \varepsilon}}$

Now $\sum_{v \in \Sigma} \frac{1}{(Nv)^{\tau - \varepsilon}}$ converges since $\prod_{v \in \Sigma} \frac{1}{1 - (Nv)^{\tau - \varepsilon}}$

converges by assumption (a classical theorem tells us that this

sum and this product converge or diverge simultaneously).

Q.E.D.

Back to the proof of lemma 2.1.4.

$$\sum_{v \in \Sigma} \sum_{m \geq 2} \frac{\log Nv}{|(Nv)^{ms}|} = \sum_{m \geq 2} \sum_{v \in \Sigma} \frac{\log Nv}{|(Nv)^{ms}|}$$

Now lemma 2.1.5 implies $\sum_{v \in \Sigma} \frac{\log Nv}{|(Nv)^{2s}|} < \infty$ for $R(s) > 1/2$.

$$\sum_{m \geq 2} \sum_{v \in \Sigma} \frac{\log Nv}{|(Nv)^{ms}|} \leq \left(\sum_{v \in \Sigma} \frac{\log Nv}{|(Nv)^{2s}|} \right) \left(1 + \frac{1}{2^{R(s)}} + \frac{1}{2^{2R(s)}} + \frac{1}{2^{3R(s)}} + \cdots \right) .$$

Q.E.D.

Step 3: applying the Wiener-Ikehara theorem and the Abel summation trick.

Using "step 1" and lemma 2.1.4, one can write

$$L'/L = - \sum_{v \in \Sigma} \frac{\chi(x_v) \log Nv}{(Nv)^s} + \phi(s)$$

where $\phi(s) = - \sum_{v \in \Sigma} \sum_{m \geq 2} \frac{\log Nv}{(Nv)^{ms}}$ converges for $R(s) > 1/2$.

Moreover from 2.1.4

$$\left| - \phi(s) \right| \leq \frac{1}{1 - \frac{1}{2^{R(s)}}} \sum_{v \in \Sigma} \frac{\log Nv}{|Nv|^{2R(s)}} .$$

Fix $\varepsilon > 0$.

Hence $\left| \phi(s) \right| \leq \frac{1}{1 - \frac{1}{2^{1/2}}} \sum_{v \in \Sigma} \frac{\log Nv}{Nv^{1 + \varepsilon}}$.

for $R(s) \geq \frac{1}{2} + \frac{\varepsilon}{2}$.

Hence the convergence is uniform on $R(s) \geq \frac{1}{2} + \frac{\varepsilon}{2}$ for any $\varepsilon > 0$.

We just proved:

Lemma 2.1.6 ϕ is holomorphic for $R(s) > 1/2$.

By the hypothesis on L:

$$L(s, \rho) \doteq \frac{g(s)}{(s - 1)^{c_\chi}}$$

where g(s) is holomorphic without zeroes or poles on $R(s) \geq 1$.

$$L'(s, \rho) = \frac{(s - 1)^{c_\chi} g'(s) - g(s) (s - 1)^{c_\chi - 1} c_\chi}{(s - 1)^{2c_\chi}}$$

$$L'/L = \frac{g'(s)}{g(s)} - \frac{c_\chi}{(s - 1)}$$

We just proved:

Lemma 2.1.7 L'/L can be extended to a meromorphic function on $R(s) \geq 1$, holomorphic except possibly for a simple pole of residue $-c_\chi$ at $s = 1$.

We state the following without proof (cf [7]):

Theorem 2.1.8 (Wiener-Ikehara)

Let $F(s) = \sum_{n=1}^{\infty} a_n/n^s$ be a Dirichlet's serie (cf [6])

with complex coefficients,

let $F^+(s) = \sum_{n=1}^{\infty} a_n^+ / n^s$ be a Dirichlet's serie with

positive coefficients,

such that:

a) $|a_n| \leq a_n^+$ for all n,

b) F^+ converges for $R(s) > 1$,

c) F^+ (resp. F') can be extended to a meromorphic function on $R(s) \geq 1$ having no poles except (resp. except possibly) for a simple pole with residue $c^+ > 0$ (resp. c).

Then $\sum_{m \leq n} a_m = c\, n + o(n)$.

Apply 2.1.8 choosing $F(s)$, $F^+(s)$ such that:

$a_n = a_n^+ = 0$ if $Nv \neq n$ for all v,

$a_n = r\, \chi(x_v) \log Nv$ where r is the number of v such that $Nv = n$ and d is the degree of the representation.

Then $F(s) = L^{1}/L - \phi(s) = - \sum_{v \in \Sigma} \frac{\chi(x_v) \log Nv}{(Nv)^s}$

$$F^{+}(s) = d \sum_{v \in \Sigma} \frac{\log Nv}{(Nv)^s}$$

as can be seen from a simple rearrangement.

From 2.1.6 and 2.1.7, F(s) verifies hypothesis c) of 2.1.8.

From the theory of compact groups (cf [12]) $\chi(x_v)$ is the sum of d complex numbers of absolute value one: hence $|\chi(x_v)| \leq d$.

Hence $|a_n| = r\,|\chi(x_v)|\,|\log Nv| \leq rd\,|\log Nv| = |a_n^{+}|$.

From 2.1.7 the residue of F(s) at s = 1 is c_χ.

From 2.1.5 $F^{+}(s)$ converges for R(s) > 1.

Applying 2.1.8:

$$\sum_{Nv \leq n} \chi(x_v) \log Nv = c_\chi\, n + o(n)$$

We state the following without proof cf [7]:

Theorem 2.1.9 (Abel summation trick)

Let $b_n \in \mathbb{C}$, $n \geq 2$

and $\Psi(N) = \sum_{n=2}^{N} b_n = \alpha N + o(N)$, $\alpha \in \mathbb{C}$, $N \in \mathbb{N}$.

Then $\sum_{n=2}^{N} b_n/\log n = \frac{\alpha N}{\log N} + o\left(\frac{N}{\log N}\right)$

Applying this to the preceeding result:

$$\sum_{Nv \leq n} \chi(x_v) = \frac{c_\chi n}{\log n} + o\left(\frac{n}{\log n}\right)$$

which is theorem 2.1.3 b).

In particular if $\chi = 1$,

we get theorem 2.1.3 a):

$$\sum_{Nv \leq n} 1 = n/\log n + o\,(n/\log n).$$

2.2 A sufficient condition for equidistribution.

We now have in hand all the information we need to get the theorem which is the "raison d'être" of this thesis: a theorem providing us with a sufficient condition for equidistribution, from which result applied to a finite Galois group the whole chapter 3 will follow. All through this section our terminology from 2.1 holds. We keep assumptions 2.1.1 and 2.1.2 but we complete them with:

Assumption 2.2.1 There exists a constant C such that for every integer n the number of $v \in \Sigma$ with $Nv = n$ is less than or equal to C.

We cannot talk about equidistribution yet since the sequence we are looking at, namely $(x_v)_{v \in \Sigma}$, is not indexed by natural numbers as in our definition 1.1.1. But we do have a function

$$\Sigma \to \mathbb{N} - \{0, 1\}$$
$$v \to Nv$$

that will enable us to do so, with the help of assumption 2.2.1.

Rearrange the elements of Σ as a sequence $(v_i)_{i \geq 1}$ so that $i \leq j$ implies $Nv_i \leq Nv_j$. In general this is possible in many ways: if there are many v's mapped to a given n, one can permute them. Let $(v_i)_{i \geq 1}$ and $(w_i)_{i \geq 1}$ be two such rearrangements and let $f \in C(X)$, $K = \sup_{x \in X} |f(x)|$ since f is continuous on a compact. For any n the sets $\{v_i\}_{i=1,\ldots n}$

and $\{w_i\}_{i=1,\ldots,n}$ are the same except possibly for at most C elements, namely v_i's and w_i's for which Nv_i and Nw_i are maximal. Therefore:

$$\left| \frac{1}{n} \sum_{i=1}^{n} f(x_{v_i}) - \frac{1}{n} \sum_{i=1}^{n} f(x_{w_i}) \right| \leq \frac{2CK}{n} \to 0.$$

We just proved:

Lemma 2.2.2 $\left(\frac{1}{n} \sum_{i=1}^{n} f(x_{v_i}) \right)_{n \geq 1}$ and $\left(\frac{1}{n} \sum_{i=1}^{n} f(x_{w_i}) \right)_{n \geq 1}$ converge (to the same limit) or diverge simultaneously where $(v_i)_{i \geq 1}$ and $(w_i)_{i \geq 1}$ are rearrangements of Σ of the type described above: one can then talk about equidistribution for $(x_v)_{v \in \Sigma}$.

The next lemma will make that even clearer:

Lemma 2.2.3 Let $(v_i)_{i \geq 1}$ be a suitable rearrangement of Σ and take $f \in C(X)$ with $K = \sup_{x \in X} |f(x)|$.

Then $$\lim_{n \to \infty} \left| \frac{\sum_{Nv \leq n} f(x_v)}{\sum_{Nv \leq n} 1} - \frac{\sum_{i=1}^{n} f(x_{v_i})}{n} \right| = 0.$$

Proof: put $k_n = \sum_{Nv \leq n} 1$,

$$s_n = \left(\sum_{i=1}^{n} f(x_{v_i}) \right) / n$$

and $$s_{k_n} = \frac{\sum_{Nv \leq n} f(x_v)}{\sum_{Nv \leq n} 1}.$$

Obviously $(s_{k_n})_{n \geq 1}$ is a subsequence of $(s_n)_{n \geq 1}$.

Clearly for every n there exists a unique m such

$$k_m \leq n < k_{m+1}.$$

Define $s_n^1 = s_{k_m}$; then $|s_n^1 - s_{k_n}| \to 0$

since $(s_n^1)_{n \geq 1}$ is obtained from $(s_{k_n})_{n \geq 1}$

by repeating every value of it at most C times after itself.

Now if $n = k_m$ (note that $n - k_m \leq C$)

$$|s_n^1 - s_n| = |s_{k_m} - s_n| = \frac{|n \sum_{i=1}^{k_m} f(x_{v_i}) - k_m \sum_{i=1}^{n} f(x_{v_i})|}{nk_m}$$

$$\leq \frac{(n - k_m) \left| \sum_{i=1}^{k_m} f(x_{v_i}) \right| + k_m \left| \sum_{i=k_m+1}^{n} f(x_{v_i}) \right|}{nk_m}$$

$$\leq \frac{Ck_m K + k_m CK}{nk_m} \to 0.$$

Now $|s_n - s_{k_n}| \leq |s_n^1 - s_{k_n}| + |s_n^1 - s_n| \to 0.$

Q.E.D.

We can now proceed to our "new" definition of equidistribution:

Definition 2.2.4 With the notation of 2.1, under the assumption 2.2.1, we say that $(x_v)_{v \in \Sigma}$ is μ-equidistributed if

$$\lim_{n \to \infty} \frac{\sum_{Nv \leq n} f(x_v)}{\sum_{Nv \leq n} 1} = \mu(f) \text{ for any } f \in C(X).$$

In the light of the above definition and under assumptions 2.1.1 and 2.1.2 we state:

Theorem 2.2.5 The sequence $(x_v)_{v \in \Sigma}$ is μ-equidistributed in X with $\mu(\chi) = c_\chi$ for any irreducible character χ of G.

Proof: theorem 2.1.3 a) and b) yields:

$$\lim_{n \to \infty} \frac{\sum_{Nv \leq n} \chi(x_v)}{\sum_{Nv \leq n} 1} = c_\chi$$

Applying theorem 1.3.2, $\mu(\chi) = c_\chi$.

Q.E.D.

We get as an immediate corollary (by 1.3.3):

Theorem 2.2.6 The elements x_v, $v \in \Sigma$ are equidistributed with respect to the normalized Haar measure if and only if $c_\chi = 0$ for every irreducible character $\chi \neq 1$ of G, if and only if, the L-functions relative to the non trivial irreducible characters of G are holomorphic and non zero at $s = 1$.

2.3 The case of a finite Galois group.

We want now to apply our sufficient condition for equidistribution (cf section 2.2) to the case of a finite Galois group. The results needed for this have standard, sometimes long and technical proofs: we will restrict ourselves to outlines and references not to make this section too heavy. The techniques required for our purpose resort to basic algebraic number theory, Galois theory, the theory of complex variables (mainly analytic continuation), the theory of representations of groups. We assume all the basic concepts of algebraic number theory (cf [1], [7] or [10]).

Let G be the Galois group of a finite normal extension K of the number field k (cf [10]). We first want to define a topology on G. In the general case where G can be infinite, Artin defines the following quantities (see [1], page 104): given $\tau \in G$ and a subfield E of K, put

$N_E(\tau) = \{\omega \in G \text{ with } \tau(e) = \omega(e) \text{ for any } e \in E\}$.

Intuitively $N_E(\tau)$ is a typical open neighborhood of τ, for any τ. In an obvious manner one can build a topology which turns out to be the discrete topology (i.e. the set of all the subsets of G) in the finite case.

This motivates the following trivial result which we do not prove.

Lemma 2.3.1 Under our hypothesis, the set of all the subsets of G is a compact Hausdorff topology making the characters of G continuous.

Denote by Σ the set of unramified prime ideals of k. Define:

$$N: \Sigma \to \mathbb{N} - \{0,1\}$$

$$v \to N_{k/Q}(v) = \text{card}(A_k/v)$$

where $N_{k/Q}$ is the "norm" function, Q is the set of rational numbers, card is the "cardinality" function and A_k is the ring of integers of k.

Lemma 2.3.2 There exists a constant C such that $\sum_{Nv=n} 1 \leq C$ for all n.

Note that this implies that Σ is denumerable since card $\Sigma \leq C$ card $\mathbb{N}$.

Proof: A most clear reference for this proof is: Algebraic Numbers, Paulo Ribenboim, Wiley, 1972, pages 109, 122, 162, 164. Since A_k is a Dedekind ring, any non-zero prime v is maximal and A_k/v is a finite field. Hence $Nv = p^s$ where p is a prime and $s \geq 1$ an integer. If n cannot be put under the form p^s, then $\sum_{Nv=n} 1 = 0$.

If $n = p^s$, let $A_k p = \prod_{i=1}^{g} v_i^{e_i}$ be the decomposition of $A_k p$ into

product of prime ideals v_i of A_k. Let m be the degree of the extension k of Q. Then by the properties of the norm (which we accept) we have (see Ribenboim page 164):

$$N(A_k p) = p^m = \prod_{i=1}^{g} (N(v_i))^{e_i}.$$

Denote $N(v_i) = p^{f_i}$. We have $m = \sum_{i=1}^{g} e_i f_i$ and the number of f_i equal to s is less than or equal to m. Since all the v's such that $Nv = p^s$ must appear in the above decomposition (they must divide $A_k p^s$ by Ribenboim, page 122), their number is less than or equal to m. Taking C = m finishes the proof.

We have not yet described the sequence $(x_v)_{v \in \Sigma}$ we want to consider. Fix $v \in \Sigma$. Let w be a prime factor of v in K. Put $D = \{\tau \in G, \text{ with } \tau(w) = w\}$; D is called the decomposition subgroup of w. It is known (cf [10], page 107 or [7], page 57) that D is cyclic and that there exists a generator τ with $\tau(a) - a^n \in w$ for all $a \in A_K$, where A_K is the ring of the integers of K, $n = \text{card}(A_k/v)$. τ is called the Frobenius automorphism of w, is denoted $(w, K/k) = \tau$. It is then easy to show $(\omega(w), K/k) = \omega(w, K/k)\omega^{-1}$ for any $\omega \in G$.

Let w_1, w_2 be two prime factors of v in K. Then there exists ω, with $\omega(w_1) = w_2$. Calling x_v the conjugacy class of the Frobenius automorphism of any of the prime factors of v in K, we get a sequence of elements of X (the space of conjugacy classes of G) which we denote $x_v = (v, K/k)$.

Our L-functions will take the following shape:

$$L(s, \rho) = \prod_{v \in \Sigma} \frac{1}{\det\left(1 - \rho\left((v, K/k)\right) N_{k/Q}(v)^{-s}\right)} .$$

Now comes the biggest gap in this thesis: the proofs for assumptions 2.1.1 and 2.1.2 are clearly beyond the scope of our work and we have also to prove that $c_\chi = 0$ for all irreducible $\chi \neq 1$. We do provide an outline though. One first considers the case where G is abelian: using analytic continuation, a functional equation (cf [3], page 209) one gets the result.

There is still a problem with the abelian case though: the reference just given uses "Dirichlet characters", defined on prime ideals of k and not on elements of G as we would need to be consistent with our exposure. A first way to deal with this is to show the relation between the two kinds of characters, using a "reciprocity law" (see [3], page 165). Another way is to go back to Artin's original work (see [2], page 113, in German) where the "right kind" of characters is used.

The general case has to be brought down to the abelian case (see [2], pages 105-121 or [3], pages 218-225). One first shows that $L(s, \rho)$ really depends only on the character χ attached to ρ (see [3], page 220). Accordingly from now on we will talk about $L(s, \chi)$. Suppose Ω is an intermediate field between K and k, normal over k. Let $H = Gal(K/\Omega)$. Then it can be proved that

$$L(s, \chi, K/k) = L(s, \chi, \Omega/k)$$

where χ is a character of G/H that can be regarded as a character of G and where the notation for the L-series has been completed in an obvious manner. With the above result, one can break $L(s, \chi)$ into a product of L-functions attached to abelian characters (see [3], page 225).

All the considerations of this section merely put us in the hypothesis of theorems 2.2.5 and 2.2.6: all together we get:

Theorem 2.3.3 The conjugacy classes of a finite Galois group containing Frobenius automorphisms are equidistributed with respect to the Haar measure.

CHAPTER 3

EQUIDISTRIBUTION

APPLIED TO

NUMBER

THEORY

3.1 The prime number theorem.

Before applying the powerful result of section 2.3 to get density theorems, we make explicit the statement of the prime number theorem hidden in the preceeding pages.

Let everything be as at the beginning of section 2.3 except that we particularize in the following way: we put $K = k = Q$. Then $v \in \Sigma$ means $v = Zp$ where p is a prime number in the ring Z of the integers and $N_{Q/Q}(v) = \text{card } Z/Zp = p$. Put $\pi(n) = \sum_{Nv \leq n} 1$: $\pi(n)$ is the number of primes of Z less then or equal to n. From the fact that theorem 2.1.3 a) holds in this case, we get:

Theorem 3.1.1 (The prime number theorem)

$$\lim_{n \to \infty} \frac{\pi(n) - n/\log n}{n/\log n} = 0.$$

3.2 The notion of density.

Let k be a number field and Σ_k, the set of its prime ideals v. Put $Nv = N_{k/Q}(v) = \text{card } A_k/v$ where A_k is the ring of integers of k. For any $P \subseteq \Sigma_k$, and n natural $n \geq 2$, $a_n(P)$ stands for the number of $v \in P$ such that $Nv \leq n$.

Definition 3.2.1 $P \subseteq \Sigma_k$ is said to have density a, where a is real, if

$$\lim_{n \to \infty} \frac{a_n(P)}{a_n(\Sigma_k)} = a.$$

It is seen at once that a finite set has measure zero. As easily we notice that for $k = Q$ the density has a very intuitive meaning, since it gives "the number of elements in a set of primes of Z divided by the number of primes in Z" (so to speak): this density is called natural following the terminology in Serre, Cours d'Arithmétique, Presses Universitaires de France, 1970, page 126. In the same book, on page 121 another kind of density, called "analytical", is defined:

Definition 3.2.2 $P \subseteq \Sigma_Q$ has density a where a is real if

$$\left(\sum_{p \in P} \frac{1}{p^s}\right) \Big/ \left(\log \frac{1}{s-1}\right) \text{ tends to a as s tends to 1 in the}$$

complex plane.

It can be shown that if P has a natural density, it has the same analytical density but the converse is not true.

For different uses of the word "density", in a close way to ours though, see Addition Theorems, Mann, Wiley, 1965: there are defined the Schnirelmann, Besicovitch and asymptotic densities, the first two using the "greatest lower bound" function. Our

concern in the next sections will be about the "natural" density.

3.3 Cebotarev's theorem.

We now prove a fairly general density theorem, assuming the conditions of section 2.3.

Let G be the Galois group of the finite extension K of the number field k. Let Y be a subset of G, stable by conjugation. Denote by P_Y the set of the prime ideals v of k, unramified in K such that the Frobenius class x_v is contained in Y (cf [7] or [10] or 2.3). Then

Theorem 3.3.1 (Cebotarev's theorem)

$$\lim_{n\to\infty} \frac{a_n(P_Y)}{a_n(\Sigma_k)} = \frac{\text{card } Y}{\text{card } G}$$

according to the notation introduced in 3.2, i.e. P_Y has "natural density" (cf 3.2.1) $\frac{\text{card } Y}{\text{card } G}$.

Proof:

The characteristic function χ_Y of Y is trivially continuous since the topology of G is discrete.

Applying 2.3.3 we get:

$$\lim_{n\to\infty} \frac{\sum_{Nv\leq n} \chi_Y(x_v)}{\sum_{Nv\leq n} 1} = \mu(\chi_Y)$$

where μ is the normalized Haar measure on G, and v varies through the set Σ of all the unramified prime ideals of k. Since there are only a finite number of ramified primes (see [7] or [10]), our result still holds if v runs througn Σ_k, the set of all prime ideals of k.

Conclusion: describing the Haar measure.

If $v \in \Sigma_k$, then v belongs to P_Y if and only if $x_v \subseteq Y$ and v is unramified. Thus: $\sum_{Nv \leq n} \chi_Y(x_v) = a_n(P_Y)$ and $\sum_{Nv \leq n} 1 = a_n(\Sigma_k)$.

On the other hand for any part P of G define $\mu(P) = \frac{\text{card } P}{\text{card } G}$.

Viewed as a set function this is clearly a translation invariant measure on G with $\mu(G) = 1$: by unicity of the normalized Haar measure, μ is the measure and χ_Y being continuous we have

$$\mu(\chi_Y) = \frac{\text{card } Y}{\text{card } G}.$$

Combining these results with "step 2" we get:

$$\lim_{n \to \infty} \frac{a_n(P_Y)}{a_n(\Sigma_k)} = \frac{\text{card } Y}{\text{card } G}.$$

Q.E.D.

3.4 Dirichlet's theorem on primes in an arithmetic progression.

From section 3.3 we now get a most classical result.

Theorem 3.4.1 (Dirichlet's theorem).

Let a, m be relatively prime integers of Q (the rational numbers).

Then there exists infinitely many primes p congruent to a mod m.

More precisely the density of the set of the prime ideals generated by the p's is $\frac{1}{\phi(m)}$ where ϕ is the Euler function

(cf [4]).

Proof: a good reference for the terminology we need now is van der Waerden, Modern Algebra, Frederick Ungar Pub. Co., 1964, page 160. Let ζ be a primitive m-th root of unity and let $Q(\zeta)$ be the smallest field containing Q and ζ. In reference to 3.3 put $K = Q(\zeta)$, $k = Q$ and $Y = \{\tau_a\}$ where $\tau_a(\zeta) = \zeta^a$. Since G is abelian (cf van der Waerden, page 162), Y is stable by conjugation, each class consisting of one element.

Now every $\tau \in G$ is of the form:

$\tau(\zeta) = \zeta^b$ where b and m are relatively prime.

Let us identify the Frobenius automorphism, to be denoted τ_p, attached to a prime p unramified in K (unicity coming from the fact that G is abelian). According to the definition

given right after lemma 2.3.2 we must have:

$$\tau_p(a) - a^n \in w \quad \text{for any} \quad a \in A_K$$

where w is a prime factor of p in $Q(\zeta)$,

$$n = \text{card}(A_k/p) = p,$$

A_K is the ring of integers of $K = Q(\zeta)$.

Since ζ is an integer and $0 \in w$,

$$\tau_p(\zeta) - \zeta^p = 0 \quad \text{meets our needs.}$$

The above equation in turn defines an element of G when p and m are relatively prime.

Since the Frobenius element of a prime p unramified in K is assumed to exist, it takes the form we just derived.

The P_Y of theorem 3.3.1 becomes the set of prime numbers p relatively prime to m such that $\tau_p = \tau_a$, i.e. the set of prime numbers congruent to a mod m.

Applying 3.3.1 we get:

The density of P_Y is $\frac{\text{card } Y}{\text{card } G} = \frac{1}{\phi(m)} \neq 0.$

Hence P_Y is infinite and we get our most classical result.

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