Measurable combinatorics and orbit equivalence relations

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Abstract

This thesis comprises three research projects.

In Chapter 2 we study the problem of complexity of orbit equivalence relations induced by Borel actions of Polish groups. We prove that the homeomorphism relation of locally connected continua is complete. This answers a question of Chang and Gao [2].

In Chapter 3 we are interested in measurable equidecompositions. Our main result provides a combinatorial condition for measurable equidecomposability a.e. of equidistributed sets with respect to actions of finitely generated abelian groups. This confirms a special case of Gardner's conjecture [18]. As a corollary we obtain a generalization of a result of Grabowski, Máthé and Pikhurko on measurable circle squaring [20].

In Chapter 4 we consider the problem of lifting invariant measures. We give a fairly general sufficient condition for lifts to exist. This answers a question of Feliks Przytycki.

Abrégé

Cette thèse comprend trois projets de recherche.

Dans le chapitre 2, nous étudions le problème de complexité des relations d'équivalences orbitales, induites par des actions boréliennes des groupes polonais. Nous prouvons que la relation d'homéomorphisme des continus localement connexes est la relation complète. Cela rèpond au question de Chang et Gao [2].

Dans le chapitre 3, nous nous intéressons aux équidécompositions mesurables. Le résultat principal fournit une condition combinatoire pour l'équidécomposabilité mesurable presque partout des ensembles équidistrubués par rapport aux actions des groupes abéliens de type fini. Cela confirme le cas special d'une conjecture de Gardner [18]. En conséquence nous obtenons une généralisation du résultat de Grabowski, Máthé and Pikhurko sur la quadrature mesurable du cercle [20].

Dans le chapitre 4, nous considérons le problème du relèvement des mesures invariantes. Nous donnons une condition assez générale pour qu'un relèvement existe. Cela répond à une question de Feliks Przytycki.

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Contents

Preface

Descriptive set theory studies "definable" sets, i.e. sets which can be "nicely" described in terms of simpler objects. For instance, Borel sets are the sets that can be defined from open sets using the operations of complements and countable unions. There are also other classes of definable sets such as analytic sets (continuous images of Borel sets), coanalytic sets (complements of analytic sets), or sets measurable with respect to a Borel measure. Descriptive set theory has a lot of interesting applications in other branches of mathematics such as topology, functional analysis, measure theory, ergodic theory and mathematical logic.

This thesis comprises three research projects in which we use descriptive set theoretic methods. We study orbit equivalence relations, measurable equidecompositions and lifts of invariant measures.

In Chapter 1 we present necessary background which is beyond the scope of standard undergraduate courses. Section 1.1 is a brief introduction to descriptive set theory. We focus only on notions that are used in later chapters. For a comprehensive introduction to descriptive set theory we refer the interested reader to [24]. In Subsection 1.1.1 we discuss Polish spaces. Subsection 1.1.2 covers hyperspaces of compact subsets of Polish spaces. Subsection 1.1.3 provides basic information on Borel sets. In Subsection 1.1.4 we define the space of probability measures on a Polish space.

In Section 1.2 we define orbit equivalence relations and the notion of reducibility of equivalence relations. Section 1.3 is an introduction to decomposition theory. In Section 1.4 we discuss the notion of amenability. Section 1.5 is devoted to Mokobodzki's medial means.

Next chapters contain the author's contribution to mathematics. Results presented in Chapters 2 and 4 are based on the author's solo papers [3] and [4], respectively. In Chapter 3 we present results from a joint paper of the author and Sabok [5].

The main result of Chapter 2 is that the homeomorphism relation of locally connected continua is a complete orbit equivalence relation in the class of orbit equivalence relations induced by actions of Polish groups. This answers a question of Chang and Gao [2] in the affirmative.

In Chapter 3 the problem of measurable equidecompositions is studied. The main result states that for free pmp actions of finitely generated abelian groups the existence of measurable equidecompositions a.e. of equidistributed sets is equivalent to the Hall condition.

In Chapter 4 we consider Feliks Przytycki's question concerning lifting of invariant measures. We prove that invariant measures admit invariant lifts in a broad context of amenable group actions.

Chapter 1

Preliminaries

1.1 Basics of descriptive set theory

1.1.1 Polish spaces

Definition 1.1. We say that a topological space is *Polish* if it is separable and completely metrizable.

The simplest examples of Polish spaces are finite and countably infinite discrete spaces, such as $\{0, 1\}$, $\{0, 1, 2, \ldots, n-1\}$, $\omega = \{0, 1, 2, 3, \ldots\}$. Other familiar examples of Polish spaces are: the real line $\mathbb{R}$, the circle $\mathbb{T}$, the closed unit interval $[0, 1]$, separable Banach spaces.

The class of Polish spaces is closed under finite and countable products, so in particular the following spaces are Polish: $\mathbb{R}^n$, $[0, 1]^n$ where $n \in \omega$, $\{0, 1\}^\omega$, $[0, 1]^\omega$, ω^ω, $\mathbb{R}^\omega$. The space $\{0, 1\}^\omega$ is called *the Cantor space* and is homeomorphic to the ternary Cantor set $\left\{\sum_{k=1}^{\infty} \varepsilon_k \cdot 3^{-k} \colon \varepsilon_k \in \{0, 2\}\right\}$. The space ω^ω is called *the Baire space* and is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$ with Euclidean topology; for this reason sequences of non-negative integers are sometimes referred to as irrationals. The space $[0,1]^\omega$ is called *the Hilbert cube.* It has a remarkable property: all metrizable separable spaces homeomorphically embed into the Hilbert cube.

It is worth pointing out that the class of Polish spaces is not closed under taking subsets. For example, $\mathbb{Q}$ with the usual topology is not a Polish space. Indeed, note that $\mathbb{Q} = \bigcup_{q\in\mathbb{Q}}\{q\}$ and the sets $\{q\}$ are closed and nowhere dense in $\mathbb{Q}$. Hence the Baire category theorem fails for $\mathbb{Q}$ and therefore $\mathbb{Q}$ is not a Polish space.

One can give a full characterization of subspaces of a Polish space which are Polish: if X is a Polish space and $Y \subset X$ is its subspace then Y is Polish if and only if it Y is a G_δ subset of X (i.e. Y is the intersection of countably many open sets). This result is known as Alexandrov's theorem.

What are possible cardinalities of Polish spaces? Clearly it can be any finite number and $\aleph_0$ as witnessed by discrete Polish spaces. It turns out that the cardinality of any uncountable Polish space is $\mathfrak{c}$. The upper bound is easy to prove: if $\{x_n : n \in \omega\}$ is dense in X then every $x \in X$ is the limit of some sequence of the form $x_{n_0}, x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ and there are only $\aleph_0^{\aleph_0} = \mathfrak{c}$ such sequences. This shows that if X is Polish then $|X| \leq \mathfrak{c}$. The lower bound follows from the following interesting fact.

Proposition 1.2. *If X is an uncountable Polish space then the Cantor space embeds homeomorphically in X.*

Sketch of proof. Fix a compatible complete metric d on X bounded by 1. Construct a Cantor scheme, i.e. a family of sets A_s indexed by $s \in \{0,1\}^{<\omega}$ with the following properties:

- A_s is an uncountable open set,
- $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$,
- $\overline{A_{s^\frown i}} \subset A_s$,

- $\operatorname{diam}(A_s) \leq 2^{-\operatorname{lh}(s)}$.

We let $A_{\langle\ \rangle} = X$. If A_s was already defined, choose two distinct points $x, y \in A_s$ so that their neighbourhood bases consist of uncountable sets. Choose $r > 0$ so that

$$r < \min\left\{\frac{1}{2}d(x,y), \operatorname{dist}(x, X \setminus A_s), \operatorname{dist}(y, X \setminus A_s), 2^{-\operatorname{lh}(s)-2}\right\}$$

and define $A_{s^\frown 0} = B(x,r)$, $A_{s^\frown 1} = B(y,r)$.

By construction, for all $z = \langle z_0, z_1, z_2, \ldots\rangle \in \{0,1\}^\omega$ the set $\bigcap_{n\in\omega} A_{\langle z_0, z_1, \ldots, z_n\rangle}$ consists of a single point, which we shall denote by x_z. Then $f\colon \{0,1\}^\omega \to X$ given by $f(z) = x_z$ embeds homeomorphically the Cantor space into X. □

1.1.2 The hyperspace of compact sets

Given a topological space X denote by $K(X)$ the collection of all compact subsets of X. We endow $K(X)$ with Vietoris topology, i.e. with the topology generated by sets of the form

$$\{K \in K(X)\colon K \cap U \neq \emptyset\} \quad \text{and} \quad \{K \in K(X)\colon K \subset U\}$$

where $U \subset X$ is open.

If $D \subset X$ is dense in X then $\{K \subset D\colon K \text{ is finite}\}$ is dense in $K(X)$. In particular, if X is separable then so is $K(X)$.

Suppose now that X is metrizable. Fix a compatible metric d on X bounded by 1. If $x \in X$ and $A \subset X$, define $d(x,A) = \inf\{d(x,a)\colon a \in A\}$. For $K_1, K_2 \in K(X)$ define $\delta(K_1, K_2) = \max_{x\in K_1} d(x, K_2)$ and

$$d_H(K_1,K_2) = \begin{cases} 0 & \text{if } K_1 = K_2, \\ 1 & \text{if } K_1 = \emptyset \neq K_2 \text{ or } K_1 \neq \emptyset = K_2, \\ \max(\delta(K_1,K_2), \delta(K_2,K_1)) & \text{otherwise.} \end{cases}$$

Then d_H is a metric on $K(X)$ called the Hausdorff metric. The Hausdorff metric is compatible with Vietoris topology. Hence $K(X)$ equipped with Vietoris topology is metrizable. Moreover, if d is complete then d_H is also complete.

It follows that if X is Polish then $K(X)$ is Polish as well. Moreover, if X is a compact Polish space then so is $K(X)$.

We conclude this subsection by stating and proving [24, 4.29 v)]. We shall use it later in the proof of the main result of Chapter 4.

Proposition 1.3. *Let X be metrizable. Let $\mathcal{K} \subset K(X)$ be a compact subset of $K(X)$. Then the set $\bigcup \mathcal{K}$ is a compact subset of X.*

Proof. Let $\mathcal{U}$ be an open cover of $\bigcup \mathcal{K}$. Note that $\mathcal{U}$ is an open cover of K for any $K \in \mathcal{K}$. By compactness of K, there exists a finite collection $\mathcal{U}' \subset \mathcal{U}$ so that $K \subset \bigcup \mathcal{U}'$. It follows that the collection

$$\left\{ \left\{ K \in K(X) \colon K \subset \bigcup \mathcal{U}' \right\} : \mathcal{U}' \text{ is a finite subset of } \mathcal{U} \right\}$$

is an open cover of $\mathcal{K}$. By compactness of $\mathcal{K}$, there exist finite subsets $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of $\mathcal{U}$ so that $\{\{K \in K(X) \colon K \subset \bigcup \mathcal{U}_i\} : i = 1, 2, \ldots, n\}$ is an open cover of $\mathcal{K}$. Then $\bigcup \mathcal{K}$ is covered by $\bigcup_{i=1}^{n} \mathcal{U}_i$, which is a finite subcover of $\mathcal{U}$. Hence $\bigcup \mathcal{K}$ is compact. □

1.1.3 Borel sets

Given a topological space X we denote by $\mathcal{B}(X)$ the σ-algebra generated by open sets. This means: $\mathcal{B}(X)$ is the smallest family of subsets of X containing all open sets which is closed under taking complements and countable unions. The elements of $\mathcal{B}(X)$ are called Borel subsets of X.

Note that we do not obtain any "new" Borel sets (i.e. sets that are not open) by taking countable union of open sets. This is because open sets are closed under arbitrary unions.

However, in general the complement of an open set does not need to be open. This yields Borel subsets of X different from the open sets. The complements of open sets are called closed sets. Note that complements of closed sets are open so this does not define "new" Borel sets. However, countable union of closed sets usually is neither open neither closed. This leads to "new" Borel sets: countable unions of closed sets (these are called F_σ sets). Continuing this process, one can define G_δ sets — the complements of F_σ sets (which by de Morgan laws happen to be countable intersections of open sets). Countable unions of G_δ sets are called $F_{\sigma\delta}$ sets, complements of $F_{\sigma\delta}$ sets are called $G_{\delta\sigma}$ sets, and so on. Continuing this process we eventually obtain all Borel sets. Note that this process has to go for more than ω steps since in general the countable union of a G_δ set, $G_{\delta\sigma}$ set, $G_{\delta\sigma\delta}$ set, $G_{\delta\sigma\delta\sigma}$ set, $\ldots$ is a "new" Borel set. We make it precise in forthcoming paragraphs.

For all ordinals $1 \leq \alpha < \omega_1$ we define by transfinite recursion classes $\mathbf{\Sigma}^0_\alpha(X)$ and $\mathbf{\Pi}^0_\alpha(X)$ of subsets of X. We denote by $\mathbf{\Sigma}^0_1(X)$ the class of open sets. If $\mathbf{\Sigma}^0_\alpha(X)$ is already defined, we let

$$\mathbf{\Pi}^0_\alpha(X) = \left\{X \setminus A \colon A \in \mathbf{\Sigma}^0_\alpha(X)\right\}.$$

If for all $\beta < \alpha$ classes $\mathbf{\Pi}^0_\beta(X)$ are defined, we let

$$\mathbf{\Sigma}^0_\alpha(X) = \left\{\bigcup_{n\in\omega} A_n \colon \forall n \in \omega\ A_n \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)\right\}.$$

Hence, $\mathbf{\Pi}^0_1(X)$ is the class of closed sets, $\mathbf{\Sigma}^0_2(X)$ is the class of F_σ sets, $\mathbf{\Pi}^0_2(X)$ is the class of G_δ sets, and so on.

Since for all $A \in \mathbf{\Pi}^0_\alpha(X)$ one has $A = \bigcup_{n\in\omega} A_n$ where $A_n = A$ for all $n \in \omega$, it follows that $A \in \mathbf{\Sigma}^0_{\alpha+1}(X)$, i.e. $\mathbf{\Pi}^0_\alpha(X) \subset \mathbf{\Sigma}^0_{\alpha+1}$. Consequently, if $A \in \mathbf{\Sigma}^0_\alpha(X)$ then $X \setminus A \in \mathbf{\Pi}^0_\alpha(X)$, so $X \setminus A \in \mathbf{\Sigma}^0_{\alpha+1}(X)$, and $A = X \setminus (X \setminus A) \in \mathbf{\Pi}^0_{\alpha+1}(X)$. So $\mathbf{\Sigma}^0_\alpha(X) \subset \mathbf{\Pi}^0_{\alpha+1}(X)$. This shows that $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha(X)$. Note that

$$\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha(X).$$

The $\supset$ inclusion follows from the observation that every element of $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X)$ can be obtained by applying countably many operations of taking countable unions and intersections starting from the open sets. The $\subset$ inclusion follows from the observation that $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X)$ is a σ-algebra of subsets of X containing all open sets.

Note that if X is Polish and $|X| \leq \aleph_0$ then $\mathcal{B}(X) = \mathcal{P}(X)$, i.e. every subset of X is Borel. The reason is that the singletons are closed, so all subsets of X already appear in $\mathbf{\Sigma}^0_2(X)$. However, if X is an uncountable Polish space then $\mathcal{B}(X) \neq \mathcal{P}(X)$. Surprisingly, all uncountable Polish spaces share the same Borel structure. This means that if X and Y are uncountable Polish spaces then there is a Borel bijection $f\colon X \to Y$ so that $A \in \mathcal{B}(X)$ if and only if $f(A) \in \mathcal{B}(Y)$. This motivates the following definition.

Definition 1.4. Let X be a set and Σ be a σ-algebra of subsets of X. We say that (X, Σ) is a *standard Borel space* if there exists a Polish topology on X so that Σ is its Borel σ-algebra.

So, any two standard Borel spaces are isomorphic. The notion of standard Borel space is very useful. For instance, if we only care about the Borel structure we can impose a topology which is most convenient for us: we may want to work with the Cantor space or with the Baire space or with $[0,1]$ or any other Polish space to make things simpler.

1.1.4 Space of measures

Let $\mathfrak{M}$ be a σ-algebra of subsets of a set X. A function $\mu\colon \mathfrak{M} \to [0, +\infty]$ is called a measure on $(X, \mathfrak{M})$ if $\mu(\emptyset) = 0$ and μ is countably additive, i.e. whenever $A_0, A_1, A_2, A_3, \ldots \in \mathfrak{M}$ are pairwise disjoint sets, $\mu(\bigcup_{n=0}^\infty A_n) = \sum_{n=0}^\infty \mu(A_n)$. If μ is a measure on $(X, \mathfrak{M})$ then we let $\mathcal{N}(\mu)$ denote the σ-ideal of μ-null sets, i.e.

$$\mathcal{N}(\mu) = \{N\colon N \subset A \text{ for some } A \in \mathfrak{M} \text{ with } \mu(A) = 0\}.$$

A set of the form $A \triangle N$ with $A \in \mathfrak{M}$, $N \in \mathcal{N}(\mu)$ is called μ-measurable. (Recall that $\triangle$ denotes symmetric difference, $X \triangle Y = (X \setminus Y) \cup (Y \setminus X) = (X \cup Y) \setminus (X \cap Y)$.) We denote the collection of μ-measurable sets by $\Sigma(\mu)$. Note that $\Sigma(\mu)$ is a σ-algebra.

If X is a Polish space and μ is a measure on $(X, \mathcal{B}(X))$ then we say that μ is a Borel measure on X. Recall that μ extends uniquely to a measure $\tilde{\mu}$ on $(X, \Sigma(\mu))$ and that $\tilde{\mu}$ satisfies $\mathcal{N}(\tilde{\mu}) = \mathcal{N}(\mu)$ and $\Sigma(\tilde{\mu}) = \Sigma(\mu)$. The measure $\tilde{\mu}$ is called the completion of μ. Usually the tilde is skipped and the same symbol is used to denote both a Borel measure and its completion.

Given a compact Polish space X, denote by $P(X)$ the collection of all Borel probability measures on X. We equip $P(X)$ with the topology of pointwise convergence on $C(X) = \{f \colon X \to \mathbb{R} \colon f \text{ is continuous}\}$, i.e., with the topology whose basic neighbourhoods of $\mu \in P(X)$ are the sets

$$\left\{ \nu \in P(X) \colon \left| \int_X f_j d\mu - \int_X f_j d\nu \right| < \varepsilon \text{ for } j = 1, 2, \ldots, n \right\}$$

where $f_1, f_2, \ldots, f_n \in C(X)$. Due to the Riesz representation theorem, there is an identification of Borel probability measures on X with $M_1^+(X)$, the space of non-negative functionals φ on $C(X)$ so that $\varphi(\chi_X) = 1$. This identification is a homeomorphism when $M_1^+(X)$ is equipped with the weak* topology.

It turns out that if X is a compact Polish space then so is $P(X)$.

1.2 Orbit equivalence relations

An orbit equivalence relation is an equivalence relation induced by an action of a group, i.e. a relation whose equivalence classes are precisely the orbits of the action. In other words, if a is an action of G on X, then a defines a relation E_a given by $x E_a y \iff \exists g \in G \; x = gy$.

From a descriptive set theoretic point of view, it is interesting to consider orbit equivalence relations induced by actions with some additional properties. For instance, G could be a Polish group (i.e. a group with a Polish topology so that the group operations are continuous), X could be a Polish space or a standard Borel space, and the action could be continuous or Borel.

Let us give some examples of actions of Polish groups.

(i) Any group G acts on itself by left multiplication. The equivalence relation induced by this action consists of a single orbit G.

(ii) Let G be a countable group. Consider $\{0,1\}^G$, the set of binary sequences indexed by the elements of G. The *shift action* of G on $\{0,1\}^G$ is defined by $g(x_h)_{h\in G} = (x_{g^{-1}h})_{h\in G}$.

(iii) View binary sequences $x = (x_0, x_1, x_2, x_3, \ldots) \in \{0,1\}^\omega$ as infinite binary strings $\ldots x_3x_2x_1x_0$. One can define addition of such strings by

$$\ldots x_3x_2x_1x_0 + \ldots y_3y_2y_1y_0 = \ldots z_3z_2z_1z_0$$

where z_n is the n-th digit in binary expansion of the number $\sum_{k=0}^n x_k 2^k + \sum_{k=0}^n y_k 2^k$. This corresponds to addition with carries of binary "numbers" having infinitely many digits. The group $\mathbb{Z}$ acts on $\{0,1\}^\omega$ by adding $\ldots 000001$. This is called *the odometer action.* The orbit of an $x \in \{0,1\}^\omega$ in the orbit equivalence relation induced by the odometer action is $\{y\colon x \text{ and } y \text{ differ on finitely many coordinates}\}$ if x contains infinitely many zeros and ones, and $\{y\colon y \text{ is eventually constant}\}$ otherwise.

(iv) Given a Polish space X let $\mathrm{Homeo}(X) = \{h\colon X \to X\colon h \text{ is a homeomorphism}\}$. Then $\mathrm{Homeo}(X)$ is a Polish group which acts on X via $hx = h(x)$.

Given a relation E on X and a relation F on Y we say that E is reducible to F if there exists a function $f\colon X\to Y$ with the following property:

$$\forall x,y\in X\ xEy \iff f(x)Ff(y).$$

We say that f reduces E to F and write $E\leq F$. The idea is that F is at least as complicated as E since given $x,y\in X$ one may tell whether x and y are E-related by examining whether their images by f are F-related. Note that $\leq$ is a pre-order (i.e. it is reflexive and transitive).

If X and Y are topological spaces then one may define the notion of continuous reducibility (denoted by $\leq_c$) by requiring that f is a continuous map. Similarly, if X and Y have Borel structures, we may consider Borel reducibility (denoted by $\leq_B$ by requiring the map f to be Borel. Clearly, both $\leq_c$, $\leq_B$ are pre-orders as well.

As an example of a Borel reduction we describe Gromov's result that the isometry relation of infinite compact metric spaces (defined below) is Borel reducible to the equality relation $\mathrm{id}(2^\omega)$ on 2^ω. For a more detailed discussion of this example we refer the reader to Sections 14.1 and 14.2 of [15]. If (X,d) is an infinite Polish metric space and $\{x_n\colon n\in\omega\}$ is its dense subset, we define $r_X\in\mathbb{R}^{\omega\times\omega}$ by $r_X=(d(x_i,x_j))_{i\in\omega,j\in\omega}$. Note that $r_X\in\mathbb{X}$ where

$$\mathbb{X}=\{(r_{i,j})_{i\in\omega,j\in\omega}\in\mathbb{R}^{\omega\times\omega}\colon \forall i,j,k\in\omega\ r_{i,j}=r_{j,i}\geq 0,\ r_{i,j}=0\iff i=j,\ r_{i,j}+r_{j,k}\geq r_{i,k}\}.$$

Conversely, every $r\in\mathbb{X}$ encodes a unique infinite Polish space X_r — the completion of the metric space (ω,d_r) where $d_r(i,j)=r_{i,j}$. Hence, one may think of the class of infinite Polish metric spaces as $\mathcal{X}$ (even though different rs may encode isometric Polish spaces). Denote by $\mathbb{X}_{\mathrm{cpt}}$ the set of all $r\in\mathbb{X}$ encoding compact metric spaces. Then $\mathbb{X}_{\mathrm{cpt}}$ is a Borel subset of $\mathbb{R}^{\omega\times\omega}$ and hence inherits a Borel structure from $\mathbb{R}^{\omega\times\omega}$. Define the isometry relation $\cong_i$ on $\mathbb{X}_{\mathrm{cpt}}$ by

$$r\cong_i s \iff r \text{ and } s \text{ encode isometric spaces.}$$

It turns out that every compact metric space (X,d) is uniquely determined by the compact sets $\Phi_n(X)=\{(d(x_i,x_j))_{0\leq i,j\leq n}\in\mathbb{R}^{(n+1)\times(n+1)}\colon x_0,x_1,\ldots,x_n\in X\}$ for $n\in\omega$ and that the map $\mathbb{X}_{\mathrm{cpt}} \ni r \mapsto \Phi_n(X_r)$ is Borel for all $n \in \omega$. It follows that if we let $f\colon \prod_{n\in\omega} K(\mathbb{R}^{(n+1)\times(n+1)}) \to 2^\omega$ be any Borel isomorphism then the map $g\colon \mathbb{X}_{\mathrm{cpt}} \to 2^\omega$ given by

$$\mathbb{X}_{\mathrm{cpt}} \ni r \mapsto f\left((\Phi_n(X_r))_{n\in\omega}\right) \in 2^\omega$$

is Borel and has the property that $r \cong_i s \iff g(r) = g(s)$. Therefore $\cong_i \leq_B \mathrm{id}(2^\omega)$ as witnessed by g.

From now on we narrow down our interest to orbit equivalence relations induced by a Borel action of a Polish group acting on a Polish space or on a standard Borel space. We are also interested only in Borel reducibility.

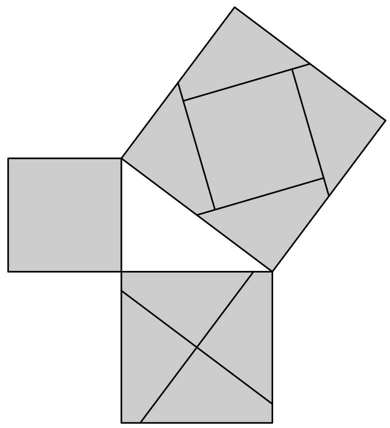
One may ask whether there is the most complicated orbit equivalence relation. This means: is there an orbit equivalence relation E so that every other orbit equivalence relation is Borel reducible to E? Such a relation is usually called *complete* or *universal*. It turns out that this is true: there exists a complete orbit equivalence relation. This follows from the existence of universal Polish groups and the Mackey-Hjorth theorem [15, Theorem 3.5.2] on extensions of actions of Polish groups. For a complete proof of this we refer the reader to [15, Theorem 5.19]. On the other hand, the first natural example of a complete orbit equivalence relation is the isometry relation of Polish metric spaces as proved by Gao and Kechris [17] and Clemens [6]. Later on many other orbit equivalence relations naturally arising in mathematics were proved to be complete. We discuss the topic further in Chapter 2.

1.3 Equidecompositions

Given two sets, A and B, can we partition A into certain number of pieces and rearrange them to obtain the set B? The answer to this general question depends on many factors:

what are the sets A and B, how many pieces we allow in the decomposition, how regular the pieces should be and what we mean by 'rearrangements'.

The idea of equidecompositions was known to ancient Greek mathematicians: one of their proofs of Pythagorean theorem is based on cutting two small squares into five pieces and building a big square using these pieces.



Later on it was observed by Bolyai and Gerwien that for any two polygons P and Q of the same area one can cut P into finitely many polygons, move them by isometries to obtain a partition of Q. On the other hand, Max Dehn proved that a similar result does not hold for three-dimensional polyhedra: it is not the case that if P and Q are arbitrary polyhedra of the same volume then P can be cut into finitely many polyhedra which can be rearranged to obtain a partition of Q. This fails even for cube and regular pyramid. This answered Hilbert's third problem.

Let us also mention that a result of Dubins, Hirsch and Karush [10] implies that a disc cannot be cut into finitely many pieces homeomorphic to discs so that the pieces can be rearranged by isometries to obtain a partition of a square.

Note that in the examples above we ignore the boundaries of pieces. We introduce now a precise set-theoretic notion of equidecomposability.

Definition 1.5. Let G be a group acting on X. We say that sets A and B are G-equidecomposable if there exist pairwise disjoint sets $A_1, A_2, \ldots, A_n$, pairwise disjoint sets $B_1, B_2, \ldots, B_n$ and group elements $g_1, g_2, \ldots, g_n$ so that $A = \bigcup_{i=1}^n A_i$, $B = \bigcup_{i=1}^n B_i$, and $B_i = g_i A_i$ for all i.

A classical theorem by Banach and Tarski, known as the Banach-Tarski paradox, states if $d \geq 3$ then one can partition the unit ball in d-dimensional space into finitely many pieces and move them by isometries to obtain a partition of two unit balls. The proof uses axiom of choice.

Theorem 1.6. *Let $d \geq 3$. The unit ball B in $\mathbb{R}^d$ is equidecomposable (with respect to the group of isometries) with two copies of B.*

It can be proved that the minimal number of pieces in paradoxical decomposition of the three-dimensional unit ball is five.

A more general version of the Banach-Tarski paradox says:

Theorem 1.7. *Let $d \geq 3$ and $A, B \subset \mathbb{R}^d$ be two bounded sets with nonempty interior. Then A and B are equidecomposable with respect to the group of isometries of $\mathbb{R}^d$.*

Sketch of proof. Since A contains a ball $C \subset A$ and B is bounded, B can be covered by a finite number of copies of C. It follows from Theorem 1.6 that C (and thus A) is equidecomposable with a superset of B. Similarly, B is equidecomposable with a superset of A. The conclusion follows from a variant of the Cantor-Schröder-Bernstein theorem. □

A natural question arises: is the analogue of the Banach-Tarski paradox true in dimensions $d < 3$? The answer is negative: if two Lebesgue-measurable sets in the plane are equidecomposable then they have the same Lebesgue measure. The reason is that there exists an isometry-invariant finitely additive measure μ defined on all subsets of $\mathbb{R}^2$ which extends the Lebesgue measure λ. So, if $A_1, \dots, A_n$ and $B_1, \dots, B_n$ are partitions of A and B, respectively, witnessing that A and B are equidecomposable then

$$\lambda(A) = \mu(A) = \sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{n} \mu(B_i) = \mu(B) = \lambda(B).$$

It turns out that non-existence of such a finitely additive measure is the only obstruction preventing the existence of paradoxical decompositions.

Theorem 1.8 (Tarski)**.** *Suppose G acts on X and $E \subset X$. The following are equivalent:*

(i) There exists a finitely additive, G-invariant measure $\mu\colon \mathcal{P}(X) \to [0, \infty]$ with $\mu(E) = 1$

(ii) E is not G-paradoxical, i.e. there exist no partition of E into pairwise disjoint subsets $E_1, \dots, E_n$ and group elements $g_1, \dots, g_n$ so that for some k the sets $g_1E_1, g_2E_2, \dots, g_kE_k$ are pairwise disjoint, the sets $g_{k+1}E_{k+1}, \dots, g_nE_n$ are pairwise disjoint, and $E = \bigcup_{i=1}^{k} g_iE_i = \bigcup_{i=k+1}^{n} g_iE_i$.

1.4 Amenability

In this section we briefly review the notion of amenable groups and semigroups. For more information on amenability the reader may wish to consult [41].

Definition 1.9. We say that a group G is amenable if there exists a finitely additive measure $\mu\colon \mathcal{P}(G) \to [0, 1]$ so that $\mu(G) = 1$ and μ is left-invariant, i.e. $\mu(gA) = \mu(A)$ for all $A \subset G$ and $g \in G$.

All finite groups are amenable, as witnessed by the measure $\mu(A) = \frac{|A|}{|G|}$.

The group of integers is amenable; we outline the sketch of proof of this. For any n let $F_n \subset \mathbb{Z}$ be a finite set of consecutive integers so that $\lim_{n\to\infty} |F_n| = \infty$. Define a measure ν_n on $\mathbb{Z}$ by $\nu_n(A) = \frac{|A \cap F_n|}{|F_n|}$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and define

$\nu(A) = \lim_{n\to\mathcal{U}} \nu_n(A)$. It turns out that ν is a left-invariant finitely additive measure on $\mathbb{Z}$ with $\nu(\mathbb{Z}) = 1$. Essentially the same argument shows that a countable group G is amenable provided it satisfies *the Følner condition*: there exists an increasing sequence of finite sets F_n so that $\bigcup_{n\in\omega} F_n = G$ and $\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0$ for all $g \in G$. In fact, the Følner condition is equivalent to amenability for countable groups. All countable abelian groups satisfy the Følner condition, hence countable abelian groups are amenable.

Note that F_2, the free group on two generators, is not amenable since it admits paradoxical decompositions. Neither is any group G containing F_2 as a subgroup. It was a long-standing problem whether there exists a non-amenable group none of whose subgroup is isomorphic to F_2. This was answered in the affirmative by Ol'shanskii [40].

It turns out that there is a bijection between finitely additive measures μ on G with $\mu(G) = 1$ and continuous linear functionals m on $\ell_\infty(G)$ so that $\|m\| = 1$ (we call such functionals *means*); this bijection is given by $\mu(E) = m(\chi_E)$. An equivalent definition of an amenable group follows: a group G is amenable if and only if there exists a left invariant mean m on G. A mean m is left invariant if $m(\varphi x) = m(\varphi)$ where for $\varphi \in \ell_\infty(G)$ and $x \in G$ we define φx by $\varphi x(y) = \varphi(xy)$ for $y \in G$. Note that this definition also makes sense for semigroups; we say that a semigroup is left amenable if it admits a left invariant mean. Let us also remark that the Følner condition is not equivalent to left amenability of semigroups.

Let S be a semigroup. Let K be a compact convex subset of a locally convex space E. The set K is called an *affine left S-set* if S acts on K from the left and for all $s \in S$ the map $K \ni x \mapsto sx \in K$ is affine. The following theorem is a fundamental result in theory of amenable semigroups.

Theorem 1.10 (Day's fixed point theorem)**.** *Let S be a left amenable semigroup and K an affine left S-set. Then the action of S on K has a fixed point.*

1.5 Mokobodzki's medial means

A *mean* is a linear functional $\mathfrak{m}\colon \ell_\infty \to \mathbb{R}$ so that:

(i) $\mathfrak{m}$ is positive, i.e. if $f \geq 0$ then $\mathfrak{m}(f) \geq 0$,

(ii) $\mathfrak{m}$ is normalized, i.e. $\mathfrak{m}(\mathbf{1}) = 1$,

We say that a mean $\mathfrak{m}$ is *shift-invariant* if for all $f \in \ell_\infty$ one has $\mathfrak{m}(Sf) = \mathfrak{m}(f)$, where $S\colon \ell_\infty \to \ell_\infty$ is the shift map: $Sf(n) = f(n+1)$. A *Banach limit* is a shift-invariant mean. Note that Banach limits extend the notion of limit to arbitrary bounded sequences, i.e. if $\mathfrak{m}$ is a Banach limit then for all $f \in \ell_\infty$ so that $\lim_{n\to\infty} f(n)$ exists we have $\mathfrak{m}(f) = \lim_{n\to\infty} f(n)$.

How strong set theoretic axioms are needed to prove the existence of Banach limits? There exists a model of $\mathsf{ZF} + \mathsf{DC}$ in which there are no Banach limits (see e.g. [23, Theorem 44.]). Hence, in order to prove that a Banach limit exists one has to work in a theory stronger than $\mathsf{ZF} + \mathsf{DC}$. Denote by HB the statement that the Hahn-Banach theorem holds. Since one can prove that Banach limits exist using the Hahn-Banach theorem for ℓ_∞ [14], it follows that HB is not a theorem of $\mathsf{ZF} + \mathsf{DC}$. Interestingly enough, HB implies the Banach-Tarski paradox [42].

Below we shall prove working in $\mathsf{ZF} + \mathsf{DC} + \mathsf{HB}$ that if μ is a Borel probability measure on $[0,1]^\omega$ then there exists a Banach limit whose restriction to $[0,1]^\omega$ is μ-measurable. We shall also present a proof in $\mathsf{ZFC} + \mathsf{CH}$ that there exists a Banach limit so that its restriction to $[0,1]^\omega$ is universally measurable (i.e. measurable with respect to all Borel probability measures on $[0,1]^\omega$). This was first proved by Mokobodzki [39]; for this reason such Banach limits are often called *Mokobodzki's medial limits* or *Mokobodzki's medial means*. Our exposition of these results is based on [37].

Let E be a locally convex topological vector space. Let X be a convex compact metrizable subset of E. For a convex function $u\colon X \to \mathbb{R}$ let $A_u = \{(x,t)\colon u(x) \leq t\}$. This is a convex set. Moreover, u is lower semicontinuous if and only if A_u is closed, and if this is the case then for all $s \in \mathbb{R}$ the set $A_u^s = \{(x,t)\colon u(x) \leq t \leq s\}$ is compact and convex. By the Hahn-Banach theorem such a u is the supremum of a family of continuous affine functions, hence u satisfies Jensen's inequality:

$$u(b) \leq \int_X u(x)\mathrm{d}\mu(x)$$

for all probability measures μ with barycenter b.

Let $u_1, u_2\colon X \to \mathbb{R}$ be two convex lower semicontinuous functions bounded from above by $s \in \mathbb{R}$. Note that

$$\mathrm{conv}(A_{u_1} \cup A_{u_2}) = \mathrm{conv}(A_{u_1}^s \cup A_{u_2}^s) \cup (X \times [s, \infty))$$

where $\mathrm{conv}(Z)$ denotes the convex hull of Z. Since $A_{u_i}^s$ are compact, the set $\mathrm{conv}(A_{u_1}^s \cup A_{u_2}^s)$ is compact as well, hence $\mathrm{conv}(A_{u_1} \cup A_{u_2})$ is closed as it is the union of a compact set and a closed set. It follows that $\mathrm{conv}(A_{u_1} \cup A_{u_2}) = A_u$ for some convex lower semicontinuous function $u\colon X \to \mathbb{R}$. We denote $u = u_1 \underset{\mathrm{conv}}{\wedge} u_2$. Clearly, the operation $\underset{\mathrm{conv}}{\wedge}$ is associative. Note that if $v\colon X \to \mathbb{R}$ is an arbitrary convex function satisfying $v \leq u_1$, $v \leq u_2$ then $v \leq u_1 \underset{\mathrm{conv}}{\wedge} u_2$. This is analogous to a property of $f \wedge g = \min(f,g)$: if $h \leq f$ and $h \leq g$ then $h \leq f \wedge g$, and justifies the choice of the symbol $\underset{\mathrm{conv}}{\wedge}$.

Let Γ_+ be the class of bounded convex functions $v\colon X \to \mathbb{R}$ so that $v = \inf_{n\in\omega} v_n$ for some convex lower semicontinuous functions $v_n\colon X \to \mathbb{R}$. Note that such a v is the limit of a decreasing sequence of bounded convex lower semicontinuous functions $u_n = s \underset{\mathrm{conv}}{\wedge} v_1 \underset{\mathrm{conv}}{\wedge} v_2 \underset{\mathrm{conv}}{\wedge} \ldots \underset{\mathrm{conv}}{\wedge} v_n$ (where s is an upper bound of v). It follows that v is universally measurable and that it also satisfies Jensen's inequality. We let $\Gamma_- = -\Gamma_+$. This class consists of bounded concave functions $v\colon X \to \mathbb{R}$ so that $v = \sup_{n\in\omega} v_n$ with v_n being concave and upper semicontinuous and, analogously, such a v is the limit of an increasing sequence of bounded concave upper semicontinuous functions.

Let us make an easy observation that if $a_1, a_2, a_3, \ldots$ are continuous affine functions then the convex function $\limsup_n a_n = \inf_n \sup_{m\geq n} a_m$ belongs to Γ_+ since the functions $\sup_{m\geq n} a_m$ are convex and lower semicontinuous. Similarly, $\liminf_n a_n \in \Gamma_-$.

Lemma 1.11 ($\mathsf{ZF}+\mathsf{DC}+\mathsf{HB}$)**.** *Let X be a convex compact metrizable subset of a locally convex topological vector space E. Let $u \in \Gamma_-$ and $v \in \Gamma_+$ satisfy $u \leq v$. Let μ be a Borel probability measure on X. Then there exist functions $u' \in \Gamma_-$, $v' \in \Gamma_+$ so that $u \leq u' \leq v' \leq v$ and $u' = v'$ μ-a.e.*

Proof. Let (u_n) be an increasing sequence of bounded concave upper semicontinuous functions converging pointwise to u. Let (v_n) be a decreasing sequence of bounded convex lower semicontinuous functions converging pointwise to v. Let s be a lower bound of $u_n - \frac{1}{n}$ and let s' be an upper bound of $v_n + \frac{1}{n}$. Note that the sets $K_n = \{(x,t) \in X\times\mathbb{R} \colon s \leq t \leq u_n(x) - \frac{1}{n}\}$ and $L_n = \{(x,t) \in X\times\mathbb{R} \colon s' \geq t \geq v_n(x) + \frac{1}{n}\}$ are disjoint convex compact subsets of $E\times\mathbb{R}$. Using the Hahn-Banach theorem we obtain a continuous linear functional $H \colon E\times\mathbb{R} \to \mathbb{R}$ and a $\gamma \in \mathbb{R}$ so that K_n and L_n are separated by the hyperplane $\{(x,t) \colon H(x,t) = \gamma\}$. Then $h \colon E \to \mathbb{R}$ given by $h(x) = \frac{\gamma - H(x,0)}{H(\mathbf{0},1)}$ is a continuous affine function so that $u_n(x) - \frac{1}{n} \leq h(x) \leq v_n(x) + \frac{1}{n}$ for all $x \in X$. Hence, for all n the set

$$B_n = \left\{h \colon h \text{ is a continuous affine function so that } u_n - \frac{1}{n} \leq h \leq v_n + \frac{1}{n}\right\} \subset L^1(X,\mu)$$

is nonempty. Let $\overline{B_n}$ be the closure of B_n in weak topology. Note that $\overline{B_n}$ is compact. Also, the sequence $\overline{B_n}$ is decreasing since the sequence B_n decreases. Hence the intersection of all $\overline{B_n}$ is nonempty. Let $a \in \bigcap_{n\in\omega} \overline{B_n}$. Note that for any n the set B_n is convex, so by Mazur's theorem $\overline{B_n}$ coincides with the closure of B_n in norm topology. Hence for all n there is $a_n \in B_n$ with $\|a_n - a\|_{L^1(X,\mu)} \leq 2^{-n}$. The sequence a_n converges to a μ-a.e., so the functions $u' = \liminf_{n\in\omega} a_n$, $v' = \limsup_{n\in\omega} a_n$ are as required. $\square$

Theorem 1.12 ($\mathsf{ZF}+\mathsf{DC}+\mathsf{HB}$)**.** *Let μ be a Borel probability measure on $[0,1]^\omega$. Then there exists a Banach limit $\mathfrak{m}\colon \ell_\infty \to \mathbb{R}$ so that the restriction of $\mathfrak{m}$ to $[0,1]^\omega$ is μ-measurable.*

Proof. Let $E=\ell_\infty$, $X=[0,1]^\omega$, $u=\liminf_n \xi_n$, and $v=\limsup_n \xi_n$ where $\xi_n\colon \ell_\infty\to\mathbb{R}$ is the n-th coordinate function. Use Lemma 1.11 to obtain functions $u'\in\Gamma_-$, $v'\in\Gamma_+$ so that $u\le u'\le v'\le v$ and $u'=v'$ μ-a.e. Let $A=\{x\in X\colon u'(x)=v'(x)\}$ and define $w\colon A\to\mathbb{R}$ by $w=u'|_A=v'|_A$. Note that w is linear since $\mathbf{0}\in A$ and it is both concave and convex. Extend w to $\tilde{w}\colon \operatorname{span}(A)\to\mathbb{R}$ by linearity and note that $u\le\tilde{w}\le v$. Use the Hahn-Banach theorem to obtain an extension of $\tilde{w}$ to a linear functional $\ell\colon\ell_\infty\to\mathbb{R}$ so that $\ell\le v$.

Note that ℓ satisfies $u\le\ell$ since for all $x\in\ell_\infty$ we have $u(x)=-v(-x)\le-\ell(-x)=\ell(x)$. So, if $x\in\ell_\infty$ and $x\ge 0$ then $\ell(x)\ge u(x)\ge 0$. This shows that ℓ is positive. Note that if $x=(x_0,x_1,\ldots)$ is a convergent sequence then $u(x)=v(x)=\lim_n x_n$, hence $\ell(x)=\lim_n x_n$. In particular, $\ell(\mathbf{1})=1$ which proves that ℓ is normalized. Therefore ℓ is a mean. Also note that the restriction of ℓ to $[0,1]^\omega$ is μ-measurable as it agrees with $v'\in\Gamma_+$ on a set of full measure, namely A.

Finally, define $\mathfrak{m}\colon\ell_\infty\to\mathbb{R}$ by

$$\mathfrak{m}(x_0,x_1,x_2,x_3,\ldots)=\ell\left(x_0,\frac{x_0+x_1}{2},\frac{x_0+x_1+x_2}{3},\frac{x_0+x_1+x_2+x_3}{4},\ldots\right). \tag{1.1}$$

Since ℓ is positive, normalized and μ-measurable, it follows that $\mathfrak{m}$ has these properties as well. We claim that $\mathfrak{m}$ is shift-invariant. Indeed, for any $x=(x_0,x_1,\ldots)\in\ell_\infty$

$$\begin{aligned}\mathfrak{m}(x)-\mathfrak{m}(Sx)&=\mathfrak{m}(x_0-x_1,x_1-x_2,x_2-x_3,\ldots)\\&=\ell\left(x_0-x_1,\frac{x_0-x_2}{2},\frac{x_0-x_3}{3},\ldots\right)\\&=0\end{aligned}$$

since $\lim_{n\to\infty}\frac{x_0-x_n}{n}=0$. This finishes the proof. □

Before we formulate and prove the second aforementioned result we shall introduce a definition and prove a more general theorem.

Definition 1.13. We say that a bounded universally measurable function $w\colon X \to \mathbb{R}$ is *strongly affine* if $w(b) = \int_X w(x)\mathrm{d}\mu(x)$ for all probability measures μ with barycenter b.

Proposition 1.14. *Every strongly affine function is affine.*

Proof. Let $w\colon X \to \mathbb{R}$ be a strongly affine function. Let $x, y \in X$, $0 < \alpha < 1$. Consider the measure $\mu = \alpha\delta_x + (1-\alpha)\delta_y$. Its barycenter is $\alpha x + (1-\alpha)y$. Hence

$$w(\alpha x + (1-\alpha)y) = \int_X w(x)\mathrm{d}\mu(x) = \alpha w(x) + (1-\alpha)w(y).$$

Hence w is affine. □

Clearly, all continuous affine functions are strongly affine. However, there exist Borel affine functions which are not strongly affine.

Theorem 1.15 (ZFC + CH)**.** *Let X be a convex compact metrizable subset of a locally convex topological vector space E. Let $u \in \Gamma_-$ and $v \in \Gamma_+$ satisfy $u \le v$. Then there exists a strongly affine function w with $u \le w \le v$.*

Proof. Since X is metrizable compact, it is Polish. Hence, $P(X)$ is a compact Polish space and therefore has cardinality $\mathfrak{c}$. Hence, using CH, we can enumerate the elements of $P(X)$ with countable ordinals: $P(X) = \{\mu_\alpha\colon \alpha < \omega_1\}$.

We define by transfinite recursion functions u_α, v_α so that:

- $u_\alpha \in \Gamma_-$, $v_\alpha \in \Gamma_+$ for all $\alpha < \omega_1$,
- $u \le u_\alpha \le u_\beta \le v_\beta \le v_\alpha \le v$ for all $\alpha < \beta < \omega_1$
- for all α the equality $u_\alpha = v_\alpha$ holds μ_α-a.e.

We obtain u_0, v_0 by applying Lemma 1.11 for u, v and μ_0. Suppose $0 < \beta < \omega_1$ and u_α, v_α are defined for all $\alpha < \beta$. Consider the function $\sup_{\alpha<\beta} u_\alpha$. Clearly, it is bounded. It is also concave: this follows from concavity of u_αs and from the fact that the sequence u_α is increasing. Since each u_α belongs to Γ_-, $u_\alpha = \sup_{n\in\omega} u_{\alpha,n}$ for some concave upper semicontinuous functions $u_{\alpha,n}$. It follows that $\sup_{\alpha<\beta} u_\alpha = \sup_{(\alpha,n)\in\beta\times\omega} u_{\alpha,n}$, hence it is the supremum of a countable family of concave upper semicontinuous functions. Therefore, $\sup_{\alpha<\beta} u_\alpha$ belongs to Γ_-. Similarly, $\inf_{\alpha<\beta} v_\alpha$ belongs to Γ_+.

We use Lemma 1.11 for $\sup_{\alpha<\beta} u_\alpha$, $\inf_{\alpha<\beta} v_\alpha$, and μ_β. We obtain functions $u_\beta \in \Gamma_-$, $v_\beta \in \Gamma_+$ so that $\sup_{\alpha<\beta} u_\alpha \leq u_\beta \leq v_\beta \leq \inf_{\alpha<\beta} v_\alpha$ and $u_\beta = v_\beta$ μ_β-a.e. Hence u_β and v_β satisfy all required conditions.

Note that for all $x \in X$ there is $\alpha < \omega_1$ so that $\delta_x = \mu_\alpha$. Hence $u_\alpha(x) = v_\alpha(x)$, and so for all $\beta > \alpha$ we have $u_\alpha(x) = u_\beta(x) = v_\beta(x) = v_\alpha(x)$. Hence the sequences $\{u_\alpha\}_{\alpha<\omega_1}$, $\{v_\alpha\}_{\alpha<\omega_1}$ converge pointwise to a common function, call it w. Clearly, $u \leq w \leq v$. Moreover, for all α the function w agrees with u μ_α-a.e., and hence w is μ_α-measurable. Finally, if μ is a measure on X with barycenter b then for some α we have $\frac{1}{2}\delta_b + \frac{1}{2}\mu = \mu_\alpha$. Then b is the barycenter of μ_α as well, hence

$$v_\alpha(b) \leq \int_X v_\alpha(x)\mathrm{d}\mu_\alpha(x) = \int_X w(x)\mathrm{d}\mu_\alpha(x) = \frac{1}{2}w(b) + \frac{1}{2}\int_X w(x)\mathrm{d}\mu(x).$$

Similarly,

$$u_\alpha(b) \geq \int_X u_\alpha(x)\mathrm{d}\mu_\alpha(x) = \int_X w(x)\mathrm{d}\mu_\alpha(x) = \frac{1}{2}w(b) + \frac{1}{2}\int_X w(x)\mathrm{d}\mu(x).$$

Since $u_\alpha(b) \leq v_\alpha(b)$, it follows that equality holds in the above inequalities. In particular, $u_\alpha(b) = v_\alpha(b) = \frac{1}{2}w(b) + \frac{1}{2}\int_X w(x)\mathrm{d}\mu(x)$. As shown earlier $u_\alpha(b) = v_\alpha(b)$ implies $u_\alpha(b) = v_\alpha(b) = w(b)$. It follows that $w(b) = \int_X w(x)\mathrm{d}\mu(x)$. Hence w is strongly affine. □

Mokobodzki's result easily follows from Theorem 1.15 and arguments used in the proof of Theorem 1.12.

Theorem 1.16 (ZFC + CH)**.** *There exists a Mokobodzki's medial mean.*

Proof. Let $E = \ell_\infty$, $X = [0,1]^\omega$, $u = \liminf_n \xi_n$, and $v = \limsup_n \xi_n$ where $\xi_n \colon \ell_\infty \to \mathbb{R}$ is the n-th coordinate function. Theorem 1.15 provides a strongly affine (and hence universally measurable) function $w \colon [0,1]^\omega \to \mathbb{R}$ so that $u \leq w \leq v$. Since $u(\mathbf{0}) = v(\mathbf{0}) = 0$, we have $w(\mathbf{0}) = 0$ and so w is linear. We extend w to a function $\ell \colon \ell_\infty \to \mathbb{R}$ by linearity. We define $\mathfrak{m} \colon \ell_\infty \to \mathbb{R}$ by formula 1.1. Note that the proof of Theorem 1.12 shows that $\mathfrak{m}$ is positive, normalized and shift-invariant. Universal measurability of the restriction of $\mathfrak{m}$ to $[0,1]^\omega$ follows from universal measurability of w. □

Chapter 2

Complexity of the homeomorphism relation of locally connected continua

In this chapter we prove that the homeomorphism relation of locally star-convex continua is a complete orbit equivalence relation. This implies that the homeomorphism relation of locally connected continua is complete. This answers a question posed by Chang and Gao in [2].

2.1 Introduction

A Borel action a of a Polish group G on a standard Borel space X determines an equivalence relation E_a given by $xE_ay \iff \exists g \in G\ gx = y$. In other words, xE_ay if and only if x and y are in the same orbit of the action a. Such relations are called *orbit equivalence relations*. Note that every orbit equivalence relation is analytic, i.e. the set $\{(x, y) \in X \times X : xE_ay\}$ is an analytic subset of the product $X \times X$.

Given two orbit equivalence relations E and F on standard Borel spaces X and Y, respectively, we say that a Borel map $f\colon X \to Y$ *reduces E to F* if and only if for every $x, y \in X$

$$xEy \iff f(x)Ff(y).$$

If this is the case we say that E is *Borel reducible* to F.

If E is Borel reducible to F and F is Borel reducible to E then we say that E and F are *Borel bireducible*. Roughly speaking, this means that E and F are of the same complexity.

If E is an orbit equivalence relation such that every orbit equivalence relation F is reducible to E then we say that E is *complete* (or *universal*) orbit equivalence relation. Complete orbit equivalence relations are, in a sense, the most complex objects in the class of orbit equivalence relations. It is known that complete orbit equivalence relations exist, on abstract grounds. This follows from the existence of universal Polish groups and the Mackey-Hjorth theorem [15][Theorem 3.5.2] on extensions of actions of Polish groups. On the other hand, the first natural example of a complete orbit equivalence relation is the isometry relation of Polish metric spaces as proved by Gao and Kechris [17] and Clemens [6]. Interestingly enough, recently Melleray [36] proved that there exists a Polish metric space whose group of isometries with its natural action on the space induces a complete orbit equivalence relation.

In recent years there has been a considerable amount of research on the classification program of separable C*-algebras from a descriptive set-theoretic point of view. This began with the work of Farah, Toms and Törnquist [12] and later Elliott, Farah, Paulsen, Rosendal, Toms and Törnquist [11] and led to the question of the complexity of the isometry relation of separable C*-algebras. This problem has been solved by Sabok [44] who showed that the isometry relation of separable C*-algebras is a complete orbit equivalence relation. Soon thereafter, Zielinski [50], using Sabok's result, solved the long-standing problem whether the homeomorphism relation of compact metric spaces is a complete orbit equivalence relation.

The latter result was subsequently improved by Chang and Gao [2] who showed that the homeomorphism relation of continua (connected compact metric spaces) is also a complete orbit equivalence relation.

These results lead to a number of open questions.

Problem 2.1 (Zielinski, [50])**.** *Is the homeomorphism relation of homogeneous compact metric spaces a complete orbit equivalence relation?*

This problem seems to be very difficult as there are not so many known ways to construct homogeneous spaces.

Problem 2.2 (Chang, Gao, [2])**.** *Is the homeomorphism relation of locally connected continua a complete orbit equivalence relation?*

In this paper we prove that the answer to Problem 2.2 is affirmative. In fact, we prove the following stronger theorem.

Theorem 2.3. *The homeomorphism relation of locally star-convex continua is a complete orbit equivalence relation.*

Recall that every compact metric space embeds in the Hilbert cube $\mathcal{Q} = [0,1]^{\mathbb{N}}$ and the family $K(\mathcal{Q})$ of all compact subsets of $\mathcal{Q}$ has a natural Borel structure steming from the Vietoris topology. Kuratowski proved that the set of locally connected subcontinua of $\mathcal{Q}$ is an $F_{\sigma\delta}$ subset of $K(\mathcal{Q})$ [25]. His proof actually gives the following stronger result: if $\mathcal{C}$ is a closed subset of $K(\mathcal{Q})$ then the set of locally-$\mathcal{C}$ subcontinua of $\mathcal{Q}$ is $F_{\sigma\delta}$ in $K(\mathcal{Q})$. Choosing $\mathcal{C}$ as the set of star-convex continua we see that the set of locally star-convex continua is $F_{\sigma\delta}$ in $K(\mathcal{Q})$. This gives Borel structures on the collection of locally connected continua and on the collection of locally star-convex continua. It is worth noting that local connectedness and local path-connectedness are equivalent in the class of continua [21]. Such continua are also called Peano continua, as they are continuous images of the interval.

Section 2.2 is devoted to a description of coding spaces. They are used in the last section in which we prove Theorem 2.3.

2.2 The coding spaces

Let d be a metric on $\mathcal{Q}$ given by the formula $d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} \frac{|x_n - y_n|}{2^n}$. Let d' be a metric on $\mathcal{Q}\times\mathcal{Q}$ given by $d'((x,y),(z,t)) = d(x,z) + d(y,t)$. We also denote $\mathbf{0} = (0,0,0,\ldots) \in \mathcal{Q}$ and $e_i = (\underbrace{0,0,\ldots,0}_{i \text{ times}},1,0,0,\ldots) \in \mathcal{Q}$.

In this section we consider locally star-convex continua $X, Y \subset \mathcal{Q}$ and non-empty families (finite or countably infinite) $\mathcal{A} = \{A_n\colon n < |\mathcal{A}|\}, \mathcal{C} = \{C_n\colon n < |\mathcal{C}|\}$ of non-empty closed convex subsets of $\mathcal{Q}$ such that $\bigcup\mathcal{A}$ is a closed subset of X and $\bigcup\mathcal{C}$ is a closed subset of Y.

For every $A \in \mathcal{A}$ let $a_0^A, a_1^A, \ldots$ be an enumeration of a dense subset of A in which every element appears infinitely many times. Define $b_k^A = (a_k^A, e_{\langle n,k\rangle}) \in X \times \mathcal{Q}$, where n is such that $A = A_n$ and $\langle\cdot,\cdot\rangle$ is a bijection between $\mathbb{N}\times\mathbb{N}$ and $\mathbb{N}$.

We define

$$X' = X \times \{\mathbf{0}\} \cup \{b_k^A\colon A \in \mathcal{A},\ k \in \mathbb{N}\}.$$

The idea is that for every $A \in \mathcal{A}$ we introduce a set of new isolated points whose boundary is precisely the set A. Note that X' is a compact space.

Similarly, for every $C \in \mathcal{C}$ we consider an enumeration with infinite repetitions $c_0^C, c_1^C, \ldots$ of a dense subset of C and we define $d_k^C = (c_k^C, e_{\langle n,k\rangle}) \in Y \times \mathcal{Q}$, where n is such that $C = C_n$. We define

$$Y' = Y \times \{\mathbf{0}\} \cup \{d_k^C\colon C \in \mathcal{C},\ k \in \mathbb{N}\}.$$

A standard back-and-forth construction yields the following

Proposition 2.4. *If $f\colon X \to Y$ is a homeomorphism such that $\{f[A]\colon A \in \mathcal{A}\} = \mathcal{C}$ then there is a homeomorphism $g\colon X' \to Y'$ extending f such that $g[\{b_k^A\colon k \in \mathbb{N}\}] = \{d_k^{f[A]}\colon k \in$*

$\mathbb{N}\}$ *for every* $A \in \mathcal{A}$.

For every $A \in \mathcal{A}$ and $k \in \mathbb{N}$ we define $X_k^A = \{t \cdot x + (1-t)b_k^A \colon 0 \leq t \leq 1,\ x \in A \times \{\mathbf{0}\}\}$, i.e. X_k^A is the cone with base $A \times \{\mathbf{0}\}$ and apex b_k^A. Further, let

$$X'' = X \times \{\mathbf{0}\} \cup \bigcup_{A \in \mathcal{A}} \bigcup_{k \in \mathbb{N}} X_k^A.$$

That is, for every $A \in \mathcal{A}$ we build a sequence of cones with base A such that the boundary of the set of apexes of these cones is A. Moreover, every two cones have no common points lying outside $X \times \{\mathbf{0}\}$.

We prove now that X'' is a locally star-convex continuum. Compactness easily follows from the assumption that all sets $A \in \mathcal{A}$ are convex. Connectedness is clear. Therefore X'' is a continuum. For the proof of local star-convexity, consider a point $(x, r) \in X''$. If $r \neq \mathbf{0}$ then clearly (x, r) has arbitrarily small convex neighbourhoods, thus proving local star-convexity at (x, r). Now assume that $r = \mathbf{0}$. Observe that any set of the form $(U \times V) \cap X''$ where $x \in U \subset X$, U is star-convex neighbourhood of x, and $V \subset \mathcal{Q}$ is a basic neighbourhood of $\mathbf{0}$, is a star-convex neighbourhood of $(x, \mathbf{0})$ in X''. This is because if $(y, s) \in (U \times V) \cap X''$ then $(y, s) \in X_k^A$ for some $A \in \mathcal{A}$ and $k \in \mathbb{N}$ and since X_k^A and V are convex, the segment connecting $(x, \mathbf{0})$ with (y, s) is contained in $X_k^A \cap V$. Moreover $y \in U$ and therefore the segment with endpoints x and y is contained in U. Finally, the segment with endpoints $(x, \mathbf{0})$, (y, s) is contained in $(U \times V) \cap X_k^A \subset (U \times V) \cap X''$. Now, since x has arbitrarily small star-convex neighbourhoods in X, $(x, \mathbf{0})$ has arbitrarily small neighbourhoods in X''.

We use the same notation for Y, and for every $C \in \mathcal{C}$ and $k \in \mathbb{N}$ let Y_k^C be the cone with base $C \times \{\mathbf{0}\}$ and apex d_k^C and let Y'' be the union of $Y \times \{\mathbf{0}\}$ and all the cones Y_k^C.

Proposition 2.5. *If* $f \colon X \to Y$ *is a homeomorphism such that* $\{f[A] \colon A \in \mathcal{A}\} = \mathcal{C}$ *and for every* $A \in \mathcal{A}$ *the restriction* $f|_A$ *is affine then there is a homeomorphism* $h \colon X'' \to Y''$ *extending* f *such that for every* $A \in \mathcal{A}$ $f[\{b_k^A \colon k \in \mathbb{N}\}] = \{d_k^{f[A]} \colon k \in \mathbb{N}\}$ *and for every* $A \in \mathcal{A}$ *and* $k \in \mathbb{N}$ *the restriction of* h *to* X_k^A *is affine.*

Proof. Let $g\colon X' \to Y'$ be a homeomorphism extending f constructed in the previous proposition. We define $h\colon X'' \to Y''$ by $h|_{X'} = g$, and for every $A \in \mathcal{A}$, $x \in A$, $k \in \mathbb{N}$, $0 < t < 1$

$$h(t \cdot (x, \mathbf{0}) + (1-t) \cdot b_k^A) = t \cdot g(x, \mathbf{0}) + (1-t) \cdot g(b_k^A).$$

This map is a bijection between compact spaces, so to prove that h is a homeomorphism we only have to show that h is continuous. It is also clear from the definition of h that $h|_{X_k^A}$ is affine for every $A \in \mathcal{A}$ and $k \in \mathbb{N}$.

Let $t_j(x_j, \mathbf{0}) + (1-t_j)b_{k_j}^{A_{n_j}}$, where $k_j, n_j \in \mathbb{N}, x_j \in A_{n_j}, t_j \in [0,1]$ be a sequence of elements of X'' converging to some $t(x,0) + (1-t)b_k^{A_n}$ (where $x \in A_n, t \in [0,1], k, n \in \mathbb{N}$). If $t < 1$ then for sufficiently large j we have $k_j = k$, $n_j = n$ and also $\lim\limits_{j\to\infty} x_j = x$, $\lim\limits_{j\to\infty} t_j = t$. Therefore

$$\begin{aligned} h(t_j(x_j, \mathbf{0}) + (1-t_j)b_{k_j}^{A_{n_j}}) &= t_j g(x_j, \mathbf{0}) + (1-t_j) g(b_{k_j}^{A_{n_j}}) \\ &\xrightarrow{j\to\infty} t g(x, \mathbf{0}) + (1-t) g(b_k^{A_n}) = h(t(x, \mathbf{0}) + (1-t) b_k^{A_n}). \end{aligned}$$

If $t = 1$ then $t_j(x_j, \mathbf{0}) + (1-t_j)b_{k_j}^{A_{n_j}} \xrightarrow{j\to\infty} (x, \mathbf{0})$. Fix $\varepsilon > 0$. Pick an integer N so large that if $\langle n, k\rangle > N$ and $g(b_k^{A_n}) = d_l^{C_m}$ then $d(c_l^{C_m}, f(a_k^{A_n})) < \varepsilon/4$ and $2^{-\langle m,l\rangle} < \varepsilon/4$. Pick an integer N' such that whenever $j > N'$ then $(1-t_j)2^{-\langle n_j, k_j\rangle} < \varepsilon 2^{-N-2}$ and $d(f(x), t_j f(x_j) + (1-t_j) f(a_{k_j}^{A_{n_j}})) < \varepsilon/4$.

We have $h(x, \mathbf{0}) = g(x, \mathbf{0}) = (f(x), \mathbf{0})$ and

$$\begin{aligned} h(t_j(x_j, \mathbf{0}) + (1-t_j)b_{k_j}^{A_{n_j}}) &= t_j g(x_j, \mathbf{0}) + (1-t_j) g(b_{k_j}^{A_{n_j}}) \\ &= (t_j f(x_j) + (1-t_j) f(a_{k_j}^{A_{n_j}}), \mathbf{0}) \\ &\quad + (1-t_j)(c_{l_j}^{C_{m_j}} - f(a_{k_j}^{A_{n_j}}), e_{\langle m_j, l_j\rangle}), \end{aligned}$$

therefore, if $j > N'$ then

$$\begin{aligned}
&d'(h(x,\mathbf{0}),h(t_j(x_j,\mathbf{0})+(1-t_j)b_{k_j}^{A_{n_j}}))\\
&\qquad\qquad\qquad \le d(f(x),t_jf(x_j)+(1-t_j)f(a_{k_j}^{A_{n_j}}))\\
&\qquad\qquad\qquad\quad +(1-t_j)d(c_{l_j}^{C_{m_j}},f(a_{k_j}^{A_{n_j}}))+(1-t_j)2^{-\langle m_j,l_j\rangle}\\
&\qquad\qquad\qquad \le \varepsilon/4+(1-t_j)d(c_{l_j}^{C_{m_j}},f(a_{k_j}^{A_{n_j}}))+(1-t_j)2^{-\langle m_j,l_j\rangle}=(*).
\end{aligned}$$

If $1-t_j \ge \varepsilon/4$ then $\varepsilon 2^{-\langle n_j,k_j\rangle-2} \le (1-t_j)2^{-\langle n_j,k_j\rangle} < \varepsilon 2^{-N-2}$, i.e. $\langle n_j,k_j\rangle > N$, so $d(c_l^{C_m}, f(a_k^{A_n})) < \varepsilon/4$ and it follows that $(*) < \varepsilon/4+(1-t_j)\varepsilon/4+\varepsilon 2^{-N-2} \le \varepsilon$.

Otherwise $1-t_j < \varepsilon/4$ and $(*) \le \varepsilon/4+\varepsilon/4\cdot\sup d'+\varepsilon/4=\varepsilon$.

It follows that h is a continuous function. This finishes the proof.

□

Now, for every $k\in\mathbb{N}$ and $A=A_n\in\mathcal{A}$ let $\hat b_k^A$ and $\tilde b_k^A$ be two distinct points in $\mathcal{Q}\times\{e_{\langle n,k\rangle}\}$ such that $d'(b_k^A,\hat b_k^A)=d'(b_k^A,\tilde b_k^A)=\frac{1}{2+\langle n,k\rangle}$. We denote $\hat I_k^A=\{tb_k^A+(1-t)\hat b_k^A\colon 0\le t\le 1\}$ and $\tilde I_k^A=\{tb_k^A+(1-t)\tilde b_k^A\colon 0\le t\le 1\}$. We define

$$T(X,\mathcal{A})=X''\cup\bigcup_{A\in\mathcal{A}}\bigcup_{k\in\mathbb{N}}\hat I_k^A\cup\tilde I_k^A.$$

In other words, we consider the space X'' and for every $k\in\mathbb{N}$ and $A\in\mathcal{A}$ we attach two short segments $\hat I_k^A$ and $\tilde I_k^A$ to the apex of X_k^A. The key property of points b_k^A in $T(X,\mathcal{A})$ is that $T(X,\mathcal{A})\setminus\{b_k^A\}$ consists of three connected components.

Note that $T(X,\mathcal{A})$ is a locally star-convex continuum. Compactness of $T(X,\mathcal{A})$ is proved similarly as of X''. It is clear that $T(X,\mathcal{A})$ is connected. Local star-convexity of $T(X,\mathcal{A})$ easily follows from local star-convexity of X''.

We define similarly $\hat d_k^C$ and $\tilde d_k^C$ as points at the distance $1/(2+\langle n,k\rangle)$ from d_k^C, where $C=C_n$, we denote the segment with endpoints $d_k^C,\hat d_k^C$ as $\hat I_k^C$ and the segment with endpoints $d_k^C,\tilde d_k^C$ as $\tilde I_k^C$. We define $T(Y,\mathcal{C})$ as the union of Y'' and all the segments $\hat I_k^C$, $\tilde I_k^C$.

Proposition 2.6. *If $f\colon X \to Y$ is a homeomorphism such that $\{f[A]\colon A \in \mathcal{A}\} = \mathcal{C}$ and for every $A \in \mathcal{A}$ the restriction $f|_A$ is affine then there is a homeomorphism $h'\colon T(X,\mathcal{A}) \to T(Y,\mathcal{C})$ extending f such that for every $A \in \mathcal{A}$ $h'[\{b_k^A\colon k \in \mathbb{N}\}] = \{d_k^{f[A]}\colon k \in \mathbb{N}\}$ and for every $A \in \mathcal{A}$ and $k \in \mathbb{N}$ the restrictions of h' to X_k^A, $\hat{I}_k^A$, and $\tilde{I}_k^A$ are affine.*

Proof. We simply extend the homeomorphism $h\colon X'' \to Y''$ constructed in the previous proposition to $h'\colon T(X,\mathcal{A}) \to T(Y,\mathcal{C})$ by the formula $h'(tb_k^A + (1-t)\hat{b}_k^A) = td_l^C + (1-t)\hat{d}_l^C$ and $h'(tb_k^A + (1-t)\tilde{b}_k^A) = td_l^C + (1-t)\tilde{d}_l^C$, where $h(b_k^A) = d_l^C$. □

We will also need a variant of the space $T(X,\mathcal{A})$. Consider a set $B \in \mathcal{A}$. Consider the space $T(X,\mathcal{A})$. For every $k \in \mathbb{N}$ let $\check{b}_k^B$ be a point distinct from $\hat{b}_k^B, \tilde{b}_k^B$ with $d'(b_k^B, \check{b}_k^B) = d'(b_k^B, \hat{b}_k^B)$ and $\check{b}_k^B - b_k^B + (a_k^B, \mathbf{0}) \in \mathcal{Q} \times \{\mathbf{0}\}$. Denote the closed segment with endpoints b_k^B, $\check{b}_k^B$ by $\check{I}_k^B$. We define

$$T'(X,B,\mathcal{A}) = T(X,\mathcal{A}) \cup \bigcup_{k\in\mathbb{N}} \check{I}_k^B,$$

that is, we attach an extra segment to the apex of every cone with base B, so removing the apex results in four connected components instead of three.

Clearly $T'(X,B,\mathcal{A})$ is a locally star-convex continuum.

Proposition 2.7. *If $f\colon X \to Y$ is a homeomorphism such that $\{f[A]\colon A \in \mathcal{A}\} = \mathcal{C}$ and for every $A \in \mathcal{A}$ the restriction $f|_A$ is affine, and $B \in \mathcal{A}, D \in \mathcal{C}$ are such that $f[B] = D$, then there is a homeomorphism $h''\colon T'(X,B,\mathcal{A}) \to T'(Y,D,\mathcal{C})$ extending f.*

Proof. Using Proposition 2.6 we get a homeomorphism $h'\colon T(X,\mathcal{A}) \to T(Y,\mathcal{C})$. We extend it by putting $h''(tb_k^B + (1-t)\check{b}_k^B) = td_l^D + (1-t)\check{d}_l^D$, where $h'(b_k^B) = d_l^D$. Then h'' clearly is a homeomorphism. □

2.3 Homeomorphism relation of locally connected continua is complete

In this section we will prove the main result.

Recall that the space $\mathbb{K}_{\mathrm{Choq}}$ of metrizable Choquet simplices is a Borel subset of $K(\mathcal{Q})$ and that the relation $\approx_a$ of affine homeomorphism on $\mathbb{K}_{\mathrm{Choq}}$ is complete (this is due to Sabok [44]).

Consider a relation $\cong_{(3)}$ introduced by Zielinski in [50] defined on the space $\{(X,R) \in K(\mathcal{Q}) \times K(Q^3)\colon R \subset X^3\}$, where $(X,R) \cong_{(3)} (Y,S)$ if and only if there is a homeomorphism $f\colon X \to Y$ with $f^3[R] = S$. Here, f^3 means $f^3(x,y,z) = (f(x), f(y), f(z))$. Consider the map $\Gamma\colon \mathbb{K}_{\mathrm{Choq}} \to K(\mathcal{Q}^3)$ given by the formula

$$\Gamma(X) = \{(x,y,z) \in X^3\colon \frac{1}{2}x + \frac{1}{2}y = z\}.$$

The following is [50, Proposition 2]

Proposition 2.8. *For every $X, Y \in \mathbb{K}_{\mathrm{Choq}}$ the following equivalence holds: $X \approx_a Y \iff (X, \Gamma(X)) \cong_{(3)} (Y, \Gamma(Y))$.*

Note that $\Gamma(X)$ is convex for every Choquet simplex X.

We recall another relation from [50]. Let $\cong_{\mathrm{perm}}$ be defined on $K(\mathcal{Q})^{\mathbb{N}}$ in the following way: $(A_1, A_2, \ldots) \cong_{\mathrm{perm}} (B_1, B_2, \ldots)$ if and only if there exists a homeomorphism $h\colon \mathcal{Q} \to \mathcal{Q}$ and a permutation σ of $\mathbb{N}$ such that $h(A_n) = B_{\sigma(n)}$ for any n.

For a Choquet simplex X consider the space $\tilde{X} = T(X, \{X\})$ and write b_k instead of b_k^X.

Define for any $n \in \mathbb{N}$,

$$\begin{aligned} B_n &= \{b_n\} \times \tilde{X}^2 \\ C_n &= \tilde{X} \times \{b_n\} \times \tilde{X} \\ D_n &= \tilde{X}^2 \times \{b_n\} \\ E_n &= B_n \cup C_n \\ F_n &= B_n \cap D_n. \end{aligned}$$

Let $\Psi\colon \mathbb{K}_{\text{Choq}} \to K(\mathcal{Q}^3)^{\mathbb{N}}$ be the function

$$\Psi(X) = (\tilde{X}^3, \Gamma(X), B_1, C_1, D_1, E_1, F_1, B_2, C_2, D_2, E_2, F_2, \ldots).$$

The proof of the following proposition is similar to the proof of [50, Proposition 3].

Proposition 2.9. *For every $X, Y \in \mathbb{K}_{\text{Choq}}$ the following equivalence holds: $X \approx_a Y \iff \Psi(X) \cong_{\text{perm}} \Psi(Y)$.*

Proof. It follows from the proof of [50, Proposition 3] (where instead of using [50, Proposition 1] we use Proposition 2.6) that $X \approx_a Y \implies \Psi(X) \cong_{\text{perm}} \Psi(Y)$.

For the implication in the other direction, suppose that $\Psi(X) \cong_{\text{perm}} \Psi(Y)$ witnessed by $\sigma\colon \mathbb{N} \to \mathbb{N}$ and $h\colon \tilde{X}^3 \to \tilde{Y}^3$. Write

$$\Psi(Y) = (\tilde{Y}^3, \Gamma(Y), H_1, I_1, J_1, K_1, L_1, H_2, I_2, J_2, \ldots)$$

and let $d_n \in \tilde{Y}$ be such that $H_n = \{d_n\} \times \tilde{Y}^2$. Again, the proof of [50, Proposition 3] shows that $h[\tilde{X}^3] = \tilde{Y}^3$, $h[\Gamma(X)] = \Gamma(Y)$, and that there is a permutation $\tau\colon \mathbb{N} \to \mathbb{N}$ with $h[B_n] = H_{\tau(n)}$, $h[C_n] = I_{\tau(n)}$, $h[D_n] = J_{\tau(n)}$, $h[E_n] = K_{\tau(n)}$, $h[F_n] = L_{\tau(n)}$.

Denoting $X' = X \cup \{b_n\colon n \in \mathbb{N}\}$ and $Y' = Y \cup \{d_n\colon n \in \mathbb{N}\}$ the proof of [50, Proposition 3] shows that $(X, \Gamma(X)) \cong_{(3)} (Y, \Gamma(Y))$. It follows from the Proposition 2.8 that $X \approx_a Y$. □

Let

$$\mathcal{X} = \{(x, y) \in \mathcal{Q}^2 \colon \forall m \neq n, y_m = 0 \vee y_n = 0\}.$$

Note that $\mathcal{X}$ is a locally star-convex continuum.

For every $(A_1, A_2, \ldots) \in K(\mathcal{Q})^{\mathbb{N}}$ we define

$$\Xi(A_1, A_2, \ldots) = \{(x, y) \in \mathcal{X} \colon \forall n\ y_n = 0 \vee x \in A_n\}.$$

We identify $\mathcal{Q}$ with $\mathcal{Q} \times \{\mathbf{0}\}$.

Recall the definition of the relation $\cong_{(1,1)}$ from [50]. If $A \subset B \subset X$ and $C \subset D \subset Y$ then $(X, B, A) \cong_{(1,1)} (Y, D, C)$ if and only if there exists a homeomorphism $f \colon X \to Y$ such that $f[A] = C$ and $f[B] = D$.

The following is [50, Proposition 4].

Proposition 2.10. *Let $\vec{A} = (A_1, A_2, \ldots) \in K(\mathcal{Q})^{\mathbb{N}}$ and $\vec{B} = (B_1, B_2, \ldots) \in K(\mathcal{Q})^{\mathbb{N}}$. Then $\vec{A} \cong_{\text{perm}} \vec{B}$ if and only if $(\mathcal{X}, \Xi(\vec{A}), \mathcal{Q}) \cong_{(1,1)} (\mathcal{X}, \Xi(\vec{B}), \mathcal{Q})$. Moreover, if $f \colon X \to Y$ and $\sigma \colon \mathbb{N} \to \mathbb{N}$ witness that $\vec{A} \cong_{\text{perm}} \vec{B}$ then $f \times h_{\sigma^{-1}}|_{\mathcal{X}}$ witnesses that $(\mathcal{X}, \Xi(\vec{A}), \mathcal{Q}) \cong_{(1,1)} (\mathcal{X}, \Xi(\vec{B}), \mathcal{Q})$, where $h_\tau \colon \mathcal{Q} \to \mathcal{Q}$ is the homeomorphism given by $h_\tau(x_1, x_2, \ldots) = (x_{\tau(1)}, x_{\tau(2)}, \ldots)$.*

Identifying $\mathcal{Q}^3$ with $\mathcal{Q}$ in an obvious way we may treat $\Psi(X)$ as an element of $K(\mathcal{Q})^{\mathbb{N}}$. Therefore it makes sense to consider $\Xi(\Psi(X))$. Note that $\Xi(\Psi(X))$ can be written as a union of a countable family of convex closed subsets of $\mathcal{X}$. Indeed, let $\mathbb{X}$ denote the family of cones and segments $\tilde{X}$ is the union of. Also let $[0,1]_n = \{te_n \colon t \in [0,1]\} \subset \mathcal{Q}$. Then $\Xi(\Psi(X))$ is the union of the following family:

$$\begin{aligned}\mathcal{F}_X = \{\mathcal{Q} \times \{\mathbf{0}\}\} &\cup \{(S_1 \times S_2 \times S_3) \times [0,1]_0 \colon S_1, S_2, S_3 \in \mathbb{X}\} \cup \{\Gamma(X) \times [0,1]_1\} \\ &\cup \bigcup_{n\in\mathbb{N}} \{(\{b_n\} \times S_1 \times S_2) \times [0,1]_{5n+2} \colon S_1, S_2 \in \mathbb{X}\} \\ &\cup \bigcup_{n\in\mathbb{N}} \{(S_1 \times \{b_n\} \times S_2) \times [0,1]_{5n+3} \colon S_1, S_2 \in \mathbb{X}\} \\ &\cup \bigcup_{n\in\mathbb{N}} \{(S_1 \times S_2 \times \{b_n\}) \times [0,1]_{5n+4} \colon S_1, S_2 \in \mathbb{X}\} \\ &\cup \bigcup_{n\in\mathbb{N}} \{(\{b_n\} \times S_1 \times S_2) \times [0,1]_{5n+5} \colon S_1, S_2 \in \mathbb{X}\} \\ &\cup \bigcup_{n\in\mathbb{N}} \{(S_1 \times \{b_n\} \times S_2) \times [0,1]_{5n+5} \colon S_1, S_2 \in \mathbb{X}\} \\ &\cup \bigcup_{n\in\mathbb{N}} \{(\{b_n\} \times S_1 \times \{b_n\}) \times [0,1]_{5n+6} \colon S_1, S_2 \in \mathbb{X}\}.\end{aligned}$$

The following proposition gives an explicit reduction of $\approx_a$ to the homeomorphism relation of locally connected continua. Borelness of this reduction follows from a routine verification. As a consequence we get Theorem 2.3.

Proposition 2.11. *The map $X \mapsto T'(\mathcal{X}, \mathcal{Q}, \mathcal{F}_X)$ is a reduction of $\approx_a$ to the homeomorphism relation of locally connected continua.*

Proof. Let $X, Y \in \mathbb{K}_{\text{Choq}}$. Assume that $f \colon X \to Y$ is an affine homeomorphism.

We use Proposition 2.6 for $\mathcal{A} = \{X\}$ and $\mathcal{C} = \{Y\}$ and we get a homeomorphism $g \colon T(X, \{X\}) \to T(Y, \{Y\})$ which is affine on the sets X_k^X, $\hat{I}_k^X$, $\tilde{I}_k^X$ for every $k \in \mathbb{N}$. Note that g determines a permutation $\sigma \colon \mathbb{N} \to \mathbb{N}$ such that g^3 and σ witness that $\Psi(X) \cong_{\text{perm}} \Psi(Y)$. Then, by proposition 2.10, $g^3 \times h_{\sigma^{-1}}|_{\mathcal{X}}$ witnesses that $(\mathcal{X}, \Xi(\Psi(X)), \mathcal{Q}) \cong_{(1,1)} (\mathcal{X}, \Xi(\Psi(Y)), \mathcal{Q})$ and since g is affine on every set X_k^X, $\hat{I}_k^X$, $\tilde{I}_k^X$, it follows that $g^3 \times h_{\sigma^{-1}}|_{\mathcal{X}}$ is affine on every set from $\mathcal{F}_X$. Therefore, using proposition 2.7, $T'(\mathcal{X}, \mathcal{Q}, \mathcal{F}_X)$ is homeomorphic to $T'(\mathcal{X}, \mathcal{Q}, \mathcal{F}_Y)$.

Conversely, assume that $T'(\mathcal{X}, \mathcal{Q}, \mathcal{F}_X)$ is homeomorphic to $T'(\mathcal{X}, \mathcal{Q}, \mathcal{F}_Y)$ and let f be a homeomorphism witnessing that. Let $S_k^X \subset T'(\mathcal{X}, \mathcal{Q}, \mathcal{F}_X)$ be the set of points x such that $T'(\mathcal{X}, \mathcal{Q}, \mathcal{F}_X) \setminus \{x\}$ consists of exactly k connected components. Note that this property is preserved by homeomorphisms. By construction of $T'(\mathcal{X}, \mathcal{Q}, \mathcal{F}_X)$, the boundary of the set S_4^X is $\mathcal{Q}$. Therefore, since $f[S_4^X] = S_4^Y$, the image of the boundary of S_4^X is equal to the boundary of S_4^Y, i.e. $f[\mathcal{Q}] = \mathcal{Q}$. Analogously, by construction we know that the boundary of the set S_3^X is $\bigcup \mathcal{F}_X = \Xi(\Psi(X))$. We conclude that $f[\Xi(\Psi(X))] = \Xi(\Psi(Y))$. The points $x \in \mathcal{X} \setminus \Xi(\Psi(X))$ are characterized by the following property: the connected component of x in the space $T'(\mathcal{X}, \mathcal{Q}, \mathcal{F}_X) \setminus \Xi(\Psi(X))$ is disjoint from $S_3^X \cup S_4^X$. Analogous statement holds for Y. It follows that $f[\mathcal{X} \setminus \Xi(\Psi(X))] = \mathcal{X} \setminus \Xi(\Psi(Y))$. Finally,

$$\begin{aligned} f[\mathcal{X}] = f[(\mathcal{X} \setminus \Xi(\Psi(X))) \cup \Xi(\Psi(X))] = f[\mathcal{X} \setminus \Xi(\Psi(X))] \cup f[\Xi(\Psi(X))] \\ = (\mathcal{X} \setminus \Xi(\Psi(Y))) \cup \Xi(\Psi(Y)) = \mathcal{X}. \end{aligned}$$

Therefore $(\mathcal{X}, \Xi(\Psi(X)), \mathcal{Q}) \cong_{(1,1)} (\mathcal{X}, \Xi(\Psi(Y)), \mathcal{Q})$. By Proposition 2.10, $\Psi(X) \cong_{\text{perm}} \Psi(Y)$. In view of Proposition 2.9 this is equivalent to $X \approx_a Y$. This finishes the proof. $\square$

Chapter 3

Measurable Hall's theorem for actions of abelian groups

3.1 Introduction

In 1925 Tarski famously asked if the unit square and the disk of the same area are equidecomposable by isometries of the plane, i.e. if one can partition one of them into finitely many pieces, rearrange them by isometries and obtain the second one. This problem became known as the Tarski circle squaring problem.

The question whether two sets of the same measure can be partitioned into congruent pieces has a long history. At the beginning of the 19th century Wallace, Bolyai and Gerwien showed that any two polygons in the plane of the same area are congruent by dissections (see [48, Theorem 3.2]) and Tarski [46] ([48, Theorem 3.9]) showed that such polygons are equidecomposable using pieces which are polygons themselves. Hilbert's 3rd problem asked if any two polyhedra of the same volume are equidecomposable using polyhedral pieces. The latter was solved by Dehn (see [1]). Banach and Tarski showed that in dimension at least 3, any two bounded sets in $\mathbb{R}^n$ with nonempty interior, are equidecomposable, which leads to the famous Banach–Tarski paradox on doubling the ball. Back in dimension 2, the situation is somewhat different, as any two measurable subsets equidecomposable by isometries must have the same measure (see [48]) and this was one of the motivation for the Tarski circle squaring problem. Using isometries was also essential as von Neumann [49] showed that the answer is positive if one allows arbitrary area-preserving transformations. The crucial feature that makes the isometries of the plane special is the fact that the group of isometries of $\mathbb{R}^2$ is amenable. Amenability was, in fact, introduced by von Neumann in the search of a combinatorial explanation of the Banach–Tarski paradox.

The first partial result on the Tarski circle squaring was a negative result of Dubins, Hirsch and Karush [10] who showed that pieces of such decompositions cannot have smooth boundary (which means that this cannot be performed using scissors). However, the full positive answer was given by Laczkovich in his deep paper [27]. In fact, in [30] Laczkovich proved a stronger result saying that whenever A and B are two bounded measurable subsets of $\mathbb{R}^n$ of positive measure such that the upper box dimension of the boundaries of A and B is less than n, then A and B are equidecomposable. The assumption on the boundary is essential since Laczkovich [28] (see also [29]) found examples of two measurable sets of the same area which are not equidecomposable even though their boundaries have even the same Hausdorff dimension. The proof of Laczkovich, however, did not provide any regularity conditions on the pieces used in the decompositions. Given the assumption that A and B have the same measure, it was natural to ask if the pieces can be chosen to be measurable. Moreover, the proof of Laczkovich used the axiom of choice.

A major breakthrough was achieved recently by Grabowski, Máthé and Pikhurko [20] who showed that the pieces in Laczkovich's theorem can be chosen to be measurable: whenever A and B are two bounded subsets of $\mathbb{R}^n$ of positive measure such that the upper box dimension of the boundaries of A and B are less than n, then A and B are equidecomposable using measurable pieces. Another breakthrough came even more recently when Marks and Unger [35] showed that for Borel sets, the pieces in the decomposition can be even chosen to be Borel, and their proof did not use the axiom of choice.

The goal of the present paper is to give a combinatorial explanation of these phenomena. There are some limitations on how far this can go because already in Laczkovich's theorem there is a restriction on the boundary of the sets A and B. Therefore, we are going to work in the measure-theoretic context and provide sufficient and necessary conditions for two sets to be equidecomposable almost everywhere. Recently, there has been a lot of effort to develop methods of the measurable and Borel combinatorics (see for instance the upcoming monograph by Marks and Kechris [32]) and we would like to work within this framework.

The classical Hall marriage theorem provides sufficient and necessary conditions for a bipartite graph to have a perfect matching. Matchings are closely connected with the existence of equidecompositions and both have been studied in this context. In 1996 Miller [38, Problem 15.10] asked whether there exists a Borel version of the Hall theorem. The question posed in such generality has a negative answer as there are examples of Borel graphs which admit perfect matchings but do not admit measurable perfect matchings. One example is provided already by the Banach-Tarski paradox and Laczkovich [26] constructed a closed graph which admits a perfect matching but does not have a measurable one. In the Baire category setting, Marks and Unger [33] proved that if a bipartite Borel graph satisfies a stronger version of Hall's condition with an additional $\varepsilon > 0$, i.e. if the set of neighbours of a finite set F is bounded from below by $(1 + \varepsilon)|F|$, then the graph admits a perfect matching with the Baire property (see also [34] and [8] for related results on matchings in this context). On the other hand, in all the results of Laczkovich [30], Grabowski, Máthé and Pikhurko [20] and Marks and Unger [35] on the circle squaring, a crucial role is played by the strong discrepancy estimates, with an $\varepsilon > 0$ such that the discrepancies of both sets are bounded by $C\frac{1}{n^{1+\varepsilon}}$ (for definitions see Section 3.2). Recall that given a finitely generated group Γ

generated by a symmetric set S and acting freely on a space X, the *Schreier graph* of the action is the graph connecting two points x and y if $\gamma \cdot x = y$ for one of the generators $\gamma \in S$.

Definition 3.1. Suppose $\Gamma \curvearrowright (X, \mu)$ is a free pmp action of a finitely generated group on a space X. Let S be a finite symmetric set of generators of Γ. Write G for the Schreier graph of the action. A pair of sets A, B satisfies the *k-Hall condition* (μ-a.e.) with respect to S if for every (μ-a.e., resp.) $x \in X$ and for every finite subset F of $\Gamma \cdot x$ we have

$$|F \cap A| \leq |\operatorname{ball}_k(F) \cap B|, \quad |F \cap B| \leq |\operatorname{ball}_k(F) \cap A|,$$

where $\operatorname{ball}_k(F)$ means the k-ball around F in the graph G.

We say that A, B satisfy the *Hall condition* (μ-a.e.) if the above holds for some choice of generators S and some $k > 0$. If the set S of generators is understood and A, B satisfy the k-Hall condition (μ-a.e.) with respect to S then we simply say that A, B satisfy the k-Hall condition (μ-a.e.).

We will work under the assumption that both sets A, B are equidistributed (for definition see Section 3.2).

Our main result is the following.

Theorem 3.2. *Let Γ be a finitely generated abelian group and $\Gamma \curvearrowright (X, \mu)$ be a free pmp action. Suppose $A, B \subseteq X$ are two measurable Γ-equidistributed sets of the same positive measure. The following are equivalent:*

1. *the pair A, B satisfies the Hall condition μ-a.e.,*

2. *A and B are Γ-equidecomposable μ-a.e. using μ-measurable sets,*

3. *A and B are Γ-equidecomposable μ-a.e.*

This result gives a positive answer to Miller's question, in the measurable setting for Schreier graphs of actions of finitely generated abelian groups and equidistributed sets, and removes the ε from the earlier results mentioned above. As a consequence, it gives the following.

Corollary 3.3. *Suppose Γ is a finitely generated abelian group and $\Gamma \curvearrowright (X, \mu)$ is a free pmp Borel action on a standard Borel probability space. Let $A, B \subseteq X$ be measurable Γ-equidistributed sets. If A and B are Γ-equidecomposable, then A and B are Γ-equidecomposable using measurable pieces.*

This generalizes the recent measurable circle squaring result [20] as already in Laczkovich's proof, he constructs an action of $\mathbb{Z}^d$ satisfying the conditions above, for a suitably chosen d (big enough, depending on the box dimensions of the boundaries).

In fact, in 1991 Gardner [18, Conjecture 6] conjectured that whenever A, B are measurable subsets of $\mathbb{R}^n$ which are Γ-equidecomposable using isometries from an amenable group Γ, then A and B are Γ-equidecomposable using measurable pieces. The above corollary confirms this conjecture in case of an abelian group Γ and Γ-equidistributed sets.

The main new idea in this paper is an application of Mokobodzki's medial means, which are measurable averaging functionals on sequences of reals. They are used together with a recent result of Conley, Jackson, Kerr, Marks, Seward and Tucker-Drob [7] on tilings of amenable group actions in averaging sequences of measurable matchings. This allows us to avoid using Laczkovich's discrepancy estimates that play a crucial role in both proofs of the measurable and Borel circle squaring. We also employ the idea of Marks and Unger in constructing bounded measurable flows. More precisely, following Marks and Unger we construct bounded integer-valued measurable flows from bounded real-valued measurable flows. However, instead of using Timár's result [47] for specific graphs induced by actions of $\mathbb{Z}^d$, we give a self-contained simple proof of the latter result, which works in the measurable setting for the natural Cayley graph of $\mathbb{Z}^d$. This is the only part of the paper which deals with abelian groups and we hope it could be generalized to a more general setting.

While this paper deals with abelian groups (the crucial and only place which works under these assumptions is Section 3.5), a positive answer to the following question would confirm Gardner's conjecture [18, Conjecture 6] for amenable groups.

Question 1. *Is the measurable version of Hall's theorem true for free pmp actions of finitely generated amenable groups?*

3.2 Equidistribution and discrepancy

Both proofs of Grabowski, Máthé and Pikhurko and of Marks and Unger use a technique that appears in Laczkovich's paper [27] and is based on discrepancy estimates. Laczkovich constructs an action of a group of the form $\mathbb{Z}^d$ for d depending on the upper box dimension of the boundaries of the sets A and B such that both sets are very well equidistributed on orbits on this action. To be more precise, given an action $\mathbb{Z}^d \curvearrowright (X, \mu)$ and a measurable set $A \subseteq X$, the *discrepancy* of A with respect to a finite subset F of an orbit of the action is defined as

$$D(F, A) = \left| \frac{|A \cap F|}{|F|} - \mu(A) \right|.$$

It is meaningful to compute the discrepancy with respect to finite cubes, i.e. subsets of orbits which are of the form $[0, n]^d \cdot x$, where $x \in X$ and $[0, n]^d \subseteq \mathbb{Z}^d$ is the d-dimensional cube with side $\{0, \ldots, n\}$. The cube $[0, n]^d$ has boundary, whose relative size with respect to the size of the cube is bounded by $c \cdot \frac{1}{n}$ for a constant c. This motivates the following definition.

Definition 3.4. Given a Borel free pmp action of a finitely generated abelian group $\mathbb{Z}^d \times \Delta \curvearrowright (X, \mu)$ with Δ finite, and a measurable set $A \subseteq X$, we say that A is *equidistributed* if there exists a constant c such that for every n the discrepancy

$$D([0,n]^d \times \Delta \cdot x, A) \leq c\frac{1}{n}$$

for μ-a.e. x.

Note that if the above holds then for every d-tuple $g_1, g_2, \ldots, g_d \in \mathbb{Z}^d$ of generators of $\mathbb{Z}^d$ there exists a constant c' such that for all n and μ-a.a. $x \in X$ the following estimate holds:

$$D(\{n_1 g_1 + \ldots n_d g_d \colon n_i \in \{0,1,\ldots,n\}\} \times \Delta \cdot x, A) \leq c'\frac{1}{n}.$$

Thus if Γ is a finitely generated abelian group and $\Gamma \curvearrowright (X,\mu)$ is a Borel free pmp action then it makes sense to say that A is equidistributed if Definition 3.4 holds for some (and therefore for every) representation of Γ in the form $\mathbb{Z}^d \times \Delta$.

A crucial estimation that appears in Laczkovich's paper is that the action of $\mathbb{Z}^d$ is such that for both sets A and B the discrepancy is actually estimated as

$$D([0,n]^d \cdot x, A), D([0,n]^d \cdot x, B) \leq c\frac{1}{n^{1+\varepsilon}},$$

for some $\varepsilon > 0$ and some $c > 0$, which means that the discrepancies of both sets on cubes decay noticeably faster than the sizes of the boundaries of these cubes.

In particular, this means that A and B satisfy the following property: for every x and cube $F = [0,n]^d \cdot x$ we have

$$||A \cap F| - |B \cap F|| \leq c\frac{1}{n^{1+\varepsilon}}$$

for some $\varepsilon > 0$ and some $c > 0$. Again, since the ratio of the boundary of the cube F to its size is at most $2d/n$, any two equidecomposable subsets must satisfy

$$||A \cap F| - |B \cap F|| \leq c\frac{1}{n}$$

for some constant c and the above condition with positive ε is not necessary for the existence of an equidecomposition. Also, for examples not satisfying this condition, see [31].

3.3 Mokobodzki's medial means

We will be working under the additional assumption of the Continuum Hypothesis. This is mainly for the purpose of the use of Mokobodzki's universally measurable medial means which exist under this (or even slightly weaker) assumption.

Definition 3.5. A *medial mean* is a linear functional $\mathrm{m} : \ell_\infty \to \mathbb{R}$ which is positive, i.e. $\mathrm{m}(f) \geq 0$ if $f \geq 0$, normalized, i.e. $\mathrm{m}(1_\mathbb{N}) = 1$ and shift invariant, i.e. $\mathrm{m}(Sf) = \mathrm{m}(f)$ where $Sf(n+1) = f(n)$.

Mokobodzki showed that under the assumption of the Continuum Hypothesis there exists a medial mean which is universally measurable as a function on $[0,1]^\mathbb{N}$, see Section 1.5. For a proof the reader can also consult the textbook of Fremlin [13, Theorem 538S]. As CH can be always true in a forcing extension or in $L[r]$ (for a suitable real r coding the Borel sets we are dealing with), the admissibility of this assumption follows from the following absoluteness lemma.

Recall that Borel sets can be coded using a $\boldsymbol{\Pi}^1_1$ set (of Borel codes) $\mathrm{BC} \subseteq 2^\mathbb{N}$ in a $\boldsymbol{\Delta}^1_1$ way, i.e. there exists a subset $C \subseteq \mathrm{BC} \times X$ such that the family $\{C_x : x \in \mathrm{BC}\}$ consists of all Borel subsets of X and the set C can be defined using both $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$ definitions. For details the reader can consult the textbook of Jech [22, Chapter 25].

Given a Borel probability measure μ on X and a subset $P \subseteq X \times Y$, we write $\forall^\mu\, xP(x,y)$ to denote that $\mu(\{x \in X : P(x,y)\}) = 1$. It is well known [24, Chapter 29E] that if P is $\boldsymbol{\Sigma}^1_1$, then $\{y \in Y : \forall^\mu x\ P(x,y)\}$ is $\boldsymbol{\Sigma}^1_1$.

Lemma 3.6. *Let $V \subseteq W$ be models of $\mathsf{ZF} + \mathsf{DC}$. Suppose in V we have a standard Borel space X with a Borel probability measure μ, two Borel subsets $A, B \subseteq X$ and $\Gamma \curvearrowright (X,\mu)$ is a Borel pmp action of a countable group Γ. The statement that the sets A and B are Γ-equidecomposable μ-a.e. using μ-measurable pieces is absolute between V and W.*

Proof. Suppose that in W the sets A and B are Γ-equidecomposable μ-a.e. Then there exist disjoint Borel subsets $A_1, \dots A_n$ of A and disjoint Borel subsets $B_1, \dots, B_n$ of B such that $\mu(A \setminus \bigcup_{i=1}^n A_i) = 0$, $\mu(B \setminus \bigcup_{i=1}^n B_i) = 0$ and $\gamma_i A_i = B_i$ for some $\gamma_1, \dots, \gamma_n \in \Gamma$. This statement can be written as

$$\exists x_1, \dots, x_n \bigwedge_{i \leq n} \mathrm{BC}(x_i) \wedge \bigwedge_{i \neq j} C_{x_i} \cap C_{x_j} = \emptyset$$
$$\wedge \ \forall^\mu x \ (x \in A \leftrightarrow \bigvee_{i=1}^n x \in C_{x_i}) \wedge \ \forall^\mu x \ (x \in B \leftrightarrow \bigvee_{i=1}^n x \in \gamma_i C_{x_i})$$

and thus is it $\mathbf{\Sigma}^1_2$. By Shoenfield's absoluteness theorem [22, Theorem 25.20], it is absolute.

□

3.4 Measurable flows in actions of amenable groups

Given a standard Borel space X, a Borel graph G on X and $f : X \to \mathbb{R}$, a function $\varphi : G \to \mathbb{R}$ is an *f-flow* if $\varphi(x, y) = -\varphi(y, x)$ for every $(x, y) \in G$ and $f(x) = \sum_{(x,y)\in G} \varphi(x, y)$ for every $x \in X$.

Let Γ be a finitely generated amenable group. Let $\gamma_1, \dots, \gamma_d$ be a finite symmetric set of generators of Γ. Let X be a standard Borel space and let μ be a Borel probability measure on X. Let $\Gamma \curvearrowright (X, \mu)$ be a free pmp action. Recall that by the *Schreier graph* of the action we mean the graph $\{(x, \gamma_i x) : x \in X, 1 \leq i \leq d\} \subseteq X \times X$.

Definition 3.7. For finite sets $F, K \subseteq \Gamma$ and $\delta > 0$ we say that F is *(K, δ)-invariant* if $|KF \triangle F| < \delta |F|$.

In the following lemma we assume that there exists a universally measurable medial mean m, which, by the remarks in the previous section, we can assume throughout this paper.

In order to make it a bit more general, let us define the Hall condition for functions: a function $f : X \to \mathbb{Z}$ satisfies the *k-Hall condition* if for every finite set F contained in an orbit of $\Gamma \curvearrowright X$ we have that

$$\sum_{x \in F : f(x) \geq 0} f(x) \leq \sum_{x \in \mathrm{ball}_k(F) : f(x) \leq 0} -f(x), \quad \sum_{x \in F : f(x) \leq 0} -f(x) \leq \sum_{x \in \mathrm{ball}_k(F) : f(x) \geq 0} f(x).$$

Note that a pair of sets A, B satisfies the k-Hall condition if and only if $f = \chi_A - \chi_B$ satisfies the k-Hall condition.

Proposition 3.8. *Let Γ be a finitely generated amenable group and $\Gamma \curvearrowright (X, \mu)$ be a Borel free pmp action. Suppose $f : X \to \mathbb{Z}$ is a measurable function such that*

- $|f| \leq l$
- *f satisfies the k-Hall condition*

for some $k, l \in \mathbb{N}$. Then there exists a Γ-invariant measurable subset $X' \subseteq X$ of measure 1 and a measurable real-valued f-flow φ on the Schreier graph of $\Gamma \curvearrowright X'$ such that

$$|\varphi| \leq l \cdot d^k,$$

where d is the number of generators of Γ.

Proof. First, we are going to assume that $|f| \leq 1$, i.e. that $f = \chi_A - \chi_B$ for two measurable subsets $A, B \subseteq X$. Indeed, consider the space $X \times l$ and the action of Γ on $X \times l$ given by $\gamma(x, i) = (\gamma x, i)$. Consider the projection $\pi \colon X \times l \to X$. Then we can find two subsets $A, B \subseteq X \times l$ such that $f(x) = |\pi^{-1}(\{x\}) \cap A| - |\pi^{-1}(\{x\}) \cap B|$. We can also induce the graph structure on $X \times l$ by taking as edges all the pairs $((x, i), (y, j))$ such that (x, y) forms an edge in X as well as all pairs $((x, i), (x, j))$ for $i \neq j$. Then A and B satisfy the k-Hall condition in $X \times l$ for the above graph.

Let $K = \{\gamma \in \Gamma \colon d(e, \gamma) \leq k\}$. Fix $\delta > 0$. Use the Conley–Jackson–Kerr–Marks–Seward–Tucker-Drob tiling theorem [7, Theorem 3.6] for K and δ to get a μ-conull Γ-invariant Borel set $X' \subseteq X$, a collection $\{C_i : 1 \leq i \leq m\}$ of Borel subsets of X', and a collection $\{F_i : 1 \leq i \leq m\}$ of (K, δ)-invariant subsets of Γ such that $\mathcal{F} = \{F_i c : 1 \leq i \leq m, c \in C_i\}$ partitions X'.

For a finite set $F \subseteq \Gamma$ define $F(K) = \{f \in F : Kf \subseteq F\}$. Note that if $F'x = F''y$ where F', F'' are finite subsets of Γ and $x, y \in X$ then $F'(K)x = F''(K)y$. If $F \subset X$ is a finite subset of a single orbit then we let $F(K) = F'(K)x$ where $F' \subset \Gamma$ and $x \in X$ satisfy $F = F'x$. This definition does not depend on the choice of representation $F = F'x$ by the previous remark. Note that if $F \subseteq X$ is a (K, δ)-invariant set lying in a single orbit then

$$|F(K)| \geq |F| - |KF \triangle F| \cdot |K| > |F| \cdot (1 - \delta|K|).$$

Write

$$H = \{(x, \gamma x) \in A \times B \colon x \in F_i(K) \cdot c \text{ for some } 1 \leq i \leq m \text{ and } c \in C_i,\ \gamma \in K\}.$$

Then H is a locally finite Borel graph satisfying Hall's condition as A, B satisfy the k-Hall condition. By the Hall theorem, there exists a Borel injection

$$h \colon A \cap \bigcup_{F \in \mathcal{F}} F(K) \to B \cap \bigcup_{F \in \mathcal{F}} F.$$

Write G for the Schreier graph of $\Gamma \curvearrowright X$. For every $x \in \operatorname{dom} h$ let $p_x = \{(x_0, x_1), (x_1, x_2), \ldots, (x_{j-1}, x_j)\}$ be the shortest lexicographically smallest path in the graph G connecting $x_0 = x$ with $x_j = h(x)$. Let $\mathcal{P} = \{p_x \colon x \in \operatorname{dom} h\}$.

Define $\varphi \colon G \to \mathbb{R}$ by the formula

$$\varphi(x, \gamma x) = |\{p \in \mathcal{P} \colon (x, \gamma x) \in p\}| - |\{p \in \mathcal{P} \colon (\gamma x, x) \in p\}|.$$

Note that φ is Borel (by definition). Also, $|\varphi|$ is bounded by d^k (the number of paths of length not greater than k passing through a given edge in the graph G). By definition, φ is a $(\chi_{\operatorname{dom} h} - \chi_{\operatorname{im} h})$-flow.

Define

$$X'' = \bigcup_{F \in \mathcal{F}} F(K) \setminus (B \setminus h(A)).$$

Note that for every $x \in X''$ we have $\chi_A(x) - \chi_B(x) = \chi_{\operatorname{dom} h}(x) - \chi_{\operatorname{im} h}(x)$.

For every $1 \leq i \leq m$ let $\{(A_{1,i}, B_{1,i}, h_{1,i}), (A_{2,i}, B_{2,i}, h_{2,i}), \ldots, (A_{n_i,i}, B_{n_i,i}, h_{n_i,i})\}$ be the set of all triples (A', B', h') consisting of sets $A', B' \subseteq F_i$ and a bijection $h' \colon A' \to B'$. For $1 \leq j \leq n_i$ define

$$C_{j,i} = \{c \in C_i \colon (\operatorname{dom} h_{j,i})c = A \cap (F_i c) \wedge \forall \gamma \in \operatorname{dom} h_{j,i}\ h_{j,i}(\gamma)c = h(\gamma c)\}.$$

Then $\{C_{1,i}, C_{2,i}, \ldots, C_{n_i,i}\}$ is a partition of C_i into Borel sets.

Observe that for every $F \in \mathcal{F}$ we have

$$|h(A) \cap F(K)| \geq |F(K) \cap A| - |F \setminus F(K)|$$

and

$$|B \cap F(K)| \leq |A \cap F| \leq |A \cap F(K)| + |F \setminus F(K)|.$$

Therefore

$$\begin{aligned}
|F(K) \cap (B \setminus h(A))| &= |(F(K) \cap B) \setminus (F(K) \cap h(A))| \\
&= |F(K) \cap B| - |F(K) \cap h(A)| \\
&\leq |A \cap F(K)| + |F \setminus F(K)| - (|F(K) \cap A| - |F \setminus F(K)|) \\
&= 2|F \setminus F(K)|.
\end{aligned}$$

It follows that

$$|F(K) \setminus (B \setminus h(A))| = |F(K)| - |F(K) \cap (B \setminus h(A))|$$
$$\geq |F(K)| - 2|F \setminus F(K)| = 3|F(K)| - 2|F| > |F|(1 - 3\delta|K|).$$

Therefore

$$\begin{aligned}\mu(X'') &= \mu(\bigcup_{i=1}^{m}\bigcup_{j=1}^{n_i}(F_i(K)C_{j,i} \setminus (B \setminus h(A)))) = \sum_{i=1}^{m}\sum_{j=i}^{n_i}|F_i(K) \setminus (B_{j,i} \setminus A_{j,i})|\mu(C_{j,i}) \\ &> \sum_{i=1}^{m}\sum_{j=i}^{n_i}|F_i|(1-3\delta|K|)\mu(C_{j,i}) = \sum_{i=1}^{m}|F_i|(1-3\delta|K|)\mu(C_i) \\ &= 1-3\delta|K|.\end{aligned}$$

Now, for every n pick $\delta_n > 0$ so that $1 - 3\delta_n|K| > 1 - \frac{1}{2^n}$. Denote $h_n = h$, $\varphi_n = \varphi$ and $X_n = X''$ where h, φ and X'' are constructed above for this particular δ_n.

Let $Y = \liminf X_n = \bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty} X_n$. Then $\mu(Y) = 1$. We can assume that Y is Γ-invariant (by taking its subset if needed). Denote by G the Schreier graph of $\Gamma \curvearrowright Y$. Write $\varphi_\infty = (\varphi_n)_{n\in\mathbb{N}} : G \to \ell_\infty$. Define

$$\varphi(x,y) = \mathrm{m}(\varphi_\infty(x,y)),$$

where m denotes the medial mean. Then for $x \in Y$ we have

$$\begin{aligned}\sum_{y\colon (x,y)\in G}\varphi(x,y) &= \sum_{y\colon (x,y)\in G}\mathrm{m}((\varphi_n(x,y))_{n\in\mathbb{N}}) = \mathrm{m}((\sum_{y\colon (x,y)\in G}\varphi_n(x,y))_{n\in\mathbb{N}}) \\ &= \mathrm{m}((\chi_{\operatorname{dom} h_n}(x) - \chi_{\operatorname{im} h_n}(x))_{n\in\mathbb{N}}) = \chi_A(x) - \chi_B(x)\end{aligned}$$

as the sequence $\chi_{\operatorname{dom} h_n}(x) - \chi_{\operatorname{im} h_n}(x)$ is eventually constant and equal to $\chi_A(x) - \chi_B(x)$.

Therefore φ is a $(\chi_A - \chi_B)$-flow in the Schreier graph G of $\Gamma \curvearrowright Y$. Moreover, $|\varphi|$ is bounded by d^k, which is a common bound for the flows φ_n. For measurability of φ, write $\mu' = \varphi_*(\mu \times \mu)$ for the pushforward to $[-d^k, d^k]^{\mathbb{N}}$ of the measure $\mu \times \mu$ on the graph G and note that since m is μ'-measurable, it follows that φ is μ-measurable.

□

3.5 Flows in $\mathbb{Z}^d$

In this section we prove a couple of combinatorial lemmas which lead to a finitary procedure of changing a real-valued flow on a cube in $\mathbb{Z}^d$ to an integer-valued flow on a cube in $\mathbb{Z}^d$. This gives an alternative proof of [35, Lemma 5.4] in the measurable setting. Also, this is the only part of the paper which deals with the groups $\mathbb{Z}^d$ as opposed to arbitrary amenable groups.

Let

$$G = \{(x, x') \in \mathbb{Z}^d \times \mathbb{Z}^d \colon x' - x \in \{\pm e_1, \pm e_2, \dots, \pm e_d\}\}$$

be the Cayley graph of $\mathbb{Z}^d$. An edge (x, x') is called *positively oriented* if $x' - x = e_j$ for some j.

Definition 3.9. For a set $A \subseteq \mathbb{Z}^d$ we define:

$$\mathrm{edges}(A) = \{(x, x + e_j) \colon j \in \{1, 2, \dots, d\},\ \{x, x + e_j\} \subseteq A\},$$

$$\mathrm{edges}^+(A) = \{(x, x + e_j) \colon j \in \{1, 2, \dots, d\},\ \{x, x + e_j\} \cap A \neq \emptyset\},$$

$$\mathrm{ball}(A) = \{x + y \colon\ x \in A,\ y \in \{-1, 0, 1\}^d\}.$$

So, $\mathrm{edges}(A)$ is the set of positively oriented edges whose both endpoints are in A, $\mathrm{edges}^+(A)$ is the set of positively oriented edges whose at least one endpoint is in A, and $\mathrm{ball}(A)$ is the 1-neighbourhood of A (in the sup-norm).

Definition 3.10. We say that a subset C of $\mathbb{Z}^d$ is a *cube* if C is of the form

$$\{n_1, n_1 + 1, \dots, n_1 + k_1\} \times \dots \times \{n_d, n_d + 1, \dots, n_d + k_d\}$$

for some $n_1, \dots, n_d, k_1, \dots, k_d \in \mathbb{Z}$ with $k_1, \dots, k_d \geq 0$. By the *upper face* of C we mean

$$\{n_1, n_1 + 1, \dots, n_1 + k_1\} \times \dots \times \{n_d + k_d\}.$$

Definition 3.11. For any $x_1, x_2, x_3, x_4 \in \mathbb{Z}^d$ which are consecutive vertices of a unit square, and a real number s we define a 0-flow $\square_s^{x_1,x_2,x_3,x_4}$ by the following formula:

$$\square_s^{x_1,x_2,x_3,x_4}(y,z) = \begin{cases} s & \text{for } (y,z) \in \{(x_1,x_2),(x_2,x_3),(x_3,x_4),(x_4,x_1)\}, \\ -s & \text{for } (y,z) \in \{(x_2,x_1),(x_3,x_2),(x_4,x_3),(x_1,x_4)\}, \\ 0 & \text{otherwise.} \end{cases}$$

That is, $\square_s^{x_1,x_2,x_3,x_4}$ is a flow sending s units through the path $x_1 \to x_2 \to x_3 \to x_4 \to x_1$.

Note that if $\varphi\colon G \to \mathbb{R}$ is an f-flow and $s = \varphi(x_1,x_4) - \lfloor \varphi(x_1,x_4) \rfloor$ then $\psi = \varphi + \square_s^{x_1,x_2,x_3,x_4}$ is an f-flow such that $|\varphi - \psi| < 1$ and $\psi(x_1,x_4)$ is an integer.

We will now prove a couple of lemmas stating that one can modify a flow so that it becomes integer-valued on certain sets of edges.

Lemma 3.12. *Let $f\colon \mathbb{Z}^d \to \mathbb{R}$. Let $\varphi\colon G \to \mathbb{R}$ be a bounded f-flow. Let*

$$C = \{n_1, n_1+1, \ldots, n_1+k_1\} \times \ldots \times \{n_{d-1}, n_{d-1}+1, \ldots, n_{d-1}+k_{d-1}\} \times \{n_d, n_d+1\}$$

for some $n_1, \ldots, n_d, k_1, \ldots, k_{d-1} \in \mathbb{Z}$ with $k_1, \ldots, k_{d-1} \geq 0$. Then for every $1 \leq \ell < d$ there is an f-flow ψ such that:

- $\operatorname{supp}(\varphi - \psi) \subseteq \operatorname{edges}(C)$,
- *for every $x = (x_1, \ldots, x_{d-1}, n_d) \in C$ such that $n_\ell \leq x_\ell < n_\ell + k_\ell$ we have*

  $$\psi(x, x+e_d) \in \mathbb{Z}.$$

- $|\varphi - \psi| < 2$.

Proof. Without loss of generality we may assume that $n_1 = n_2 = \ldots = n_d = 0$.

For every $j \leq k_\ell$ define $C_j = \{(x_1, \ldots, x_{d-1}, 0) \in C \colon x_\ell = j\}$. We will define a sequence of f-flows $\varphi_0, \varphi_1, \ldots, \varphi_{k_\ell}$ such that

$$\varphi_j(x, x+e_d) \in \mathbb{Z} \quad \text{and} \quad \operatorname{supp}(\varphi - \varphi_j) \subseteq \operatorname{edges}(C).$$

for all $x \in \bigcup_{i<j} C_i$.

So, let $\varphi_0 = \varphi$. Given φ_j we define φ_{j+1} in the following way. For every $x \in C_j$ let $\square_x = \square_s^{x,y,z,t}$ where $y = x + e_\ell$, $z = y + e_d$, $t = z - e_\ell = x + e_d$ and

$$s = \varphi_j(x,t) - \lfloor \varphi_j(x,t) \rfloor.$$

We define

$$\varphi_{j+1} = \varphi_j + \sum_{x \in C_j} \square_x.$$

Note that $\operatorname{supp}(\square_x)$ for $x \in C_j$ are disjoint from $\{(x, x+e_d) \colon x \in \bigcup_{i<j} C_i\}$. Therefore, $\varphi_{j+1}(x, x+e_d) = \varphi_j(x, x+e_d) \in \mathbb{Z}$ for $x \in \bigcup_{i<j} C_i$. Also, the sets $\operatorname{supp}(\square_x)$ are pairwise disjoint for $x \in C_j$, and therefore, by definition of φ_{j+1} we have for $x \in C_j$

$$\varphi_{j+1}(x, x+e_d) = \varphi_j(x, x+e_d) + \square_x(x, x+e_d) = \lfloor \varphi_j(x, x+e_d) \rfloor \in \mathbb{Z}.$$

It is also clear that $\operatorname{supp}(\square_x) \subseteq \operatorname{edges}(C)$, so

$$\operatorname{supp}(\varphi - \varphi_{j+1}) \subseteq \operatorname{supp}(\varphi - \varphi_j) \cup \bigcup_{x \in C_j} \operatorname{supp}(\square_x) \subseteq \operatorname{edges}(C).$$

Therefore φ_{j+1} satisfies all required properties.

We put $\psi = \varphi_{k_\ell}$. It remains to check that $|\varphi - \psi| < 2$. This is because $\psi = \varphi + \sum_{j=0}^{k_\ell - 1} \sum_{x \in C_j} \square_x$, $|\square_x| < 1$ and for every edge (y,z) there are at most two $x \in \bigcup_{j<k_\ell} C_j$ for which $\square_x(y,z) \neq 0$. $\square$

Lemma 3.13. *Let $f \colon \mathbb{Z}^d \to \mathbb{R}$. Let $\varphi \colon G \to \mathbb{R}$ be a bounded f-flow. Let*

$$C = \{n_1, n_1+1, \ldots, n_1+k_1\} \times \ldots \times \{n_{d-1}, n_{d-1}+1, \ldots, n_{d-1}+k_{d-1}\} \times \{n_d, n_d+1\}$$

for some $n_1, \ldots, n_d, k_1, \ldots, k_{d-1} \in \mathbb{Z}$ *with* $k_1, \ldots, k_{d-1} \geq 0$*. Then there is an* f*-flow* ψ *such that:*

- $\mathrm{supp}(\varphi - \psi) \subseteq \mathrm{edges}(C)$,
- *if* $x = (x_1, \ldots, x_{d-1}, n_d) \in C \setminus \{(n_1 + k_1, n_2 + k_2, \ldots, n_{d-1} + k_{d-1}, n_d)\}$*, then*
$$\psi(x, x + e_d) \in \mathbb{Z},$$
- $|\varphi - \psi| < 2d$.

Proof. Without loss of generality we may assume that $n_1 = n_2 = \ldots = n_d = 0$.

Define

$$C_j = \{k_1\} \times \ldots \times \{k_{\ell-1}\} \times \{0, 1, \ldots, k_\ell\} \times \ldots \times \{0, 1, \ldots, k_{d-1}\} \times \{0, 1\}$$

and

$$D_j = \{(x_1, \ldots, x_{d-1}, 0) \colon (x_1, \ldots, x_j) \neq (k_1, \ldots, k_j)\}.$$

By induction, construct f-flows $\varphi_0, \varphi_1, \ldots, \varphi_{d-1}$ such that

(i) $\mathrm{supp}(\varphi - \varphi_j) \subseteq \mathrm{edges}(C)$,

(ii) $\varphi_j(x, x + e_d) \in \mathbb{Z}$ for every $x \in D_j$,

(iii) $|\varphi_j - \varphi_{j-1}| < 2$.

We define $\varphi_0 = \varphi$. Given φ_{j-1}, we obtain φ_j by applying Lemma 3.12 for φ_{j-1}, f, $\ell = j$ and C_j. Then φ_j satisfies (i) as

$$\mathrm{supp}(\varphi - \varphi_j) \subseteq \mathrm{supp}(\varphi - \varphi_{j-1}) \cup \mathrm{supp}(\varphi_{j-1} - \varphi_j) \subseteq \mathrm{edges}(C) \cup \mathrm{edges}(C_j) = \mathrm{edges}(C).$$

For (ii) observe that

$$D_j = D_{j-1} \cup \{(k_1, \ldots, k_{j-1}, x_j, \ldots, x_{d-1}, 0) \in C \colon x_j < k_j\}.$$

By Lemma 3.12, φ_j agrees with φ_{j-1} on $\{(x, x+e_d) \colon x \in D_{j-1}\}$, thus $\varphi_j(x, x+e_d) \in \mathbb{Z}$ for $x \in D_{j-1}$. Moreover, $\varphi_j(x, x+e_d) \in \mathbb{Z}$ for $x \in D_j \setminus D_{j-1}$ again by Lemma 3.12. Also (iii) is immediate by Lemma 3.12. Therefore φ_j satisfies the required properties.

We define $\psi = \varphi_{d-1}$. By construction, ψ satisfies the first two conditions. For the third condition note that

$$|\varphi - \psi| \leq \sum_{j=1}^{d-1} |\varphi_j - \varphi_{j-1}| < 2d.$$

□

Lemma 3.14. *Let C be a cube. Let $\mathcal{C}$ be a collection of cubes such that:*

- $\operatorname{ball}(C') \subseteq C$ *for every* $C' \in \mathcal{C}$,
- $\operatorname{ball}(C') \cap \operatorname{ball}(C'') = \emptyset$ *for every distinct* $C', C'' \in \mathcal{C}$.

Write

$$E = \operatorname{edges}^+(C) \setminus \bigcup \{\operatorname{edges}(\operatorname{ball}(C')) \colon C' \in \mathcal{C}\}.$$

Let $f \colon \mathbb{Z}^d \to \mathbb{Z}$. Let $\varphi \colon G \to \mathbb{R}$ be a bounded f-flow. Then there exists an f-flow $\psi \colon G \to \mathbb{R}$ such that:

- $\operatorname{supp}(\varphi - \psi) \subseteq \operatorname{edges}(\operatorname{ball}(C))$,
- $\operatorname{supp}(\varphi - \psi)$ *is disjoint from* $\operatorname{edges}^+(C')$ *for every* $C' \in \mathcal{C}$,
- $\psi(e)$ *is integer for every edge* $e \in E$,
- $|\varphi - \psi| < 6d$.

Proof. Without loss of generality we may assume that

$$C = \{1, 2, \ldots, k_1\} \times \ldots \times \{1, 2, \ldots, k_d\}$$

for some positive integers $k_1, \ldots, k_d$. Then

$$\mathrm{ball}(C) = \{0, 1, \ldots, k_1 + 1\} \times \ldots \times \{0, 1, \ldots, k_d + 1\}.$$

For any $0 \leq k \leq k_d$ let $H_k = \mathbb{Z}^{d-1} \times \{k\}$. Let

$$E_{2k} = \{(x, x + e_d) \in E \colon x \in H_k\}$$

be the set of vertical edges from E having their starting point in H_k and

$$E_{2k+1} = \{(x, x + e_j) \in E \colon x \in H_k, j < d\}$$

be the set of edges from E having both endpoints in H_k.

We construct a sequence $\varphi_0, \varphi_1, \ldots, \varphi_{2k_d}$ of f-flows so that

- $\mathrm{supp}(\varphi - \varphi_k) \subseteq \mathrm{edges}(\mathrm{ball}(C))$ for every $0 \leq k \leq 2k_d$,
- $\mathrm{supp}(\varphi - \varphi_k)$ is disjoint from $\mathrm{edges}^+(C')$ for every $C' \in \mathcal{C}$ and $0 \leq k \leq 2k_d$,
- $\varphi_k(y, z)$ is integer for every $0 \leq k \leq 2k_d$ and $(y, z) \in \bigcup_{i \leq k} E_i$.

In the end we put $\psi = \varphi_{2k_d}$.

To define φ_0 we use Lemma 3.12 for φ, f, $\ell = 1$, and the cube

$$\{1, 2, \ldots, k_1 + 1\} \times \{1, 2, \ldots, k_2\} \times \{1, 2, \ldots, k_3\} \times \ldots \times \{1, 2, \ldots, k_{d-1}\} \times \{0, 1\}.$$

Suppose that φ_{2k} is defined. Now we define φ_{2k+1} (cf. Fig. 3.1). For every edge $(x, y) \in E_{2k+1}$ let $z = y + e_d$, $t = x + e_d$, $s = -\varphi_{2k}(x, y) + \lfloor \varphi_{2k}(x, y) \rfloor$ and $\square_{(x,y)} = \square_s^{x,y,z,t}$.

Define $\varphi_{2k+1} = \varphi_{2k} + \sum \square_{(x,y)}$ where the summation goes over all $(x, y) \in E_{2k+1}$.

Note that φ_{2k+1} assumes integer values on all $(x, y) \in E_{2k+1}$. Indeed, if $(x', y') \in E_{2k+1}$ is distinct from (x, y) then $\square_{(x',y')}(x, y) = 0$ and so

$$\varphi_{2k+1}(x, y) = \varphi_{2k}(x, y) + \square_{(x,y)}(x, y) = \lfloor \varphi_{2k}(x, y) \rfloor \in \mathbb{Z}.$$

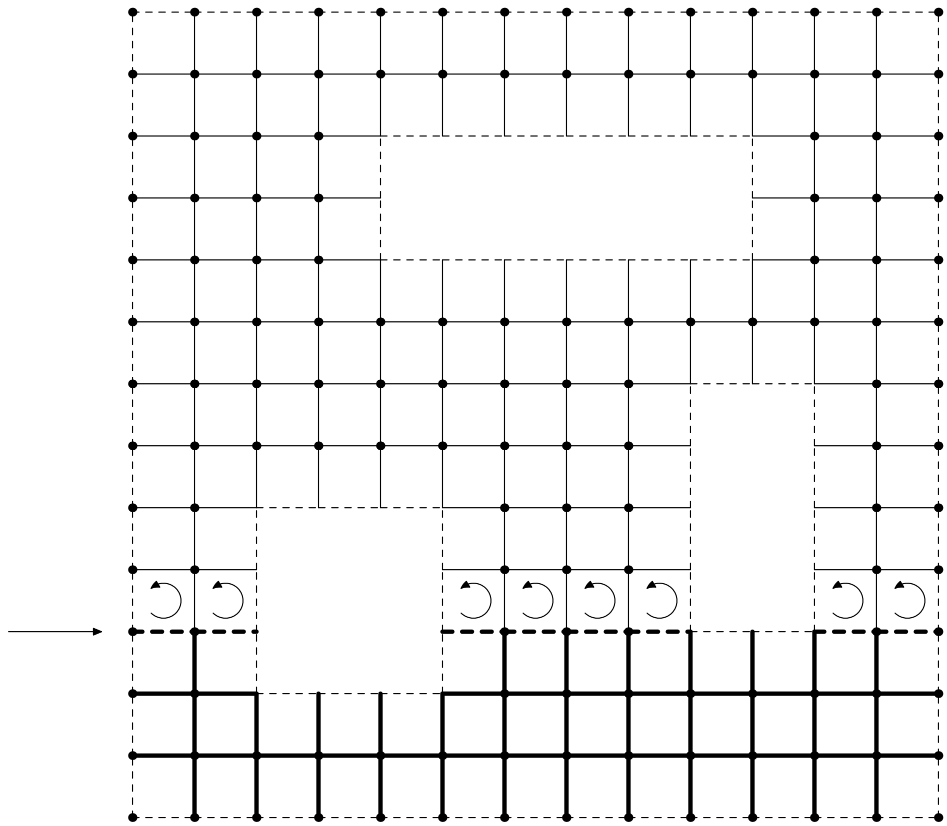


Figure 3.1: Construction of φ_{2k+1}.

Moreover, by definition, φ_{2k+1} agrees with φ_{2k} on $\bigcup_{i\leq 2k} E_i$. It follows that φ_{2k+1} is integer-valued on $\bigcup_{i\leq 2k+1} E_i$.

Since for every $(x,y)\in E_{2k+1}$ we have $\operatorname{supp}(\square_{(x,y)})\subseteq \operatorname{edges}(\operatorname{ball}(C))$ and $\operatorname{supp}(\square_{(x,y)})\cap \operatorname{edges}^+(C')=\emptyset$ for every $C'\in\mathcal{C}$, and φ_{2k} satisfies these as well by inductive hypothesis, we see that φ_{2k+1} also has these properties.

Thus φ_{2k+1} is as required.

Now suppose that φ_{2k+1} is defined. We construct φ_{2k+2} (cf. Fig. 3.2). Let $D=\{x\colon (x,x+e_d)\in E_{2k+2}\}$. Note that every $x\in D$ is either an element of $C\setminus\bigcup\{\operatorname{ball}(C')\colon C'\in\mathcal{C}\}$ or lies on the upper face of some cube $\operatorname{ball}(C')$ for $C'\in\mathcal{C}$. We also note that if $C'\in\mathcal{C}$ then the upper face of $\mathrm{ball}(C')$ is either contained in D or disjoint from D. So, let $C_1, C_2, \ldots C_n$ be all elements of $\mathcal{C}$ such that the upper faces $D_1, D_2, \ldots, D_n$ of $\mathrm{ball}(C_1), \mathrm{ball}(C_2), \ldots, \mathrm{ball}(C_n)$ are subsets of D.

Let $(x, x+e_d) \in E_{2k+2}$. Then either $x \in D_j$ for some $j \leq n$ or $x \in D \setminus \bigcup_{j \leq n} D_j$.

First we deal with the case $x \in D \setminus \bigcup_{j \leq n} D_j$. Then $(x - e_d, x) \in E_{2k}$ and $(x, x+e_i), (x - e_i, x) \in E_{2k+1}$ for every $1 \leq i \leq d-1$. By the inductive hypothesis

$$\varphi_{2k+1}(x, x \pm e_1), \varphi_{2k+1}(x, x \pm e_2), \ldots, \varphi_{2k+1}(x, x \pm e_{d-1}), \varphi_{2k+1}(x, x - e_d) \in \mathbb{Z}.$$

Since $f(x) \in \mathbb{Z}$ and

$$f(x) = \sum_{i=1}^{d} \varphi_{2k+1}(x, x \pm e_i),$$

it follows that $\varphi_{2k+1}(x, x+e_d) \in \mathbb{Z}$.

Next we deal with the case $x \in D_j$ for some $j \leq n$. Each $D_j, j \leq n$ is dealt with separately. For every $j \leq n$ we obtain an f-flow φ'_j by applying Lemma 3.13 for φ_{2k+1}, f and the cube

$$D'_j = D_j \cup (D_j + e_d) = \{n'_1, \ldots, n'_1 + k'_1\} \times \ldots \times \{n'_{d-1}, \ldots, n'_{d-1} + k'_{d-1}\} \times \{n'_d, n'_d + 1\}.$$

Then φ'_j agrees with φ_{2k+1} outside of $\mathrm{edges}(D'_j)$, and φ'_j is also integer-valued on all edges of the form $(x, x+e_d)$ with $x \in D_j \setminus \{x'\}$, where

$$x' = (n'_1 + k'_1, n'_2 + k'_2, \ldots, n'_{d-1} + k'_{d-1}, n'_d).$$

The only problematic edge is the one $(x', x'+e_d)$ We claim that $\varphi'_j(x', x'+e_d)$ is integer as well.

Indeed, observe that

$$\sum_{x \in \mathrm{ball}(C_j)} f(x) = \sum_{(x,y) \in E, x \in \mathrm{ball}(C_j), y \notin \mathrm{ball}(C_j)} \varphi'_j(x, y)$$

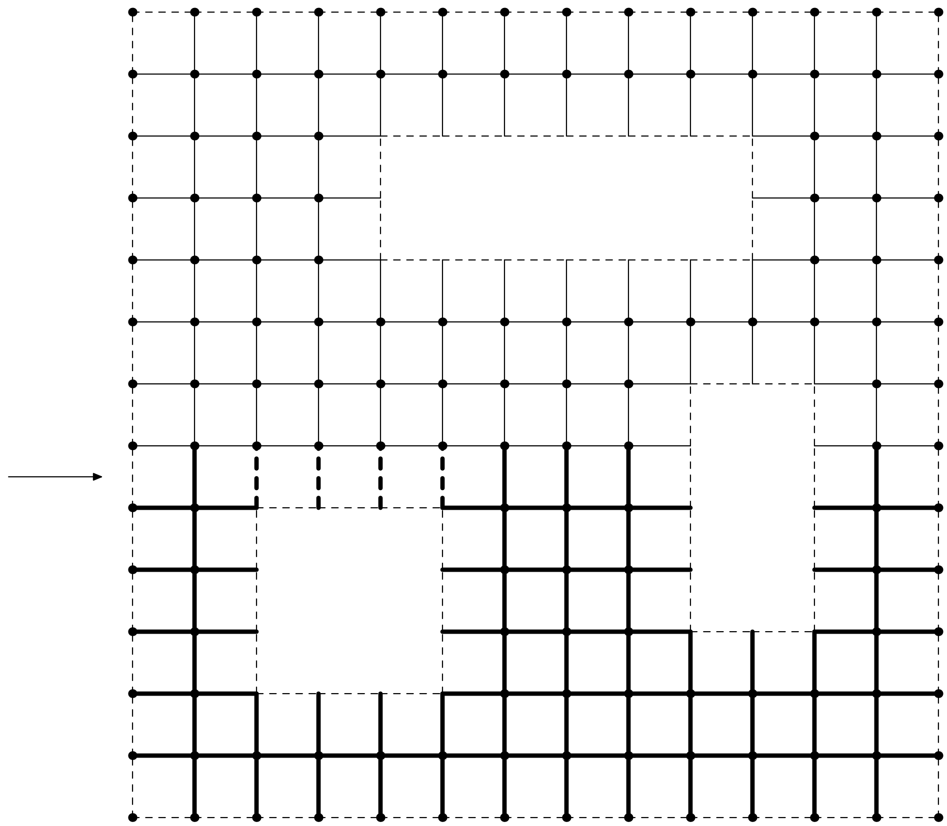


Figure 3.2: Construction of φ_{2k}.

Since $f(x) \in \mathbb{Z}$ for every x and, by the properties of φ'_j, we have that $\varphi'_j(x,y) \in \mathbb{Z}$ for all $(x,y) \neq (x', x'+e_d)$ with $x \in \operatorname{ball}(C_j)$ and $y \notin \operatorname{ball}(C_j)$, it follows that $\varphi'_j(x', x'+e_d) \in \mathbb{Z}$ as well.

We define φ_{2k+2} by the formula

$$\varphi_{2k+2}(x,y) = \begin{cases} \varphi'_j(x,y) & \text{if } (x,y) \in \operatorname{edges}(D'_j) \text{ or } (y,x) \in \operatorname{edges}(D'_j) \text{ for some } j, \\ \varphi_{2k+1}(x,y) & \text{otherwise.} \end{cases}$$

φ_{2k+2} is well-defined because $\operatorname{edges}(D'_j)$ are pairwise disjoint. By definition, it is integer-valued on $\bigcup_{i \leq 2k+2} E_i$, and the conditions on $\operatorname{supp}(\varphi - \varphi_{2k+2})$ are clearly satisfied. Thus

φ_{2k+2} is as required.

We put $\psi = \varphi_{2k_d}$. It remains to check that $|\varphi - \psi| < 6d$. This follows from the fact that the value on every edge was modified at most three times by at most $2d$. □

3.6 Measurable bounded $\mathbb{Z}$-flows a.e.

In this section we show how to turn a measurable bounded real-valued flow into a measurable bounded integer-valued flow on a set of measure 1. We only use Lemma 3.14 proved in the previous section and the Gao–Jackson tiling theorem for actions of $\mathbb{Z}^d$.

Suppose $\mathbb{Z}^d \curvearrowright (X, \mu)$ is a free pmp action. We follow the notation from the previous section in the context of the action.

Definition 3.15. We say that a finite subset of X is a *cube* if it is of the form

$$(\prod_{i=1}^{d} k_i) \cdot x = (\{0, 1, \ldots, k_1\} \times \ldots \times \{0, 1, \ldots, k_d\}) \cdot x$$

for some positive integers $k_1, \ldots, k_d$ and $x \in X$. We refer to the numbers $k_1, \ldots, k_d$ as to the lengths of the sides of the cube. A family of cubes $\{(\prod_{i=1}^{d} k_i(x)) \cdot x : x \in C\}$ is *Borel* if the set C is Borel and the functions k_i are Borel. A family of cubes $\{C_x : x \in C\}$ is a *tiling* of X if it forms a partition of X.

Definition 3.16. Let $\mathcal{C} \subseteq [X]^{<\infty}$ be a collection of cubes. We say that it is *nested* if for every distinct $C, C' \in \mathcal{C}$:

- if $C \cap C' = \emptyset$ then $\mathrm{ball}(C) \cap \mathrm{ball}(C') = \emptyset$,
- if $C \cap C' \neq \emptyset$ then either $\mathrm{ball}(C) \subseteq C'$ or $\mathrm{ball}(C') \subseteq C$.

Definition 3.17. Given a cube of the form

$$C = \{(n_1, \ldots, n_d) \cdot x : 0 \leq n_i \leq N_i\},$$

by its *interior* we mean the cube

$$\operatorname{int} C = \{(n_1, \ldots, n_d) \cdot x : 1 \leq n_i \leq N_i - 1\}$$

and its *boundary* is

$$\operatorname{bd} C = C \setminus \operatorname{int} C.$$

Lemma 3.18. *Suppose $\mathbb{Z}^d \curvearrowright (X, \mu)$ is a free pmp action. Then there is a sequence of familes F_n of cubes such that each F_n consists of disjoint cubes, $\bigcup F_n$ is nested and covers X up to a set of measure zero.*

Proof. If S and T are families of sets, define

$$S \sqcap T = \{C \cap C' \colon C \in S,\ C' \in T,\ C \cap C' \neq \emptyset\}.$$

Note that $\bigcup(S \sqcap T) = (\bigcup S) \cap (\bigcup T)$. Also note that if S and T are families of cubes then $S \sqcap T$ is a family of cubes as well. We also write $\operatorname{int} S = \{\operatorname{int} C \colon C \in S\}$ and int^k for the k-th iterate of int.

Use the Gao–Jackson theorem [16] to obtain a sequence of partitions $S_1, S_2, \ldots$ of X so that S_n consists of cubes with sides n^3 or n^3+1. Define $S_n^1 = \operatorname{int} S_n$ and $S_n^k = S_n^{k-1} \sqcap \operatorname{int}^k S_{n+k}$ for $k > 1$. Note that each S_n^k consists of pairwise disjoint cubes.

Define

$$F_n = \liminf_m S_n^m = \{C \colon \exists m_0 \forall m \geq m_0\ C \in S_n^m\}.$$

Note that if $C \in F_n$ then there exist unique cubes $C_n \in S_n, C_{n+1} \in S_{n+1}, \ldots$ such that $C = \bigcap_{k \geq 0} \operatorname{int}^{k+1} C_{n+k}$. Also note that $\bigcup F_n = \bigcap_{k=0}^{\infty} \bigcup \operatorname{int}^{k+1} S_{n+k}$.

We claim that $F = \bigcup_n F_n$ is nested and covers a set of measure 1.

For nestedness, consider cubes $C, C' \in F$. Then $C \in F_n$, $C' \in F_m$ for some n, m. We may assume that $m \geq n$. Write $C = \bigcap_{k \geq 0} \operatorname{int}^{k+1} C_{n+k}$ and $C' = \bigcap_{k \geq 0} \operatorname{int}^{k+1} C_{m+k}$ with $C_k, C'_k \in S_k$.

If $m = n$ and $C_k = C'_k$ for all $k \geq m$ then $C = C'$.

If $m > n$ and $C_k = C'_k$ for all $k \geq m$ then

$$C \subseteq \bigcap_{k \geq m} \operatorname{int}^{k-n+1} C_k = \bigcap_{k \geq m} \operatorname{int}^{k-n+1} C'_k \subseteq \bigcap_{k \geq m} \operatorname{int}^{k-m+2} C_k,$$

so $\operatorname{ball} C \subseteq \bigcap_{k \geq m} \operatorname{int}^{k-m+1} C_k = C'$.

If $C_k \neq C'_k$ for some $k \geq m$ then $C_k \cap C'_k = \emptyset$. Note that $C \subseteq \operatorname{int}^{k-n+1} C_k \subseteq \operatorname{int} C_k$ so $\operatorname{ball} C \subseteq C_k$. Similarly, $\operatorname{ball} C' \subseteq C'_k$. Since $C_k, C'_k \in S_k$ are disjoint, C and C' are disjoint.

This shows that F is nested.

We will prove now that $\mu(\bigcup F) = 1$.

For a cube C let x_C to be the point $x \in X$ such that $C = \left(\prod_{i=1}^{d} [0, n_i]\right) \cdot x_C$. For a positive integer n write $X_n = \{x_C \colon C \in S_n\}$. Note that for any $0 \leq k < n$

$$\mu\left(\bigcup \operatorname{int}^k S_n\right) \geq (n^3 - 2k)^d \mu(X_n) \geq \frac{(n^3 - 2k)^d}{(n^3+1)^d} = \left(1 - \frac{2k+1}{n^3+1}\right)^d \geq 1 - d \cdot \frac{2k+1}{n^3+1}.$$

Since $\bigcup F_n = \bigcap_{k=0}^{\infty} \bigcup \operatorname{int}^{k+1} S_{n+k}$, we have

$$\mu\left(X \setminus \bigcup F_n\right) \leq \sum_{k=0}^{\infty} \mu\left(X \setminus \bigcup \operatorname{int}^{k+1} S_{n+k}\right) \leq d \cdot \sum_{k=0}^{\infty} \frac{2k+3}{(n+k)^3+1} \leq d \cdot \sum_{k=n}^{\infty} \frac{3}{k^2}.$$

This implies that

$$\mu\left(X \setminus \bigcup F\right) = \lim_{n \to \infty} \mu\left(X \setminus \bigcup F_n\right) = 0.$$

Hence $\mu\left(\bigcup F\right) = 1$.

□

Marks and Unger [35, Lemma 5.4] showed that for every $d \geq 2$, any Borel, bounded real-valued flow on the Schreier graph of a free Borel action of $\mathbb{Z}^d$ can be modified to a bounded Borel integer-valued flow. Below we provide a short proof for the case $d = 1$ and additionally an independent proof (based on Lemma 3.14) for $d \geq 2$ in the case of a pmp action where we consider flows defined a.e.

Proposition 3.19. *Suppose $\mathbb{Z}^d \curvearrowright (X, \mu)$ is a free pmp action and G is its Schreier graph. Let $f : X \to \mathbb{Z}$ be a bounded measurable function. For every measurable f-flow $\varphi : G \to \mathbb{R}$, there exists a measurable bounded $\psi : G \to \mathbb{Z}$ such that:*

- *ψ is an f-flow μ-a.e.,*
- *$|\psi| \leq |\varphi| + 12d$.*

Proof. First we deal with the case $d = 1$. In that case for every $e \in G$ we simply put $\psi(e) = \lfloor \varphi(e) \rfloor$. Note that since G is a graph of degree 2, for every $x \in X$, the fractional parts of the two edges which contain x are equal because f is integer-valued. Thus, ψ is also an f-flow.

Now suppose $d \geq 2$. By Lemma 3.18, there exists an invariant subset $X' \subseteq X$ of measure 1 and a sequence of families F_n of cubes such that $\bigcup_{n \in \mathbb{N}} F_n$ is nested, each F_n consists of disjoint cubes, $\bigcup_{n \in \mathbb{N}} F_n$ covers X'. By induction on n we construct measurable f-flows φ_n such that $\varphi_0 = \varphi$ and

- $\mathrm{supp}(\varphi_{n+1} - \varphi_n) \subseteq \bigcup \{\mathrm{edges}(\mathrm{ball}(C)) : C \in F_n\}$,
- $\varphi_m = \varphi_{n+1}$ for every $m > n$ on every $\mathrm{edges}^+(C)$ for $C \in F_n$,
- $|\varphi_n| \leq |\varphi| + 12d$.

Given the flow φ_n we apply Lemma 3.14 on each cube $C \in F_n$ to obtain the flow φ_{n+1}. The bound on φ_n follows from the fact that the value of the folow on each edge is changed at most twice by at most $6d$ along this construction.

The sequence φ_n converges pointwise on the edges of X' to a measurable f-flow φ_∞, which is integer-valued on all edges in X' except possibly for the edges in $\mathrm{bd}\, C$ for cubes $C \in \bigcup_n F_n$. However, the family $\{\mathrm{bd}\, C : C \in \bigcup_n F_n\}$ consists of pairwise disjoint finite sets. By the integral flow theorem for finite graphs, we can further correct φ_∞ on each of these finite subgraphs without changing the bound $|\varphi|+12d$ to obtain a measurable integer-valued f-flow ψ, which is equal to φ_∞ on all edges from $G \setminus \bigcup\{\mathrm{edges}(\mathrm{bd}\, C) : C \in \bigcup_{n\in\mathbb{N}} F_n\}$. □

3.7 Hall's theorem

In this section we prove Theorem 3.2. The proof of (1)⇒(2) is based on an idea of Marks and Unger [35].

Proof of Theorem 3.2. (2)⇒(3) is obvious.

(3)⇒(1) is true for every finitely generated group Γ. In general, if A and B are Γ-equidecomposable, and the group elements used in the decomposition are $\gamma_1, \ldots, \gamma_n$, then A and B satisfy the k-Hall condition for k greater than the word lengths of the group elements $\gamma_1, \ldots, \gamma_n$. If $X' \subseteq X$ is a set of measure 1 such that $A \cap X'$ and $B \cap X'$ are Γ-equidecomposable, then $A \cap X'$ and $B \cap X'$ satisfy the k-Hall condition.

(1)⇒(2). Without loss of generality assume that the k-Hall condition is satisfied everywhere and the equidistribution condition

$$D([0,n]^d \times \Delta \cdot x, A), D([0,n]^d \times \Delta \cdot x, B) \leq c\frac{1}{n}$$

holds for all x. Write $\alpha = \mu(A) = \mu(B)$. Let $\Gamma = \mathbb{Z}^d \times \Delta$ where Δ is a finite group and $d \geq 0$.

If $d = 0$, then the group Γ is finite and the action has finite orbits (the discrepancy condition trivializes and we do not need to use it). On each orbit the Hall condition is satisfied, so on each orbit there exists a bijection between A and B on that orbit. Thus, the sets A and B are Δ-equidecomposable using a Borel choice of bijections on each orbit separately.

Thus, we can assume for the rest of the proof that $d \geq 1$. Since Δ is finite, we can quotient by its action and get a standard Borel space $X' = X/\Delta$ with the probability measure induced by the quotient map $\pi : X \to X'$. We then have a free pmp action of $\mathbb{Z}^d \curvearrowright X'$. Consider the function $f : X' \to \mathbb{Z}$ defined by

$$f(x') = |A \cap \pi^{-1}(\{x'\})| - |B \cap \pi^{-1}(\{x'\})|.$$

Note that f is bounded by $|\Delta|$. Using Proposition 3.8 and Proposition 3.19 we get an invariant subset $Y' \subseteq X'$ of measure 1 and an integer-valued measurable f-flow ψ on the edges of the Schreier graph G of $\mathbb{Z}^d \curvearrowright Y'$ on Y' such that $|\psi| \leq |\Delta|\, d^k + 12d$. Again, without loss of generality, we can assume $Y' = X'$ by replacing X with $Y = \pi^{-1}(Y')$, if needed.

Note that there exists a constant K, depending only on d such that for every tiling of $\mathbb{Z}^d$ with cubes with sides n or $n+1$, every cube is adjacent to at most K many other cubes in the tiling.

Note that equidistribution implies that

$$|A \cap D|, |B \cap D| \geq \alpha(n+1)^d |\Delta| - c\,|\Delta| \frac{(n+1)^d}{n}.$$

Now, let n be such that

$$\alpha(n+1)^d |\Delta| - c\,|\Delta| \frac{(n+1)^d}{n} \geq K(n+1)^{d-1}(|\Delta|\, d^k + 12d). \tag{$*$}$$

Using the Gao–Jackson theorem [16], find a Borel tiling T' of X' with cubes of sides n or $n+1$. Pulling back the tiling to X via π, we get a Borel tiling T of X with cubes of the form $D = (C \times \Delta) \cdot x$ where C has sides of length n or $n+1$. Note that the assumption that both A and B are equidistributed in X with constant c and the estimate $(*)$ imply that for every tile D in T we have

$$|A \cap D|, |B \cap D| \geq K(n+1)^{d-1}(|\Delta|\, d^k + 12d). \tag{$**$}$$

Let H be the graph on T where two cubes are connected with an edge if they are adjacent and similarly let H' be the graph on T' with two cubes connected with an edge if they are adjacent. We have two functions $F' : T' \to \mathbb{Z}$ defined as $F'(C) = \sum_{x' \in C} f(x')$ and $F : T \to \mathbb{Z}$ defined as

$$F(C) = |A \cap C| - |B \cap C|.$$

Define an F'-flow Ψ' on H' as $\Psi'(C, D) = \sum_{(x_1', x_2') \in G, x_1' \in C, x_2' \in D} \psi(x_1', x_2')$ and let Ψ be an F-flow on H obtained by pulling back Ψ' via π. Note that any adjacent cubes in T' are connected by at most $(n+1)^{d-1}$ edges, so both Ψ and Ψ' are bounded by $|\Psi|, |\Psi'| \leq (n+1)^{d-1}(|\Delta| \, d^k + 12d)$.

Note that each vertex in H' has degree at most K and the same is true in H.

Thus, by $(**)$, for each $C \in T$ and $D \in T$ which are connected with an edge in H, we can find pairwise disjoint sets $A(C, D), B(C, D) \subseteq C$ of size at least $(n+1)^{d-1}(|\Delta| \, d^k + 12d)$ such that $A(C, D) \subseteq A \cap C$, $B(C, D) \subseteq B \cap C$.

Now, the function which witnesses the equidecomposition is defined in two steps. First, for each C, D if $\Psi(C, D) > 0$, then move $\Psi(C, D)$ points from $B(C, D)$ to $A(C, D)$ and if $\Psi(D, C) > 0$, then move $\Psi(D, C)$ points from $B(D, C)$ to $A(D, C)$. After this step, for each $C \in T$ we have $|A \cap C| = |B \cap C|$ and we can find a measurable bijection which within each C maps points of $A \cap C$ onto $B \cap C$. Since ψ and hence Ψ' and Ψ are measurable, in each of the two steps, the bijections can be chosen measurable and they move points by at most $2(|\Delta| + (n+1)^d)$ in the Schreier graph distance. Thus, their composition witnesses that A and B are equidecomposable using measurable pieces. □

3.8 Measurable circle squaring

In this section we comment on how Corollary 3.3 follows from Theorem 3.2. We use an argument which appears in a preprint of Grabowski, Máthé and Pikhurko [19] and provide a short proof for completeness.

Lemma 3.20. *Suppose $\Gamma \curvearrowright (X,\mu)$ is a free pmp action of a countable group Γ. If $A, B \subseteq X$ are Γ-equidecomposable and $X' \subseteq X$ is Γ-invariant, then $A \cap X'$ and $B \cap X'$ are also equidecomposable. If X' is additionally μ-measurable and A and B are Γ-equidecomposable using μ-measurable pieces, then $A \cap X'$ and $B \cap X'$ are Γ-equidecomposable using μ-measurable pieces.*

Proof. The proof is the same in both cases. Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be partitions of A and B such that $\gamma_i A_i = B_i$ for some $\gamma_i \in \Gamma$. Put $A_i' = A_i \cap X'$ and $B_i' = B_i \cap X'$. Then $\gamma_i A_i' = B_i'$, so A_i' and B_i' witness that $A \cap X'$ and $B \cap X'$ are equidecomposable. □

Lemma 3.21. *Let μ be a probability measure on X and $\Gamma \curvearrowright X$ be a Borel pmp action of a countable group Γ. Suppose $A, B \subseteq X$ are Γ-equidecomposable and there exists a measurable set $Y \subseteq X$ of measure 1 such that $A \cap Y, B \cap Y$ are equidecomposable using μ-measurable piecces. Then A, B are equidecomposable using μ-measurable pieces.*

Proof. Write $X' = \bigcap_{\gamma \in \Gamma} \gamma X$. Note that $\mu(X') = 1$ and $\gamma X' = X'$ for all $\gamma \in \Gamma$. By Lemma 3.20, $A' = A \cap X'$ and $B' = B \cap X'$ are Γ-equidecomposable using μ-measurable pieces. Write $X'' = X \setminus X'$ and note that $\gamma X'' = X''$ for all $\gamma \in \Gamma$. By the previous lemma again, $A'' = A \cap X''$ and $B'' = B \cap X''$ are Γ-equidecomposable. However, all pieces in the latter decomposition all μ-null, hence μ-measurable. This shows that $A = A' \cup A''$ and $B = B' \cup B''$ are Γ-equidecomposable using μ-measurable pieces.

□

Finally, we give a proof of Corollary 3.3.

Proof of Corollary 3.3. Suppose $\Gamma \curvearrowright (X,\mu)$ is a free pmp action of a finitely generated abelian group Γ and A and B are two measurable Γ-equidistributed sets which are Γ-equidecomposable. Note that since Γ is amenable, A and B must have the same measure (see [48, Corollary 10.9]). Let $\gamma_1, \ldots, \gamma_n$ be the elements of Γ used in the equidecomposition and let k be bigger than the lengths of γ_i. Then A and B satisfy the k-Hall condition. In particular, A and B satisfy the k-Hall condition μ-a.e., so by Theorem 3.2 there is a Γ-invariant measurable set $X' \subseteq X$ of measure 1 such that $A \cap X'$ and $B \cap X'$ are Γ-equidecomposable using μ-measurable pieces. By Lemma 3.21, A and B are Γ-equidecomposable using μ-measurable pieces as well. $\square$

Chapter 4

Lifting of invariant measures

4.1 Introduction

In this chapter we address the following question asked by Feliks Przytycki:

Question. *Let X be a compact metric space and Y a Polish space. Let $T\colon X \to X$, $S\colon Y \to Y$ be continuous maps. Let $p\colon Y \to X$ be a Borel surjection with $p \circ S = T \circ p$. Let μ be a T-invariant Borel probability measure on X. When does μ lift to an S-invariant Borel probability measure on Y?*

The answer is affirmative under the assumption that S is injective, fibers of p are finite, and the sets $\{x \in X\colon |p^{-1}(x)| = n\}$ are T-invariant (for instance, this holds if S and T are homeomorphisms). A special case of this ($|p^{-1}(x)| \leq 2$ for all $x \in X$) appeared in the proof of [43, Corollary 10.2]. An obvious modification of Przytycki's argument shows that one can lift μ to an S-invariant measure ν where ν is defined by

$$\nu(A) = \int_X \frac{|A \cap p^{-1}(x)|}{|p^{-1}(x)|} \mathrm{d}\mu(x).$$

It is also known that if Y is compact and p is continuous then μ lifts to an S-invariant measure ν. Note that p induces the push-forward map $p_*\colon P(Y) \to P(X)$ (here, $P(Y)$ and $P(X)$ denote the spaces of all Borel probability measures on Y and X, respectively) which is a continuous surjection, so the preimage of μ is a non-empty compact subset K of $P(Y)$. Clearly, K is convex. Since μ is T-invariant and $p \circ S = T \circ p$, we obtain $S_*(K) \subset K$. Hence by Schauder's fixed-point theorem there exists $\nu \in K$ with $\nu = S_*(\nu)$. This means: ν is a lift of μ which is S-invariant.

On the other hand, if the assumption on compactness of fibers of p is dropped then it may happen that μ does not lift to an S-invariant measure even if Y is compact, T is the identity map and S is a homeomorphism. For instance, let $X = \{0, 1\}$ and $Y = \mathbb{Z} \cup \{\infty\}$ be the one-point compactification of the countable discrete space $\mathbb{Z}$. Let $T = \mathrm{id}_X$, $S(n) = n+1$ for $n \in \mathbb{Z}$, $S(\infty) = \infty$, $p(n) = 0$ for $n \in \mathbb{Z}$, $p(\infty) = 1$, and $\mu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. Suppose that ν is an S-invariant measure on Y. By S-invariance, $\nu(\{n\}) = \nu(\{0\})$ for all $n \in \mathbb{Z}$. If $\nu(\{0\}) = 0$ then $\nu(\mathbb{Z}) = \sum_{n\in\mathbb{Z}} \nu(\{n\}) = 0$ and if $\nu(\{0\}) > 0$ then $\nu(\mathbb{Z}) = \sum_{n\in\mathbb{Z}} \nu(\{n\}) = \infty$. In both cases $\nu(\mathbb{Z}) \neq \frac{1}{2}$, hence μ does not lift to an S-invariant measure.

We shall work in a more general context. We drop the assumption on compactness of X and continuity of T. The following result generalizes both special cases discussed above.

Theorem 4.1. *Let X be a standard Borel space with a Borel probability measure μ and let $T : X \to X$ be a μ-measurable map preserving μ. Let Y be a Polish space and let $S\colon Y \to Y$ be a continuous map. Let $p\colon Y \to X$ be a Borel map such that $p \circ S = T \circ p$ and $\mu(p(Y)) = 1$. Suppose that for μ-a.a. $x \in X$ the set $p^{-1}(x)$ is compact. Then there exists a Borel probability measure ν on Y which is S-invariant and $p_*(\nu) = \mu$.*

One can prove even more general result: instead of single maps S and T one can work with a left amenable semigroup Γ (for instance, an abelian semigroup) acting on Y by continuous maps and acting on X by measure-preserving maps so that the actions of Γ on Y and X commute with p.

Theorem 4.2. *Let X be a standard Borel space with a Borel probability measure μ. Let Y be a Polish space. Let $p\colon Y \to X$ be a Borel map with $\mu(p(Y)) = 1$ and such that the set $p^{-1}(x)$ is compact for μ-a.a. $x \in X$. Let Γ be a left amenable semigroup. Consider actions $\Gamma \curvearrowright Y$, $\Gamma \curvearrowright X$ so that:*

- *Γ acts on Y by continuous maps, i.e. for all $\gamma \in \Gamma$ the map $S_\gamma\colon Y \to Y$, $S_\gamma(y) = \gamma y$ is continuous,*
- *μ is Γ-invariant, i.e. for all $\gamma \in \Gamma$ the map $T_\gamma\colon X \to X$, $T_\gamma(x) = \gamma x$ preserves μ,*
- *The actions of Γ on Y and X commute with p, i.e. $p \circ S_\gamma = T_\gamma \circ p$ for all $\gamma \in \Gamma$.*

Then there exists a Γ-invariant Borel probability measure ν on Y such that $p_*(\nu) = \mu$.

Clearly, Theorem 4.1 is a special case of Theorem 4.2; to see this just take $\Gamma = (\mathbb{N}, +)$ with actions on X and Y given by $\mathbb{N} \times X \ni (n, x) \mapsto T^n x \in X$ and $\mathbb{N} \times Y \ni (n, y) \mapsto S^n y \in Y$, respectively. Therefore it is enough to prove Theorem 4.2. Nevertheless, we provide a proof of Theorem 4.1 which avoids using tools from theory of amenable semigroups.

4.2 Preliminaries

In this section we recall some definitions and useful facts.

A standard Borel space is an uncountable set X with a σ-algebra Σ of subsets of X such that there exists a Polish (i.e. separable, completely metrizable) topology τ on X whose Borel σ-algebra is Σ.

Given a topological space Y we denote by $K(Y)$ the collection of all compact subsets of Y. The set $K(Y)$ can be endowed with Vietoris topology, i.e. the topology generated by sets

$$\{K \in K(Y)\colon K \cap U \neq \emptyset\} \quad \text{and} \quad \{K \in K(Y)\colon K \subset U\}$$

where $U \subset Y$ is open. If Y is Polish, compact, then $K(Y)$ is Polish, compact, respectively.

For a Polish space Y we denote by $P(Y)$ the set of all Borel probability measures on Y endowed with the weak* topology, i.e. the topology generated by sets of the form

$$\left\{\sigma \in P(Y)\colon \left|\int_Y f \mathrm{d}\sigma - \int_Y f \mathrm{d}\sigma_0\right| < \varepsilon\right\}$$

where $\sigma_0 \in P(Y)$, $f\colon Y \to \mathbb{R}$ is continuous and bounded, and $\varepsilon > 0$. Traditionally, a somewhat erroneous terminology is in use: a sequence of measures convergent in the weak* topology is sometimes said to converge weakly. If Y is a compact metric space then $P(Y)$ is a compact metric space.

Recall that semigroup Γ is called left amenable if there exists a left invariant mean for Γ. For more on amenability of semigroup the reader may wish to consult [41, 0.18].

4.3 Proof of Theorems 4.1 and 4.2

We start with the following key lemma.

Lemma 4.3. *Let X be a standard Borel space with a Borel probability measure μ. Let Y be a Polish space. Let $p\colon Y \to X$ be a Borel map such that $\mu(p(Y)) = 1$. Let $M \subset P(Y)$ be the set of all measures σ with $p_*(\sigma) = \mu$. If for μ-a.a. $x \in X$ the set $p^{-1}(x)$ is compact then M is a non-empty convex compact subset of $P(Y)$.*

Proof. Suppose additionally that Y is compact. The general case will be considered later.

First of all, the set M is non-empty. For instance, by [24, 18.3] there exists a μ-measurable function $u\colon p(Y) \to Y$ with $u(x) \in p^{-1}(x)$ for all $x \in p(Y)$. Define a measure $\sigma \in P(Y)$ by $\sigma(B) = \int_{p(Y)} \delta_{u(x)}(B) \mathrm{d}\mu(x)$. Then $\sigma \in M$. Secondly, it is clear that M is convex. It remains to prove that M is compact. Let $\nu_1, \nu_2, \nu_3, \dots$ be a sequence of elements of M convergent to some $\nu \in P(Y)$. We shall prove that $\nu \in M$, i.e. that $p_*(\nu) = \mu$.

Claim 1. Let $A \subset X$ be a Borel set. Then $p_*(\nu)(A) \geq \mu(A)$.

Proof of Claim 1. This is trivial if $\mu(A) = 0$, so let us assume that $\mu(A) > 0$. Fix $\varepsilon > 0$. Endow X with a Polish topology giving X its Borel structure. Let $X' \subset X$ be a Borel set of full measure such that for all $x \in X'$ the set $p^{-1}(x)$ is compact.

Let $f\colon X' \to K(Y)$ be given by $f(x) = p^{-1}(x)$. We shall prove that f is Borel. Recall that the Borel structure of $K(Y)$ is generated by sets of the form $B = \{K \in K(Y)\colon K \cap U \neq \emptyset\}$ where $U \subset Y$ is open (see [24, 12.C]). Therefore it is enough to prove that the set $f^{-1}(B)$ is Borel whenever B is of the aforementioned form. Note that

$$\begin{aligned} f^{-1}(B) &= \{x \in X'\colon f(x) \in B\} = \{x \in X'\colon f(x) \cap U \neq \emptyset\} \\ &= \{x \in X'\colon \exists y \in U\ p(y) = x\} = \pi_X(\mathrm{graph}(p) \cap (U \times X')), \end{aligned}$$

which is Borel by 1.3. Hence f is Borel.

By Lusin's Theorem there exists a non-empty compact subset $K \subset A \cap X'$ such that $\mu(K) > \mu(A) - \varepsilon$ and the function $f|_K\colon K \to K(Y)$ is continuous. Then the set $\{f(x)\colon x \in K\}$ is compact in $K(Y)$, as it is a continuous image of a compact set. By [24, 4.29], the set $f(K) = \bigcup\{f(x)\colon x \in K\} = p^{-1}(K)$ is a compact subset of Y.

Since ν_n converges to ν weakly and $p^{-1}(K)$ is compact, we have by Portmanteau lemma

$$p_*(\nu)(K) = \nu(p^{-1}(K)) \geq \limsup_{n\to\infty} \nu_n(p^{-1}(K)) = \limsup_{n\to\infty} \mu(K) = \mu(K).$$

It follows that $p_*(\nu)(A) \geq p_*(\nu)(K) \geq \mu(K) \geq \mu(A) - \varepsilon$. Since $\varepsilon > 0$ can be chosen arbitrarily, the claim follows. □

Claim 2. Let $A \subset X$ be a Borel set. Then $p_*(\nu)(A) \leq \mu(A)$.

Proof of Claim 2. Claim 1 for the set $X \setminus A$ gives $p_*(\nu)(X \setminus A) \geq \mu(X \setminus A)$. This can be rewritten as $1 - p_*(\nu)(A) \geq 1 - \mu(A)$, hence $p_*(\nu)(A) \leq \mu(A)$. □

Claims 1 and 2 imply that $p_*(\nu)(A) = \mu(A)$ for all Borel sets $A \subset X$. Therefore $p_*(\nu) = \mu$, which proves that M is closed in $P(Y)$ and hence compact. This finishes the proof in the case when Y is compact.

It remains to consider the case when Y is non-compact. Recall that any Polish space embeds homeomorphically into the Hilbert cube $[0,1]^{\mathbb{N}}$ as a G_δ subset. Write $Y' = [0,1]^{\mathbb{N}}$ for brevity and view Y as a subspace of Y'. Let Σ be the Borel σ-algebra of X. Let $X' = X \cup \{*\}$. Let $\Sigma' = \Sigma \cup \{A \cup \{*\} \colon A \in \Sigma\}$. Then Σ' gives X' a structure of standard Borel space. Let μ' be a Borel probability measure on X' given by $\mu'(B) = \mu(B \cap X)$ for any $B \in \Sigma'$. Let $p' \colon Y' \to X'$ be given by $p'(y) = p(y)$ if $y \in Y$ and $p'(y) = *$ otherwise. Note that p' is Borel. Let $M' \subset P(Y')$ be the set of all measures σ' with $p_*(\sigma') = \mu'$. Then X', Y', μ', p', and M' satisfy the hypotheses of the lemma and in addition Y' is compact, so M' is a non-empty convex subset of $P(Y')$. It is clear that the map $M \ni \sigma \mapsto \sigma' \in P(Y')$ given by $\sigma'(B) = \sigma(B \cap Y)$ maps M onto M' homeomorphically. Therefore M is a non-empty compact subset of $P(Y)$, which obviously is convex as well. $\square$

We prove Theorem 4.1 using the averaging trick.

Proof of Theorem 4.1. Let $M \subset P(Y)$ be the set of all measures σ with $p_*(\sigma) = \mu$. By Lemma 4.3, M is non-empty, convex and compact.

Pick an arbitrary $\sigma \in M$. For all positive integers n define

$$\nu_n = \frac{1}{n} \sum_{i=0}^{n-1} (S^i)_*(\sigma).$$

Note that for all i

$$p_*((S^i)_*(\sigma)) = (p \circ S^i)_*(\sigma) = (T^i \circ p)_*(\sigma) = (T^i)_*(p_*(\sigma)) = (T^i)_*(\mu) = \mu$$

so $(S^i)_*(\sigma) \in M$ for all i and since M is convex $\nu_n \in M$ for all n. So, by compactness of M there exists a subsequence $\nu_{n_1}, \nu_{n_2}, \nu_{n_3}, \ldots$ convergent to some $\nu \in M$. Then ν is

S-invariant by the proof of the Bogolyubov-Krylov theorem (see [45, Theorem 1.1]). Hence ν is as required. □

The averaging trick can be used to prove Theorem 4.2 provided Γ admits a Følner sequence, i.e. an increasing sequence of finite sets $F_n \subset \Gamma$ such that $\Gamma = \bigcup_{n\in\mathbb{N}} F_n$ and $\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0$ for all $g \in \Gamma$. This is the case for instance for amenable groups and for abelian semigroups. However, there exist amenable semigroups admitting no Følner sequences, so we need a different method to prove Theorem 4.2.

Proof of Theorem 4.2. Let $M \subset P(Y)$ be the set of all measures σ satisfying $p_*(\sigma) = \mu$. By Lemma 4.3, M is a non-empty convex compact subset of $P(Y)$.

Note that the action $\Gamma \curvearrowright Y$ induces an action $\Gamma \curvearrowright P(Y)$ by push-forwards: $\gamma\sigma = (S_\gamma)_*(\sigma)$. Also, $\Gamma M \subset M$. Indeed, for any $\gamma \in \Gamma$ and $\sigma \in M$

$$p_*(\gamma\sigma) = p_*((S_\gamma)_*(\sigma)) = (p \circ S_\gamma)_*(\sigma) = (T_\gamma \circ p)_*(\sigma) = (T_\gamma)_*(p_*(\sigma)) = (T_\gamma)_*(\mu) = \mu.$$

Hence by Day's fixed-point theorem [9] there exists $\nu \in M$ with $\nu = (S_\gamma)_*(\nu)$ for all $\gamma \in \Gamma$. □

Bibliography

[1] Vladimir G. Boltianskiĭ. *Hilbert's third problem.* V. H. Winston & Sons, Washington, D.C.; Halsted Press [John Wiley & Sons], New York-Toronto-London, 1978. Translated from the Russian by Richard A. Silverman, With a foreword by Albert B. J. Novikoff, Scripta Series in Mathematics.

[2] C. Chang and S. Gao. The complexity of the classification problem of continua. *Proc. Amer. Math. Soc.* to appear.

[3] Tomasz Cieśla. Completeness of the homeomorphism relation of locally connected continua. *Proceedings of the American Mathematical Society*, 147(3):1269–1276, 2019.

[4] Tomasz Cieśla. On lifting invariant probability measures. *arXiv preprint arXiv:1910.12341*, 2019.

[5] Tomasz Cieśla and Marcin Sabok. Measurable Hall's theorem for actions of abelian groups. *arXiv preprint arXiv:1903.02987*, 2019.

[6] John D. Clemens. Isometry of Polish metric spaces. *Ann. Pure Appl. Logic*, 163(9):1196–1209, 2012.

[7] Clinton T. Conley, Steve C. Jackson, David Kerr, Andrew S. Marks, Brandon Seward, and Robin D. Tucker-Drob. Følner tilings for actions of amenable groups. *Math. Ann.*, 371(1-2):663–683, 2018.

[8] Clinton T. Conley and Benjamin D. Miller. Measurable perfect matchings for acyclic locally countable Borel graphs. *J. Symb. Log.*, 82(1):258–271, 2017.

[9] Mahlon M Day et al. Fixed-point theorems for compact convex sets. *Illinois Journal of Mathematics*, 5(4):585–590, 1961.

[10] Lester Dubins, Morris W. Hirsch, and Jack Karush. Scissor congruence. *Israel J. Math.*, 1:239–247, 1963.

[11] G.A. Elliott, I. Farah, V. Paulsen, C. Rosendal, A. Toms, and A. Törnquist. The isomorphism relation for separable C*-algebras. *Mathematical Research Letters*. to appear.

[12] Ilijas Farah, Andrew S. Toms, and Asger Törnquist. Turbulence, orbit equivalence, and the classification of nuclear C^*-algebras. *J. Reine Angew. Math.*, (688):101–146, 2014.

[13] D. H. Fremlin. *Measure theory. Vol. 5. Set-theoretic measure theory. Part I.* Torres Fremlin, Colchester, 2015. Corrected reprint of the 2008 original.

[14] Alex Furman. Banach limit(s) using Hahn-Banach theorem. Available at `http://homepages.math.uic.edu/~furman/4students/Banach-LIM.pdf`.

[15] Su Gao. *Invariant descriptive set theory*, volume 293 of *Pure and Applied Mathematics (Boca Raton)*. CRC Press, Boca Raton, FL, 2009.

[16] Su Gao and Steve Jackson. Countable abelian group actions and hyperfinite equivalence relations. *Invent. Math.*, 201(1):309–383, 2015.

[17] Su Gao and Alexander S. Kechris. On the classification of Polish metric spaces up to isometry. *Mem. Amer. Math. Soc.*, 161(766), 2003.

[18] R. J. Gardner. Measure theory and some problems in geometry. *Atti Sem. Mat. Fis. Univ. Modena*, 39(1):51–72, 1991.

[19] Łukasz Grabowski, András Máthé, and Oleg Pikhurko. Measurable equidecompositions for group actions with an expansion property. 2016. `arXiv:1601.02958`.

[20] Łukasz Grabowski, András Máthé, and Oleg Pikhurko. Measurable circle squaring. *Ann. of Math. (2)*, 185(2):671–710, 2017.

[21] J.G. Hocking and G.S. Young. *Topology*. Addison-Wesley series in mathematics. Dover Publications, 1988.

[22] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[23] Asaf Karagila. Zornian functional analysis or: How i learned to stop worrying and love the axiom of choice. Available at `http://karagila.org/wp-content/uploads/2016/10/axiom-of-choice-in-analysis.pdf`.

[24] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[25] Casimir Kuratowski. Evaluation de la classe borélienne ou projective d'un ensemble de points à l'aide des symboles logiques. *Fundamenta Mathematicae*, 17(1):249–272, 1931.

[26] M. Laczkovich. Closed sets without measurable matching. *Proc. Amer. Math. Soc.*, 103(3):894–896, 1988.

[27] M. Laczkovich. Equidecomposability and discrepancy; a solution of Tarski's circle-squaring problem. *J. Reine Angew. Math.*, 404:77–117, 1990.

[28] M. Laczkovich. Discrepancy estimates for sets with small boundary. *Studia Sci. Math. Hungar.*, 30(1-2):105–109, 1995.

[29] M. Laczkovich. Equidecomposability of Jordan domains under groups of isometries. *Fund. Math.*, 177(2):151–173, 2003.

[30] Miklós Laczkovich. Decomposition of sets with small boundary. *J. London Math. Soc. (2)*, 46(1):58–64, 1992.

[31] Miklós Laczkovich. Uniformly spread discrete sets in $\mathbf{R}^d$. *J. London Math. Soc. (2)*, 46(1):39–57, 1992.

[32] Andrew Marks and Alexander Kechris. *Descriptive graph combinatorics.* upcoming monograph.

[33] Andrew Marks and Spencer Unger. Baire measurable paradoxical decompositions via matchings. *Adv. Math.*, 289:397–410, 2016.

[34] Andrew S. Marks. A determinacy approach to Borel combinatorics. *J. Amer. Math. Soc.*, 29(2):579–600, 2016.

[35] Andrew S. Marks and Spencer T. Unger. Borel circle squaring. *Ann. of Math. (2)*, 186(2):581–605, 2017.

[36] Julien Melleray. A Polish metric space whose group of isometries induces a universal relation for Polish group actions. *Fund. Math.*, 239(1):43–49, 2017.

[37] Paul-André Meyer. Limites médiales, d'après Mokobodzki. In *Séminaire de probabilités VII*, pages 198–204. Springer, 1973.

[38] Arnold Miller. *Some interesting problems*. 1996. `arXiv:math/9604209`.

[39] Gabriel Mokobodzki. Ultrafiltres rapides sur ℕ. construction d'une densité relative de deux potentiels comparables. *Séminaire Brelot-Choquet-Deny. Théorie du potentiel*, 12:1–22, 1969.

[40] A Yu Ol'shanskii. On the problem of the existence of an invariant mean on a group. *Russian Mathematical Surveys*, 35(4):180, 1980.

[41] Alan LT Paterson. *Amenability*. Number 29. American Mathematical Soc., 2000.

[42] Janusz Pawlikowski. The Hahn-Banach theorem implies the Banach-Tarski paradox. *Fundamenta Mathematicae*, 138(1):21–22, 1991.

[43] Feliks Przytycki. Thermodynamic formalism methods in one-dimensional real and complex dynamics. *Proc. Int. Cong. of Math.*, 2:2081–2106, 2018.

[44] Marcin Sabok. Completeness of the isomorphism problem for separable C*-algebras. *Invent. Math.*, 204(3):833–868, 2016.

[45] Y. G. Sinai. *Dynamical systems II: Ergodic theory with applications to dynamical systems and statistical mechanics*. Volume 2 of Encyclopaedia of Mathematical Sciences. Springer-Verlag Berlin Heidelberg 1989.

[46] Alfred Tarski. O równoważności wielokątów. *Przegląd Matematyczno-Fizyczny*, 1-2, 1924.

[47] Ádám Timár. Boundary-connectivity via graph theory. *Proc. Amer. Math. Soc.*, 141(2):475–480, 2013.

[48] Grzegorz Tomkowicz and Stan Wagon. *The Banach-Tarski paradox*, volume 163 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, New York, second edition, 2016. With a foreword by Jan Mycielski.

[49] John von Neumann. Zur allgemeinen Theorie des Masses. *Fundamenta Mathematicae*, 13:73–116, 1929.

[50] Joseph Zielinski. The complexity of the homeomorphism relation between compact metric spaces. *Adv. Math.*, 291:635–645, 2016.