TWO INVESTIGATIONS IN QUANTUM GRAVITY

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Abstract

Two aspects of quantum field theory in curved spacetimes are discussed. First, the limits for applicability of the equivalence principle in the context of low energy effective field theories is considered. In particular, we find three classes of higher-derivative interactions for the gravitational and electromagnetic fields which produce dispersive photon propagation. One of these classes of interactions also produces birefringent propagation. This result is illustrated by calculating the energy-dependent contribution to the bending of light. In the second part, the divergences appearing in statistical black hole entropy are analysed. Using a Pauli-Villars regulator, it is shown that 't Hooft's approach to evaluating black hole entropy through a statistical-mechanical counting of states for a scalar field propagating outside the event horizon yields precisely the one-loop renormalization of the standard Bekenstein-Hawking formula, $S = \mathcal{A}/(4G)$, where $\mathcal{A}$ is the black hole area. The calculation also yields a constant contribution to the black hole entropy, which may be associated with the one-loop renormalization of certain higher curvature terms in the gravitational action. The calculation of black hole entropy is done for a Schwarzschild black hole as well as for a Reissner-Nordström black hole.

Résumé

Deux aspects de la théorie des champs quantiques en espace-temps courbe sont examinés. D'abord, la limite d'applicabilité du principe d'équivalence est abordée dans le contexte de la théorie effective des champs à basses énergies. En particulier, nous trouvons trois classes d'interactions d'ordres élevés qui produisent une dispersion dans la propagation des photons. Une de ces classes d'interactions produit aussi de la biréfringence. Ceci est illustré en calculant la contribution de la déflection de la lumière qui dépend de l'énergie des photons. La seconde partie analyse les divergences qui apparaissent dans le calcul de l'entropie statistique des trous noirs. En utilisant une régularisation de Pauli-Villars, nous trouvons que la méthode introduite par 't Hooft pour calculer l'entropie des trous noirs à partir d'un comptage des états d'un champ scalaire qui se déplace à l'extérieur de l'horizon donne précisément la renormalisation à une boucle de la formule habituelle de Bekenstein et Hawking, $S = \mathcal{A}/(4G)$, où $\mathcal{A}$ est la surface du trou noir. Nos calculs donnent aussi une contribution constante à l'entropie du trou noir, associée à la renormalisation à une boucle de termes de l'action gravitationnelle d'ordre plus élevé en courbure. Le calcul de l'entropie est effectué pour un trou noir de Schwarzschild ainsi que pour un trou noir de Reissner-Nordström.

À Catherine, Angélique, Marie et Geneviève

Preface

I want to express all my gratitude to my supervisor Prof. Robert Myers who guided me in the world of black holes and supported me through all these years. His financial support was also essential to the completion of the present work. I also want to thank Prof. Cliff Burgess for valuable discussions and Dr Jean-Guy Demers for interesting collaborations in black hole physics.

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Statement of original contributions

The present thesis contains materials that were previously published in refs. [1, 2]. The chapter 1 is a general review. The chapter 2 presents the results of the paper [1] that I have done in collaboration with my supervisor Robert Myers. Sections 2.1, 2.2 and 2.3 are reviews of previous results. Sections 2.4 and 2.5, is my contribution and was discussed at each stage with my supervisor, except section 2.5.2 which was done by my supervisor.

The chapter 3 presents the results that appeared in [2]. This paper was done in collaboration with Jean-Guy Demers and Robert Myers. Sections 3.1, 3.2 and 3.3 are reviews materials. The section 3.4 is my calculation, except the calculation leading to eq. (3.72) which was done originally by Robert Myers. In section 3.5, all the materials were done and discussed by the whole group but the calculations of sections 3.5.3 and 3.5.4 were originally done by Jean-Guy Demers. Finally, the last chapter is the conclusion and was done by myself, with the advice of my supervisor and the appendices contain some calculations that appeared previously.

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Chapter 1

Introduction

1.1 Purposes of the thesis

The 20^{th} century has brought two very successful theories to describe the physical world. On the one hand, quantum field theory is the foundation of the standard model of particle physics and describes the microscopic world with a great accuracy. On the other hand, General Relativity is the theory describing physics at very large scales, up to the size of the universe, and as well it explains the dynamics of stars and galaxies. It is a classical theory of gravity and spacetime. It is generally believed that the fundamental framework for the description of all fields should be quantum field theory. Therefore, classical General Relativity is an incomplete theory because it treats the gravitational field as classical. It is also incomplete because it predicts that singularities of spacetime arise in the beginning of the expansion of the universe (the Big Bang) and in the collapse of stars to form black holes* (see ref. [3]). At these singularities, General Relativity will break down. It is hoped that a quantum

*Black holes are defined in section 1.2.

theory of gravity will be able to address what will happen at these singularities.

Despite many efforts during the past fifty years, there is still no consistent quantum theory of gravity. The procedures to quantize a classical theory, which are very successful for the other interactions, have encountered fundamental difficulties. The essential difference between General Relativity and other classical theories is the lack of a background spacetime for General Relativity. Indeed, for other interactions, one quantizes the fields in a flat background. On the other hand for gravity, the spacetime is a dynamical field and one cannot use a background geometry from the start. It seems necessary to radically change our idea of particles and spacetime. At the moment, there are two leading theories to build a quantum theory of gravity: string theory [4] and non-perturbative canonical gravity[5]. In string theory, one replaces point-like particles by strings (we give some introductory materials on string theory in section 1.3). In the canonical gravity, one uses a Hamiltonian formulation of General Relativity and one is led to a quantization of the geometry.

Without a definite quantum theory of gravity, one can still look for a semiclassical approximation where the gravitational field is considered classically but the matter fields are treated quantum-mechanically. The quantum effects should become important for scales of the order of the Planck length $l_P = (\hbar G/c^3)^{1/2}$. The Planck length is so small ($l_P \sim 10^{-33}$ cm, twenty powers of ten below the size of the atom nucleus) that one should be able to build a sensible semiclassical theory of gravity. This theory is called quantum field theory in curved spacetime.

The effect of quantum gravity should become important near the time of the Big Bang or near a black hole singularity. A black hole is a region of spacetime where the gravitational attraction is so strong that nothing, even light, can escape. The gravitational attraction is produced by a compact massive object. A black hole can be produced by a massive collapsing star at the late stage of its evolution. It is also believed that black holes form the center of galaxies. Recently, this claim was confirmed by the observation by the Hubble Space Telescope of massive black holes in the galaxies M87 [6] and NGC 4261 [7]. In this thesis, we are concerned with the possible applications of quantum field theory in curved spacetime to black hole physics.

The study of quantum field theory in curved spacetime has led to two important discoveries. First, Hawking discovered[8] that black holes radiate subatomic particles with a thermal spectrum. As we will see in section 3.1.1, this result establishes a strong connection between black hole dynamics and ordinary thermodynamics, that was suspected before from results of classical General Relativity[9]. In particular, it establishes the idea that black holes have entropy, proportional to the black hole area. Black hole entropy should be related to information loss during its formation and when matter falls inside the black hole [10, 11]. The idea of analysing black holes from a thermodynamical point of view is called black hole thermodynamics. The understanding of black hole entropy, though, is only within a thermodynamic framework, and despite a great deal of effort, a microphysical understanding of this entropy is still lacking. Many attempts have been made to provide a definition of black hole entropy using statistical mechanics (we review the different methods in section 3.2) but in these calculations, divergences appear in the entropy[12, 13].

The second discovery, obtained by Drummonds and Hathrell[14], is the limit of applicability of the equivalence principle in quantum field in curved spacetime. By vacuum polarization, point-like photons acquire an effective size and they are sensitive to tidal interactions. The first consequence is the existence of superluminal motion (without necessarily the implication of a causal paradox) [14, 15]. In the context of string theory, Mende[16] also observed limits of the applicability of the equivalence principle and he suggested that the extended nature of strings should imply an energy-dependent deflection of light and that it would be a clear signature of string theory.

These two effects are signatures of quantum gravity that should survive when the final theory of quantum gravity is found. With the lack of observations related to quantum gravity, these signatures may serve as guidelines for the construction of a quantum theory of gravity. The purpose of this thesis is to address two questions related to each effect. First, we want to see if it is possible to obtain an energy-dependent deflection of light within the framework of quantum field theory in curved spacetime. If so, this behavior would not be a clear signature of string theory. The second goal is to understand the divergences occuring in the statistical black hole entropy. Such an understanding is essential to make sense of the statistical black hole entropy.

In the remainder of this chapter, we introduce some background material needed in the thesis. In section 1.2, we review some notions of General Relativity and the Schwarzschild and Reissner-Nordström solutions. Then section 1.3 introduces the classical action for bosonic string theory.

The chapter 2 is concerned with the limit of applicability of the equivalence principle and the construction of an effective action that can describe dispersive photon propagation in a gravitational field. We start our study in section 2.1 by reviewing the equivalence principle in General Relativity and its implication on light propagation. We will see in this section that the equivalence principle implies the energy-independent propagation of photons. Then in section 2.2, we review Mende's idea that string theory leads to energy-dependent light deflection. The results of ref. [14] on the limit of the equivalence principle in quantum field theory in curved spacetime is reviewed in section 2.3. This section leads us to try to build an effective action that implies an energy-dependent deflection of light by a gravitational field. This is done is section 2.4 where we find three classes of effective actions leading to dispersive photon propagation. Then in section 2.5, we discuss our results with respect to the uniqueness of the effective actions, to the relation to possible string effective action and to the magnitude of the dispersion.

In the chapter 3, we study black hole entropy using statistical methods. We start by reviewing in section 3.1 black hole thermodynamics, *i.e.*, the relation between black hole dynamics and ordinary thermodynamics. This is done for General Relativity as well as for more general theories of gravitation. In section 3.2, we present the different methods that were introduced to calculate statistical black hole entropy. These calculations lead to divergences in the entropy and one needs to introduce a regularization scheme. To compare with the divergences of black hole entropy, we calculate in section 3.3 the effective action for a scalar field in a curved background. Divergences occur in this calculation and we introduce a Pauli-Villars regularization and then absorb the infinities by renormalization of Newton's constant and other coupling constants. The divergences of black hole entropy are studied in section 3.4 using the same Pauli-Villars regularization. To calculate black hole entropy, we use a modification of the method initially introduced by 't Hooft[12]. We find that the divergences are exactly what is needed to renormalize Newton's constant and the other coupling constants. Then some discussion remarks are included in section 3.5, with respect to possible generalizations of the calculation.

Finally, we conclude this thesis in chapter 4. Some technical material is included in the appendices. Throughout the thesis, we use the convention of ref. [17], where the metric has signature $(-+++)$, the Riemann tensor is given by $R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ic}\Gamma^i_{bd} - \Gamma^a_{id}\Gamma^i_{bc}$, with Γ^a_{bc} the Christoffel symbol and Einstein equation is given by eq. (1.1). We also use units where $\hbar = c = k_B = 1$ except in chapter 2 where we use units where $\hbar = c = G = 1$.

1.2 Review of General Relativity

Currently, the accepted classical theory of gravity is Einstein's theory of General Relativity. The theory was first verified by looking at three predictions (the so-called classical tests) namely, the precession of the perihelion of Mercury, the bending of light by the sun and the red-shift of light escaping from a gravitational field. These tests probe the weak-field regime. Since that time, the theory has been verified with other weak-field observations and also in the strong-field regime, with observations related to binary pulsars (for reviews on the experimental verification of General Relativity, see *e.g.*, refs. [18, 19]).

General Relativity is a theory of gravitation and spacetime. Energy generates curvature of the spacetime. Conversely, curvature interacts with the energy distribution of spacetime, establishing an equilibrium described by Einstein field equation

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab} \ , \tag{1.1}$$

where R_{ab} is the Ricci tensor, R is the curvature scalar, g_{ab} is the metric, G is Newton's constant and T_{ab} is the matter stress-energy tensor. If the energy distribution is expressed by a field action I_f, the stress tensor can be calculated by

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta I_f}{\delta g^{ab}} . \tag{1.2}$$

with g the determinant of the metric tensor g_{ab}. Eq. (1.1) can also be written

$$R_{ab} = 8\pi G\left(T_{ab} - \frac{1}{2}g_{ab}T^c{}_c\right) .$$

The motion of particles in free fall is described by timelike geodesics (or null geodesics for massless particles). A geodesic is a curve whose tangent vector T^a is parallel propagated along itself, thus T^a satisfies $T^a\nabla_a T^b = \alpha T^b$, where α is an arbitrary function of the curve. The geodesic can be reparametrized such that the tangent vector obeys

$$T^a\nabla_a T^b = 0 . \tag{1.3}$$

In that case, the parametrization is called an affine parametrization. If one introduces a coordinate system, the geodesic is mapped to a curved $x^a(\tau)$ and the tangent vector is given by

$$T^a = \frac{dx^a}{d\tau} .$$

For timelike geodesics, one has $g_{ab}T^aT^b < 0$ and for null geodesics, one has $g_{ab}T^aT^b = 0$. Eq. (1.3) becomes

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0 . \tag{1.4}$$

This is the geodesic equation. The parameter τ is the affine parameter. For timelike geodesics, the affine parameter can be interpreted as the proper time and the geodesic is the curve between two points that extremizes the proper time.

1.2.1 The Schwarzschild black hole

The empty space Einstein equation

$$R_{ab} = 0 \tag{1.5}$$

admits a spherical symmetric solution, known as the Schwarzschild solution. The metric is

$$\begin{aligned} ds^2 &= g_{ab}\, dx^a\, dx^b \\ &= -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \ , \end{aligned} \tag{1.6}$$

where (t, r, θ, ϕ) describe the Schwarzschild time, radial and angular coordinates and $r_s = 2GM$ is the Schwarzschild radius. This solution describes the geometry around a spherical distribution of matter with mass M. The Schwarzschild metric is asymptotically flat, *i.e.*, it behaves as the Minkowski metric for $r \to \infty$. A theorem due to Birkhoff states that the Schwarzschild solution is the only spherical symmetric, asymptotically flat solution of Einstein vacuum field equation (1.5)†. Thus, the field produced in the surrounding region by a spherical symmetric mass distribution may always be represented by the metric (1.6), regardless of whether the mass is static, collapsing, expanding or pulsating.

Components of the metric become zero or diverge at $r = 0$ and at $r = r_s$. Numerically, one has

$$r_s = \frac{2GM}{c^2} \cong 3\frac{M}{M_\odot}\text{km} \ .$$

For ordinary objects like the sun or the earth, the Schwarzschild radius is inside the radius of the body, where the *vacuum* Einstein equation is not valid. However, these singularities are relevant for bodies which undergo complete

†For a proof, see ref. [3].

gravitational collapse. The divergences can be produced by singularities of the spacetime or by pathologies in the coordinate system. The easiest way to identify a spacetime singularity is to look for a scalar quantity describing the geometry that becomes infinite. If a scalar quantity is infinite in a certain coordinate system, it will be infinite in all coordinate systems and the singularity will be a true singularity, independently of the coordinate system. On the other hand, apparent singularities in tensor components may be only coordinate singularities, produced by a bad choice of the coordinate system. In Schwarzschild geometry, the simplest curvature scalar is $R_{abcd}R^{abcd}$. From eq. (A.10) in appendix A, one obtains

$$R_{abcd}R^{abcd} = \frac{12r_s^2}{r^6} \; .$$

This quantity remains finite at $r = r_s$ but it becomes infinite at $r = 0$. Hence the singularity at $r = 0$ is real and $r = r_s$ is probably only a coordinate singularity. In fact, we will see shortly that the latter is the event horizon of the Schwarzschild black hole. The surface $r = r_s$ is also called the static limit surface because on this surface, $g_{tt} = 0$ and one cannot remain at rest (with $dr = d\theta = d\phi = 0$). For Schwarzschild geometry, the event horizon and the static limit surface coincide but in general they are different.

Let us remove the singularity at $r = r_s$ by using the Kruskal coordinate system[20]. We first introduce the Kruskal null coordinates

$$U = -e^{-u/2r_s}$$

$$V = e^{v/2r_s} \; ,$$

with

$$u = t - r_*$$

$$v = t + r_*$$

where r_* is the tortoise coordinate

$$r_* = r + r_s \ln\left(\frac{r}{r_s} - 1\right) .$$

The original (t, r) coordinates are mapped to $-\infty < U \leq 0$, $0 \leq V < \infty$. With this transformation, the metric (1.6) reads

$$ds^2 = -\frac{4r_s^3 e^{-r/r_s}}{r} dU\, dV + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) ,$$

where r is to be viewed as a function of U and V. The metric is no longer singular at $r = r_s$ ($U = 0$ or $V = 0$). It is possible to extend the Schwarzschild solution by allowing U and V to take all values compatible with the restriction $r > 0$, that is, $-\infty < U, V < \infty$. We then make the final transformation $T = (U + V)/2$ and $X = (V - U)/2$. In this way, we obtain the maximal extension of the Schwarzschild geometry, given by the metric

$$ds^2 = \frac{4r_s^3}{r} e^{-r/r_s}(-dT^2 + dX^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) . \qquad (1.7)$$

This geometry is illustrated by the two-dimensional spacetime diagram in fig. 1.1. By construction, the radial null geodesics are 45° lines in the Kruskal geometry. Each point represents a two-sphere with radius r. The relation between the old coordinates (t, r) and the new coordinates (T, X) is given by

$$\left(\frac{r}{r_s} - 1\right) e^{r/r_s} = X^2 - T^2$$
$$\frac{t}{r_s} = \ln\frac{T + X}{T - X} .$$

In fig. 1.1, the curves $t =$ const. are lines through the origin and the curves $r =$ const. are hyperbolae. This figure illustrates the bad behavior of the (t, r) coordinate system at $T = X$. The singularity $r = 0$ becomes the hyperbola $T = \pm\sqrt{X^2 + 1}$. In fig. 1.1, there are four different regions. The region labeled I is the original asymptotically flat region $r > r_s$. It can be interpreted as the spacetime outside a spherical body. However, a radially infalling observer can

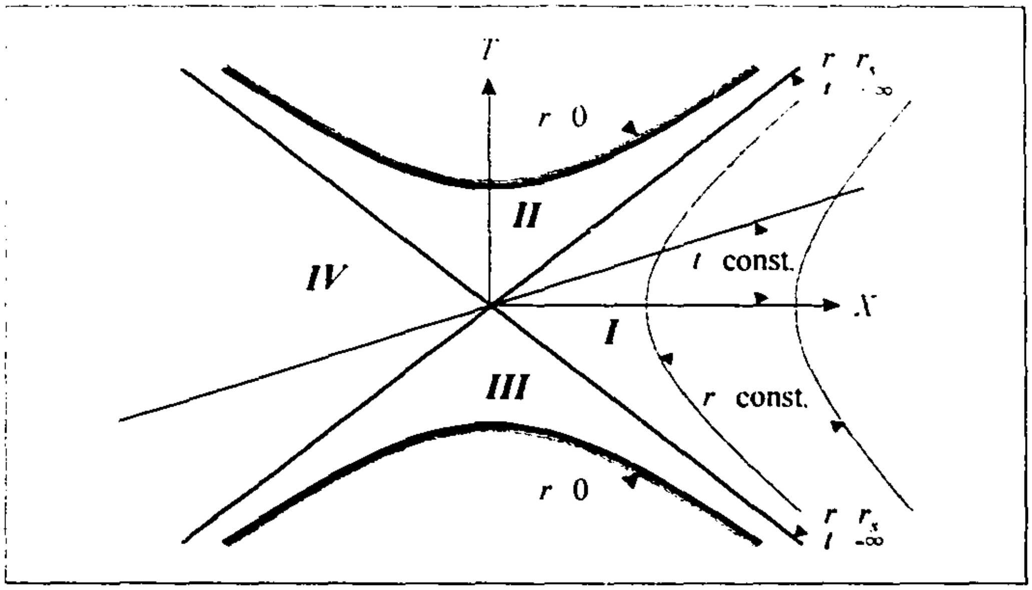

Figure 1.1: Spacetime diagram for the Kruskal extension of Schwarzschild geometry.

cross the null line $T = X$ and enter in region II. Once this observer has entered in this region, he cannot escape from it. Within a finite proper time, he will fall into the singularity $r = 0$. Even light cannot escape from this region and will fall also into the singularity. Hence, the region II is a black hole. The null surface $X = T$ (corresponding to $r = r_s$) is a one-way membrane. It is called the future event horizon because observers in the region I cannot be causally influenced by an event that takes place beyond this surface. The region III has the same time-reversed properties of the region II and it is called a white hole. Any observer should have started his existence near the singularity $T = -\sqrt{X^2+1}$ and must leave the region within a finite proper time. Finally, region IV is another asymptotically flat region, which is causally disconnected from region I.

The complete extended Schwarzschild solution cannot represent the spacetime resulting from a gravitational collapse because one does not initially have

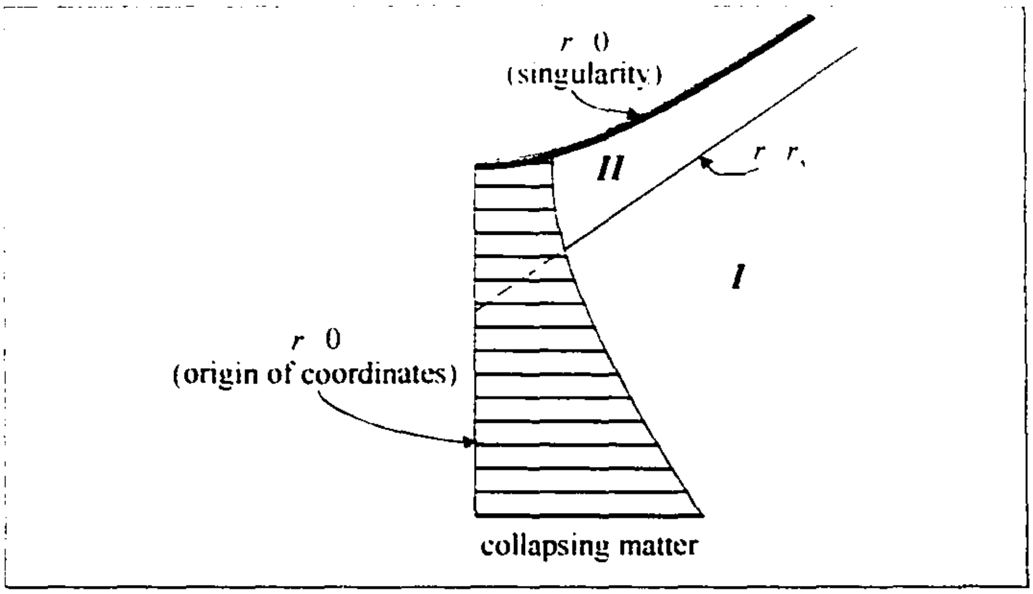

Figure 1.2: Spacetime diagram for a complete gravitational collapse of a spherical body.

two asymptotically flat regions with an initial singularity connecting them. One can represent a spherical collapse by the spacetime diagram of fig. 1.2. The regions III and IV are unphysical but part of region II is produced and a black hole is formed. The complete gravitational collapse of a spherical body always produces a Schwarzschild black hole.

1.2.2 The Reissner-Nordström black hole

The Schwarzschild solution describes the gravitational field surrounding a spherical distribution of mass which is electrically neutral. For an electrically charged distribution of matter, one has to solve the coupled Einstein-Maxwell equations.

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab}$$

$$\nabla_b F^{ab} = J^a$$

$$\nabla_{[a} F_{bc]} = 0 \ ,$$

where J^a is the current density and T_{ab} is the stress-energy tensor for the electromagnetic field. From the Maxwell action

$$I_{em} = -\frac{1}{4} \int d^4x \sqrt{-g} F_{ab} F^{ab} \ , \tag{1.8}$$

and eq. (1.2), one may calculate T_{ab}

$$T_{ab} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd} \ .$$

A spherical solution can be found. It is the Reissner-Nordström solution, with the metric

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2}\right) dt^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2}\right)^{-1} dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \tag{1.9}$$

and the field strength

$$F_{ab} = 2E(r)\, \delta_{0[a} \delta_{b]r} \ , \tag{1.10}$$

where the square brackets indicate that the expression is antisymmetrized in a and b with a factor of 1/2 and

$$E(r) = \frac{Q}{4\pi r^2} \tag{1.11}$$

is the electric field. This solution describes the geometry around a spherical distribution of matter with mass M and charge Q. It reduces to the

Schwarzschild solution when $Q = 0$. The metric has divergences for $r = 0$ and $r = r_\pm$, where

$$r_\pm = GM \pm \sqrt{M^2G^2 - \frac{GQ^2}{4\pi}} \ . \tag{1.12}$$

As for the Schwarzschild case, only the singularity at $r = 0$ is a space-time singularity. This can be seen by calculating the scalar $R_{ab}R^{ab}$. Using eq. (A.17), one obtains the scalar quantity

$$R_{ab}R^{ab} = \frac{G^2Q^4}{4\pi^2r^8}$$

which diverges only at $r = 0$. For $GM^2 > Q^2/(4\pi)$, the null surface $r = r_+$ is the event horizon and the null surface $r = r_-$ is another horizon inside the black hole. When $GM^2 = Q^2/(4\pi)$, $r_+ = r_-$ and the two horizons coincide. This black hole is called extremal. Finally, for $Q^2/(4\pi) > GM^2$, there are no horizons, only a singularity at $r = 0$. A singularity without horizons is called a naked singularity. For the Reissner-Nordström black hole, the configuration $Q^2/(4\pi) > GM^2$ is unstable because the electric repulsion becomes bigger than the gravitational attraction. For general collapse, it is generally believed that a naked singularity cannot be produced by a gravitational collapse (this is the cosmic censorship conjecture[21]).

1.3 The Bosonic string action

String theory is a theory that incorporates gravity with the other interactions. String theory has a better ultraviolet behavior than the ordinary field theory and the scattering amplitude should be finite to all orders of the coupling constant. In this way, no infinite renormalization should be needed. In this thesis, we are studying quantum gravity in a semi-classical approximation. In this line, we need to introduce the classical action for bosonic strings (we restrict ourselves to bosonic string for simplicity) to compare some of our results with those coming from string theory.

We begin with the action principle that describes the motion of a point particle of mass m in a curved spacetime. This action is proportional to the invariant length of the world line

$$I = -m \int ds = -m \int \sqrt{-g_{ab}\, dx^a\, dx^b}$$

where g_{ab} is the spacetime metric. If the trajectory is described by the curve $x^a(\tau)$, with τ an affine parameter, the action takes the form

$$I = -m \int d\tau \sqrt{-g_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau}} \,. \tag{1.13}$$

This action is invariant under reparametrization $\tau \to \tau'(\tau)$ of the particle trajectory and so does not depend on the coordinate system. The action (1.13) leads to two difficulties. First, it is non-polynomial and thus it is difficult to work with this kind of action. Next, it does not apply for massless particles. To overcome these difficulties, one may introduce an auxiliary field $h(\tau)$. The action becomes

$$I = \frac{1}{2} \int d\tau \sqrt{h}\left(h^{-1} g_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} - m^2\right) \,. \tag{1.14}$$

h plays the role of a world line metric. One may recover eq. (1.13) by solving the equation of motion for h

$$g_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} + m^2 h = 0 \,.$$

The point-particle can be generalized to a one-dimensional object. To parametrize the string, one introduces a spacelike parameter σ^1. As it moves, the string sweeps a two-dimensional surface called the world sheet, parametrized by σ^1 and by a timelike parameter σ^0 (which plays the same role as τ in the point particle case). Then, the position of the string is given by a function $X^a(\sigma^0,\sigma^1)$. With this, the action (1.14) is generalized to the string action

$$I_{\mathrm{P}} = -\frac{T}{2}\int d^2\sigma\sqrt{-h}h^{\hat{\mu}\hat{\nu}}(\sigma)g_{ab}(X)\partial_{\hat{\mu}}X^a\partial_{\hat{\nu}}X^b\ . \qquad (1.15)$$

Usually, string theory is formulated in a flat background and the action is given by

$$I_{\mathrm{P}} = -\frac{T}{2}\int d^2\sigma\sqrt{-h}h^{\hat{\mu}\hat{\nu}}(\sigma)\eta_{ab}\partial_{\hat{\mu}}X^a\partial_{\hat{\nu}}X^b\ .$$

This is the Polyakov action [22]. The hatted greek indices refer to the world sheet coordinates (σ^0,σ^1), $h_{\hat{\mu}\hat{\nu}}$ is the world sheet metric and T is the string tension, with dimensions of $(\text{mass})^2$. It can be related to the Regge slope parameter α' by $T = 1/(2\pi\alpha')$. The action (1.15) is invariant under reparametrization of the world sheet $\sigma^0 \to \sigma'^0(\sigma^0,\sigma^1)$, $\sigma^1 \to \sigma'^1(\sigma^0,\sigma^1)$. It is also invariant classically under Weyl rescaling of the world sheet metric $h_{\hat{\mu}\hat{\nu}} \to \Lambda(\sigma^0,\sigma^1)h_{\hat{\mu}\hat{\nu}}$.

Chapter 2

Dispersive photon propagation

We begin our study of quantum gravity by looking at limits for applicability of the equivalence principle. After reviewing the effect of vacuum polarization on photon propagation in curved spacetime, we will show that it is possible to construct effective actions leading to dispersive propagation, *i.e.*, propagation that is energy dependent. Throughout this chapter, we use units where $\hbar = c = G = 1$.

2.1 The equivalence principle

The equivalence principle is part of the foundation of Einstein's theory of General Relativity. It states that at every point, one can build a local Lorentz frame and that for an arbitrarily small region, the laws of physics are described by the laws of Special Relativity. This is the Einstein equivalence principle. It implies that all local inertial frames are equivalent and it imposes a minimal coupling between the gravitational field and the external fields, *i.e.*, one can obtain the laws of physics from Special Relativity by replacing the Minkowski metric by a curved metric and by replacing ordinary derivatives by covariant derivatives. There are no couplings between the fields and the Riemann tensor. We will see in this chapter that when we consider interacting quantum fields in curved spacetime, the Einstein equivalence principle is violated. One can still build a local Lorentz frame at each point but the local frames need not all be equivalent. This is the weak equivalence principle, which is the real foundation of General Relativity, that is, spacetime is a pseudo-Riemannian manifold. In more physical terms, the weak equivalence principle states that the inertial mass is equal to the gravitational mass. In the following, we refer to the Einstein equivalence principle.

To illustrate the equivalence principle in General Relativity, we consider the deflection of light by a gravitational field. The deflection of light is one of the classical tests of General Relativity proposed by Einstein to verify his theory. Its observation in 1919 [23] was one of the first verifications of the theory. The deflection of light was measured many times since then and the observations are still in accord with the theory.* Because of this success, the deflection of light has become a useful tool in astrophysics. Gravitational lenses are used for many purposes (see,*e.g.*, ref. [25]) and particularly for the possible observation of the dark matter[26, 27].

2.1.1 Geometric optics approximation

To study photon propagation in a gravitational field, we start with the Maxwell action (1.8)

$$I_0 = -\frac{1}{4}\int d^4x\,\sqrt{-g}F_{ab}F^{ab}\ , \tag{2.1}$$

*For the results of the latest measurement, see ref. [24].

where $F_{ab} = \nabla_a A_b - \nabla_b A_a$ is the field strength and A_a is the gauge potential. The variation of I_0 leads to the equation of motion

$$\nabla_a F^{ab} = 0 \, . \tag{2.2}$$

The field strength also obeys the Bianchi identity

$$\nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0 \, . \tag{2.3}$$

In order to know the properties of photon propagation, it is sufficient to take the geometric optics approximation (for more details, see appendix B). In the leading order, one writes the field strength as the product of a slowly varying amplitude and a rapidly varying phase

$$F_{ab} = f_{ab} \, e^{i\Theta} \, . \tag{2.4}$$

The wave vector is defined by $k_a = \nabla_a \Theta$. Light rays follow curves $x^a(\tau)$ normal to the wave fronts $\Theta = \text{const}$. So the wave vector is tangent to x^a. Given an affine parameter τ, one has

$$\frac{dx^a}{d\tau} = k^a = g^{ab} \nabla_b \Theta \, . \tag{2.5}$$

Such curves are called integral curves. In the quantum interpretation, the wave vector becomes the photon momentum and the light rays become the photon trajectories.

Let us introduce eq. (2.4) in the Bianchi identity (2.3). In the leading order of the geometric optics approximation, all derivatives act on the phase

$$k_a f_{bc} + k_b f_{ca} + k_c f_{ab} = 0 \, . \tag{2.6}$$

This constrains f_{ab} to take the form

$$f_{ab} = k_a \mathrm{a}_b - k_b \mathrm{a}_a \tag{2.7}$$

for some vector a_a. The direction of a_a indicates the photon polarization. The field strength is an antisymmetric tensor of rank two, with six components. Eq. (2.6) imposes three independent constraints and so f_{ab} has three independent components. From eq. (2.7), we see that a polarization a_a parallel to the wave vector yields $f_{ab} = 0$ and thus, it is unphysical. So in fact, the field strength has only two independent components.

Now, let us introduce eq. (2.4) in the equation of motion (2.2). To leading order, one obtains:

$$k_a f^{ab} = 0 \ . \tag{2.8}$$

This equation, together with the Bianchi identity, implies that light rays are null geodesics. Indeed, by multiplying eq. (2.8) by k^c

$$\begin{aligned} 0 &= k_a k^c f^{ab} \\ &= -k_a k^a f^{bc} - k_a k^b f^{ca} \\ &= -k_a k^a f^{bc} \ . \end{aligned}$$

Therefore,

$$k^2 = 0 \tag{2.9}$$

and k_a is a null vector. Then, taking a covariant derivative on k^2, one obtains

$$0 = \nabla_a (k^2) = 2k^b \nabla_a k_b = 2k^b \nabla_b k_a \ ,$$

where we use the fact that k_a is the gradient of a scalar function Θ, for which $\nabla_a k_b = \nabla_b k_a$. The geodesic equation is found by substituting eq. (2.5)

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0 \ . \tag{2.10}$$

An important property of the geometric optics results is that the light-cone condition (2.9) is invariant under the rescaling of k_a, *i.e.*, under $k_a \to \Lambda k_a$.

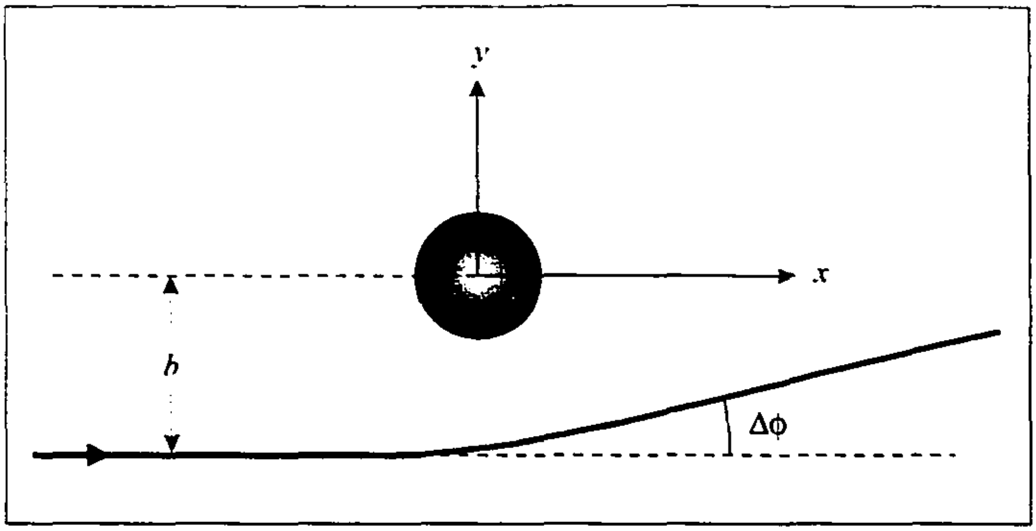

Figure 2.1: Bending of light.

Therefore, in this approximation the propagation of light is independent of the photon frequency. This may be understood as a direct implication of the Einstein equivalence principle. Photons fall freely along geodesics, independently of their frequency.

2.1.2 Bending of light

To illustrate the frequency-independent propagation of light, we consider the bending of light in a spherical symmetric spacetime. Consider a Schwarzschild spacetime described by the metric (1.6)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \, d\Omega^2 \, , \tag{2.11}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$ is the angular line element for a unit two-sphere. For large values of r, the Schwarzschild metric is almost flat and the coordinates (r, θ, ϕ) can be identified with the usual flat space spherical coordinates. We consider photon trajectories which begin at $r \to \infty$ in the equatorial plane $\theta = \pi/2$ (the $x-y$ plane) with an impact parameter b, as illustrated in fig. 2.1. We want to find the change $\Delta\phi = \phi_\infty - \phi_{-\infty} - \pi$ in the angular coordinates

ϕ. The factor of π is subtracted because it is the change in ϕ when there is no deflection at all.

One can solve the geodesic equation (2.10) to find (see eqs. (C.7) in appendix C)

$$\dot{r} = \pm E\left[1 - \left(1 - \frac{2M}{r}\right)\frac{b^2}{r^2}\right]^{1/2}$$
$$\dot{\phi} = \frac{Eb}{r^2} ,$$

where E is the photon energy and the dot indicates a derivative with respect to the affine parameter. The minus sign describes the incoming photons and the plus sign, the outgoing photons. The spatial orbits of the photons are then given by

$$\frac{d\phi}{dr} = \frac{\dot{\phi}}{\dot{r}} = \pm\frac{b}{r^2}\left[1 - \left(1 - \frac{2M}{r}\right)\frac{b^2}{r^2}\right]^{-1/2} .$$

To obtain the deflection angle, one integrates over the entire trajectory. The integral is an even function of r and one may integrate from the distance of closest approach r_0 (where $\dot{r}$ changes sign) to infinity and multiply by 2

$$\Delta\phi + \pi = 2\int_{r_0}^{\infty} dr\,\frac{b}{r^2}\left[1 - \left(1 - \frac{2M}{r}\right)\frac{b^2}{r^2}\right]^{-1/2} .$$

The impact parameter is related to the distance of closest approach r_0 by

$$b = r_0\left(1 - \frac{2M}{r_0}\right)^{-1/2} .$$

The photon energy entered by $\dot{\phi}$ and $\dot{r}$ but scales out of $\frac{d\phi}{dr}$ and the deflection angle is independent of the photon frequency. In the solar system, the gravitational field is always weak and one can expand the deflection angle in powers of M/r. To lowest order, the impact parameter equals the distance of closest approach and the deflection angle is

$$\Delta\phi = \frac{4GM}{r_0 c^2} , \tag{2.12}$$

where we have put back the factors of G and c^2. For photons just grazing the sun, one obtains the standard result[17, 28]

$$\Delta\phi = \frac{4GM_\odot}{R_\odot c^2} = 1.75'' \ . \tag{2.13}$$

2.2 Bending of light in string theory

Point particles fall along geodesics. This result is a consequence of the equivalence principle. For photons, the trajectory is given by eq. (2.10) and the deflection angle can depend only on the impact parameter b. One might ask what would happen if photons were extended objects, as is the case in string theory.

As explained in section 1.3, to describe the propagation of strings, one replaces the world line $x^a(\tau)$ by a world sheet $X^a(\sigma^0, \sigma^1)$, where σ^0 is the affine parameter and σ^1 is a spatial coordinate describing the string. The equation (2.10) is replaced by

$$\Box X^a + \Gamma^a_{bc}(X)\partial_{\hat{\mu}} X^b \partial_{\hat{\nu}} X^c h^{\hat{\mu}\hat{\nu}} = 0 \ ,$$

where $h^{\hat{\mu}\hat{\nu}}$ is the world sheet inverse metric and $\Box$ is the world-sheet Laplacian. The hatted greek indices refer to the world sheet parameters σ^0 and σ^1.

For point particles, one can choose Riemann normal coordinates (see section 3.3.2) where Γ^a_{bc} is zero along the trajectory. This is the free fall inertial frame. The equation of motion for the particle then becomes

$$\frac{d^2x^a}{d\tau^2} = 0$$

and there is no apparent local gravitational interaction. The particle energy does not appear in this equation and so the trajectory is energy independent.

For strings, things are different because in general, there are no coordinate systems where $\Gamma^a_{bc} = 0$ over the entire world sheet. Because of their extended nature, strings always feel tidal forces. The equation of motion is non-linear. Ultimately the energy does not scale out and the trajectories should be energy dependent.

This idea led Mende[16] to suggest that string theory would yield an energy-dependent bending of light and that this would be a clear signature of string theory because there is no such effect in classical General Relativity. Of course, this effect would be very small, typically of order l_P/R, where $l_P = (G\hbar/c^3)^{1/2}$ is the Planck length and R is the typical curvature scale. Mende further argued that the energy-dependent bending of light is a consequence of the extended nature of strings and therefore it should occur independently of the details of the final string theory.

2.3 Quantum effects in photon propagation

In this section, we consider point-like photons in the context of quantum electrodynamics in curved spacetime. Because of the vacuum polarization, a photon exists part of the time as electron-positron pairs. These pairs can then be influenced by external fields. For example, it was shown by Adler[29] that electromagnetic waves passing through a strong magnetic field will exhibit birefringence, *i.e.*, the propagation would depend on the wave polarization. Similar results can be found for photons propagating in a gravitational field. The virtual pairs give the photon an effective size of $\mathcal{O}(\lambda_c)$, where λ_c is the electron Compton wavelength. Because of this effective size, photons can feel tidal forces and the Einstein equivalence principle is violated. This effect was first considered by Drummonds and Hathrell[14] who studied photon propagation in de Sitter, Robertson-Walker[†], Schwarzschild and gravitational wave backgrounds. For de Sitter space, the curvature is isotropic and Maxwell equations are modified only by a normalization factor. In Robertson-Walker geometry, the photon velocity is changed but it is independent of the direction of photon polarization. More interesting results appear for Schwarzschild and gravitational wave backgrounds for which the Riemann tensor is not isotropic. In those cases, the propagation of photons is polarization dependent. There is a gravitational birefringence effect. Recently, gravitational birefringence was also found for Reissner-Nordström background [15] and Kerr background[‡] [30]. Similar violations of the equivalence principle were also considered in refs. [31, 32]. In the following, we describe the calculation leading to birefringence in the bending of light for the Schwarzschild geometry, as presented in ref. [14].

2.3.1 Effective action for QED in curved spacetime

The contribution of virtual particle loops in photon propagation can be obtained by replacing the Maxwell action I_0 by a one-loop effective action $I = I_0 + I_1$. The action I_1 incorporates electron loops. It is given by

$$I_1 = \sum_{n,\mathrm{even}} \frac{1}{n!} \int \Big[\prod_n d^4x_n \, A_{a_n}(x_n)\Big] G^{a_1 \cdots a_n}(x_1, \dots, x_n) \, , \tag{2.14}$$

where $G^{a_1 \cdots a_n}(x_1, \dots, x_n)$ is the sum over one-particle-irreducible Feynman diagrams. Because we are concerned with the propagation of photons, it is

[†]De Sitter and Robertson-Walker backgrounds are two idealized cosmological models describing homogenous and isotropic universes. The stress-energy is provided in the former by a non-zero cosmological constant and in the latter by a perfect fluid.

[‡]The Kerr metric describes rotating black holes.

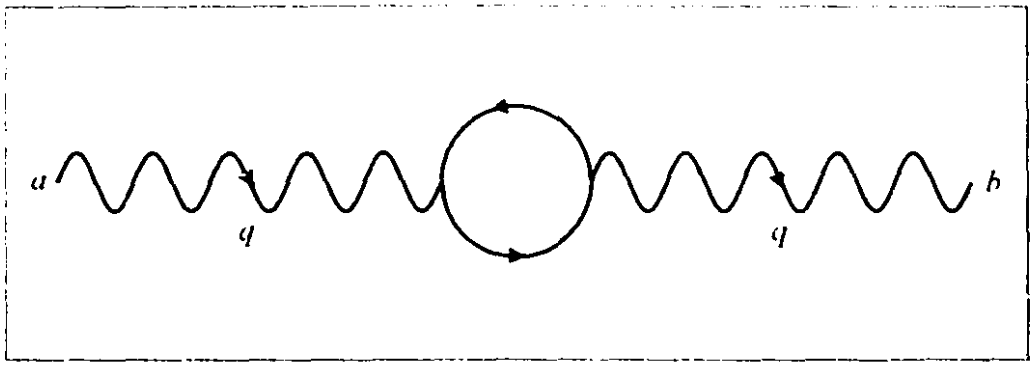

Figure 2.2: Diagram for the one-loop vacuum polarization in flat spacetime.

sufficient to consider only contributions which are quadratic in $A_a(x)$. One may do an expansion of (2.14) in the number of derivatives. The first modification to the Maxwell action has four-derivative interactions. There are four gauge invariant and coordinate invariant interactions. They may be written

$$I_1 = \frac{1}{m^2}\int d^4x\,\sqrt{-g}\Big[aRF_{ab}F^{ab} + bR_{ab}F^{ac}F^{b}{}_{c} + cR_{abcd}F^{ab}F^{cd} + d\nabla_a F^{ab}\nabla_c F^{c}{}_{b}\Big]\ , \qquad (2.15)$$

where m is the electron mass and a, b, c, d are dimensionless coefficients. To determine the latter, one may compare the scattering amplitudes given by I_1 with the one-loop results calculated for a weak gravitational field theory. The interaction proportional to d can be found by considering the flat spacetime vacuum polarization amplitude of fig. 2.2. The on-shell renormalized amplitude is given to fourth order in derivatives by (see, *e.g.*, ref. [33])

$$\Pi^{ab} = \left(q^a q^b - \eta^{ab}q^2\right)\left(1 + \frac{\alpha q^2}{15\pi m^2}\right) \qquad (2.16)$$

where $\eta^{ab} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric and $\alpha = e^2/(4\pi)$ is the fine structure constant. The effective action will yield the same scattering amplitude to $\mathcal{O}(e^2)$ if

$$d = \frac{\alpha}{30\pi m^2}\ .$$

Similarly, one may calculate the coefficients a, b and c by looking at the diagrams of fig. 2.3. Using the Feynman rules found in ref. [34], Drummonds and Hathrell obtained, to $\mathcal{O}(e^2)$ [14][§]

$$a = \frac{5\alpha}{720\pi} \qquad b = -\frac{13\alpha}{360\pi} \qquad c = \frac{\alpha}{360\pi} .$$

These coefficients may also be obtained by calculating the Green function for electrons in an external gravitational and electromagnetic background using DeWitt-Schwinger techniques [14], similar to the methods presented in section 3.3.

The equation of motion is given by

$$\begin{aligned} 0 &= \frac{\delta I}{\delta A_a(x)} \\ &= \nabla_a F^{ab} - \frac{1}{m^2}\nabla_a\Big[4aRF^{ab} + 2b(R^a{}_cF^{cb} - R^b{}_cF^{ca}) + 4cR^{ab}{}_{cd}F^{cd}\Big] . \end{aligned} \tag{2.17}$$

The interaction proportional to d has been omitted because $\nabla_a F^{ab}$ is of order e^2, so $d\nabla_a F^{ab}$ is of order e^4. Eq. (2.17) is valid for photon wavelengths $\lambda > \lambda_c = \frac{1}{m}$.

2.3.2 Superluminal velocity in Schwarzschild geometry

The Schwarzschild geometry is described by the metric (2.11).

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2 .$$

It is a Ricci-flat spacetime with $R = R_{ab} = 0$. The equation of motion (2.17) reduces to

$$\nabla_a F^{ab} - \xi^2\Big[\nabla_a R^{ab}{}_{cd}F^{cd} + R^{ab}{}_{cd}\nabla_a F^{cd}\Big] = 0 ,$$

[§]The actual coefficients in ref. [14] have opposite signs because they use $(+ - - -)$ for the metric signature.

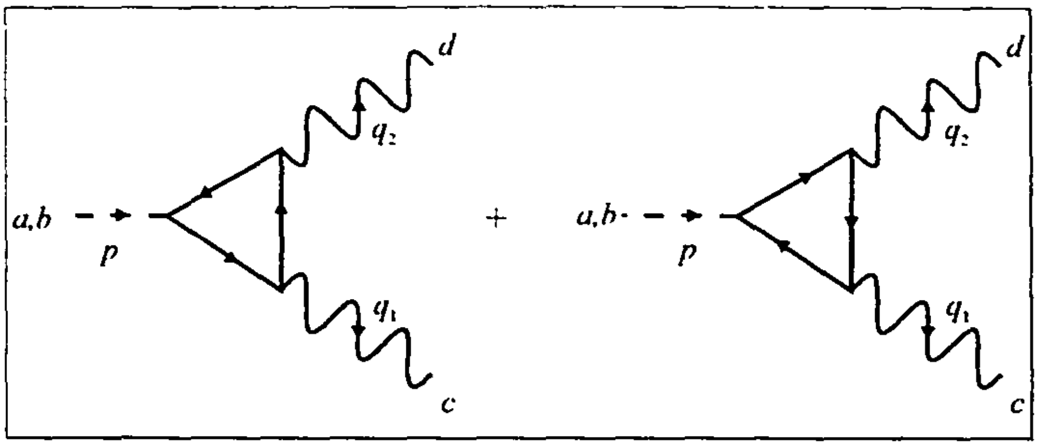

Figure 2.3: Diagrams for the one-loop gravitational vacuum polarization.

with $\xi^2 = \alpha/(90\pi m^2)$. Using the Bianchi identity for the Riemann tensor, one obtains

$$\begin{aligned}\nabla_a R^{ab}{}_{cd} F^{cd} &= \left[-\nabla_c R^{ab}{}_{da} - \nabla_d R^{ab}{}_{ac}\right] F^{cd} \\ &= \left[\nabla_c R^b{}_d - \nabla_d R^b{}_c\right] F^{cd} = 0\end{aligned}$$

for Ricci-flat geometries. Hence

$$\nabla_a F^{ab} - \xi^2 R^{abcd} \nabla_a F_{cd} = 0 \ .$$

Let us examine this equation in the geometric optics approximation (2.4). As in section 2.1.1, all derivatives act on the phase in the leading order which yields

$$k_a f^{ab} - \xi^2 R^{abcd} k_a f_{cd} = 0 \ .$$

This equation is invariant under rescaling $k_a \to \Lambda k_a$. Therefore, the trajectory will be frequency independent, just as in General Relativity. However, the $R^{abcd} k_a f_{cd}$ term does violate the Einstein equivalence principle.

To solve the equation of motion, it is useful to work in a local orthonormal frame by introducing the vierbein $e^\mu{}_a = \mathrm{diag}(U, 1/U, r, r\sin\theta)$ satisfying

${e^\mu}_a {e^\nu}_b \eta_{\mu\nu} = g_{ab}$, where $U = \left(1 - \frac{2M}{r}\right)^{1/2}$. The greek letters represent the orthonormal coordinates $(0,1,2,3)$, associated with (t,r,θ,ϕ). We also introduce the bivector

$$U^{ab}_{\mu\nu} = {e_\mu}^a {e_\nu}^b - {e_\mu}^b {e_\nu}^a \ .$$

One may choose the three independent components of the field strength to be $f_{01} = \frac{1}{2} f_{ab} U^{ab}_{01}$ and similarly f_{23} and f_{03}. Then, the equation of motion can be put in a matrix form

$$\begin{pmatrix} k^2 + \epsilon l^2 & 0 & 0 \\ 0 & k^2 - \epsilon m^2 & 0 \\ -\epsilon\, l\cdot p & -\epsilon\, m\cdot p & k^2 \end{pmatrix} \begin{pmatrix} f_{01} \\ f_{23} \\ f_{03} \end{pmatrix} = 0 \ , \qquad (2.18)$$

where

$$\epsilon = \frac{6\xi^2 M}{r^3}\left(1 + \frac{2M\xi^2}{r^3}\right)^{-1} = \frac{6\xi^2 M}{r^3} + \mathcal{O}(e^4) \ , \qquad (2.19)$$

$$l^b = U^{ab}_{01} k_a \qquad m^b = U^{ab}_{23} k_a \qquad p^b = U^{ab}_{03} k_a \ . \qquad (2.20)$$

In terms of local frame components, we have

$$l^2 = k_0^2 - k_1^2 \qquad m^2 = k_2^2 + k_3^2 \qquad p^2 = k_0^2 - k_3^2 \qquad (2.21a)$$

$$l\cdot m = 0 \qquad l\cdot p = -k_1 k_3 \qquad m\cdot p = k_0 k_2 \ . \qquad (2.21b)$$

To have a non-trivial solution, the determinant of the square matrix in eq.(2.18) must vanish

$$(k^2 + \epsilon l^2)(k^2 - \epsilon m^2) k^2 = 0 \ . \qquad (2.22)$$

Each root yields a modified light-cone condition, and a specific polarization is associated to each. Eq. (2.18) determines the polarization vector to $\mathcal{O}(e^0)$ and so it is sufficient to consider the classical propagation equation for the polarization vector. From the results of appendix B, the polarization vector is parallel transported along the rays and thus its orientation remains constant. For a given trajectory, two of the light-cone conditions describe physical polarizations while the third is unphysical. To understand the implication of these modified light cones, we consider two limiting cases.

(i) First, we look at radial trajectories given by $k^\theta = k^\phi = 0$. From eq. (2.21a), one has $m^2 = k_2^2 + k_3^2 = 0$ and $l^2 = k_0^2 - k_1^2 = -k^2$. The first root yields $k^2 + \epsilon l^2 = k^2(1-\epsilon) = 0$ and by eq. (2.18) it describes photons with radial polarization $\mathrm{a}_a = \delta_{a1}$, which are unphysical. The second root and the third root are degenerate and yield the unmodified light-cone condition $k^2 = 0$. They describe the two independent physical polarizations. Hence, the photon velocity remains unity for both polarizations

$$\left|\frac{k_0}{k_1}\right| = 1 \ .$$

(ii) Next, we consider the opposite case where $k^r = k^\theta = 0$ and $k^\phi \neq 0$. Using eq. (2.21a), the first root yields the light cone

$$(1-\epsilon)(-k_0^2) + k_3^2 = 0$$

and the photon velocity is

$$\left|\frac{k_0}{k_3}\right| = \frac{1}{\sqrt{1-\epsilon}} = 1 + \frac{\epsilon}{2} + \mathcal{O}(e^4) \ ,$$

where $\epsilon = \mathcal{O}(e^2)$. From eq. (2.18), this describes radial polarization. In the same way, the second root yields

$$-k_0^2 + (1-\epsilon)k_3^2 = 0$$

which leads to a photon velocity

$$\left|\frac{k_0}{k_3}\right| = \sqrt{1-\epsilon} = 1 - \frac{\epsilon}{2} + \mathcal{O}(e^4)$$

and describes the transverse polarization $a_b = \delta_{b\theta}$. The third root $k^2 = 0$ descibes the unphysical polarization $a_b = \delta_{b\phi}$.

Substituting (2.19), one obtains

$$\left|\frac{k_0}{k_3}\right| = \begin{cases} 1 + \frac{\alpha M}{30m^2r^3} & \text{for radial polarization,} \\ 1 - \frac{\alpha M}{30m^2r^3} & \text{for } \theta \text{ polarization.} \end{cases}$$

Hence, the propagation of light is polarization dependent and one obtains gravitational birefringence.

The two previous cases are useful to analyse a general motion of propagation. In fact, one can always choose a coordinate system such that the trajectories remain entirely in the equatorial plane $\theta = \pi/2$ with $k^\theta = k_2 = 0$ (see appendix C). Then the cases analysed previously represent the two limiting cases. Now in general, one has $l^2 \neq 0$ and $m^2 \neq 0$ and the root of eq. (2.22) $k^2 = 0$ describes the unphysical polarization $a_b = \Lambda k_b$, which yields $f_{ab} = 0$. Moreover, the velocity of light is greater than unity for the polarization tangent to the equatorial plane and it is less than unity for the polarization normal to the equatorial plane. It is only for purely radial motion that the light cone remains $k^2 = 0$ and the velocity unity.

The velocity is greater than unity for the polarization tangent to the equatorial plane. The existence of superluminal motion suggests the possibility of violations of causality. It is well known that if an observer A sends a spacelike signal (with $k^2 > 0$) to B, then there exists an observer C for whom B happens before A. Then by the equivalence of Lorentz frames, B can send back a signal to A that arrives before the emission of the initial signal. In the present case, the effect is produced by a tidal effect and so all Lorentz frames are not equivalent. Therefore the second step in the construction of the causal paradox is missing, and hence, the violation of the Einstein equivalence principle allows "faster than light" motion without necessarily implying a causal paradox.

2.3.3 Birefringence in the bending of light

To show that there are real physical consequences behind the previous analysis, one may consider the bending of light produced by the new equation of motion. For photons with polarization lying in the x-y plane (the equatorial plane $\theta = \pi/2$), the light-cone condition reads

$$k^2 + \epsilon l^2 = 0 \; .$$

In a Schwarzschild geometry, the equation of motion reduces to

$$-\left(1-\frac{2M}{r}\right)\left(1-\frac{\alpha M}{15m^2r^3}\right)k^t k^t + \left(1-\frac{2M}{r}\right)^{-1}\left(1-\frac{\alpha M}{15m^2r^3}\right)k^r k^r$$
$$+ r^2 k^\theta k^\theta + r^2 \sin^2\theta \, k^\phi k^\phi = 0 \; ,$$

where we are using k^a with spacetime coordinate indices. The easiest way to determine the deflection angle is to consider the wave vector k^a as a null vector in an effective metric

$$ds^2 = -B(r)\, dt^2 + A(r)\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

with

$$A(r) = \left(1-\frac{2M}{r}\right)^{-1}\left(1-\frac{\alpha M}{15m^2r^3}\right) \tag{2.22a}$$

$$B(r) = \left(1-\frac{2M}{r}\right)\left(1-\frac{\alpha M}{15m^2r^3}\right) \; . \tag{2.22b}$$

Expanding $A(r)$ and $B(r)$ in powers of α and keeping only the linear term, one obtains the modification to the deflection angle for photons with a polarization lying in the equatorial plane

$$\delta\Delta\phi_{x-y} = -\frac{\alpha}{45\pi}\left(\frac{\lambda_c}{r_0}\right)^2 \frac{4M}{r_0} \; .$$

where r_0 is the distance of closest approach and $\lambda_c = 1/m$ is the electron's Compton wavelength. For the other polarization parallel to the z-axis, the magnitude of the modification is the same but the sign is the opposite.

$$\delta\Delta\phi_z = \frac{\alpha}{45\pi}\left(\frac{\lambda_c}{r_0}\right)^2 \frac{4M}{r_0} .$$

Putting back factors of c, $\hbar$ and G, the two polarizations acquire a separation angle

$$\delta\phi = \frac{2\alpha}{45\pi}\left(\frac{\hbar}{mcr_0}\right)^2 \frac{4GM}{r_0c^2} . \tag{2.23}$$

For solar parameters, this angle is unmeasurably small:

$$\delta\phi = 3\times 10^{-47}\Delta\phi .$$

To get an observable angle, one would need a small black hole with radius of the order of the electron's Compton wavelength. The deflection of light is still frequency independent. In ref. [34], they reported a frequency-dependent deflection angle for this effective action but it was an erroneous result, as described by ref. [14].

2.4 Dispersive photon propagation

The first quantum corrections to photon propagation produces gravitational birefringence but the bending of light remains independent of the photon frequency. In this section, we want to find if it is possible to build an effective action which yields dispersive bending of light in the context of interacting quantum field theory. This is important in order to understand whether string theory is the only possible theory with this prediction.

2.4.1 Six-derivative interactions

To have dispersive results then, the leading-order equation should not be invariant under scaling of k_a. Thus one must consider interactions with more derivatives, and in particular, the interactions must contribute to the electromagnetic equations of motion with more derivatives of the field strength. With this in mind, a natural extension of eq. (2.15) is

$$I_2 = -\frac{\beta\lambda^4}{4}\int d^4x\,\sqrt{-g}\;R^{abcd}\nabla_e F_{ab}\nabla^e F_{cd} \tag{2.24}$$

where β is a dimensionless coupling constant, and λ is the (length) scale associated with the effective interaction. This action may be a curved spacetime modification to the effective action coming from the diagram of fig. 2.2. Combined with the Maxwell action (2.1), this new term leads to an equation of motion

$$\nabla_a F^{ab} - \beta\lambda^4\nabla_a\nabla_e\left(R^{abcd}\nabla^e F_{cd}\right) = 0$$

and in the leading order of the geometric optics approximation, one finds

$$k_a f^{ab} - \beta\lambda^4 R^{abcd} k_a k^2 f_{cd} = 0\ . \tag{2.25}$$

Multiplying by k^g and antisymmetrizing over b and g, one finds

$$k^2\left(f^{gb} - 2\beta\lambda^4 k^{[g|} R^{a|b]cd} k_a f_{cd}\right) = 0\ , \tag{2.26}$$

where the square brackets indicate that the expression is antisymmetrized in b and g with a factor of 1/2. We will assume that the effective action is constructed *perturbatively* in the coupling β. Within such a framework, even though eq. (2.26) is not invariant under scaling of k_a, the light-cone condition remains $k^2 = 0$. The second factor in brackets would define spurious characteristics which are nonperturbative in β. Alternatively, one may say that perturbatively we wish to calculate the modification of eq. (2.9) at order β, and so expect to find $k^2 = O(\beta)$ in general. Substituting the latter into the term proportional to β in eq. (2.25), we in fact have $k^2 = O(\beta^2)$ and so there is no perturbation of the light cone to the order which we are calculating.

There are other six-derivative interactions similar to I_2 where the indices are contracted in different ways, but in the equations of motion, the higher-derivative terms are proportional to k^2 or to $k_a f^{ab}$, which are both higher order corrections. Thus we found that there are no six-derivative interactions which will produce a dispersive light-cone condition. To obtain an energy-dependent result, one needs to consider interactions with both more derivatives and more background curvatures in order to avoid the above contractions.

2.4.2 Dispersive interaction without birefringence

From the last subsection then, we have learned that in order to produce a dispersive modification of the light-cone condition, we need an interaction which is quadratic in the field strength, has two derivatives of the field strength, and has more than one background curvature or derivatives of the background curvature. In the remainder of this section following these criteria, we construct a number of eight-derivative interactions, and show that they lead to energy-dependent photon propagation.

We begin with a simple extension of eq. (2.24), where a second curvature tensor is introduced.

$$I_3 = -\frac{\beta\lambda^6}{4}\int d^4x\,\sqrt{-g}\;R^{cdab}R_c{}^{egh}\nabla_d F_{ab}\nabla_e F_{gh} \qquad (2.27)$$

where β and λ are the coupling and scale, as above. The equation of motion for the electromagnetic field becomes

$$\nabla_a F^{ab} - \beta\lambda^6 \nabla_a \nabla_d \left(R^{cdab} R_c{}^{egh} \nabla_e F_{gh}\right) = 0 \ .$$

Introducing the geometric optics approximation,

$$k_a f^{ab} + \beta\lambda^6 R^{cdab} R_c{}^{egh} k_a k_d k_e f_{gh} = 0 \ . \tag{2.28}$$

As desired, these equations are not invariant under scaling of the wave vector, and the higher order term is not proportional to k^2 or to $k_a f^{ab}$. Hence these equations produce dispersion, and should lead to energy-dependent light deflection in a gravitational potential.

We now turn to the Schwarzschild background to display such dispersive deflection of light. Using the method of section 2.3, we introduce the vierbein $e^{\mu}{}_a = \mathrm{diag}(U, 1/U, r, r\sin\theta)$. The Riemann tensor can be conveniently expressed as (see appendix A)

$$R^{abcd} = -\frac{M}{r^3}\left[g^{ac}g^{bd} - g^{ad}g^{bc}\right] - \frac{3M}{r^3}U^{ab}_{01}U^{cd}_{01} + \frac{3M}{r^3}U^{ab}_{23}U^{cd}_{23} \ ,$$

with the bivector

$$U^{ab}_{\mu\nu} = e_{\mu}{}^{a} e_{\nu}{}^{b} - e_{\mu}{}^{b} e_{\nu}{}^{a} \ .$$

The equation of motion (2.28) becomes

$$k_a f^{ab} + \zeta\left[l^2 l^b f_{01} + m^2 m^b f_{23}\right] = 0 \ , \tag{2.29}$$

where $\zeta = \frac{18\beta\lambda^6 M^2}{r^6}$ and we have dropped terms of the form $\zeta k^2 = \mathcal{O}(\beta^2)$ which are higher order terms.

We choose f_{01}, f_{23} and f_{03} as independent components of f_{ab}. One can project eq. (2.29) over these components by multiplying successively by l_b, m_b

and p_b, defined in eq. (2.20). The equations can then be put in a matrix form

$$\begin{pmatrix} k^2+\zeta l^4 & 0 & 0 \\ 0 & k^2+\zeta m^4 & 0 \\ \zeta l^2\, l\cdot p & \zeta m^2\, m\cdot p & k^2 \end{pmatrix} \begin{pmatrix} f_{01} \\ f_{23} \\ f_{03} \end{pmatrix} = 0 \ . \tag{2.30}$$

The determinant condition is

$$(k^2+\zeta l^4)(k^2+\zeta m^4)k^2 = 0 \ .$$

To find the polarization associated to each light-cone condition, we consider a general motion (with $l^2 \neq 0$ and $m^2 \neq 0$) in the equatorial plane, with $k^\theta = k_2 = 0$.

(i) Taking the root $k^2+\zeta l^4 = 0$, eq. (2.30) leads to

$$\mathrm{a}_b^{(1)} \propto -k_3\delta_{br} + k_1\delta_{b\phi} \tag{2.31}$$

which is the polarization in the x-y plane.

(ii) The root $k^2+\zeta m^4 = 0$ describes the polarization in the z direction.

$$\mathrm{a}_b^{(2)} \propto \delta_{b\theta} \tag{2.32}$$

(iii) For the root $k^2 = 0$, the solution of eq. (2.30) is

$$\mathrm{a}_b^{(3)} \propto k_b \ . \tag{2.33}$$

This polarization is not physical because the field strength is then identically zero.

In fact, both light-cone conditions are equivalent because $m^2 = k_2^2 + k_3^2 = l^2 + k^2 = l^2 + \mathcal{O}(\beta)$. Hence, the interaction I_3 produces no birefringence.

Now we calculate the modifications to the deflection angle. In the Schwarzschild background using spacetime coordinates, the generalized light-cone condition is

$$-\left(1-\frac{2M}{r}\right)(1-\zeta l^2)k^t k^t + \left(1-\frac{2M}{r}\right)^{-1}(1-\zeta l^2)k^r k^r$$
$$+ r^2 k^\theta k^\theta + r^2 \sin^2\theta\, k^\phi k^\phi = 0 \ .$$

Working perturbatively in ζ, it is sufficient to use the classical value for $l^2 = k_0 k_0 - k_1 k_1 = E^2 b^2/r^2$ above, where E is the photon energy and b, the impact parameter (see appendix C). The deflection angle may be found by considering the wave vector k^a as a null vector in an effective metric

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

with

$$A(r) = \left(1-\frac{2M}{r}\right)^{-1}\left(1-\frac{\zeta E^2 b^2}{r^2}\right)$$
$$B(r) = \left(1-\frac{2M}{r}\right)\left(1-\frac{\zeta E^2 b^2}{r^2}\right) \ .$$

The deflection angle is then given by eq. (C.9)

$$\Delta\phi + \pi = 2\int_{r_0}^{\infty}\frac{dr}{r}\left[\frac{A(r)}{\frac{r^2}{r_0^2}\frac{B(r_0)}{B(r)} - 1}\right]^{1/2}$$

where r_0 is the distance of closest approach. Expanding $A(r)$ and $B(r)$ in powers of ζ and keeping only the linear terms, one obtains an integral for the modification of the deflection angle $\delta\Delta\phi$

$$\delta\Delta\phi = \int_{r_0}^{\infty}\frac{dr}{r}\left[\frac{\delta A(r)}{(\frac{r^2}{r_0^2}-1)^{1/2}} - \frac{r^2}{r_0^2}\frac{\delta B(r_0)-\delta B(r)}{(\frac{r^2}{r_0^2}-1)^{3/2}}\right] \ .$$

Inserting $\delta A(r) = \delta B(r) = -18\beta\lambda^6 M^2 E^2 b^2/r^8$, one obtains for both polarizations

$$\delta\Delta\phi = \frac{2205\pi}{128}\frac{\beta\lambda^6 M^2 E^2}{r_0^6} \ . \qquad (2.34)$$

Note that this result is a leading order expression, corrected by terms which are higher order in M/r_0.

A second interaction which produces similar dispersive results is

$$I_4 = -\frac{\beta\lambda^6}{4}\int d^4x\,\sqrt{-g}\,R^{cdea}R_{cgeh}\nabla_d F_{ab}\nabla^g F^{hb}\ . \qquad (2.35)$$

In this case, the equation of motion for the electromagnetic field becomes

$$\nabla_a F^{ab} - \beta\lambda^6\nabla_a\nabla_d\Big(R_{cgeh}R^{cde[a|}\nabla^g F^{h|b]}\Big) = 0\ .$$

Inserting the geometric optics approximation (2.4), one obtains

$$k_a f^{ab} + \beta\lambda^6 R_{cgeh}R^{cde[a|}k_a k_d k^g f^{h|b]} = 0\ .$$

One may now follow the procedure used above to determine the modified light-cone conditions for photons in a Schwarzschild background. A simpler approach for this specific case yields a general light-cone condition, namely use the Bianchi identity (2.6) on $k^{[a}f^{h]b}$ to find

$$f^{ab}\Big(k_a + \frac{\beta\lambda^6}{2}R_{cgeh}R^{cde}{}_a k_d k^g k^h\Big) = 0\ .$$

To have a non-trivial solution, the expression inside brackets must vanish, and this leads to a general light-cone condition describing all polarizations

$$k^2 + \frac{\beta\lambda^6}{2}R_{cgeh}R^{cdea}k_a k_d k^g k^h = 0\ . \qquad (2.36)$$

Therefore, there is no birefringence in any background. In the Schwarzschild background, this light cone becomes

$$k^2 + \frac{\zeta}{4}\left[l^4 + m^4\right]\ , \qquad (2.37)$$

where as above, we use $\zeta = \frac{18\beta\lambda^6 M^2}{r^6}$, $l^2 = k_0^2 - k_1^2$, and $m^2 = k_2^2 + k_3^2$. To leading order, $k^2 = \mathcal{O}(\beta)$ and $m^2 = l^2 + \mathcal{O}(\beta)$ and so eq. (2.37) reduces to $k^2 + \frac{\zeta}{2}l^4 = 0$. Hence up to a factor of two, we have recovered precisely the same dispersive light cone as in the analysis of the interaction I_3 (for the physical polarizations). The modification to the deflection of light is therefore one half the angle obtained in eq. (2.34).

2.4.3 Dispersive and birefringent interaction

A final eight-derivative interaction which produces dispersive light propagation is found by extending eq. (2.24) by introducing extra background derivatives, rather than an extra curvature tensor

$$I_5 = -\frac{\beta\lambda^6}{4}\int d^4x\,\sqrt{-g}\nabla^{(c}\nabla^{d)}R^{abeg}\nabla_c F_{ab}\nabla_d F_{eg} \tag{2.38}$$

where $\nabla^{(c}\nabla^{d)} = (\nabla^c\nabla^d + \nabla^d\nabla^c)/2$. After adding I_5 to the Maxwell action (2.1), the equation of motion for the electromagnetic field becomes

$$\nabla_a F^{ab} - \beta\lambda^6\,\nabla_a\nabla_c\left(\nabla^{(c}\nabla^{d)}R^{abeg}\nabla_d F_{eg}\right) = 0\;.$$

We then insert the geometric optics ansatz (2.4) to obtain:

$$k_a f^{ab} + \beta\lambda^6\nabla^c\nabla^d R^{abeg}k_a k_c k_d f_{eg} = 0\;. \tag{2.39}$$

To calculate the light-cone conditions in the Schwarzschild background, we followed the same method that was used in the analysis of I_3 above. If we consider photon trajectories in the plane $\theta = \pi/2$ and we apply $k^2 = \mathcal{O}(\beta)$ in the second term of eq. (2.39), we find:

$$\begin{pmatrix} k^2 + \eta l^2\mathsf{A} & 0 & 2\eta k_1 k_3 l^2\mathsf{B} \\ 0 & k^2 - \eta l^2\mathsf{C} & 0 \\ -\eta k_1 k_3\mathsf{A} & 0 & k^2 - 2\eta k_1^2 l^2\mathsf{B} \end{pmatrix}\begin{pmatrix} f_{01} \\ f_{23} \\ f_{03} \end{pmatrix},$$

where $\eta = 60\beta\lambda^6 M/r^5$ and

$$\begin{aligned} \mathsf{A} &= k_0^2\left(1 - \frac{3M}{r}\right) - k_1^2\left(5 - \frac{11M}{r}\right) \\ \mathsf{B} &= \left(1 - \frac{2M}{r}\right) \\ \mathsf{C} &= k_0^2\left(1 - \frac{3M}{r}\right) - k_1^2\left(7 - \frac{15M}{r}\right) . \end{aligned}$$

The determinant condition yields the modified light cone

$$(k^2 + \eta l^2\mathsf{C})(k^2 - \eta l^2\mathsf{C})k^2 = 0 .$$

As before, the root $k^2 = 0$ describes the unphysical polarization $a_b = \Lambda k_b$. The second root $k^2 + \eta l^2\mathsf{C} = 0$ describes photons with the polarization (2.31), tangent to the x-y plane and the second root $k^2 - \eta l^2\mathsf{C} = 0$ describes the polarization (2.32), in the z direction. Since the light cones for the polarizations $a_b^{(1)}$ and $a_b^{(2)}$ differ, this last case provides an example of gravitational birefringence. Calculating the deflection angle as above, we find an energy-dependent contribution

$$\delta\Delta\phi = \pm 504\frac{\beta\lambda^6 M E^2}{r_0^5} \tag{2.40}$$

where the plus sign corresponds to the polarization in the x-y plane $a_b^{(1)}$ and the minus sign, to the polarization in the z direction $a_b^{(2)}$. Note that this result is one order lower in the M/r_0 expansion than the previous result (2.34). This reduction occurs since the present interaction (2.38) involves a single Riemann tensor, while the previous interactions have two curvature tensors.

2.5 Discussion

2.5.1 Uniqueness of the interactions

We have found some explicit field theory interactions that produce dispersive photon propagation, in the context of an effective field theory where the Maxwell action is modified by higher-derivative terms. Such dispersion was not observed in earlier studies simply because the effective actions considered previously did not include sufficiently high numbers of derivatives. The final case also provides a new example of gravitational birefringence.

If one considers the post-geometric modification to the General Relativity deflection (2.12), one also finds a dispersive scattering. This deflection is the usual wave-like effect of diffraction. Contrary to the present results, diffractive scattering is proportional to the photon wavelength ($\lambda_{\rm ph} \propto 1/E$) instead of to the photon energy. Therefore, the two results have the opposite energy dependence. Moreover, the post-geometric modifications are small when the photon wavelength is much smaller than the typical curvature scale and also much smaller than the scale of variation for the amplitude of the wave front.

One may ask whether there will be other eight-derivative interactions which will produce dispersion, and clearly the answer is yes. However, the three interactions that we have considered are representatives of three classes of interactions, which produce the same leading order equations in the geometric optics approximation. It is not difficult to verify that eqs. (2.28), (2.36) and (2.39) are unique. For instance, there is only a single way to contract three wave vectors and one field strength with a double derivative of the background curvature tensor, and this is the combination appearing in eq. (2.39). Thus,

$$I_5' = -\frac{\beta\lambda^6}{2}\int d^4x\,\sqrt{-g}\nabla_{(a}\nabla_{b)}R_{cdeg}\nabla^a F^{bc}\nabla^d F^{eg}$$

leads to precisely eq. (2.39) as the leading order equations of motion. The two interactions, I_5 and I_5', differ by total derivatives, and also terms which do not contribute to these leading order contributions (*i.e.*, they do not contribute to the dispersion).

2.5.2 String effective action

In eqs. (2.34) and (2.40), we have found contributions to the deflection angle of light rays, which depend on the square of the photon energy. This behavior is the same as that found for string theory by Mende. One may then ask if interactions of the form discussed here appear amongst the higher dimension interactions included in the low-energy effective string action.

There are two alternative approaches to build these low-energy actions. The first method starts with a string theory in an external background[4, 35]. For example, in the case of the bosonic string, one works with the action (1.15)

$$I_{\rm str} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{-h}\,\partial_{\hat\mu}X^a\partial_{\hat\nu}X^b h^{\hat\mu\hat\nu}g_{ab}(X^{\hat\sigma})\ . \qquad (2.41)$$

The two-dimensional field theory based on the action (2.41) is called the non-linear σ model. Classically, the action (2.41) is invariant under the Weyl rescaling of the two-dimensional metric, $h^{\hat\mu\hat\nu} \to \Lambda(\sigma)h^{\hat\mu\hat\nu}$. To be consistent, the σ model as a quantum field theory should remain locally scale invariant. This requirement is non-trivial and defines the low-energy string equation of motion. Unfortunately, σ model calculations involving background metric and gauge fields have not been carried out to sufficient order to detect terms of the form suggested here.

Alternatively, one can use string theory to compute the tree-level scattering amplitudes for the massless particles and then build an effective action which reproduces the S-matrix[4, 36]. This method is similar to the method used to find the QED effective action, presented in section 2.3.1. The calculation can be done perturbatively in powers of $\alpha' p^2$, where p^2 represents a typical momentum from the scattering process. Interactions of the form (2.27), (2.35) or (2.38) would contribute to a scattering amplitude of two photons and two gravitons — the contribution of I_5 to a two-photon and one-graviton amplitude vanishes on-shell. A sufficiently detailed study of the low-energy effective action for heterotic strings has been made to detect terms of the form discussed here[36], but unfortunately, one finds that these terms do not appear in this action. This suggests that Mende's dispersive effect, which should be universal to all string theories[16], must be produced by an interaction at an even higher order in the α' expansion (or the expansion in numbers of derivatives) than considered in the present paper. So one would expect that the dependence on the radius of closest approach is even more dramatic than the r_0^{-6} appearing in eq. (2.34). Additional Riemann tensors would also increase the power of the central mass appearing in the dispersive contribution to the deflection angle.

If one considers studies of low-energy string actions, there is one eight-derivative interaction which is known to be universal to all string theories[37, 38]

$$I_6 = \frac{\zeta(3)\alpha'^3}{512\pi} \int d^D x \sqrt{-g} \left(2 R_{abkl} R_i{}^{bk}{}_j R^{acdi} R^j{}_{cd}{}^l + R_{abkl} R_{ij}{}^{kl} R^{acdi} R^j{}_{ca}{}^b \right)$$

where $\zeta(s)$ is the Riemann zeta function. We have also indicated that this effective action is in D dimensions, since typically string theories are constructed for $D > 4$. If the spacetime is then compactified down to four dimensions via a Kaluza-Klein ansatz[39, 40], then new vector particles will appear in the effective theory arising from off-diagonal components of the metric, which mix the four-dimensional spacetime with the compact directions (*e.g.*, $g_{a5} \simeq A_a$). The D-dimensional Einstein action provides the standard Maxwell action (2.1) for these vectors upon compactification. Similarly one finds that upon compactification the above interaction yields interactions of the form of eqs. (2.27) and (2.35) (*e.g.*, using $R_{5cab} \simeq -\frac{1}{2}\nabla_c F_{ab} + \ldots$). Therefore the above string interaction produces dispersive propagation as described in our present analysis for these Kaluza-Klein vector fields. The latter, of course, correspond to particular modes in the string spectrum.

2.5.3 Magnitude of the dispersion

Finally, we consider the magnitude of the deflection angles that we have calculated. Ultimately, we expect that this dispersion would only be observable in very exotic circumstances, but to begin let us evaluate eq. (2.40) with solar parameters for which the leading order deflection angle of General Relativity is given by eq. (2.13): $\Delta\phi = 1.75''$. The length scale λ is the microphysical scale associated with the processes that induce our effective interaction. Here, we will choose the interaction scale to correspond to the Compton wavelength of the electron (*i.e.*, $\lambda = \lambda_e \simeq 2.4 \times 10^{-12}$ m) as it would be if eq. (2.38) arose as a higher order term in the derivative expansion of the one-loop effective action for QED — clearly the effect will be more suppressed if we choose a shorter length scale, *e.g.*, the Planck scale in a string effective action. In this case, it is natural to choose the dimensionless coupling constant to be of the order of the fine structure constant (*i.e.*, $\beta \simeq \alpha$). With these choices, the dispersive deflection angle for a photon with polarization tangent to the x-y plane grazing over the limb of the sun (*i.e.*, $r_0 \simeq 7 \times 10^8$ m) is given by

$$\frac{\delta\Delta\phi}{\Delta\phi} \simeq \frac{10^{-89}}{\lambda_{\rm ph}^2} \tag{2.42}$$

where $\lambda_{\rm ph}$ is the wavelength of the photon measured in angstroms. So the visible spectrum ranging from four- to seven-thousand angstroms would be spread over an angle of about 6×10^{-98} arcseconds. Clearly as such, dispersive propagation of light would be unobservable.

Now we also wish to consider situations in which the dispersion would become more pronounced. If we consider eq. (2.34) or eq. (2.40) with λ and β fixed as above for the QED (*i.e.*, $\lambda = \lambda_e$ and $\beta \simeq \alpha$), there are three options: increase the photon energy, decrease the radius of closest approach or increase the central mass. With any of these options, we are limited by the approximations entering into our calculations. The deflection of much higher energy photons is certainly greater, but one must remember that the applicability of the effective action is limited to photon wavelengths greater than the interaction scale λ, which we are here considering to be the Compton wavelength of the electron. Thus one could only consider photons up to the X-ray portion of the spectrum. The deflection is also increased with a reduction in the radius of closest approach r_0. This radius would be minimized by considering a black hole for which one might achieve $r_0 \simeq M$. Such a scenario, though, runs into conflict with another approximation made in our scattering angle calculations, namely $M/r_0 \ll 1$. In principle, one could carry out those calculations in more detail if one wished to consider $M/r_0 \simeq 1$. With this choice then, one would actually want to decrease, rather than increase, the mass, M. Here the limitation is the validity of the geometric optics approximation, which requires that the photon wavelength be much smaller than the radius of curvature of the spacetime geometry. In a Schwarzschild geometry then, one demands that $\lambda_{ph}^2 < r_0^3/M \simeq M^2$. Thus at least M must be greater than λ_e, which was a lower bound on the photon wavelength. Certainly, one could imagine then that dramatic dispersion would be produced for X-rays by a black hole of $M \simeq 10^{18}$g, for which the gravitational radius would be of the order of the Compton wavelength of electron. It seems, though, that such an object (with $M \simeq 10^{-15} M_\odot$) and the dispersed X-rays are unlikely to be observed. It may also be interesting though to consider photon propagation beyond the geometric optics approximation. It may be that effective interactions of lower dimension than considered in section 3 could produce dispersion in situations with large and rapidly varying curvatures, as could possibly be created by gravitational collapse. In conclusion, while the dispersive photon propagation appearing in the present analysis in principle presents a violation of the equivalence principle, it appears to be beyond the practical limits of observation.

Chapter 3

Statistical black hole entropy

The final part of our study of quantum gravity is concerned with black hole entropy in the context of statistical physics. This entropy should be related to information stored inside the black hole horizon but its exact statistical interpretation is still lacking. Previous calculations in statistical mechanics of black hole entropy produced divergent results[12, 13]. My goal is to relate these divergences to the divergences that appear in the gravitational effective action.

3.1 Black hole thermodynamics

3.1.1 Laws of black hole dynamics

Intensive work on General Relativity and specifically on black holes in the 60's and 70's culminated in the four laws of black hole mechanics [9, 41].

Starting with κ the surface gravity*, the zeroth law states that the surface gravity is constant over the horizon of a stationary black hole. The first law is concerned with how the black hole mass changes when a small stationary axisymmetric change is made in the solution:

$$\delta M = \frac{\kappa}{8\pi G}\,\delta\mathcal{A} + \Omega\,\delta\mathcal{J}\ , \tag{3.1}$$

where M is the black hole mass, G is Newton's constant, $\mathcal{A}$ is the black hole area, Ω is the angular velocity and $\mathcal{J}$ is the angular momentum of the black hole. The first law has the nice property of relating variations of quantities measured at infinity (the mass, the angular momentum) to the variation of a geometric property of the horizon (the area). The second law is known as the area law [42]:

$$\delta\mathcal{A} \geq 0\ , \tag{3.2}$$

namely, the area of a black hole never decreases in any physical process. Finally, the third law states that it is impossible to achieve $\kappa = 0$ by a physical process.

These laws are very similar to the ordinary laws of thermodynamics (see table 3.1). The analog quantities are: $E \leftrightarrow M$, work terms $\leftrightarrow \Omega\,\delta\mathcal{J}$, $T \leftrightarrow \alpha\kappa$ and $S \leftrightarrow 1/(8\pi G\alpha)\mathcal{A}$, where α is a constant. In fact, the black hole mass represents physically the total energy of the system and the term $\Omega\,\delta\mathcal{J}$ corresponds to a work term for a rotating body. Bekenstein took seriously the idea that black holes have an intrinsic entropy given by $S_{\rm bh} = \gamma\mathcal{A}/l_{\rm P}^2$ [10, 11], where γ is a constant and $l_{\rm P} = \sqrt{G\hbar/c^3}$ is the Planck length. He suggested to replace the ordinary second law of thermodynamics by a generalized second

*The surface gravity can be interpreted as the acceleration of a fiducial observer moving just outside the horizon. A more precise definition appears in eq. (3.23).

law

$$\delta S + \delta S_{\mathrm{bh}} \geq 0 \ . \tag{3.3}$$

This is motivated by the following idea. Having fallen through an event horizon, a system cannot interact with the outside universe. If a system with a non-zero entropy is dropped into a black hole, the entropy is then unobservable from the outside and so the entropy decreases for that part of the universe. However, because of the area theorem, the area of the black hole increases at the same time and one might expect the combination (3.3) increases for γ of the order of unity [10]. At this stage, the physical interpretation breaks down because classically the black hole absorbs everything and emits nothing. Hence one must interpret its temperature as being exactly zero.

Further insight came from Hawking who showed that a black hole emits thermal radiation when one takes into account quantum field theory in the black hole background[8]. This discovery revealed that the laws of black holes dynamics are probably just the ordinary laws of thermodynamics applied for black holes. Hawking's result was that black holes emit thermal radiation with

Law	Thermodynamics	Black holes
Zeroth	In thermal equilibrum, T is constant throughout the body	For stationary black hole, κ is constant over the horizon
First	$\delta E = T\,\delta S +$ work terms	$\delta M = \frac{\kappa}{8\pi G}\,\delta \mathcal{A} + \Omega\,\delta \mathcal{J}$
Second	$\delta S \geq 0$ in any process	$\delta \mathcal{A} \geq 0$ in any process
Third	Impossible to achieve $T = 0$ in physical process	Impossible to achieve $\kappa = 0$ in any process

Table 3.1: Analogy between thermodynamics and black hole dynamics.

temperature $T = \kappa/(2\pi)$. Taking eq. (3.1) as the ordinary first law, one then obtains the Bekenstein-Hawking entropy

$$S_{\rm BH} = \frac{\mathcal{A}}{4l_{\rm P}^2} = \frac{\mathcal{A}}{4G} \ . \tag{3.4}$$

The generalized second law (3.3) should also hold for processes involving Hawking radiation. In that case, the energy needed to produce the radiation comes from the black hole mass which decreases and so the black hole area decreases but the radiation emitted is thermal. Hence, the black hole entropy decreases but the entropy of the surroundings increases at the same time [43, 44].

3.1.2 Euclidean methods

To go further in the understanding of black hole entropy, one needs to consider the statistical mechanical origin of these thermodynamic relations. For this, it is useful to introduce a partition function. For gravity, this step was introduced by Gibbons and Hawking [45, 46] using Euclidean methods.

In statistical physics, the starting point is the definition of a partition function at temperature $T = 1/\beta$

$$Z = {\rm Tr}\, e^{-\beta\hat{H}} = \sum_n < \phi_n | e^{-\beta\hat{H}} | \phi_n > \ , \tag{3.5}$$

where $\hat{H}$ is the Hamiltonian. The usefulness of Euclidean methods comes from the analogy between ordinary field theory at zero temperature and finite temperature field theory. In ordinary field theory, the amplitude for a configuration ϕ_1 at time t_1 to propagate to a configuration ϕ_2 at time t_2 can be written in the Schrödinger picture

$$\begin{aligned} <\phi_2, t_2 | \phi_1, t_1> &= < \phi_2 | e^{-i\hat{H}(t_2 - t_1)} | \phi_1 > \\ &= \int D[\phi]\, e^{iI[\phi]} \ , \end{aligned} \tag{3.6}$$

where $I[\phi]$ is the action. If we rotate the time axis to imaginary time $\tau = -it$ and we put $t_2 - t_1 = -i\beta$, $\phi_2 = \phi_1 = \phi$ then sum over a complete set of states ϕ_n, we obtain

$$\sum_n < \phi_n|^{-\beta \hat{H}}|\phi_n >= Z \ .$$

where we used eq. (3.5). From eq. (3.6), the partition function has the path integral representation

$$Z = \int_{\text{periodic}} D[\phi]\, e^{-I_{\text{E}}[\phi]} \ .$$

with $I_{\text{E}} = -iI$ the Euclidean action and the integral is over fields periodic in imaginary time τ with period β.

To illustrate this approach for gravity, we introduce the Schwarzschild metric (2.11). (The same method can be used for other black hole solutions [45].)

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \ ,$$

with $r_s = 2GM$ and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. The Euclidean Schwarzschild metric is found by setting $\tau = -it$

$$ds^2 = \left(1 - \frac{r_s}{r}\right) d\tau^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \ . \tag{3.7}$$

Using the change of coordinates $x = 2r_s\sqrt{1 - r_s/r}$, the metric becomes

$$ds^2 = x^2 \left(\frac{d\tau}{2r_s}\right)^2 + \left(\frac{r^2}{r_s^2}\right)^2 dx^2 + r^2 d\Omega^2 \ . \tag{3.8}$$

Near $r = r_s$, the $x - \tau$ part of eq. (3.8) is like the origin of polar coordinates if one identifies the coordinate τ with period $4\pi r_s = 2\pi/\kappa^{\dagger}$. With this choice, the metric (3.8) is free of singularities for $x \geq 0$, $0 \leq \tau \leq \beta$. In the Euclidean formalism, the periodicity in τ refers to the inverse temperature of the system.

†For Schwarzschild metric, $\kappa^{-1} = 2r_s$.

Therefore, the Euclidean formalism leads naturally to $T = \kappa/2\pi$ for the black hole temperature.

The next step is to calculate the partition function

$$Z = \int D[g,\phi]\, e^{-I_{\rm E}[g,\phi]} , \tag{3.9}$$

where ϕ represents some matter fields. One expects that the main contribution to eq. (3.9) should come from metrics and fields which are near a metric g_0 and fields ϕ_0 which are solutions of the equations of motion. From this point of view, one may expand the Euclidean action as a Taylor series in the fluctuation fields $\bar{g}_{ab}$ and $\bar{\phi}$

$$I_{\rm E}[g,\phi] = I_{\rm E}[g_0,\phi_0] + I_2[\bar{g},\bar{\phi}] + \cdots ,$$

where $g_{ab} = g_{0ab} + \bar{g}_{ab}$, $\phi = \phi_0 + \bar{\phi}$ and I_2 is quadratic in $\bar{g}_{ab}$ and $\bar{\phi}$. The linear term is absent because the first derivative of the action yields the equations of motion and the background fields are solutions of these equations of motion. The free energy is

$$F = -\frac{1}{\beta}\ln Z = \frac{1}{\beta} I_{\rm E}[g_0,\phi_0] - \frac{1}{\beta}\ln\int D[\bar{g},\bar{\phi}]e^{-I_2[\bar{g},\bar{\phi}]} + \ldots$$

The first term is the contribution of the background to the free energy and the second term is the one-loop contribution. As a first approximation, we keep only the zero-loop contribution of the Schwarzschild background with $\phi_0 = 0$. The free energy reduces to the action of the background

$$F = \frac{1}{\beta} I_{\rm E}[g_0] . \tag{3.10}$$

The action for General Relativity in Euclidean space is the Euclidean Hilbert action

$$I_{\rm H}^{\rm E} = -\frac{1}{16\pi G}\int d^4x\,\sqrt{g}R , \tag{3.11}$$

where R is the curvature scalar. This action contains terms which are linear in the second-order derivatives of the metric. To obtain Einstein equation from eq. (3.11), the metric and the normal derivative of the metric have to vanish at the boundary. In field theory, the condition on the normal derivative of the metric is too strong to be implemented. The best way to take care of this problem is to add a surface term that will cancel the contribution coming from the normal derivative at the boundary. This surface term can be calculated by taking variation of eq. (3.11) under variations of the metric g^{ab} (see *e.g.*, ref. [47])

$$\delta I_{\rm H} = -\frac{1}{16\pi G}\int d^4x\,\sqrt{g}\left[\left(R_{ab}-\frac{1}{2}g_{ab}R\right)\delta g^{ab}+g^{ab}\delta R_{ab}\right] .$$

The surface term comes from $g^{ab}\delta R_{ab}$, which can be written $\nabla^a v_a$, with

$$v_a = \nabla^b(\delta g_{ab}) - g^{cd}\nabla_a(\delta g_{cd}) .$$

Using Gauss's theorem, one can write the surface term

$$I_{\rm S} = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\,\sqrt{g}\nabla^a v_a = -\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d^3x\,\sqrt{h}n^a v_a ,$$

where $\mathcal{M}$ is the manifold, n_a is the unit normal to the boundary $\partial\mathcal{M}$ and h is the determinant of the induced metric h_{ab} on $\partial\mathcal{M}$. Using $g_{ab} = h_{ab} + n_a n_b$, we have on $\partial\mathcal{M}$

$$\begin{aligned} n^a v_a &= n^a g^{bc}\Big[\nabla_c(\delta g_{ab}) - \nabla_a(\delta g_{bc})\Big] \\ &= n^a h^{bc}\Big[\nabla_c(\delta g_{ab}) - \nabla_a(\delta g_{bc})\Big] \\ &= -n^a h^{bc}\nabla_a\left(\delta g_{bc}\right) , \end{aligned}$$

where we use $h^{bc}\nabla_c\left(\delta g_{ab}\right) = 0$ because $h^{bc}\nabla_c$ is the derivative along the boundary and $\delta g_{ab} = 0$ on the boundary. The right-hand side is related to the variation of the trace of the extrinsic curvature of the boundary, $K = K^a{}_a =$

$h^{ab}\nabla_b n_a$

$$\begin{aligned}\delta K &= h^a{}_b \delta(\nabla_a n^b) \\ &= \frac{1}{2} h^a{}_b g^{bd} \Big[\nabla_a(\delta g_{cd}) + \nabla_c(\delta g_{ad}) - \nabla_d(\delta g_{ac})\Big] n^c \\ &= \frac{1}{2} h^{ad} n^c \nabla_c (\delta g_{ad}) \ ,\end{aligned}$$

where we used the symmetry of the induced metric $h_{ab} = h_{ba}$. Therefore, the complete gravity action is

$$I_g = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\,\sqrt{g}R - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\,\sqrt{h}K \ . \tag{3.12}$$

The extremization of I_g yields the Einstein equation when one imposes $\delta g_{ab} = 0$ at the boundary, without any conditions on the normal derivative of the metric. This action is well defined for spatially compact geometries but diverges for non-compact ones. To define an action in the latter case, one must choose a reference background g_0, which is a static solution to the field equations [48]. The physical action is then

$$I_P = I_g[g] - I_g[g_0] \ .$$

This action is finite for a class of fields g which asymptotically approach g_0. For asymptotically flat spacetime, the appropriate background is flat space, with $R = 0$, and I_P reduces to

$$I_P[g] = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\,\sqrt{g}R - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{h}\,(K - K^0) \ , \tag{3.13}$$

where $K^0 = K(\eta_{ab})$, is the extrinsic curvature of the boundary when it is embedded in flat spacetime.

We are now ready to calculate the free energy for the Schwarzschild black hole. For the Schwarzschild geometry, $R = 0$ and the free energy is given by eq. (3.10)

$$F = -\frac{1}{8\pi G\beta}\int_{\partial\mathcal{M}} d^3x\,\sqrt{h}\,(K - K^0) \ . \tag{3.14}$$

At this stage, we see that the surface term is essential, otherwise the free energy would be zero and there would be no thermodynamics from this zero-loop approximation. To evaluate eq. (3.14), we choose $S^2 \otimes [0, \beta]$ as the boundary, where S^2 is a sphere with radius $r \gg r_s$. The integral of K is given by the derivative of the surface as each point of $\partial\mathcal{M}$ is moved an equal distance along n [45], the outward unit normal to $\partial\mathcal{M}$

$$\int_{\partial\mathcal{M}} d^3x\, \sqrt{h} K = \frac{\partial}{\partial n} \int_{\partial\mathcal{M}} d^3x\, \sqrt{h}\ . \tag{3.15}$$

For a sphere, the unit normal n is along the r-axis

$$\frac{\partial}{\partial n} = \left(1 - \frac{r_s}{r}\right)^{1/2} \frac{\partial}{\partial r} \tag{3.16}$$

and the area is

$$\int_{\partial\mathcal{M}} d^3x\, \sqrt{h} = \int_0^{\beta} d\tau \int d\Omega\, r^2 \left(1 - \frac{r_s}{r}\right)^{1/2} = 4\pi\beta r^2 \left(1 - \frac{r_s}{r}\right)^{1/2}\ . \tag{3.17}$$

The integral of the extrinsic curvature (3.15) is given by

$$\begin{aligned}\int_{\partial\mathcal{M}} d^3x\, \sqrt{h} K &= 4\pi\beta \left(1 - \frac{r_s}{r}\right)^{1/2} \frac{\partial}{\partial r} \left[r^2 \left(1 - \frac{r_s}{r}\right)^{1/2}\right] \\ &= 4\pi\beta \left(2r - \frac{3}{2} r_s\right)\ ,\end{aligned} \tag{3.18}$$

where we use eqs. (3.16) and (3.17). In the same way,

$$\begin{aligned}\int_{\partial\mathcal{M}} d^3x\, \sqrt{h} K^0 &= 4\pi\beta \left(1 - \frac{r_s}{r}\right)^{1/2} \frac{\partial}{\partial r} r^2 \\ &= 4\pi\beta \left(1 - \frac{r_s}{r}\right)^{1/2} 2r\ .\end{aligned} \tag{3.19}$$

Using eqs. (3.18) and (3.19), the free energy is given by

$$\begin{aligned}F &= -\frac{1}{2G} \left[2r - \frac{3}{2} r_s - 2r \left(1 - \frac{r_s}{r}\right)^{1/2}\right] \\ &\simeq \frac{r_s}{4G} = \frac{\beta}{16\pi G}\ ,\end{aligned}$$

where we have used $\beta = 4\pi r_s$. Given the free energy, different thermodynamic quantities can be evaluated. By differentiating the free energy with respect to β, one obtains the expectation value of the energy

$$<E>= \frac{\partial}{\partial\beta}(\beta F) = \frac{\beta}{8\pi G} .$$

This confirms the relation between the mass of the black hole and the inverse temperature $M =<E>= \beta/(8\pi G)$. One can also calculate the entropy

$$S = \beta\Big(<E> - F\Big) = \frac{\beta^2}{16\pi G} = \frac{\mathcal{A}}{4G}$$

where $\mathcal{A} = 4\pi r_s^2$ is the area of the event horizon. The entropy and the temperature are exactly the same as was needed in section 3.1.1 to identify the laws of black hole dynamics with the laws of thermodynamics applied to black holes.

3.1.3 Thermodynamics for general theories

As described above, the relation found between black hole dynamics and thermodynamics was shown for black hole solutions to General Relativity. The final theory of quantum gravity is not known but it should lead at low energies to a general covariant effective action with higher curvature interactions. Such effective actions occur naturally in the context of renormalization of quantum field theory in curved spacetime (see section 3.3 and ref. [49]) and in string theory (see *e.g.*, ref. [4]). In that case, one might ask: Does black hole thermodynamics arise as well or is it particular to General Relativity? This is an important question that has stimulated a great deal of research recently. In our study, it is important to know if black hole entropy is defined for more general theories and if in particular the Bekenstein-Hawking formula is modified.

In general, black holes do emit thermal radiations at a temperature $T = \kappa/(2\pi)$ because Hawking's result is a property of black hole solutions, independently of the field equations. Moreover, Wald [50] found a derivation of the first law for general theories invariant under diffeomorphisms using a Lagrangian method. His first law takes the same form as eq. (3.1) but the entropy is modified. However the latter is given by an integration over a spacelike cross section of the event horizon of a function of local geometric quantities.

Before reviewing Wald's proof, we need to introduce some concepts of differential geometry. An isometry is a change of coordinates $x \to x'$ that leaves the form of the metric g_{ab} unchanged

$$g_{ab}(x) = g'_{ab}(x) = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b} g_{cd}(x') \ . \tag{3.20}$$

In general, eq. (3.20) is a complicated restriction. To simplify, one can look at the special case of an infinitesimal transformation $x'^a = x^a + \epsilon\chi^a(x)$ with $\epsilon \ll 1$. Eq. (3.20) then reduces to

$$\chi_{a;b} + \chi_{b;a} = \pounds_\chi g_{ab} = 0 \ , \tag{3.21}$$

where $\pounds_\chi$ is the Lie derivative along the vector χ (see, *e.g.*, ref. [47]). This equation is called the Killing equation and the vector χ^a is called the Killing field. In Wald's calculation, one takes also $\pounds_\chi\psi = 0$ for ψ the matter fields so that the entire solution is invariant under the Killing flow. A spacetime is said to be stationary if there exists a Killing field χ^a which is timelike at infinity. This Killing field generates a one-parameter group of isometries, whose orbits are timelike curves. This group of isometries expresses the time translational symmetry of the spacetime.

A horizon h^+ is a Killing horizon if there exists a Killing field ξ^a which is normal to h^+. In the context of General Relativity, there is a theorem that states that the event horizons of any stationary black hole are Killing horizons [3]. If in such a stationary spacetime, the Killing field χ^a is not normal to h^+, then this theorem implies that there exists another Killing field ξ^a which is. It can be shown also that a linear combination ϕ^a of ξ^a and χ^a has closed orbits. Hence, stationary black holes for which $\chi^a \neq \xi^a$ are axis-symmetric. One defines the angular velocity of the black hole Ω by

$$\xi^a = \chi^a + \Omega\phi^a \, , \tag{3.22}$$

where ϕ^a is normalized such that the orbits have period 2π and χ^a is normalized by imposing $\chi^a\chi_a = -1$ at infinity. With the Killing field ξ^a, one defines the surface gravity κ

$$\xi^a \nabla_a \xi^b = \kappa \xi^b \quad \text{on } h^+. \tag{3.23}$$

Using Einstein equations, one can show that if the generators of h^+ are geodesically complete in the past and if the surface gravity is non-zero, the Killing horizon contains a two-dimensional spacelike cross section $\mathcal{B}$, called the bifurcation surface, on which ξ^a vanishes [51]. Such a horizon is called a bifurcate horizon. The presence of such a bifurcation surface is a consequence of the zeroth law. If the surface gravity is constant on the horizon, then the horizon must be a bifurcate Killing horizon or the surface gravity vanishes.

For more general theory of gravity, there is no general proof of the zeroth law but recently, Rácz and Wald have given a proof with the condition that the twist form field ω_a is zero at the horizon [52], where $\omega_a = \epsilon_{abcd}\xi^b\nabla^c\xi^d$ and ϵ_{abcd} is the Levi-Civita tensor. As a corollary, the zeroth law holds for static black holes and stationary axis-symmetric black holes possessing a $t-\phi$ reflection isometry. These black holes have a Killing horizon that can be extended (if necessary) to a bifurcate horizon.

In the following, we review Wald's method[50] for general theories in four dimensions. (The generalization to other numbers of dimensions is trivial.) Wald starts with a Lagrangian $\boldsymbol{L}(\psi)$ describing a general theory invariant under diffeomorphisms and built out from the metric and matter fields, collectively named ψ. It is useful to take the Lagrangian as a 4-form (for a review of forms, see *e.g.*, ref. [53]) instead of the usual scalar density. Under a general field variation $\delta\psi$, the Lagrangian varies as

$$\delta \boldsymbol{L} = \boldsymbol{E}\delta\psi + \mathrm{d}\Theta \ , \tag{3.24}$$

where in the first term of the right-hand side, a summation over all fields (including contractions of tensor indices) is understood and $\boldsymbol{E} = 0$ are the equations of motion. If $\delta\psi$ is chosen as a symmetry of the Lagrangian (*i.e.*, $\delta\boldsymbol{L} = 0$) then Θ is the corresponding Noether current 3-form, locally constructed from ψ and $\delta\psi$. When the equations of motion are satisfied, the Noether current is conserved, *i.e.*, $\mathrm{d}\Theta = 0$.

Let $\hat{\delta}\psi$ be a diffeomorphism transformation generated by a Killing vector ξ^a, $\hat{\delta}\psi = \pounds_\xi \psi$. The Lagrangian variation is

$$\hat{\delta}\boldsymbol{L} = \pounds_\xi \boldsymbol{L} = \mathrm{d}(\xi \cdot \boldsymbol{L}) \ ,$$

where we use the formula for Lie derivative on differential forms

$$\pounds_\xi \Lambda = \xi \cdot \mathrm{d}\Lambda + \mathrm{d}(\xi \cdot \Lambda) \ . \tag{3.25}$$

In eq. (3.25), the "$\cdot$" denotes the inner product. In this case, the Lagrangian variation does not vanish but it is a total derivative. The action will be invariant if the fields satisfy appropriate boundary conditions.

One can build an improved Noether current 3-form $\boldsymbol{J}$ which is conserved when the equations of motion are satisfied

$$\boldsymbol{J} = \Theta - \xi \cdot \boldsymbol{L} \tag{3.26}$$
$$\mathrm{d}\boldsymbol{J} = 0 \quad \text{when } \boldsymbol{E} = 0.$$

When $\boldsymbol{E} = 0$, one can show that there exists a globally-defined 2-form $\boldsymbol{Q}$ [54] such that the Noether current is given by $\boldsymbol{J} = \mathrm{d}\boldsymbol{Q}$. $\boldsymbol{Q}$ is called the Noether potential 2-form relative to ξ^a. The integral of $\boldsymbol{Q}$ over a closed surface yields the Noether charge Q relative to ξ^a.

Now consider a stationary black hole with a bifurcate Killing horizon h^+ and a bifurcation surface $\mathcal{B}$. Choose ξ^a to be the Killing field that vanishes on $\mathcal{B}$ and is normalized as eq. (3.22). Then $\mathcal{L}_\xi \psi = 0$. Choose ∇_a to be the covariant derivative operator for this background. Let $\delta\psi$ be a variation of the dynamical fields away from the background solution ψ such that $\delta\psi$ is an asymptotically flat solution of the linearized equations. Then the Noether current $\boldsymbol{J}$ changes as

$$\delta \boldsymbol{J} = \mathrm{d}\left(\xi \cdot \Theta\right) \ , \tag{3.27}$$

where we used eq. (3.24) and eq. (3.26) as well as $\mathcal{L}_\xi \psi = 0$.

Because $(\psi + \delta\psi)$ is still a solution of the equations of motion, $\delta\boldsymbol{J} = \mathrm{d}\delta\boldsymbol{Q}$. To obtain the first law, one integrates eq. (3.27) over $\mathcal{C}$, a spatial three-surface stretching from asymptotic infinity to an interior boundary at $\mathcal{B}$.

$$0 = \int_{\mathcal{C}} \mathrm{d}\left(\delta\boldsymbol{Q} - \xi \cdot \Theta\right) = \int_{\mathcal{B}} \left(\delta\boldsymbol{Q} - \xi \cdot \Theta\right) - \int_{\infty} \left(\delta\boldsymbol{Q} - \xi \cdot \Theta\right) \ , \tag{3.28}$$

where ∞ is a 2-sphere at infinity. At infinity, it is natural to associate the Noether charges associated with ξ^a to conserved quantities in the manner of refs. [55, 56].

$$\delta M = \int_{\infty} (\delta \boldsymbol{Q}[\chi] - \chi \cdot \Theta) \tag{3.29}$$

$$\delta \mathcal{J} = - \int_{\infty} \delta \boldsymbol{Q}[\phi] \; . \tag{3.30}$$

In eq. (3.30), $\phi \cdot \Theta = 0$ because ϕ^a is taken to be tangent to the sphere at infinity. Using eqs. (3.28), (3.29) and (3.30) and the fact that $\xi^a = 0$ at the bifurcate surface, one obtains

$$\delta \int_{\mathcal{B}} \boldsymbol{Q} = \delta M - \Omega \delta \mathcal{J} \; . \tag{3.31}$$

$\boldsymbol{Q}$ is a local functional of the local fields (the metric and other matter fields), and also of the Killing field ξ^a and its derivatives. Higher derivatives of ξ^a may be eliminated using the identity for Killing fields $\nabla_a \nabla_b \xi_c = -R_{abc}{}^d \xi_d$, leaving $\boldsymbol{Q}$ as a linear combination of ξ^a and $\nabla_a \xi_b$. One can eliminate all the dependence of the Killing field at the bifurcation surface $\mathcal{B}$ since there $\xi^a = 0$ and $\nabla_a \xi_b = \kappa \epsilon_{ab}$, ϵ_{ab} being the binormal to $\mathcal{B}$. Thus on the bifurcate surface, the Noether potential 2-form depends only on the local fields.

The first law is found by expressing $\boldsymbol{Q}$ as $\mathbf{Q} = \kappa \tilde{\boldsymbol{Q}}$ [50], where $\tilde{\boldsymbol{Q}}$ is the Noether potential 2-form build from the Killing field $\hat{\xi}^a$ normalized to have unit surface gravity. Using this and eq. (3.31), one obtains the first law of black hole dynamics for general theory invariant under diffeomorphisms

$$\frac{\kappa}{2\pi} \delta S = \delta M - \Omega \delta \mathcal{J} \; , \tag{3.32}$$

with

$$S = 2\pi \int_{\mathcal{B}} \tilde{\boldsymbol{Q}} \; . \tag{3.33}$$

Thus, the first law still relates variations of quantities measured at infinity to variations of geometric properties of the horizon.

There are three kinds of ambiguities in the definition of the Noether potential 2-form. First, an exact form can be added to the Lagrangian. Second, an exact form can be added to Θ in eq. (3.24). Finally, one can add a closed form to the Noether potential. These ambiguities do not change the entropy (3.33) for stationary black holes [57]. It can also be shown that for stationary black holes, one can integrate over any spacelike cross section of the horizon and the entropy is still a local function of the metric and other matter fields [57].

To calculate the entropy for a generic theory invariant under diffeomorphisms, one needs to calculate the Noether potential. An inductive algorithm has been given in ref. [54] and the entropy has been calculated for a wide class of theories using Wald's method [57, 58] and Euclidean method [59]. For a theory described by a Lagrangian scalar density L built out from the metric, from m derivatives of the Riemann tensor and from l derivatives of the matter field ϕ (with symmetrized derivatives)

$$L = L\Big(g_{ab}, R_{bcde}, \nabla_a R_{bcde}, \ldots, \nabla_{(a_1} \cdots \nabla_{a_m)} R_{bcde}, \phi, \nabla_a \phi, \ldots, \nabla_{(a_1} \cdots \nabla_{a_l)} \phi\Big),$$

the entropy is given by

$$S = -2\pi \int E^{abcd} \epsilon_{ab} \epsilon_{cd} \ ,$$

where

$$E^{abcd} = \frac{\partial L}{\partial R_{abcd}} - \nabla_{a_1} \frac{\partial L}{\partial(\nabla_{a_1} R_{abcd})} + \cdots + (-1)^l \nabla_{(a_1} \cdots \nabla_{a_l)} \frac{\partial L}{\partial(\nabla_{(a_1} \cdots \nabla_{a_l)} R_{abcd})} \ .$$

g_{ab} and ∇_a are kept fixed and the integral is taken over any spacelike cross section of the horizon.

We finish this section by an example of the calculation of black hole entropy for a theory described by the action

$$I = \int d^4x\, \sqrt{-g}\left[\frac{1}{16\pi G}R + \frac{\alpha}{4\pi}R^2 + \frac{\beta}{4\pi}R^{ab}R_{ab} + \frac{\gamma}{4\pi}R^{abcd}R_{abcd}\right] .$$

This action occurs in the context of renormalization of quantum fields in curved spacetime (see section 3.3). The entropy is given by

$$S = -\int d^2x\, \sqrt{h}\left[\frac{1}{8G}g^{ac}g^{bd} + \alpha R g^{ac}g^{bd} + \beta R^{ac}g^{bd} + \gamma R^{abcd}\right]\epsilon_{ab}\epsilon_{cd} .$$

Introduce a unit timelike vector n_a and a unit spacelike vector v_a, both normal to the horizon with $n\cdot n = -1$, $v\cdot v = 1$ and $n\cdot v = 0$. The binormal is given by $\epsilon_{ab} = n_a v_b - n_b v_a$ and the entropy reduces to

$$S = \frac{A}{4G} + \int d^2x\, \sqrt{h}\left[2\alpha R + \beta R_{ab}g_{\perp}^{ab} - \gamma R^{abcd}\epsilon_{ab}\epsilon_{cd}\right] , \qquad (3.34)$$

where we used $\epsilon^{ab}\epsilon_{ab} = -2$ and $\epsilon_{ab}\epsilon_c{}^b = n_a n_c - v_a v_c = -g_{\perp ac}$, $g_{\perp}^{ab}$ is the metric in the normal subspace to the cross section of the horizon. The first term in (3.34) is the usual Bekenstein-Hawking formula and the second term is the modification due to the higher-derivative interactions. It will be important to take into account these modifications in black hole entropy when we consider the statistical entropy.

3.2 Statistical entropy

The concept of black hole entropy described in the previous section is in terms of thermodynamics. Black hole entropy comes into play from the first and the second laws of black hole dynamics and the analogy with ordinary thermodynamics. One would also like to understand black hole entropy from a statistical point of view. Despite a great deal of effort, this understanding is still incomplete. Various methods have been proposed to calculate the entropy by some counting of microstates of the black hole system.

York considered a dynamical black hole with no external fields [60]. Because of the fluctuations of the geometry, the black hole emits radiation. These fluctuations induce a shift of the event horizon inside the static limit surface. (We recall that the static limit surface is the surface where an observer cannot be at rest and the event horizon is a null surface defined by the outermost locus traced by outgoing photons that can never reach infinity.) The region between the two surfaces is called the quantum ergosphere. York proposed that the entropy is then given by the logarithm of the number of ways that the quantum ergosphere can be excited during the black hole evaporation. This model is unsatisfactory because it is non-local in time. The entropy depends on the entire future evolution of the black hole.

In the membrane paradigm [61, 62], the entropy is hypothesized to be the logarithm of the amount of information that one loses under the stretched horizon. The entropy is located at the event horizon. Put differently, the entropy is the logarithm of the number of quantum mechanically distinct ways to generate the black hole total mass, angular momentum and charge by injecting quanta.

A similar method was introduced by 't Hooft [12], where the entropy is obtained by counting the number of states of a quantum field propagating just outside the horizon of a fixed black hole. The model was initially defined in four dimensions but it was generalized for other dimensions in ref. [63]. The number of states a particle can occupy diverges at the horizon because of an infinite blue shift factor. To avoid this problem, 't Hooft introduces a "brick wall", where the field vanishes, at a coordinate distance h from the horizon.

The entropy found becomes

$$S = \frac{r_s}{360h} \; . \tag{3.35}$$

If one replaces the distance h by an invariant proper distance ϵ

$$\epsilon = \int_{r_s}^{r_s+h} ds = \int_{r_s}^{r_s+h} \frac{dr}{\sqrt{1-\frac{r_s}{r}}} \approx 2\sqrt{r_s h} \; ,$$

the entropy is

$$S = \frac{r_s^2}{90\epsilon^2} = \frac{A}{4}\frac{1}{90\pi\epsilon^2} \; .$$

The entropy is equal to the Bekenstein-Hawking entropy if one chooses $\epsilon = l_P/\sqrt{90\pi} = \sqrt{G/(90\pi)}$. It is natural to take a cut-off of the order of the Planck length. However, the entropy diverges if one allows the cut-off to vanish, *i.e.*, $\epsilon \to 0$.

Another method to define statistical black hole entropy is the idea of entanglement entropy [13, 64]. In this approach, entropy is due to loss of information in a region outside the observation. One can study this in flat spacetime by introducing an imaginary sphere Ω that mimics the black hole. One starts with the ground state

$$|0> = \Psi[\phi_{\rm in}, \phi_{\rm out}] \; ,$$

where $\phi_{\rm in}$ and $\phi_{\rm out}$ represent the degrees of freedom inside and outside the sphere Ω, respectively. For a black hole, we cannot observe the degrees of freedom inside the event horizon and we must trace over these degrees of freedom and build a density matrix

$$\rho[\phi_{out}, \phi'_{out}] = \int D[\phi_{in}]\, \Psi[\phi_{in}, \phi_{out}]\Psi^*[\phi_{in}, \phi'_{out}] \; .$$

Alternatively, one can trace over the degrees of freedom outside Ω [65]. The results should be equivalent if one starts with a global pure state [13]. By diagonalizing the density matrix

$$\int D[\phi']\,\rho[\phi,\phi']f_n[\phi'] = p_n f_n[\phi] ,$$

one can calculate the entanglement entropy

$$S = -\sum_n p_n \ln p_n . \tag{3.36}$$

The entropy (3.36) diverges and one needs to introduce a cut-off. Ref. [13] uses a lattice and obtains for the entropy

$$S = 0.30\frac{r_s^2}{a^2} , \tag{3.37}$$

where a is the lattice spacing. The entropy is proportional to the area and if one takes the cut-off to be of the order of the Planck length, eq. (3.37) is of the order of the Bekenstein-Hawking formula. The entanglement entropy (3.36) has a thermal character, independent of the quantum field theory [66]. The latter implies that eq. (3.37) is equivalent to the entropy found using 't Hooft's brick wall model, although the actual divergent coefficient is scheme dependent.

Instead of taking the cut-off to be of the order of the Planck length, one can interpret these entropy formulae as the one-loop correction to the Bekenstein-Hawking formula [67, 68, 69]. In this sense, Susskind and Uglum suggested [67] that the divergence in the entropy could be absorbed in the renormalization of Newton's constant. This is an interesting suggestion because both divergences are quadratic (for the divergence of Newton's constant, see section 3.3).

In the rest of this chapter, we are concerned with this suggestion. Because the value of the divergent coefficient is scheme dependent, it is important to do both calculations (the divergences in G and in S) with the same regulator. With this concern, we will first use Pauli-Villars regularization in the calculation of the divergences that appear in Newton's constant in section 3.3. Then we will use 't Hooft's method to calculate black hole entropy in section 3.4 but we will replace his brick wall by the same Pauli-Villars regulator.

3.3 Renormalization of the effective action

The divergence of Newton's constant appears in the context of quantum field theory in curved spacetime, where one considers quantum fields that propagate in a fixed curved background [49]. It is a semi-classical approximation to the unknown theory of quantum gravity where the gravitational field is treated classically and the matter fields quantum mechanically.

3.3.1 The effective action

To quantize the matter fields, one may use the path-integral formalism. The starting point is the generating functional $Z[J]$ that gives the transition amplitude from the initial vacuum $|0, \text{in}>$ to the final vacuum $|0, \text{out}>$ in the presence of external sources $J(x)$:

$$Z[J] = <0, \text{out}|0, \text{in}>^J = \int D[\psi] \exp\left[iI[\psi] + i\int d^4x\, J(x)\psi(x)\right] ,$$

where $\psi(x)$ represents collectively the matter fields and the metric and I is the total action. The connected time-ordered Green functions are obtained by differentiation

$$< 0, \text{out}|\mathrm{T}\big(\phi(x_1)\cdots\phi(x_n)\big)|0, \text{in} >_c = \frac{1}{i^n}\frac{\partial^n \ln Z[J]}{\partial J(x_1)\cdots\partial J(x_n)}\bigg|_{J=0} ,$$

where T is the time-ordering operator. In flat spacetime, $Z[0] = <0|0>$ is a constant that can be normalized to unity. However, particles can be created in curved spacetime by the background gravitational field and $|0, \text{in}> \neq |0, \text{out}>$ in general, even without the external sources $J(x)$.

In the study of the divergence of Newton's constant, we may set $J(x)=0$ and study the generating functional $Z \equiv Z[0]$ for a fixed background g and a scalar field ϕ:

$$Z[g] = e^{iI_g[g]} \int D[\phi] e^{iI_m[g,\phi]} ,$$

where I_g is the gravity action and I_m is the matter action. One defines the effective action by

$$I_{\text{eff}}[g] = -i \ln Z[g] = I_g[g] + W[g] , \tag{3.38}$$

where $W[g]$ is the scalar effective action

$$W = -i \ln \int D[\phi] e^{iI_m[g,\phi]} . \tag{3.39}$$

The effective action represents the gravity action after taking into account the contribution of the matter fields to the geometry. The gravity action is modified. Let us illustrate this by choosing the Hilbert action as the gravity action and neglecting the surface term

$$I_H = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R .$$

The equation of motion for the gravitational field is obtained by taking variation of I_{eff} with respect to the inverse metric g^{ab}

$$\begin{aligned}\delta I_{\text{eff}} &= 0 \\ &= \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R_{ab} - \frac{1}{2} g_{ab} R \right) \delta g^{ab} \\ &\quad + \frac{1}{<0,\text{out}|0,\text{in}>} \int D[\phi] \frac{\delta I_m}{\delta g^{ab}} e^{iI_m[g,\phi]} \delta g^{ab} .\end{aligned}$$

The variation of the matter action yields the matter stress tensor $T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta I_m}{\delta g^{ab}}$. Therefore, the equation of motion is

$$\begin{aligned}R_{ab} - \frac{1}{2} g_{ab} R &= \frac{8\pi G}{<0,\text{out}|0,\text{in}>} \int D[\phi]\, T_{ab}\, e^{iI_m[g,\phi]} \\ &= 8\pi G \frac{< 0,\text{out}|T_{ab}|0,\text{in} >}{<0,\text{out}|0,\text{in}>} = 8\pi G <T_{ab}> .\end{aligned}$$

Hence, the variation of the effective action (3.38) leads to the semi-classical Einstein equation, with the stress tensor replaced by its expectation value.

We want to calculate the scalar effective action (3.39) for a single free scalar particle described by the action

$$I_m = -\frac{1}{2}\int d^4x\,\sqrt{-g}\Big[g^{ab}\nabla_a\phi\nabla_b\phi + m^2\phi^2\Big]\ , \tag{3.40}$$

where m is the mass of the scalar particle. For the gravity action, one needs to add to the Hibert action a cosmological constant and higher-order interactions because these terms may be induced by the scalar effective action W. Thus, we start with the gravity action

$$I_g = \int d^4x\,\sqrt{-g}\left[-\frac{\Lambda_B}{8\pi G_B} + \frac{R}{16\pi G_B} + \frac{\alpha_B}{4\pi}R^2 \right.$$
$$\left. + \frac{\beta_B}{4\pi}R^{ab}R_{ab} + \frac{\gamma_B}{4\pi}R^{abcd}R_{abcd} + \cdots\right]\ , \tag{3.41}$$

where Λ_B is the cosmological constant, G_B is Newton's constant and α_B, β_B and γ_B are dimensionless coupling constants. The subscript B indicates that the quantities are bare ones that will be renormalized shortly. The ellipsis indicates that the action may contain other higher-order interactions but only the present terms will be of interest in the present calculation. The scalar field effective action W is a gaussian integral that can be evaluated exactly

$$W = -i\ln\int D[\phi]\exp\Big\{-\frac{i}{2}\int d^4x\,\sqrt{-g(x)}\Big[g^{ab}\nabla_a\phi\nabla_b\phi + m^2\phi^2\Big]\Big\}\ . \tag{3.42}$$

Using an integration by parts, the exponent can be written

$$-\frac{i}{2}\int d^4x\,d^4x'\,\sqrt{-g(x)}\sqrt{-g(x')}\phi(x)K(x,x')\phi(x')\ , \tag{3.43}$$

where $K(x,x')$ is a symmetric operator

$$K(x,x') = \frac{1}{\sqrt{-g(x)}}\left(-\Box_x + m^2\right)\delta(x-x')\ . \tag{3.44}$$

The operator $K(x,x')$ is the inverse of the Feynman Green function G_F, where G_F is defined by

$$\left(\Box_x - m^2\right) G_F(x,x') = -\frac{\delta(x-x')}{\sqrt{-g(x)}} . \tag{3.45}$$

Using eq. (3.43) and eq. (3.44) as well as $G_F = K^{-1}$, eq. (3.42) becomes

$$\begin{aligned} W &= -i \ln \int D[\phi] \exp\left\{-\frac{i}{2}\int d^4x\, d^4y\, \sqrt{-g(x)}\sqrt{-g(y)}\phi(x)K(x,y)\phi(y)\right\} \\ &= -\frac{i}{(\det K)^{1/2}} = -\frac{i}{2}\,\mathrm{Tr}\ln\left(G_F\right) . \end{aligned} \tag{3.46}$$

3.?.2 Asymptotic expansion of the Green function

The scalar effective action (3.46) is only a formal result. To obtain meaningful information, one needs to introduce a representation for the Green function. In flat spacetime, the behavior of Green functions are studied using momentum space techniques. In curved spacetime, the homogeneity required for a global momentum space is lacking in general. However, it is possible to introduce a local momentum space to study the ultraviolet behavior of Green functions [70] using Riemann normal coordinates [71]. Riemann normal coordinates are valid in normal neighborhoods of the origin in which the geodesics from the origin do not intersect. They should describe the ultraviolet regime because the latter involves arbitrarily short distances scales.

Consider the Green function defined by eq. (3.45) and introduce Riemann normal coordinates y^a with origin at the point x'. One has [71]

$$\begin{aligned} g_{ab} = \eta_{ab} - \frac{1}{3}R_{afbg}\, y^f y^g - \frac{1}{6}R_{afbg;h}\, y^f y^g y^h \\ + \left(-\frac{1}{20}R_{afbg;hj} + \frac{2}{45}R_{fagl}R^l{}_{hbj}\right) y^f y^g y^h y^j + \cdots \end{aligned} \tag{3.47}$$

and

$$g = 1 - \frac{1}{3}R_{ab}\,y^a y^b - \frac{1}{6}R_{ab;c}y^a y^b y^c$$
$$+ \left(\frac{1}{18}R_{ab}R_{cd} - \frac{1}{90}R_{lab}{}^{k}R_{lcdk} - \frac{1}{20}\nabla_d\nabla_c R_{ab}\right) y^a y^b y^c y^d + \cdots \ . \quad (3.48)$$

where the coefficients are evaluated at $y = 0$ and η_{ab} is the Minkowski metric. All indices in the right-hand side of eqs. (3.47) and (3.48) are raised with the Minkowski metric.

One may define

$$\mathcal{G}(x,x') = (-g(x))^{1/4}G_{\rm F}(x,x') \quad (3.49)$$

and its Fourier transform

$$\mathcal{G}(x,x') = \int \frac{d^4k}{(2\pi)^4}e^{ik\cdot y}\mathcal{G}(k) \ , \quad (3.50)$$

where $k \cdot y = \eta^{ab}k_a y_b$. With this, one can expand eq. (3.45) in normal coordinates using eqs. (3.49) and (3.50) to obtain an equation for $\mathcal{G}(k)$ that can be solved by iteration. This yields

$$\begin{aligned}\mathcal{G}(k) \approx (k^2+m^2)^{-1} &+ \frac{1}{6}R(k^2+m^2)^{-2} + \frac{i}{12}R_{;a}\frac{\partial}{\partial k_a}(k^2+m^2)^{-2} \\ &+ \frac{1}{3}a_{ab}\frac{\partial^2}{\partial k_a \partial k_b}(k^2+m^2)^{-2} + \left[\frac{1}{36}R^2 - \frac{2}{3}a^b{}_b\right](k^2+m^2)^{-3} \ ,\end{aligned} \quad (3.51)$$

where

$$a_{ab} = -\frac{3}{40}R_{;ab} - \frac{1}{40}\Box R_{ab} + \frac{1}{30}R_a{}^dR_{db} - \frac{1}{60}R_{daeb}R^{de} - \frac{1}{60}R^{def}{}_aR_{defb} \ ,$$

and $\approx$ means that this is an asymptotic expansion. The Green function (3.49) corresponds to a time-ordered product if one uses the usual replacement $m^2 \to m^2 - i\epsilon$. Substituting eq. (3.51) in (3.50), one obtains

$$\begin{aligned}\mathcal{G}(x,x') \approx \int \frac{d^4k}{(2\pi)^4}e^{ik\cdot y}\Bigg[a_0(x,x') &+ a_1(x,x')\left(-\frac{\partial}{\partial m^2}\right) \\ &+ a_2(x,x')\left(\frac{\partial}{\partial m^2}\right)^2\Bigg]\frac{1}{k^2+m^2-i\epsilon}\end{aligned} \quad (3.52)$$

where

$$a_0(x,x') = 1 \tag{3.53a}$$

and, to fourth order in derivatives of the metric

$$a_1(x,x') = \frac{1}{6}R + \frac{1}{12}R_{;a}y^a - \frac{1}{3}a_{ab}y^a y^b \,, \tag{3.53b}$$

$$a_2(x,x') = \frac{1}{72}R^2 - \frac{1}{3}a^b{}_b \,. \tag{3.53c}$$

To evaluate the integral in eq. (3.52), it is useful to introduce the integral representation

$$\frac{1}{k^2+m^2-i\epsilon} = i\int_0^\infty ds\, e^{-is(k^2+m^2-i\epsilon)} \,.$$

The integral on d^4k is then a gaussian integral. It can be evaluated to obtain

$$\mathcal{G}(x,x') = \frac{i}{(4\pi)^2}\int_0^\infty \frac{i\,ds}{(is)^2}\exp\left[-im^2 s - \frac{\sigma}{2is}\right] F(x,x';is) \,, \tag{3.54}$$

where $\sigma = \frac{1}{2}y_a y^a$ is half the geodesic distance between x and x' and $m^2 \to m^2 - i\epsilon$ is understood. To fourth order in derivatives of the metric

$$F(x,x';is) = a_0(x,x') + is a_1(x,x') + (is)^2 a_2(x,x') \,. \tag{3.55}$$

The Feynman Green function is found using eqs. (3.49) and (3.54)

$$G_F(x,x') = \frac{i\Delta(x,x')}{(4\pi)^2}\int_0^\infty \frac{i\,ds}{(is)^2}\exp\left[-im^2 s - \frac{\sigma}{2is}\right] F(x,x';is) \,, \tag{3.56}$$

where $\Delta(x,x')$ is the van Vleck determinant

$$\Delta(x,x') = -\frac{1}{\sqrt{-g(x)}}\det\left[-\partial_a\partial_b\sigma(x,x')\right]\frac{1}{\sqrt{-g(x')}}$$

which reduces to $\Delta(x,x') = 1/\sqrt{-g(x)}$ in normal coordinates. The representation (3.56) for the Green function was originally derived by DeWitt [72, 73] following the work of Schwinger [74]. The representation is exact, but the function $F(x, x'; is)$ is not known exactly. It can be expanded in an asymptotic series

$$F(x, x'; is) = \sum_{n=0}^{\infty} a_n(x, x')(is)^n \, .$$

The coefficients $a_n(x, x')$ are given by recursion relations [75].

3.3.3 DeWitt-Schwinger expansion

With the representation for the Green function $G_F(x, x')$, we can find a representation for the scalar effective action $W[g]$. In eq. (3.46), G_F can be viewed as an operator acting in the space of vectors $|x>$ such that $G_F(x, x') = \, <x| \, G_F \, |x'>$. The vectors $|x>$ are normalized by $<x|x'>= \delta(x - x')/\sqrt{-g(x)}$ and the trace of an operator O is given by

$$\mathrm{Tr}\, O = \int d^4x \, \sqrt{-g(x)} \lim_{x \to x'} < x|O|x' > \, . \tag{3.57}$$

One may introduce the proper time representation of the Green function

$$G_F(x, x') = K^{-1}(x, x') = i \int_0^{\infty} ds \, e^{-isK(x,x')} \, . \tag{3.58}$$

If one compares eq. (3.58) with eq. (3.56), one obtains

$$e^{-isK(x,x')} = \frac{-i\Delta(x, x')}{16\pi^2 s^2} \exp\left[-im^2 s - \frac{\sigma}{2is}\right] F(x, x'; is) \, . \tag{3.59}$$

Then, assume that K has a small negative imaginary part (the $-i\epsilon$) and consider the integral

$$\int_{\delta}^{\infty} \frac{ds}{s} e^{-isK} = E_1(i\delta K) \, , \tag{3.60}$$

for $\delta \ll 1$. $E_1(x)$ is the exponential integral. For small values of the argument, $E_1(x)$ has the expansion (see, *e.g.*, ref. [76])

$$E_1(x) = -\gamma - \ln(x) + \mathcal{O}(x) \, . \tag{3.61}$$

Take $\delta \to 0$ and discard a metric independent infinite constant to obtain the logarithm of the Green function

$$\ln G_{\rm F}(x,x') = -\ln K(x,x') = \int_0^\infty \frac{ds}{s} e^{-isK(x,x')} , \qquad (3.62)$$

where eqs. (3.60) and (3.61) have been used. Substitute (3.59) in (3.62) to obtain

$$\ln G_{\rm F}(x,x') = -i\frac{\Delta(x,x')}{16\pi^2}\int_0^\infty \frac{ds}{s^3} \exp\left[-im^2 s - \frac{\sigma}{2is}\right] F(x,x';is) . \qquad (3.63)$$

The DeWitt-Schwinger expansion of the scalar effective action is found using eqs. (3.57) and (3.63) in (3.46)

$$W = -\frac{1}{32\pi^2}\int d^4x\,\sqrt{-g(x)}\lim_{x'\to x}\int_0^\infty \frac{ds}{s^3}\Delta(x,x') \times \exp\left[-im^2 s - \frac{\sigma}{2is}\right] F(x,x';is) .$$

For $x \neq x'$, the integral over s converges because for $s \to 0$, the factor $-\sigma/s$ in the exponential plays the role of a damping factor and for $s \to \infty$, the damping factor comes from the replacement of m^2 by $m^2 - i\epsilon$. Taking the limit $x' \to x$, and using the asymptotic expansion for $F(x,x';is)$, the scalar effective action is

$$W = -\frac{1}{32\pi^2}\int d^4x\,\sqrt{-g(x)}\int_0^\infty \frac{ds}{s^3}e^{-im^2 s}\sum_{n=0}^\infty a_n(x)(is)^n , \qquad (3.64)$$

where $a_n(x) = \lim_{x'\to x} a_n(x,x')$ and $\Delta(x,x) = 1$. The integral over s diverges for $n \leq 2$ because there is no damping factor for $s \to 0$. Therefore, one needs to introduce a regularization to properly define W.

3.3.4 Regularization and renormalization

In the following, we are only interested in the divergent parts of W to see how Newton's constant is renormalized. Therefore, we consider the three first terms in the expansion of W

$$W_{\rm div} = -\frac{1}{32\pi^2}\int d^4x\,\sqrt{-g}\int_0^\infty \frac{ds}{s^3}\left[a_0(x) + isa_1(x) + (is)^2a_2(x)\right] \ ,$$

where the coefficients $a_n(x)$ are given by

$$a_0(x) = 1 \tag{3.65a}$$

$$a_1(x) = \frac{1}{6}R \tag{3.65b}$$

$$a_2(x) = \frac{1}{180}R^{abcd}R_{abcd} - \frac{1}{180}R^{ab}R_{ab} + \frac{1}{30}\Box R + \frac{1}{72}R^2 \ , \tag{3.65c}$$

where we used eqs. (3.53).

The effective action may be regulated using many different methods [49], but in the present calculation we adopt a Pauli-Villars regularization scheme [77, 78, 79]. In general, such a scheme involves the introduction of a number of fictitious fields with very large masses set by some ultraviolet cut-off scale. Some of these regulator fields are also quantized with the "wrong" statistics, so that their contributions in loops tend to cancel those of the remaining fields. The number, statistics and masses of the regulator fields are chosen in order to render all of the ultraviolet divergences finite. The potentially divergent terms are functions of the fictitious particle masses. To avoid the production of fictitious particles and to remove the regularization, the masses are allowed to go to infinity at the end of the calculation.

In the present four-dimensional scalar field theory, one introduces five regulator fields: ϕ_1 and ϕ_2, which are two anticommuting fields with mass $m_{1,2} = \sqrt{\mu^2 + m^2}$; ϕ_3 and ϕ_4, which are two commuting fields with mass $m_{3,4} = \sqrt{3\mu^2 + m^2}$; and ϕ_5, which is an anticommuting field with mass $m_5 = \sqrt{4\mu^2 + m^2}$. The total action for the matter fields is

$$I_{\rm m} = -\frac{1}{2}\sum_{i=0}^{5}\int d^4x\,\sqrt{-g}\left[g^{ab}\nabla_a\phi_i\nabla_b\phi_i + m_i^2\phi_i^2\right] \ , \tag{3.66}$$

where the original scalar is included as $\phi_0 = \phi$ with mass $m_0 = m$. Now, each field makes a contribution to the effective action as discussed in section 3.3.3, except that as a result of the anticommuting statistics for ϕ_1, ϕ_2 and ϕ_5, their contribution to the effective action has the opposite sign, *i.e.*, $W(g) \simeq \frac{i}{2}\operatorname{Tr}\log[G_F(g, m_i^2)]$. The divergent part of the scalar effective action is then

$$
\begin{aligned}
W_{\rm div} &= -\frac{1}{32\pi^2}\int d^4x\,\sqrt{-g}\int_0^\infty \frac{ds}{s^3}\,\left[a_0(x) + isa_1(x) + (is)^2a_2(x)\right] \\
&\qquad \times \left[e^{-im^2s} - 2e^{-i(\mu^2+m^2)s} + 2e^{-i(3\mu^2+m^2)s} - e^{-i(4\mu^2+m^2)s}\right] \\
&= \frac{1}{32\pi^2}\int d^4x\,\sqrt{-g}\,\left[-C\,a_0(x) + B\,a_1(x) + Aa_2(x)\right] \ .
\end{aligned} \tag{3.67}
$$

In this expression, A, B and C are constants which depend on m and μ, and which diverge for $\mu \to \infty$:

$$
A = \ln\frac{4\mu^2+m^2}{m^2} + 2\ln\frac{\mu^2+m^2}{3\mu^2+m^2} \tag{3.68a}
$$

$$
B = \mu^2\left[2\ln\frac{3\mu^2+m^2}{\mu^2+m^2} + 4\ln\frac{3\mu^2+m^2}{4\mu^2+m^2}\right] + m^2\left[\ln\frac{m^2}{4\mu^2+m^2} + 2\ln\frac{3\mu^2+m^2}{\mu^2+m^2}\right] \tag{3.68b}
$$

$$
\begin{aligned}
C = \mu^4&\left[8\ln\frac{3\mu^2+m^2}{4\mu^2+m^2} + \ln\frac{3\mu^2+m^2}{\mu^2+m^2}\right] + 2m^2\mu^2\left[\ln\frac{3\mu^2+m^2}{\mu^2+m^2} + 2\ln\frac{3\mu^2+m^2}{4\mu^2+m^2}\right] \\
&+ \frac{m^4}{2}\left[\ln\frac{m^2}{4\mu^2+m^2} + 2\ln\frac{3\mu^2+m^2}{\mu^2+m^2}\right] \ .
\end{aligned} \tag{3.68c}
$$

Combining the scalar one-loop action with the original bare action in eq. (3.41), we can identify the renormalized coupling constants in the effective gravitational action

$$
\begin{aligned}
I_{\rm eff} &= I_g + W \\
&= \int d^4x\sqrt{-g}\left[-\frac{1}{8\pi}\left(\frac{\Lambda_B}{G_B} + \frac{C}{4\pi}\right) + \frac{R}{16\pi}\left(\frac{1}{G_B} + \frac{B}{12\pi}\right) + \frac{R^2}{4\pi}\left(\alpha_B + \frac{A}{576\pi}\right)\right. \\
&\left. + \frac{1}{4\pi}R_{ab}R^{ab}\left(\beta_B - \frac{A}{1440\pi}\right) + \frac{1}{4\pi}R_{abcd}R^{abcd}\left(\gamma_B + \frac{A}{1440\pi}\right) + \cdots\right] \ ,
\end{aligned} \tag{3.69}
$$

where in this action, we discard the total derivative term $\Box R$ occurring in a_2. In particular from eq. (3.69), we obtain the renormalized Newton's constant

$$\frac{1}{G_{\mathrm{R}}} = \frac{1}{G_{\mathrm{B}}} + \frac{B}{12\pi} \; . \tag{3.70}$$

In eq. (3.69), divergent renormalizations also occur for the cosmological constant Λ_{B} and the quadratic-curvature coupling constants α_{B}, β_{B} and γ_{B}.

$$\frac{\Lambda_{\mathrm{R}}}{G_{\mathrm{R}}} = \frac{\Lambda_{\mathrm{B}}}{G_{\mathrm{B}}} + \frac{C}{4\pi} \quad , \quad \alpha_{\mathrm{R}} = \alpha_{\mathrm{B}} + \frac{A}{576\pi} \tag{3.71a}$$

$$\beta_{\mathrm{R}} = \beta_{\mathrm{B}} - \frac{A}{1440\pi} \quad , \quad \gamma_{\mathrm{R}} = \gamma_{\mathrm{B}} + \frac{A}{1440\pi} \; . \tag{3.71b}$$

For large values of μ, the constants A, B and C grow to leading order as $\ln(\mu/m)$, μ^2 and μ^4 respectively, but they also contain subleading and finite contributions. The higher order bare coupling constants (beyond those explicitly shown) would receive finite renormalizations from the finite terms in the one-loop action (3.64), but they will play no role in the present analysis.

3.4 Renormalization of the entropy

In this section, we calculate statistical black hole entropy and we identify the divergences with the divergences due to renormalization of Newton's constant. To calculate the black hole entropy, we follow the work of 't Hooft [12], but we replace his brick wall by a Pauli-Villars regularization. This method calculates the entropy by counting the number of states of a scalar field propagating just outside a fixed black hole horizon. We consider a Schwarzschild black hole given by the metric (1.6)

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \; .$$

In section 3.3, we saw that one has to introduce higher-order interactions in the Lagrangian to be able to absorb the infinities that arise from the scalar effective action. However, as seen in section 3.1.3, these higher-order interactions modify the black hole entropy. From eq. (3.34), the modified entropy for a Schwarzchild black hole is

$$S_B = \frac{A}{4G_B} - \gamma_B \oint d^2x\sqrt{h}R^{abcd}\epsilon_{ab}\epsilon_{cd} ,$$

where we use the fact that for Schwarzschild black holes, $R = R_{ab} = 0$. Also the integration is over a spatial cross section of the event horizon $r = r_s$. We can introduce the unit timelike vector n_a and the unit vector v_a such that the binormal ϵ_{ab} is given by $\epsilon_{ab} = n_a v_b - n_b v_a$. Using the symmetry of the Riemann tensor, we obtain

$$R^{abcd}\epsilon_{ab}\epsilon_{cd} = 4R^{abcd}n_a v_b n_c v_d = 4R^{0101} ,$$

where $(0,1)$ correspond to the Lorentz coordinates associated with (t,r) (see appendix A). From results of appendix A, eq. (A.9) yields $R^{0101} = -r_s/r^3$. Therefore, the bare entropy is

$$S_B = \frac{A}{4G_B} + 16\pi\gamma_B . \tag{3.72}$$

3.4.1 The density of states

We calculate the entropy in three steps. First, we obtain the density of states. Then, we define the free energy of a canonical ensemble. We then obtain the entropy by differentiation of the free energy.

In 't Hooft's method, one starts with the Klein-Gordon equation

$$(\Box - m^2)\phi(x) = 0 , \tag{3.73}$$

where

$$\Box\phi(x) = g^{ab}\nabla_a\nabla_b\phi(x) = \frac{1}{\sqrt{-g}}\partial_a\left[\sqrt{-g}g^{ab}\partial_b\phi(x)\right] .$$

Expanding the field in spherical coordinates $\phi = e^{iEt} f(r) Y_{\ell m}(\theta,\phi)$, the Klein-Gordon equation becomes

$$\left(1-\frac{r_s}{r}\right)^{-1} E^2 f(r) + \frac{1}{r^2}\partial_r\left[r^2\left(1-\frac{r_s}{r}\right)\partial_r f(r)\right] - \left[\frac{\ell(\ell+1)}{r^2}+m^2\right] f(r) = 0 \,. \tag{3.74}$$

This equation can be solved in the WKB approximation, which is similar to the geometric optics approximation used in chapter 2. In this approximation, one sets $f(r) = \rho(r) e^{iS(r)}$, where $\rho(r)$ is a slowly varying amplitude and $S(r)$ is a rapidly varying phase. To leading order, the real part of eq. (3.74) yields the radial wave number $k(r,\ell,E) \equiv \partial_r S$:

$$k(r,\ell,E) = \left(1-\frac{r_s}{r}\right)^{-1}\left[E^2 - \left(1-\frac{r_s}{r}\right)\left(\frac{\ell(\ell+1)}{r^2}+m^2\right)\right]^{1/2} , \tag{3.75}$$

and the imaginary part yields a differential equation for the amplitute $\rho(r)$

$$2\frac{\partial_r \rho}{\rho} + \frac{\partial_r k}{k} + \frac{1}{r}\left(2-\frac{r_s}{r}\right)\left(1-\frac{r_s}{r}\right) = 0 \,,$$

that can be solved to yield

$$\rho(r) = \frac{\mathcal{C}}{r}\left[E^2 - \left(1-\frac{r_s}{r}\right)\left(\frac{\ell(\ell+1)}{r^2}+m^2\right)\right]^{-1/4} ,$$

with $\mathcal{C}$ a constant of integration. The amplitude is finite at the horizon but because of the infinite blue shift, the phase diverges at the horizon. To avoid this, 't Hooft introduces a brick wall cut-off at a distance h to the horizon

$$\phi(x) = 0 \qquad \text{for } x \leq r_s + h \,, \tag{3.76}$$

with $h \ll r_s$. To also remove infrared divergences, one puts the black hole in a box such that at some large distance L, the field also vanishes

$$\phi(x) = 0 \qquad \text{for } x \geq L \,.$$

The radial modes number n is found by counting the number of nodes in the radial wave function

$$\pi n = \int_{r_s+h}^{L} dr\, k(r, \ell, E) \ .$$

To obtain the total number of modes with energy less than E, one sums over the angular degeneracy of the radial modes

$$g(E) \equiv \sum_{\ell} (2\ell + 1) \int_{r_s+h}^{L} \frac{dr}{\pi} k(r, \ell, E) \ .$$

If we replace the sum over ℓ by an integral (which is a good approximation because we are in the large quantum number regime) and we use eq. (3.75), the number of states becomes

$$\begin{aligned} g(E) = \frac{1}{\pi} \int d\ell (2\ell + 1) \int_{r_s+h}^{L} dr \left(1 - \frac{r_s}{r}\right)^{-1} & \\ \times \left[E^2 - \left(1 - \frac{r_s}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + m^2\right)\right]^{1/2} & \ , \qquad (3.77) \end{aligned}$$

where the integral over ℓ ranges for the values of ℓ for which the square root is real.

3.4.2 The free energy

We consider the free energy of a thermal ensemble of scalar particles with an inverse temperature β

$$\begin{aligned} \beta F &= \sum_{N} \ln\left(1 - e^{-\beta E_N}\right) \\ &= \int_0^{\infty} dE\, \frac{dg}{dE} \ln\left(1 - e^{-\beta E}\right) \ , \end{aligned}$$

where $\frac{dg}{dE}$ is the density of states with energy E. Integrating by parts and using eq. (3.77) to determine the density of states, we obtain

$$\begin{aligned}
F &= -\int_0^\infty \frac{dE\, g(E)}{e^{\beta E}-1} \\
&= -\frac{1}{\pi}\int_0^\infty \frac{dE}{e^{\beta E}-1}\int_{r_s+h}^{L} dr \left(1-\frac{r_s}{r}\right)^{1}\int d\ell\,(2\ell+1) \\
&\qquad\qquad \times \left[E^2-\left(1-\frac{r_s}{r}\right)\left(\frac{\ell(\ell+1)}{r^2}+m^2\right)\right]^{1/2} .
\end{aligned}$$

The integration over ℓ can be evaluated to yield

$$F = -\frac{2}{3\pi}\int_0^\infty \frac{dE}{e^{\beta E}-1}\int_{r_s+h}^{L} dr\, r^2 \left(1-\frac{r_s}{r}\right)^{-2}\left[E^2-\left(1-\frac{r_s}{r}\right)m^2\right]^{3/2} ,$$

where the remaining integrals are still taken for values where the square root is real. To examine the divergences, we introduce a new variable, $s = 1 - r_s/r$. The horizon corresponds to $s = 0$ and $r \to \infty$ corresponds to $s = 1$. In terms of s, the free energy is

$$F = -\frac{2r_s^3}{3\pi}\int_0^\infty \frac{dE}{e^{\beta E}-1}\int_{h'}^{L'} \frac{ds}{s^2(1-s)^4}\left[E^2-m^2 s\right]^{3/2} , \tag{3.78}$$

where $L' = 1 - r_s/L$ and $h' = h/(r_s+h) \simeq h/r_s$. The necessity of the brick wall cut-off is clear at this point since the integrand diverges with a double pole at the event horizon. Thus, for small values of s, we have $\int_{h'} ds/s^2 \simeq -1/h'$, which diverges as the brick wall is pulled back to the horizon, *i.e.*, as $h' \to 0$.

The integrand also diverges as $r \to \infty$ *i.e.*, $s \to 1$. Taking $s \approx L'$, the main contribution in the infrared regime is

$$F_{\mathrm{IR}} \approx -\frac{2L^3}{9\pi}\int_m^\infty \frac{dE}{e^{\beta E}-1}\left[E^2-m^2\right]^{3/2} .$$

This result is just the usual free energy of scalar particles propagating in flat spacetime and confined to a box of volume L^3. The subleading divergences are modifications due to the curvature of spacetime.

In order to separate the two types of divergences, we write the first factor in the integrand as

$$\frac{1}{s^2(1-s)^4} = \frac{1+4s}{s^2} + \frac{10-20s+15s^2-4s^3}{(1-s)^4} \, .$$

Because we are only interested with divergences that come from the horizon, we drop the volume-dependent contribution which has no poles at the horizon, and we take the limit of the infinite volume $L \to \infty$. The part of the free energy that diverges at the horizon is

$$F_{\mathrm{UV}} = -\frac{2r_s^3}{3\pi} \int_0^\infty \frac{dE}{e^{\beta E}-1} \int_{h'}^1 ds \, \frac{1+4s}{s^2} \Big[E^2 - m^2 s\Big]^{3/2} \, .$$

One can then integrate over s and E to find the brick wall regulated free energy, which then yields the 't Hooft entropy (3.35).

Now, rather than considering a single scalar field, we repeat 't Hooft's calculation for the Pauli-Villars regulated field theory introduced in eq. (3.66). Each of the fields makes a contribution to the free energy as in eq. (3.78), and the total free energy becomes

$$\bar{F} = -\frac{2{r_s}^3}{3\pi} \sum_{i=0}^{5} \Delta_i \int_0^\infty \frac{dE}{e^{\beta E}-1} \int_{h'}^1 ds \, \frac{1+4s}{s^2} \Big[E^2 - s m_i^2\Big]^{3/2} \, , \qquad (3.79)$$

where $\Delta_0 = \Delta_3 = \Delta_4 = +1$ for the commuting fields, and $\Delta_1 = \Delta_2 = \Delta_5 = -1$ for the anticommuting fields. The free energy of the anticommuting regulator fields comes with a minus sign with respect to the commuting fields, as is required since the role of these fields is to cancel contributions of very high energy modes in the regulated theory. The integral is taken for the values of E and s for which the square root is real. The exact domain of integration is presented in fig. 3.1.

Consider the free energy F_i that comes from the field ϕ_i with mass m_i. The

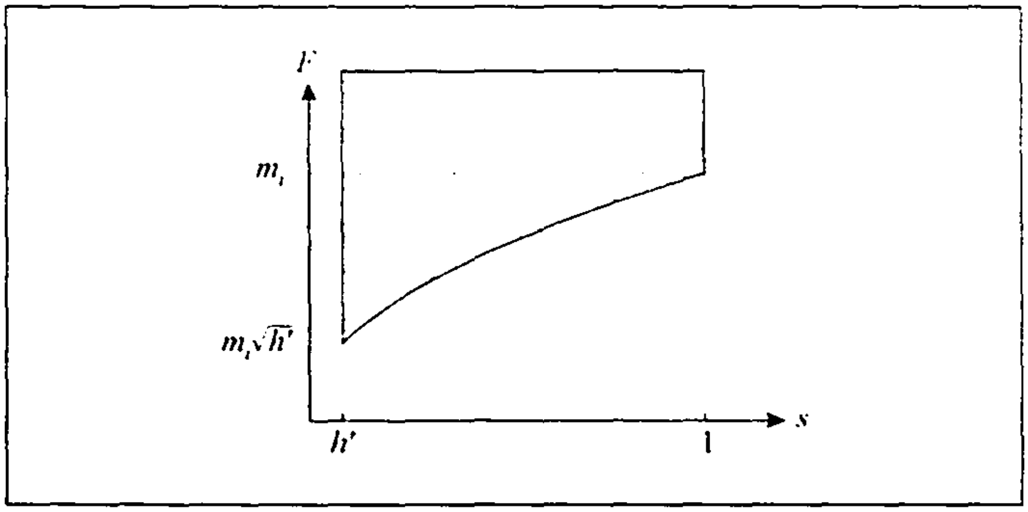

Figure 3.1: Integration domain for the free energy.

integration over s can be evaluated to yield

$$\begin{aligned} F_i = &-\frac{2r_s^3\Delta_i}{3\pi}\int_{m_i\sqrt{h'}}^{\infty}\frac{dE}{e^{\beta E-1}}\left\{\frac{(E^2-m_i^2h')^{5/2}}{E^2h'}-\left(4-\frac{3m_i^2}{2E^2}\right)\right. \\ &\times\left.\left[\frac{2}{3}(E^2-m_i^2h')^{3/2}+2E^2(E^2-m_i^2h')^{1/2}+E^3\ln\frac{E-\sqrt{E^2-m_i^2h'}}{E+\sqrt{E^2-m_i^2h'}}\right]\right\} \\ &-\frac{2r_s^3\Delta_i}{3\pi}\int_{m_i}^{\infty}\frac{dE}{e^{\beta E-1}}\left\{\frac{(E^2-m_i^2)^{5/2}}{E^2}-\left(4-\frac{3m_i^2}{2E^2}\right)\right. \\ &\times\left.\left[\frac{2}{3}(E^2-m_i^2)^{3/2}+2E^2(E^2-m_i^2)^{1/2}+E^3\ln\frac{E-\sqrt{E^2-m_i^2}}{E+\sqrt{E^2-m_i^2}}\right]\right\}. \end{aligned} \tag{3.80}$$

The second integral in eq. (3.80) is independent of h' and therefore, it does not diverge when one takes the limit $h' \to 0$. It also becomes vanishingly small for the Pauli-Villars fields in the limit $\mu \to 0$. Hence, we drop it. The first integral has linear and logarithmic divergences, as $h' \to 0$. To isolate the form of the divergences, one may expand the integrand in a Laurent series around $h' = 0$.

$$F_i = -\frac{2r_s^3\Delta_i}{3\pi}\int_{m_i\sqrt{h'}}^{\infty}\frac{dE}{e^{\beta E}-1}\left[\frac{E^3}{h'}-\left(4-\frac{3m_i^2}{2E^2}\right)E^3\ln\frac{m_i^2h'}{E^2}+\mathcal{O}(h'^0)\right]. \tag{3.81}$$

Neglecting $O(h'^0)$ contributions, one is free to integrate from 0 instead of integrating from $m\sqrt{h'}$, without changing the divergent parts. One can then sum over the scalar fields. The linear divergence cancels out because $\sum_{i=0}^{5}\Delta_i = 0$. The logarithmic divergence also cancels out because $\sum_{i=0}^{5}\Delta_i m_i^2 = 0$. Therefore, we are free to remove 't Hooft's brick wall by taking the limit $h' \to 0$. The sum over i yields the same renormalization constants introduced in eqs. (3.68):

$$\sum_{i=0}^{5}\Delta_i m_i^2 \ln\frac{m_i^2}{E^2} = B$$
$$\sum_{i=0}^{5}\Delta_i \ln\frac{m_i^2}{E^2} = -A\ .$$

Note that the energy E drops out of these sums. Hence,

$$\begin{aligned}\bar{F} &= -\frac{2r_s^3}{3\pi}\int_0^\infty \frac{dE}{e^{\beta E}-1}\left[\frac{3}{2}BE + 4AE^3\right]\\ &= -r_s^3\left[\frac{\pi}{6\beta^2}B + \frac{8\pi^3}{45\beta^4}A\right]\ . \end{aligned} \tag{3.82}$$

We emphasize that eq. (3.82) neglects contributions to the integral which do not diverge as $\mu \to \infty$.

3.4.3 Statistical black hole entropy

Given the free energy of the black hole system, the entropy may be calculated using the standard formula

$$S = \beta^2\frac{\partial\bar{F}}{\partial\beta} = r_s^3\left[\frac{\pi}{3\beta}B + \frac{32\pi^3}{45\beta^3}A\right]\ . \tag{3.83}$$

Choosing the inverse temperature β to correspond to the Hawking temperature of a Schwarzchild black hole, we set

$$\beta = 4\pi r_s\ ,$$

upon which the entropy (3.83) becomes

$$S = \frac{A}{4}\frac{B}{12\pi} + \frac{A}{90} \tag{3.84}$$

where $A = 4\pi r_s^2$ is the surface area of the event horizon. Thus we see that the entropy contains the constants A and B, which give precisely the dependence on the regulator mass μ appearing in the renormalization of Newton's constant and of the quadratic-curvature coupling constants. In fact we see that eq. (3.84) can be interpreted as the one-loop correction to the bare entropy (3.72). The total entropy is then the sum

$$\begin{aligned} S_{\text{total}} &= S_B + S \\ &= \frac{A}{4}\left(\frac{1}{G_B} + \frac{B}{12\pi}\right) + 16\pi\left(\gamma_B + \frac{A}{1440\pi}\right) \\ &= \frac{A}{4G_R} + 16\pi\gamma_R \ , \end{aligned} \tag{3.85}$$

where we have used eq. (3.70) for the renormalized Newton's constant and eq. (3.71b) for the coupling constant γ. Thus both terms in the scalar field entropy (3.84) account for precisely the scalar one-loop renormalization of the full black hole entropy.

A priori, one might not have expected the Pauli-Villars scheme to regulate 't Hooft's entropy calculation at all. In fact, though, not only do we find that the Pauli-Villars scheme regulates the latter calculation, our results are in complete agreement with the suggestion of Susskind and Uglum. The divergences appearing in 't Hooft's statistical-mechanical calculation of black hole entropy are precisely the quantum field theory divergences associated with the renormalization of the coupling constants appearing in the expressions of the entropy. This identification includes both the divergent and finite contributions in the renormalization of the couplings, G_B and γ_B. This precise equality, including the finite terms, occurs because the combinations of masses

$\sum \Delta_i m_i^2 \ln m_i^2$ and $\sum \Delta_i \ln m_i^2$ arise naturally in both calculations. We have not considered here any of the remaining finite contributions arising in the free energy (3.79). It should be possible to identify the corresponding contributions to the black hole entropy with finite renormalizations of the higher curvature terms arising from finite terms in the one-loop action (3.64).

3.5 Discussion

In this last section, we discuss our results and look for possible extensions of the above calculations.

3.5.1 Definition of the density of states

We have defined the density of states of scalar fields by imposing a brick wall near the horizon. We have then removed the brick wall or more specifically, we have pulled back the brick wall to the horizon. The free energy has a smooth limit in this process. It is implicit that there is still a brick wall, but at the horizon. We still have to impose $\phi(r = r_s) = 0$ to produce a well-defined density of states. We assume that the results from this limiting procedure coincide with those arising within the canonical quantization of the Pauli-Villars regulated theory.

3.5.2 Robustness

One would like to know whether the present results hold for arbitrary field theories coupled to gravity, rather than for just a minimally coupled scalar field. One simple extension of our calculations would be to consider a non-minimally coupled scalar field. The original matter action in eq. (3.40) is then extended to

$$I'_m = -\frac{1}{2}\int d^4x\,\sqrt{-g}\left[g^{ab}\nabla_a\phi\nabla_b\phi + m^2\phi^2 + \xi R\phi^2\right] \ .$$

It is well known [49] that the additional coupling of the scalar field to the curvature modifies the adiabatic expansion coefficients in eqs. (3.65), and therefore it affects the renormalizations of the bare coupling constants. For example, eq. (3.70) for the renormalized Newton's constant is replaced by

$$\frac{1}{G_R} = \frac{1}{G_B} + \frac{B}{2\pi}\left(\frac{1}{6} - \xi\right) \ . \tag{3.86}$$

On the other hand, if we repeat the calculation of section 3.4 for the new scalar field theory, we find that the resulting entropy is completely unchanged. The new coupling constant ξ enters the new equation of motion, $(\Box - m^2 - \xi R)\phi = 0$, which replaces eq. (3.73). The remainder of the calculation is unmodified, though, because $R = 0$ for the background Schwarzschild metric. Given that Newton's constant is renormalized as in eq. (3.86), the entropy in eq. (3.84), which is independent of ξ, does not properly account for the renormalization of the Bekenstein-Hawking formula.

To cure this, one probably has to take into account the degrees of freedom at the horizon using the methods introduced by Fursaev and Solodukhin [80, 81, 82]. We saw in eq. (3.8) that the Schwarzschild Euclidean action is similar to polar coordinates when one identifies the imaginary time τ with a period $2\pi/\kappa = \beta_H$. Without this identification, there is a conical singularity at the horizon and the curvature scalar is not zero but it is given by

$$R = 4\pi\left(1 - \frac{\beta}{\beta_H}\right)\delta_\Sigma \ ,$$

where δ_Σ is a δ function normalized as

$$\int_M f\delta_\Sigma = \int_\Sigma f$$

and integration over Σ is an integration over a spacelike cross section of the horizon. To calculate the entropy, one needs to take derivatives with respect to β and therefore one has to consider metrics with inverse temperature slightly different than β_H. Because the curvature is zero except at the horizon for conical metrics, it is plausible that the needed renormalization comes from behavior of the horizon.

In our calculations, the conical singularity does not come into play because the brick wall fixes the scalar field away from the horizon. To see the conical singularity, we may replace the conical metric with a regularized metric and consider 't Hooft's calculation in this background. This problem may also have some relation with the definition of the density of states, *i.e.*, that we impose $\phi(r = r_*) = 0$. In any event, these questions require further study.

One may also want to verify the calculations with fields of higher spins. For this purpose, we need to solve the problem associated with the non-minimal coupling of the fields with the curvature because there is always such a coupling for fields with higher spins. For example, the Dirac equation in curved spacetime is

$$\left(i\gamma^i\nabla_i + m\right)\psi = 0 , \tag{3.87}$$

where γ^i are the curved Dirac matrices, obeying $\{\gamma^i, \gamma^j\} = 2g^{ij}$ and ∇_i is the covariant derivative for the spinor ψ. Multiplying eq. (3.87) by $(i\gamma^j\nabla_j + m)$, one obtains

$$\left(-\Box + m^2 + \frac{1}{4}R\right)\psi = 0 .$$

This is Klein-Gordon equation with $\xi = 1/4$. In the same way, Maxwell equation in curved spacetime can be written

$$\left(\Box\delta^i_j - R^i{}_j\right)A^j = 0 ,$$

where A^μ is the spin-one field.

3.5.3 Reissner-Nordström background

Another place where one may want to generalize the present calculation is to do it in a different background. In this sense, we can consider our calculation in a Reissner-Nordström (RN) background. The background includes a metric and a U(1) gauge potential. Therefore, the gravity action should be supplemented with a Maxwell term and in general, additional higher-derivative terms, like in chapter 2:

$$I_{\mathrm{U}(1)} = \int d^4x \sqrt{-g}\left[-\frac{1}{4}F_{ab}F^{ab} + \delta_{\mathrm{B}}\left(F_{ab}F^{ab}\right)^2 + \lambda_{\mathrm{B}} R_{ab}F^{ac}F_c{}^b + \cdots\right] .$$

Despite introducing a background gauge field, we consider only a neutral scalar field as above, and therefore, in the effective action, the gauge field interactions are completely unaffected by the scalar one-loop contributions. (An obvious extension of the present analysis would be to repeat the calculations for a complex scalar field which couples to the gauge potential.)

The RN metric (1.9) can be written

$$ds^2 = -\left(1-\frac{r_-}{r}\right)\left(1-\frac{r_+}{r}\right)dt^2 + \left[\left(1-\frac{r_-}{r}\right)\left(1-\frac{r_+}{r}\right)\right]^{-1}dr^2 + r^2 d\Omega^2 .$$

This black hole has an event horizon at $r_+ = GM + \sqrt{G^2M^2 - GQ^2/(4\pi)}$ and an inner horizon at $r_- = GM - \sqrt{G^2M^2 - GQ^2/(4\pi)}$, where Q is the black hole charge. In this background, the Ricci tensor is non-zero and the bare entropy (3.34) is given by

$$S = \frac{A}{4G_{\mathrm{B}}} + \int d^2x \sqrt{h}\left[\beta_{\mathrm{B}} R^{ab} g_{\perp ab} - \gamma_{\mathrm{B}} R^{abcd}\epsilon_{ab}\epsilon_{cd}\right] .$$

As before, introducing the unit timelike vector n_a and the unit spacelike vector r_a, the binormal is $\epsilon_{ab} = n_a r_b - n_b r_a$ and the metric $g_{\perp ab} = -n_a n_b + r_a r_b$. The entropy is given by

$$S = \frac{A}{4G_B} - 8\pi u\beta_B + 16\pi(1-2u)\gamma_B \ . \qquad (3.88)$$

where we use eqs. (A.13) and (A.16) and $u = r_-/r_+ = GQ^2/(4\pi r_+^2)$.

Now, we repeat 't Hooft's calculation as described in section 3.4. We consider a scalar field, which satisfies the Klein-Gordon equation (3.73) and we introduce a brick wall near the event horizon by setting $\phi(x) = 0$ for $r \leq r_+ + h$ to define the density of states. Then, the radial Klein-Gordon equation is solved within the WKB approximation and we obtain the number of modes with energy not exceeding E:

$$g(E) = \frac{1}{\pi}\int_{r_++h}^{L} dr\left[\left(1-\frac{r_-}{r}\right)\left(1-\frac{r_+}{r}\right)\right]^{-1}\int d\ell\,(2\ell+1)$$
$$\times\left[E^2-\left(1-\frac{r_-}{r}\right)\left(1-\frac{r_+}{r}\right)\left(\frac{\ell(\ell+1)}{r^2}+m^2\right)\right]^{1/2} .$$

One can introduce the free energy of a thermal ensemble of scalar particles at inverse temperature β

$$F = \int dE\,\frac{dg}{dE}\ln\left(1-e^{\beta E}\right) \ ,$$

and introduce the same Pauli-Villars regularization as previously. Then, we are free to remove 't Hooft's brick wall and the free energy becomes

$$\bar{F} = -\frac{2r_+^3}{3\pi}\int_0^\infty \frac{dE}{e^{\beta E}-1}\int_0^{L'}\frac{ds}{s^2(1-s)^4(1-u+us)^2}$$
$$\sum_{i=0}^{5}\Delta_i\left[E^2-s(1-u+us)m_i^2\right]^{3/2} \ . \qquad (3.89)$$

Now, integrating over s and E, we focus only on the divergent contributions at the horizon and find

$$\bar{F} \simeq -r_+^3\left[\frac{\pi}{6(1-u)\beta^2}B + \frac{4\pi^3(2-3u)}{45(1-u)^3\beta^4}A\right]$$

where A and B are the same constants given in eqs. (3.68). The entropy is then given by

$$S = \beta^2 \frac{\partial F}{\partial \beta} = r_+^3 \left[\frac{\pi}{3(1-u)\beta} B + \frac{16(2-3u)\pi^3}{45(1-u)^3\beta^3} A \right] \quad . \tag{3.90}$$

Choosing the inverse temperature β to correspond to the Hawking temperature of a non-extremal RN black hole, we set

$$\beta = \frac{4\pi r_+}{1-u} \; ,$$

upon which the entropy (3.90) becomes

$$S = \frac{\mathcal{A}}{4} \frac{B}{12\pi} + \frac{(2-3u)A}{180} \tag{3.91}$$

where $\mathcal{A} = 4\pi r_+^2$ is the surface area of the event horizon. Combining eq. (3.91) and eq. (3.88), we obtain the total entropy

$$\begin{aligned} S_{\text{total}} &= \frac{\mathcal{A}}{4} \left(\frac{1}{G_{\text{B}}} + \frac{B}{12\pi} \right) - 8\pi u \left(\beta_{\text{B}} - \frac{A}{1440\pi} \right) + 16\pi(1-2u) \left(\gamma_{\text{B}} + \frac{A}{1440\pi} \right) \\ &= \frac{\mathcal{A}}{4G_{\text{R}}} + 8\pi u \beta_{\text{R}} + 16\pi(1-2u)\gamma_{\text{R}} \; , \end{aligned}$$

where eqs. (3.71b) have been used. In this case, both terms in the scalar entropy account for precisely the renormalization of the black hole entropy, including the contribution of the Ricci tensor squared in the bare entropy (3.88). In the RN background, the divergences appearing in 't Hooft's statistical-mechanical calculation of black hole entropy are precisely the quantum field theory divergences associated with the renormalization of the coupling constants appearing in the expressions of the entropy, including both the divergent and finite contributions in the renormalization of the couplings, G_{B}, β_{B} and γ_{B}. Thus the RN background allows for a more sensitive comparison between the renormalization of the effective action and 't Hooft's entropy calculation than the Schwarzschild background.

3.5.4 Extremal Reissner-Nordström

It is not difficult to repeat our calculations for the case of an extremal RN black hole with $r_+ = r_-$. In this case, 't Hooft's brick wall cut-off leads to ill-defined results [83]. The problem is that the coordinate cut-off, h, cannot be converted to a proper length cut-off because any point which is a fixed coordinate distance outside of the extremal horizon is in fact an infinite proper distance from the horizon (on a constant time hypersurface). No such problem arises with the covariant Pauli-Villars regulator. However, precisely at the extremal limit $u = 1$, the structure of the small s divergences in eq. (3.89) changes, and hence we must re-evaluate the integral. We find that the divergent part of the free energy is given by

$$\bar{F}_{\rm ext} \simeq -r_+^3 \left[\frac{\pi}{3\beta^2}B + \frac{4\pi^3}{9\beta^4}A\right] ,$$

and the entropy which follows is

$$S_{\rm ext} = r_+^3 \left[\frac{2\pi}{3\beta}B + \frac{16\pi^3}{9\beta^3}A\right] . \tag{3.92}$$

Here A and B are the same divergent coefficients (3.68) that appear in the scalar one-loop action and in the non-extremal entropy. Hence, with a covariant regulator, we find that the extremal entropy has no stronger divergences than appear in the non-extremal case. In fact, the entire result has essentially the same form as the non-extremal entropy in eq. (3.90).

To proceed further, one must fix the inverse temperature in eq. (3.92). Using the standard formula $T = \kappa/(2\pi)$ [8], one finds that the temperature is zero since the surface gravity vanishes for the extremal RN black hole. Thus, the inverse temperature β diverges, and we find that S_{ext} vanishes. The same result is found when using the brick wall regulator [83, 84]. This result is in accord with the recent discovery [85, 86] that extremal black holes should have vanishing entropy, since one then expects that the renormalization contribution must also vanish; since the value of entropy is independent of the coupling constants, the renormalized value of zero is still zero.

On the other hand, the recent investigations of extremal black holes [85, 86] also suggest that an extremal black hole can be in equilibrium with a heat bath of an arbitrary temperature. Hence, one might consider leaving the inverse temperature arbitrary in eq. (3.92). In this case, one has the curious result that S_{ext} appears to represent the renormalization of some finite entropy expression for an extremal RN black hole. For example, the first term in eq. (3.92) would represent the renormalization of $S = \frac{A}{4G}\frac{8\pi r_+}{\beta}$. Previous calculations have given no indication that such an entropy arises for extremal black holes, and so one may conclude that one must use $\beta \to \infty$ in this case. Alternatively, it may be that 't Hooft's model does not capture the full physics of extremal black holes as for non-minimal couplings, and that the correct result should still be $S_{ext} = 0$ even with a nonvanishing temperature.

3.5.5 On-shell versus off-shell

Most of the discussions and derivations of black hole entropy focus on black hole backgrounds which are solutions to the equations of motion. For example, the method of Noether charge presented in section 3.1.3 calculates the entropy using the equations of motion. In our method, we calculate the first quantum correction to this entropy and therefore, it is an off-shell calculation. We do not refer to any equations of motion for the backgrounds, even if the backgrounds used may be solutions of the bare equations of motion. However, we use the usual entropy expression (3.34) to assign a black hole entropy to the backgrounds, within both bare and renormalized theories. Hence, we suppose that this formula is valid off-shell. This is suggested in ref. [82], where they demonstrate eq. (3.34) without any reference to equations of motion.

Chapter 4

Conclusion

In this thesis, we have considered quantum gravity from a semi-classical point of view, where gravity is analysed classically and the matter fields quantum mechanically. With the small size of the Planck length with respect to other length scales of nature, the semi-classical treatment of gravity should yield sensible results, even if it is not the complete final theory of quantum gravity. In this scheme, interesting results have emerged and especially one finds that the equivalence principle is violated and also that black holes can be analysed in terms of thermodynamical quantities.

The Einstein equivalence principle states that all Lorentz frames are equivalent and there is no coupling between the matter fields and the Riemann tensor. It implies that photons fall freely along null geodesics, independently of their frequency. We illustrate in section 2.3 with a review of the results of ref. [14] that this principle is violated when one considers interacting quantum field theory in curved spacetime. In particular, birefringence appears. In section 2.4, we consider effective action for the electromagnetic field in curved spacetime with higher-order interactions to obtain new effects. In this way, we are able to build three classes of eight-derivative interactions that produce energy-dependent light deflection. One class also produces birefringent propagation. The first important conclusion of this thesis is that it is possible to produce energy-dependent deflection of light in the context of quantum field theory in curved spacetime. Hence, the claim of ref. [16] saying that such behavior would be a clear signature of string theory is false.

Having given the possible dispersive interactions, the next task would be to calculate the one-loop effective action for QED in curved spacetime up to eight derivatives to see if these interactions are generated and if so, what their actual coefficient is.

The magnitude of the dispersive deflection found is unmeasurably small for the solar parameters which of course is no surprise. With the weakness of the gravitational field, one cannot hope to observe quantum gravity effects in the solar system. Like all of the quantum gravity predictions, one would need small black holes to observe the effect of dispersive propagation. In that regime, our calculation is not directly applicable due to the approximations we have used. However in principle, we should be able to do the calculation for the strong-field regime as well.

The second investigation in this thesis was concerned with the statistical interpretation of black hole entropy. The thermodynamic interpretation of black hole entropy is well established. It comes into play from the first law and the second law of black hole dynamics and the entropy is proportional to the black hole area. However, the statistical interpretation remains unclear. One problem that arises in this context is the appearance of divergences in the entropy. The understanding of these divergences is essential to make sense of the statistical black hole entropy calculation. The suggestion by ref. [67] that statistical black hole entropy should be viewed as the one-loop modification to the Bekenstein-Hawking entropy $S_{\rm BH} = A/(4G)$ is interesting. In this way, the divergences may come from the renormalization of Newton's constant that appears in the Bekenstein-Hawking formula.

With this in mind, the renormalization of the gravitational effective action is presented in section 3.3. To regularize the calculation, we introduce a Pauli-Villars regularization. This regularization is manifestly covariant. To absorb all the infinities that arise at one-loop, one needs to introduce a bare action with a cosmological constant and with four-derivative interactions. As noted by refs. [57, 58, 59], these higher-order interactions modify the black hole entropy by a constant.

The statistical black hole entropy is calculated in section 3.4. We use the method introduced by ref. [12] but with an important modification. We replace the brick wall regulator by the same Pauli-Villars regularization introduced in section 3.4. In this way, the regulator is manifestly covariant and we can compare directly the divergences appearing in the entropy with the divergences of the effective action.

The second important conclusion of this thesis is that the divergences appearing in the statistical entropy are the divergences needed to renormalize the Newton's constant and the coupling constants of the higher-order interactions. We have done the calculation for the Schwarzschild geometry in section 3.4 and the calculation was generalized to Reissner-Nordström geometry in section 3.5.

In the near future, I would like to generalize the calculation to non-minimally coupled scalar fields and for fields of higher spin. For this purpose, it is important to consider the degrees of freedom at the horizon. The methods of the conical singularity introduced by refs. [80, 81, 82] might be useful. One strategy would be to smear the horizon curvature over an extended region so its effects could be felt at the brick wall, even when the latter is still away from the horizon and then pull back the brick wall to the horizon after a Pauli-Villars regularization. It might also be useful to consider the effects on the density of states arising from the horizon boundary condition which we impose on the field.

Appendix A

Calculation of the Riemann tensor

In this appendix, we calculate the components of the Riemann tensor for the Schwarzschild geometry and for the Reissner-Nordström geometry. We use the method of the orthonormal frame (see, *e.g.*, refs. [17] or [47]).

A.1 Spherical symmetric manifold

Consider a spherically symmetric manifold described by the metric

$$ds^2 = -U^2(r)\,dt^2 + \frac{1}{U^2(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\ . \tag{A.1}$$

This metric is not the most general metric for spherical symmetric manifolds but it is general enough to describe both Schwarzschild and Reissner-Nordström geometries. We introduce the vierbein

$$e^{\mu}{}_{a} = \mathrm{diag}(U, 1/U, r, r\sin\theta)$$

satisfying $\eta_{\mu\nu}e^{\mu}{}_{a}e^{\nu}{}_{b} = g_{ab}$. The greek letters describe the orthonormal components $(0, 1, 2, 3)$ and the roman letters describe the spacetime components (t, r, θ, ϕ). We also introduce the dual basis $\hat{\boldsymbol{\theta}}^{\mu} = e^{\mu}{}_{a}\, dx^{a}$:

$$\hat{\boldsymbol{\theta}}^{0} = U(r)\, dt \tag{A.2a}$$

$$\hat{\boldsymbol{\theta}}^{1} = \frac{1}{U(r)}\, dr \tag{A.2b}$$

$$\hat{\boldsymbol{\theta}}^{2} = r\, d\theta \tag{A.2c}$$

$$\hat{\boldsymbol{\theta}}^{3} = r \sin\theta\, d\phi\ . \tag{A.2d}$$

With this basis, the metric reads

$$ds^2 = \eta_{\mu\nu} \hat{\boldsymbol{\theta}}^{\mu} \otimes \hat{\boldsymbol{\theta}}^{\nu}$$

The computation of the Riemann tensor is done in two steps. First, one calculates the connection one-form $\omega^{\mu}{}_{\nu}$ using the Cartan torsion-free structure equation

$$\mathrm{d}\hat{\boldsymbol{\theta}}^{\mu} + \omega^{\mu}{}_{\nu} \wedge \hat{\boldsymbol{\theta}}^{\nu} = 0\ . \tag{A.3}$$

The connection one-form is related to the connection in the orthonormal frame $\Gamma^{\mu}_{\lambda\nu}$

$$\omega^{\mu}{}_{\nu} = \Gamma^{\mu}_{\lambda\nu} \hat{\boldsymbol{\theta}}^{\lambda}\ . \tag{A.4}$$

The condition that the covariant derivative is compatible with the metric (*i.e.*, $\nabla_a g_{bc} = 0$) implies that the connection one-form is antisymmetric $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

For the metric (A.1), one obtains

$$\mathrm{d}\hat{\boldsymbol{\theta}}^{0} = U'(r)\, dr \wedge dt$$

$$\mathrm{d}\hat{\boldsymbol{\theta}}^{1} = 0$$

$$\mathrm{d}\hat{\boldsymbol{\theta}}^{2} = dr \wedge d\theta$$

$$\mathrm{d}\hat{\boldsymbol{\theta}}^{3} = \sin\theta\, dr \wedge d\phi + r\cos\theta\, d\theta \wedge d\phi\ .$$

By comparing this result with eq. (A.3) and using the definition (A.2) and the antisymmetry of the connection one-form, we obtain

$$\omega^0{}_1 = \omega^1{}_0 \quad = U(r)U'(r)\, dt \tag{A.5a}$$

$$\omega^2{}_1 = -\omega^1{}_2 = U(r)\, d\theta \tag{A.5b}$$

$$\omega^3{}_1 = -\omega^1{}_3 = U(r) \sin\theta\, d\phi \tag{A.5c}$$

$$\omega^3{}_2 = -\omega^2{}_3 = \cos\theta\, d\phi\ . \tag{A.5d}$$

For the second step of the calculation, one calculates the curvature two-form $\boldsymbol{R}^{\mu}{}_{\nu}$ using the second Cartan structure equation

$$\boldsymbol{R}^{\mu}{}_{\nu} = \mathrm{d}\boldsymbol{\omega}^{\mu}{}_{\nu} + \boldsymbol{\omega}^{\mu}{}_{\rho} \wedge \boldsymbol{\omega}^{\rho}{}_{\nu}\ . \tag{A.6}$$

The Riemann tensor in local coordinates is readily obtained by using the identification

$$\boldsymbol{R}^{\mu}{}_{\nu} = R^{\mu}{}_{\nu[\rho\sigma]} \hat{\boldsymbol{\theta}}^{\rho} \wedge \hat{\boldsymbol{\theta}}^{\sigma}\ . \tag{A.7}$$

From eqs. (A.5), we calculate the curvature two-form

$$\begin{aligned}
\boldsymbol{R}^0{}_1 &= -(U'^2 + UU'')\hat{\boldsymbol{\theta}}^0 \wedge \hat{\boldsymbol{\theta}}^1 \\
\boldsymbol{R}^0{}_2 &= -\frac{UU'}{r}\hat{\boldsymbol{\theta}}^0 \wedge \hat{\boldsymbol{\theta}}^2 \\
\boldsymbol{R}^0{}_3 &= -\frac{UU'}{r}\hat{\boldsymbol{\theta}}^0 \wedge \hat{\boldsymbol{\theta}}^3 \\
\boldsymbol{R}^1{}_2 &= -\frac{UU'}{r}\hat{\boldsymbol{\theta}}^1 \wedge \hat{\boldsymbol{\theta}}^2 \\
\boldsymbol{R}^1{}_3 &= -\frac{UU'}{r}\hat{\boldsymbol{\theta}}^1 \wedge \hat{\boldsymbol{\theta}}^3 \\
\boldsymbol{R}^2{}_3 &= \frac{1}{r^2}(1 - U^2)\hat{\boldsymbol{\theta}}^2 \wedge \hat{\boldsymbol{\theta}}^3\ .
\end{aligned}$$

Using eq. (A.7), one obtains

$$\begin{gathered}
R^0{}_{101} = -(U'^2 + UU'') \\
R^0{}_{202} = R^0{}_{303} = R^1{}_{212} = R^1{}_{313} = -\frac{UU'}{r} \\
R^2{}_{323} = \frac{1}{r^2}(1 - U^2)\ .
\end{gathered}$$

The other non-zero components are found using the symmetry of the Riemann tensor

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} \, . \tag{A.8}$$

A.2 Schwarzchild geometry

For the Schwarzschild metric, one has $U(r) = \sqrt{1 - \frac{2GM}{r}}$ and the Riemann tensor reads in local coordinates

$$R^{0101} = -R^{2323} = -\frac{2GM}{r^3} \tag{A.9a}$$

$$R^{0202} = R^{0303} = -R^{1212} = -R^{1313} = \frac{GM}{r^3} \, . \tag{A.9b}$$

Using eqs. (A.9), one may calculate the scalar

$$R^{abcd} R_{abcd} = \frac{48 G^2 M^2}{r^6} \, . \tag{A.10}$$

The Riemann tensor can be expressed in a useful way by introducing the bivector $U^{\mu\nu}_{\zeta\eta} = \delta^\mu_\zeta \delta^\nu_\eta - \delta^\mu_\eta \delta^\nu_\zeta$

$$R^{\mu\nu\rho\sigma} = -\frac{GM}{r^3}\left[\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right] - \frac{3GM}{r^3} U^{\mu\nu}_{01} U^{\rho\sigma}_{01} + \frac{3GM}{r^3} U^{\mu\nu}_{23} U^{\rho\sigma}_{23} \, .$$

This relation can be transformed in spacetime components with the help of the vierbein

$$\begin{aligned} R^{abcd} &= e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d R^{\mu\nu\rho\sigma} \\ &= -\frac{GM}{r^3}\left[g^{ac}g^{bd} - g^{ad}g^{bc}\right] - \frac{3GM}{r^3} U^{ab}_{01} U^{cd}_{01} + \frac{3GM}{r^3} U^{ab}_{23} U^{cd}_{23} \, , \end{aligned} \tag{A.11}$$

with

$$U^{ab}_{\mu\nu} = e_\mu{}^a e_\nu{}^b - e_\mu{}^b e_\nu{}^a \, . \tag{A.12}$$

A.3 Reissner-Nordström geometry

The non-zero components of the Riemann tensor for the Reissner-Nordström metric with

$$U(r) = \sqrt{1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2}}$$

is given by, in local components

$$R^{0101} = -\frac{2GM}{r^3} + \frac{3GQ^2}{4\pi r^4} \tag{A.13a}$$

$$R^{0202} = R^{0303} = -R^{1212} = -R^{1313} = \frac{GM}{r^3} - \frac{GQ^2}{4\pi r^4} \tag{A.13b}$$

$$R^{2323} = \frac{2GM}{r} - \frac{GQ^2}{4\pi r^4}\,. \tag{A.13c}$$

The other non-zero components are found using the symmetry of the Riemann tensor (A.8). The Riemann tensor can be expressed as

$$R^{abcd} = -(A-B)\left[g^{ac}g^{bd} - g^{ad}g^{bc}\right] - (3A-4B)U^{ab}_{01}U^{cd}_{01} + (3A-2B)U^{ab}_{23}U^{cd}_{23} \tag{A.14}$$

with

$$A = \frac{GM}{r^3} \quad , \quad B = \frac{GQ^2}{4\pi r^4}\,. \tag{A.15}$$

The Ricci tensor can also be calculated. The non-zero components are

$$R_{00} = -R_{11} = R_{22} = R_{33} = \frac{GQ^2}{4\pi r^4}\,. \tag{A.16}$$

From eqs. (A.13), we can calculate the scalar

$$R^{ab}R_{ab} = \frac{G^2Q^4}{4\pi^2 r^8}\,. \tag{A.17}$$

A.4 Derivatives of the Riemann tensor

In section 2.4.3, we need the second covariant derivative of the Riemann tensor for the Schwarzschild geometry. In local frame, the first covariant derivative is given by

$$\nabla_\lambda R_{\mu\nu\rho\sigma} = e_\lambda{}^a \partial_a R_{\mu\nu\rho\sigma} - \Gamma^\kappa_{\lambda\mu} R_{\kappa\nu\rho\sigma} - \Gamma^\kappa_{\lambda\nu} R_{\mu\kappa\rho\sigma} - \Gamma^\kappa_{\lambda\rho} R_{\mu\nu\kappa\sigma} - \Gamma^\kappa_{\lambda\sigma} R_{\mu\nu\rho\kappa} \ . \tag{A.18}$$

The non-zero components of the spin connection are found using eqs. (A.4) and (A.5)

$$\Gamma^0_{01} = \Gamma^1_{00} = \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1/2} \tag{A.19a}$$

$$\Gamma^2_{21} = -\Gamma^1_{22} = \frac{1}{r}\left(1 - \frac{2GM}{r}\right) \tag{A.19b}$$

$$\Gamma^3_{31} = -\Gamma^1_{33} = \frac{1}{r}\left(1 - \frac{2GM}{r}\right) \tag{A.19c}$$

$$\Gamma^3_{32} = -\Gamma^2_{33} = \frac{\cot\theta}{r} \ . \tag{A.19d}$$

Thus, using eqs. (A.9) and (A.19), one obtains the non-zero components of the covariant derivative of the Riemann tensor in local frame

$$\nabla_1 R_{0101} = -\nabla_1 R_{2323} = \frac{6GM}{r^4}\left(1 - \frac{2GM}{r}\right)^{1/2} \tag{A.20a}$$

$$\nabla_1 R_{0202} = \nabla_1 R_{0303} = -\nabla_1 R_{1212} = -\nabla_1 R_{1313} = -\frac{3GM}{r^4}\left(1 - \frac{2GM}{r}\right)^{1/2} \tag{A.20b}$$

$$\nabla_2 R_{1332} = \nabla_3 R_{1223} = -\nabla_2 R_{0102} = -\nabla_3 R_{0103} = \frac{3GM}{r^4}\left(1 - \frac{2GM}{r}\right)^{1/2} \ . \tag{A.20c}$$

The second covariant derivative may be calculated in the same way

$$\begin{aligned}\nabla_\kappa \nabla_\lambda R_{\mu\nu\rho\sigma} = {} & e_\kappa{}^a \partial_a \left(\nabla_\lambda R_{\mu\nu\rho\sigma}\right) - \Gamma^\eta_{\kappa\lambda} \nabla_\eta R_{\mu\nu\rho\sigma} - \Gamma^\eta_{\kappa\mu} \nabla_\lambda R_{\eta\nu\rho\sigma} \\ & - \Gamma^\eta_{\kappa\nu} \nabla_\lambda R_{\mu\eta\rho\sigma} - \Gamma^\eta_{\kappa\rho} \nabla_\lambda R_{\mu\nu\eta\sigma} - \Gamma^\eta_{\kappa\sigma} \nabla_\lambda R_{\mu\nu\rho\eta} \ .\end{aligned}$$

From eqs. (A.20) and (A.19), the second covariant derivative can be found. The result is a lengthy expression that we will not explicitly write.

Appendix B

Geometric optics approximation in curved spacetime

In this appendix, we present some results of the geometric optics approximation in General Relativity. More details can be found in ref. [17]. We analyse the approximation in terms of the vector potential. One can easily obtain the field strength from this analysis.

Consider the wavelength of the electromagnetic wave, λ, as measured by a typical Lorentz frame. Let L be the typical length over which the amplitude, polarization and wavelength vary (like the radius of the wave front for exemple). Consider also R the typical radius of the curvature of the spacetime through which the waves propagate. Geometric optics approximation is valid when $\lambda \ll \mathsf{R}$ and $\lambda \ll \mathsf{L}$. Then the waves are locally plane waves propagating through spacetime of negligible curvature.

The vector potential can be written as the real part of the product of a slowly varying complex amplitude and a rapidly varying real phase. If one holds fixed the scale of the amplitude variation L and the scale of the spacetime curvature R while making the wavelength shorter and shorter, the phase will vary more and more rapidly but the amplitude can remain almost unchanged. Thus, one may write

$$A^a = \left[(\mathrm{a}^a + \varepsilon \mathrm{b}^a + \varepsilon^2 \mathrm{c}^a) e^{i\Theta/\varepsilon}\right] \ . \qquad \text{(B.1)}$$

where the real part is understood. The coefficients b^a and c^a are post-geometric optics corrections (which would be necessary to realize the full wavelike character of the solution, like diffraction and interference) and ε is a dummy expansion parameter that keeps track of how rapidly various terms change as $\lambda/L \to 0$, with L the minimum of L and R.

We define the wave vector $k_a = \nabla_a \Theta$, the scalar amplitude $\mathrm{a} = (\mathrm{a}_a \bar{\mathrm{a}}^a)^{1/2}$ and the polarization vector $\mathrm{f}^a = \mathrm{a}^a/\mathrm{a}$, where $\bar{\mathrm{a}}^a$ is the complex conjugate of the amplitude. Light rays are defined to be the curves $x^a(\tau)$ normal to the surface of constant phase Θ. Since $k_a = \nabla_a \Theta$ is normal to this surface, the light rays are

$$\frac{dx^a}{d\tau} = k^a(x) = g^{ab} \nabla_b \Theta \ .$$

Consider the source-free Maxwell equation (2.2)

$$\nabla_a F^{ab} = 0 \ .$$

If one introduces the vector potential defined by

$$F_{ab} = \nabla_a A_b - \nabla_b A_a \ , \qquad \text{(B.2)}$$

one obtains the wave equation for the vector potential

$$-\Box A^a + R^a{}_b A^b = 0 \ . \qquad \text{(B.3)}$$

We use also the Lorentz gauge condition

$$\nabla \cdot A = 0 \ . \qquad \text{(B.4)}$$

To leading order, the field strength is given by

$$F_{ab} = ik_a \mathrm{a}_b - ik_b \mathrm{a}_a \ .$$

If we insert eq. (B.1) in eq. (B.4), we obtain

$$0 = \left\{ \left[\frac{i}{\varepsilon} k_a \left(\mathrm{a}^a + \varepsilon \mathrm{b}^a \right) + \nabla_a \mathrm{a}^a + \mathcal{O}(\varepsilon) \right] e^{i\Theta/\varepsilon} \right\} \ .$$

The leading order yields

$$k_a \mathrm{a}^a = 0 \tag{B.5}$$

or similarly

$$k_a \mathrm{f}^a = 0 \ .$$

Hence the polarization vector is orthogonal to the wave vector. The post-geometric optics modifies the orthogonality between the amplitude and the wave vector

$$k_a \mathrm{b}^a = i \nabla_a \mathrm{a}^a \ .$$

Next, we insert eq. (B.1) in the wave equation

$$0 = \left\{ \left[\frac{1}{\varepsilon^2} k^2 \left(\mathrm{a}^a + \varepsilon \mathrm{b}^a + \varepsilon^2 \mathrm{c}^a \right) - \frac{2i}{\varepsilon} k^b \nabla_b \left(\mathrm{a}^a + \varepsilon \mathrm{b}^a \right) \right. \right. \\ \left. \left. - \frac{i}{\varepsilon} \nabla \cdot k \left(\mathrm{a}^a + \varepsilon \mathrm{b}^a \right) - \Box \mathrm{a}^a + R^a{}_b \mathrm{a}^b + \mathcal{O}(\varepsilon) \right] \right\} .$$

To leading order, one obtains

$$k^2 = 0 \ . \tag{B.6}$$

The wave vector is a null vector. To order $1/\varepsilon$, we have

$$k^b \nabla_b \mathrm{a}^a + \frac{1}{2} (\nabla \cdot k) \, \mathrm{a}^a = 0 \ . \tag{B.7}$$

The order ε^0 yields the first post-geometric optics modification

$$-2i\left(k^c\nabla_c \mathrm{b}^a + \frac{1}{2}(\nabla \cdot k)\,\mathrm{b}^a\right) - \Box \mathrm{a}^a + R^a{}_c \mathrm{a}^c = 0 \ .$$

The eqs. (B.6) and (B.7), together with eq. (B.5) are the basis of the geometric optics approximation in curved spacetime. Eq. (B.6) leads to the geodesic equation for the wave vector (see section 2.1.1)

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} \ .$$

Writing $\mathrm{a}^a = \mathrm{a}\mathrm{f}^a$, one may obtain propagation equations for a and f^a separately. Using eq. (B.7), one obtains

$$\begin{aligned} 2\mathrm{a}k^a\nabla_a \mathrm{a} = k^a\nabla\left(\mathrm{a}^2\right) &= \mathrm{a}_b k^a\nabla_a \bar{\mathrm{a}}^b + \bar{\mathrm{a}}_b k^a \nabla_a \mathrm{a}^b \\ &= -\frac{1}{2}\nabla\cdot k\,(\,\bar{\mathrm{a}}\cdot\mathrm{a} + \mathrm{a}\cdot\bar{\mathrm{a}}\,) = -a^2\,\nabla\cdot k \ . \end{aligned}$$

Hence, the propagation equation for the amplitude is

$$k^a\nabla_a \mathrm{a} = -\frac{1}{2}(\nabla\cdot k)\,\mathrm{a} \ . \tag{B.8}$$

For the wave vector, one obtains, using eqs. (B.7) and (B.8)

$$0 = k^a\nabla_a\left(\mathrm{a}\mathrm{f}^a\right) + \frac{1}{2}(\nabla\cdot k)\,\mathrm{a}\mathrm{f}^a = \mathrm{a}k^a\nabla_a \mathrm{f}^b \ .$$

Thus, the polarization vector is parallel transported along the trajectory

$$k^a\nabla_a \mathrm{f}^b = 0 \ . \tag{B.9}$$

Therefore, if we impose the conditions $k^2 = 0$, $\mathrm{f}^2 = 1$ and $k\cdot\mathrm{f} = 0$ at one point, they will be satisfied along the entire trajectory because from eqs. (B.6) and (B.9) as both vectors are parallel transported along the trajectory.

Appendix C

Null geodesics in spherical symmetric geometry

In this appendix, we find the null geodesics of spherical symmetric spacetime. They are used in chapter 2 to calculate light deflection in a gravitational field. We use the methods presented in ref. [87].

In the geometric optics approximation, one obtains that light rays are null geodesics satisfying $k^2 = 0$, where $k_a = \nabla_a \Theta$ is the derivative of the wave phase. For a general asymptotically flat spherical spacetime described by the metric

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\ , \tag{C.1}$$

the light-cone condition becomes

$$-\frac{1}{B(r)}\left(\frac{\partial\Theta}{\partial t}\right)^2 + \frac{1}{A(r)}\left(\frac{\partial\Theta}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\Theta}{\partial\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial\Theta}{\partial\phi}\right)^2 = 0\ . \tag{C.2}$$

We have assumed that the metric (C.1) is independent of t and ϕ. These isometries imply the existence of two Killing vectors $\xi^a = \delta^a_t$ and $\chi^a = \delta^a_\phi$

which give rise to two conserved quantities

$$E = -g_{ab}\xi^a k^b = -\frac{\partial\Theta}{\partial t} \tag{C.3a}$$

$$\ell = g_{ab}\psi^a k^b = \frac{\partial\Theta}{\partial\phi} \; . \tag{C.3b}$$

The quantity E has the physical interpretation of the total energy and the quantity ℓ, the azimuthal angular momentum, as measured by a static observer at infinity. Introducing eqs. (C.3), the light-cone condition becomes

$$-\frac{1}{B(r)}E^2 + \frac{1}{A(r)}\left(\frac{\partial\Theta}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\Theta}{\partial\theta}\right)^2 + \frac{\ell^2}{r^2\sin^2\theta} = 0 \; . \tag{C.4}$$

The geodesics can be found by solving eq. (C.4) by a separation of variables. Writing $\Theta = \Theta_r(r) + \Theta_\theta(\theta)$, one can introduce a separation variable L such that

$$\left(\frac{\partial\Theta}{\partial\theta}\right)^2 + \frac{\ell^2}{\sin^2\theta} = L^2 \tag{C.5a}$$

$$\frac{1}{A(r)}\left(\frac{\partial\Theta}{\partial r}\right)^2 - \frac{1}{B(r)}E^2 = -\frac{L}{r^2} \; . \tag{C.5b}$$

The constant L has the physical interpretation of the total angular momentum, as measured by a static observer at infinity. Using eqs. (C.5) and (C.3), one obtains the equations of motion $k^i = \dot{x}^i = g^{ij}k_j$, where the dot represents derivative with respect to an affine parameter.

$$\dot{t} = k^t = \frac{E}{B(r)} \tag{C.6a}$$

$$\dot{r} = k^r = \pm\left[\frac{E^2}{A(r)B(r)} - \frac{L^2}{r^2A(r)}\right]^{1/2} \tag{C.6b}$$

$$\dot{\theta} = k^\theta = \pm\frac{1}{r^2}\left[L^2 - \frac{\ell^2}{\sin^2\theta}\right]^{1/2} \tag{C.6c}$$

$$\dot{\phi} = k^\phi = \frac{\ell}{r^2\sin^2\theta} \; . \tag{C.6d}$$

The equations of motion (C.6) apply for any photon trajectories. We can simplify the analysis by restricting the trajectory to the equatorial plane $\theta =$

$\pi/2$. In that case the azimuthal angular momentum equals the total angular momentum and we have $k^\theta = \dot{\theta} = 0$. Therefore, the entire photon trajectory lies in the equatorial plane. The trajectory can be specified by the impact parameter $b = L/E$ and the energy E. The photon momentum is then given by

$$k^t = EB(r) \tag{C.7a}$$

$$k^r = \pm E \left[\frac{1}{A(r)B(r)} - \frac{b^2}{r^2 A(r)} \right]^{1/2} \tag{C.7b}$$

$$k^\phi = \frac{Eb}{r^2} \,. \tag{C.7c}$$

The light deflection angle is found by integrating along the trajectory

$$\begin{aligned}\Delta\phi + \pi &= \int dr \, \frac{d\phi}{dr} = \int dr \, \frac{k^\phi}{k^r} \\ &= 2 \int_{r_0}^{\infty} dr \, \frac{b}{r^2} \left[\frac{1}{B(r)A(r)} - \frac{b^2}{r^2 A(r)} \right]^{1/2} ,\end{aligned}$$

where r_0 is the distance of closest approach, where k^r vanishes. Hence

$$b^2 = \frac{r_0^2}{B(r_0)} \,. \tag{C.8}$$

Replacing b by r_0, one obtains[28]

$$\Delta\phi + \pi = 2 \int_{r_0}^{\infty} \frac{dr}{r} \left[\frac{A(r)}{\frac{r^2 B(r_0)}{r_0^2 B(r)} - 1} \right]^{1/2} \,. \tag{C.9}$$

For the special case of Schwarzschild metric, one has $B(r) = A^{-1}(r) = (1 - 2GM/r)$ and the deflection angle is

$$\Delta\phi + \pi = 2 \int_{r_0}^{\infty} dr \, \frac{b}{r^2} \left[1 - \left(1 - \frac{2GM}{r} \right) \frac{b^2}{r^2} \right]^{-1/2} , \tag{C.10}$$

with

$$b = r_0 \left(1 - \frac{2GM}{r_0} \right)^{-1/2} \,. \tag{C.11}$$

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