Localization, Completion and Duality in HNP rings

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ABSTRACT

This thesis is a study of localization, completion and duality in HNP rings with results extended to FBN hereditary rings where possible. Chapter 1 contains a review of localization and completion in Noetherian rings with some special results for FBN hereditary rings. In chapter 2 is given a new proof of a theorem of Singh on indecomposable injectives over HNP rings. This result is then extended to FBN hereditary rings followed by a discussion of duality over these rings. A complete semilocal Noetherian hereditary ring has Morita duality (Theorem 2.14). The presence of this duality is a powerful tool in Chapter 3 where the author investigates the endomorphism rings of certain injectives over FBN hereditary rings and shows that if R is a complete semilocal Noetherian hereditary ring and $J(R)$ is the intersection of a clan, $R \cong \mathrm{End}_R(E(R/J(R)))$. This leads to a new proof of a theorem of Michler on the structure of semiperfect Noetherian hereditary rings.

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LOCALISATION, COMPLETION ET DUALITE DANS LES ANNEAUX HNP

RESUME

Dans cette thèse, nous étudions la localisation, la complétion et la dualité pour les anneaux héréditaires Noethériens et premiers (HNP). Chapitre 1 contient quelques résultats sur la localisation et la complétion dans les anneaux Noethériens et les T-anneaux. Dans chapitre 2 nous donnons une nouvelle preuve d'un théorème de Singh sur les R-modules injectifs indécomposables où R est un anneau HNP. Ce résultat peut être étendu aux T-anneaux héréditaires (Théorème 2.6). Ensuite, nous étudions la dualité générale et la dualité de Morita dans les T-anneaux héréditaires: un anneau Noethérien, héréditaire, semi-local et complet possède la dualité de Morita (Théorème 2.14). Cette dualité est un outil efficace dans chapitre 3 où nous étudions l'anneau d'endomorphisme d'un module injectif sur un T-anneau héréditaire. Si R est un anneau HNP et N un idéal semi-premier et inversible, $\hat{R}_N \cong \mathrm{End}_R(E(R/N))$. Enfin, nous donnons une nouvelle preuve d'un théorème de Michler sur la structure d'un anneau héréditaire, Noethérien et semi-parfait.

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PREFACE

In recent years, localization of noncommutative rings has been studied exhaustively by Goldie, Lambek, Michler, Stenstrom, Jategaonkar and many others. A particularly "nice" method of localization has been developed for a semiprime ideal N in a Noetherian ring R such that $\mathcal{C}(N) = \{ c \in R \mid [c]_N$ is regular in $R/N \}$ satisfies the right Ore condition. A good deal is known about the "localized" ring R_N but little is known in general about its completion in the $J(R_N)$-adic topology, $\hat{R}_N$. The main results of this thesis are concerned with the structure and properties of $\hat{R}_N$ where R is a fully bounded Noetherian (FBN) hereditary ring and N is a localizable intersection of non-minimal prime ideals. The problem can be reduced to the case where R is a bounded (non-Artinian) HNP ring and N is a maximal invertible ideal. Our most useful tool is the fact that $\hat{R}_N$ has Morita duality with $\mathrm{End}_R(E(R/N))$.

Chapter 1, §1 contains a review of some localization techniques. Those proofs which are given are original. §2 answers a question of Müller for FBN hereditary rings and §3 is devoted to a few results on completion of which Lemma 1.12 is believed to be original.

The main result of Chapter 2 §1 on the structure of certain indecomposable injectives over an HNP ring is due to S. Singh but we give an independent proof (Theorem 2.4) and show how the result extends to FBN hereditary rings (Theorem

2.6). It is fundamental to all later results. It is used in Chapter 2 §2 to study Morita duality over FBN hereditary rings and the more general duality theory of Lambek and Rattray [22, 23] as applied to these rings. In Theorem 2.14 we prove that if R is an FBN hereditary ring and N a localizable intersection of non-minimal prime ideals of R, $\hat{R}_N$ is a Morita ring. Surprisingly, Morita duality (over any ring) cannot be described in terms of the Lambek-Rattray theory as applied to discrete modules (Lemma 2.16).

Theorem 2.6 and the resulting Morita duality form the basis for our methods in Chapter 3. In §1, we investigate the properties of the endomorphism ring, K, of a suitable injective R-module and use these to establish some properties of $\hat{R}_N$. Then, concentrating on the situation where R is a bounded HNP ring and N a maximal invertible ideal, we show $\hat{R}_N \cong \mathrm{End}_R(E(R/N))$ (Theorem 3.11). It is not difficult to extend to the case where N is a localizable intersection of non-minimal prime ideals in a FBN hereditary ring (Theorem 3.13). This result generalizes Matlis' well-known theorem for commutative Noetherian rings. Using the same methods, in §2 we determine the structure of $K = \mathrm{End}_R(E(R/N))$ and we obtain from this a new proof of a theorem of Michler on the structure of an arbitrary semiperfect Noetherian hereditary ring (Theorem 3.20).

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NOTATION AND TERMINOLOGY

-- All rings have 1 and all modules are unitary. Morphisms are always written opposite scalars.

-- $N \leq M$ means N is a submodule of M; $N \leq' M$ means N is an essential submodule of M.

-- $E(M)$ denotes the injective hull of the module M.

-- If I is an indecomposable injective R-module, Ass I denotes the associated prime ideal of I.

-- If M is a right R-module, $X \leq M$ a submodule and $A \leq R$ a right ideal of R, $\mathrm{Ann}_R X = \{r \in R \mid Xr=0\}$; $\mathrm{Ann}_M A = \{m \in M \mid mA=0\}$.

-- "Ideal" always means a two-sided ideal and ring properties written without the prefix "left" or "right" are understood to mean two-sided.

-- A ring is semi-local if $R/J(R)$ is semi-simple Artinian and $\bigcap_{n=1}^{\infty} J(R)^n = 0$.

Chapter 1

§1. Localization of right Noetherian rings at semiprime ideals

In this section, we summarize those definitions and results on localization in right Noetherian rings which will be needed later. Proofs are given only where they cannot readily be found in the literature.

Given a semiprime ideal N in a right Noetherian ring R, let $\mathcal{C}(N) = \{c \in R \mid [c]_N$ is regular in $R/N\}$. There is an idempotent filter $\mathcal{D}_N$ associated with $\mathcal{C}(N)$:

$$\mathcal{D}_N = \{ I \leq R \mid \forall r \in R \ r^{-1}I \cap \mathcal{C}(N) \neq \emptyset \}.$$

One may also look at the idempotent filter associated with E(R/N):

$$\mathcal{D}_N' = \{ I \leq R \mid \mathrm{Hom}_R(R/I, E(R/N)) = 0 \}.$$

Lambek and Michler [21] have shown that $\mathcal{D}_N = \mathcal{D}_N'$ and hence they determine the same torsion theory, called the N-torsion theory. For any R-module M, $T_N(M) = \{m \in M \mid \exists \ I \in \mathcal{D}_N \ mI = 0\}$ is the N-torsion submodule of M. M is <u>N-torsion</u> if $T_N(M) = M$, <u>N-torsion free</u> if $T_N(M) = 0$. $M' \leq M$ is an <u>N-dense</u> submodule if M/M' is N-torsion. M' is an <u>N-closed</u> submodule if M/M' is N-torsion free. Equivalently, M is N-torsion if $\mathrm{Hom}_R(M, E(R/N)) = 0$; M is N-torsion free if M is cogenerated by E(R/N); $M' \leq M$ is N-dense if $\mathrm{Hom}_R(M/M', E(R/N)) = 0$; $M' \leq M$ is N-closed if M/M' is cogenerated by E(R/N). The

<u>N-closure</u> of M' in M is $\{m \in M \mid \exists\ I \in \mathcal{D}_N\ mI \leq M'\}$. M is <u>N-divisible</u> if $E(M)/M$ is N-torsion free. This is the same as saying that whenever A is an N-dense submodule of B, every R-homomorphism $f: A \longrightarrow M$ extends to some $g: B \longrightarrow M$. Every module M has an <u>N-divisible hull</u>, $D_N(M)$, defined by

$$D_N(M)/M = T_N(E(M)/M).$$

The module of quotients of M with respect to N is given by

$$M_N = Q_N(M) = D_N(M/T_N(M))$$

$R_N = Q_N(R)$ is a ring, the <u>ring of quotients of R at N</u>, or the N-localization of R. It is clear that a module M is N-torsion free and divisible iff $Q_N(M) = M$.

There is a more general notion of localization which can be applied in any complete additive category (Lambek [20]): if I is an object of a complete additive category <u>A</u>, consider the pair of functors

$$\text{Mod-R} \underset{U_I = I^{-} = \text{Hom}(\_,\ I)}{\overset{F_I = \text{Hom}_R(\_,\ I)}{\rightleftarrows}} \mathcal{S} = \underline{\text{Sets}}$$

The natural transformation η_I defined by $\eta_I(A)(a)(f) = f(a)$ $\forall A$, $\forall a \in A$, $\forall f \in \text{Hom}_R(A, I)$ satisfies the universal property of a front adjunction. Let $Q_I \xrightarrow{\kappa} U_I F_I = S_I$ be the equalizer of $\eta_I S_I$, $S_I \eta_I$: $S_I \rightrightarrows S_I^{\,2}$. The following characterization of κ is very useful:

<u>Lemma 1.1</u> (Lambek and Rattray [22])

$\kappa(A): Q_I(A) \longrightarrow S_I(A)$ is the joint equalizer of all pairs of maps $\varphi, \psi : S_I(A) \rightrightarrows I$ for which

$$\varphi . \eta_I(A) = \psi . \eta_I(A).$$

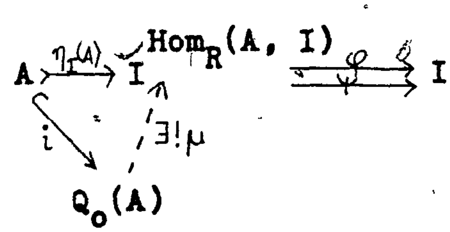
By the naturality of η_I, $\eta_I S_I . \eta_I = S_I \eta_I . \eta_I$ hence $\exists! \ \lambda : id \longrightarrow Q_I$ such that $\kappa \lambda = \eta_I$.
Lemma 1.2 shows that this localization agrees with the more usual localization in Mod-R.

<u>Lemma 1.2</u>

Let I be an injective R-module. Let Q_o be the localization functor obtained from the usual I-torsion theory on Mod-R and Q_I the functor defined above. Then $\forall A \in$ Mod-R, $Q_o(A) = Q_I(A)$.

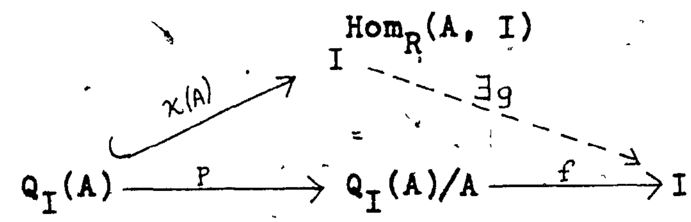
<u>Proof:</u>

If $\tau(A)$ denotes the torsion submodule of A, we know $Q_o(A) = Q_o(A/\tau(A))$. Also $Hom_R(\tau(A), I) = 0$, hence $Hom_R(A/\tau(A), I) = Hom_R(A, I)$, and so $U_I F_I(A) = U_I F_I(A/\tau(A))$ from which follows $Q_I(A) = Q_I(A/\tau(A))$. Hence, without loss of generality, we may assume $\tau(A) = 0$. By Lemma 1.1 $Q_I(A)$ is the joint equalizer of all pairs $\varphi, \psi : S_I(A) \rightrightarrows I$ such that $\varphi . \eta_I(A) = \psi . \eta_I(A)$. Since A is I-torsion free, A is a submodule of $Q_o(A)$ and $\eta_I(A)$ is a monomorphism. By the injectivity of I, $\exists! \mu$ completing the following:



μ is unique since $Q_o(A)/A$ is I-torsion. If $\varphi\mu \neq \psi\mu$ then since $\varphi\circ\eta_I(A) = \psi\cdot\eta_I(A)$, $\varphi\mu - \psi\mu$ induces a nonzero homomorphism $h: Q_o(A)/A \longrightarrow S_I(A)$. But this is impossible since $Q_o(A)/A$ is I-torsion. Hence (up to isomorphism) $Q_o(A) \subseteq Q_I(A)$.

For the reverse inclusion, note that $\varkappa(A)\lambda(A) = \eta_I(A) \Rightarrow \lambda(A): A \longrightarrow Q_I(A)$ is a monomorphism and may be thought of as inclusion. If $0 \neq f \in \mathrm{Hom}_R(Q_I(A)/A, I)$ and $p: Q_I(A) \longrightarrow Q_I(A)/A$ is the canonical projection



By construction, $g\varkappa(A) \neq 0$. Now $g\varkappa(A)\lambda(A) = fp\lambda(A) = 0$ — i.e. $g\circ\eta_I(A) = 0\circ\eta_I(A)$. Since $\varkappa(A): Q_I(A) \longrightarrow S_I(A)$ equalizes the pair $g, 0: S_I(A) \rightrightarrows I$, $g\circ\varkappa(A) = 0$, contradiction. Hence $Q_I(A)/A$ is I-torsion. Since λ is a monomorphism, $Q_I(A)$ is therefore clearly an essential extension of A. It follows immediately from the definition of divisible hull that (up to isomorphism) $Q_I(A)/A \subseteq Q_o(A)/A$. Hence $Q_I(A) = Q_o(A)$.

<u>Definition</u>: The semiprime ideal $N \leqslant R$ is <u>right localizable</u> if $\forall r \in R\ \forall c \in \mathscr{C}(N)\ \exists r' \in R\ \exists c' \in \mathscr{C}(N)\ rc' = cr'$.

When N is a right localizable semiprime ideal, R_N

takes a classical form — i.e. there is a ring homomorphism $h: R \longrightarrow R_N$ such that every element of R_N can be written in the form $h(r)h(c)^{-1}$ for some $r \in R$, $c \in \mathcal{C}(N)$, and $h(r) = 0 \Rightarrow \exists c \in \mathcal{C}(N)$ $rc = 0$. There are many ways of characterizing right localizable semiprime ideals. The following are only some:

<u>Theorem 1.3</u> ([16], [9])

Let N be a semiprime ideal of a right Noetherian ring R. Then the following are equivalent:

(a) For any R-module A, if A has an essential and N-dense submodule B such that B is a non-singular R/N-module, then $AN = 0$;

(b) For any cyclic R-module A, if A has an essential and N-dense submodule B isomorphic to a uniform R/N-right ideal, then $AN = 0$;

(c) Every maximal N-closed right ideal of R contains N;

(d) For all maximal N-closed right ideals $I \leqslant R$, $I \cap \mathcal{C}(N) = \emptyset$;

(e) The elements of $\mathcal{C}(N)$ operate regularly on $E(R/N)$;

(f) N is right localizable.

<u>Proof</u>:

(a) $\Rightarrow$ (b) trivially.

For the implications (e) $\Rightarrow$ (f) and (f) $\Rightarrow$ (a), see Jategaonkar [16].

(b) $\Rightarrow$ (c): Let K be a maximal closed right ideal of R. Let $A = R/K$. A is N-torsion free but every proper factor module of A is N-torsion. Since A is cogenerated by $E(R/N)$ $\exists\, 0 \neq f: B \longrightarrow R/N$ for some uniform $B \leqslant A$. If f is not a monomorphism, $f(B) \cong B/\ker(f)$ is N-torsion, contradiction. Hence B is (isomorphic to) a uniform R/N right ideal. A/B is N-torsion, therefore B is N-dense in A. If $B' \cap B = 0$ for some $B' \leqslant A$ then $B \hookrightarrow A/B'$ which is N-torsion. But B is torsion free. Hence $B' = 0$ and B is essential in A. By (b), $AN = 0$ $\therefore K \geqslant N$.

(c) $\Rightarrow$ (d): Let I be a maximal closed right ideal of R and suppose $I \cap \mathcal{C}(N) \neq \emptyset$. By (c), $I \geqslant N$. Let $N \leqslant N + cR \leqslant I$ where $c \in \mathcal{C}(N)$. By Goldie's Theorem applied to R/N, N+cR is N-dense. Hence I is N-dense, contradiction.

(d) $\Rightarrow$ (e): If $ec = 0$ for some $0 \neq e \in E$, $c \in \mathcal{C}(N)$, then $\mathrm{Ann}_R e \cap \mathcal{C}(N) \neq \emptyset$. Hence $\mathrm{Ann}_R e$ is not contained in any maximal closed right ideal, and therefore $\mathrm{Ann}_R e \in \mathcal{D}_N$ and $T_N(E(R/N)) \neq 0$, contradiction.

Proposition 1.4 [32]

Let N be a right localizable semiprime ideal of a right Noetherian ring R and let $N = \bigcap_{i=1}^{n} P_i$ be its unique representation as a finite irredundant intersection of prime ideals. Let $h: R \longrightarrow R_N$ define the ring of quotients of R at N. Then

(a) R_N is right Noetherian;

(b) $J(R_N) = h(N)R_N$ and $R_N/J(R_N)$ is semisimple Artinian;

(c) $\mathcal{C}(N) = \mathcal{C}(P_1) \cap \ldots \cap \mathcal{C}(P_n)$ and if $c_i \in \mathcal{C}(P_i)$

$\forall\ i = 1, 2, \ldots, n$ then $\exists r_i \in R\ \sum_{i=1}^{n} c_i r_i \in \mathcal{C}(N)$;

(d) The prime ideals of R_N are exactly those $h(Q)R_N$ such that Q is a prime ideal of R and $Q \leqslant \bigcup_{i=1}^{n} P_i$.

The "nicest" localizations are those which most closely parallel the commutative situation. This leads one to impose further conditions on N.

Definition (Müller): Let N be a right localizable semiprime ideal of the right Noetherian ring R. Then N is right classical if $NR_N = h(N)R_N$ has the right AR-property — i.e. for every right ideal A of R_N, $\exists n \in \mathbb{N}$ $A \cap (NR_N{}^n) \subseteq ANR_N$.

Note before Theorem 1.5: The canonical monomorphism $R/N \hookrightarrow R_N/NR_N$ is essential in Mod-R. Hence $E_R(R_N/NR_N) \cong E_R(R/N)$. The latter is an R_N-module which is an R_N-essential extension of R_N/NR_N; therefore, in Mod-R_N, $E_R(R_N/NR_N) \subseteq E_{R_N}(R_N/NR_N)$. Since $E_{R_N}(R_N/NR_N)$ is clearly an R-essential extension of R_N/NR_N, $E_R(R/N) \cong E_{R_N}(R_N/NR_N)$.

Theorem 1.5 [16 , 21]

For a right localizable semiprime ideal N in a right Noetherian ring R, the following are equivalent:

(a) N is right classical;

(b) $E = E(R/N) = \bigcup_{n=1}^{\infty} \mathrm{Ann}_E N^n$;

(c) For any cyclic R-module A, if A has an essential submodule B which is isomorphic to a uniform R/N-right ideal, then $\exists n \quad AN^n = 0$;

(d) For any cyclic R_N-module A', if A' has an essential submodule B' which is isomorphic to a uniform R_N/NR_N-right ideal, then $\exists n \quad A'NR_N{}^n = 0$;

(e) Every right ideal of R_N is closed in the NR_N-adic topology: $\bigcap_{n=1}^{\infty}(A + N^nR_N) = A \ \forall A \leqslant R_N$.

<u>Proof:</u>

For the equivalence of (a), (b) and (e) see Lambek and Michler [21].

(b) $\Rightarrow$ (c): see Jategaonkar [16].

(c) $\Rightarrow$ (d): Suppose A' is a cyclic R_N-module containing a submodule B' satisfying the assumptions of (d). If $A' = aR_N$, consider the submodule $aR \leqslant A$. Applying (b) to $B' \cap aR \leqslant aR$ $\exists n \quad aRN^n = 0 \therefore aN^n = 0$. Hence $a(NR_N)^n = 0$, therefore $A(NR_N)^n = 0$.

(d) $\Rightarrow$ (e): If not all right ideals of R_N are closed in the NR_N-adic topology, let A be maximal among right ideals which are not (since R_N is right Noetherian). Let $B = \bigcap_{n=1}^{\infty}(A + NR_N{}^n) \supsetneq A$. If $C \supsetneq A$ then $C = \bigcap_{n=1}^{\infty}(C + NR_N{}^n) \geqslant \bigcap_{n=1}^{\infty}(A + NR_N^n) = B$. Hence B/A is a simple R_N-module and R_N/A is uniform since A is meet-irreducible. Now R_N/NR_N is semisimple Artinian, hence B/A is (isomorphic to) an R_N/NR_N right ideal which is essential in the cyclic R_N-module R_N/A. By (d), $\exists n$ $[1]_A NR_N{}^n = 0$ which implies $NR_N{}^n \leqslant A$, contradiction.

In keeping with our conventions, a semiprime ideal in a Noetherian ring R is localizable and classical if it is right and left localizable, right and left classical. The terms "localizable" and "classical" may also be applied to the set $\{P_1, \ldots, P_n\}$ where $N = \bigcap_{i=1}^{n} P_i$ is the unique representation of N as a finite, irredundant intersection of prime ideals and N is localizable and classical. A minimal localizable, classical set of prime ideals is called a <u>clan</u>. When R is an HNP ring, there is also the notion of a "cycle" of prime ideals introduced by Eisenbud and Robson [12]. They showed that when R is HNP, X is maximal among invertible ideals of R iff X is the intersection of a cycle (see Theorems 2.4 – 2.6 of [12]). (Recall that all non-zero primes in an HNP ring are maximal [2]). Müller has shown that for an HNP ring, the notion of a clan coincides with that of a cycle. In fact, a semiprime ideal N in an HNP ring is right localizable iff it is invertible. In that case it is localizable and classical [32]. In general, two questions naturally arise: (i) are different clans disjoint? and (ii) does every prime ideal belong to a clan? Both questions have been answered for HNP rings: for any HNP ring, different clans are disjoint and if R is right bounded, every prime ideal belongs to a clan (Eisenbud and Robson [12]; Lenagan [24]). An example is known of an HNP ring which is not right bounded and for which the second question has a negative answer (Robson [34]).

Definition: A right Noetherian ring R is right fully bounded (r. FBN) if every prime factor ring is right bounded. Equivalently (Krause [17]), the map $I \longmapsto \mathrm{Ass}\,I$ of isomorphism classes of indecomposable injective right R-modules to prime ideals of R is bijective.

Classically, in Noetherian commutative rings, one of the uses of localization is to deduce information about R from facts known about all R_P (P a prime ideal). The techniques used can be carried over to non-commutative FBN rings with enough clans (i.e. every prime ideal belongs to a clan). Since all proper factor rings of an HNP ring are Artinian, any bounded HNP ring is an FBN ring. Thus we have some FBN rings with enough clans. This leads naturally to the question: does every FBN ring have enough clans? In view of Robson's example, we cannot expect to drop the condition of fully boundedness. The next section provides a partial answer to this question.

§2. Localization of Noetherian hereditary rings

A Noetherian hereditary ring is a direct sum of indecomposable ideals each of which as a ring is either HNP or Artinian hereditary [3]. This prompts us to investigate localization in a finite product of rings. Let $R = R_1 \oplus R_2$

and let $N = \bigcap_{i=1}^{n} P_i$ be a semiprime ideal. Each prime P_i contains one of R_1 or R_2 and if $P_i \not\supseteq R_j$, $P_i \cap R_j$ is a prime ideal of R_j (possibly 0).

Proposition 1.6

Let N be a right ideal of R, $N_i = N \cap R_i$. Then

(a) $N = N_1 \oplus N_2$;

(b) $R/N \cong R_1/N_1 \oplus R_2/N_2$;

(c) $E_R(R/N) \cong E_{R_1}(R_1/N_1) \oplus E_{R_2}(R_2/N_2)$ where each direct summand on the right hand side is given an appropriate R-module structure.

Proof:

(a) is clear and (b) follows immediately from (a).

(c): Let $y \in E_{R_i}(R_i/N_i)$ and $r \in R$. Write $r = r_1 + r_2$ where $r_i \in R_i$. Define $yr = yr_i$. This agrees with the R-module structure already operating on R_i/N_i. For any right ideal D of R and any $f \in Hom_R(D, E_{R_i}(R_i/N_i))$ there exists a $g \in Hom_{R_i}(R, E_{R_i}(R_i/N_i))$ extending f. If g is right R-linear, we are done. Let s, $r \in R$. Write $r = r_1 + r_2$, $s = s_1 + s_2$, $g(s)r = g(s)r_i = g(sr_i) = g(s_ir_i)$. On the other hand, $g(sr) = g(s_1r_1) + g(s_2r_2)$. Now for $j \neq i$, if $g(s_jr_j) \neq 0$, $\exists t_i \in R_i$ $g(s_jr_j)t_i$ is a non-zero element of R_i/N_i. But $g(s_jr_j)t_i = g(s_jr_jt_i) = g(0) = 0$, contradiction. Hence $g \in Hom_R(R, E_{R_i}(R_i/N_i))$; $E_{R_i}(R_i/N_i)$ is R-injective and is clearly an R-essential extension of R_i/N_i.

<u>Lemma 1.7</u>

Let $N = \bigcap_{i=1}^{n} P_i$ be a semiprime ideal of $R = R_1 \oplus R_2$. Let $N_i = N \cap R_i$. If both $N_i \neq R_i$, then ($x = x_1 + x_2$ with $x_i \in R_i$ and $x \in \mathcal{C}(N)$) $\Leftrightarrow$ ($x_i \in \mathcal{C}_{R_i}(N_i)$ for $i = 1, 2$). If $N_2 = R_2$ then ($x = x_1 + x_2 \in \mathcal{C}(N)$) $\Leftrightarrow$ ($x_1 \in \mathcal{C}_{R_1}(N_1)$).

<u>Proof:</u>

Assume both $N_i \neq R_i$ and suppose $x = x_1 + x_2 \in \mathcal{C}(N)$. If for some $r_i \in R_i$, $x_i r_i \in N_i$ then $xr_i = x_i r_i \in N_i \leq N \Rightarrow r_i \in N_i$. Hence $x_i \in \mathcal{C}_{R_i}(N_i)$. Conversely, if both $x_i \in \mathcal{C}_{R_i}(N_i)$ and $xr \in N$, then $x_i r_i \in N_i$ for $i = 1, 2$, hence $r \in N$.

If $N_2 = R_2$, a similar proof works.

<u>Corollary</u>

(a) If $N_i \neq R_i$ for $i = 1, 2$, then N is right localizable iff each N_i is right localizable in R_i.

(b) If $N_2 = R_2$, N is right localizable in R iff N_1 is right localizable in R_1.

<u>Proof:</u>

(a): Assume N is right localizable. Let $c_1 \in \mathcal{C}(N_1)$ and $r_1 \in R_1$. By the right Ore condition for $\mathcal{C}(N)$, $\exists\, c' \in \mathcal{C}(N)$ and $r' \in R$ such that $(c_1 + e_2)r' = r_1 c'$ (where $1 = e_1 + e_2$). By the uniqueness of representations, $c_1 r_1' = r_1 c_1'$ and by the lemma $c_1' \in \mathcal{C}_{R_1}(N_1)$.

Conversely, assuming each N_i is right localizable in R_i, let $c \in \mathcal{C}(N)$ and $r \in R$. For each i, find $c_i' \in \mathcal{C}_{R_i}(N_i)$ and $r_i' \in R_i$ such that $c_i r_i' = r_i c_i'$. Clearly $r' = r_1' + r_2'$

and $c' = c_1' + c_2'$ will do.

(b) If N is right localizable and $c_1 \in \mathscr{C}_{R_1}(N_1)$, $r_1 \in R_1$, let $c' \in \mathscr{C}(N)$ and $r' \in R$ be such that $c_1 r' = r_1 c'$. Then clearly, if $c' = c_1' + c_2'$ and $r' = r_1' + r_2'$, we have $c_1 r_1' = r_1 c_1'$. Conversely, assuming N_1 is right localizable in R_1, given $c \in \mathscr{C}(N)$ and $r \in R$, write $c = c_1 + c_2$ and $r = r_1 + r_2$. Then $c_1 \in \mathscr{C}(N_1)$ and $\exists\, r_1' \in R_1$ and $c_1' \in \mathscr{C}(N_1)$ such that $c_1 r_1' = r_1 c_1'$. Putting $r' = r_1' + 0$ and $c' = c_1' + 0$, we have $c' \in \mathscr{C}(N)$ and $cr' = rc'$.

<u>Lemma 1.8</u>

(a) If the right ideal D is N-dense in R, then for each i, $D_i = D \cap R_i$ is N_i- dense in R_i.

(b) If for each i, D_i is an N_i-dense right ideal of R_i, then the right ideal $D = D_1 \oplus D_2$ is N-dense in R.

<u>Proof</u>:

(a): Assume $r^{-1}D \cap \mathscr{C}(N) \neq \emptyset$ $\forall r \in R$. Given $r_i \in R_i$ $\exists c \in \mathscr{C}(N)$ $r_i c = r_i c_i \in D \cap R_i$ (where $c = c_1 + c_2$). Hence $r_i^{-1} D_i \cap \mathscr{C}(N_i) \neq \emptyset$.

(b): Let $r = r_1 + r_2$. Let $r_i c_i \in D_i$ where $c_i \in \mathscr{C}_{R_i}(N_i)$. Then clearly $r(c_1 + c_2) = r_1 c_1 + r_2 c_2 \in D$.

<u>Corollary 1</u>

If $N_i \neq R_i$ for $i = 1, 2$, then $T_N(R) = T_{N_1}(R_1) \oplus T_{N_2}(R_2)$ and $R/T_N(R) \cong R_1/T_{N_1}(R_1) \oplus R_2/T_{N_2}(R_2)$. If $N_2 = R_2$, then $T_N(R) = T_{N_1}(R_1) \oplus R_2$ and $R/T_N(R) \cong R_1/T_{N_1}(R_1)$.

Corollary 2

If $N_i \neq R_i$, $T_{N_i}(R_i) = T_N(R_i)$.

Localization preserves finite direct sums. Hence

$$D_N(R/T_N(R)) \cong D_N(R_1/T_{N_1}(R_1)) \oplus D_N(R_2/T_{N_2}(R_2)).$$

Lemma 1.9

In Mod-R_i, $D_N(R_i/T_{N_i}(R_i)) = D_{N_i}(R_i/T_{N_i}(R_i))$.

Proof:

We shall show

(i) $D_N(R_i/T_{N_i}(R_i))$ is an R_i-essential extension of $R_i/T_{N_i}(R_i)$,
(ii) $D_N(R_i/T_{N_i}(R_i))/(R_i/T_{N_i}(R_i))$ is N_i-torsion,
(iii) $D_N(R_i/T_{N_i}(R_i))$ is N_i-divisible in Mod-R_i.

(i): Given $0 \neq x \in D_N(R_i/T_{N_i}(R_i))$ $\exists r \in R$ $0 \neq xr \in R_i/T_{N_i}(R_i)$. In that case, $0 \neq xre_i \in R_i/T_{N_i}(R_i)$ and since $re_i \in R_i$, (i) holds.
(ii): Given $x \in D_N(R_i/T_{N_i}(R_i))$ $\exists D \in \mathcal{D}_N$ such that $xD \subseteq R_i/T_{N_i}(R_i)$. Then $x(D \cap R_i) \subseteq R_i/T_{N_i}(R_i)$ and $D \cap R_i \in \mathcal{D}_{N_i}$.
(iii): Let D_1 be an N_1-dense right ideal of R_1. Then $D = D_1 \oplus R_2 \in \mathcal{D}_N$. $R/D \cong R_1/D_1 \Rightarrow D_1$ is N-dense in R_1 (considered as R-modules). Let $f: D_1 \longrightarrow D_N(R_1/T_{N_1}(R_1))$ be R_1-linear. D_1 is an R-module in the obvious way. The R-structure on $D_N(R_1/T_{N_1}(R_1))$ has the property that for $x \in D_N(R_1/T_{N_1}(R_1))$ and $r_2 \in R_2$, $xr_2 = 0$. It follows that f is R-linear, hence extends to an R-linear $g: R_1 \longrightarrow D_N(R_1/T_{N_1}(R_1))$. Since g is obviously R_1-linear, the result is proved.

Corollary

If $R_i \neq N_i$ for $i = 1, 2$, then $R_N \cong R_{1N_1} \oplus R_{2N_2}$ and if $R_2 = N_2$, $R_N \cong R_{1N_1}$.

Proposition 1.10

If rings R_1 and R_2 have enough clans, then $R = R_1 \oplus R_2$ does also.

Proof:

Let P be a prime ideal of R. Assume $P \supseteq R_2$. Let $P_1 = P \cap R_1$. Then P_1 is a prime ideal of R_1 so belongs to a clan — $P_1 \in \{P_1, Q_2, \ldots, Q_n\}$. Then $P = P_1 \oplus R_2$ belongs to the clan $\{P_1 \oplus R_2, Q_2 \oplus R_2, \ldots, Q_n \oplus R_2\}$.

Corollary

A fully bounded Noetherian hereditary ring has enough clans.

Proof:

If R is bounded HNP and P is a non-zero prime ideal, we know P belongs to a clan. The prime ideal 0 belongs to the clan $\{0\}$ by Goldie's theorem. If R is Artinian hereditary and indecomposable, J(R) is the only localizable semiprime ideal [32] so every nonzero prime belongs to the clan whose intersection is J(R). Finally, if R is Artinian hereditary and already prime, Goldie's theorem shows that $\{0\}$ is a clan. Hence if R is a fully bounded Noetherian hereditary ring, it is a direct sum of rings with enough clans, hence has enough clans.

§3. Completions

Assume that R is Noetherian and $N = \bigcap_{i=1}^{n} P_i$ is a localizable classical semiprime ideal. It follows from the AR-property of NR_N that $\bigcap_{n=1}^{\infty} NR_N{}^n = 0$. Hence the NR_N-adic topology on R_N, whose basic open neighbourhoods of 0 are $\{NR_N{}^n\}_{n=1}^{\infty}$, is Hausdorff. Let $\hat{R}_N$ denote the completion of R_N in this topology.

<u>Theorem 1.11</u> (Müller [32])

Let $N = \bigcap_{i=1}^{n} P_i$ be a localizable classical semiprime ideal of R written as a finite irredundant intersection of prime ideals. There is a one-one correspondence between localizable subsets of $\{P_1, \ldots, P_n\}$ and central idempotents of $\hat{R}_N$ given by

$\{P_{i_1}, \ldots, P_{i_s}\} \longleftrightarrow e$ where $T\hat{R}_N + J(\hat{R}_N) = e\hat{R}_N$ and $T = P_{i_1} \cap \ldots \cap P_{i_s}$

<u>Corollary</u>

If $\mathcal{S} = \{P_1, \ldots, P_n\}$ is a localizable classical set of prime ideals of R then $\mathcal{S}$ is the disjoint union of clans in a unique way. A subset $\mathcal{J} \subseteq \mathcal{S}$ is localizable iff it is the union of some of these clans.

In particular, when $\{P_1, \ldots, P_n\}$ is a clan, $\hat{R}_N$ is ring-directly indecomposable.

Müller's theorem shows that in some sense the decomposition of $\hat{R}_N$ as a sum of indecomposable rings reflects the decomposition of $\{P_1, \ldots, P_n\}$ into clans. In a similar vein, Lemma 1.12 shows that if R is a direct sum of rings, $R = R_1 \oplus \ldots \oplus R_n$, then in some sense $\hat{R}_N$ also reflects that direct sum decomposition. As in section 2, let $R=R_1 \oplus R_2$, $N= N_1 \oplus N_2$ where $N_i = N \cap R_i$.

Lemma 1.12

The NR_N-adic topology on R_N coincides with the product topology induced by the $N_i {R_i}_{N_i}$-adic topologies on R_1, R_2 respectively.

Proof:

A typical basic open neighbourhood of 0 in the NR_N-adic topology on R_N is some power $N^s R_N = (NR_N)^s$. A typical element of such a neighbourhood is a finite sum of elements of the form $r_1 r_2 \ldots r_s$ where each $r_j \in NR_N$. For each j, let $r_j = r_{1j} + r_{2j}$ with $r_{ij} \in {R_i}_{N_i}$. Since ${R_1}_{N_1} \cap {R_2}_{N_2} = 0$ and because of the way each is made into an R_N-R_N bimodule, it is easy to see that $r_1 r_2 \ldots r_s = r_{11} r_{12} \ldots r_{1s} + r_{21} r_{22} \ldots r_{2s}$ which is an element of $N_1 {R_1}_{N_1}{}^s + N_2 {R_2}_{N_2}{}^s$. i.e. we have shown $NR_N{}^s \subseteq N_1 {R_1}_{N_1}{}^s \oplus N_2 {R_2}_{N_2}{}^s$. The reverse inclusion is obvious. It follows that the NR_N-adic topology is contained in the product topology.

Conversely, a basic product-topology neighbourhood of the form $N_1 R_{1N_1}^{s_1} \oplus N_2 R_{2N_2}^{s_2}$ contains $N_1 R_{1N_1}^{s} \oplus N_2 R_{2N_2}^{s} = NR_N^{s}$ for $s \geq s_1, s_2$. Hence the product topology is contained in the NR_N-adic.

Corollary

$\hat{R}_N \cong \hat{R}_{1N_1} \oplus \hat{R}_{2N_2}$ where $\hat{R}_{iN_i}$ is understood to be the completion of R_{iN_i} in the $N_i R_{iN_i}$-adic topology and both $N_i \neq R_i$. If $N_2 = R_2$, then $\hat{R}_N \cong \hat{R}_{1N_1}$.

Proof:

Assume $N_i \neq R_i$ for i = 1, 2. First we show every Cauchy sequence in R_N has a limit in $\hat{R}_{1N_1} \oplus \hat{R}_{2N_2}$. If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in R_N, write $x_n = x_{1n} + x_{2n}$ with $x_i \in R_{iN_i}$. Given s, for sufficiently large n and m, $x_n - x_m \in NR_N^{s}$, hence $x_{in} - x_{im} \in N_i R_{iN_i}^{s}$. Thus $\{x_{1n}\}_{n=1}^{\infty}$ and $\{x_{2n}\}_{n=1}^{\infty}$ are Cauchy sequences with limits x_1 and x_2 in $\hat{R}_{1N_1}$ and $\hat{R}_{2N_2}$ respectively. Clearly $x = x_1 + x_2 = \lim_{n\to\infty} x_n$. Hence $\hat{R}_N \subseteq \hat{R}_{1N_1} \oplus \hat{R}_{2N_2}$. The reverse inclusion is trivial.

If $N_2 = R_2$, the result is obvious.

It turns out that $\hat{R}_N$ coincides with the bicommutator of E = E(R/N) [19]. Since E is N-torsion free divisible, there is a unique way of making E an R_N-module and because of the way this is done, it is easy to verify that E is an H-R_N-bimodule where H = $\mathrm{End}_R(E)$. R_N is then naturally embedded in S = Bic(E) = $\mathrm{End}_H(E)$. There are three topologies on R_N: the NR_N-adic described above; the E-adic, whose basic open neighbourhoods of 0 are of the form ker(f) where $f: R_N \longrightarrow E^n$ for some n; and the finite, whose basic open neighbourhoods of 0 are of the form $\{q \in R_N \mid i_1 q = \ldots = i_n q = 0\}$ for some set $\{i_1, \ldots, i_n\} \subseteq E$. By Lambek [19, Proposition 3], the finite topology agrees with the E-adic on S, hence on R_N. By Lambek and Michler [21 , Proposition 4.3], since N is localizable and classical, the E-adic topology agrees with the NR_N-adic topology on R_N.

Lemma 1.13

Every N-torsion free factor module of $(R_N)_R$ is divisible.

Proof:

(i): Every N-torsion free factor module of $(R_N)_R$ is an R_N-module. Indeed, let $f: R_N \longrightarrow M$ be an R-epimorphism. Let $h: R \longrightarrow R_N$ define the ring of quotients of R at N. Define $mh(a)h(c)^{-1} = f(qh(a)h(c)^{-1})$ where $f(q) = m$. This is well defined since if $f(q') = 0$ and $f(q'h(a)h(c)^{-1}) \neq 0$ for some $h(a)h(c)^{-1}$ then $f(q'h(a)) \neq 0$ since M is N-torsion free and N is localizable. Hence $f(q')h(a) \neq 0$ so $f(q') \neq 0$. It is straightforward to check that this makes M into an R_N-module.

(ii): Every N-torsion free R_N-module is divisible as an R-module: Since every element of $\mathcal{C}(N)$ becomes invertible in R_N we have $\forall D \in \mathcal{D}_N$, $h(D)R_N = R_N$. Let M be an R_N-module which is N-torsion free as an R-module and suppose $f \in Hom_R(D, M)$ for some $D \in \mathcal{D}_N$. Extend f to $g: h(D)R_N \longrightarrow M$ by defining

$g(h(d)h(c)^{-1}) = f(d)h(c)^{-1}$. $h(d)h(c)^{-1} = 0 \Rightarrow h(d) = 0 \Rightarrow$ $\exists\, c' \in \mathscr{C}(N)$ such that $dc' = 0 \Rightarrow f(d)c' = 0 \Rightarrow f(d) \in T_N(M) = 0$. Hence g is well defined. Since $h(1) \in h(D)R_N$, $g(h(1))$ is defined. Define an extension $\tilde{f}$ of f by $\tilde{f}(r) = g(h(r))\ \forall r \in R$.

<u>Corollary</u>

R_N is dense in S.

<u>Proof</u>:

The conditions of Lambek [19, Prop. 2] are satisfied.

It follows immediately that $\hat{R}_N = S = Bic(E)$. Notice that if I is any injective R-module which is embedded in a finite direct sum of copies of E and which also cogenerates the N-torsion theory, then the same arguments show that $\hat{R}_N = Bic(I)$.

Chapter 2

§1. Indecomposable injectives over bounded HNP rings

Assume now that R is a bounded, not primitive, HNP ring. Then R is fully bounded. Let I_1 be an indecomposable injective R-module and let $P_1 = \mathrm{Ass} I_1 \neq 0$. By Proposition 1.10, P_1 belongs to a clan $\{P_1, P_2, \dots, P_n\}$ whose intersection is an invertible semiprime ideal N. Let I_i be the unique (up to isomorphism) indecomposable injective with associated prime ideal P_i. The main theorem of this section concerns the structure of each I_i.

Lemma 2.1

Each $\mathrm{Ann}_{I_i} N$ is simple and non-zero.

Proof:

Let M_i be a maximal right ideal of R containing P_i. Then $I_i \cong E(R/M_i)$. Since $(R/M_i)N = 0$, $\mathrm{Ann}_{I_i} N \neq 0$. Since $\mathrm{Ann}_{I_i} N$ is an R/N-module and R/N is semisimple Artinian, $\mathrm{Ann}_{I_i} N$ is a direct sum of simple R-modules. But I_i is uniform. Hence $\mathrm{Ann}_{I_i} N$ must be simple.

Lemma 2.2

(a) $I_i = \bigcup_{n=1}^{\infty} \mathrm{Ann}_{I_i} N^n$

(b) $\mathrm{Ann}_{I_i} N^m \subsetneqq \mathrm{Ann}_{I_i} N^{m+1}$ for all $m \geqslant 0$.

Proof:

(a) follows immediately from Theorem 1.5

(b): Suppose $\mathrm{Ann}_{I_i} N^m = \mathrm{Ann}_{I_i} N^{m+1}$. Then for all $s \geqslant m+1$, $(\mathrm{Ann}_{I_i} N^s)(N^{s-m-1})(N^{m+1}) = 0$. Therefore,

$$(\mathrm{Ann}_{I_i} N^s)(N^{s-m-1}) \subseteq \mathrm{Ann}_{I_i} N^{m+1} = \mathrm{Ann}_{I_i} N^m$$

$$(\mathrm{Ann}_{I_i} N^s)(N^{s-m-1})N^m = 0$$

$$(\mathrm{Ann}_{I_i} N^s)(N^{s-1}) = 0$$

ETC.

Proceeding in this fashion one shows $I_i N^m = 0$. By Goldie's theorem, let $x \in N^m$ be a regular element. Given any $0 \neq j \in I_i$ map $xR \longrightarrow I_i$ by $xr \longmapsto jr$. Extend this to a map $h: R \longrightarrow I_i$ and let $h(1) = y$. Then $yx = 0$ because $yx \in I_i N^m$. But $yx \neq 0$ because $yx = h(x) = j$, contradiction!

Lemma 2.3

(a) The submodules of I_i are linearly ordered (cf. Singh [38, theorem 4])

(b) $I_i/\mathrm{Ann}_{I_i} N$ is an indecomposable injective R-module isomorphic to one of $\{I_j\}_{j=1}^{n}$.

(c) $\{I_1, \ldots, I_n\} = \{I_1/\mathrm{Ann}_{I_1} N, I_2/\mathrm{Ann}_{I_2} N, \ldots, I_n/\mathrm{Ann}_{I_n} N\}$.

Proof:

(a): It is enough to show that if $x, y \in I_i$ then $xR \subseteq yR$ or $yR \subseteq xR$. The ring $\overline{R} = R/\mathrm{Ann}(xR+yR)$ is a proper factor ring of R, hence is serial (Eisenbud and Griffith [10]) which implies that the $\overline{R}$-module $xR+yR$ is a direct sum of uniserial modules. But $xR+yR \subseteq I_i$ is uniform. Therefore,

$xR+yR$ is uniserial and either $xR \subseteq yR$ or $yR \subseteq xR$.

(b): By (a), $Ann_{I_i}N$ is meet-irreducible. Hence $I_i/Ann_{I_i}N$ is uniform. But over a right hereditary ring, factor modules of injective modules are injective. Hence $I_i/Ann_{I_i}N$ is an indecomposable injective. Since $(Ann_{I_i}N^2/Ann_{I_i}N)N = 0$ and $\{P_1, \ldots, P_n\}$ are the only primes containing N, the associated prime ideal of $I_i/Ann_{I_i}N$ must be P_j for some $j \in \{1, \ldots, n\}$. Since R is FBN, $I_i/Ann_{I_i}N \cong I_j$.

(c): By (b), $I_1/Ann_{I_1}N \cong I_1$ or I_j for some $j \neq 1$. In the second case, re-number if necessary so that $I_1/Ann_{I_1}N \cong I_2$. By (b) again, $I_2/Ann_{I_2}N \cong I_1$ or I_2 or (possible after re-numbering) I_3. Thus it is possible to re-number the I_j so that:

$I_1/Ann_{I_1}N \cong I_2$, $I_2/Ann_{I_2}N \cong I_3, \ldots, I_s/Ann_{I_s}N \cong I_1$ for some $s \leq n$. Suppose $s \neq n$. We shall show that $S = P_1 \cap \ldots \cap P_s$ is localizable by showing that condition (b) of Theorem 1.3 is satisfied. A uniform R/S-right ideal U/S is certainly a uniform R-module. Hence $E_R(U/S)$ is an indecomposable injective. Since $(U/S)S = 0$, the associated prime ideal of $E_R(U/S)$ is one of $P_1, P_2, \ldots, P_s$. Hence $E_R(U/S) \cong I_j$ for some $j \in \{1, 2, \ldots, s\}$. U/S is (isomorphic to) a finitely generated R-submodule of I_j so by (a), $0 \neq U/S \cong yR \subseteq I_j$. If U/S is an essential and dense submodule of a cyclic R-module xR, we may assume $x \in I_j$. Now $yS = 0$ implies $yN = 0 \Rightarrow yR = Ann_{I_j}N$ by Lemma 2.1. If $xR \neq yR$ then, by (b), $I_j/Ann_{I_j}N \supseteq xR/yR \supseteq Ann_{I_j}N^2/Ann_{I_j}N$. Hence $0 \neq xR/yR$ is N-torsion free, contradiction. It follows that $xR = yR$ and $xRS = 0$. Thus the condition for right localizability is satisfied and this implies S is invertible. Since $\{P_1, \ldots, P_n\}$ is a clan, $n = s$ and (c) is proved.

Theorem 2.4

Let R be a bounded HNP ring, I_1 the indecomposable injective with associated prime ideal $P_1 \neq 0$. Let P_1 belong to the clan $\{P_1, \ldots, P_n\}$ and $N = \bigcap_{i=1}^{n} P_i$. Then I_1 is the union of submodules $0 \subsetneq B_1 \subsetneq B_2 \subsetneq \ldots \subsetneq B_n \subsetneq B_{n+1} \subsetneq \ldots$ where

(a) $B_i = \mathrm{Ann}_{I_1} N^i$;

(b) each B_i/B_{i-1} is simple;

(c) each B_i is cyclic;

(d) there are no other submodules of I_1;

(e) $B_i/B_{i-1} \cong B_j/B_{j-1}$ iff $i \equiv j \pmod n$.

Note: S. Singh has obtained a similar theorem but the present proof was obtained independently.

Proof:

(a): If $B_i = \mathrm{Ann}_{I_1} N^i$, we know $I_1 = \bigcup_{n=1}^{\infty} B_n$ and $\forall i$, $B_i \subsetneq B_{i+1}$.

(b) follows from the fact that B_i/B_{i-1} is an R/N-module, therefore a finite direct sum of simple R-modules. But at the same time, B_i/B_{i-1} is a submodule of one of the I_j, and these are all uniform.

(c) and (d): We proceed by induction. Any $0 \neq y \in B_1$ must generate B_1 by Lemma 2.1. Assume $B_{i-1} = y_{i-1}R$ and select any $y_i \in B_i \setminus B_{i-1}$. By Lemma 2.3(a), since $y_iR \not\subseteq y_{i-1}R$, we

must have $y_{i-1}R \subseteq y_iR$. $y_iR = B_i$ now follows from (b).
(e) follows from Lemma 2.3(c). Indeed $B_i/B_{i-1} \cong B_j/B_{j-1}$ iff $I_1/B_{i-1} \cong I_1/B_{j-1}$ and this happens iff $i \equiv j \pmod n$.

Corollary 1

Let I be any finite direct sum of copies of the I_i such that each I_i appears at least once. Then I is Artinian. I is also a self-cogenerator — i.e. every submodule, C, of a factor module of some I^m is cogenerated by I. In particular, $Hom_R(C, I) \neq 0$.

Proof:

That I is Artinian follows immediately from the theorem. I^m is a direct sum $\bigoplus_{k=1}^{m'} E_k$ where $E_k \cong I_i$ for some $i \in \{1, 2, \ldots, n\}$. If $Y \leqslant I^m$ then I^m/Y is injective since R is hereditary. It is also Artinian, hence is a finite direct sum of indecomposable injectives: say $I^m/Y = I_1' \oplus \ldots \oplus I'_t$. Let $\pi_j : I^m/Y \longrightarrow I_j'$ be the canonical projection and $\varkappa_k : E_k \longrightarrow I^m$ the canonical injection with $p : I^m \longrightarrow I^m/Y$ representing the canonical surjection. If $\pi_1 p \varkappa_k = 0 \ \forall k$ then $\pi_1 p = 0 \therefore \pi_1 = 0$. Hence $\exists k$ such that $Hom_R(E_k, I'_1) \neq 0$. Since I'_1 is indecomposable, since any homomorphic image if E_k is injective, and since the only submodules of E_k are the $Ann_{E_k} N^s$ $\exists s$ $I'_1 \cong E_k/Ann_{E_k} N^s \cong I_i$ for some $i \in \{1, 2, \ldots, n\}$. Similarly each $I'_j \cong I_{i(j)}$ for some $i(j) \in \{1, 2, \ldots, n\}$. Hence $I^m/Y \hookrightarrow I^x$ for some $x \in \mathbb{N}$. It follows that any submodule, C, of I^m/Y is cogenerated by I and

$Hom_R(C, I) \neq 0$.

Corollary 2

$\forall X \leqslant I_1$, $Hom_R(I_1/X, I_1) \neq 0$.

Proof:

By the theorem, $X = Ann_{I_1} N^s$ for some s, hence $I_1/X \cong I_i$ for some $i \in \{1, 2, \ldots, n\}$. Then $I_i/Ann_{I_i} N^t \cong I_1$ for some t and the composite $I_1/X \cong I_i \longrightarrow I_i/Ann_{I_i} N^t \cong I_1$ is a non-zero homomorphism: $I_1/X \longrightarrow I_1$.

These results can be extended to fully bounded Noetherian hereditary rings as long as we restrict our attention to non-minimal prime ideals. Recall that in a Noetherian hereditary ring, any chain of prime ideals consists of at most two elements [2].

Lemma 2.5

Let R be a Noetherian hereditary ring and assume

$$R = R_1 \oplus R_2 \oplus \ldots \oplus R_m$$

where each R_i is either HNP or Artinian hereditary. Let $\{P_1, \ldots, P_n\}$ be a clan of prime ideals.

(a) If $P_1 \not\supseteq R_1$ then $P_i \not\supseteq R_1$ for all $i = 1, 2, \ldots, n$.

(b) If P_1 is a minimal prime, so are all the P_i.

(c) If P_1 is a maximal ideal, so are all the P_i.

(d) If P_1 is both minimal and maximal, so are all the P_i.

Proof:

(a): Suppose $P_1 \not\supseteq R_1$. Let $R' = R_2 \oplus \ldots \oplus R_m$. By the corollary to Lemma 1.7, $N \cap R_1 = \bigcap_{i=1}^{n}(P_i \cap R_1)$ is localizable. Suppose that only $P_1, P_2, \ldots, P_s \not\supseteq R_1$. Then each contains R', so $\bigcap_{i=1}^{n}(P_i \cap R_1) = \bigcap_{i=1}^{s}(P_i \cap R_1)$ is localizable and $\bigcap_{i=1}^{s}(P_i \cap R')$ = R'. By Lemma 1.7, $\{P_1, \ldots, P_s\}$ is localizable and by Theorem 1.11, it follows that s = n.

(b): Now assume P_1 is minimal and suppose $P_2 \supsetneqq Q$ where Q is a prime ideal. Since $P_2 \not\supseteq R_1$, $\therefore Q \not\supseteq R_1$ $\therefore Q \supseteq R'$. Hence $P_2 = (P_2 \cap R_1) \oplus R'$ and $Q = (Q \cap R_1) \oplus R'$ $\Rightarrow$ $P_2 \cap R_1 \supsetneqq Q \cap R_1$. Since all primes of an Artinian ring are maximal, it follows that R must be HNP and in that case, $Q \cap R_1 = 0$ because all non-zero primes of an HNP ring are maximal. But then $P_1 \cap R_1 \supseteq Q \cap R_1$ $\therefore P_1 = (P_1 \cap R_1) \oplus R' \supseteq Q$. By minimality of P_1, $P_1 = Q \subsetneqq P_2$. This contradicts the irredundancy of the set $\{P_1, \ldots, P_n\}$.

(c): Assume P_1 is maximal and suppose $P_2 \subsetneqq Q$ where Q is a (proper) prime ideal of R. Since $P_2 \supseteq R'$ $\therefore Q \supseteq R'$ $\quad Q \not\supseteq R_1$. Then in R_1 we have primes $P_2 \cap R_1 \subsetneqq Q \cap R_1$ and this forces R_1 to be HNP and $P_2 \cap R_1 = 0$ as in (b). Then $P_2 = R' \subseteq P_1$ which again contradicts the irredundancy.

(d) follows immediately from (b) and (c).

Since we are assuming $R = R_1 \oplus \ldots \oplus R_m$ is a fully bounded Noetherian hereditary ring, a non-minimal prime ideal P_1 will contain all the direct summands to R except for (say) R_1, and $P_1 \cap R_1$ must be a non-zero prime of R_1. Hence R_1 must be bounded HNP. If P_1 belongs to the clan

$\{P_1, P_2, \ldots, P_n\}$ and $N = \bigcap_{i=1}^{n} P_i$, we know $E_R(R/N) \cong E_{R_1}(R_1/N_1)$ (Prop. 1.6). Applying Lemma 2.3 to $E_{R_1}(R_1/N_1)$ we conclude that if I_1 is the indecomposable injective R-module with associated prime ideal P_1 it is also the indecomposable injective R_1-module with associated prime ideal $P_1 \cap R_1 \neq 0$, so has as only R_1-submodules the chain

$0 \subsetneqq \mathrm{Ann}_{I_1} N \cap R_1 \subsetneqq \ldots \subsetneqq \mathrm{Ann}_{I_1} (N \cap R_1)^n \subsetneqq \mathrm{Ann}_{I_1} (N \cap R_1)^{n+1} \subsetneqq \ldots.$

As R-modules, these coincide with the chain

$0 \subsetneqq \mathrm{Ann}_{I_1} N \subsetneqq \ldots \subsetneqq \mathrm{Ann}_{I_1} N^n \subsetneqq \mathrm{Ann}_{I_1} N^{n+1} \subsetneqq \ldots.$ The arguments used in Lemma 2.3(b) and (c) still work because Lemma 2.5 assures that all the P_i are non-minimal and because over a FBN ring, every localizable semiprime ideal is classical [32]. Hence we have

<u>Theorem 2.6</u>

Let R be a FBN hereditary ring and P_1 a non-minimal prime ideal of R. Let I_1 be the indecomposable injective with associated prime ideal P_1. Then P_1 belongs to a clan $\{P_1, P_2, \ldots, P_n\}$ of non-minimal prime ideals whose intersection is a localizable (classical) semiprime ideal N. The only submodules of I_1 are $0 \subsetneqq \mathrm{Ann}_{I_1} N \subsetneqq \ldots \subsetneqq \mathrm{Ann}_{I_1} N^n \subsetneqq \ldots$; $I_1 = \bigcup_{n=1}^{\infty} \mathrm{Ann}_{I_1} N^n$; each factor $\mathrm{Ann}_{I_1} N^n / \mathrm{Ann}_{I_1} N^{n+1}$ is simple; each $\mathrm{Ann}_{I_1} N^n$ is cyclic; and the factors $\mathrm{Ann}_{I_1} N^{s+1} / \mathrm{Ann}_{I_1} N^s$ and $\mathrm{Ann}_{I_1} N^{r+1} / \mathrm{Ann}_{I_1} N^r$ are isomorphic iff $s \equiv r \pmod{n}$.

<u>Corollary</u>

If I_i is the indecomposable injective R-module with associated prime ideal P_i and $I = I_1^{s_1} \oplus \ldots \oplus I_n^{s_n}$ for some $s_i \geqslant 1$, the corollaries to Theorem 2.4 still hold.

§2. Duality in FBN hereditary rings

<u>Definition</u>: Consider arbitrary rings S and T and a bimodule ${}_S I_T$. For any T-module M (resp. S-module M), define $M^* = \mathrm{Hom}_T(M, I)$ ($M^* = \mathrm{Hom}_S(M, I)$). There is a natural homomorphism $M \xrightarrow{\wedge} M^{**}$ where $(f)\hat{m} = f(m)$. Call an S-module M (a T-module M_T) <u>reflexive</u> if $M^{**} \cong M$. If $S = \mathrm{End}_T(I_T)$, $T = \mathrm{End}_S({}_S I)$ and ${}_S I$ and I_T are injective cogenerators of S-Mod and Mod-T respectively, then the functors $\mathrm{Hom}_S(\_, I)$ and $\mathrm{Hom}_T(\_, I)$ induce a duality between the categories of reflexive S-modules and reflexive T-modules. We say that ${}_S I_T$ induces a <u>Morita duality</u> between S and T and call them <u>Morita</u> rings. The reflexive subcategories are closed under submodules and factor modules and contain ${}_S S$ and T_T respectively [29].

Let R be a fully bounded Noetherian hereditary ring and N a localizable semiprime ideal which is an intersection of non-minimal prime ideals $P_1, \ldots, P_n$. Let I_i be the unique (up to isomorphism) indecomposable injective with associated prime P_i and let $I_o = \overset{n}{\underset{i=1}{\oplus}} I_i$. Denote by I any finite direct sum of copies of the I_i such that $I_o \leq I$ — i.e. $I = I_1^{s_1} \oplus \ldots \oplus I_n^{s_n}$ for some $s_i \geqslant 1$. Let $K_o = \mathrm{End}_R(I_o)$ and $K = \mathrm{End}_R(I)$. Let R_N be the localization of R at N and

$\hat{R}_N$ the completion of R_N in the NR_N-adic topology. I can be viewed as an R_N-module in the following way: if $c \in \mathscr{C}(N)$ then cR is N-dense. It is then easy to check that if g extends the map $f(cr) = ir$ to all of R, $ic^{-1} = g(1)$ is well defined. Now any element $q \in \hat{R}_N$ can be thought of as $q = \lim_{n\to\infty} q_n$ where $q_n \in R_N$. Given $0 \neq i \in I$, since N is classical, $\exists\, n_0$ such that $iN^n = iNR_N{}^n = 0\ \forall n \geqslant n_0$. Since $q = \lim_{n\to\infty} q_n$, $\exists\, n_1$ such that $q_n - q_m \in NR_N{}^{n_0}\ \forall m, n \geqslant n_1$. Clearly $i\hat{q} = iq_{n_1} = iq_n$ $\forall n \geqslant n_1$, and $\mathrm{Hom}_{\hat{R}_N}(I, I) = \mathrm{Hom}_{R_N}(I, I) = \mathrm{Hom}_R(I, I) = K$. By the remarks following Lemma 1.13, $\hat{R}_N = \mathrm{End}_K({}_K I)$. Hence one of the conditions for a Morita duality between K-Mod and Mod-$\hat{R}_N$ is satisfied.

<u>Proposition 2.7</u>

${}_K I$ is an injective cogenerator for K-Mod.

<u>Proof</u>:

<u>Step 1</u>: I is semi-injective (Sandomierski [37]).

Let B be a finitely generated left ideal of K. For some m there is a surjection $K^m \longrightarrow B \longrightarrow 0$. Let $K^m \xrightarrow{\alpha} (K^m)^{**}$ and $B \xrightarrow{\beta} B^{**}$ be the natural homomorphisms. Since the sequence $0 \longrightarrow B^* \longrightarrow (K^m)^*$ is exact and I_R is injective, the following diagram is commutative with exact rows:

$$\begin{array}{ccccc} K^m & \longrightarrow & B & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \\ (K^m)^{**} & \longrightarrow & B^{**} & \longrightarrow & 0 \end{array}$$

But α is an isomorphism, hence β is an epimorphism. On the other hand, ${}_K B$ is cogenerated by I — indeed $B \subseteq K = I^* \subseteq I^I$ — and it follows that β is a monomorphism. Hence $B \cong B^{**}$. To show ${}_K I$ is semi-injective, it is enough to show that $K^* \longrightarrow B^* \longrightarrow 0$ is exact in Mod-R. If not, let C be such that $K^* \longrightarrow B^* \longrightarrow C \longrightarrow 0$ is exact. Then

$$\begin{array}{ccccccc} 0 \longrightarrow & C^* & \longrightarrow & B^{**} & \longrightarrow & K^{**} \\ & & & \wr\| & & \wr\| \\ & & 0 \longrightarrow & B & \longrightarrow & K \end{array}$$

has exact rows and commutes. Hence $C^* = \mathrm{Hom}_R(C, I) = 0$. But C_R is a factor module of a submodule, B^*, of $(K^m)^* = I^m$. Since I_R is a self-cogenerator, $C = 0$.

<u>Step 2</u>: I is injective as a K-module. Let B be any left ideal of K, $g \in \mathrm{Hom}_K(B, I)$. Let $\{B_j\}_{j \in J}$ be the family of all finitely generated submodules of B. Each $g|_{B_j}$ has an extension to $g_j \in \mathrm{Hom}_K(K, I)$. Now $\mathrm{Ann}_I B = \mathrm{Ann}_I(\sum_{j \in J} B_j) = \bigcap_{j \in J} \mathrm{Ann}_I B_j = \bigcap_{j \in F} \mathrm{Ann}_I B_j$ for some finite subset $F \subseteq J$ since I_R is Artinian. Hence $\mathrm{Ann}_I B = \bigcap_{j \in F} \mathrm{Ann}_I B_j = \mathrm{Ann}_I(\sum_{j \in F} B_j) = \mathrm{Ann}_I B_{j_0}$ for some $j_0 \in J$. Clearly, if $B_j \supseteq B_{j_0}$, then since g_j and g_{j_0} both agree with g on B_{j_0}, we have

$$B_{j_0}((1)g_j - (1)g_{j_0}) = 0.$$

But $\mathrm{Ann}_I B_{j_0} = \mathrm{Ann}_I B \subseteq \mathrm{Ann}_I B_j \subseteq \mathrm{Ann}_I B_{j_0}$. Thus $B_j((1)g_j - (1)g_{j_0}) = 0$. i.e. $g_j|_{B_j} = g_{j_0}|_{B_j}$. Since $B = \sum_{B_j \supseteq B_{j_0}} B_j$, it follows that $g_{j_0}|_B = g$.

<u>Step 3</u>: Every simple K-module is cogenerated by ${}_K I$. It is sufficient to show that if L is a proper maximal left ideal of K then $\mathrm{Hom}_K(K/L, I) \neq 0$. Now $f \longleftrightarrow f[1]$ defines an

isomorphism between $Hom_K(K/L, I)$ and $Ann_I L$. $Ann_I L = 0$ $\Rightarrow \bigcap_{f \in L} ker(f) = 0 \Rightarrow \bigcap_{i=1}^{t} ker(f_i) = 0$ for some $f_i \in L$ since I_R is Artinian. Hence $m: I \longrightarrow I^t$ defined by $m(y) = (f_1(y), f_2(y), \ldots, f_t(y))$ is a monomorphism. Let $j = (j_1, j_2, \ldots, j_t)$ be a map from I^t to I such that $jm = id$ ($j_i \in K$). For any $y \in I$ we have $y = jm(y) = \sum_{i=1}^{t} j_i f_i(y)$. Hence $1 = \sum_{i=1} j_i f_i \in L$, contradiction. Therefore $Ann_I L \neq 0$ and $Hom_K(K/L, I) \neq 0$.

Note that step 3 really shows that if I_R is any Artinian injective module then every simple K-module is contained in I. If additionally $_K I$ is injective, it is an injective cogenerator. We shall see that the conclusion of Proposition 2.7 is also true for I_1 and $K_1 = End_R(I_1)$. We first need the following Lemma:

Lemma 2.8

K_1 is a local domain. $J(K_1) = K_1 q_1$ for some $q_1 \in K_1$ and the only left ideals of K_1 are $K_1 q_1^m \ \forall m$.

Proof:

K_1 is local since it is the endomorphism ring of an indecomposable injective. For any $f \in End_R(I_1)$, $f(I_1)$ is an injective $\subseteq I_1$. Hence $f(I_1) = I_1$. If $gf = 0$ and $f \neq 0$, then $g(I_1) = gf(I_1) = 0 \Rightarrow g = 0$. Hence K_1 is a domain. If $f \in K_1$ is not an isomorphism, it is not a monomorphism; therefore $ker(f) \supseteq Ann_{I_1} N$, hence f induces a map $f^{(1)}: I_1/Ann_{I_1} N \longrightarrow I_1$.

Since $\mathrm{Ann}_{I_1} N^2/\mathrm{Ann}_{I_1} N$ is simple and not isomorphic to $\mathrm{Ann}_{I_1} N$, $\ker(f^{(1)}) \supseteq \mathrm{Ann}_{I_1} N^2/\mathrm{Ann}_{I_1} N$ so induces a map $f^{(2)}: I_1/\mathrm{Ann}N^2 \longrightarrow I_1$. Proceeding in this way, we see that in fact, $\ker(f) \supseteq \mathrm{Ann}_{I_1} N^n$. Hence, if $q_1: I_1 \longrightarrow I_1/\mathrm{Ann}_{I_1} N^n \cong I_1$ denotes the composite, $\ker(f) \supseteq \ker(q_1)$. We may therefore define a map $h: I_1 \longrightarrow I_1$ by $h(q_1(y)) = f(y)$ $\forall y \in I_1$. Thus we have proved $J(K_1) = K_1 q_1$. Given any $b \in K_1$, if b is not already a unit, we can write $b = uq_1{}^m$ for some unit $u \in K_1$ and some $m \in \mathbb{N}$. Hence $K_1 b = K_1 q_1{}^m$. The proof is now complete.

Lemma 2.9

I_1 is injective as a K_1-module.

Proof:

If L is any left ideal of K_1, by Lemma 2.8, $L = K_1 q_1{}^m \cong K_1$ for some m. To show ${}_{K_1}I_1$ is injective, we need to show $\mathrm{Hom}_{K_1}(K_1, I_1) \longrightarrow \mathrm{Hom}_{K_1}(L, I_1) \longrightarrow 0$ is exact. If it is not, let $K_1{}^* \longrightarrow L^* \longrightarrow C \longrightarrow 0$ be exact where $X^* = \mathrm{Hom}_{K_1}(X, I_1)$. Then as in the proof of Proposition 2.7, $C^* = 0$. But C is a factor module of $L^* \cong K_1{}^* \cong I_1$. By the Corollary to Theorem 2.6, $\mathrm{Hom}_R(C, I_1) = C^* \neq 0$, contradiction.

Corollary

Over an FBN hereditary ring R, if I_1 is an indecomposable injective with non-minimal associated prime ideal P_1 and $K_1 = \mathrm{End}_R(I_1)$, then ${}_{K_1}I_1$ is an injective cogenerator.

Lemma 2.10 (cf. Matlis [26])

With the same assumptions on R, for any submodule $B_R \leq I_R$

(a) $\mathrm{Ann}_K B \cong \mathrm{Hom}_R(I/B, I)$,

(b) $K/\mathrm{Ann}_K B \cong \mathrm{Hom}_R(B, I)$,

(c) $\mathrm{Ann}_I \mathrm{Ann}_K B = B$.

Proof:

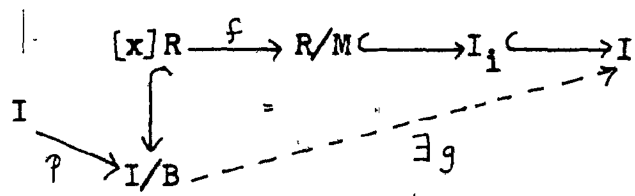
(a): Define $\varphi : \mathrm{Ann}_K B \longrightarrow \mathrm{Hom}_R(I/B, I)$ by $\varphi(k)[i] = ki$. Conversely, given $f \in \mathrm{Hom}_R(I/B, I)$, if $p : I \longrightarrow I/B$ is the canonical projection, define $\psi(f) = fp \in \mathrm{Ann}_K B$. We have $\varphi(fp)[i] = fpi$ and $\psi(\varphi(k)) = \varphi(k)p = k \ \forall k \in \mathrm{Ann}_K B$, and clearly, φ and ψ are K-homomorphisms.

(b): Since I_R is injective, the following diagram has exact rows and commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(I/B, I) & \longrightarrow & \mathrm{Hom}_R(I, I) & \longrightarrow & \mathrm{Hom}_R(B, I) & \longrightarrow & 0 \\ & & \wr\| & & \wr\| & & & & \\ 0 & \longrightarrow & \mathrm{Ann}_K B & \longrightarrow & K & \longrightarrow & K/\mathrm{Ann}_K B & \longrightarrow & 0 \end{array}$$

The two known isomorphisms induce an isomorphism between $K/\mathrm{Ann}_K B$ and $\mathrm{Hom}_R(B, I)$.

(c): Obviously $\mathrm{Ann}_I \mathrm{Ann}_K B \geq B$. If $x \in \mathrm{Ann}_I \mathrm{Ann}_K B \setminus B$, $\mathrm{Ann}_R [x]_B$ is contained in some maximal right ideal M of R. Since $xN^m = 0$ for some m, one of $P_1, \ldots, P_n \leq M$. Since R is fully bounded and the P_i are maximal, $E(R/M) \cong I_i$ for some $i \in \{1, 2, \ldots, n\}$. Consider



where $f([x]_B r) = [r]_M$ and $g|_{[x]R} = f$. On the one hand, $gp(x) = fp(x) \neq 0$ but on the other, since $gp \in \mathrm{Ann}_K B$ and $x \in \mathrm{Ann}_I \mathrm{Ann}_K B$, $gp(x) = 0$, contradiction.

Similarly one proves:

<u>Lemma 2.11</u>

For any right ideal $A \leq R$,

(a) $\mathrm{Ann}_I A \cong \mathrm{Hom}_R(R/A, I)$,

(b) $I/\mathrm{Ann}_I A \cong \mathrm{Hom}_R(A, I)$,

(c) If $A \geq N^m$ for some m, $\mathrm{Ann}_R \mathrm{Ann}_I A = A$.

<u>Lemma 2.12</u>

For any left submodule ${}_K C \leq {}_K I$

(a) $\mathrm{Ann}_{\hat{R}_N} C \cong \mathrm{Hom}_K(I/C, I)$,

(b) $\hat{R}_N/\mathrm{Ann}_{\hat{R}_N} C \cong \mathrm{Hom}_K(C, I)$,

(c) $\mathrm{Ann}_I \mathrm{Ann}_{\hat{R}_N} C = C$.

<u>Lemma 2.13</u>

For any left ideal $L \leq K$

(a) $\mathrm{Ann}_I L \cong \mathrm{Hom}_K(K/L, I)$,

(b) $I/\mathrm{Ann}_I L \cong \mathrm{Hom}_K(L, I)$,

(c) $\mathrm{Ann}_K \mathrm{Ann}_I L = L$.

Consequently, there are one-one order-inverting correspondences between: (a) right ideals of R containing N^m

and left K-submodules of $\mathrm{Ann}_I N^m$ $\forall m$; and (b) left ideals of K and right R-submodules of I. In particular, $\forall m$ ${}_K(\mathrm{Ann}_I N^m)$ is Artinian and K is left Noetherian. Then in K-Mod, ${}_K I = \sum^{\oplus}_{\alpha \in A} Y_\alpha$ where $\forall \alpha$ Y_α is indecomposable injective. If A is infinite and $\forall i = \langle y_\alpha \rangle_{\alpha \in A} \in I$ (where all but finitely many y_α are zero) we let $\langle y \rangle e_\alpha = \langle 0, \ldots, y_\alpha, 0 \ldots \rangle$, $\{[e_\alpha]_{J(\hat{R}_N)}\}$ is an infinite set of orthogonal idempotents in $\hat{R}_N / J(\hat{R}_N) \cong R_N / NR_N$ which is a Noetherian ring — contradiction. Hence ${}_K I$ is a finite direct sum of indecomposable injectives, ${}_K I = Y_1 \oplus \ldots \oplus Y_s$. By the corollary to Lemma 1.9 and since for an HNP ring Q with invertible semiprime ideal S Q_S is HNP, R_N is a finite direct sum of HNP rings. Hence $\hat{R}_N / J(\hat{R}_N)^t \cong R_N / NR_N{}^t$ is serial $\forall t$. From this we see that the submodules of $e_i \hat{R}_N$ are linearly ordered $\forall i = 1, \ldots, s$. By Lemmas 2.12 and 2.13, the K-submodules of Y_i are also linearly ordered $\forall i$. Since $\forall i$, $\forall y \in Y_i$, Ky is Artinian (for $y \in \mathrm{Ann}_I N^m$ for some m), ${}_K I$ is Artinian. Applying the argument of Proposition 2.7 to ${}_K I$ one sees that $I_{\hat{R}_N}$ is an injective cogenerator of Mod-$\hat{R}_N$ and hence we have

Theorem 2.14

Let R be an FBN hereditary ring, N a localizable intersection of non-minimal prime ideals, I, K, etc. as before. Then ${}_K I_{\hat{R}_N}$ induces a Morita duality between K-Mod and Mod-$\hat{R}_N$.

Definition (Sandomierski [37]): Let X_R be an R-module (R any ring) and $\{X_j\}_{j \in J}$ a collection of submodules of X. Then the system of congruences $\{x \equiv x_j \pmod{X_j}\}$ is <u>finitely solvable</u> if for every finite subset $F \subseteq J$ $\exists x_F \in X$ such that $\forall j \in F$, $x_F - x_j \in X_j$. The system is solvable if $\exists x_0 \in X$ such that $x_0 - x_j \in X_j$ $\forall j \in J$. X is called <u>linearly compact</u> if every finitely solvable system of congruences in X is solvable.

It is known that in the presence of a Morita duality between rings S and T, the reflexive modules are exactly the linearly compact ones. In particular, in our present situation, $\hat{R}_N$ is right linearly compact and K is left linearly compact. Also there are dualities between Noetherian right $\hat{R}_N$-modules and Artinian left K-modules and between Artinian right $\hat{R}_N$-modules and Noetherian left K-modules.

In the following example, we see that such a Morita duality does not necessarily exist for all FBN hereditary rings.

<u>Example</u>: P. M. Cohn [8] has shown that there exist division rings $D_1 \subseteq D_2$ such that $[D_2 : D_1]_r = 2$ and $[D_2 : D_1]_\ell = \infty$.

Put $R = \begin{pmatrix} D_1 & 0 \\ D_2 & D_2 \end{pmatrix}$. R is an Artinian hereditary ring, hence is certainly FBN hereditary. As a left R-module,

$$\begin{pmatrix} 0 & D_2 \\ 0 & D_2 \end{pmatrix} = E_R \begin{pmatrix} 0 & 0 \\ 0 & D_2 \end{pmatrix}$$ is indecomposable injective.

But it is not finitely generated as a left R-module. Hence no injective cogenerator for R-Mod can be Noetherian. If R has a Morita duality with a ring S induces by ${}_RE_S$, then since R is Artinian, $J(R)^n = 0$ for some n. It follows that $J(S)^n = \mathrm{Ann}_S\mathrm{Ann}_E J(R)^n = 0$ (see Müller [29, Lemma 6]). Since the category of reflexive S-modules is closed under submodules and factor modules, $J(S)/J(S)^2$, $J(S)^2/J(S)^3, \ldots$ are reflexive and semi-simple. But reflexive modules must be finite dimensional (Sandomierski [37]). Hence S_S is Artinian and by the duality, ${}_RE$ is Noetherian, contradiction.

Lambek and Rattray have recently developed a categorical approach to duality which encompasses many classical duality theorems [20, 22, 23]. Their results are generally important as tools for finding new examples of duality. A concept fundamental to this approach is that of a co-small $\varkappa$-injective object:

Definition: Let $\underline{A}$ be a complete additive category, I an object of $\underline{A}$. Consider the functors

$$\underline{A} \underset{U_I = I^{-}}{\overset{F_I = \mathrm{Hom}(\_,\, I)}{\rightleftarrows}} \mathcal{S} = \underline{\mathrm{Sets}}$$

Let $(Q_I, \varkappa)$ be the equalizer of $\eta_I U_I F_I,\ U_I F_I \eta_I : U_I F_I \rightrightarrows (U_I F_I)^2$. I is called $\underline{\varkappa\text{-injective}}$ if $\forall f: Q_I(A) \longrightarrow I$ there exists $g: U_I F_I(A) \longrightarrow I$ such that $g\varkappa(A) = f$. I is <u>co-small</u> if F_I takes products in $\underline{A}$ to coproducts in $(E\text{-Mod})^{op}$ where $E = \underline{A}(I, I)$. Equivalently, I is co-small if $\forall f: \prod_{x \in X} A_x \longrightarrow I$ in $\underline{A}$, there exists a finite subset $F \subseteq X$ and $\exists f': \prod_{x \in F} A_x \longrightarrow I$

such that $f = f'\pi_F$ (where π_F is the canonical projection).

In Cont-R, the category whose objects are topological $\mathbb{Z}$-modules such that multiplication by elements of R is continuous and whose maps are continuous R-homomorphisms, any quasi-injective R-module equipped with the discrete topology is co-small and $\varkappa$-injective (Lambek [20, Prop. 4.3]). The following Proposition then applies to the situation we have been studying.

<u>Proposition 2.15</u> (Lambek and Rattray [23])

Let I be an injective Artinian right R-module with the discrete topology. Let $E = End_R(I)$. Then the adjoint pair

$$\text{Cont-R} \underset{U = Hom_E(\_,\ I)}{\overset{F = Cont_R(\_,\ I)}{\rightleftarrows}} (\text{E-Mod})^{op}$$

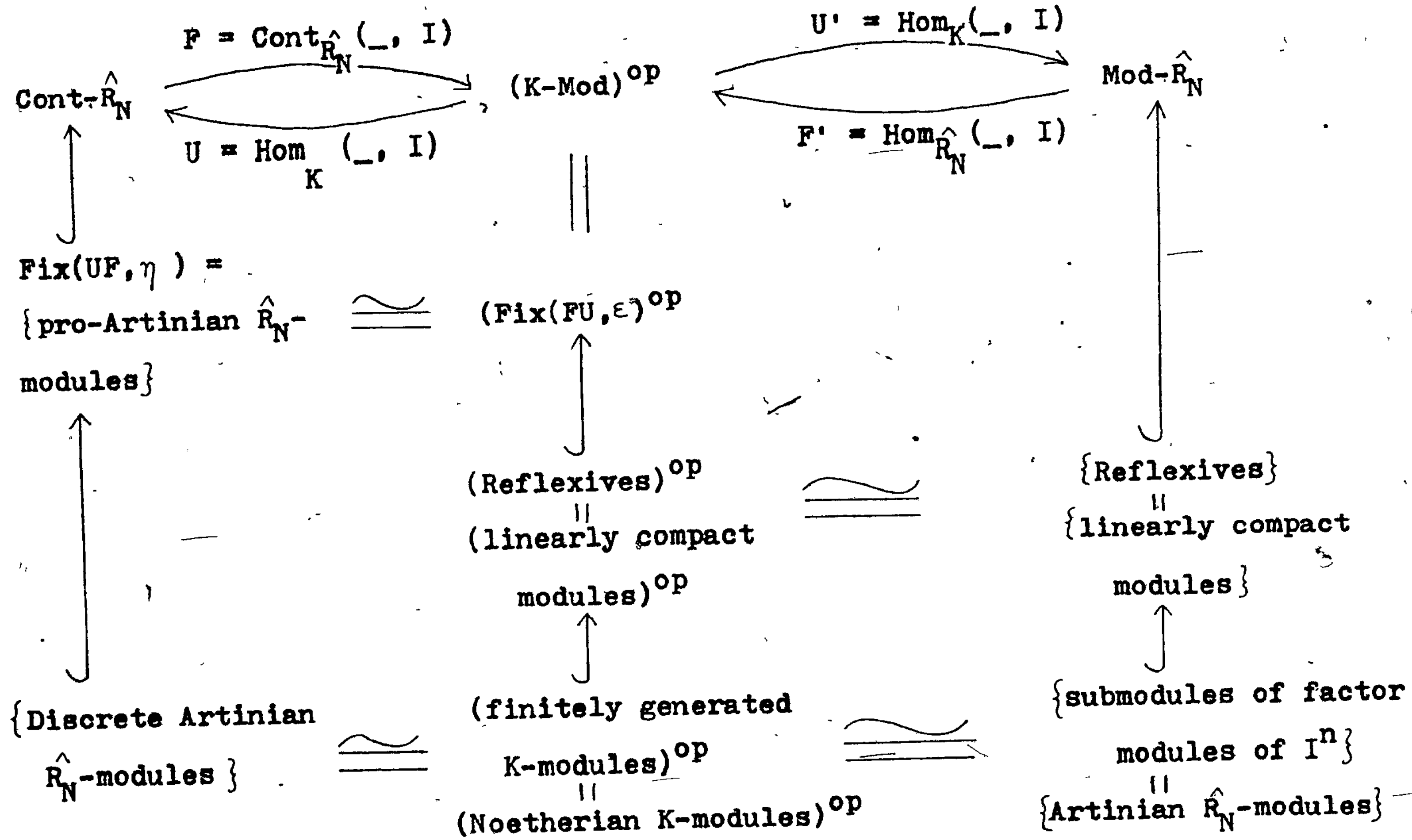
induces a duality between discrete Artinian R-modules which are I-torsion free divisible and finitely generated E-modules which are cogenerated by ${}_E I$. Moreover, for any $A \in$ Cont-R, the following are equivalent:

(a) A is a filtered limit of discrete Artinian modules which are I-torsion free divisible;

(b) A is a limit of discrete Artinian modules which are I-torsion free divisible;

(c) $A \in \mathcal{L}(I)$, the smallest subcategory of Cont-R containing I and closed under limits.

<u>Example</u>: Let R be an FBN hereditary ring, N a localizable intersection of non-minimal prime ideals. Let I_i, I, K etc. be as before. By Theorem 2.6, I is Artinian. Give it the discrete topology. Then there is a duality between the limit closure of I in Cont-R and the full subcategory of (K-Mod) cogenerated by $_KI$. But by Proposition 2.7, the latter is all of K-Mod. Also, by Proposition 2.15, $\mathcal{L}(I)$ is the set of all limits of discrete Artinian modules which are I-torsion free divisible. But $\{R_N\text{-modules}\} = \{R\text{-modules which are I-torsion free divisible}\}$. Hence we have a duality between K-Mod and pro-Artinian R_N-modules (i.e. R_N-modules which are filtered limits of discrete Artinian modules). Further, if A is a discrete Artinian R_N-module, $\exists\ n\ K^n \longrightarrow F(A) \longrightarrow 0$, hence $A \cong UF(A) \hookrightarrow I^n$ and so $\forall a \in A\ \exists m\ \ aJ(\hat{R}_N)^m = 0$. Thus every discrete Artinian R_N-module can be viewed as an $\hat{R}_N$-module and as such, is still discrete Artinian. Conversely every discrete Artinian $\hat{R}_N$-module is a discrete Artinian R_N-module because every R_N-submodule is already an $\hat{R}_N$-submodule. Thus we also have a duality between K-Mod and pro-Artinian $\hat{R}_N$-modules. Similarly, since $_KI$ is a cogenerator, all K-modules are $_KI$-torsion free divisible and there is a duality between pro-Artinian K-modules and Mod-$\hat{R}_N$. The findings of this chapter on duality for FBN hereditary rings (and in particular, for bounded HNP rings) are summarized on p. 41.


F = Cont_{R̂_N}(_, I)
U = Hom_K (_, I)
U' = Hom_K(_, I)
F' = Hom_{R̂_N}(_, I)
Cont-R̂_N
(K-Mod)^op
Mod-R̂_N
Fix(UF, η) =
{pro-Artinian R̂_N-modules}
(Fix(FU, ε))^op
(Reflexives)^op
(linearly compact modules)^op
{Reflexives}
{linearly compact modules}
{Discrete Artinian R̂_N-modules}
(finitely generated K-modules)^op
(Noetherian K-modules)^op
{submodules of factor modules of I^n}
{Artinian R̂_N-modules}

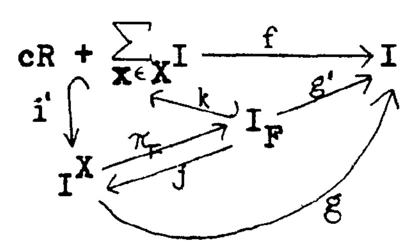
Our final result in this chapter shows that, strangely enough, Morita duality in Mod-R cannot be obtained by these methods. In fact there are no non-trivial co-small injectives in any module category.

<u>Lemma 2.16</u>

Over any ring R, if I is co-small and weakly injective (i.e. $\forall B \subseteq I^X$, $\mathrm{Hom}_R(I^X, I) \longrightarrow \mathrm{Hom}_R(B, I)$ is onto), then I=0.

<u>Proof</u>:

Let $0 \neq i \in I$. Consider the element $c \in I^X$ defined by $\pi_x(c) = i \ \forall x \in X$ where X is an infinite set and π_x is the projection associated with x. Define $f: cR + \sum_{x \in X} I \longrightarrow I$ by $f(cr + y) = ir$. This is well defined since if $cr \in \sum_{x \in X} I$, say $y = cr$, then $\pi_x(y) = 0$ for all but finitely many x, hence $ir = 0$. By the weak injectivity of I, extend f to $g: I^X \longrightarrow I$ and since I is co-small, factor g as $g = g'\pi_F$ where F is a finite subset of X and π_F is the canonical projection. Let j be the canonical injection $I^F \longrightarrow I^X$, k the canonical injection $I^F \longrightarrow cR + \sum_{x \in X} I$ and i' the inclusion of $cR + \sum_{x \in X} I$ in I^X. We have



On the one hand, $gi'k = gj = g'\pi_F j = g' \neq 0$ since $g = g'\pi_F \neq 0$. On the other hand, $gi'k = 0$ by construction, contradiction.

Chapter 3

§1. Properties of $K = End_R(I)$

Assume that R is an FBN hereditary ring, $N = \bigcap_{i=1}^{n} P_i$ is a localizable intersection of non-minimal prime ideals, I, I_o, K, K_o, etc. are as in Chapter 2. Using the Morita duality between K-Mod and Mod-$\hat{R}_N$ we shall investigate the properties of the rings K and $\hat{R}_N$. Both turn out to be semiperfect, fully bounded Noetherian hereditary rings which are complete and Hausdorff in the Jacobson radical topology. In fact, K is Morita equivalent to $\hat{R}_N$.

<u>Proposition 3.1</u> (Miller and Turnidge [28])

Let R be a ring, I_R an injective self-cogenerator and $K = End_R(I)$. Then the following are equivalent:

(a) K is left semihereditary;

(b) whenever $Y \subseteq I$ is such that I/Y is embedded in a finite product of copies of I, I/Y is injective.

<u>Corollary 1</u>

If R is an FBN hereditary ring and $N = \bigcap_{i=1}^{n} P_i$ is a localizable intersection of non-minimal prime ideals, then K is left hereditary and $\hat{R}_N$ is right hereditary.

<u>Proof</u>:

Since R is hereditary, condition (b) of Proposition 3.1

is clearly true for I_R. Hence K is left semihereditary. By Lemma 2.13, since I_R is Artinian, K is left Noetherian, hence K is left hereditary. In that case, condition (b) of Proposition 3.1 holds in K-Mod and $_K I$ is an injective cogenerator. Then as above, $\hat{R}_N$ is right hereditary.

Corollary 2

$\hat{R}_N$ is left Noetherian and left hereditary.

Proof:

Because of the symmetric assumptions on R, $R_N = {}_N R$, hence $\hat{R}_N = {}_N\hat{R}$ which is left Noetherian and left hereditary.

Proposition 3.2

With the same assumptions on R, N, I and K,

$$\mathrm{Ann}_I N^n = \mathrm{Ann}_I J(K)^n = \mathrm{Ann}_I J(\hat{R}_N)^n \;\forall n.$$

Proof: (Müller [29, Lemma 6])

The proof is by induction. When n = 1, by Lemma 2.11, since $R/N \cong \bigoplus_{i=1}^{t} R/M_i$ for some maximal right ideals M_i of R, $\mathrm{Ann}_I N \cong \mathrm{Hom}_R(R/N,\ I) \cong \bigoplus_{i=1}^{t}\mathrm{Hom}_R(R/M_i,\ I) \cong \bigoplus_{i=1}^{t}\mathrm{Ann}_I M_i$. Each $\mathrm{Ann}_I M_i$ is either simple or zero as a K-module and $\mathrm{Ann}_I N$ is a finite direct sum of simple K-modules. Hence $\mathrm{Ann}_I N \subseteq \mathrm{Ann}_I J(K)$. Similarly $\mathrm{Ann}_I J(K) \subseteq \mathrm{Ann}_I N$.

Assume $\mathrm{Ann}_I J(K)^n = \mathrm{Ann}_I N^n$. Then $J(K)^n \subseteq \mathrm{Ann}_K \mathrm{Ann}_I N^n$ and $N^n \subseteq \mathrm{Ann}_R \mathrm{Ann}_I J(K)^n$. A typical element of $J(K)^{n+1}$ is a finite sum of elements of the form st where $s \in J(K)^n$ and $t \in J(K)$. Then $t(\mathrm{Ann}_I N^{n+1})N^n \subseteq t(\mathrm{Ann}_I N) = 0$ and so

$t(\mathrm{Ann}_I N^{n+1}) \subseteq \mathrm{Ann}_I N^n$. Since $s \in J(K)^n \subseteq \mathrm{Ann}_K \mathrm{Ann}_I N^n$,

$$st(\mathrm{Ann}_I N^{n+1}) \subseteq s(\mathrm{Ann}_I N^n) = 0$$

$$\therefore J(K)^{n+1} \subseteq \mathrm{Ann}_K \mathrm{Ann}_I N^{n+1} \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Similarly, $N^{n+1} \subseteq \mathrm{Ann}_R \mathrm{Ann}_I J(K)^{n+1} \ldots\ldots\ldots\ldots\ldots$ (2).

By (1), $\mathrm{Ann}_I J(K)^{n+1} \supseteq \mathrm{Ann}_I \mathrm{Ann}_K \mathrm{Ann}_I N^{n+1} \supseteq \mathrm{Ann}_I N^{n+1}$.

By (2), $\mathrm{Ann}_I N^{n+1} \supseteq \mathrm{Ann}_I \mathrm{Ann}_R \mathrm{Ann}_I J(K)^{n+1} \supseteq \mathrm{Ann}_I J(K)^{n+1}$.

The statement $\mathrm{Ann}_I J(K)^n = \mathrm{Ann}_I J(\hat{R}_N)^n \; \forall n$ is proved similarly.

<u>Proposition 3.3</u>

Under the same hypotheses, K is complete and Hausdorff in the J(K)-adic topology.

<u>Proof</u>:

By Proposition 1.5, $I = \bigcup_{n=1}^{\infty} \mathrm{Ann}_I N^n = \bigcup_{n=1}^{\infty} \mathrm{Ann}_I J(K)^n$. It follows that $\bigcap_{n=1}^{\infty} J(K)^n \subseteq \mathrm{Ann}_K I = 0$. Hence K is Hausdorff in the J(K)-adic topology. If $\{k_n\}_{n=1}^{\infty}$ is a Cauchy sequence in K we want to find a limit $k \in K$. Define $k(0) = 0$. If $0 \neq i \in I$, there exists a least n_o such that $i \in \mathrm{Ann}_I J(K)^{n_o}$ and there also exists a least m_o such that $n, m \geqslant m_o \Rightarrow k_n - k_m \in J(K)^{n_o}$. Define $k(i) = k_{m_o}(i)$. As in [19, Prop. 3], k is well defined and $\lim_{n\to\infty} k_n = k$.

That $\hat{R}_N$ is complete and Hausdorff in the $J(\hat{R}_N)$-adic topology was shown in Chapter 1.

We shall assume from now on that $\{P_1, P_2, \ldots, P_n\}$ is a clan. Then as we have seen, if $R = R_1 \oplus \ldots \oplus R_m$ (possibly after re-numbering), $P_1, \ldots, P_n \not\supseteq R_1$ and $R_N = R_{1_{N \cap R_1}}$. Thus we may also assume without loss of generality that R is a non-Artinian bounded HNP ring. Because of the Morita duality between K-Mod and Mod-$\hat{R}_N$, in particular, because I is an injective cogenerator of Mod-$\hat{R}_N$, Lemma 2.11 holds for a right ideal $A \leqslant \hat{R}_N$ without the assumption in (c) that $A \geqslant J(\hat{R}_N)^m$ for some m. Hence, we have

Lemma 3.4

(a) Every non-zero ideal of $\hat{R}_N$ contains some $J(\hat{R}_N)^m$.

(b) Every non-zero ideal of K contains some $J(K)^m$.

Proof:

(a): Work with $I_0 = I_1 \oplus \ldots \oplus I_n$. If $0 \neq A$ is an ideal of $\hat{R}_N$ and $A \ngeqslant J(\hat{R}_N)^m \; \forall m$, by Lemma 2.11, $\text{Ann}_{I_0} A \nleqslant \text{Ann}_{I_0} J(\hat{R}_N)^m \; \forall m$. Now $\text{Ann}_{I_0} A = \bigoplus_{i=1}^{n} \text{Ann}_{I_i} A$, and $\text{Ann}_{I_i} J(\hat{R}_N)^m = \bigoplus_{i=1}^{n} \text{Ann}_{I_i} J(\hat{R}_N)^m$ so it follows that for some i, $I_i A = 0$. Say $I_1 A = 0$. Let $p_i: I_1 \longrightarrow I_1/\text{Ann}_{I_1} N^{i-1} \cong I_i$ be the composition and let $f_i \in K_0$ be defined by the rule $f_i(x_1, \ldots, x_n) = (0, \ldots, p_i(x_1), \ldots, 0)$ $\forall i$. By its construction, $f_i(I_1) = I_i$. Hence $I_1 A = 0 \Rightarrow f_i(I_1 A) = 0 \Rightarrow f_i(I_1)A = 0 \; \forall i \Rightarrow IA = 0 \Rightarrow A = 0$.

(b): If $0 \neq B$ is an ideal of K then $\text{Ann}_I B$ is a proper K-$\hat{R}_N$-submodule of I, hence $\text{Ann}_{\hat{R}_N} \text{Ann}_I B$ is a non-zero ideal of $\hat{R}_N$. By (a), $\text{Ann}_{\hat{R}_N} \text{Ann}_I B \geqslant J(\hat{R}_N)^m$ for some m. Hence $\text{Ann}_I B = \text{Ann}_I \text{Ann}_{\hat{R}_N} \text{Ann}_I B \leqslant \text{Ann}_I J(K)^m \; \therefore \; B = \text{Ann}_K \text{Ann}_I B \geqslant J(K)^m$.

Corollary

If R is an FBN hereditary ring and $N = \bigcap_{i=1}^{n} P_i$ is the intersection of a clan of non-minimal prime ideals then $\hat{R}_N$ and K are prime, each proper factor ring of K is left Artinian and $\hat{R}_N$ is HNP.

Proof:

By the Lemma, any product of two non-zero ideals of $\hat{R}_N$ (resp. K) contains a power of $J(\hat{R}_N)$ (resp. $J(K)$), hence is non-zero. Every proper factor ring of K is a factor ring of $K/J(K)^m$ for some m and these are left Artinian by the duality (since $\mathrm{Ann}_I J(K)^m$ is a Noetherian R-module).

Note that $\hat{R}_N$ is not primitive, hence is bounded HNP [12, 24].

Lemma 3.5

Write $I = \overset{m}{\underset{s=1}{\oplus}} E_s$ where each $E_s \cong I_i$ for some $i \in \{1, \ldots, n\}$. If L is an essential left ideal of K, then $\mathrm{Ann}_{E_s} L \neq E_s \; \forall s$.

Proof:

Let $e_s \in K$ be defined by $e_s(x_1, \ldots, x_m) = (0, \ldots, x_s, \ldots, 0)$. Since L is essential in K, $\exists k \in K \; 0 \neq ke_s \in L$. If $LE_s = 0$, then we have $0 \neq ke_s(I) \leqslant LE_s = 0$, contradiction.

Corollary

K is left fully bounded.

Proof:

Every proper prime factor ring of K is left Artinian, hence left bounded and K itself is left bounded since $L \leqslant' K$ $\Rightarrow \mathrm{Ann}_I L \subsetneq \mathrm{Ann}_I J(K)^m$ for some m $\Rightarrow L \geqslant J(K)^m$.

These conditions on K are sufficient for us to prove directly that K is HNP but the following yields a better result. We specialize further to the case where $I = E = E(R/N)$ and $H = \mathrm{End}_R(E)$. We have already noted that $E_R(R/N) = E_{R_N}(R_N/NR_N)$. Since $R_N/NR_N \cong \hat{R}_N/J(\hat{R}_N)$ and $E_{\hat{R}_N}$ is injective, we conclude that $E_{\hat{R}_N} = E_{\hat{R}_N}(\hat{R}_N/J(\hat{R}_N))$. We shall therefore assume without loss of generality that R is bounded HNP, complete and Hausdorff in the J(R)-adic topology and $N = J(R)$ is the intersection of a clan - i.e. $\hat{R}_N = R$. R has Morita duality with H induced by ${}_HE_R$. In particular, $X \longmapsto \mathrm{Ann}_{(\ )}X$ describes a one-one, order-inverting correspondence between left ideals of H and R-submodules of E and between right ideals of R and H-submodules of E. Since ${}_HE$ is Artinian and H is left Noetherian, we may write

$$ {}_HE = Y_1 \oplus Y_2 \oplus \dots \oplus Y_s $$

where the ${}_H(Y_j)$ are indecomposable injective H-modules. As each Y_j is also Artinian, ${}_H(Y_j)$ has unique simple submodule ${}_H(C_j)$. If necessary, re-number the Y_j so that $C_1 \cong \dots \cong C_{s_1}$, $C_{s_1+1} \cong \dots \cong C_{s_2}$, ... , $C_{s_{m-1}+1} \cong \dots \cong C_{s_m}$ and such that the C_{s_i} are pairwise non-isomorphic for i = 1, 2, ..., m. (Since $Y_j = E_H(C_j)\ \forall j$, we have: $Y_1 \cong \dots \cong Y_{s_1}$, $Y_{s_1+1} \cong \dots \cong Y_{s_2}$, ... $Y_{s_{m-1}+1} \cong \dots \cong Y_{s_m}$). For each i = 1, 2, ..., m, $M_i = \mathrm{Ann}_R C_{s_i}$ is a maximal right ideal of R. If we let $D_i = C_{s_{i-1}+1} \oplus \dots \oplus C_{s_i}$, since $R = \mathrm{End}_H(E)$, we have $C_j r \cong C_j$ or $C_j r = 0$ for all $r \in R$ and all j = 1, ..., s. Hence $D_i R = D_i$. It follows that $\mathrm{Ann}_R D_i$ is a two-sided ideal of R contained in

M_i.

Lemma 3.6

$\forall i = 1, 2, \ldots, m$, $Ann_I D_i$ is a primitive ideal of R and these are all the non-zero prime ideals of R.

Proof:

Let $B \leqslant M_i$ be any two-sided ideal. Then $Ann_E B$ is a right R-submodule of E containing C_{s_i}. Hence $Ann_E B \geqslant D_i$. By Lemma 2.11, $B = Ann_R Ann_E B \leqslant Ann_R D_i$. Therefore $Q_i = Ann_R D_i$ is the largest ideal contained in M_i, so is primitive. Now $J(H)(\bigoplus_{j=1}^{s} C_j) = 0$, hence $\bigoplus_{j=1}^{s} C_j \leqslant Ann_E J(H) = Ann_E J(R)$. Also $Ann_E J(H) \leqslant Soc_H E = \bigoplus_{j=1}^{s} C_j$, hence $\bigoplus_{j=1}^{s} C_j = Ann_E J(H) = Ann_E J(R)$. We have $\bigcap_{i=1}^{m} Q_i \leqslant Ann_R(\bigoplus_{j=1}^{s} C_j) = Ann_R Ann_E J(R) = J(R)$ and conversely, $\bigcap_{i=1}^{m} Q_i \geqslant J(R)$. Since all non-zero primes of R are maximal, it follows that $\{Q_i\}_{i=1}^{m}$ is the set of all non-zero prime ideals of R. Henceforth we shall assume $P_i = Q_i = Ann_R D_i$.

Corollary

If $N = \bigcap_{i=1}^{n} P_i$ there are exactly n isomorphism classes of simple H-modules.

Proof:

It follows from the lemma that $m = n$ and since ${}_H E$ is a cogenerator, it contains a copy of every simple H-module.

Let $(y_1, \ldots, y_s) f_j = (0, \ldots, y_j, \ldots, 0)$ where $y_j \in Y_j$. Then $\{f_1, f_2, \ldots, f_s\}$ is a set of local orthogonal idempotents of R whose sum is 1. By passing to the factor rings R/J^m for all m and remembering that proper factor rings of an HNP ring are serial [12], one sees that the only submodules of f_jR are $f_jR \supsetneqq f_jJ \supsetneqq \dots \supsetneqq f_jJ^m \supsetneqq f_jJ^{m+1} \supsetneqq \dots \ \forall j$.

Lemma 3.7

$$\forall k,\ J^{kn}/J^{kn+1} \cong R/J \text{ and } E(R/J^{kn+1}) \cong E(R/J).$$

Proof:

$R/J^{kn+1} \cong f_1R/f_1J^{kn+1} \oplus \dots \oplus f_sR/f_sJ^{kn+1}$. Since f_1R/f_1J^{kn+1} is uniserial, it has unique simple submodule f_1J^{kn}/f_1J^{kn+1} which is annihilated by J. Now R is FBN and $\{P_i \mid i=1,\dots,n\}$ is the set of all non-zero prime ideals of R hence $E(f_1R/f_1J^{kn+1}) \cong I_1$ (possibly after re-numbering). In particular $f_1R/f_1J^{kn+1} \cong \mathrm{Ann}_{I_1} J^{kn+1}$ since it is obviously contained in $\mathrm{Ann}_{I_1} J^{kn+1}$ but not contained in $\mathrm{Ann}_{I_1} J^{kn}$. We assume that the other I_i are indexed such that $I_1/\mathrm{Ann}\, J \cong I_2$, $I_2/\mathrm{Ann}\, J \cong I_3, \dots, I_n/\mathrm{Ann}\, J \cong I_1$. By restricting to f_1J^{kn}/f_1J^{kn+1} we have $f_1J^{kn}/f_1J^{kn+1} \cong \mathrm{Ann}_{I_1} J$ and we also have induced isomorphisms $f_1R/f_1J^{kn} \cong \mathrm{Ann}_{I_1} J^{kn+1}/\mathrm{Ann}_{I_1} J \cong \mathrm{Ann}_{I_2} J^{kn}$

$$f_1R/f_1J^{kn-1} \cong \mathrm{Ann}_{I_2} J^{kn}/\mathrm{Ann}_{I_2} J \cong \mathrm{Ann}_{I_3} J^{kn-1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$f_1R/f_1J^{(k-1)n+1} \cong \mathrm{Ann}_{I_n} J^{(k-1)n+2}/\mathrm{Ann}\, J \cong \mathrm{Ann}_{I_1} J^{(k-1)n+1}$$

Proceeding in this fashion, we eventually reach

$$f_1R/f_1J \cong \mathrm{Ann}_{I_n} J^2/\mathrm{Ann}_{I_n} J \cong \mathrm{Ann}_{I_1} J$$

Hence $f_1J^{kn}/f_1J^{kn+1} \cong f_1R/f_1J$ and this is independent of k.

The same argument works for all the f_j. It follows that

$$\forall k \quad R/J \cong f_1R/f_1J \oplus \ldots \oplus f_sR/f_sJ$$
$$\cong f_1J^{kn}/f_1J^{kn+1} \oplus \ldots \oplus f_sJ^{kn}/f_sJ^{kn+1}$$
$$= J^{kn}/J^{kn+1}$$

and

$$E(R/J) \cong E(f_1J^{kn}/f_1J^{kn+1}) \oplus \ldots \oplus E(f_sJ^{kn}/f_s\,J^{kn+1})$$
$$\cong E(f_1R/f_1J^{kn+1}) \oplus \ldots \oplus E(f_sR/f_sJ^{kn+1})$$
$$\cong E(R/J^{kn+1}).$$

<u>Corollary</u>

$$J^{kn}/J^{kn+1} = \text{Soc}(R/J^{kn+1}).$$

<u>Proof:</u>

By the lemma, J^{kn}/J^{kn+1} is essential in R/J^{kn+1}, hence contains $\text{Soc}(R/J^{kn+1})$. On the other hand, J^{kn}/J^{kn+1} is a direct sum of simple modules, hence is contained in $\text{Soc}(R/J^{kn+1})$.

<u>Lemma 3.8</u>

Under the same assumptions, $\exists\, p \in R$ such that $J^n = pR$.

<u>Proof:</u>

By Lemma 3.7, since $J^n/J^{n+1} \cong R/J$ $\exists\, p \in R$ such that $J^n = pR + J^{n+1}$. Then $J^{n+1} = pJ + J^{n+2}$, $J^{n+2} = pJ^2 + J^{n+3}$ etc. Hence $J^n = pR + J^{n+1} = pR + pJ + J^{n+2} = pR + J^{n+2} = pR + pJ^2 + J^{n+3} = pR + J^{n+3} = \ldots$ — i.e. $J^n = \bigcap_{m=1}^{\infty}(pR + J^m)$. But J is localizable and classical so by Theorem 1.5, we have

$$J^n = \bigcap_{m=1}^{\infty}(pR + J^m) = pR.$$

Note that p is regular since J^n is essential. Hence

$J^n = pR \cong R$. By symmetry $\exists p' \in R$ such that $J^n = Rp' \cong R$. Also note that $J^{kn} = p(Rp)(Rp) \ldots (Rp)R \leqslant p(pR) \ldots pR \leqslant p^kR \leqslant J^{nk}$.

Lemma 3.9

(a) $\forall k,\ R/J^{kn+1} \cong \mathrm{Ann}_E J^{kn+1}$

(b) $\forall k,\ J^n/J^{kn+1} \cong R/J^{(k-1)n+1}$

Proof:

(a): As in Lemma 3.7, $R/J^{kn+1} \cong f_1R/f_1J^{kn+1} \oplus \ldots \oplus f_sR/f_sJ^{kn+1}$. $\forall j = 1, \ldots, s \ \exists i(j)$ such that $E(f_jR/f_jJ^{kn+1}) \cong I_{i(j)}$. Clearly $f_jR/f_jJ^{kn+1} \subseteq \mathrm{Ann}_{I_{i(j)}} J^{kn+1}$ and since $(f_jR/f_jJ^{kn+1})J^{kn} \neq 0$ it follows that $f_jR/f_jJ^{kn+1} \cong \mathrm{Ann}_{I_{i(j)}} J^{kn+1}$. The result now follows from Lemma 3.7

(b): Consider $f: J^n/J^{kn+1} \longrightarrow R/J^{(k-1)n+1}$ where $f[pr] = [r]$. If $pr \in J^{kn+1} = (J^n)^kJ = p^kRJ = p^kJ$ then $r \in p^{k-1}J = (J^n)^{k-1}J = J^{(k-1)n+1}$. Thus f is well defined. It is clearly a surjection and $r \in J^{(k-1)n+1} = (J^n)^{k-1}J = p^{k-1}J \Rightarrow pr \in p^kJ = J^{kn+1}$. Hence f is an isomorphism.

Lemma 3.10

$\exists \{z_k \mid k = 0, 1, \ldots\} \subseteq E$ such that $z_kR = Hz_k = \mathrm{Ann}_E J^{kn+1}\ \forall k$ and $z_kp = z_{k-1}\ \forall k \geqslant 1$, $z_op = 0$.

Proof:

The proof is by induction. When $k = 0$, Lemma 3.8 shows $\exists z_o \in E$ such that $z_oR = \mathrm{Ann}_E J$ and $\mathrm{Ann}_R z_o = J$. Since $\mathrm{Ann}_R z_o = \mathrm{Ann}_R Hz_o = J$, we have $Hz_o = {}_H(\mathrm{Ann}_E J)$. Clearly $z_op = 0$.

Assume that $z_o, \ldots, z_{k-1}$ have been found with the desired properties. Consider the composite

$$t_{k-1}:\ J^n/J^{kn+1} \longrightarrow R/J^{(k-1)n+1} \longrightarrow \mathrm{Ann}_E J^{(k-1)n+1}$$
$$[pr] \longmapsto [r] \longmapsto z_{k-1}r$$

Extend t_{k-1} to $g_k: R/J^{kn+1} \longrightarrow E$. Then $\mathrm{Im}\ g_k \subseteq \mathrm{Ann}_E J^{kn+1}$. Since J^n/J^{kn+1} is an essential submodule of R/J^{kn+1}, if g_k is not a monomorphism we have $\ker g_k \cap J^n/J^{kn+1} = \ker t_{k-1} \neq 0$, contradiction. Now $\mathrm{Ann}_E J^{kn+1}$ is the R/J^{kn+1}-injective hull of R/J so by Lemma 3.9, $g_k(R/J^{kn+1}) \cong R/J^{kn+1}$ is a direct summand of $\mathrm{Ann}_E J^{kn+1}$. If g_k is not a surjection, this contradicts the Krull-Remak-Schmidt-Azumaya theorem. Hence g_k is an isomorphism. Define $z_k = g_k[1]$. Then by construction $z_k p = g_k[p] = z_{k-1}$ and since $\mathrm{Ann}_R z_k = \mathrm{Ann}_R H z_k = J^{kn+1}$, $Hz_k = {}_H\mathrm{Ann}_E J^{kn+1}$.

<u>Notation</u> For all m, denote $\mathrm{Ann}_E J^m$ by A_m.

<u>Corollary</u>

$J(R)^n = pR = Rp$.

<u>Proof</u>:

$J(R)^n = pR$ was shown in Lemma 3.8. Choose any $pr \in J^n$. The map $h: z_o s \longmapsto z_o rs$ of $A_1 \longrightarrow A_1$ is well defined since $\mathrm{Ann}_R z_o = J = \mathrm{Ann}_R z_o R \subseteq \mathrm{Ann}_R z_o r$. Extend h to $h': E \longrightarrow E$. Since $h'(A_{n+1}) \subseteq A_{n+1} = z_1 R$, $\exists r_1 \in R$ such that $h(z_1) = z_1 r_1$. Then $z_1 pr = z_o r = h(z_o) = h(z_1 p) = z_1 r_1 p$. Hence $pr - r_1 p \in \mathrm{Ann}_R z_1 = J^{n+1}$. Thus we have shown $J^n = pR \subseteq Rp + J^{n+1}$. Since J is left localizable and left classical, by Theorem 1.5, we may conclude $J^n = \bigcap_{m=1}^{\infty}(Rp + J^m) = Rp$.

Theorem 3.11

If R is an FBN hereditary ring and $0 \neq N = \bigcap_{i=1}^{n} P_i$ is the intersection of a clan of non-minimal prime ideals, $E = E(R/N)$ and $H = \mathrm{End}_R(E)$, then $H \cong \hat{R}_N$.

Proof:

By the corollary to Lemma 1.12 and the earlier results of this chapter, we may assume without loss of generality that $R = \hat{R}_N$, a bounded HNP ring complete in the J(R)-adic topology and $N = J = J(R)$ is the intersection of a clan. We shall construct a consistent system of isomorphisms

$$\lambda_k : H/J(H)^{kn+1} \longrightarrow R/J(R)^{kn+1} \qquad \forall k \geqslant 0.$$

Given $h \in H$, $\forall k\ h(z_k) \in z_k R$, hence $\exists \{r_k | k=0,1,\ldots\}$ such that $h(z_k) = z_k r_k$.

Define $\lambda_0[h]_{J(H)} = [r_0]_{J(R)}$. $h - h' \in J(H)$ and $h'(z_0) = z_0 s_0$ $\Rightarrow r_0 - s_0 \in J(R)$. Hence λ_0 is independent of the choice of representative of $[h]_{J(H)}$ and of the choice of r such that $h(z_0) = z_0 r$. Obviously λ_0 preserves 0, 1, +, -. If $h(z_0) = z_0 r_0$ and $h''(z_0) = z_0 s_0'$ then $hh''(z_0) = h(z_0 s_0') = z_0 r_0 s_0'$. Hence λ_0 preserves products. $\lambda_0[h] = 0 \Rightarrow h \in J(H) = \mathrm{Ann}_H z_0$, hence λ_0 is a monomorphism. Finally, given $[r] \in R/J$, define a map $h_0 : z_0 R \longrightarrow z_0 R$ by $h(z_0 s) = z_0 rs$ and extend h_0 to a map $h : E \longrightarrow E$. Clearly $\lambda_0[h] = [r]$, hence λ_0 is a ring isomorphism. To construct $\lambda_k\ \forall k \geqslant 1$, since $pR = Rp$, let $r_k p = ps_{k-1}^{(k)}$, $s_{k-1}^{(k)} p = ps_{k-2}^{(k)}$ $\ldots s_1^{(k)} p = ps_0^{(k)}$. Define $\lambda_k[h]_{J(H)^{nk+1}} = [s_0^{(k)}]_{J(R)^{nk+1}}$.

We must show that λ_k is independent of the choice of representative of $[h]_{J(H)^{nk+1}}$, independent of the choice of r such that $h(z_k) = z_k r$ and independent of the choice of all the $s_i^{(k)}$; that it is a ring isomorphism; and that the following diagram commutes:

$$\begin{array}{ccc} H/J(H)^{(k-1)n+1} & \xrightarrow{\lambda_{k-1}} & R/J(R)^{(k-1)n+1} \\ {\scriptstyle \pi'_{k-1}}\uparrow & & \uparrow{\scriptstyle \pi_{k-1}} \\ H/J(H)^{kn+1} & \xrightarrow[\lambda_k]{} & R/J(R)^{kn+1} \end{array}$$

Now if $[h]_{J(H)^{nk+1}} = [h']_{J(H)^{nk+1}}$ and $h'(z_k) = z_k t_k$ then $(h-h')(z_k) = z_k(r_k - t_k) = 0 \Rightarrow (r_k - t_k) \in J^{kn+1}$. Let $s_0^{(k)}$ be chosen as indicated. Then $r_k p^k = p^k s_0^{(k)}$. Select <u>any</u> u_0 such that $t_k p^k = p^k u_0$. Then $(r_k - t_k) \in J^{kn+1} \Rightarrow$ $(r_k - t_k)p^k \in J^{2kn+1} \Rightarrow p^k(s_0^{(k)} - u_0) \in J^{2kn+1} = p^{2k}J$. This implies $(s_0^{(k)} - u_0) \in p^k J = J^{kn+1}$. Thus λ_k is well defined. λ_k clearly preserves +, -, 0, 1. If $h''(z_k) = z_k t'_k$ and $t'_k p = pu_{k-1}^{(k)}$, ..., $u_1^{(k)} p = pu_0^{(k)}$, we have $r_k t'_k p = r_k pu_{k-1}^{(k)} = ps_{k-1}^{(k)} u_{k-1}^{(k)}$, ..., $s_1^{(k)} u_1^{(k)} p = ps_0^{(k)} u_0^{(k)}$. Hence $\lambda_k[h][h''] = \lambda_k[h]\lambda_k[h'']$. Thus λ_k is a well defined ring homomorphism. If $\lambda_k[h] = [s_0^{(k)}] = 0$ then $s_0^{(k)} \in J^{kn+1}$ and $ps_0^{(k)} \in J^{(k+1)n+1}$. Hence $s_1^{(k)}p$, $s_2^{(k)}p$, ... $r_k p \in J^{(k+1)n+1} = J^{kn+1}p$. It follows that $r_k \in J^{kn+1}$ (since p is regular) and λ_k is a ring monomorphism. Given any $[s] \in R/J(R)^{kn+1}$, let $t_1 p = ps$, $t_2 p = pt_1$, ..., $t_k p = pt_{k-1}$ (since $pR = Rp = J^n$). Define $h_0(z_k) = z_k t_k : z_k R \longrightarrow z_k R$ (h_0 is well defined since $\mathrm{Ann}_R z_k = J^{kn+1} \subseteq \mathrm{Ann}_R z_k t_k$). Extend h_0 to some $h \in H$. Then clearly $\lambda_k[h] = [s]$ and so λ_k is a ring isomorphism.

Finally we check that $\pi_{k-1}\lambda_k = \lambda_{k-1}\pi'_{k-1}$. Now $z_{k-1}s_{k-1}^{(k)} = z_k p s_{k-1}^{(k)} = z_k r_k p = h(z_k p) = h(z_{k-1}) = z_{k-1}r_{k-1}$. Hence $r_{k-1} - s_{k-1}^{(k)} \in J^{(k-1)n+1} \Rightarrow (r_{k-1} - s_{k-1}^{(k)})p \in J^{kn+1} \Rightarrow p(s_{k-2}^{(k-1)} - s_{k-2}^{(k)}) \in J^{kn+1} \Rightarrow \ldots \Rightarrow p(s_0^{(k-1)} - s_0^{(k)}) \in J^{kn+1} = pJ^{(k-1)n+1} \Rightarrow (s_0^{(k-1)} - s_0^{(k)}) \in J^{(k-1)n+1}$ since p is regular. Hence we have $\pi_{k-1}\lambda_k[h]_{J(H)^{kn+1}} = \pi_{k-1}[s_0^{(k)}] = [s_0^{(k)}]_{J^{(k-1)n+1}} = [s_0^{(k-1)}]_{J^{(k-1)n+1}} = \lambda_{k-1}\pi_{k-1}[h]_{J(H)^{kn+1}}$.

Since both H and R are complete, $H = \varprojlim_k H/J(H)^{kn+1}$ and $R = \varprojlim_k R/J(R)^{kn+1}$ and the λ_k induce a ring isomorphism $H \cong R$. Specifically, given $h \in H$, we know $[h]_{J(H)^{kn+1}} \longleftrightarrow [s_k]_{J^{kn+1}}$ (for some s_k) and it is easy to verify that $\{s_k\}_{k=0}^{\infty}$ is a Cauchy sequence in R. It has a limit $r \in R$. Then $h \longmapsto r$ defines the isomorphism.

We would like to extend these results to any localizable classical intersection of non-minimal prime ideals in an FBN hereditary ring. If $N = \bigcap_{i=1}^{n} P_i$ is such an intersection, then $\mathcal{S} = \{P_1, \ldots, P_n\}$ is uniquely a disjoint union of clans $\mathcal{S}_1, \ldots, \mathcal{S}_t$ with intersections $S_1, \ldots, S_t$, and $E(R/N) \cong \bigoplus_{k=1}^{t} E(R/S_k)$.

<u>Lemma 3.12</u>

(a) With the same notation as before, $H(E(R/S_k)) \subseteq E(R/S_k)$ $\forall k = 1, 2, \ldots, t$ and as an H-module ${}_HE \cong {}_HE(R/S_1) \oplus \ldots \oplus {}_HE(R/S_t)$.

(b) If $H_k = \mathrm{End}_R(E(R/S_k))$, then $H \cong H_1 \oplus \ldots \oplus H_t$ and each H_t is a semilocal bounded non-Artinian HNP ring complete and Hausdorff in the $J(H_k)$-adic topology.

Proof:

(a): If I_i is a direct summand to $E(R/S_k)$ then $\forall h \in H$, $\exists n_i$ such that $h(I_i) \cong I_i/\ker h \cong I_i/\mathrm{Ann}_{I_i} S_k^{n_i}$ by Theorem 2.6, and the latter is also a direct summand to $E(R/S_k)$. Hence $\forall k$, $H(E(R/S_k)) \subseteq E(R/S_k)$.

(b): Consider primes P_i and P_j and suppose $f: I_i \longrightarrow I_j$ is non-zero. Then $f(I_i) \cong I_i/\ker(f)$ is injective and since I_j is indecomposable, $f(I_i) = I_j \cong I_i/\ker(f)$. By the proof of Lemma 3.2, this would imply P_i and P_j belong to the same clan. Hence if P_i and P_j belong to different clans, $\mathrm{Hom}_R(I_i, I_j) = 0$. It follows that $\mathrm{Hom}_R(E(R/S_k), E(R/S_m)) = 0$ $\forall k \neq m \in \{1, \ldots, t\}$ Hence we have

$$\begin{aligned} H &= \mathrm{End}_R(E(R/N)) \\ &\cong \mathrm{End}_R\Big(\bigoplus_{k=1}^{t} E(R/S_k)\Big) \\ &\cong \bigoplus_{k=1}^{t} \mathrm{End}_R(E(R/S_k)) \\ &\cong H_1 \oplus \ldots \oplus H_t \end{aligned}$$

where $h \longmapsto (h|_{E(R/S_1)}, \ldots, h|_{E(R/S_t)})$. It is easy to verify that this is a ring isomorphism. That each H_k is semilocal bounded HNP, complete and Hausdorff in the $J(H_k)$-adic topology follows from Theorem 3.11 and the corollary to Lemma 3.4. Proposition 1.8 shows that

$$_H E \cong {}_{H_1}E(R/S_1) \oplus \ldots \oplus {}_{H_t}E(R/S_t) \cong {}_H E(R/S_1) \oplus \ldots \oplus {}_H E(R/S_t).$$

Proposition 3.13

Under the same assumptions on R, $\hat{R}_N \cong \bigoplus_{k=1}^{t} \hat{R}_{S_k}$

Proof:

We know $\hat{R}_N = \text{End}_H(E(R/N))$ and $\forall k = 1, \ldots, t$, $\hat{R}_{S_k} = \text{End}_{H_k}(E(R/S_k))$. Now if for $k \neq m$, $f \in \text{Hom}_H(E(R/S_k), E(R/S_m))$ and if the identity of H is written $1 = e_1 + \ldots + e_t$ with $e_k \in H_k$, we have for any $y \in E(R/S_k)$ $yf = e_m(yf) = (e_m y)f = (0)f = 0$ and it follows that $\text{Hom}_H(E(R/S_k), E(R/S_m)) = 0$ for $k \neq m$. Hence $r \longmapsto (r|_{E(R/S_1)}, \ldots, r|_{E(R/S_t)})$ is an isomorphism of $\hat{R}_N \longrightarrow \bigoplus_{k=1}^{t} \text{End}_H(E(R/S_k))$ as in Lemma 3.12. The result follows.

Corollary

If R is an FBN hereditary ring and N is a localizable intersection of non-minimal prime ideals, let $E = E(R/N)$ and $H = \text{End}_R(E)$. Then $H \cong \hat{R}_N$.

Proof:

Let $\{P_1, \ldots, P_n\}$ be the disjoint union of clans $\mathcal{S}_1, \ldots, \mathcal{S}_t$ with intersections $S_1, \ldots, S_t$ respectively. By Lemma 3.12 $H \cong H_1 \oplus \ldots \oplus H_t$ and by Theorem 3.11, $\forall k$, $H_k \cong \hat{R}_{S_k}$. In Proposition 3.13 we have just shown that $\hat{R}_N \cong \bigoplus_{k=1}^{t} \hat{R}_{S_k}$. The Corollary follows.

It was originally our purpose to study the properties of $K = \text{End}_R(I)$ where $I = I_1^{s_1} + \ldots + I_n^{s_n}$ for some natural numbers $s_i \geq 1$. In addition to the properties already found in Propositions 3.2 - 3.5, we can now say that K is hereditary Noetherian and (when N is a clan) prime, as a consequence of the following theorem due to Vámos [39].

Theorem 3.14

If R has Morita duality with rings K and H then K is Morita equivalent to H.

§2. Structure of K and $\hat{R}_N$

Let R be a bounded HNP ring and $N = \bigcap_{i=1}^{n} P_i$ the intersection of a clan of prime ideals ($P_i \neq 0$). As always, I_i denotes the indecomposable injective with associated prime ideal P_i, $I = I_1^{s_1} \oplus \ldots \oplus I_n^{s_n}$ for some $s_i \geqslant 1$ and $K = \mathrm{End}_R(I)$. Our aim in this section is to describe the structure of K (Theorem 3.19). This will give as corollary the structure of $\hat{R}_N$ for any FBN hereditary ring and any localizable intersection of non-minimal prime ideals of R. We shall then extend our results to arbitrary semiperfect HNP rings thus obtaining a new proof of a theorem of Michler (Theorem 3.20).

Lemma 3.15

Let Q be a local ring. Suppose $Qa = J(Q)$ for some $a \in Q$ and that J(Q) is finitely generated as a right ideal. Then $Qa = aQ = J(Q)$.

Proof:

Consider the right Q-module Qa/aQ. Given any $qa \in Qa$, either $q \in J(Q)$ or $1-q \in J(Q)$. If $q \in J(Q)$ then qa and $aq \in J(Q)^2 = QaQa \subseteq Qa^2 \subseteq J(Q)^2$. Hence $qa - aq = q'a^2$ for some $q' \in Q$ —

$[qa]_{aQ} = [q'a^2]_{aQ}$. If $1-q \in J(Q)$ then $(1-q)a = a-qa \in J(Q)^2 = Qa^2$ — i.e. $a-qa = q''a^2$ for some $q'' \in Q$ and so $[qa]_{aQ} = [-q''a^2]_{aQ}$. We have proved $Qa/aQ = (Qa/aQ)(J(Q))$. Since Qa is finitely generated as a right ideal, by Nakayama's lemma $Qa=aQ=J(Q)$.

Let $I = E_1 \oplus \ldots \oplus E_m$ where for each $s = 1,2,\ldots, m$, $E_s \cong I_i$ for some $i \in \{1,2,\ldots, n\}$. Let $e_s(x_1,\ldots, x_n) = (0,\ldots, x_s,\ldots, 0)$. Clearly $e_sKe_s \cong \mathrm{End}_R(E_s)$. Since E_s is indecomposable, e_sKe_s is local. In fact $\{e_1,\ldots, e_m\}$ is a set of local orthogonal idempotents of K whose sum is 1.

Lemma 3.16

$\forall s$, e_sKe_s is a complete, local Noetherian domain whose only one-sided ideals are the $J(e_sKe_s)^m \ \forall m$. Further $\exists q_s \in e_sKe_s$ such that $J(e_sKe_s) = q_se_sKe_s = e_sKe_sq_s$.

Proof:

e_sKe_s is hereditary by [34, Lemma 4.4]. By Lemma 2.8 we already know that e_sKe_s is a local domain whose Jacobson radical is principal as a left ideal and whose only left ideals are $\{J(e_sKe_s)^m\}_{m=1}^{\infty}$. Given a right ideal $C \leqslant e_sKe_s$, Let $Ce_sK = x_1K + \ldots + x_nK$ since K is right Noetherian. For each i, we can write $x_i = \sum_{k=1}^{n_i} c_{ik}e_st_k$ for some $t_k \in K$. Then clearly $\{c_{ik} \mid i = 1,\ldots, n;\ k = 1,\ldots, n_i\}$ generates C in e_sKe_s. Hence e_sKe_s is Noetherian. If $J(e_sKe_s) = (e_sKe_s)q_s$, we see by Lemma 3.15 that $J(e_sKe_s) = q_s(e_sKe_s)$ and by the argument used in Lemma 2.8, the only right ideals of e_sKe_s are $\{J(e_sKe_s)^m\}_{m=1}^{\infty}$.

In particular, $End_R(I_1)$ satisfies these properties. In the terminology of Lambek and Michler [21], it is a complete discrete rank-one valuation ring which we shall henceforth denote by D. Consider now the ring T of all matrices of the form

$$\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1m} \\ f_{21} & f_{22} & \cdots & f_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ f_{m1} & f_{m2} & \cdots & f_{mm} \end{pmatrix} \quad \text{where } f_{st} \in Hom_R(E_t, E_s)\ \forall\ s, t.$$

Multiplication can be defined in the usual way since if $f_{su} \in Hom_R(E_u, E_s)$ and $g_{ut} \in Hom_R(E_t, E_u)$ then the composite $f_{su}g_{ut}: E_t \longrightarrow E_u \longrightarrow E_s$ is defined and the usual matrix product $(f_{st})(g_{st}) = (\sum_{u=1}^{m} f_{su}g_{ut})$ is again of the same form.

<u>Proposition 3.17</u>

With R, I and K as usual and T as above, $K \cong T$.

<u>Proof</u>:

Let $\kappa_t: E_t \longrightarrow I$ and $\pi_s: I \longrightarrow E_s$ be the t^{th} injection and s^{th} projection respectively. Consider the map φ, where $\varphi(k) = (\pi_s k \kappa_t)$. Clearly φ preserves 0, +, 1. To check whether or not φ preserves products consider $\varphi(k)\varphi(k') =$

$$\begin{pmatrix} \pi_1 k \kappa_1 & \pi_1 k \kappa_2 & \cdots & \pi_1 k \kappa_m \\ \vdots & \vdots & & \vdots \\ \pi_m k \kappa_1 & \pi_m k \kappa_2 & \cdots & \pi_m k \kappa_m \end{pmatrix} \begin{pmatrix} \pi_1 k' \kappa_1 & \cdots & \pi_1 k' \kappa_m \\ \vdots & & \vdots \\ \pi_m k' \kappa_1 & \cdots & \pi_m k' \kappa_m \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{u=1}^{m} \pi_1 k \varkappa_u \pi_u k' \varkappa_1 & \sum_{u=1}^{m} \pi_1 k \varkappa_u \pi_u k' \varkappa_2 & \cdots & \sum_{u=1}^{m} \pi_1 k \varkappa_u \pi_u k' \varkappa_m \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{u=1}^{m} \pi_m k \varkappa_u \pi_u k' \varkappa_1 & \sum_{u=1}^{m} \pi_m k \varkappa_u \pi_u k' \varkappa_2 & \cdots & \sum_{u=1}^{m} \pi_m k \varkappa_u \pi_u k' \varkappa_m \end{pmatrix}$$

Since $\sum_{u=1}^{m} \varkappa_u \pi_u = 1$, we have $\varphi(k)\varphi(k') = \varphi(kk')$. If $k \neq 0$, then $k\varkappa_t \neq 0$ for some t and $\pi_s k \varkappa_t \neq 0$ for some s. Hence φ is a ring monomorphism. Finally, if $(f_{st}) \in T$, define $k \in K$ by the rules $\pi_s k \varkappa_t = f_{st}$ and the universal properties of $\{\pi_s\}, \{\varkappa_t\}$. Then clearly $\varphi(k) = (f_{st}) \in T$ and φ is a ring isomorphism.

<u>Proposition 3.18</u>

(a) $\forall\, i = 1, \ldots, n$ $End_R(I_i) \cong End_R(I_1)$ as rings;

(b) if $1 \leq i < j \leq n$, $Hom_R(I_i, I_j) \cong End_R(I_1)$ as Abelian groups;

(c) if $1 \leq j < i \leq n$, $Hom_R(I_i, I_j) \cong J(D)$ as Abelian groups (where $D = End_R(I_1)$).

<u>Proof</u>:

(a): $\forall\, i$, let $B_i = Ann_{I_1} N^i$ and let $\pi_i: I_1/B_{i-1} \longrightarrow I_1/B_i$ be the canonical projection. Any map $f: I_1/B_{i-1} \longrightarrow I_1/B_{i-1}$ induces a map $f': I_1/B_i \longrightarrow I_1/B_i$ where $f'\pi_i = \pi_i . f$. Since π_i is an epimorphism, f' is uniquely determined by f. Clearly $0' = 0$, $1' = 1$, $(f+g)' = f'+g'$. Further $(fg)\pi_i = \pi_i(fg) = f'\pi_i g = f'g'\pi_i \Rightarrow f'g' = (fg)'$. Assume that the I_i are indexed such that $I_1/B_{i-1} \cong I_i$. Then $f \longmapsto f'$ induces a ring homomorphism: $End_R(I_i) \longrightarrow End_R(I_{i+1})\ \forall\, i$. If $f'(I_1/B_i) = 0$ then $f(I_1/B_{i-1}) \subseteq B_i/B_{i-1} \Rightarrow f(I_1/B_{i-1})N = 0 \Rightarrow I_1 N^i = 0$ which is impossible by Lemma 2.2. Hence $f \longmapsto f'$ is a ring monomorphism of $End_R(I_i) \longrightarrow End_R(I_{i+1})$ $\forall i$.

The composite $End_R(I_1) \rightarrowtail End_R(I_2) \to \ldots \to End_R(I_1/B_n)$ is a ring monomorphism which takes $f \longmapsto \overline{f}$ where

$$\overline{f}\pi_n \ldots \pi_1 = \pi_n \ldots \pi_1 f.$$

If $\alpha : I_1 \longrightarrow I_1/B_n$ denotes the known isomorphism, recall that $q_1 = \alpha^{-1}\pi_n \ldots \pi_1 : I_1 \longrightarrow I_1/B_n \longrightarrow I_1$ generates $J(D)$ as a left and as a right ideal (2.8 and 3.16). An R-homomorphism $g : I_1/B_n \longrightarrow I_1/B_n$ gives rise to $\tilde{g} = \alpha^{-1}g\alpha : I_1 \longrightarrow I_1$. By Lemma 3.16 $\exists g^*$ $q_1 g^* = \tilde{g} q_1$. Then

$$\alpha^{-1}\pi_n \ldots \pi_1 g^* = q_1 g^* = \tilde{g} q_1 = \alpha^{-1} g \alpha \alpha^{-1} \pi_n \ldots \pi_1$$
$$\pi_n \ldots \pi_1 g^* = g \pi_n \ldots \pi_1$$
$$g = \overline{g^*}$$

This implies $f \longmapsto f'$ is a ring isomorphism of

$$End_R(I_n) \cong End_R(I_1/B_{n-1}) \longrightarrow End_R(I_1/B_n) \cong End_R(I_1).$$

Similarly $End_R(I_i) \cong End_R(I_{i+1}) \cong \ldots \cong End_R(I_1)$ $\forall i$.

(b) and (c): For any $1 \leq i, j \leq n$, consider

$$\begin{array}{ccc} I_1/B_{i-1} & \xrightarrow{\ f\ } & I_1/B_{j-1} \\ {\scriptstyle \pi_n \ldots \pi_i} \downarrow & & \downarrow {\scriptstyle \pi_n \ldots \pi_j} \\ I_1/B_n & \dashrightarrow & I_1/B_n \end{array}$$

By the argument used in Lemma 2.8, since $\ker \pi_n \ldots \pi_j f \supseteq B_n/B_{i-1}$, f induces $\overline{f} : I_1/B_n \longrightarrow I_1/B_n$ such that $\overline{f}\pi_n \ldots \pi_i = \pi_n \ldots \pi_j f$. By (a), $\exists!$ $f' : I_1 \longrightarrow I_1$ such that $\pi_n \ldots \pi_1 f' = \overline{f}\pi_n \ldots \pi_1$. $f \longmapsto f'$ clearly preserves 0, +, - so that it is an Abelian group homomorphism of $Hom_R(I_i, I_j)$ into $End_R(I_1)$. If $f' = 0$

then $\bar{f} = 0$, hence $\pi_n \ldots \pi_j f = 0$ — i.e. $(I_1/B_{j-1})N^{n-j} = f(I_1/B_{i-1})N^{n-j} = 0$ which is impossible by Lemma 2.2. Hence $f \longmapsto f'$ is a monomorphism: $\mathrm{Hom}_R(I_i, I_j) \longrightarrow \mathrm{End}_R(I_1) = D$.

Now assume $1 \leq i < j \leq n$ and $g: I_1 \longrightarrow I_1$ is any R-homomorphism. Then g induces a homomorphism $g^*: I_1/B_n \longrightarrow I_1/B_n$ such that $g^*\pi_n \ldots \pi_1 = \pi_n \ldots \pi_1 g$. If $g^* = \bar{f}$ for some homomorphism $f: I_1/B_{i-1} \longrightarrow I_1/B_{j-1}$ we will know $f \longmapsto f'$ is an isomorphism of Abelian groups: $\mathrm{Hom}_R(I_i, I_j) \longrightarrow \mathrm{End}_R(I_1)$. Define f by $f\pi_{i-1} \ldots \pi_1 = \pi_{j-1} \ldots \pi_1 g$. This is well defined since $x \in B_{i-1} \Rightarrow g(x) \in B_{i-1} \subseteq B_{j-1}: \pi_{j-1} \ldots \pi_1 g(x) = 0$. Since $\pi_n \ldots \pi_j f \pi_{i-1} \ldots \pi_1 = \pi_n \ldots \pi_j \pi_{j-1} \ldots \pi_1 g = g^*\pi_n \ldots \pi_i \pi_{i-1} \ldots \pi_1$ and since $\pi_{i-1} \ldots \pi_1$ is an epimorphism, $\pi_n \ldots \pi_j f = g^*\pi_n \ldots \pi_i$. Hence $g^* = \bar{f}$. It follows that if $i < j$, $f \longmapsto \bar{f} \longmapsto f'$ is an Abelian group isomorphism: $\mathrm{Hom}_R(I_i, I_j) \longrightarrow \mathrm{End}_R(I_1) = D$.

If $1 \leq j < i \leq n$ and $g: I_1 \longrightarrow I_1$ is in $J(D)$ then $\ker g \supseteq B_n$. Hence g induces $g^+: I_1/B_{i-1} \longrightarrow I_1$ such that $g^+\pi_{i-1} \ldots \pi_1 = g$. Let $f = \pi_{j-1} \ldots \pi_1 g^+: I_1/B_{i-1} \longrightarrow I_1/B_{j-1}$. If $g^*\pi_n \ldots \pi_1 = \pi_n \ldots \pi_1 g$ then

$$
\begin{aligned}
g^*\pi_n \ldots \pi_1 &= \pi_n \ldots \pi_j \pi_{j-1} \ldots \pi_1 g \\
&= \pi_n \ldots \pi_j g^+ \pi_{i-1} \ldots \pi_1 \\
&= \pi_n \ldots \pi_j \pi_{j-1} \ldots \pi_1 g^+ \pi_{i-1} \ldots \pi_1 \\
&= \pi_n \ldots \pi_j f \pi_{i-1} \ldots \pi_1 .
\end{aligned}
$$

Since $\pi_{i-1} \ldots \pi_1$ is an epimorphism

$$
\begin{aligned}
g^*\pi_n \ldots \pi_i &= \pi_n \ldots \pi_j f = \bar{f}\pi_n \ldots \pi_i \\
g^* &= \bar{f}.
\end{aligned}
$$

Since for $j < i$, $\mathrm{Im}(f \longmapsto f') \subseteq J(D)$, $f \longmapsto f'$ is an Abelian group isomorphism of $\mathrm{Hom}_R(I_i, I_j) \longrightarrow J(D) = J(\mathrm{End}_R(I_1))$.

<u>Theorem 3.19</u>

Assuming $R = \hat{R}_N$ and $N = \bigcap_{i=1}^{n} P_i$, and assuming $\{P_1, \ldots, P_n\}$ is a clan, let $I = E_1 \oplus \ldots \oplus E_m$ where

$$\begin{aligned} E_1 &\cong E_2 \cong \ldots \cong E_{s_1} \cong I_1 \\ E_{s_1+1} &\cong E_{s_1+2} \cong \ldots \cong E_{s_1+s_2} \cong I_2 \\ &\vdots \\ E_{s_{n-1}+1} &\cong E_{s_{n-1}+2} \cong \ldots \cong E_{s_{n-1}+s_n} \cong I_n. \end{aligned}$$

Let $K = \mathrm{End}_R(I)$. Then

$$K \cong \begin{pmatrix} D_{s_1 x s_1} & J(D)_{s_1 x s_2} & J(D)_{s_1 x s_3} & \ldots & J(D)_{s_1 x s_n} \\ D_{s_2 x s_1} & D_{s_2 x s_2} & J(D)_{s_2 x s_3} & \ldots & J(D)_{s_2 x s_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D_{s_n x s_1} & D_{s_n x s_2} & D_{s_n x s_3} & \ldots & D_{s_n x s_n} \end{pmatrix}$$

<u>Proof:</u>

Let $\psi_{st}: \mathrm{Hom}_R(E_t, E_s) \longrightarrow D$ (or $J(D)$) be the Abelian group isomorphism found in Proposition 3.18 — i.e. $\psi_{st}(f) = f'$ where

$$\begin{array}{ccccc} I_1/B_{i-1} = I_i \xrightarrow[\sim]{\beta} E_t \xrightarrow{f} E_s \xrightarrow{\delta^{-1}} I_j = I_1/B_{j-1} \\ \pi_n \ldots \pi_i \searrow \qquad\qquad = \qquad\qquad \swarrow \pi_n \ldots \pi_j \\ I_1/B_n \xrightarrow{\quad \bar{f} \quad} I_1/B_n \\ \pi_n \ldots \pi_1 \uparrow \qquad = \qquad \uparrow \pi_n \ldots \pi_1 \\ I_1 \xrightarrow{\quad f' \quad} I_1 \end{array}$$

Define $\psi(f_{st}) = (\psi_{st}(f_{st}))$ (recall that $f_{st}: E_t \longrightarrow E_s$).

Obviously ψ preserves 0, 1, -, +.

$$\psi((f_{st})(g_{st})) = \psi(\sum_{u=1}^{m} f_{su}g_{ut})$$
$$= (\psi_{st}(\sum_{u=1}^{m} f_{su}g_{ut}))$$
$$= (\sum_{u=1}^{m} \psi_{st}(f_{su}g_{ut})).$$

Now $(f_{su}g_{ut})' = (f_{su})'(g_{ut})'$ iff $(\overline{f_{su}g_{ut}}) = \overline{f}_{su}\overline{g}_{ut}$.

$$\begin{array}{ccccccccccccc}
I_1/B_{i-1} \cong I_i & \xrightarrow{\beta} & E_t & \xrightarrow{g_{ut}} & E_u & \xrightarrow{\gamma^{-1}} & I_k & \xrightarrow{\gamma} & E_u & \xrightarrow{f_{su}} & E_s & \xrightarrow{\delta^{-1}} & I_j \cong I_1/B_{j-1} \\
\downarrow \pi_n\cdots\pi_i & & & & & & \downarrow \pi_n\cdots\pi_k & & & & & & \downarrow \pi_n\cdots\pi_j \\
I_1/B_n & & & \xrightarrow{\overline{g}_{ut}} & & & I_1/B_n & & & \xrightarrow{\overline{f}_{su}} & & & I_1/B_n
\end{array}$$

By definition, $(\overline{f_{su}g_{ut}})\pi_n\cdots\pi_i = \pi_n\cdots\pi_j\delta^{-1}f_{su}g_{ut}\beta$

$$= \pi_n\cdots\pi_j\delta^{-1}f_{su}\gamma\gamma^{-1}g_{ut}\beta$$
$$= \overline{f}_{su}\pi_n\cdots\pi_k\gamma^{-1}g_{ut}\beta$$
$$= \overline{f}_{su}\overline{g}_{ut}\pi_n\cdots\pi_i$$

Hence $(\overline{f_{su}g_{ut}}) = \overline{f}_{su}\overline{g}_{ut}$. Hence $\psi((f_{st})(g_{st})) = (\sum_{u=1}^{m}\psi_{st}(f_{su}g_{ut}))$ $= (\sum_{u=1}^{m}\psi_{su}(f_{su})\psi_{ut}(g_{ut})) = \psi(f_{st})\psi(g_{st})$ — i.e. ψ is a ring homomorphism. By Proposition 3.18, if $1 \leq s \leq s_1$, $1 \leq t \leq s_1$ then $\psi_{st}(\mathrm{Hom}_R(E_t, E_s)) = D$. If $1 \leq s \leq s_1$, $s_1+1 \leq t \leq s_1+s_2$, then $E_t \cong I_2$ and $E_s \cong I_1$ so $\psi_{st}(\mathrm{Hom}_R(E_t, E_s)) = J(D)$. Similarly, $\forall\, t > s_1$ and $1 \leq s \leq s_1$, $\psi_{st}(\mathrm{Hom}_R(E_t, E_s)) = J(D)$. If $s_1+1 \leq s \leq s_1+s_2$ and $1 \leq t \leq s_1$ then $E_t \cong I_1$ and $E_s \cong I_2$, therefore $\psi_{st}(\mathrm{Hom}_R(E_t, E_s)) \cong D$. If $s_1+1 \leq t \leq s_1+s_2$, then $E_t \cong I_1 \cong E_s$ and so $\psi_{st}(\mathrm{Hom}_R(E_t, E_s)) = D$. However, if $s_1+1 \leq s \leq s_1+s_2$ and $t > s_2$ then $E_t \cong I_i$ for some $i > 2$ while $E_s \cong I_2$ so by Proposition 3.18, $\psi_{st}(\mathrm{Hom}_R(E_t, E_s)) = J(D)$. Proceeding thus, we see

$$K \cong \begin{pmatrix} D_{s_1 x s_1} & J(D)_{s_1 x s_2} & \cdots & J(D)_{s_1 x s_n} \\ \vdots & \vdots & & \vdots \\ D_{s_n x s_1} & D_{s_n x s_2} & \cdots & D_{s_n x s_n} \end{pmatrix}$$

(That ψ is an isomorphism follows immediately from Proposition 3.18).

Corollary

Let R be an FBN hereditary ring and $N = \bigcap_{i=1}^{n} P_i$ the intersection of a clan of non-minimal prime ideals. Let I_i be the unique indecomposable injective with associated prime ideal P_i. Suppose $E = E(R/N) = I_1^{t_1} + \ldots + I_n^{t_n}$. Then

$$\hat{R}_N \cong \begin{pmatrix} D_{t_1 \times t_1} & J(D)_{t_1 \times t_2} & J(D)_{t_1 \times t_3} & \cdots & J(D)_{t_1 \times t_n} \\ D_{t_2 \times t_1} & D_{t_2 \times t_2} & J(D)_{t_2 \times t_3} & \cdots & J(D)_{t_2 \times t_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D_{t_n \times t_1} & D_{t_n \times t_2} & D_{t_n \times t_3} & \cdots & D_{t_n \times t_n} \end{pmatrix}$$

where $D = \mathrm{End}_R(I_1)$ is a complete rank-one discrete valuation ring. If $\{P_1, \ldots, P_n\}$ is the disjoint union of clans $\mathcal{S}_1, \ldots, \mathcal{S}_r$ then $\hat{R}_N$ is a product of r such matrix rings.

Extension to an arbitrary semiperfect HNP ring R:

We sketch here a new proof of a theorem of Michler [27] on the structure of an arbitrary semiperfect HNP ring R. If R is primitive, it is already semi-simple Artinian, so is a full ring of nxn matrices over some division ring. If R is not primitive, it is FBN, hence $\bigcap_{n=1}^{\infty} J(R)^n = 0$ and $R \hookrightarrow \hat{R}$. Let $1 = f_1 + \ldots + f_m$ where the f_j are local orthogonal idempotents in R. Then the f_j remain local orthogonal idempotents in $\hat{R}$ because $R/J(R) \cong \hat{R}/J(\hat{R})$. As usual, let $E = E(R/J)$ and $H = \mathrm{End}_R(E)$.

Since E_R is Artinian, $\hat{R}$ has Morita duality with °H induced by ${}_HE_R$ (Theorem 2.14). Considered as an H-module, E is a finite direct sum of indecomposable injective H-modules, $E = Y_1 \oplus \ldots \oplus Y_s$ which give rise to local orthogonal idempotents $\{e_i \mid i=1,\ldots,s\}$ in $\hat{R}$ whose sum is 1. Then $D_1=e_1\hat{R}e_1 \cong \mathrm{End}_H(Y_1)$ is a complete discrete rank-one valuation ring by Proposition 3.16, and by 3.18, $D_1 \cong e_i\hat{R}e_i\ \forall i = 1, 2,\ldots, s$. From the general theory of semiperfect rings ([18, Prop.3, p. 77]) we also know s=m and $D_1 \cong f_j\hat{R}f_j\ \forall j=1, 2,\ldots, m$. Let $D=f_1Rf_1$. Clearly D_1 is (isomorphic to) the completion of D in the J(D)-adic topology. It is equally clear that $\forall i,j = 1,\ldots, m$, $f_i\hat{R}f_j \cap R = f_iRf_j$. Because of the one-one correspondence between right ideals of $\hat{R}$ and left H-submodules of E, $\mathrm{Ann}_E(\oplus_{j\neq i} e_j\hat{R})$ is an indecomposable direct summand of ${}_HE$ so we may assume without loss of generality that $e_i=f_i$ and $Y_i=\mathrm{Ann}_E(\oplus_{j\neq i} f_j\hat{R})\ \forall i = 1,\ldots, m$. Given $r \in f_1Rf_1$, it may be viewed as the H-homomorphism $y_1 \longmapsto y_1r$ of $Y_1 \longrightarrow Y_1$ which induces $[y_1] \mapsto [y_1r]$ from $Y_1/\mathrm{Ann}\ J(H)^j \longrightarrow Y_1/\mathrm{Ann}\ J(H)^j\ \forall j$. In other words, if $Y_i \cong Y_1/\mathrm{Ann}\ J(H)^{n_i}$ and $\varphi_i : f_1\hat{R}f_1=\mathrm{End}_H(Y_1) \xrightarrow{\sim} \mathrm{End}_H(Y_i) \cong f_i\hat{R}f_i$ is the isomorphism found in Proposition 3.18,

$$\mathrm{Im}(\varphi_i|_{f_1Rf_1}) \subseteq R \cap f_i\hat{R}f_i = f_iRf_i\ \forall i.$$

Hence the isomorphisms $f_1\hat{R}f_1 \cong f_i\hat{R}f_i$ induce isomorphisms $f_1Rf_1 \cong f_iRf_i\ \forall i = 1,\ldots, m$. Similarly one sees that for $1 \leq n_i < n_j \leq m$ $f_iRf_j \cong f_1Rf_1$ (as Abelian groups) and for $1 \leq n_j < n_i \leq m$ $f_iRf_j \cong J(f_1Rf_1)$. It is then a matter of straightforward verification (cf. Proposition 3.17, 3.19) to see that for some t_i, $Y_1 \cong \ldots \cong Y_{t_1}$, $Y_{t_1+1} \cong \ldots \cong Y_{t_1+t_2}$, . . . , $Y_{t_{r-1}+1} \cong \ldots \cong Y_{t_{r-1}+t_r}$ and that

$$R \cong \int \begin{pmatrix} D_{t_1 x t_1} & J(D)_{t_1 x t_2} & \cdots & J(D)_{t_1 x t_r} \\ D_{t_2 x t_1} & D_{t_2 x t_2} & \cdots & J(D)_{t_2 x t_r} \\ \vdots & \vdots & \vdots & \vdots \\ D_{t_r x t_1} & D_{t_r x t_2} & \cdots & D_{t_r x t_r} \end{pmatrix}$$

D is a hereditary local Noetherian domain (cf. the proof of Lemma 3.16) and since e_1 is a local idempotent, by passing to the factor rings $D/J(D)^n$ it is easy to show that D has as only one-sided ideals $D \supsetneq J(D) \supsetneq J(D)^2 \supsetneq \ldots \supsetneq J(D)^m \supsetneq \ldots$. It follows from this that $J(D)^n = Da^n = a^nD$ for some (any) $a \in J(D) \setminus J(D)^2$. Thus we have proved

<u>Theorem 3.20</u> (Michler)

Let R be a semiperfect Noetherian hereditary ring. Then R is a finite product of indecomposable rings R_i where R_i is either a full ring of nxn matrices over some division ring V_i or R_i has a $D_i:J(D_i)$-upper triangular matrix structure described above for some discrete valuation ring D_i. If R is complete, each D_i may be chosen to be complete.

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