A priori estimates of the degenerate Monge-Ampère equation on compact Kähler manifolds

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ABSTRACT

The regularity theory of the degenerate complex Monge-Ampère equation is studied. First, the equation is considered on a compact Kähler manifold (M, g) without boundary of dimension m. Accordingly, some background information on Kähler geometry is presented.

Given a solution φ of the degenerate complex Monge-Ampère equation, it is shown that its oscillation and gradient can be bounded independently of φ. The Laplacian of φ is also estimated. There is a slight improvement from the literature on the conditions required in order to obtain the estimate on the Laplacian of φ, however the estimates developed only hold in the case of non-negative bisectional curvature of M.

As an application, a Dirichlet problem in $\mathbb{C}^m$ is considered. The obtained estimates are used to show existence and uniqueness of pluri-subharmonic solutions to the degenerate complex Monge-Ampère equation in a domain in $\mathbb{C}^m$.

ABRÉGÉ

La question de la régularité des solutions de l'équation complexe Monge-Ampère dégénérée est étudiée. Premièrement, l'équation est considérée sur une variété compacte Kähler (M, g) sans frontière de dimension m. Une revue des concepts clés de la géométrie Kähler est présentée.

Soit φ une solution de l'équation complexe Monge-Ampère dégénérée. Il est démontré que la différence entre la borne supérieure et la borne inférieure de φ est sous controle, et ainsi pour le gradient de φ. Le Laplacien de φ est également bornée. Cette borne du Laplacien est une légerte amélioration de ce qui a été établi dans la littérature jusqu'à présent, mais par contre, l'argument tient seulement sous la condition que M a une courbure non-négative.

Les résultats sont appliqués à un problème de Dirichlet dans $\mathbb{C}^m$. L'existence et l'unicité d'une solution pluri-subharmonique de l'équation complexe Monge-Ampère dégénérée dans un domaine dans $\mathbb{C}^m$ est démontré.

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CHAPTER 1
Introduction

We will be looking at the regularity theory of the degenerate complex Monge-Ampere equation. Let us consider the equation on a compact Kähler manifold (M, g) without boundary of dimension m. The problem of solving the complex Monge-Ampere equation on M was first motivated by the Calabi conjecture. The Calabi conjecture states that given a closed $(1, 1)$ form $\tilde{R}$ in the Chern class of M, then there exists a metric $\tilde{g}$ on M such that the Ricci form of $(M, \tilde{g})$ is $\tilde{R}$. This problem was reduced to solving a non-degenerate Monge-Ampere equation, and the question was completely solved by S.T. Yau in [12]. The solution of the Calabi conjecture had significant geometric implications, in particular, leading to the theory of Calabi-Yau manifolds, which now play a central role in string theory and complex geometry.

Although the Calabi conjecture deals with a non-degenerate Monge-Ampere equation, Yau's paper also treated the degenerate Monge-Ampere equation, with an application to holomorphic sections of line bundles over M. More recently, the degenerate Monge-Ampere equation has resurfaced in complex geometry through the works of Donaldson [7]. Here, the degenerate Monge-Ampere equation appears in trying to find a geodesic joining two Kähler potentials in the space of Kähler metrics. It was shown by Chen in [6] that there exists $C^{1,1}$ geodesics, and the proof involved the regularity theory of a degenerate Monge-Ampere equation.

In this thesis, we shall consider the following complex Monge-Ampere equation:

$$\det(\varphi_{i\bar{j}} + g_{i\bar{j}}) = f \det g_{i\bar{j}}, \tag{1.1}$$

where $f : M \to \mathbb{R}$. We shall assume $f > 0$, however, the goal will be to develop an estimate that does not depend on the infimum of f. This will allow the *a priori* estimates to be applicable to the degenerate case when $f \geq 0$ via a limiting process.

The objective is the following: given a solution φ to (1.1) such that $(\varphi_{i\bar{j}} + g_{i\bar{j}})$ is positive-definite, we seek an estimate $|\Delta\varphi| \leq C$ depending only on (M, g), $\sup f$ and the infimum of $\Delta f^{\frac{1}{m-1}}$.

This problem is motivated by a similar result obtained for the real Monge-Ampere equation by P. Guan in [8]. Previously, a bound on the Laplacian of φ was obtained by Blocki in [3] while assuming that $f^{\frac{1}{m-1}}\Delta(\log f)$ is bounded below. We obtain the estimate $|\Delta\varphi| \leq C$ depending only on (M, g), $\sup f$ and the infimum of $\Delta f^{\frac{1}{m-1}}$, but only in the case when M is non-negatively curved. For manifolds of arbitrary curvature, a result is only obtained in the case $m = 2$, and the estimate also depends on $|\nabla f^{1/m}|$.

As an application of the estimates, we solve the Dirichlet problem for the degenerate complex Monge-Ampere equation on a domain Ω in $\mathbb{C}^m$. This type of question was investigated by Bedford and Taylor in [2], and Caffarelli, Kohn, Niremberg and Spruck in [5]. Here the problem is of the following form:

$$\det u_{i\bar{j}}(z) = f(z) \;\text{ in } \Omega,$$

$$u = 0 \;\text{ on } \partial\Omega.$$

We assume $f \geq 0$, $f^{\frac{1}{m}}$ is Lipschitz, and $\Delta f^{\frac{1}{m-1}}$ is bounded below, and show that there exists a unique solution u such that $(u_{i\bar{j}})$ is positive-definite, and $|\Delta u| \leq C$.

CHAPTER 2
Preliminaries

2.1 Notions from Kähler Geometry

In this section, we will recall concepts and notation from the geometry of Kähler manifolds. Let M be a complex manifold with local coordinates $(z^1, \dots, z^m)$. These coordinates are related to the coordinates of the real manifold of dimension $2m$ via $z^k = x^k + iy^k$, where $i = \sqrt{-1}$. In these local coordinates, we have the 1-forms

$$dz^k = dx^k + idy^k, \qquad d\bar{z}^k = dx^k - idy^k.$$

We denote the set of k forms by $\Omega^k M$. The space $\Omega^k M$ is spanned locally by forms of the type

$$\eta(z) = f(z)dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q},$$

where f is a smooth function defined locally, $k = p+q$ where p and q are non-negative integers, and $i_1, \dots, i_p, j_1, \dots, j_q$ are elements of $\{1, 2, \dots, m\}$. We write

$$\Omega^k M = \bigoplus_{p+q=k} \Omega^{p,q} M,$$

where $\Omega^{p,q} M$ consists of forms which are locally spanned by the basis consisting of the wedge product of p forms from $\{dz^1, \dots, dz^m\}$ and q forms from $\{d\bar{z}^1, \dots, d\bar{z}^m\}$.

With respect to these coordinates, we define the following basis for the tangent space of M

$$\frac{\partial}{\partial z^k} := \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right),$$
$$\frac{\partial}{\partial \bar{z}^k} := \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right).$$

For a function $f : M \to \mathbb{C}$, we will use any of the following notation interchangeably: $\frac{\partial f}{\partial z^k}$, $\partial_k f$, and f_k. Similarly, we will sometimes denote $\frac{\partial f}{\partial \bar{z}^k}$ by $\bar{\partial}_k f$ or $f_{\bar{k}}$.

We let the operators $\partial : \Omega^{p,q}M \to \Omega^{p+1,q}M$ and $\bar{\partial} : \Omega^{p,q}M \to \Omega^{p,q+1}M$ act on η in the following way. We remind the reader that the Einstein summation convention is always assumed.

$$\partial\eta = \frac{\partial f}{\partial z^\gamma} dz^\gamma \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}.$$

$$\bar{\partial}\eta = \frac{\partial f}{\partial \bar{z}^\gamma} dz^\gamma \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}.$$

We can compute that

$$(\partial + \bar{\partial})(f) = \frac{\partial f}{\partial x^j} dx^j + \frac{\partial f}{\partial y^j} dy^j = df,$$

and similarly for forms, so we have $\partial + \bar{\partial} = d$.

A *Hermitian metric* on M is an expression of the form

$$g_{j\bar{k}}(z) dz^j \otimes d\bar{z}^k,$$

such that $(g_{j\bar{k}}(z))$ is a Hermitian, positive-definite matrix at each point z and varies smoothly with z.

For future reference, we compute the derivative of $g^{i\bar{j}}$. We begin with the definition:

$$g^{i\bar{j}} g_{i\bar{k}} = \delta^{\bar{j}}_{\bar{k}}.$$

Next, we take the derivative of both sides:

$$\frac{\partial g^{i\bar{j}}}{\partial z^k} g_{i\bar{k}} = -g^{i\bar{j}} \frac{\partial g_{i\bar{k}}}{\partial z^k}.$$

Contracting with $g^{l\bar{k}}$ yields

$$g^{l\bar{k}} g_{i\bar{k}} \frac{\partial g^{i\bar{j}}}{\partial z^k} = -g^{l\bar{k}} g^{i\bar{j}} \frac{\partial g_{i\bar{k}}}{\partial z^k}.$$

Hence we obtain

$$\frac{\partial g^{l\bar{j}}}{\partial z^k} = -g^{l\bar{k}} g^{i\bar{j}} \frac{\partial g_{i\bar{k}}}{\partial z^k}. \tag{2.1}$$

Definition 2.1.1. *The Kähler form Ω of (M, g) is defined as*

$$\Omega = \frac{i}{2} g_{j\bar{k}} dz^j \wedge d\bar{z}^k.$$

The Kähler form gives us the notion of a Kähler manifold.

Definition 2.1.2. *Let (M, g) be a complex manifold with a Hermitian metric. We say M is a Kähler manifold if the Kähler form is closed: $d\Omega = 0$.*

The simple property $d\Omega = 0$ yields a surprising amount of nice consequences. We will now list the ones that will be used in the subsequent sections. These results are all standard, and their proofs can be found in [10].

Theorem 2.1.1. (Existence of Normal Holomorphic Coordinates) *A Hermitian manifold (M, g) is Kähler if and only if at each point $p \in M$, there exists holomorphic normal coordinates: coordinates such that $g_{j\bar{k}}(p) = \delta_{jk}$ and*

$$\frac{\partial g_{j\bar{k}}(p)}{\partial z^l} = 0, \quad \frac{\partial g_{j\bar{k}}(p)}{\partial \bar{z}^l} = 0,$$

for all indices j, k, l.

Theorem 2.1.2. (Global $\partial\bar{\partial}$ lemma) *Let M be a compact Kähler manifold. If ω is a $(1,1)$ form on M such that $\omega = d\alpha$ for some one-form α, then there exists a function F on M such that $\omega = \partial\bar{\partial}F$.*

There is a relationship between the Kähler form Ω and the integration volume element $dVol$ of the real Riemannian manifold identified with M of dimension $2m$. We note that by definition of dz^k and $d\bar{z}^k$, we have $(i/2)dz^k \wedge d\bar{z}^k = dx^k \wedge dy^k$.

Theorem 2.1.3. *Let (M, g) be a Kähler manifold of dimension m. Then the Kähler form Ω satisfies*

$$\begin{aligned}\Omega^m &= m!(i/2)^m \det(g_{j\bar{k}})\ dz^1 \wedge d\bar{z}^1 \wedge \ldots dz^m \wedge d\bar{z}^m \\ &= m!\ dVol.\end{aligned}$$

The condition $d\Omega = 0$ also implies some nice properties for the Christoffel symbols. The closedness property means that

$$\frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial g_{k\bar{j}}}{\partial z^i}, \quad \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^l} = \frac{\partial g_{i\bar{l}}}{\partial \bar{z}^j}. \tag{2.2}$$

From the representation $\Gamma^A_{BC} = \frac{1}{2} g^{AD}(g_{BD,C} + g_{CD,B} - g_{BC,D})$, we can see that many Christoffel symbols must vanish.

Proposition 2.1.1. *Let (M, g) be a Kähler manifold. Then the only non-vanishing Christoffel symbols are of the form $\Gamma^{\alpha}_{\beta\gamma}$ or $\Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}$. In other words, all Christoffel symbols involving both barred and unbarred indices are identically zero.*

The Laplace operator Δ takes a particularly nice form on a Kähler manifold due to the identities (2.2).

Proposition 2.1.2. *On a Kähler manifold (M, g), the Laplace operator takes the following form*

$$\Delta = g^{j\bar{k}} \frac{\partial^2}{\partial z^j \partial \bar{z}^k}.$$

The Riemannian curvature tensor R also has a simplified coordinate expression on a Kähler manifold. For our purposes, we may take the following as a definition:

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \sum_{p,q} g^{p\bar{q}} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}^l} \frac{\partial g_{i\bar{q}}}{\partial z^k}. \tag{2.3}$$

The Bianchi identity and the symmetries of the derivatives of g recorded in (2.2) yield the following symmetries for the curvature tensor

$$R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}}, \ R_{i\bar{j}k\bar{l}} = R_{k\bar{l}i\bar{j}}. \tag{2.4}$$

We end this review on Kähler geometry with a technical definition. A lower bound for the bisectional curvature of M is a constant B such that at each point we have

$$R_{i\bar{j}k\bar{l}} a^i \overline{a^j} b^k \overline{b^l} \geq B|a|^2|b|^2, \quad a, b \in \mathbb{C}^m. \tag{2.5}$$

If $B \geq 0$, we say that M has non-negative bisectional curvature.

2.2 Useful Identities and Inequalities

Before ending this section, we mention some properties that will be useful when considering equation (1.1). We will use the convention $\varphi_{i\bar{j}} = \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}$; as opposed to

[12], subscripts do not indicate covariant derivatives. We shall denote

$$g'_{i\bar{j}} := g_{i\bar{j}} + \varphi_{i\bar{j}}. \tag{2.6}$$

Since we are looking for a solution φ of (1.1) such that $(g_{i\bar{j}} + \varphi_{i\bar{j}})$ is positive-definite, we have that g' defines a metric on M. Moreover, since

$$d\left(g'_{j\bar{k}} dz^j \wedge d\bar{z}^k\right) = d\left(g_{i\bar{j}} dz^j \wedge d\bar{z}^k\right) + d\left(\varphi_{j\bar{k}} dz^j \wedge d\bar{z}^k\right) = 0,$$

we have that g' is a Kähler metric on M. We shall use the notation Δ' to denote the Laplacian of (M, g'). While proving the second order estimate, we shall encounter an expression involving $\Delta'\Delta\varphi$. As it will be useful later on, now is as good of a time as any to embark on a side-calculation and explicitly compute $\Delta'\Delta\varphi$. First, by direct computation, we have

$$\begin{aligned}
\Delta'(\Delta\varphi) &= g'^{k\bar{l}} \frac{\partial^2}{\partial z^k \partial \bar{z}^l} \left(g^{i\bar{j}} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right) \\
&= g'^{k\bar{l}} \frac{\partial^2 g^{i\bar{j}}}{\partial z^k \partial \bar{z}^l} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} + g'^{k\bar{l}} \left(\frac{\partial g^{i\bar{j}}}{\partial \bar{z}^l} \frac{\partial^3 \varphi}{\partial z^i \partial \bar{z}^j \partial z^k} + \frac{\partial g^{i\bar{j}}}{\partial z^k} \frac{\partial^3 \varphi}{\partial z^i \partial \bar{z}^j \partial \bar{z}^l} \right) \\
&+ g'^{k\bar{l}} g^{i\bar{j}} \frac{\partial^4 \varphi}{\partial z^i \partial \bar{z}^j \partial z^k \partial \bar{z}^l}.
\end{aligned}$$

We choose coordinates guaranteed by Theorem 2.1.1 such that at the point in consideration, we have $g_{i\bar{j}} = \delta_{ij}$, $\partial g_{i\bar{j}}/\partial z^k = 0$ and $\partial g_{i\bar{j}}/\partial \bar{z}^l = 0$. Furthermore, we can rotate this orthonormal basis in order to diagonalize the Hermitian matrix $(\varphi)_{i\bar{j}}$. Hence we further assume that $\varphi_{i\bar{j}} = \delta_{ij}\varphi_{j\bar{j}}$. With this convenient choice of coordinates, the previous expression becomes

$$\Delta'(\Delta\varphi) = g'^{k\bar{l}} \frac{\partial^2 g^{i\bar{j}}}{\partial z^k \partial \bar{z}^l} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} + g'^{k\bar{l}} \delta^{i\bar{j}} \frac{\partial^4 \varphi}{\partial z^i \partial \bar{z}^j \partial z^k \partial \bar{z}^l}.$$

From (2.1), we substitute the expression for $\frac{\partial g'^{i\bar{j}}}{\partial z^k}$, noting that all first order derivatives of the metric vanish.

$$\Delta'(\Delta\varphi) = -g'^{k\bar{l}} g^{i\bar{t}} g^{n\bar{j}} \frac{\partial^2 g_{n\bar{t}}}{\partial z^k \partial \bar{z}^l} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} + g'^{k\bar{l}} \delta^{i\bar{j}} \frac{\partial^4 \varphi}{\partial z^i \partial \bar{z}^j \partial z^k \partial \bar{z}^l}.$$

Exploiting symmetries $g_{i\bar{j}} = g_{j\bar{i}}$ and using (2.3), we obtain our final expression.

$$\Delta'(\Delta\varphi) = g'^{k\bar{l}} \delta^{i\bar{j}} R_{i\bar{j}k\bar{l}} \varphi_{i\bar{j}} + g'^{k\bar{l}} \delta^{i\bar{j}} \varphi_{k\bar{l}i\bar{j}}. \tag{2.7}$$

Before moving on, it needs to be shown that equation (1.1) is well-defined on M. We need to show that $(\det g'_{i\bar{j}})(\det g_{i\bar{j}})^{-1}$ is independent of choice of coordinates and is thus globally defined.

Indeed, let $\{w^1, \ldots, w^m\}$ be another set of coordinates, and express the metric g in these coordinates as $h_{j\bar{k}}$ and the metric g' in these coordinates as $h'_{j\bar{k}}$. The change of coordinates formula gives us $g_{k\bar{l}}(z) = h_{\alpha\bar{\beta}}(w) \frac{\partial w^\alpha}{\partial z^k} \frac{\partial w^{\bar{\beta}}}{\partial \bar{z}^l}$. Hence

$$\det(g_{k\bar{l}}) = \det\left(\sum_{\alpha,\beta} \frac{\partial w^\alpha}{\partial z^k} h_{\alpha\bar{\beta}} \frac{\partial w^{\bar{\beta}}}{\partial \bar{z}^l}\right) = \det\left(\frac{\partial w^k}{\partial z^l}\right) \det\left(h_{k\bar{l}}\right) \det\left(\frac{\partial w^{\bar{k}}}{\partial \bar{z}^l}\right).$$

We used the property $\det(ABC) = \det A \cdot \det B \cdot \det C$. An analogous identity holds for $\det(g'_{k\bar{l}})$, and since the coordinate transformation determinants cancel, we obtain

$$\frac{\det(g'_{k\bar{l}})}{\det(g_{k\bar{l}})} = \frac{\det(h'_{k\bar{l}})}{\det(h_{k\bar{l}})}.$$

We now state a few handy estimates. Since $g'_{i\bar{j}} = g_{i\bar{j}} + \varphi_{i\bar{j}}$ is positive definite, we have $0 < Tr(g_{i\bar{j}} + \varphi_{i\bar{j}})$. At any point $p \in M$, we may choose coordinates such that $g_{i\bar{j}} = \delta_{i\bar{j}}$. Thus

$$0 < m + \Delta\varphi. \tag{2.8}$$

We next notice the following inequality

$$\left(\sum_{i=1}^{m} \frac{1}{B_i}\right)^{m-1} \geq \frac{\sum_{i=1}^{m} B_i}{\prod_{i=1}^{m} B_i},$$

for $B_i > 0$. The inequality comes from clearing the denominators and $(\sigma_{m-1}(B))^{m-1} \geq (\prod_i^m B_i)^{m-2}(\sum_i^m B_i)$ where $\sigma_{m-1}(B)$ is the elementary symmetric polynomial of order $m-1$ in the m variables $B_1, \ldots, B_m$. Since $g^{i\bar{i}} > 0$, we thus have

$$\sum_i g'^{i\bar{i}} \geq \left(\frac{\sum_i g'_{i\bar{i}}}{\prod_i g'_{i\bar{i}}}\right)^{1/(m-1)}.$$

Since $(\det g'_{i\bar{j}}) = f(\det g_{i\bar{j}})$, if our coordinates at $p \in M$ are such that $g_{i\bar{j}} = \delta_{i\bar{j}}$ and $\varphi_{i\bar{j}}$ is diagonal, then $\prod_i g'_{i\bar{i}} = f$. Thus, with these coordinates,

$$\sum_i g'^{i\bar{i}} \geq \left(\frac{m + \Delta\varphi}{f}\right)^{\frac{1}{m-1}}. \tag{2.9}$$

Another estimate that will be used many times is the inequality of arithmetic and geometric means: for any list of m non-negative numbers B_i, we have

$$\frac{B_1 + B_2 + \cdots + B_m}{m} \geq (B_1 \cdot B_2 \cdots B_m)^{1/m}.$$

If we set $B_i = g'^{i\bar{i}}$, assuming the coordinates as above, we get

$$\sum_i g'^{i\bar{i}} \geq \left(\frac{m}{f^{1/m}}\right). \tag{2.10}$$

There is one more often-used identity to be shown in this section. As before, we assume our coordinates at $p \in M$ are such that $g_{i\bar{j}} = \delta_{i\bar{j}}$, $\partial_k g_{j\bar{k}}(p) = 0$ and $\varphi_{i\bar{j}}$ is diagonal. Let $\alpha > 0$, and compute

$$\begin{aligned}
\partial_k (f \det g_{i\bar{j}})^\alpha &= \partial_k (\det g'_{i\bar{j}})^\alpha \\
&= \alpha (\det g'_{i\bar{j}})^{\alpha-1} \partial_k \det g'_{i\bar{j}} \\
&= \alpha (\det g'_{i\bar{j}})^{\alpha} g'^{\gamma\bar{\gamma}} \partial_k g'_{\gamma\bar{\gamma}} \\
&= \alpha (\det g'_{i\bar{j}})^{\alpha} g'^{\gamma\bar{\gamma}} \varphi_{\gamma\bar{\gamma}k}.
\end{aligned}$$

The last line used the formula for the derivative of the determinant $\partial_k \det A_{ij} = \sum_{i,j} c(A_{ij}) \partial_k A_{ij} = \det A_{ij} \sum_{i,j} A^{ji} \partial_k A_{ij}$ for a matrix A. Thus

$$g'^{\gamma\bar{\gamma}} \varphi_{\gamma\bar{\gamma}k} = \frac{\partial_k f^\alpha}{\alpha f^\alpha}. \tag{2.11}$$

CHAPTER 3
Estimates on Kähler Manifolds

3.1 Overview

The main result of this section is the following:

Theorem 3.1.1. *Let (M, g) be a compact Kähler manifold with non-negative bisectional curvature and empty boundary. Let $f > 0$ be a positive function on M such that* $\inf_M \Delta f^{\frac{1}{m-1}} \geq -A$ *for some constant A. For all $\varphi \in C^4(M)$ satisfying* (1.1) *such that $(\varphi_{i\bar{j}} + g_{i\bar{j}})$ is positive-definite, we have*

$$(\sup_M \varphi - \inf_M \varphi) + ||\nabla\varphi||_\infty + ||\Delta\varphi||_\infty \leq C,$$

where C depends on (M, g), A, and $\sup f$.

Even though we assume $f > 0$, this sort of estimate is useful for the degenerate Monge-Ampère equation, since it does not depend on the lower bound of f.

If the assumption on the bisectional curvature of M is removed, results were only obtained for the case when the dimension of M is $m = 2$. The estimate also depends on $\nabla f^{1/m}$. The result is the following:

Theorem 3.1.2. *Let (M, g) be a compact Kähler manifold without boundary of dimension $m = 2$. Let $f > 0$ be a positive function on M such that* $\inf_M \Delta f^{\frac{1}{m-1}} \geq -A$ *for some constant A, and $f^{\frac{1}{m}}$ is Lipschitz continuous. For all $\varphi \in C^4(M)$ satisfying* (1.1) *such that $(\varphi_{i\bar{j}} + g_{i\bar{j}})$ is positive-definite, we have*

$$(\sup_M \varphi - \inf_M \varphi) + ||\nabla\varphi||_\infty + ||\Delta\varphi||_\infty \leq C,$$

where C depends on (M, g), A, the Lipschitz constant of $f^{\frac{1}{m}}$, and $\sup f$.

The only new work appearing in the proof of these theorems are the estimates on the Laplacian of φ. We estimate the Laplacian directly from the L^∞ estimate, and therefore in the proof of Theorem 3.1.1, we obtain the estimate $|\Delta\varphi| \leq C$ depending only on (M, g), $(\sup \varphi - \inf \varphi)$, $\sup f$ and the infimum of $\Delta f^{\frac{1}{m-1}}$. However, a direct gradient estimate can also be useful in some applications, for example when dealing with a manifold with boundary. For the sake of completeness, we include the known proofs of the L^∞ estimate and direct gradient estimate.

3.2 L^∞ Estimate

We assume φ is a real-valued function in $C^4(M)$ such that $(g_{j\bar{k}} + \varphi_{j\bar{k}})dz^j \otimes d\bar{z}^k$ defines a Kähler metric on M, and $\det(g_{j\bar{k}} + \partial^2\varphi/\partial z^j \partial \bar{z}^k) = f \det g_{j\bar{k}}$. Furthermore, since we are only interested in controlling the difference $(\sup_M \varphi - \inf_M \varphi)$, we may shift φ by a constant without loss of generality. Thus for the remainder of this section, we assume

$$\int_M \varphi = 0. \tag{3.1}$$

(i) The Supremum Estimate

The first step will be to estimate the supremum of φ, which will be quick and painless with the help of Green's functions. The estimate of the infimum of φ is more involved.

We begin by recalling the properties of Green's functions. The reference for the following theorem is [1].

Theorem 3.2.1. (Green's Functions) *Let M be a compact Riemannian manifold. Then there exists a function $G(p,q)$ with the following properties:*

(1) For all $\varphi \in C^2$, we have

$$\varphi(p) = (\mathrm{Vol}(M))^{-1} \int_M \varphi(q)\ dVol(q) - \int_M G(p,q)\Delta\varphi(q)\ dVol(q).$$

(2) The Green's function is defined up to a constant, and can be chosen such that $G(p,q) \geq 0$.

(3) $\int_M G(p,q)\ dVol(q) = const$.

The function $G(p,q)$ is called the Green's function of the Laplacian Δ. Let $G(p,q)$ be the Green's function of the Laplacian Δ on M, chosen such that $G(p,q) \geq 0$. Then since $\int_M \varphi = 0$, we have

$$\varphi(p) = -\int_M G(p,q)\Delta\varphi(q)dVol(q).$$

From (2.8), we have

$$\varphi(p) \leq m \int_M G(p,q)dVol(q).$$

By Theorem 3.2.1, $\int_M G(p,q) = C$ where C is independent of $p \in M$. Therefore

$$\sup_M \varphi \leq mC.$$

We can also use the Green's function to estimate $\int_M |\varphi|$.

$$\int_M |\varphi| \leq \int_M |\sup_M \varphi - \varphi| + \int_M |\sup_M \varphi| = \left|\int_M \sup_M \varphi - \int_M \varphi\right| + \int_M |\sup_M \varphi|.$$

Since $\int_M \varphi = 0$ and $\sup_M \varphi$ is bounded, we have

$$\int_M |\varphi| \leq 2mC\mathrm{Vol}(M). \tag{3.2}$$

(ii) The L^{p-1} Reduction Estimate

The key inequality to be shown in this section is the following:

$$\int_M |\nabla|\varphi|^{p/2}|^2 \leq p\ C_1 \int_M |\varphi|^{p-1}. \tag{3.3}$$

Assuming this estimate, we can obtain an estimate on $\int_M |\varphi|^2$. Indeed, first using the Poincaré inequality and (3.1), and afterwards using (3.3) and the estimate on $\int_M |\varphi|$ given by (3.2), we get

$$\int_M |\varphi|^2 \leq C_1 \int_M |\nabla\varphi|^2 \leq C_2 \int_M |\varphi| \leq C_3. \tag{3.4}$$

We now prove (3.3), following the argument given in [11]. Since $\frac{d}{dx}x|x|^{p-2} = (p-1)|x|^{p-2}$, we have $\partial(\varphi|\varphi|^{p-2}) = (p-1)|\varphi|^{p-2}\partial\varphi$. Denoting by Ω as the Kähler form of M, we compute

$$\begin{aligned} d(\varphi|\varphi|^{p-2}\Omega^{m-\nu}\sqrt{-1}\ \bar\partial\varphi(\sqrt{-1}\partial\bar\partial\varphi)^{\nu-1})) &= \varphi|\varphi|^{p-2}\Omega^{m-\nu}(\sqrt{-1}\partial\bar\partial\varphi)^{\nu} \\ &\quad + (p-1)|\varphi|^{p-2}\Omega^{m-\nu}\sqrt{-1}\partial\varphi \wedge \bar\partial\varphi(\sqrt{-1}\partial\bar\partial\varphi)^{\nu-1}. \end{aligned}$$

Hence by Stokes's theorem, we obtain

$$\int_M \varphi|\varphi|^{p-2}\Omega^{m-\nu}(\sqrt{-1}\partial\bar{\partial}\varphi)^{\nu} = -\int_M (p-1)|\varphi|^{p-2}\Omega^{m-\nu}\sqrt{-1}\partial\varphi\wedge\bar{\partial}\varphi(\sqrt{-1}\partial\bar{\partial}\varphi)^{\nu-1}. \tag{3.5}$$

To obtain the bound (3.3), we look at the quantity

$$\int_M ((\Omega+\sqrt{-1}\partial\bar{\partial}\varphi)^m - \Omega^m)\varphi|\varphi|^{p-2}.$$

. Using (3.5), we compute the following:

$$\begin{aligned}\int_M ((\Omega+\sqrt{-1}\partial\bar{\partial}\varphi)^m - \Omega^m)\varphi|\varphi|^{p-2} &= \int_M \sum_{\nu=1}^{m}\binom{m}{\nu}\Omega^{m-\nu}(\sqrt{-1}\partial\bar{\partial}\varphi)^{\nu}\varphi|\varphi|^{p-2}\\ &= -\int_M \sum_{\nu=1}^{m}(p-1)|\varphi|^{p-2}\binom{m}{\nu}\Omega^{m-\nu}\sqrt{-1}\partial\varphi\wedge\bar{\partial}\varphi(\sqrt{-1}\partial\bar{\partial}\varphi)^{\nu-1}.\end{aligned} \tag{3.6}$$

We fix a point $q \in M$ and choose coordinates such that $g_{i\bar{j}} = \delta_{ij}$ (hence $\Omega_{i\bar{j}}$ is diagonal) and $\varphi_{i\bar{j}}$ is diagonal at the point in consideration. Taking a look at the previous expression, the form under the integral at the point q looks like

$$\sum_{i=1}^{m}|\partial_i\varphi|^2\chi_i,$$

where χ_i is given by

$$\chi_i = \sum_{\nu=1}^{m}(p-1)|\varphi|^{p-2}\binom{m}{\nu}\Omega^{m-\nu}\sqrt{-1}dz^i\wedge d\bar{z}^i(\sqrt{-1}\partial\bar{\partial}\varphi)^{\nu-1}.$$

This follows from

$$dz^i \wedge d\bar{z}^j \wedge \Omega^{m-\nu} \wedge (\sqrt{-1}\partial\bar{\partial}\varphi)^{\nu-1} = \delta_{ij} dz^i \wedge d\bar{z}^i \wedge \Omega^{m-\nu} \wedge (\sqrt{-1}\partial\bar{\partial}\varphi)^{\nu-1}.$$

We now estimate χ_i. As before, we denote $g'_{i\bar{j}} = g_{i\bar{j}} + \varphi_{i\bar{j}}$, and Ω' will denote the Kähler form associated to $(M, g'_{i\bar{j}})$. First, we compute

$$\begin{aligned}
\chi_i &= (p-1)|\varphi|^{p-2} \sum_{\nu=1}^{m} \frac{m}{\nu} \binom{m-1}{\nu-1} \Omega^{m-\nu} \sqrt{-1} dz^i \wedge d\bar{z}^i (\sqrt{-1}\partial\bar{\partial}\varphi)^{\nu-1} \\
&= (p-1)|\varphi|^{p-2} \sum_{\nu=0}^{m-1} \frac{m}{\nu+1} \binom{m-1}{\nu} \Omega^{m-\nu-1} \sqrt{-1} dz^i \wedge d\bar{z}^i (\sqrt{-1}\partial\bar{\partial}\varphi)^{\nu} \\
&= (p-1)|\varphi|^{p-2} m \sum_{\nu=0}^{m-1} \left(\int_0^1 t^\nu dt \right) \binom{m-1}{\nu} \Omega^{m-\nu-1} \sqrt{-1} dz^i \wedge d\bar{z}^i (\sqrt{-1}\partial\bar{\partial}\varphi)^{\nu} \\
&= (p-1)|\varphi|^{p-2} m \int_0^1 (t\sqrt{-1}\partial\bar{\partial}\varphi + \Omega)^{m-1} \sqrt{-1} dt \; dz^i \wedge d\bar{z}^i \\
&= (p-1)|\varphi|^{p-2} m \int_0^1 ((1-t)\Omega + t\Omega')^{m-1} \sqrt{-1} dt \; dz^i \wedge d\bar{z}^i.
\end{aligned}$$

We now use the fact that Ω and Ω' are positive definite.

$$\begin{aligned}
\chi_i &= (p-1)|\varphi|^{p-2} m \int_0^1 ((1-t)\Omega + t\Omega')^{m-1} \sqrt{-1} dt \; dz^i \wedge d\bar{z}^i \\
&\geq (p-1)|\varphi|^{p-2} m \int_0^1 (1-t)^{m-1} \Omega^{m-1} \sqrt{-1} dt \; dz^i \wedge d\bar{z}^i \\
&= (p-1)|\varphi|^{p-2} \Omega^{m-1} \sqrt{-1} dz^i \wedge d\bar{z}^i.
\end{aligned}$$

Using the volume form as given in Theorem 2.1.3, we have $dVol = (\frac{i}{2})^m \det g_{j\bar{k}} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^m$. Since our coordinates at q are such that $g_{j\bar{k}} = \delta_{jk}$, we obtain

$$\begin{aligned}\sum_{i=1}^{m} |\partial_i \varphi|^2 \chi_i &\geq \sum_{i=1}^{m} |\partial_i \varphi|^2 (p-1)|\varphi|^{p-2} \Omega^{m-1} dz^i \wedge d\bar{z}^i \\ &= |\nabla\varphi|^2 (p-1)|\varphi|^{p-2} m! \frac{(\sqrt{-1})^{m-1}}{2^{m-1}} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^m \\ &= \frac{2}{\sqrt{-1}} |\nabla\varphi|^2 (p-1)|\varphi|^{p-2} dVol.\end{aligned}$$

Therefore, we have shown

$$\sum_{\nu=1}^{m} (p-1)|\varphi|^{p-2} \binom{m}{\nu} \Omega^{m-\nu} \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi (\sqrt{-1}\partial\bar{\partial}\varphi)^{\nu-1} \geq \frac{2}{\sqrt{-1}} |\nabla\varphi|^2 (p-1)|\varphi|^{p-2} dVol.$$

Integrating and using (3.6) yields

$$\left| \int_M (\Omega + \sqrt{-1}\partial\bar{\partial}\varphi)^m - \Omega^m)\varphi|\varphi|^{p-2} \right| \geq 2 \int_M |\nabla\varphi|^2 (p-1)|\varphi|^{p-2} dVol. \tag{3.7}$$

On the other hand, from Theorem 2.1.3 , we have

$$(\Omega + \sqrt{-1}\partial\bar{\partial}\varphi)^m - \Omega^m)\varphi|\varphi|^{p-2} = m!(\det g'_{j\bar{k}} - \det g_{j\bar{k}})\varphi|\varphi|^{p-2} \frac{i^m}{2^m} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^m.$$

Therefore,

$$(\Omega + \sqrt{-1}\partial\bar{\partial}\varphi)^m - \Omega^m)\varphi|\varphi|^{p-2} = m! \left(\frac{\det g'_{j\bar{k}}}{\det g_{j\bar{k}}} - 1 \right) \varphi|\varphi|^{p-2} dVol.$$

Since φ satisfies the Monge-Ampere equation (1.1), we can integrate and obtain

$$\int_M (\Omega + \sqrt{-1}\partial\bar{\partial}\varphi)^m - \Omega^m)\varphi|\varphi|^{p-2} = m! \int_M (f-1)\varphi|\varphi|^{p-2} dVol.$$

From now on, we omit writing out the volume form $dVol$ when integrating scalar functions. Thus we obtain the estimate

$$\left| \int_M (\Omega + \sqrt{-1}\partial\bar{\partial}\varphi)^m - \Omega^m)\varphi|\varphi|^{p-2} \right| \leq m! \sup_M f \int_M |\varphi|^{p-1}.$$

Combining this estimate with (3.7), we see that

$$\int_M |\nabla\varphi|^2|\varphi|^{p-2} \leq \frac{C}{p-1} \int_M |\varphi|^{p-1}.$$

Since $|\nabla|\varphi|^{p/2}|^2 = (p^2/4)|\varphi|^{p-2}|\nabla\varphi|^2$, we can conclude

$$\int_M |\nabla|\varphi|^{p/2}|^2 \leq p\ C_1 \int_M |\varphi|^{p-1}.$$

We have thus shown (3.3).

(iii) The Moser Iteration

The obtain the next estimate, we use the Sobolev inequality. By the Sobolev inequality with $q = 2$, $q^* = 2m/(m-1)$, we have

$$||\ |\varphi^{p/2}|\ ||_{2\beta} \leq C(||\ |\varphi|^{p/2}\ ||_2 + ||\nabla|\varphi|^{p/2}\ ||_2),$$

where $\beta = m/(m-1)$. (Equation (1.1) is trivial when the dimension of the Kähler manifold is $m = 1$, so we assume dimension $m \in \mathbb{N}$, $m > 1$.) We note that $1 < \beta \leq 2$. By squaring and applying (3.3), we can obtain

$$\left(\int |\varphi|^{\beta p} \right)^{1/\beta} \leq C_2 \left(\int |\varphi|^p + \int |\nabla|\varphi|^{p/2}|^2 \right) \leq C_2 \left(\int |\varphi|^p + pC_1 \int |\varphi|^{p-1} \right).$$

We use Holder's inequality with conjugate exponents $p/(p-1)$ and p:

$$\int_M |\varphi|^{p-1} \leq Vol(M)^{1/p} \left(\int_M |\varphi|^p \right)^{\frac{p-1}{p}} \leq Vol(M)^{1/p} \max\{1, \int_M |\varphi|^p\}.$$

Combining this with the previous inequality yields

$$\left(\int |\varphi|^{\beta p} \right)^{1/\beta} \leq p\, C_3 \ \max\{1, \int |\varphi|^p\}.$$

Rewritten in a more convenient notation, we have

$$||\varphi||_{p\beta} \leq C_3^{1/p} p^{1/p} \max\{1, ||\varphi||_p\}. \tag{3.8}$$

Let $p_0 = 2$ and recursively apply (3.8) by replacing p with $p\beta$ after each step. One obtains

$$||\varphi||_{p\beta^k} \leq \left(\prod_{k=0}^{k-1} (2C_3\beta^k)^{\frac{1}{2\beta^k}} \right) \max\{1, ||\varphi||_2\}. \tag{3.9}$$

Taking the limit as $k \to \infty$, we obtain

$$||\varphi||_\infty \leq C_4 \max\{1, ||\varphi||_2\}. \tag{3.10}$$

The convergence of the product follows from the fact that

$$\sum_{k=0}^{\infty} \log(2C_3\beta^k)^{\frac{1}{2\beta^k}} = \sum_{k=0}^{\infty} \left(\frac{\log(2C_3)}{2\beta^k} + \frac{k \log \beta}{2\beta^k} \right), \tag{3.11}$$

converges. By (3.4), $||\varphi||_2$ is under control and hence $||\varphi||_\infty \leq C$.

3.3 Gradient Estimate

The following result was proved independently by Blocki [3] and P. Guan [9].

Theorem 3.3.1. *Let (M, g) be a compact Kähler manifold without boundary. Let $f > 0$ be a positive function on M such that $f^{\frac{1}{m}}$ is Lipschitz continuous. For all $\varphi \in C^4(M)$ satisfying (1.1) such that $(\varphi_{i\bar{j}} + g_{i\bar{j}})$ is positive-definite, we have*

$$||\nabla\varphi||_\infty \leq C,$$

where C depends on (M, g), $(\sup\varphi - \inf\varphi)$, the Lipschitz constant of $f^{\frac{1}{m}}$, and $\sup f$.

Proof. We consider the test function

$$H = |\nabla\varphi|^2 e^{-\alpha(\varphi)},$$

where $\alpha(x) : [2, \lambda] \to \mathbb{R}$ is a function that will be specified later. In view of the L^∞ estimate, we may shift φ by a constant and assume that $\varphi(p) \in [2, \lambda]$ for all $p \in M$. As they will be needed later, the first thing to do is to compute the first two derivatives of H.

$$H_\gamma = (g^{i\bar{j}})_\gamma \varphi_i \varphi_{\bar{j}} e^{-\alpha(\varphi)} + g^{i\bar{j}} \varphi_{i\gamma} \varphi_{\bar{j}} e^{-\alpha(\varphi)} + g^{i\bar{j}} \varphi_i \varphi_{\bar{j}\gamma} e^{-\alpha(\varphi)} - \alpha' g^{i\bar{j}} \varphi_i \varphi_{\bar{j}} \varphi_\gamma e^{-\alpha(\varphi)}.$$

$$
\begin{aligned}
H_{\gamma\bar\gamma} &= (g^{i\bar j})_{\gamma\bar\gamma}\varphi_i\varphi_{\bar j}e^{-\alpha(\varphi)} + (g^{i\bar j})_{\gamma}(\varphi_i\varphi_{\bar j}e^{-\alpha(\varphi)})_{\bar\gamma} + (g^{i\bar j})_{\bar\gamma}(\varphi_{i\gamma}\varphi_{\bar j} + \varphi_i\varphi_{\bar j\gamma} - \alpha'\varphi_i\varphi_{\bar j}\varphi_\gamma)e^{-\alpha(\varphi)} \\
&\quad + g^{i\bar j}\left(\varphi_{i\gamma\bar\gamma}\varphi_{\bar j} + \varphi_{\bar j\gamma\bar\gamma}\varphi_i\right)e^{-\alpha(\varphi)} + g^{i\bar j}\left(\varphi_{i\gamma}\varphi_{\bar j\bar\gamma} + \varphi_{i\bar\gamma}\varphi_{\gamma\bar j} - \alpha''\varphi_i\varphi_{\bar j}\varphi_\gamma\varphi_{\bar\gamma} - \alpha'\varphi_i\varphi_{\bar j}\varphi_{\gamma\bar\gamma}\right)e^{-\alpha(\varphi)} \\
&\quad + g^{i\bar j}\left(-\alpha'\varphi_{\bar\gamma}(\varphi_{i\gamma}\varphi_{\bar j} + \varphi_{\gamma\bar j}\varphi_i) - \alpha'\varphi_\gamma\overline{(\varphi_{j\gamma}\varphi_{\bar i} + \varphi_{\gamma\bar i}\varphi_j)} + (\alpha')^2\varphi_i\varphi_{\bar j}\varphi_\gamma\varphi_{\bar\gamma}\right)e^{-\alpha(\varphi)}.
\end{aligned}
$$

Let $p \in M$ be the point where H achieves its maximum. We choose coordinates as in Theorem 2.1.1 such that at the maximal point p of H, we have $g_{i\bar j}(p) = \delta_{ij}$, $\frac{\partial}{\partial z^k}g_{i\bar j}(p) = 0$. Furthermore, we may rotate these coordinates by a unitary transformation and assume $\varphi_{i\bar j}(p) = \delta_{ij}\varphi_{i\bar j}(p)$. The vanishing of the first derivative $\nabla H(p) = 0$ implies the following equality at the point in consideration:

$$
\sum_i \varphi_{i\gamma}\varphi_{\bar i} + \varphi_\gamma\varphi_{\gamma\bar\gamma} = \alpha'|\nabla\varphi|^2\varphi_\gamma. \tag{3.12}
$$

By the maximum principle, $0 \geq g'^{\gamma\bar\gamma}H_{\gamma\bar\gamma}$ at p, where g' was defined in (2.6). Using (3.12), (2.3), and the properties of our choice of coordinates, we can simplify some terms in the expansion of $H_{\gamma\bar\gamma}$ computed above, and get:

$$
\begin{aligned}
0 &\geq \sum_{\gamma,i} g'^{\gamma\bar\gamma}|\varphi_i|^2R_{i\bar i\gamma\bar\gamma} + 2\sum_{\gamma,i} g'^{\gamma\bar\gamma}Re(\varphi_{\bar i\gamma\bar\gamma}\varphi_i) + \sum_\gamma g'^{\gamma\bar\gamma}(|\varphi_{\gamma\bar\gamma}|^2 + \sum_i |\varphi_{i\gamma}|^2) \\
&\quad - \alpha'\sum_\gamma |\nabla\varphi|^2 g'^{\gamma\bar\gamma}\varphi_{\gamma\bar\gamma} - (\alpha'' + (\alpha')^2)\sum_\gamma g'^{\gamma\bar\gamma}|\varphi_\gamma|^2|\nabla\varphi|^2.
\end{aligned} \tag{3.13}
$$

By the definition of $g'_{i\bar j}$, we have

$$
\sum_\gamma g'^{\gamma\bar\gamma}\varphi_{\gamma\bar\gamma} = \sum_\gamma \frac{\varphi_{\gamma\bar\gamma}}{1+\varphi_{\gamma\bar\gamma}} = \sum_\gamma \frac{1+\varphi_{\gamma\bar\gamma}-1}{1+\varphi_{\gamma\bar\gamma}} = m - \sum_\gamma g'^{\gamma\bar\gamma}. \tag{3.14}
$$

Using the definition of bisectional curvature (2.5), (2.11), and (3.14), we manipulate (3.13) and obtain

$$0 \leq (B-\alpha')|\nabla\varphi|^2 \sum_\gamma g'^{\gamma\bar\gamma} + (\alpha'' + (\alpha')^2)|\nabla\varphi|^2 \sum_\gamma g'^{\gamma\bar\gamma}|\varphi_\gamma|^2 + \alpha' m|\nabla\varphi|^2$$
$$- 2mf^{-1/m} Re(\sum_\gamma (f^{1/m})_{\bar\gamma}\varphi_\gamma) - \sum_\gamma g'^{\gamma\bar\gamma}(|\varphi_{\gamma\bar\gamma}|^2 + \sum_i |\varphi_{i\gamma}|^2). \tag{3.15}$$

Next, we use Cauchy-Bunyakowsky-Schwarz and obtain

$$-\sum_i |\varphi_{i\gamma}|^2 \leq -\left|\sum_i \varphi_{i\gamma}\varphi_i\right|^2 |\nabla\varphi|^{-2}.$$

From (3.12), we see that

$$-\sum_i |\varphi_{i\gamma}|^2 \leq -\frac{|\alpha'|\nabla\varphi|^2\varphi_\gamma - \varphi_\gamma\varphi_{\gamma\bar\gamma}|^2}{|\nabla\varphi|^2}.$$
$$= -(\alpha')^2|\nabla\varphi|^2|\varphi_\gamma|^2 + 2\alpha'|\varphi_\gamma|^2\varphi_{\gamma\bar\gamma} - |\nabla\varphi|^{-2}|\varphi_\gamma|^2\varphi_{\gamma\bar\gamma}^2.$$

We use (3.14) again to obtain

$$-\sum_\gamma\sum_i g'^{\gamma\bar\gamma}|\varphi_{i\gamma}|^2 \leq -(\alpha')^2|\nabla\varphi|^2 \sum_\gamma g'^{\gamma\bar\gamma}|\varphi_\gamma|^2 + 2\alpha'|\nabla\varphi|^2 - 2\alpha' \sum_\gamma g'^{\gamma\bar\gamma}|\varphi_\gamma|^2$$
$$- \sum_\gamma g'^{\gamma\bar\gamma}\varphi_{\gamma\bar\gamma}^2|\nabla\varphi|^{-2}|\varphi_\gamma|^2.$$

Substituting this identity in (3.15) and dividing out by $|\nabla\varphi|^2$ yields

$$0 \leq (B-\alpha')\sum_\gamma g'^{\gamma\bar\gamma} + \alpha'(m+2) + \frac{2m|\nabla f^{1/m}|}{|\nabla\varphi| f^{1/m}}$$
$$- \sum_\gamma \frac{g'^{\gamma\bar\gamma}\varphi_{\gamma\bar\gamma}^2}{|\nabla\varphi|^2}(1 + \frac{|\varphi_\gamma|^2}{|\nabla\varphi|^2}) - \frac{2\alpha'}{|\nabla\varphi|^2}\sum_\gamma g'^{\gamma\bar\gamma}|\varphi_\gamma|^2 + \alpha''\sum_\gamma g'^{\gamma\bar\gamma}|\varphi_\gamma|^2.$$

We select $\alpha(x) : [2, \lambda] \to \mathbb{R}$ such that α' is large and positive, and α'' is negative. For example, we could use $\alpha(x) = Ex - x^2$, where $E \geq 2\lambda + B + 2m||\nabla f^{1/m}||_\infty$. Dropping negative terms and using (2.10),

$$0 \leq (B + \frac{2m|\nabla f^{1/m}|}{|\nabla \varphi|} - \alpha') \sum_\gamma g'^{\gamma\bar{\gamma}} + \alpha'(m+2) + \alpha'' \sum_\gamma g'^{\gamma\bar{\gamma}} |\varphi_\gamma|^2. \tag{3.16}$$

By our choice of $\alpha(x)$, we have that α' dominates the first term. The objective is to bound $|\nabla \varphi(p)| \leq C$. If $|\nabla \varphi(p)| \leq 1$, we are done. Therefore, we assume $|\nabla \varphi(p)| \geq 1$. Throwing out the last term of (3.16), we obtain

$$(\alpha' - B - 2m|\nabla f^{1/m}|) \sum_\gamma g'^{\gamma\bar{\gamma}} \leq \alpha'(m+2). \tag{3.17}$$

Thus

$$\frac{1}{1 + \varphi_{\gamma\bar{\gamma}}} = g'^{\gamma\bar{\gamma}} \leq C_1. \tag{3.18}$$

It follows that

$$1 + \varphi_{i\bar{i}} = \frac{f}{\prod_{\gamma \neq i}(1 + \varphi_{\gamma\bar{\gamma}})} \leq C_1^{m-1} \sup f. \tag{3.19}$$

We have thus found an upper bound $(m + \Delta\varphi) \leq C_2$ at the point p. Returning to (3.16), we throw out the first term this time and obtain

$$-\alpha'' \sum_\gamma g'^{\gamma\bar{\gamma}} |\varphi_\gamma|^2 \leq \alpha'(m+2). \tag{3.20}$$

Since α'' is negative, we have

$$-\alpha'' \frac{|\nabla\varphi|^2}{m + \Delta\varphi} \leq -\alpha'' \sum_\gamma g'^{\gamma\bar{\gamma}} |\varphi_\gamma|^2. \tag{3.21}$$

Therefore, we obtain

$$|\nabla\varphi|^2 \leq \frac{\alpha'(m+2)(m+\Delta\varphi)}{(-\alpha'')} \leq \frac{\alpha'(m+2)C_2}{(-\alpha'')}. \tag{3.22}$$

Thus we have bounded $|\nabla\varphi(p)|^2 \leq C$. It follows that for all $z \in M$, we have

$$|\nabla\varphi(z)|^2 \leq |\nabla\varphi(p)|^2 e^{\alpha(\varphi(z))-\alpha(\varphi(p))} \leq Ce^{E\lambda+\lambda^2}.$$

□

3.4 Laplacian Estimate

Before stating and proving our estimate, we recall the previously known results. The first Laplacian estimate was obtained by S.T. Yau in [12]. It was shown that for all $\varphi \in C^4(M)$ satisfying (1.1) such that $(\varphi_{i\bar{j}} + g_{i\bar{j}})$ is positive-definite, we have $|\Delta\varphi| \leq C$, where C depends on (M, g), a lower bound of $\Delta \log f$, an upper bound of f, and an upper bound of $(\sup \varphi - \inf \varphi)$.

Another estimate was obtained by Blocki in [4]. It was shown that for all $\varphi \in C^4(M)$ satisfying (1.1) such that $(\varphi_{i\bar{j}}+g_{i\bar{j}})$ is positive-definite, we have $|\Delta\varphi| \leq C$, where C depends on (M, g), an upper bounds of $||f^{\frac{1}{m-1}}||_{C^{1,1}}$, and an upper bound of $(\sup \varphi - \inf \varphi)$. Later, this result was revisited by Blocki in [3] by replacing the upper bound of $||f^{\frac{1}{m-1}}||_{C^{1,1}}$ with a lower bound of $f^{\frac{1}{m-1}} \Delta \log f$ and an upper bound of f. We replace the lower bound of $f^{\frac{1}{m-1}} \Delta \log f$ with a lower bound of $\Delta f^{\frac{1}{m-1}}$. This is an improvement because

$$f^{\frac{1}{m-1}} \Delta \log f = \frac{f^{\frac{1}{m-1}}}{f^2}(f\Delta f - |\nabla f|^2), \text{ and}$$

$$\Delta f^{\frac{1}{m-1}} = \frac{f^{\frac{1}{m-1}}}{f^2}(f\Delta f - \frac{m-2}{m-1}|\nabla f|^2).$$

We present the following proposition, which is the main result of this thesis.

Proposition 3.4.1. *Let (M, g) be a closed, compact Kähler manifold with non-negative bisectional curvature. Let $f > 0$ be a positive function on M such that $\inf_M \Delta f^{\frac{1}{m-1}} \geq -A$ for some constant A. For all $\varphi \in C^4(M)$ satisfying (1.1) such that $(\varphi_{i\bar{j}} + g_{i\bar{j}})$ is positive-definite, we have*

$$|\Delta\varphi| \leq C,$$

where C depends on (M, g), $(\sup\varphi - \inf\varphi)$, A, and $\sup f$.

Proof. From (2.8), we have $-m < \Delta\varphi$ and thus we only need to bound $\Delta\varphi$ from above.

We will estimate the maximum value of the following test function

$$H = (m + \Delta\varphi)e^{-\alpha(\varphi)}, \tag{3.23}$$

where $\alpha : [2, \lambda] \to \mathbb{R}$ is a function that will be specified later. In view of the L^∞ estimate, we may shift φ by a constant and assume that $\varphi(p) \in [2, \lambda]$ for all $p \in M$. We start by computing the first two derivatives of H.

$$H_\gamma = (\Delta\varphi)_\gamma e^{-\alpha(\varphi)} - \alpha'(m + \Delta\varphi)\varphi_\gamma e^{-\alpha(\varphi)},$$

$$\begin{aligned} H_{\gamma\bar{\gamma}} &= \left((\Delta\varphi)_{\gamma\bar{\gamma}} - \alpha'(m + \Delta\varphi)\varphi_{\gamma\bar{\gamma}} - \alpha''(m + \Delta\varphi)\varphi_\gamma\varphi_{\bar{\gamma}}\right) e^{-\alpha(\varphi)} \\ &\quad + \left(-\alpha'\left((\Delta\varphi)_\gamma\varphi_{\bar{\gamma}} + \overline{(\Delta\varphi)_\gamma}\ \varphi_\gamma\right) + (\alpha')^2(m + \Delta\varphi)\varphi_\gamma\varphi_{\bar{\gamma}}\right) e^{-\alpha(\varphi)}. \end{aligned} \tag{3.24}$$

Let $p \in M$ be the point where H achieves its maximum value. We choose coordinates such that at p we have $g_{i\bar{j}} = \delta_{ij}$, $\frac{\partial}{\partial z^k} g_{i\bar{j}} = 0$ and $\varphi_{i\bar{j}} = \delta_{ij}\varphi_{i\bar{j}}$.

At p, the gradient of H is equal to zero, and hence

$$(\Delta\varphi)_j = \alpha'\varphi_j(m + \Delta\varphi). \tag{3.25}$$

We recall the notation $g'_{i\bar{j}} := g_{i\bar{j}} + \varphi_{i\bar{j}}$. Since g' defines a Kähler metric on M, we denote $\Delta' = g'^{a\bar{b}}\partial_a\bar{\partial}_b$ to be the Laplacian of (M, g'). By the maximum principle, $\Delta' H(p) \leq 0$, hence if we use this fact while substituting the gradient equation (3.25) into (3.24), we obtain

$$\Delta'\Delta\varphi \leq \alpha'(m + \Delta\varphi)\Delta'\varphi + (\alpha')^2(m + \Delta\varphi)g'^{i\bar{i}}\varphi_i\varphi_{\bar{i}} + \alpha''(m + \Delta\varphi)g'^{i\bar{i}}\varphi_i\varphi_{\bar{i}}. \tag{3.26}$$

Next, we take the first derivative of $(\det(g_{i\bar{j}} + \varphi_{i\bar{j}}))^{\frac{1}{m-1}} = (f \det g_{i\bar{j}})^{\frac{1}{m-1}}$. This yields

$$(\det g'_{i\bar{j}})^{\frac{1}{m-1}} g'^{i\bar{j}}(\partial_\gamma g_{i\bar{j}} + \varphi_{i\bar{j}\gamma}) = (m-1)(\det g_{i\bar{j}})^{\frac{1}{m-1}}\partial_\gamma f^{\frac{1}{m-1}} + f^{\frac{1}{m-1}}(\det g_{i\bar{j}})^{\frac{1}{m-1}} g^{i\bar{j}}\partial_\gamma g_{i\bar{j}}.$$

We then take another derivative of the previous expression.

$$\begin{aligned}
&(m-1)\bar{\partial}_\gamma(\det g_{i\bar{j}})^{\frac{1}{m-1}}\partial_\gamma f^{\frac{1}{m-1}} + (m-1)(\det g_{i\bar{j}})^{\frac{1}{m-1}}\partial_\gamma\bar{\partial}_\gamma f^{\frac{1}{m-1}} \\
&+ \left(\bar{\partial}_\gamma((f \det g_{i\bar{j}})^{\frac{1}{m-1}} g^{i\bar{j}})\right)\partial_\gamma g_{i\bar{j}} + (f \det g_{i\bar{j}})^{\frac{1}{m-1}} g^{i\bar{j}}\partial_\gamma\bar{\partial}_\gamma g_{i\bar{j}} \\
&= \frac{1}{(m-1)}(\det g'_{i\bar{j}})^{\frac{1}{m-1}} g'^{k\bar{l}}(\bar{\partial}_\gamma g_{k\bar{l}} + \varphi_{k\bar{l}\bar{\gamma}})g'^{i\bar{j}}(\partial_\gamma g_{i\bar{j}} + \varphi_{i\bar{j}\gamma}) \\
&+ (\det g'_{i\bar{j}})^{\frac{1}{m-1}}(g'^{i\bar{j}}(\partial_\gamma\bar{\partial}_\gamma g_{i\bar{j}} + \varphi_{i\bar{j}\gamma\bar{\gamma}}) + (\partial_\gamma g_{i\bar{j}} + \varphi_{i\bar{j}\gamma})\bar{\partial}_\gamma g'^{i\bar{j}}).
\end{aligned}$$

At the point p, our choice of coordinates greatly simplifies the expression and we are left with the following:

$$\begin{aligned}&(m-1)\partial_\gamma\bar{\partial}_\gamma f^{\frac{1}{m-1}} + f^{\frac{1}{m-1}}\delta^{i\bar{j}}\partial_\gamma\bar{\partial}_\gamma g_{i\bar{j}}\\ &= \frac{1}{m-1} f^{\frac{1}{m-1}} g'^{i\bar{i}} g'^{j\bar{j}} \varphi_{i\bar{i}\gamma}\varphi_{j\bar{j}\bar{\gamma}} + f^{\frac{1}{m-1}}(g'^{i\bar{i}}(\partial_\gamma\bar{\partial}_\gamma g_{i\bar{i}} + \varphi_{i\bar{i}\gamma\bar{\gamma}}) + \varphi_{i\bar{j}\gamma}\bar{\partial}_\gamma g'^{i\bar{j}}).\end{aligned}$$

From (2.1) and (2.3), we obtain

$$\begin{aligned}&(m-1)f^{\frac{-1}{m-1}}\partial_\gamma\bar{\partial}_\gamma f^{\frac{1}{m-1}} - \delta^{i\bar{j}} R_{i\bar{j}\gamma\bar{\gamma}}\\ &= \frac{1}{m-1} g'^{i\bar{i}} g'^{j\bar{j}} \varphi_{i\bar{i}\gamma}\varphi_{j\bar{j}\bar{\gamma}} - g'^{i\bar{i}} R_{i\bar{i}\gamma\bar{\gamma}} + g'^{i\bar{i}}\varphi_{i\bar{i}\gamma\bar{\gamma}} - g'^{i\bar{i}} g'^{j\bar{j}}\varphi_{i\bar{j}\gamma}\varphi_{j\bar{i}\bar{\gamma}}.\end{aligned} \tag{3.27}$$

Also, using (2.7), at the point in consideration we have

$$\Delta'\Delta\varphi = g'^{k\bar{l}}\partial_k\bar{\partial}_l(g^{i\bar{j}}\varphi_{i\bar{j}}) = g'^{\gamma\bar{\gamma}}\varphi_{i\bar{i}}R_{i\bar{i}\gamma\bar{\gamma}} + g'^{i\bar{i}}\varphi_{i\bar{i}\gamma\bar{\gamma}}.$$

After summing the γ in (3.27) and substituting the previous identity, one obtains the following at the point p:

$$\begin{aligned}(m-1)f^{\frac{-1}{m-1}}\Delta f^{\frac{1}{m-1}} &= \Delta'\Delta\varphi + \sum_k \frac{1}{m-1} g'^{i\bar{i}} g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}} - \sum_k g'^{i\bar{i}} g'^{j\bar{j}}\varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}}\\ &\quad - \sum_k g'^{i\bar{i}}(1+\varphi_{k\bar{k}})R_{i\bar{i}k\bar{k}} + \sum_{i,k} R_{i\bar{i}k\bar{k}}.\end{aligned}$$

We shall define

$$S := \sum_{i,k} R_{i\bar{i}k\bar{k}}.$$

We substitute (3.26), the definition of S, and use the definition (2.5) of bisectional curvature B to obtain

$$\begin{aligned}(m-1)f^{\frac{-1}{m-1}}\Delta f^{\frac{1}{m-1}} \leq\ & \alpha' m(m+\Delta\varphi) - \alpha'(m+\Delta\varphi)\left(\sum_i g'^{i\bar{i}}\right) + (\alpha')^2(m+\Delta\varphi)g'^{i\bar{i}}\varphi_i\varphi_{\bar{i}} \\ & + \alpha''(m+\Delta\varphi)g'^{i\bar{i}}\varphi_i\varphi_{\bar{i}} + \sum_k \frac{1}{m-1}g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}} - \sum_k g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}} \\ & - B(m+\Delta\varphi)\left(\sum_i g'^{i\bar{i}}\right) + S. \end{aligned} \tag{3.28}$$

If M has non-negative bisectional curvature, then $B \geq 0$ and the term involving B can be thrown out. We are left with

$$\begin{aligned}(m-1)f^{\frac{-1}{m-1}}\Delta f^{\frac{1}{m-1}} \leq\ & \alpha' m(m+\Delta\varphi) + S - \alpha'(m+\Delta\varphi)\left(\sum_i g'^{i\bar{i}}\right) \\ & + (\alpha'' + (\alpha')^2)(m+\Delta\varphi)g'^{i\bar{i}}\varphi_i\varphi_{\bar{i}} \\ & + \sum_k \frac{1}{m-1}g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}} - \sum_k g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}}. \end{aligned} \tag{3.29}$$

The trouble terms are those involving third order derivatives, and we shall follow the argument of P. Guan in [8] to control the following quantity for a fixed k:

$$\frac{1}{m-1}g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}} - g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}}. \tag{3.30}$$

First, we drop mixed terms $|\varphi_{\bar{i}jk}|^2$ for $i \neq j$ and obtain

$$\frac{1}{m-1}g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}} - g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}} \leq \frac{1}{m-1}\left|g'^{i\bar{i}}\varphi_{i\bar{i}k}\right|^2 - (g'^{i\bar{i}})^2|\varphi_{i\bar{i}k}|^2.$$

We recall that since $\overline{\varphi_{i\bar{i}}} = \varphi_{\bar{i}i} = \varphi_{i\bar{i}}$, we have that $\varphi_{i\bar{i}}(z)$ is a locally defined real-valued function. Also, $\varphi_{i\bar{i}k} = \partial_{z^k}\varphi_{i\bar{i}}$ where $\partial_{z^k} = \frac{1}{2}(\partial_{x^k} - i\partial_{y^k})$. Thus

$$|\varphi_{i\bar{i}k}|^2 = \frac{1}{4}(\varphi_{i\bar{i}x}^2 + \varphi_{i\bar{i}y}^2),$$

where we write f_x for $\partial_{x^k} f$, and there is no confusion since k is fixed. Thus we get

$$\begin{aligned} 4\left(\frac{1}{m-1} g'^{i\bar{i}} g'^{j\bar{j}} \varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}} - g'^{i\bar{i}} g'^{j\bar{j}} \varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}}\right) \leq \frac{1}{m-1}\left(\sum_i g'^{i\bar{i}}\varphi_{i\bar{i}x}\right)^2 - \sum_i (g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 \\ + \frac{1}{m-1}\left(\sum_i g'^{i\bar{i}}\varphi_{i\bar{i}y}\right)^2 - \sum_i (g'^{i\bar{i}}\varphi_{i\bar{i}y})^2. \end{aligned} \tag{3.31}$$

We shall show how to control the terms containing real derivatives in the x direction. Let $I = \{1 \leq i \leq m : \varphi_{i\bar{i}x}(p) > 0\}$ and $J = \{1 \leq i \leq m : \varphi_{i\bar{i}x}(p) < 0\}$. We consider the two following cases. Case 1: I and J are both non-empty, or case 2: either I or J is empty.

In case 1, we have $|I| \leq m-1$ and $|J| \leq m-1$. From the Cauchy-Bunyakowsky-Schwarz inequality, we know that

$$(\sum_i^n a_i)^2 \leq n \sum a_i^2,$$

for $a_i \geq 0$. By letting $n = m-1$, we will see that this case is easy to handle.

Indeed, we can compute the following

$$
\begin{aligned}
&\frac{1}{m-1}(\sum_i g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 - \sum_i (g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 \\
&= \frac{1}{m-1}\left((\sum_I g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 + (\sum_J g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 + 2(\sum_I g'^{i\bar{i}}\varphi_{i\bar{i}x})(\sum_J g'^{i\bar{i}}\varphi_{i\bar{i}x})\right) \\
&- \sum_I (g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 - \sum_J (g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 \\
&\leq \frac{1}{m-1}\left(\sum_I g'^{i\bar{i}}\varphi_{i\bar{i}x}\right)^2 - \sum_I (g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 + \frac{1}{m-1}\left(\sum_J g'^{i\bar{i}}\varphi_{i\bar{i}x}\right)^2 - \sum_J (g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 \\
&\leq 0.
\end{aligned}
$$

Case 2 is a little bit more delicate. Without loss of generality, we assume that $J = \emptyset$. Therefore, $\varphi_{i\bar{i}x}(p) > 0$ for all i. Using (3.25), we obtain the following at p:

$$\varphi_{i\bar{i}x} \leq \sum_{j=1}^{m} \varphi_{j\bar{j}x} = 2Re(\frac{\partial}{\partial z^k}\Delta\varphi) \leq 2|(\Delta\varphi)_k| \leq 2\alpha'|\nabla\varphi|(m+\Delta\varphi). \tag{3.32}$$

We now compute

$$
\begin{aligned}
&\frac{1}{m-1}(\sum_i g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 - \sum_i (g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 \\
&= \frac{1}{m-1}(\sum_{i=1}^{m-1} g'^{i\bar{i}}\varphi_{i\bar{i}x} + g'^{m\bar{m}}\varphi_{m\bar{m}x})^2 - \sum_{i=1}^{m}(g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 \\
&= \frac{2}{m-1}g'^{m\bar{m}}\varphi_{m\bar{m}x}\sum_{i=1}^{m-1} g'^{i\bar{i}}\varphi_{i\bar{i}x} + \frac{1}{m-1}(\sum_{i=1}^{m-1} g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 - \sum_{i=1}^{m-1}(g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 \\
&+ \frac{1}{m-1}(g'^{m\bar{m}}\varphi_{m\bar{m}x})^2 - (g'^{m\bar{m}}\varphi_{m\bar{m}x})^2.
\end{aligned}
$$

Without loss of generality, we can assume $\varphi_{m\bar{m}}(p) \geq \varphi_{i\bar{i}}(p)$ for all i. Therefore, we have

$$\begin{aligned}
\frac{1}{m-1}(\sum_i g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 - \sum_i (g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 &\leq \frac{2}{m-1} g'^{m\bar{m}}\varphi_{m\bar{m}x}\sum_{i=1}^{m-1} g'^{i\bar{i}}\varphi_{i\bar{i}x} \\
&\leq \frac{8}{m-1}(\alpha')^2|\nabla\varphi|^2(m+\Delta\varphi)^2 g'^{m\bar{m}}\sum_{i=1}^{m-1} g'^{i\bar{i}} \\
&\leq \frac{8m}{m-1}(\alpha')^2|\nabla\varphi|^2(m+\Delta\varphi)\sum_{i=1}^{m} g'^{i\bar{i}}.
\end{aligned}$$

The last line is justified as follows. Let $B_1, B_2, \ldots, B_m$ be such that $B_i > 0$ and $B_m \geq B_i$ for all $i \in \{1, 2, \ldots, m\}$. Then it is easy to see that

$$\frac{1}{B_m}\left(\sum_{i=1}^{m} B_i\right)\sum_{i=1}^{m-1}\frac{1}{B_i} \leq m\sum_{i=1}^{m}\frac{1}{B_i}.$$

The terms involving y derivatives in (3.31) can be controlled in the same way as the x derivatives. Thus combining both cases and (3.31), we obtain

$$\frac{1}{m-1} g'^{i\bar{i}} g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}} - g'^{i\bar{i}} g'^{j\bar{j}}\varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}} \leq \frac{4m}{m-1}(\alpha')^2|\nabla\varphi|^2(m+\Delta\varphi)\sum_{i=1}^{m} g'^{i\bar{i}}. \quad (3.33)$$

We substitute (3.33) into (3.29) and obtain

$$\begin{aligned}
(m-1)f^{\frac{-1}{m-1}}\Delta f^{\frac{1}{m-1}} &\leq \alpha' m(m+\Delta\varphi) + S - \alpha'(m+\Delta\varphi)(\sum_i g'^{i\bar{i}}) \\
&\quad + \left(\alpha'' + (\alpha')^2(1 + \frac{4m^2}{m-1})\right)(m+\Delta\varphi)|\nabla\varphi|^2(\sum_i g'^{i\bar{i}}). \quad (3.34)
\end{aligned}$$

Denote $C_0 := 1 + 4m^2/(m-1)$. Following an idea of Blocki in his gradient estimate [3], we pick $\alpha(x) = (C_0)^{-1}\log x$. We know that $\alpha(\varphi)$ is well-defined, since φ was renormalized such that $2 \le \varphi \le \lambda$. This choice of α yields $\alpha'' + C_0(\alpha')^2 = 0$ and hence we are left with

$$(m-1)f^{\frac{-1}{m-1}}\Delta f^{\frac{1}{m-1}} \le \frac{1}{2C_0}m(m+\Delta\varphi) + S - \frac{1}{C_0\lambda}(m+\Delta\varphi)(\sum_i g'^{i\bar{i}}). \tag{3.35}$$

By (2.9), we obtain

$$(m-1)\Delta f^{\frac{1}{m-1}} \le \frac{1}{2C_0}f^{\frac{1}{m+1}}m(m+\Delta\varphi) + Sf^{\frac{1}{m+1}} - \frac{1}{C_0\lambda}(m+\Delta\varphi)^{1+\frac{1}{m+1}}.$$

From the definition of A, we get

$$A(m-1) \ge \frac{1}{C_0\lambda}(m+\Delta\varphi)^{1+\frac{1}{m+1}} - \left(\frac{m}{2C_0}\sup_M f^{\frac{1}{m+1}}\right)(m+\Delta\varphi) - S\sup_M f^{\frac{1}{m+1}}.$$

Thus there are constants C_1,C_2 under control such that

$$(m+\Delta\varphi(p))^{1+1/(m-1)} \le C_1(m+\Delta\varphi(p)) + C_2.$$

Therefore either

$$(m+\Delta\varphi(p))^{1+1/(m-1)} \le 2C_1(m+\Delta\varphi(p)), \quad \text{or} \quad (m+\Delta\varphi(p))^{1+1/(m-1)} \le 2C_2.$$

It follows that there exists a constant C_3 under control such that

$$m+\Delta\varphi(p) \le C_3.$$

Now that we have control of $(m + \Delta\varphi)$ at p, we will show that we have control of $(m + \Delta\varphi)$ at all $z \in M$. Indeed,

$$(m + \Delta\varphi(z))e^{-\alpha(\varphi(z))} \leq (m + \Delta\varphi(p))e^{-\alpha(\varphi(p))} \leq C_3 e^{-\alpha(\varphi(p))}.$$

Since $\alpha(x) = C_0^{-1} \log x$, we have

$$(m + \Delta\varphi(z)) \leq C_3(\frac{\lambda}{2})^{1/C_0}.$$

□

To complete the proof of Theorem 3.1.1, it remains to show that the bound on the Laplacian implies the bound on the gradient. The direct gradient estimate required $f^{\frac{1}{m}}$ to be Lipschitz, so instead of using it, we shall use Schauder estimates. If we look at $\Delta\varphi(z) := G(z)$, then $|G(z)| \leq C$. Then by the Schauder estimates

$$\sup_M |\nabla\varphi| \leq C_0(||G||_\infty + ||\varphi||_2) \leq C. \tag{3.36}$$

We now move to complete the proof of Theorem 3.1.2. By dropping the assumption on the bisectional curvature of M, the curvature terms wreak havoc on the previous argument. To attempt to control these curvature terms, we strengthen our hypothesis to match those of the gradient estimate (Theorem 3.3.1): we assume $f^{1/m}$ is Lipschitz continuous. In the case $m = 2$, this additional assumption makes dealing with the terms (3.30) particularly easy, and we can thus obtain the following estimate.

Proposition 3.4.2. *Let (M,g) be a closed, compact Kähler manifold of dimension $m=2$. Let $f>0$ be a positive function on M such that $\inf_M \Delta f \geq -A$ for some constant A, and $f^{\frac{1}{2}}$ is Lipschitz. For all $\varphi \in C^4(M)$ satisfying (1.1) such that $(\varphi_{i\bar{j}}+g_{i\bar{j}})$ is positive-definite, we have*

$$|\Delta\varphi| \leq C, \tag{3.37}$$

where C depends on (M,g), $(\sup\varphi - \inf\varphi)$, A, the Lipschitz constant of $f^{\frac{1}{2}}$, and $\sup f$.

Proof. We run the same argument as the proof of Proposition 3.4.1 up until equation (3.28). In this case, we can simply let $\alpha(x)=x$. Our test function is then $H=(m+\Delta\varphi)\exp(-\alpha_0\varphi)$, where $0<\alpha_0$ is a constant. Equation (3.28) becomes

$$\begin{aligned} f^{\frac{-1}{m-1}}\Delta(f^{\frac{1}{m-1}}) \leq{}& \alpha_0 m(m+\Delta\varphi)+S-(\alpha_0+B)(m+\Delta\varphi)\left(\sum_i g'^{i\bar{i}}\right) \\ &+(\alpha_0)^2(m+\Delta\varphi)g'^{i\bar{i}}\varphi_i\varphi_{\bar{i}}+\sum_k \frac{1}{m-1}g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}}-\sum_k g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}}. \end{aligned}$$

We see that if we choose $\alpha_0>B$, the coefficient on the third term is negative. To eliminate the α_0^2 term, we substitute the gradient equation (3.25): $(\Delta\varphi)_j = \alpha_0(m+\Delta\varphi)\varphi_j$.

$$\begin{aligned} f^{\frac{-1}{m-1}}\Delta(f^{\frac{1}{m-1}}) \leq{}& \alpha_0 m(m+\Delta\varphi)+S-(\alpha_0+B)(m+\Delta\varphi)\left(\sum_i g'^{i\bar{i}}\right) \\ &+(m+\Delta\varphi)^{-1}g'^{i\bar{i}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{i}}+\sum_k \frac{1}{m-1}g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}}-\sum_k g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}} \end{aligned}$$

Using Cauchy-Bunyakowsky-Schwarz, we obtain

$$
\begin{aligned}
(m+\Delta\varphi)^{-1}g'^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}} &= (m+\Delta\varphi)^{-1}\sum_i g'^{i\bar{i}}(\sum_k \varphi_{k\bar{k}i})(\sum_k \varphi_{k\bar{k}\bar{i}}) \\
&= \sum_i \frac{g'^{i\bar{i}}}{m+\Delta\varphi}\left(\sum_k \frac{\varphi_{k\bar{k}i}(1+\varphi_{k\bar{k}})^{1/2}}{(1+\varphi_{k\bar{k}})^{1/2}}\right)\left(\sum_k \frac{\overline{\varphi_{k\bar{k}i}}(1+\varphi_{k\bar{k}})^{1/2}}{(1+\varphi_{k\bar{k}})^{1/2}}\right) \\
&\leq (m+\Delta\varphi)^{-1}\sum_i g'^{i\bar{i}}\left(\sum_k \frac{|\varphi_{k\bar{k}i}|^2}{(1+\varphi_{k\bar{k}})}\right)\left(\sum_k (1+\varphi_{k\bar{k}})\right) \\
&= \sum_{i,k} g'^{i\bar{i}}g'^{k\bar{k}}|\varphi_{k\bar{k}i}|^2 \\
&\leq \sum_{i,j,k} g'^{i\bar{i}}g'^{j\bar{j}}|\varphi_{i\bar{j}k}|^2.
\end{aligned}
$$

We are left with

$$
\begin{aligned}
f^{\frac{-1}{m-1}}\Delta(f^{\frac{1}{m-1}}) \leq \alpha_0 m(m+\Delta\varphi) + S - (\alpha_0+B)(m+\Delta\varphi)\left(\sum_i g'^{i\bar{i}}\right) \\
+ \sum_k \frac{1}{m-1} g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}}.
\end{aligned}
$$

It is at this point that we use the hypotheses that $m = 2$ and $f^{1/2}$ is Lipschitz continuous. From (2.11), we see that $g'^{\gamma\bar{\gamma}}\varphi_{\gamma\bar{\gamma}k} = 2f^{-1/2}\partial_k f^{1/2}$. Therefore,

$$
f^{-1}\Delta f \leq 2\alpha_0(2+\Delta\varphi) + S - (\alpha_0+B)(2+\Delta\varphi)\left(\sum_i g'^{i\bar{i}}\right) + 4\frac{|\nabla f^{1/2}|^2}{f}. \quad (3.38)
$$

Since we choose α_0 such that $\alpha_0 + B > 0$, we use (2.9) and get

$$
f^{-1}\Delta f \leq 2\alpha_0(2+\Delta\varphi) + S - f^{-1}(\alpha_0+B)(2+\Delta\varphi)^2 + 4f^{-1}|\nabla f^{1/2}|^2. \quad (3.39)
$$

Therefore,

$$-A \leq S \sup_M f + 4||\nabla f^{1/2}||_\infty^2 + 2\alpha_0(\sup_M f)(2 + \Delta\varphi) - (\alpha_0 + B)(2 + \Delta\varphi)^2. \quad (3.40)$$

As shown in the previous argument, it follows that $(m + \Delta\varphi) \leq C$.

$\square$

CHAPTER 4
Application to the Dirichlet Problem in $\mathbb{C}^m$

4.1 Overview

As an application of the estimates shown previously, we shall solve a Dirichlet problem in $\mathbb{C}^m$, following the footsteps of [5]. We first establish some terminology.

We say a real-valued function u is *pluri-subharmonic* if $(u_{i\bar{j}})$ is positive semi-definite. We say a real-valued function u is *strictly pluri-subharmonic* if $(u_{i\bar{j}})$ is positive definite. Following the terminology of [4], if $|\Delta u|$ is bounded, we say u is almost $C^{1,1}$. A domain $\Omega \subset \mathbb{C}^m$ with smooth boundary $\partial\Omega$ is called *strongly pseudo-convex* if there exists a smooth real-valued function r defined on a neighbourhood of $\bar{\Omega}$ such that $r < 0$ in Ω, $r = 0$ on $\partial\Omega$, $r > 0$ outside of $\bar{\Omega}$, $dr \neq 0$, and $(r_{i\bar{j}}(z))$ is positive-definite at each point in its domain.

Theorem 4.1.1. *Let Ω be a strongly pseudo-convex domain in $\mathbb{C}^m$. Let $f : \Omega \to \mathbb{R}$ be a function such that $f \geq 0$, $|\nabla f^{1/m}| \leq A_1$, and $\Delta f^{\frac{1}{m-1}} \geq -A_2$. Then there exists a unique pluri-subharmonic, almost $C^{1,1}$ solution u such that*

$$\det u_{i\bar{j}}(z) = f(z) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega.$$

Furthermore, $||u||_{C^1(\bar{\Omega})} + ||\Delta u||_\infty \leq C$, *where* C *depends only on* A_1,A_2, $\sup(f)$ *and* Ω.

In order to solve this problem, we shall make use of *a priori* estimates, which we now state.

Theorem 4.1.2. *Let* Ω *be a strongly pseudo-convex domain in* $\mathbb{C}^m$*. Let* $f : \Omega \to \mathbb{R}$ *be a function such that* $f > 0$, $|\nabla f^{1/m}| \leq A_1$, *and* $\Delta f^{\frac{1}{m-1}} \geq -A$. *Suppose there exists a strictly pluri-subharmonic solution* $u \in C^\infty(\bar{\Omega})$ *such that*

$$\det u_{i\bar{j}}(z) = f(z) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega.$$

Then there exists a constant C *which depends only on* Ω, $\sup(f)$, A_1 *and* A_2 *such that*

$$||u||_{C^1(\bar{\Omega})} + ||\Delta u||_\infty \leq C.$$

4.2 Proof of the Main Theorem

Assuming Theorem 4.1.2, we shall now prove Theorem 4.1.1. The strategy will be to solve the non-degenerate Dirichlet problem for $f > 0$, and then use a limiting process. Let $g_\varepsilon = f^{\frac{1}{m-1}} + \varepsilon$, with $\varepsilon > 0$. We extend f such that it is defined on all of $\mathbb{C}^m$. Let $\varphi_\rho = \varphi(|z|/\rho)$, where $\varphi : \mathbb{C}^m \to \mathbb{R}$ is a C^∞ function of compact support such that $0 \leq \varphi \leq 1$, $\int \varphi = 1$. We define $h_{\varepsilon,\rho} : \Omega \to \mathbb{R}$ in the following way:

$$h_{\varepsilon,\rho}(x) = (g_\varepsilon * \varphi_\rho(x))^{m-1} = \left(\int g_\varepsilon(y) \varphi_\rho(x-y) dy \right)^{m-1}.$$

We remark the following

$$\nabla f^{\frac{1}{m-1}} = \frac{m}{m-1} f^{\frac{1}{m(m-1)}} \nabla f^{\frac{1}{m}}. \tag{4.1}$$

Denote $\beta := \frac{1}{m(m-1)}$. Since $\bar{\Omega}$ is compact, we know that $g_\varepsilon * \varphi_\rho \to g_\varepsilon$ uniformly on $\bar{\Omega}$, and $f^\beta * \varphi_\rho \to f^\beta$ uniformly on $\bar{\Omega}$. Let $\rho_0 > 0$ be small enough such that for all $0 < \rho < \rho_0$, $|g_\varepsilon * \varphi_\rho - g_\varepsilon| < \varepsilon/2$ and $|f^\beta * \varphi_\rho - f^\beta| < \varepsilon/2$. We compute

$$\begin{aligned}
|\nabla h_{\varepsilon,\rho}^{1/m}| &= \frac{m-1}{m} \left| (g_\varepsilon * \varphi_\rho)^{-1/m} \left((\nabla g_\varepsilon) * \varphi_\rho \right) \right| \\
&= \left| \frac{(f^\beta \nabla f^{1/m}) * \varphi_\rho}{(g_\varepsilon * \varphi_\rho)^{1/m}} \right| \\
&\leq A_1 \frac{|f^\beta * \varphi_\rho|}{|g_\varepsilon * \varphi_\rho|^{1/m}} \\
&\leq A_1 \frac{f^\beta + \varepsilon/2}{(f^{\frac{1}{m-1}} + \varepsilon - \varepsilon/2)^{1/m}}.
\end{aligned}$$

If $f^\beta \leq \varepsilon^{1/m}$, then for all ε small enough

$$\frac{f^\beta + \varepsilon/2}{(f^{\frac{1}{m-1}} + \varepsilon/2)^{1/m}} \leq \frac{\varepsilon^{1/m} + \varepsilon/2}{(\varepsilon/2)^{1/m}} \leq 2.$$

On the other hand, if $f^\beta \geq \varepsilon^{1/m}$, then for all ε small enough

$$\frac{f^\beta + \varepsilon/2}{(f^{\frac{1}{m-1}} + \varepsilon/2)^{1/m}} \leq \frac{1 + (\varepsilon/2) f^{-\beta}}{(1 + (\varepsilon/2) f^{-\frac{1}{m-1}})^{1/m}} \leq 1 + \frac{\varepsilon^{1-1/m}}{2} \leq 2.$$

Therefore, for all $\varepsilon > 0$ small enough, then for all $0 < \rho < \rho_0(\varepsilon)$ we have

$$|\nabla h_{\varepsilon,\rho}^{1/m}| \le 2A_1.$$

We also notice

$$\Delta h_{\varepsilon,\rho}^{1/(m-1)} = \Delta(g_\varepsilon * \varphi_\rho) = (\Delta g_\varepsilon) * \varphi_\rho \ge -A_2 * \varphi_\rho = -A_2. \tag{4.2}$$

Now, we consider the non-degenerate Monge-Ampere Dirichlet problem

$$\det(u_{\varepsilon,\rho})_{i\bar{j}} = h_{\varepsilon,\rho} \text{ in } \Omega,$$

$$u_{\varepsilon,\rho} = 0 \text{ on } \partial\Omega.$$

By [5], since $h_{\varepsilon,\rho}$ is smooth, we know there exists a smooth strictly pluri-subharmonic solution $u_{\varepsilon,\rho}$. For all ε and ρ small enough, we have $|\nabla h_{\varepsilon,\rho}^{1/(m)}| \le 2A_1$ and $\Delta h_{\varepsilon,\rho}^{1/(m-1)} \ge -A_2$. Therefore, by Theorem 4.1.2, we have

$$||u_{\varepsilon,\rho}||_{C^1\bar(\Omega)} + ||\Delta u_{\varepsilon,\rho}||_\infty \le C,$$

for some constant C independent of ε and ρ. We let $\rho \to 0$ and obtain a strictly pluri-subharmonic solution u_ε of

$$\det(u_\varepsilon)_{i\bar{j}} = (f^{\frac{1}{m-1}} + \varepsilon)^{m-1} \text{ in } \Omega,$$

$$u_\varepsilon = 0 \text{ on } \partial\Omega.$$

such that $||u_\varepsilon||_{C^1(\bar\Omega)} + ||\Delta u_\varepsilon||_\infty \le C$. Finally, we let $\varepsilon \to 0$ and obtain a pluri-subharmonic solution u of (4.1.1) such that $||u||_{C^1(\bar\Omega)} + ||\Delta u||_\infty \le C$. Uniqueness will follow from Lemma 4.3.2 which will be shown below.

4.3 C^1 Estimate

We now proceed to the proof of Theorem 4.1.2. Most of the work has been done in the previous section, however we must still control the derivatives $u_{i\bar{j}}$ near the boundary $\partial\Omega$. All the estimates from the compact Kähler case can be applied; indeed, we can let $M = K$ where K is a compact subset $K \subset \Omega$, and $g_{i\bar{j}}(z) = \delta_{i\bar{j}}$ for all $z \in K$. All curvature terms vanish, and we obtain estimates for ∇u and Δu which reduce the problem to estimating these quantities on the boundary $\partial\Omega$.

To prove Theorem 4.1.2, we follow the arguments given in [5]. We let φ be a strictly pluri-subharmonic function on Ω such that $\varphi = 0$ on $\partial\Omega$ and

$$\det(\varphi_{i\bar{j}}) > \sup_{\bar{\Omega}} f.$$

Such a φ may be constructed by multiplying the function r associated with Ω by a large enough constant. To give a C^0 estimate on u, we use a maximum principle.

Lemma 4.3.1. *Let u, v be smooth real-valued functions on a bounded domain $\Omega \subset \mathbb{C}^m$ such that v is strictly pluri-subharmonic, u is pluri-subharmonic,* $\det v_{i\bar{j}} \geq \det u_{i\bar{j}}$ *in Ω and $v \leq u$ on $\partial\Omega$. Then $v \leq u$ on $\bar{\Omega}$.*

Proof. We consider

$$\begin{aligned} 0 \leq \det v_{i\bar{j}} - \det u_{i\bar{j}} &= \int_0^1 \frac{d}{dt}\det(tv + (1-t)u)_{i\bar{j}} dt \\ &= \int_0^1 \sum_{i,j} c(tv_{i\bar{j}} + (1-t)u_{i\bar{j}})(v-u)_{i\bar{j}} dt. \end{aligned}$$

Since $(u_{i\bar{j}})$ and $(v_{i\bar{j}})$ are Hermitian matrices and $(v_{i\bar{j}})$ is positive-definite, we can simultaneously diagonalize them by choosing the right coordinates. Thus the cofactor $c(tv_{i\bar{j}} + (1-t)u_{i\bar{j}})$ becomes

$$c(tv_{i\bar{j}} + (1-t)u_{i\bar{j}}) = \prod_{k \neq i}(tv_{k\bar{k}} + (1-t)u_{k\bar{k}}).$$

Since $v_{k\bar{k}} > 0$, $u_{k\bar{k}} \geq 0$, we have a uniformly elliptic operator $L(v-u) \geq 0$, where

$$L = \sum_{i,j}\left(\int_0^1 c(tv_{i\bar{j}} + (1-t)u_{i\bar{j}})dt\right)\partial_i\bar{\partial}_j.$$

By the maximum principle for linear operators, $v \leq u$. □

Lemma 4.3.2. *Let u, v be smooth real-valued pluri-subharmonic functions on a bounded domain $\Omega \subset \mathbb{C}^m$ such that $\det v_{i\bar{j}} \geq \det u_{i\bar{j}}$ in Ω and $v \leq u$ on $\partial\Omega$. Then $v \leq u$ on $\bar{\Omega}$.*

Proof. Let $\tilde{v} = v + \varepsilon|z|^2 - \varepsilon \max_{\partial\Omega}|z|^2$. For $\varepsilon > 0$ small enough, we have that $\tilde{v}$ is strictly pluri-subharmonic, $\det \tilde{v}_{i\bar{j}} \geq \det u_{i\bar{j}}$ in Ω, and $u \geq \tilde{v}$ on $\partial\Omega$. By the previous lemma, $u \geq \tilde{v}$ on $\bar{\Omega}$. We let $\varepsilon \to 0$ to obtain the result. □

From the previous lemmas, we obtain $\varphi \leq u$. To get a upper bound, we solve the Laplace equation for a harmonic function h: $\Delta h = 0$ in Ω and $h = 0$ on $\partial\Omega$. Then

$$\Delta u \geq m(\det u_{i\bar{j}})^{1/n} \geq 0 = \Delta h,$$

thus $u \leq h$ by the maximum principle. From $\varphi \leq u \leq h$, we can obtain $|\nabla u(z)| \leq \max\{|\nabla \varphi(z)|, |\nabla h(z)|\}$ for all $z \in \partial\Omega$. Since φ and h depend only on Ω, we have a gradient estimate on $\partial\Omega$. The proof of Theorem 3.3.1 can push the interior gradient estimate to the boundary. Thus we obtain

$$||u||_{C^1(\bar{\Omega})} \leq C.$$

4.4 Boundary C^2 Estimate

From the proof of Proposition 3.4.1, we see that we can control Δu on the interior of Ω by its maximum on $\partial\Omega$. Thus, it only remains to show a boundary second-order estimate to complete Theorem 4.1.2.

Let $p \in \partial\Omega$. Centre coordinates $z_1, \ldots, z_m$ such that $p = 0$. Denote $t_1 = x_1$, $t_2 = y_1$, ..., $t_{2m-3} = x_{m-1}$, $t_{2m-2} = y_{m-1}$, and $t = y_m$. We will also use the notation $t' = (t_1, \ldots, t_{2m-2})$. Rotate coordinates such that $\partial_{t_1} r(0) = 0$, ..., $\partial_{t_{2m-2}} r(0) = 0$, $\partial_t r(0) = 0$, and $\partial_{x_m} r(0) = -1$.

We Taylor expand r near 0 and obtain

$$r = Re\left(-z_m + \sum_{i,j} a_{ij} z_i z_j\right) + \sum_{i,j} b_{i\bar{j}} z_i \bar{z}_j + O(|z|^3).$$

We can change coordinates $\tilde{z}_m = z_m - \sum a_{ij} z_i z_j$, $\tilde{z}_k = z_k$ for all $k \leq m-1$. Thus without loss of generality, we can can assume our coordinates near 0 are such that

$$r = -Re(z_m) + \sum_{i,j}^{m-1} c_{i\bar{j}} z_i \bar{z}_j + O(|z|^3). \tag{4.3}$$

Since $\partial\Omega$ is where $r = 0$, we can write

$$Re(z_m) = \sum_{i,j}^{m-1} c_{i\bar{j}} z_i \bar{z}_j + O(|z|^3), \quad z \in \partial\Omega. \tag{4.4}$$

$$x_m(t') = \sum_{i,j<2m} \xi_{ij} t_i t_j + O(|t'|^3), \quad (t', x_m) \in \partial\Omega. \tag{4.5}$$

Now, on $\partial\Omega$ near 0, we have $u(t', x_m(t')) = \varphi(t', x_m(t'))$. Taking ∂_{t_i} we get

$$u_{t_i} + u_{x_m} \partial_{t_i} x_m - \varphi_{t_i} - \varphi_{x_m} \partial_{t_i} x_m = 0.$$

We apply ∂_{t_j} and evaluate at 0 to get

$$u_{t_i t_j}(0) + u_{x_m}(0)\xi_{ij} - \varphi_{t_i t_j}(0) - \varphi_{x_m}(0)\xi_{ij} = 0.$$

Therefore, we have

$$|u_{t_i t_j}(0)| = |\varphi_{t_i t_j}(0) - \xi_{ij}(u_{x_m} - \varphi_{x_m})(0)| \leq C. \tag{4.6}$$

We now estimate the mixed normal-tangential derivatives $u_{t_i x_m}$. We define

$$T_i = \frac{\partial}{\partial t_i} - \frac{r_{t_i}}{r_{x_m}} \frac{\partial}{\partial x_m}, \tag{4.7}$$

for $i = 1, \ldots, 2m-1$. We can see that $T_i r = 0$, and hence T_i is tangential to the surface $r = 0$. For $\varepsilon > 0$, we define the region

$$S_\varepsilon = \{x \in \Omega : x_m \leq \varepsilon\}. \tag{4.8}$$

We consider the following test function on S_ε:

$$w := T_i(u - \varphi) + (u_t - \varphi_t)^2 - Ax_m + B|z|^2. \tag{4.9}$$

The objective is to show the two following statements for suitable constants A and B :

$$u^{p\bar{q}} w_{p\bar{q}} \geq 0 \text{ in } S_\varepsilon, \text{ and} \tag{4.10}$$

$$w \leq 0 \text{ on } \partial S_\varepsilon. \tag{4.11}$$

Assuming (4.10) and (4.11), by the maximum principle we can conclude that $w \leq 0$ in S_ε. Thus $w(0) = 0$ is a maximum, and $w_{x_m}(0) \leq 0$. We compute

$$\begin{aligned} w_{x_m} = {} & u_{t_i x_m} - \partial_{x_m}\left(\frac{r_{t_i}}{r_{x_m}}\right) u_{x_m} - \frac{r_{t_i}}{r_{x_m}} u_{x_m x_m} - \varphi_{t_i x_m} + \partial_{x_m}\left(\frac{r_{t_i}}{r_{x_m}}\right)\varphi_{x_m} \\ & + \frac{r_{t_i}}{r_{x_m}}\varphi_{x_m x_m} + 2(u_t - \varphi_t)(u_{t x_m} - \varphi_{t x_m}) - A + 2Bx_m. \end{aligned}$$

By our gradient estimate and since $r_{t_i}(0) = 0$ and $u_t(0) = \varphi_t(0)$, we obtain $u_{t_i x_m}(0) \leq C$. We can run through the argument again switching w to $\tilde{w} := -T_i(u - \varphi) + (u_t - \varphi_t)^2 - Ax_m + B|z|^2$ and get $-u_{t_i x_m}(0) \leq C$, and thus

$$|u_{t_i x_m}(0)| \leq C. \tag{4.12}$$

We now compute $u^{p\bar{q}}w_{p\bar{q}}$ inside S_ε to prove (4.10). Denote $a = r_{t_i}/r_{x_m}$. Let $z \in S_\varepsilon$, and rotate coordinates such that $u_{p\bar{q}}(z)$ is diagonal. We start with

$$\begin{aligned}\partial_p\bar{\partial}_q T_i u &= \partial_p\bar{\partial}_q(u_{t_i} + au_{x_m})\\ &= \partial_p(u_{t_i\bar{q}} + a_{\bar{q}}u_{x_m} + au_{x_m\bar{q}}\\ &= u_{p\bar{q}t_i} + a_{p\bar{q}}u_{x_m} + a_{\bar{q}}u_{x_mp} + a_pu_{x_m\bar{q}} + au_{p\bar{q}x_m}.\end{aligned}$$

Using (2.11), we get $u^{p\bar{q}}u_{p\bar{q}t_i} = mf^{-1/m}(f^{1/m})_{t_i}$. Combining this with our C^1 estimate for u, we get

$$u^{p\bar{q}}(T_iu)_{p\bar{q}} \geq \frac{-C_1}{f^{1/m}} - C_2\sum u^{p\bar{p}} + u^{p\bar{q}}a_pu_{x_m\bar{q}} + u^{p\bar{q}}a_{\bar{q}}u_{x_mp}. \tag{4.13}$$

Since

$$\frac{\partial}{\partial z_m} = \frac{1}{2}\left(\frac{\partial}{\partial x_m} - i\frac{\partial}{\partial y_m}\right),$$

we obtain $u_{x_m\bar{q}} = 2u_{m\bar{q}} + iu_{\bar{q}t}$. Using $u^{p\bar{q}}u_{m\bar{q}} = \delta_{mp}$ and the Cauchy-Bunyakowsky-Schwarz inequality, we see that

$$\begin{aligned}|u^{p\bar{q}}a_pu_{x_m\bar{q}}| &\leq 2|a_n| + |u^{p\bar{q}}a_pu_{t\bar{q}}|\\ &\leq 2|a_n| + \left(u^{p\bar{q}}a_pa_{\bar{q}}\right)^{1/2}\left(u^{p\bar{q}}u_{tp}u_{t\bar{q}}\right)^{1/2}\\ &\leq 2|a_n| + \frac{1}{2}\left(u^{p\bar{q}}a_pa_{\bar{q}} + u^{p\bar{q}}u_{tp}u_{t\bar{q}}\right).\end{aligned}$$

Similarly, we can obtain

$$|u^{p\bar{q}}a_{\bar{q}}u_{x_mp}| \leq 2|a_n| + \frac{1}{2}\left(u^{p\bar{q}}a_pa_{\bar{q}} + u^{p\bar{q}}u_{tp}u_{t\bar{q}}\right).$$

Combining these inequalities with (4.13), along with (2.10), we get

$$u^{p\bar{q}}(T_i u)_{p\bar{q}} \geq -C_3 \sum u^{p\bar{p}} - u^{p\bar{q}} u_{tp} u_{t\bar{q}}. \tag{4.14}$$

Next, we compute

$$\begin{aligned}
u^{p\bar{p}} \partial_p \bar{\partial}_p (u_t - \varphi_t)^2 &= 2u^{p\bar{p}}(|u_{tp} - \varphi_{tp}|^2 + (u_t - \varphi_t)(u_{tp\bar{p}} + \varphi_{tp\bar{p}})) \\
&= 2u^{p\bar{p}} u_{tp} u_{t\bar{p}} - 4u^{p\bar{p}} u_{pt} \varphi_{\bar{p}t} + 2u^{p\bar{p}} \varphi_{tp} \varphi_{t\bar{p}} + 2(u_t - \varphi_t)(u^{p\bar{p}} u_{tp\bar{p}} + u^{p\bar{p}} \varphi_{tp\bar{p}}) \\
&\geq 2u^{p\bar{p}} u_{tp} u_{t\bar{p}} - 4u^{p\bar{p}} u_{pt} \varphi_{\bar{p}t} - C_1 \sum u^{p\bar{p}} \\
&\geq 2u^{p\bar{p}} u_{tp} u_{t\bar{p}} - 4|\frac{u^{p\bar{p}}}{2} u_{pt} u_{\bar{p}t}|^{1/2} |(2u^{p\bar{p}}) \varphi_{pt} \varphi_{\bar{p}t}|^{1/2} - C_1 \sum u^{p\bar{p}} \\
&\geq 2u^{p\bar{p}} u_{tp} u_{t\bar{p}} - u^{p\bar{p}} u_{pt} u_{\bar{p}t} - 4u^{p\bar{p}} \varphi_{pt} \varphi_{\bar{p}t} - C_1 \sum u^{p\bar{p}} \\
&\geq u^{p\bar{p}} u_{tp} u_{t\bar{p}} - C_2 \sum u^{p\bar{p}}.
\end{aligned}$$

Combining this with (4.14), the trouble terms cancel and we obtain

$$u^{p\bar{q}} w_{p\bar{q}} \geq B \sum u^{p\bar{p}} - C_3 \sum u^{p\bar{p}} \geq 0,$$

for B large enough. We have thus shown (4.10).

We now prove (4.11). We look at ∂S_ε as two pieces. The first piece is $\partial\Omega \cap S_\varepsilon$. Here $u = \varphi$ and the tangential derivatives are equal. Furthermore, by (4.5) we have $a|z|^2 \leq x_m$ near 0, for some $a > 0$. Hence

$$w = -Ax_m + B|z|^2 \leq -Ax_m + Ba^{-1}x_m \leq 0$$

for A large. The second piece of ∂S_ε is $\{x_m = \varepsilon\} \cap \Omega$, and by our C^1 estimate, here we have

$$w \leq C - A\varepsilon + B\varepsilon^2 \leq 0.$$

We have thus shown (4.11) and (4.10). As shown in (4.12), this completes the proof of the mixed tangential-normal estimate $|u_{x_m t_i}(0)| \leq C$.

To complete the proof of Theorem 4.1.2, it remains to show that $|u_{n\bar{n}}(0)| \leq C$. Using the expression from (4.4), along $\partial\Omega$ we have $u(z', z_m(z')) = 0$. We compute second derivatives for $i, j \leq m-1$:

$$u_{i\bar{j}} + u_{ix_m}\frac{\partial z_m}{\partial \bar{z}^j} + u_{\bar{j}x_m}\frac{\partial z_m}{\partial z^i} + u_{x_m x_m}\frac{\partial z_m}{\partial \bar{z}^j}\frac{\partial z_m}{\partial z^i} + u_{x_m}\frac{\partial^2 z_m}{\partial z^i \partial \bar{z}^j} = 0.$$

At 0, we thus have

$$u_{i\bar{j}}(0) = -c_{i\bar{j}} u_{x_m}(0). \tag{4.15}$$

Since r is strictly pluri-subharmonic, $c_{i\bar{j}} > 0$. Fix $z_\alpha \in \partial\Omega$. Since Ω has the interior ball condition, let $B \subset \Omega$ be a ball such that $\overline{B} \cap \partial\Omega = \{z_\alpha\}$. We have $\Delta u \geq m f^{1/m} \geq 0$, thus by the Hopf lemma, if coordinate centred at z_α are set such that x_m points in the direction of $-\hat{n}$, we have $u_{x_m}(z_\alpha) \leq -d_\alpha < 0$. Since $\partial\Omega$ is compact, we have $d := \inf_{z_\alpha \in \partial\Omega} d_\alpha > 0$. Therefore,

$$u_{i\bar{j}}(0) \geq d\, c_{i\bar{j}}(0),$$

$$(u_{i\bar{j}}(0))_{i,j\leq m-1} \geq c_0 I,$$

for some $c_0 > 0$. Unravelling $\det u_{i\bar{j}} = f$ by cofactor expansion, we obtain

$$f = u_{m\bar{m}} c(u_{m\bar{m}}) + \sum_{i=1}^{m-1} u_{m\bar{i}} c(u_{m\bar{i}}) = u_{m\bar{m}} \det(u_{i\bar{j}}(0))_{i,j \le m-1} + R,$$

where R is a quantity involving second order partials $u_{\alpha\bar{\beta}}$, where $\beta \neq m$. By our previous estimates, R is under control. We have

$$|u_{m\bar{m}}(0)| = \left| \frac{f - R}{\det(u_{i\bar{j}}(0))_{i,j \le m-1}} \right| \le \frac{\sup f + |R|}{c_0^{m-1}}.$$

This completes the proof of Theorem 4.1.2.

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