Bifurcations towards periodic solutions in quadratic semilinear partial differential equations

Matthieu Cadiot

Master of Science

Department of Mathematics and Statistics

McGill University

Montreal,Quebec

07/2020

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of Master of Science in Mathematics and Statistics

ACKNOWLEDGEMENTS

I would like to express my deep gratitude to Professor Jean-Philippe Lessard and Professor Jean-Christophe Nave, my two research supervisors at McGill University. Their doors were always opened to discuss my interrogations and to direct me throughout the successes and failures of the project. Their patient guidance for me in the world of research has been highly appreciated.

I would also like to extend my thanks to the students of my two research groups who were always carefully attending my presentations. Their feedback was a precious asset in the progression of my research.

ABSTRACT

In this project, we present a finite-dimensional approach that will enable us to prove bifurcations towards periodic solutions in quadratic semilinear Partial Differential Equations (PDEs). To do so, we use the Banach space X^s of Fourier coefficients. A projection on the N first coefficients and a set of hypotheses allow us to use the Crandall-Rabinowitz Theorem and prove a bifurcation in the finite-projection. Then we build up a well-chosen fixed point operator in order to prove the existence of a bifurcation towards a periodic solution in the actual PDE (thanks to some regularity results). Finally, we give some applications of our main Theorem. To highlight these applications, we present a numerical method to approximate Spontaneous Periodic Orbits and a second one to mimic the bifurcations of the finite-projection. Results for Navier-Stokes equations and Kuramoto-Sivashinsky equation are presented.

ABRÉGÉ

Dans ce projet, nous présentons une approche en dimension finie qui nous permettra dans un second temps de prouver l'existence de bifurcations vers des solutions périodiques dans les Equations aux Dérivées Partielles (EDP) quadratiques et semilinéaires. Pour cela nous utilisons le Banach X^s dans l'espace des coefficients des séries de Fourier. Une projection sur les N premiers termes associée à un ensemble d'hypothèses, nous permet l'utilisation du Théorème de Crandall-Rabinowitz pour prouver l'existence de bifurcations dans la projection finie. Par la suite, nous démontrons l'existence de bifurcations vers des solutions périodiques dans l'EDP grâce à un opérateur de point fixe bien choisi et certains résultats de régularité. Pour finir, nous exposons certaines applications de notre Théorème central. Dans le but d'appuyer ces applications, nous développons une méthode numérique pour approximer les Orbites Périodiques Spontanées ainsi qu'une méthode pour approcher les bifurcations de la projection finie. Finalement, nous donnons des exemples de ces méthodes aux travers des équations de Navier-Stokes et de Kuramoto-Sivashinsky.

CONTRIBUTION

Matthieu Cadiot, Master's student in Mathematics and Statistics at McGill University, supervised by Jean-Philippe Lessard and Jean-Christophe Nave, is the only author of this document. All the chapters were completed under his own and only writing.

TABLE OF CONTENTS

LIST OF FIGURES

Introduction

Periodic solutions in Partial Differential Equations (PDEs) are a central subject in the analysis of existence of solutions. In fact, searching for periodic solutions may seem useless at first sight if one thinks about finding general solutions to a Cauchy problem. However, over the years it has been shown that periodic solutions could lead to more complex patterns that help deepen the analysis of PDEs. It is a first insight into the wider analysis of solutions. We want to emphasize the importance of this kind of solution through the examples of turbulence and chaos. These phenomena are the subject of very active investigations, in both the fields of Mathematics and Physics. Now, if we specifically look at turbulence or chaos in fluid dynamics, and especially in the Navier-Stokes equations, we encounter the famous theory of Laudau and Lifshitz [33], derived in 1959. The idea relies on a cascade of periodic solutions in which the period doubles at each step of the cascade. This reasoning helped to understand the different scales of turbulence, and made the link between macro and micro observations. Indeed, this phenomenon of period doubling also justifies the distribution of the system's energy in a similar cascading way. Moreover, [29], [7] and [20] also describe the famous period doubling route to chaos and turbulence. These articles provide theoretical details following Laudau and Lifshitz's idea of period doubling.

At this point, we want to get back to the theory of Laudau and Lifshitz, which they derived from the experimental observations of Reynolds in the 19th century. Indeed, they used the idea that the first step of turbulence is the "bifurcation" from a steady state to a periodic solution, as the Reynolds number increases. Therefore, if one detects bifurcations from steady states, it may lead to a better understanding of turbulence. Such a premise motivates our study for bifurcations in PDEs. As a consequence, one part of our study will be to provide assumptions under which we can find bifurcations from a steady state to a periodic solution in the incompressible Navier-Stokes equations (NSE). Moreover, we aim to generalize our result in NSE and apply it to the following class of PDEs :

$$\mathcal{L}u + \mathcal{G}(u,u) = f \tag{1}$$

where the linear differential operator $\mathcal{L}$ and the bilinear differential operator $\mathcal{G}$ obey some assumptions $A(1-3)$. To give an insight of our assumptions, we will suppose that the linear term has a higher differential order than the non-linearity. In terms of frequencies, this implies that the higher frequencies will be controlled by the linear part. As a consequence, (1) is a quadratic semilinear partial differential equation. From a notation point of view, f is called the forcing term or more simply the forcing.

In the Navier-Stokes equations, or in parabolic equations in general, we want to study bifurcations from a steady state u_0. We call steady state a time-independent solution of (1). The idea is to construct a branch of periodic solutions from u_0. In fact, it is also possible to impose the forcing to be stationary. Indeed, we developed a method to prove the existence of Spontaneous Periodic Orbits (SPO), which is defined as a periodic solution arising from a time-independent forcing f (see [48]). Furthermore, we give a method to numerically approximate these solutions thanks to the minimization of an appropriate energy. The interest for SPO also comes from fluid dynamics. An experiment that accurately represents Spontaneous Period Orbits is the von Kármán vortex street. It consists in making a fluid flow passed a fixed cylinder and making the velocity of the fluid evolve. If the velocity is small, then the fluid is laminar and circulates around the cylinder as we would imagine. But if the velocity is increased enough, the flow begins to oscillate and vortices are created in the wake of the cylinder. Moreover, they alternately leave the cylinder periodically from one side to the other. This phenomenon is called vortex shedding and gives birth to the so-called Kármán vortex street.

In a general setting, we construct a domain of definition of $\mathcal{L}$ so that it becomes invertible in that domain. Moreover, $\mathcal{L}$ does not particularly involve time derivatives anymore. Then we want to reproduce the previous idea by creating a branch of periodic solutions around some central solution u_0, which is not necessarily stationary. Our technique is based on the decomposition in Fourier series and on a finite-dimensional approximation of the problem. Secondly, we develop a fixed point method, based on the finite-dimensional problem. This allows us to obtain strong solutions to (1) in the infinite dimensional Banach space X^s. SPO will be a special case of our main Theorem.

Some mathematical tools of interest to our theory will be explicited in the first chapter. The second chapter of this thesis will provide the set up of the study, the hypotheses and our main Theorem. In Chapter 3, under the given hypotheses, we develop the finite-dimensional approach. The results will then be used to build up branches of solutions to problem (1) thanks to a fixed point method. In Chapter 4 we relax the hypotheses and shows how our result can be generalized to a wider range of central solutions u_0. Finally, we give some applications (in particular Spontaneous Periodic Orbits) in Chapter 5, that will be completed with numerical results in Chapter 6.

Literature review

We want to begin this literature review by recalling some known results of the Navier-Stokes equations. In fact, these equations are the perfect example for the model of problems we describe. We are interested in the complexity of the equation and moreover in the tremendous work that has been achieved in this field. In particular, there is a large study of periodic solutions in NSE. We want to give a review of this literature as it will help understand our point of view on the problem. The first one to give an insight of such solutions in NSE was Serrin in 1959 when he gave a note [43] linked to his previous article [44]. The hypothesis of Serrin was the existence of a solution for a small Reynolds number, and under some periodicity hypotheses of the domain, the forcing and the boundary condition, he gave the existence of a periodic solution. The first point of interest in this early study is the hypothesis of a small Reynolds number. This point implies in particular some stability against perturbations that Serrin proved in [44]. In [47], Teramoto gave the existence of periodic solutions under the same kind of hypothesis on the Reynolds number. However, he just needed some bound and periodicity of the forcing for the existence of solutions. In a similar way Kaniel and Shinbrot in [28] gave the existence of periodic solutions for small periodic forcing and small initial data. One interesting corollary of their result is that if the forcing is time-independent and if the initial data is small, then the periodic solution is actually time-independent as well. Therefore, this gives rise to the problematic of the Spontaneous Periodic Orbits : how can we find strictly (non-constant) periodic solutions if the forcing term is time-independent ? One necessary condition given by the work of the authors we cited is when the Reynolds number is big enough. Even if we know from Galdi and Kyed in [22] that Navier-Stokes has strong time-periodic solutions corresponding to time-periodic forcings, these result do not differentiate between a steady state and a strictly periodic solution. This justifies why we want to use bifurcation theory in order to prove the existence of Spontaneous Periodic Orbits. The existence of bifurcations in NSE has already been studied by many authors, e.g., [26], [11] [35], [14], [18] and [48]. In particular, [48] gives the existence of a Spontaneous Periodic Orbit for the Taylor-Green forcing. The authors proved a branch of solutions, far from the bifurcation, via a computer-assisted proof. The interesting connection of the article with our study is how the branch was detected. They started from a steady state that solves the stationary problem, and they numerically built the bifurcation thanks to continuation with viscosity as the continuation parameter. Our goal is to extend this kind of result to a wider range of steady states.

But as was mentioned before, we also want to generalize this idea to a larger class of problems : given a known (strictly periodic or steady) solution u_0, how can we find a branch of periodic solutions in its neighborhood ? This leads us to the study of the linearization of the differential operator around u_0 and the possible bifurcation properties. The Crandall-Rabinovitz Theorem in [15] is the fundamental tool for this kind of analysis. From this point of view, we are going to discuss in what ways our formulation can match the realm of bifurcations in PDEs by giving examples of some related works.

Our hypothesis is less restrictive than just a parabolic or an elliptic hypothesis, but we still give a review of the bifurcations in both setups. In fact we will show why they are good candidates for bifurcations and adequate sources for examples. We present this subject, in which Navier-Stokes equations take part, because there is no general study of periodic solutions in PDEs. Our choice for this specific direction is justified by the extremely broad range of literature with these hypotheses. This in part underlines why we wanted to prove general results in quadratic equations, and

not just limit ourselves with parabolic or elliptic equations. The case of hyperbolic equations is more complicated for our approach. Indeed, they can be ill-posed, as it can be seen through the wave equation.
We begin by studying the case where the equation (1) is elliptic. For this approach, we refer to [31], as the elliptic hypothesis is fully addressed in Section III. We give some details of the analysis. We begin by noticing that the Garding's inequality and the Lax-Milgram Theorem (see [19]) imply, for an elliptic operator, that the corresponding Dirichlet problem has a unique weak solution, up to a constant shift of the operator (induced by Garding's inequality). Moreover, the elliptic regularity also implies that the weak derivatives of the solution are well defined. In particular, the analysis of [31] shows that second order elliptic operators have a zero Fredholm index for suitable Hilbert spaces of definition. This special kind of operators is, as a consequence, very suitable for the use of the Crandall-Rabinowitz Theorem. In fact, we find in [31] the elliptic version of the Theorem. As it is a fitting hypothesis, a lot of research has been made on the bifurcations in elliptic partial differential equations, we give an insight of the accomplished work.

In a series of three articles, [15], [16] and [41], Crandall and Rabinowitz gave a class of PDEs in which their Theorem can easily be applied. They considered the following class of PDEs

$$L(u) + G(u) = \lambda u \tag{2}$$

with L being an elliptic linear operator and G is such that $G(0) = 0$ and $DG(0) = 0$. Then clearly for λ being an eigenvalue of L, the linearization of $L(u) + G(u) - \lambda u$ at zero is not invertible. Therefore, Crandall and Rabinowitz were able to use their Theorem of bifurcation for this kind of problem. To be more specific, they proved a bifurcation from the trivial line of solutions $\{(0, \lambda), \lambda \in \mathbb{R}\}$. Then, in his work in [37], Nirenberg gave a famous result, on the possible unboundedness of a continuum of solutions for (2). This characteristic allows global bifurcation in Crandall-Rabinowitz Theorem, where the statement is originally only local. In particular, this unbounded continuum is the one associated with the principal eigenvalue of the operator L. However, all the continua associated to the eigenvalues of L can also be bounded if they intersect each other. These results have motivated works on the extension of the local bifurcation into a global one. Amann summarized some important results in [4] on the eigenvalue problems associated to elliptic systems. He also showed the link with bifurcations. Moreover, Aliev, in [2] and [3], gave global bifurcations results for some general second order elliptic equations under growth control on the non-linearity. We can cite other works that give examples of bifurcations, sometimes global, to different kinds of elliptic equations : [9], [12], [21] and [45].

Now we want to focus on the classical work of Chafee and Infante in [13] because it is close to the idea of our approach. They studied the heat equation perturbed by a non-linear term with an amplitude λ. The result they obtained is the stability of the null solution for small λ and the apparition of two stable branches of solutions for some critical λ_1. The interesting idea is the use of a linear part which has a unique solution to the associated Dirichlet problem. This is a similar criteria as the one we impose on the linear operator $\mathcal{L}$. The non-linearity can then be seen as a perturbation of the well-posedness of the linear problem. In the same direction, we can refer to Shinbrot and Kaniel in [46] and [28] that used the heat equation in Navier-Stokes equations. They applied the Schaefer's fixed point Theorem, presented in the mathematical tools, to prove the existence of solutions.

As a consequence, we have seen that elliptic operators are suitable for the study of bifurcations in PDEs. Therefore, it is natural to search for bifurcations in parabolic equations where an elliptic operator is involved. We specifically look at time-periodic solutions in parabolic equations.

The book [39] by Vejvoda tries to catalog the main results on the subject and gathers the main techniques to prove the existence of periodic solutions. Some other insightful examples can be found in [5], [36], [32], [34], [25] and [40]. Even if this is just a very small introduction to the subject, it still motivates our approach to build a method to find bifurcations towards periodic solutions for this class of equation, and in particular to the Navier-Stokes equations.

Therefore, we want to know in what cases these bifurcations lead to periodic solutions. This brings us to the study of Hopf bifurcations, which we also present in Chapter 1. This specific bifurcation type requires some non-degeneracy hypotheses to be applicable. Therefore, a lot of research has been done in order to understand in which specific cases a Hopf bifurcation is well suited. We first cite [17] from Crandall and Rabinowitz where the authors put in place a strictly imaginary eigenvalue problem. From this they gave a range of PDEs in which we can find Hopf bifurcations. Then we give some references for Hopf bifurcations in the Navier-Stokes equations : [11], [14], [27], [30] and [35]. The last article gives the existence of Hopf bifurcations thanks to a well defined set of stationary solutions. Each element of that set implies that the hypotheses for a bifurcation will be satisfied. We distinguish our result from this one as we will use a larger and more precise set of stationary solutions. Overall, there is a need to assume some purely imaginary eigenvalues and/or some non-degeneracy criteria. This is why we chose not to study Hopf bifurcation but instead to analyse bifurcations in Fourier series. The periodicity of the solutions will be a consequence of our construction.

In order to build our approach, we need to choose the adapted space for our Fourier series. It naturally comes to mind to use of the X^s space defined in [24]. X^s is a Banach space that is well suited for the study of partial differential equations.

In particular, it gives sharp estimates for the convolution terms as it is shown in [23]. We cite [8] as an other use of that space for the study of resonant PDEs. Moreover, if we can prove some regularity of the equation, the Fourier series give an excellent way to approximate the solution. In fact, if we can prove some Gevrey regularity or analycity of the solution, then the coefficients of the series decay exponentially. For instance, it has been shown in some cases of Navier-Stokes equations in [20] and [38] or in the Kuramoto-Sivashinsky equation where we refer to [6] and [10]. In addition to the periodicity criterion, this exponential decay justifies our use of Fourier series to approximate solutions.

We refer to [24] to highlight the method used in order to find solutions. The method is based on an approximated solution upon which the authors built a branch of solutions thanks to a fixed point method. This is exactly the method we are going to use, but instead of using numerical solutions as approximations, we are going to use solutions of finite-dimensional projections (on the N first frequencies). Then we refer to [48] as a numerical method to detect bifurcations. The article shows the existence of a Spontaneous Periodic Orbits by computer-assisted proof. The authors first explicited a continuous branch of stationary solutions. Then by continuation, they numerically detected a bifurcation. Finally, by following the new branch, they proved a solution far from the bifurcation point, thanks to the radii polynomial Theorem. This is an idea we hope to use but in reverse by continuously linking the found SPO to the steady states line.

After giving the background for our study, we would now like to be more precise by explicitly giving some Mathematical tools spoken about in the literature review. These tools are central in the study of bifurcations in PDEs.

CHAPTER 1
Mathematical tools

1.1 Definitions

We begin by giving some definitions. We refer to [1], [19] and [31] for more details.

Definition 1.1.1. (multi-index and vector inequalities) Let $x, y \in \mathbb{R}^m$ and $\alpha \in \mathbb{N}^m$, we define $x^\alpha := x_1^{\alpha_1}...x_m^{\alpha_m}$, where x_i and α_i are the i_{th} component of x and α respectively. We also define $|\alpha| := \alpha_1 + ... + \alpha_m$.
Moreover, we say that $x \leq y$ if for all $i \in \{1, ..., m\}$, $x_i \leq y_i$. The same notation holds for $<, >$ and $\geq$.

Definition 1.1.2. (ball) Let $R > 0$ and $(X, ||.||_X)$ be a Banach space. Then $B_R := \{x \in X, ||x||_X \leq R\} \subset X$ denotes the closed *ball of radius* R centered at zero in X.

Definition 1.1.3 (elliptic, uniformly elliptic and parabolic operators)**.** Let $L = \sum\limits_{|\alpha|\leq n} a_\alpha \partial^\alpha$ be a linear differential operator of order n on a domain $\Omega \subset \mathbb{R}^m$. $(a_\alpha)_\alpha$ are smooth real-valued functions on Ω. Then L is said to be *elliptic* if for all $x \in \Omega$ and all $\xi \in \mathbb{R}^m$ we have

$$\sum_{|\alpha|=n} a_\alpha(x)\xi^\alpha \neq 0.$$

Moreover, L is said to be *uniformly elliptic* if $n = 2k$ and if there exists $C > 0$ such that for all $\xi \in \mathbb{R}^m$

$$(-1)^k \sum_{|\alpha|=2k} a_\alpha(x)\xi^\alpha \geq C|\xi|^{2k}.$$

Finally, if L is uniformly elliptic, we say that $\partial_t + L$ is a *parabolic* operator.

Definition 1.1.4 (semilinear PDE)**.** A partial differential equation of order n, defined on a domain $\Omega \subset \mathbb{R}^m$ is said to be *semilinear* if for $x \in \Omega$ it has the form

$$\sum_{|\alpha|=n} a_\alpha(x)\partial^\alpha u + a_0(D^{n-1}u, ..., Du, u, x) = 0$$

where $D^i := \{\partial^\alpha, |\alpha| = i\}$ for $i \in \mathbb{N}$.

Definition 1.1.5 (Fredholm operator and index)**.** The *Fredholm index* is denoted *ind*. It is defined for each bounded linear operator $T : X \to Y$ with X, Y two Banach spaces. Moreover, T needs to have a finite-dimensional kernel, a finite-dimensional cokernel and a closed range. T is called a *Fredholm operator* and *ind* T is given by

$$ind\ T := dim\ Ker(T) - dim\ coKer(T)$$

where Ker is the kernel and $coKer$ is the cokernel.

1.2 Background results

We study the following problem :

$$F(x, \lambda) = 0 \tag{1.1}$$

where the mapping F is defined as

$$F \colon U \times \mathbb{R} \to Z$$
$$(x, \lambda) \mapsto F(x, \lambda).$$

$U \subset X$ is an open set and X, Z are Banach spaces. We now give the main Theorems related to that problem.

1.2.1 Implicit function Theorem

Theorem 1.2.1 (Implicit function). *Suppose that (1.1) has a solution* $(x_0, \lambda_0) \in U \times \mathbb{R}$ *such that the Fréchet derivative of* F *with respect to* x *at* (x_0, λ_0) , $D_x F(x_0, \lambda_0)$, *is bijective. We also assume that* F *and* $D_x F(x_0, \lambda_0)$ *are continuous.*

Then there exists a neighborhood $U_1 \times V_1 \subset U \times \mathbb{R}$ *of* (x_0, λ_0) *and a mapping* f *defined from* V_1 *to* U_1 *such that* $f(\lambda_0) = x_0$ *and for all* $\lambda \in V_1$

$$F(f(\lambda), \lambda) = 0.$$

Furthermore, f *is uniquely defined and is continuous on* V_1*.*

Remark 1. *If* F *is k-times continuously (Fréchet) differentiable on* $U \times \mathbb{R}$*, then* f *is k-times continuously differentiable as well on* V_1*. Furthermore, if* F *is analytic, then* f *will be analytic.*

1.2.2 Crandall-Rabinowitz Theorem

Theorem 1.2.2 (Crandall-Rabinowitz). *We suppose the following assumptions*

- *The line* $\{(0, \lambda), \lambda \in \mathbb{R}\}$ *is called the trivial line and each point of this line solves (1.1).*
- *There exists* λ_0 *such that the dimension of the Kernel of* $D_x F(x_0, \lambda_0)$ *is 1 and* $D_x F(x_0, \lambda_0)$ *is a Fredholm operator of index zero.*
- F *is twice differentiable.*

Moreover we suppose that

$$Ker(D_x F(0, \lambda_0)) = span(v_0)$$

$$D^2_{x,\lambda} F(0, \lambda_0) v_0 \notin R(D_x F(0, \lambda_0)),$$

where Ker stands for Kernel and R for Range. Then there exists $\delta > 0$ and a non-trivial continuously differentiable curve through $(0, \lambda_0)$

$$\{(x(s), \lambda(s))|\ s \in (-\delta, \delta), (x(0), \lambda(0)) = (0, \lambda_0)\}$$

such that $F(x(s), \lambda(s)) = 0$ *for all* $s \in (-\delta, \delta)$. *Moreover, all the solutions around* $(0, \lambda_0)$ *are either on this curve or on the trivial line.*

1.2.3 Hopf Bifurcation

We look at the special case of (1.1) with a time derivative :

$$\frac{dU(x,t)}{dt} = F(U(x,t), \lambda) \tag{1.2}$$

where we still assume that the trivial line $\{(0, \lambda), \lambda \in \mathbb{R}\}$ is a line of solutions for (1.2). Now the variable $x \in \Omega \subset \mathbb{R}^m$ represents the space variable and t the time. Ω is a bounded domain with smooth boundary. We solve for U as a function of (x, t) under the parametrization of λ.

Then, we consider F to be a partial differential operator. We call I_d the identity operator and we add the following assumptions.

- There exits $\kappa_0 > 0$ such that $i\kappa_0$ is a simple eigenvalue of $D_U F(0, \lambda_0)$ with eigenvector $\psi \notin R(i\kappa_0 I_d - D_U F(0, \lambda_0))$.
- $D_U F(0, \lambda_0)$ is an elliptic partial differential operator.
- F is three times differentiable.

Then, there exists (see [31]) a continuously differentiable curve of perturbed eigenvalues $\mu(r)$ near λ_0, with $\mu(0) = \lambda_0$ and r small.

Therefore we can introduce a supplementary condition (with “Re” representing the real part) :

$$Re(\frac{d\mu(r)}{dr}) \neq 0$$

Theorem 1.2.3 (Hopf Bifurcation)**.** *Let F and $D_U F(0,\lambda_0)$ satisfy the previous assumptions. Moreover we add the following non-resonance condition*

$$\forall n \in \mathbb{Z} \setminus \{-1,1\}, \quad in\kappa_0 \textit{ is not an eigenvalue of } D_U F(0,\lambda_0).$$

Then there exists $\delta > 0$ and a non-trivial continuously differentiable curve through $(0,\lambda_0)$ that solves (1.2)

$$\{(U(s),\lambda(s))|s \in (-\delta,\delta),(x(0),\lambda(0)) = (0,\lambda_0)\}$$

such that $U(s)$ is $\frac{2\pi}{\kappa(s)}$ time-periodic with $\kappa(0) = \kappa_0$, which means that $\forall(x,t) \in \Omega \times \mathbb{R}$, $U(s)(x,t+\frac{2\pi}{\kappa(s)}) = U(s)(x,t)$. Moreover, $U(s)$ is in the $\frac{2\pi}{\kappa(s)}$ periodic Holder continuously differentiable functions space $C^{1+\alpha}_{\frac{2\pi}{\kappa(s)}}(\mathbb{R},Z)\bigcap C^{\alpha}_{\frac{2\pi}{\kappa(s)}}(\mathbb{R},X)$.

We refer to [31] for a more detailled definition of $C^{1+\alpha}_{\frac{2\pi}{\kappa(s)}}(\mathbb{R},Z)\bigcap C^{\alpha}_{\frac{2\pi}{\kappa(s)}}(\mathbb{R},X)$.

1.2.4 Fixed point method

We recall Schauder's and Banach fixed point Theorems as they are given in [19] :

Theorem 1.2.4 (Schauder's fixed point)**.** *Let X be a Banach space and suppose that $K \subset X$ is compact and convex. Then if*

$$A : K \to K$$

is continuous, then A has a fixed point in K.

Theorem 1.2.5 (Banach fixed point)**.** *Let X be a Banach space. Then if*

$$A : X \to X$$

is a contraction mapping, then A has a unique fixed point in X.

Schauder's fixed point technique

In particular, this Theorem can be useful to solve non-linear partial differential equations. We give some framework in which the Theorem can be applied.

We still consider a smooth domain $\Omega \subset \mathbb{R}^m$ and the following Cauchy problem

$$\begin{aligned} &\frac{dU}{dt} = L(U) + N(U) + f \\ &U(0) = U_0 \end{aligned}$$

where we consider $U, f \in E \subset X$ and $L(U), N(U) \in E$ with X a Banach space and E an open subspace of X. We assume that L and N are partial differential operators. Moreover, we assume that for all $V_0, f \in E$, the following Cauchy problem has a unique solution

$$\begin{aligned} &\frac{dV}{dt} = L(V) + f \\ &V(0) = V_0. \end{aligned}$$

Therefore, for all $U \in E$

$$\begin{aligned} &\frac{dV}{dt} = L(V) + N(U) + f \\ &V(0) = U(0) \end{aligned}$$

has a unique solution. Then we define the following operator

$$\begin{aligned} A \colon &E \to E \\ &U \mapsto AU = V \end{aligned}$$

where $AU = V$ is the unique solution of the previous problem. Now if we are able to find a convex and compact $K \subset E$ on which A is continuous and such that $A : K \to K$, then the Schauder Theorem can be applied and there exits $U \in K$ such that $AU = U$. Therefore, by definition of A, U solves the initial problem.

Another advantage in the case of elliptic partial differential equations is the fact that by being a solution to the PDE, V gains elliptic regularity, and in particular the weak derivatives will be defined. We will use a similar idea in the construction of our fixed point method.

We give [46] and [28] as examples of that idea in the case of the Navier-Stokes equations.

CHAPTER 2
Formulation of the problem and main result

In this second chapter, we begin by presenting our main problem as well as necessary tools and spaces, in particular the Banach space X^s defined in [23]. Afterwards we will state our central Theorem of existence.

2.1 The problem

The main goal is to study periodic solutions u of quadratic semilinear partial differential equations of the following form :

$$\mathcal{L}(u) + \mathcal{G}(u, u) = f \tag{2.1}$$

where $\mathcal{L}$ is a linear differential operator and $\mathcal{G}$ is a bilinear differential operator. Moreover u is supposed to be a smooth real function defined on the bounded squared domain $\Omega = [-\pi, \pi]^m \subset \mathbb{R}^m$. We suppose that (2.1) is purely quadratic, i.e, it cannot be transformed into a linear problem.

We recall the formulation used in [24] by introducing the weight of dimension 1 for $s_1 \in \mathbb{R}_+$ and $k_1 \in \mathbb{Z}$

$$w_{k_1}^{s_1} := \begin{cases} 1 & \text{if } k_1 = 0 \\ |k_1|^{s_1} & \text{otherwise .} \end{cases}$$

Then for $s \in \mathbb{R}_+^m$ and $k \in \mathbb{Z}^m$ we define $w_k^s := w_{k_1}^{s_1} ... w_{k_m}^{s_m}$.

Now we introduce some complementary assumptions on $\mathcal{L}$ and $\mathcal{G}$, which will give the semilinear characteristic to (2.1)

Assumption 1. *There exists* $\theta \geq (1, ..., 1)$ *such that* $\mathcal{L}u := \sum_{\alpha \leq \theta} a_\alpha \partial^\alpha u$ *with* $(a_\alpha)_\alpha$ *a sequence of constant coefficients.*

Assumption 2. *There exits* $\gamma \geq (1, ..., 1)$ *such that* $\mathcal{G}(u, v) := \mathcal{G}_1(u)\mathcal{G}_2(v) := \sum_{\alpha,\beta \leq \theta-\gamma} p_\alpha \partial^\alpha u \; q_\beta \partial^\beta v$*. Moreover, there exists* $C_g > 0$ *that gives*

$|\frac{\sum_{\alpha \leq (\theta-\gamma)} p_\alpha \xi^\alpha}{\sum_{\alpha \leq \theta} a_{\alpha_i} \xi^\alpha}| \leq \frac{C_g}{w_\xi^\gamma}$ *and* $|\frac{\sum_{\beta \leq (\theta-\gamma)} q_\beta \xi^\beta}{\sum_{\alpha \leq \theta} a_{\alpha_i} \xi_i^{\alpha_i}}| \leq \frac{C_g}{w_\xi^\gamma}$ *for all* $i \in \{1, ..., m\}$ *and* $|\xi|$ *big enough.*

Assumption 3. *The Fourier series transform of* $\mathcal{L}$*, defined as* L*, is invertible in a suitable restricted subspace* $\mathcal{D}$ *of* $X^{s+\theta}$*. Moreover, for a fixed* $U \in X^{s+\theta}$*,*

$$R(G(U, \cdot)) \subset \mathcal{D}.$$

From now on we refer to these hypotheses as $A(1-3)$ and $A(i)$ for the assumption i $(i = 1, 2, 3)$. The notations of $A(3)$ are defined in the next section.

2.2 Function spaces

At this point we need to define the space in which we expect the solution u to live in. As we search for periodic solutions, it is quite natural to look for u as a Fourier series.

Remark 2. *What we mean by periodic solution is a solution that is periodic with respect to each variable. In fact, our solutions will be defined on the whole* $\mathbb{R}^m$ *and they will be periodic in each direction of* $\mathbb{R}^m$*. In the case of parabolic operators, as we presented in the introduction and the literature review, the time-periodicity will be a direct consequence of our general semilinear formulation. Moreover, in the spatial elliptic setting, solutions correspond to spatially periodic stationary patterns (steady states).*

This leads to the space of Fourier coefficients X^s ($s \in \mathbb{R}_+^m$) defined as

$$X^s := \{U = (u_n)_{n\in\mathbb{Z}^m} | \ \forall n \in \mathbb{Z}^m \ u_n = \overline{u_{-n}} \text{ and } ||U||_s < \infty\}$$

where $||U||_s := \sup_{n\in\mathbb{Z}^m} |u_n| w_n^s$ is a weighed ℓ^∞-norm.

Similarly we introduce spaces of smooth functions that can be represented as Fourier series on Ω for some $s \in \mathbb{R}_+^m$. We denote $\mathbf{i} = \sqrt{-1}$, then for $s > (1, ..., 1)$, we can define

$$C^s_{Fou}(\Omega) := \left\{ \begin{array}{ll} u \in C^0(\Omega)| & \exists U = (u_n)_{n\in\mathbb{Z}^m} \in X^s, \forall n \in \mathbb{Z}^m \ u_n \in \mathbb{C} \text{ and } u_n = \overline{u_{-n}}, \\ & \forall x \in \Omega, u(x_1, ..., x_m) = \sum\limits_{n\in\mathbb{Z}^m} u_n e^{\mathbf{i}(n_1x_1+...+n_mx_m)} \end{array} \right\}.$$

In the previous definition, $u \in C^s_{Fou}(\Omega)$ implies in particular that $u \in C^k$ if $s > k + (1, ..., 1)$. Now if we consider $U = (u_n)_n$, the associated vector, it means that $||U||_s < \infty$. This comes from the fact that the equivalent of ∂_{x_i} in X^s is a multiplication by $\mathbf{i}n_i$. Therefore a solution u lives in $C^s_{Fou}(\Omega)$ if it can uniquely be represented in X^s and vice versa. As a consequence we can identify $u \in C^s_{Fou}(\Omega)$ and $U \in X^s$ as the same element. We keep the notation *capital letter* for elements in X^s and *lower case* letter for elements in $C^s_{Fou}(\Omega)$.

We also define

$$X := \bigcap_{s>(1,...,1)} X^s$$

as the set of vectors decaying quicker than any algebraic power. X is the equivalent of $C^\infty_{Fou}(\Omega)$ in terms of function space.

We now introduce the classical ℓ^2 inner product for series. The particular symmetry of Fourier coefficients gives a real inner product.

Proposition 2.2.1. Let $s > (1, ..., 1)$, then

$$\begin{aligned}(\cdot,\cdot)\colon X^s \times X^s &\to \mathbb{R}\\ (U,V) &\mapsto \overline{U}^T V\end{aligned}$$

defines an inner product on X^s. Moreover

$$\begin{aligned}||\cdot||_2\colon X^s &\to \mathbb{R}\\ U &\mapsto \sqrt{(U,U)}\end{aligned}$$

defines a norm on X^s.

Proof. It is a classical result that $(\cdot,\cdot)$ defines an inner product and $||\cdot||_2$ a norm for $s > (1, ..., 1)$. We prove that the ℓ^2 inner product restricted to $X^s \times X^s$ maps to $\mathbb{R}$. $\overline{U}^T V = \sum_{k\in\mathbb{Z}^m} \overline{u_k} v_k = \sum_{k\in\mathbb{Z}^m} u_{-k} v_k = \sum_{k\in\mathbb{Z}^m} \frac{u_{-k}v_k + u_k v_{-k}}{2} \in \mathbb{R}$ where we used the fact that $u_k v_{-k} = \overline{u_{-k} v_k}$ by definition of U, V and a change of indices in the third equality. □

Moreover we define a norm $|||.|||_s$ for linear operators on X^s as

$$|||A|||_s := \max_{||V||_s=1} ||AV||_s.$$

We are now ready to transform our system with Fourier series.

2.3 Transformation to Fourier series

We begin this section by defining the convolution, given by the symbol "$*$", in X^s. It will be the tool used to derived the bilinear term as it enables to represent in Fourier series the multiplication in the function space.

Definition 2.3.1 (convolution in X^s)**.** Let $s > (1, ..., 1)$ and let $a, b \in X^s$. Then the quantity $a * b := \sum_{k\in\mathbb{Z}^m} a_k b_{n-k}$ is well defined and is called the *convolution* between a and b.

Proof. [23] shows that the convolution is well-defined for elements in X^s when $s > (1, ..., 1)$. $\square$

From now on we fix some $s > (1, ..., 1)$ in order to be able to define convolutions. Then, by injecting $u \in C^s_{Fou}(\Omega)$ in the equation (2.1) we can notice that the operator $\mathcal{L}$ turns into an infinite diagonal matrix defined on X^s with coefficients L_n. Similarly $\mathcal{G}$ becomes a bilinear operator G on X^s that can be derived as a convolution $G_1(U) * G_2(V)$ for $U, V \in X^s$. Now, we give below the definitions of L and G for some $s \in \mathbb{N}^m$

$$\forall\, U, V \in X^s,$$

$$LU := (L_n u_n)_{n \in \mathbb{Z}^m}, \quad L_n := \sum_{\alpha \leq \theta} \mathbf{i}^{|\alpha|} a_\alpha n^\alpha$$

$$(G(U)V)_n := \sum_{k \in \mathbb{Z}^m} u_k v_{n-k} g(k, n), \quad g(k, n) := \frac{1}{2} \sum_{\alpha, \beta \leq \theta - \gamma} \mathbf{i}^{|\alpha+\beta|} p_\alpha q_\beta (k^\alpha (n-k)^\beta + k^\beta (n-k)^\alpha).$$

Remark 3. *This definition of $g(k, n)$ implies that $g(k, n) = g(n-k, n)$. A consequence of this change of index is that $G(U)V = G(V)U$ for $U, V \in X^s$. For a fixed U, $G(U)$ can be seen as a linear operator on X^s. We notice that the domain could have been $\Omega = \prod_{i=1}^{m} [-\frac{d_i}{2}, \frac{d_i}{2}]$ and the sizes d_i would have been integrated in the definitions of L_n and $g(k, n)$. However it is equivalent to our problem as we can change the value of the coefficients a_α, p_α and q_β to take the size into account.*

The equation (2.1) can then be turned into

$$LU + G(U)U = F \tag{2.2}$$

where F is the equivalent of f in X^s. For a well chosen subspace $\mathcal{D}$ of X^s, the Assumption 3 gives that the problem $LU = V$ has a unique solution $U \in \mathcal{D}$ for each $V \in \mathcal{D}$. As $G(U)U \in \mathcal{D}$, by choosing $F \in \mathcal{D}$, we make sense of the invertibility of L in such a problem. We do not specify the subspace in which L will be invertible because it will depend on the equation. Therefore, we will suppose that L is invertible on the full X^s and we will show in Chapter 5 how we deal with a specific $\mathcal{D}$. The overall reasoning we are about to present will not change. Therefore (2.2) is equivalent to finding $U \in X^s$ such that

$$I_d U + G(L^{-1}U)L^{-1}U = F \tag{2.3}$$

where I_d stands for the identity operator. We will usually use this formulation to prove the regularity results. When it is not mentioned, the equation (2.2) is the one considered.

At this point we need to specify the set of indices on which we are going to work with. Indeed, the symmetry $u_n = \overline{u_{-n}}$ derived from Fourier series allows to work on a subset of indices in $\mathbb{Z}^m$. We begin by defining the family of set of indices I_i

$$I_i := \{n \in \mathbb{Z}^m | \; n_i > 0 \text{ and } n_k = 0, k \in \{1, ..., i-1\}\}.$$

This enables us to introduce our three sets of indices of interest for some $N \in \mathbb{N}^m$

$$\begin{aligned}
I &:= (\bigcup_{i \in \{1,...,m\}} I_i) \bigcup \{0\} \\
I^N &:= \{n \in I | \; |n_i| \leq N_i \text{ for all } i \in \{1, ..., m\}\} \\
J^N &:= \{n \in \mathbb{Z}^m | \; |n_i| \leq N_i \text{ for all } i \in \{1, ..., m\}\}.
\end{aligned}$$

The set J^N will be of use later, now we only take care of the two first sets. We introduce the cardinality $d_N := |I^N|$ and we define π^N as the projection on the N first coefficients

$$\begin{aligned}
\pi^N \colon X^s &\to \mathbb{C}^{d_N} \\
U &\mapsto U^N := (u_n)_{n \in I^N}.
\end{aligned}$$

Similarly, for $U \in X^s$ we note $U_N := (u_n)_{n \notin I^N}$. We call π_N the corresponding projection, so that $U = (\pi^N(U), \pi_N(U))$.

Now we want to be able to easily go from finite to infinite dimension. This leads to the next definition associated with an embedding result.

Definition 2.3.2 (tilde operator)**.** Let $N \in \mathbb{N}^m$ and $U \in \mathbb{C}^{d_N}$. We first extend U by the Fourier symmetry "$u_n = \overline{u_{-n}}$" and then we pad it with infinitely many zeros. The resulting vector is an element of X^s and we denote it $\tilde{U}$. Therefore, the *tilde operator* is the natural passage from $\mathbb{C}^{d_N}$ to X^s.

Proposition 2.3.1. For all $N \in \mathbb{N}^m$ and all $s > (1, ..., 1)$, $\mathbb{C}^{d_N}$ is continuously embedded in X^s.

Proof. Let $N \in \mathbb{N}^m$ and $s \in X^s$, then we take $U \in \mathbb{C}^{d_N}$. Then clearly by construction, the corresponding $\tilde{U}$ (defined in Definition 2.3.2) respects the Fourier series symmetry and has a finite number of non-zero terms. Therefore it is in X^s. This gives the continuous embedding of $\mathbb{C}^{d_N}$ in X^s. $\square$

From this idea, we introduce the finite projections of our operators L and G ; for $N \in \mathbb{N}^m$ and $U_0 \in X^s$, we define

$$\begin{aligned} L^N \colon \mathbb{C}^{d_N} &\to \mathbb{C}^{d_N} \\ U &\mapsto \pi^N(L\tilde{U}) \\ G^N(U_0) \colon \mathbb{C}^{d_N} &\to \mathbb{C}^{d_N} \\ U &\mapsto \pi^N(G(U_0)\tilde{U}). \end{aligned}$$

As a consequence, we are going to be able to use the matrix analysis on L^N and $G^N(U_0)$ as they can be seen as squared matrices of size d_N.

Remark 4. *We can replace $\mathbb{Z}^m$ by I in the definition of the inner product of Proposition 2.2.1 and then take the real part to obtain the same result. Moreover we can* *define* $(\cdot,\cdot)_N$*, the finite inner product on the indices* I^N*. It also gives a norm* $||.||_N$ *on* $\mathbb{C}^{d_N}$ *(which is the usual* ℓ^2 *norm).*

We now build our *central* function space $\mathbb{U}_0$ (as solutions will be built upon that space). We say that $U_0 \in \mathbb{U}_0$ if it satisfies the following hypothesis :

$(H_1) : U_0 \in X^{s+\theta},\ U_0 \neq 0.$

(H_2) : There exists $\lambda^* \in \mathbb{R}^*$, $N \in \mathbb{N}^m$ such that $L^N + 2\lambda^* G^N(U_0)$ is not invertible.

(H_3) : $\mathrm{Ker}(L^N + 2\lambda^* G^N(U_0)) = \mathrm{span}\ (U^*)$ where Ker is the kernel.

(H_4) : $G^N(U_0)U^* \notin \mathrm{R}(L^N + 2\lambda^* G^N(U_0))$ where R stands for the range.

From now on we suppose that $\mathbb{U}_0$ is not empty and we will see later how we can relax the hypotheses (H_{1-4}). In fact, (H_{1-4}) are basically the hypotheses for Crandall-Rabinowitz Theorem 1.2.2 in the finite-dimensional approach.

2.4 Existence bifurcations towards periodic solutions in (2.1)

We are finally in a position to state our central Theorem of existence.

Theorem 2.4.1. *Let* $U_0 \in \mathbb{U}_0$ *and be* u_0 *the corresponding element in* $C^{s+\theta}_{Fou}(\Omega)$*. Then if* N *(from* (H_2)*) is big enough, there exists* $\delta > 0$ *and a unique (up to a phase condition) non-trivial continuous branch* l_u *of* 2π*-periodic (in each variable) solutions in* $C^{s+\theta}_{Fou}(\Omega)$ *to (2.1)*

$$l_u := \{u(r) + \lambda(r)u_0 |\ r \in (-\delta,\delta) \text{ and } (u(0),\lambda(0)) = (0,\lambda^*)\}.$$

This branch corresponds to a continuous branch l_f *of forcing terms in* $C^s_{Fou}(\Omega)$ *given by*

$$l_f := \{\lambda(r)\mathcal{L}u_0 + \lambda(r)^2\mathcal{G}(u_0,u_0), r \in (-\delta,\delta)\}.$$

CHAPTER 3
Construction of the bifurcation

3.1 Finite-dimensional approximation

In this section we build a solution for a finite number of frequencies. To do so we take $U_0 \in \mathbb{U}_0$ and we take N to be the vector of frequencies defined in (H_2). As before, we denote $d_N := |I^N|$. Then we begin by defining the operator C_f such that the solutions of the finite-dimensional problem will be the zeros of the projection of C_f on I^N

$$\begin{aligned} C_f \colon X^s \times \mathbb{U}_0 \times \mathbb{R}^* &\to X^s \\ (U, \quad U_0, \quad \lambda) &\mapsto LU + 2\lambda G(U_0)U + G(U)U. \end{aligned}$$

C_f represents the linearization of (2.2) around U_0 for a forcing term defined as $F = \lambda LU_0 + \lambda^2 G(U_0)U_0$. We will prove later the well posedness of C_f but now we work on the zeros of C_f^N that we define as $C_f^N(U, U_0, \lambda) := \pi^N(C_f(U, U_0, \lambda))$.

Theorem 3.1.1. *Let $U_0 \in \mathbb{U}_0$. Then there exists $\delta > 0$ and a non trivial continuously differentiable curve through $(0, \lambda^*) \in X^s \times \mathbb{R}$:*

$$l := \{(\tilde{U}_1(r), \lambda(r))|\ r \in (-\delta, \delta), (\tilde{U}_1(0), \lambda(0)) = (0, \lambda^*)\}$$

such that $C_f^N(\tilde{U}_1(r), U_0, \lambda(r)) = 0$ for all $r \in (-\delta, \delta)$, where $N \in \mathbb{N}^m$ and $\lambda^* \in \mathbb{R}^*$ are defined in (H_{1-4}).

Proof. Let $U_0 \in \mathbb{U}_0$ and its corresponding $N \in \mathbb{N}^m$, $\lambda^* \in \mathbb{R}^*$. We first introduce the mapping $\mathcal{C}_f^N$ that we define as

$$\begin{aligned} \mathcal{C}_f^N \colon \mathbb{C}^{d_N} \times \mathbb{U}_0 \times \mathbb{R}^* &\to \mathbb{C}^{d_N} \\ (U_1, \quad U_0, \quad \lambda) &\mapsto L^N U_1 + 2\lambda G^N(U_0)U_1 + G^N(\tilde{U}_1)U_1 \end{aligned}$$

and we look for zeros of $\mathcal{C}_f^N$ in $\mathbb{C}^{d_N} \times \mathbb{R}^*$ for the fixed U_0.

Now we know that $D_{U_1}\mathcal{C}_f^N(0, U_0, \lambda^*) = L^N + 2\lambda^* G^N(U_0)$ has a kernel of dimension 1 using (H_3). Then clearly the rank-nullity Theorem yields that $D_{U_1}\mathcal{C}_f^N(0, U_0, \lambda^*)$, as a finite-dimensional linear operator, has a zero Fredholm index. Moreover we have $D^2_{U_1,\lambda}\mathcal{C}_f^N(0, U_0, \lambda^*) = 2\lambda G^N(U_0)$, so that (H_4) gives that $2\lambda G^N(U_0)U^* \notin R(D_{U_1}\mathcal{C}_f^N(0, U_0, \lambda^*))$. Then $\mathcal{C}_f^N$ is infinitely differentiable and $(\mathbb{C}^{d_N}, ||.||_N)$ is a Banach space, where $||.||_N$ was defined in Remark 5. Therefore we can use the hypotheses (H_{1-4}) to be able to apply the Crandall-Rabinowitz Theorem 1.2.2. The Theorem gives us the existence of a curve of solutions through $(0, \lambda^*) \in \mathbb{C}^{d_N} \times \mathbb{R}^*$:

$$\{(U_1(r), \lambda(r))|\ r \in (-\delta, \delta), (U_1(0), \lambda(0)) = (0, \lambda^*)\}.$$

Now to finish the proof we denote $\tilde{U}_1(r) \in X^s$ the associated vector defined in Definition 2.3.2 for all $r \in (-\delta, \delta)$. Then by construction of $\mathcal{C}_f^N$, $\mathcal{C}_f^N(\tilde{U}_1(r), U_0, \lambda(r)) = 0$ for all $r \in (-\delta, \delta)$. □

3.2 The fixed-point operator and the proof of Theorem 2.4.1

In this section we build a solution of (2.2) in X^s from a solution of the finite-dimensional problem. To do so we build an operator T_r^N defined from X^s to X^s and we prove that it has a fixed point which will be a solution to our problem. Again we refer to the article [24] as the results we present are similarly built.

We begin by proving a result giving some regularity for the convolution.

Proposition 3.2.1. Let $s > (1, ..., 1)$ and $U, V \in X^s$. Then $G(L^{-1}U)L^{-1}V \in X^{s+\gamma}$ and there exists $C > 0$ that depends only on L, G and s such that $||G(L^{-1}U)L^{-1}V||_s \leq C||U||_s||V||_s$.

Proof. Let $s > (1, ..., 1)$ and $U, V \in X^s$. Then by Assumption 2 we can deduce that $G_1(L^{-1}U) \in X^{s+\gamma}$ and $G_2(L^{-1}V) \in X^{s+\gamma}$.

Note that $G(L^{-1}U)L^{-1}V = G_1(L^{-1}U) * G_2(L^{-1}V)$. Then we refer to [23] and to the estimate given for convolution terms in X^s to finish the proof of the Proposition. The proof of the inequality is a direct consequence. □

The previous Proposition will allow us to prove some regularity for the solutions of (2.3) and, as a consequence, to show the equivalence between (2.1) and (2.3).

Proposition 3.2.2. Let $s > (1, ..., 1)$ and $F \in X^s$. Then if $V \in X^t$, such that $(1, ..., 1) < t \leq s$, solves (2.3) for the forcing term $F \in X^s$, it implies that V is actually in X^s. Moreover $V \in X^s$ solves (2.3) if and only if there exists $u \in C^{s+\theta}_{Fou}(\Omega)$ such that u is a strong 2π-periodic solution of (2.1) for the corresponding forcing term $f \in C^s_{Fou}(\Omega)$.

Proof. We begin by proving the first assertion of the proposition using a bootstrapping reasoning similar to the one provided in [24].

Suppose that $V \in X^t$ solves (2.3) for the forcing term $F \in X^s$. Then $V = F - G(L^{-1}V)L^{-1}V$, which, considering the previous Proposition, implies in particular that $V \in X^{t+\gamma}$ if $t + \gamma \leq s$. Therefore, we can repeat this argument and we obtain that $V \in X^s$. If $t + \gamma \geq s$, then directly $V \in X^s$. This reasoning implies that the Fourier series associated to V is uniformly convergent in $C^s_{Fou}(\Omega)$, as well as the one associated to $G(L^{-1}V)L^{-1}V$.

Now we are going to prove the two sides of the equivalence :

($\Longrightarrow$) From the previous reasoning and $A(3)$, we can deduce that there exists a unique $U := L^{-1}V \in X^{s+\theta}$. This means that $U \in X^{s+\theta}$ solves (2.2) by construction. The regularity of U gives that $LU \in X^s$ and $G(U)U \in X^s$ so that the corresponding series converge uniformly in $C^s_{Fou}(\Omega)$. As a consequence, the corresponding $u \in C^{s+\theta}_{Fou}(\Omega)$ of U is actually well defined and is a strong, 2π periodic solution of (2.1).

($\Longleftarrow$) If u is a strong, 2π periodic solution of (2.1) then clearly the associated $U \in X^{s+\theta}$ solves (2.2). Then we can easily build $V = LU$ that solves (2.3). $\square$

Remark 5. *The previous proof also shows the equivalence of (2.2) and (2.3) as the solution U of $LU = V$ will be smooth as well. This proves the well posedness of C_f in the previous section. Moreover, we can obtain that $V \in X$ if $F \in X$ using the same idea.*

We define $A(r) := I_d^N + 2G^N(\tilde{U}_1(r) + \lambda(r)U_0)L^{-N}$ where $L^{-N} := (L^N)^{-1}$. We know from Crandall-Rabinowitz Theorem that the kernel of $A(r)$ is at most of dimension 1 (see [31]). Then using the continuity of the determinant, we can distinguish two cases.

$$\text{(Case 1)}: \text{ There exists } \delta > 0 \text{ such that } A(r) \text{ is invertible for all } |r| < \delta \text{ and } r \neq 0.$$

$$\text{(Case 2)}: \text{ There exists } \delta > 0 \text{ such that } Ker(A(r)) = span(V^*(r)) \text{ for all } |r| < \delta.$$

3.2.1 Case 1

We consider the following case

$$\text{(Case 1)}: \text{ There exists } \delta > 0 \text{ such that } A(r) \text{ is invertible for all } |r| < \delta \text{ and } r \neq 0.$$

Then we prove the following

Proposition 3.2.3. Let $U_0 \in \mathbb{U}_0$ and we consider the corresponding branch from Theorem 3.1.1, then there exists $C_0 > 0$ that depends on U_0, L and G such that for all $r \in (-\delta, \delta)$

$$r||(I_d^N + 2\lambda(r)G^N(\tilde{U}_1(r) + U_0)L^{-N})^{-1}||_s \leq C_0$$

where $L^{-N} := (L^N)^{-1}$

Proof. By assumption, we have that

$$L^N + 2\lambda(r)G^N(\tilde{U}_1(r) + U_0) \text{ is invertible for } |r| > 0.$$

Therefore, as λ and $\tilde{U}_1$ are smooth functions of r, we just need to prove that the limit when r goes to zero is finite.

Then,

$$\begin{aligned} &I_d^N + 2\lambda(r)G^N(\tilde{U}_1(r) + U_0)L^{-N} \\ &= I_d^N + 2(\lambda(r) - \lambda^*)G^N(\tilde{U}_1(r) + U_0)L^{-N} + 2\lambda^* G^N(\tilde{U}_1(r) + U_0)L^{-N}. \end{aligned}$$

Now we use the smoothness of the determinant and the fact that $\det(I_d^N + 2\lambda^* G^N(U_0)L^{-N}) = 0$ in order to use Taylor expansion and get that there exists $C > 0$ such that

$$\begin{aligned} &|\det(I_d^N + 2\lambda(r)G^N(\tilde{U}_1(r) + U_0)L^{-N})| \\ &= C(|\lambda^* - \lambda(r)| + ||\tilde{U}_1(r)||_s) + O(r) \\ &= Cr + O(r) \end{aligned}$$

where we used Taylor expansions of λ and $\tilde{U}_1$ on the second equality and the fact that $\frac{dU_1}{dr}(0) = U^*$ (see [31]).

For an invertible matrix A, we can use the adjoint formula to get that there exists $C^* > 0$ such that $||A^{-1}||_s \leq \frac{C^*}{|\det(A)|}$. We finish the proof by applying this inequality to our problem. $\square$

We are now ready to introduce our fixed point result. The following lemma will enable us to use the finite-dimensional bifurcation and to turn it into an actual bifurcation for the PDE.

Lemma 3.2.1. *Let $U_0 \in \mathbb{U}_0$ and consider the associated results from Theorem 3.1.1 in the Case 1. Then for all $r \in (-\delta, \delta)$ we denote $W(r) := \tilde{U}_1(r) + \lambda(r)U_0 \in X^{s+\theta}$ and we define the operator T_r^N as*

- *If $r > 0$,*

$$T_r^N \colon B_{R(r,N)} \to B_{R(r,N)}$$
$$V \mapsto \begin{cases} \pi^N(T_r^N) \\ \pi_N(T_r^N) \end{cases}$$

where $\begin{cases} \pi^N(T_r^N) := V^N - (I_d^N + 2\lambda(r)G^N(W(r))L^{-N})^{-1}\frac{C_f^N(rL^{-1}V, W(r), \lambda(r))}{r} \\ \pi_N(T_r^N) := V_N - \frac{C_{fN}(rL^{-1}V, W(r), \lambda(r))}{r} \end{cases}$

- *If $r = 0$,*

$$T_r^N \colon \{0\} \to \{0\}$$
$$0 \mapsto 0$$

with $B_{R(r,N)} \subset X^s$ and $R(r, N)$ is defined in the proof. Then T_r^N is well defined and has a unique non-trivial fixed point for all $r \in (-\delta, \delta)$ for N big enough. Moreover $R(r, N)$ goes to zero when r approaches zero.

Proof. First of all, the proof of Proposition 3.2.2 gives that T_r^N is well defined in X^s. From this, we develop a formula for $R(r, N)$ and show that T_r^N will indeed map $B_{R(r,N)}$ into itself.

Using the fact that T_r^N is a Newton-like operator and the expression of C_f, we get

$$\pi^N(T_r^N(V)) = -r(I_d^N + 2G^N(W(r))L^{-N})^{-1}\pi^N(G(L^{-1}V)L^{-1}V).$$

Then we use Propositions 3.2.1 and 3.2.3 to obtain that

$$||\pi^N\widetilde{(T_r^N(V))}||_s \leq CC_0||V||_s^2.$$

Now we search for a radius R^* such that $\pi^N\widetilde{(T_r^N(V))}$ will map $B_{R^*} \subset X^s$ into itself. From that point, we search for $||V||_s$ such that $||\pi^N\widetilde{(T_r^N(V))}||_s \leq ||V||_s$. A sufficient condition is to use the previous inequality and analyse the second order polynomial in $||V||_s$.

Then, by defining $R^* := \frac{1}{4C_0C}$ we obtain the desired result. The $\frac{1}{4}$ will make sense later.

Now we look at the second part $\pi_N(T_r^N(V))$. We have by definition

$$\pi_N(T_r^N(V)) = -\pi_N(rG(L^{-1}V)L^{-1}V + 2G(W(r))L^{-1}V + \tfrac{1}{r}[G(\tilde{U}_1(r))\tilde{U}_1(r) + 2\lambda(r)G(U_0)\tilde{U}_1(r)]).$$

Similiarly as for the tilde operator in Definition 2.3.2, $\pi_N(T_r^N(V))$ can be seen as an element of X^s if we consider the N first frequencies to be zero. To simplify the notations we keep $\pi_N(T_r^N(V))$ unchanged but it has to be seen as an element of X^s so we can take its norm.

Then we denote $\tilde{V} := L^{-1}W$, $V_0 := L^{-1}U_0$ and $\tilde{V}_1 := L^{-1}\tilde{U}_1$. Therefore as $V \in X^s$ we can use a reasoning similar to the proof of the Proposition 3.2.1 to get that

$$||\pi_N(T_r^N(V))||_s \leq \frac{C}{w_N^{s+\gamma}}\left(r||V||_s^2 + 2||V||_s||\tilde{V}(r)||_s + \frac{||\tilde{V}_1(r)||_s}{r}(2\lambda(r)||V_0||_s + ||\tilde{V}_1(r)||_s)\right).$$

Similarly as before we analyse the second order polynomial and we define

$$R^{\pm}(r,N) := \frac{w_N^{s+\gamma} - 2C||\tilde{V}(r)||_s \pm \sqrt{(w_N^{s+\gamma} - 2C||\tilde{V}(r)||_s)^2 - 4C^2||\tilde{V}_1(r)||_s(2\lambda(r)||V_0||_s + ||\tilde{V}_1(r)||_s)}}{2C}.$$

These radii will be well defined in particular if

$$w_N^{s+\gamma} \geq 2C\left(2\sqrt{||\tilde{V}_1(r)||_s(2\lambda(r)||V_0||_s + ||\tilde{V}_1(r)||_s)} + ||\tilde{V}(r)||_s\right). \tag{3.1}$$

Therefore $\pi_N(T_r^N(V))$ maps all the balls of radii between $R^-(r,N)$ and $R^+(r,N)$ into themselves. However a trivial fixed point of T_r^N would be $-\frac{\tilde{V}_1(r)}{r}$, which we want to avoid. Therefore we look for a condition such that the radius where the solution V lives will be strictly less that $||\frac{\tilde{V}_1(r)}{r}||_s$ but also between $R^-(r,N)$ and $R^+(r,N)$ and less than R^*. It is sufficient to show that $R^-(r,N) \leq ||\frac{\tilde{V}_1(r)}{2r}||_s$ and $R^-(r,N) \leq R^*$, then to define $R(r,N) := R^-(r,N)$ in order to satisfy the previous requirements.

We can begin by using the inequality $1 - \sqrt{1-x} \leq x$ for $x \in (0,1)$ to show that

$$R^-(r,N) \leq \frac{2C||\tilde{V}_1(r)||_s(2\lambda(r)||V_0||_s + ||\tilde{V}_1(r)||_s)}{w_N^{s+\gamma} - 2C||\tilde{V}(r)||_s}.$$

Therefore we will have $R^-(r,N) \leq \frac{||\tilde{V}_1(r)||_s}{2r}$ if

$$w_N^{s+\gamma} \geq 4Cr(2\lambda(r)||V_0||_s + ||\tilde{V}_1(r)||_s) + 2C||\tilde{V}(r)||_s. \tag{3.2}$$

Then we will have $R^-(r,N) \leq R^*$ if

$$w_N^{s+\gamma} \geq 8C^2C_0||\tilde{V}_1(r)||_s(2\lambda(r)||V_0||_s + ||\tilde{V}_1(r)||_s) + 2C||\tilde{V}(r)||_s. \tag{3.3}$$

To conclude, if N is big enough such that (3.1-3) are satisfied, then we call $R(r,N) := R_2(r,N)$ and T_r^N will map $B_{R(r,N)}$ into itself.

Moreover, let V_1, $V_2 \in B_{R(r,N)}$, then

$$\begin{aligned}
& ||\pi_N(T_r^N(V_2)) - \pi_N(T_r^N(V_1))||_s \\
= \; & ||\pi_N(rG(L^{-1}(V_1+V_2))L^{-1}(V_2-V_1) + 2G(W(r))L^{-1}(V_2-V_1))||_s \\
\leq & \frac{C}{w_N^{s+\gamma}}\left(r||V_1+V_2||_s||V_2-V_1||_s + 2||\tilde{V}(r)||_s||V_2-V_1||_s\right) \\
\leq & \frac{C}{w_N^{s+\gamma}}\left(2rR(r,N) + 2||\tilde{V}(r)||_s\right)||V_2-V_1||_s.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& ||\pi^N(T_r^N(V_2)) - \pi^N(T_r^N(V_1))||_s \\
\leq \; & C_0||\pi_N(G(L^{-1}(V_1+V_2))L^{-1}(V_2-V_1)||_s \\
\leq & \frac{1}{2}||V_2-V_1||_s.
\end{aligned}$$

using the fact that $R(r,N) \leq R^* = \frac{1}{4CC_0}$. Therefore, T_r^N will be a contraction if

$$w_N^{s+\gamma} > C(2rR(r,N) + 2||\tilde{V}(r)||_s). \tag{3.4}$$

Now as $\tilde{V}_1$ and λ are continuous functions of r, then we can take the maximum norm in $(3.1-4)$ over $(-\delta,\delta)$. Therefore we can find N big enough such that $(3.1-4)$ will be satisfied for all $r \in (-\delta,\delta)$.

As a consequence, for N big enough, we can apply the Banach fixed point Theorem for all $r \in (-\delta,\delta)$ to obtain that T_r^N has a unique fixed point in $B_{R(r,N)} \subset X^s$. Moreover the condition (3.2) gives us that this fixed point is not trivially $-\frac{\tilde{V}_1(r)}{r}$.

We also obtained that

$$R(r,N) \leq \frac{2C||\tilde{V}_1(r)||_s(2||V_0||_s + ||\tilde{V}_1(r)||_s)}{w_N^{s+\gamma} - 2C||\tilde{V}(r)||_s}$$

which means that $R(r,N)$ goes to zero as r goes to zero, which justifies our formulation. Therefore, if $r=0$, the only fixed point is zero. $\square$

3.2.2 Case 2

We now consider the other possible scenario

$$\text{(Case 2)} : \text{There exists } \delta > 0 \text{ such that } Ker(A(r)) = span(V^*(r)) \text{ for all } |r| < \delta.$$

We define $U^* = L^{-N}V^*$ (in particular, $U^*(0)$ is the vector defined in (H_3)). In this case, we need to add a phase condition in order to gain back some invertibility. We begin by decomposing $\mathbb{C}^{d_N} = span(U^*(r)) \bigoplus Z(r)$ such that $Z(r) = span(U^*(r))^{\perp}$. Then, we define the mapping H_r as

$$\begin{aligned} H_r \colon \mathbb{C}^{d_N} \times \mathbb{R} &\to \mathbb{C}^{d_N} \times \mathbb{R} \\ (U, \lambda) &\mapsto \begin{cases} L^N U + 2\lambda G^N(U_0)U + rG^N(\tilde{U})U = \frac{C_f^N(rU, U_0, \lambda)}{r} \\ (U - \frac{U_1(r)}{r}, U^*(r))_N \end{cases} \end{aligned}$$

Therefore, for all $|r| < \delta$, $H_r(\frac{U_1(r)}{r}, \lambda(r)) = 0$ (the limit as r goes to zero is defined as $U_1(r) = rU^* + O(r)$ for r small).

Then we compute

$$D(r) := DH_r(\frac{U_1(r)}{r}, \lambda(r)) = \begin{pmatrix} L^N + 2G^N(\lambda(r)U_0 + \tilde{U}_1(r)) & 2G^N(U_0)\frac{U_1(r)}{r} \\ (\cdot, U^*(r))_N & 0 \end{pmatrix}.$$

We define $W(r) = \lambda(r)U_0 + U_1(r)$ and we check the invertibility of $DH_r(\frac{U_1(r)}{r}, \lambda(r))$ as follows

$$\begin{aligned} &DH_r(\frac{U_1(r)}{r}, \lambda(r)) \begin{pmatrix} U \\ \lambda \end{pmatrix} = 0 \\ \iff &\begin{cases} L^N U + 2\lambda(r)G^N(W(r))U + 2\lambda G(U_0)\frac{U_1(r)}{r} = 0 \\ (U, U^*(r))_N = 0 \end{cases} \end{aligned}$$

We know that $\frac{U_1(r)}{r} \to U^*(0)$ when $r \to 0$. Moreover, by (H_4) we know that $G^N(U_0)U^*(0) \notin R(L^N + 2\lambda(0)G^N(U_0))$. Therefore, by continuity, there exists $\tilde{\delta} > 0$

such that $G^N(U_0)\frac{U_1(r)}{r} \notin R(L^N + 2\lambda(r)G^N(W(r)))$ for all $|r| < \tilde{\delta}$. We redefine δ as $min(\delta, \tilde{\delta})$.

Therefore, for $|r| < \delta$, the first equality yields $\lambda = 0$ and $U \in Z(r)^\perp$. But then the last equation gives that $U \in Z(r)$. Therefore $U = 0$ and we obtained that $DH_r(\frac{U_1(r)}{r}, \lambda(r))$ is invertible for all $|r| < \delta$.

Lemma 3.2.2. *Let $U_0 \in \mathbb{U}_0$ and consider the associated results from Theorem 3.1.1 in the Case 2. Then for all $r \in (-\delta, \delta)$ we denote $W(r) := \tilde{U}_1(r) + \lambda(r)U_0 \in X^{s+\theta}$ and we define the operator T_r^N as*

$$T_r^N : B_{R(r,N)} \to B_{R(r,N)}$$
$$(V, \lambda) \mapsto \begin{cases} \begin{pmatrix} V^N \\ \lambda \end{pmatrix} - D(r)^{-1} \begin{pmatrix} \frac{C_f^N(rL^{-1}V, W(r), \lambda)}{r} \\ (L^{-1}V - \frac{U_1(r)}{r}, U^*(r))_N \end{pmatrix} \\ V_N - \frac{C_{fN}(rL^{-1}V, W(r), \lambda(r))}{r} \end{cases}$$

with $B_{R(r,N)} \subset X^s \times \mathbb{R}$ and $R(r,N)$ is defined in the proof. Then T_r^N is well defined and has a unique non-trivial fixed point for all $r \in (-\delta, \delta)$ for N big enough. Moreover $R(r,N)$ goes to zero when r approaches zero.

Proof. The proof is analogue to the one of the Case 1 as we similarly built up a Newton-like fixed point operator. Notice that this time $\lambda(r)$ can be perturbed. We give below the changes in the needed conditions.

$D(r)^{-1}$ is continuous in r so we can define $C_0 > 0$ such that $||D(r)^{-1}||_s \leq C_0$ for all $|r| < \delta$. Therefore, we define $R^* := \frac{1}{4CC_0} \min(1; \frac{1}{2||V_0||_s})$ with $V_0 = LU_0$ and the first component maps B_{R^*} into itself.

This changes condition (3.3) into

$$w_N^{s+\gamma} \geq \frac{2C}{R^*}||\tilde{V}_1(r)||_s(2\lambda(r)||V_0||_s + ||\tilde{V}_1(r)||_s) + 2C||\tilde{V}(r)||_s.$$

In particular, with this choice of R^*, we obtain the contractive feature of the first component. As the second component does not change, we obtain the existence and the uniqueness a fixed point for T_r^N for all $|r| < \delta$. $\square$

We are now in a position to prove our main result.

Proof of Theorem 2.4.1. Let U_0 in $\mathbb{U}_0$, then we consider the branch of solutions for the finite dimensional projection given by Theorem 3.1.1. Moreover we suppose that $N \in \mathbb{N}^m$ given in (H_2) is big enough such that the conditions (3.1-4) of the previous Lemmas are satisfied. Then for all $r \in (-\delta, \delta)$, we know that T_r^N has a unique fixed point $V(r)$ in X^s for the Case 1 and a unique fixed point $(V(r), \lambda(r))$ for the Case 2.

Then by construction we obtain that $\lambda(r)U_0 + \tilde{U}_1(r) + rL^{-1}V(r) \in X^{s+\theta}$ solves (2.2) for $F(r) := \lambda(r)LU_0 + \lambda(r)^2 G(U_0)U_0 \in X^s$ and for all $r \in (-\delta, \delta)$. Moreover, Theorem 3.1.1 and the Lemmas yield that $\tilde{U}_1(0) + 0 \times L^{-1}V(0) = 0$, so that it justifies the origin of the branch. Then we use Proposition 3.2.2 to finish the proof of the Theorem and obtain the equivalence of the branch in $C^{s+\theta}_{Fou}(\Omega)$. $\square$

Remark 6. *We note that we do not need the invertibility of L. In fact, we could have just built a diagonal matrix with $(w_n^\theta)_n$ on the diagonal to match the higher derivatives of $\mathcal{L}$. This infinite matrix would have been invertible by construction and it could have replaced L^{-1} in the formulation (2.3). Therefore, the whole proof of the Theorem 2.4.1 can be done without L^{-1}. However, we are going to see in Chapter 4 why it is interesting to assume the invertibility of L.*

From a practical point of view, we notice that the conditions (3.1-4) will be easily satisfied if r is small as $V_1(r)$ goes to zero as r goes to zero. Therefore, only a few number of modes will be necessary to be able to use the previous Lemmas close to $r = 0$.

CHAPTER 4
Hypotheses (H_{1-4})

Now that our main result is set and proved, we work on the hypotheses (H_{1-4}) to relax them. We introduce the following hypothesis (H_5) for $U_0 \in X^{s+\theta}$

$$(H_5) : \text{There exists } N \in \mathbb{N}^m \text{ big enough such that } G^N(U_0) \text{ is invertible}$$

In the following work we suppose that (H_1) is satisfied, which means that we work on $X^{s+\theta}$. Then we call $H_i \subset X^{s+\theta}$ the subset of $X^{s+\theta}$ satisfying the hypothesis (H_i).

4.1 Hypothesis (H_5)

We begin by showing that H_5 is dense in $X^{s+\theta}$.

Proposition 4.1.1. Let $N \in \mathbb{N}^m$, then the set of vectors $U \in \mathbb{C}^{d_N}$ such that $G^N(\tilde{U})$ is invertible is dense in $\mathbb{C}^{d_N}$ with $d_N = |I^N|$. Moreover H_5 is dense in $X^{s+\theta}$.

Proof. Let $K := \{U \in \mathbb{C}^{d_N},\ \det(G^N(\tilde{U})) = 0\}$, then as $\mathcal{G}$ is not trivial, the set K is the set of the roots of a non-zero multivariate polynomial of order d_N and with d_N variables. Therefore K is a $(d_N - 1)$-dimensional $\mathbb{C}$-vector field, which proves the first part of the proposition.

Let $U_0 \in X^{s+\theta} - H_5$ and let $N \in \mathbb{N}^m$. Now, the previous result shows that we can find a sequence $(U_n)_n \subset \mathbb{C}^{d_N}$, such that $G^N(\tilde{U}_n)$ is invertible for all $n \in I^N$ and that converges to U_0^N. Then, as N is arbitrary, we can choose it as big as we want. Therefore we create a sequence of vectors $N_1 < ... < N_k < ...$ in $\mathbb{N}^m$ and for each k we can find a sequence $(\tilde{U}_n^k)_n \subset H_5$ that converges to $\tilde{U_0}^{N_k}$. Moreover, the coefficients of U_0 decay to the rate $s + \theta$ by definition of $X^{s+\theta}$. This means that, for each k and each $\epsilon > 0$, we can find n^* such that for all $|n| \geq |n^*|$

$$||\tilde{U}_n^k - U_0||_{s+\theta} \leq (||U_0||_{s+\theta} + \epsilon) w_{N_k}^{s+\theta}$$

As a consequence we can build a sequence $(\tilde{U}_n)_n \subset H_5$ that converges to U_0 in $X^{s+\theta}$. This proves the density result. $\square$

4.2 Hypothesis (H_2)

We can prove, under some very light criteria, that (H_2) is actually an implication of (H_5). In this section we use the set of indices J^N instead of I^N because we are going to need to use the symmetry of the Fourier series in order to make its property appear. Therefore U^N is now the projection on the indices J^N and $d_N = |J^N|$. It is clearly equivalent to work with J^N, therefore we do not want to change the notations.

Proposition 4.2.1. Let $N \in \mathbb{N}^m$ and $U \in X^{s+\theta}$ such that $u_n = \overline{u_{-n}}$ for all $n \in J^N$ with $d_N = |J^N|$ such that $G^N(U)$ is invertible, if $d_N - 1$ is divisible by 4 then (H_2) holds.

Before giving the proof of the proposition we introduce some notations and results.

We fix $V \in X^{s+\theta}$ such that $v_n = \overline{v_{-n}}$ for all $n \in J^N$ and we begin by defining the following operator D :

$$\begin{aligned} D \colon \mathbb{R} &\to \mathbb{C}^{d_N \times d_N} \\ \lambda &\mapsto L^N + \lambda G^N(V). \end{aligned}$$

We drop the index N to simplify the notations and we call $U = V^{2N}$ as only the $2N$ first frequencies are necessary. We call $G(U) := G^N(\tilde{U})$ then we notice that $G(U)$ can be written as $G(U) := \begin{pmatrix} U^T G_1 \\ \ldots \\ U^T G_N \end{pmatrix}$ with $G_n = (G_{n(i,j)})_{(i,j) \in J_{2N} \times J_N} = (\mathbb{1}_{i=n-j} g(i,n))$. Each matrix G_n is a Toeplix matrix that comes from the finite convolution induced by G.

Now we call l_n the n_{th} column of L and g_{kn} the one of G_k.

Definition 4.2.1. Let $M_1 := \{\alpha \in \mathbb{R}^{d_N \times d_{2N}} |\ \alpha_{ij} = 0 \text{ or } 1\}$.

Then for all $U \in \mathbb{C}^{d_{2N}}$ and $\alpha \in M_1$ we define U^α by

$$U^\alpha := \prod_{i \in J^N,\ j \in J^{2N}} U_j^{\alpha_{ij}}.$$

Moreover we define the following squared matrix

$$(g_\alpha, l_{\alpha^*}) := \left\{ \alpha^*_{ii} l_i + \sum_{j \in J^{2N}} \alpha_{ij} g_{ij} \right\}_{i \in J^N} \quad \text{for } \alpha + \alpha^* \in M_1.$$

Proposition 4.2.2. For all $\lambda \in \mathbb{R}$ and $d_N - 1$ divisible by 4,

$$\det(D(\lambda)) = \sum_{\alpha + \alpha^* \in M_1,\ |\alpha + \alpha^*| = d_N} \lambda^{|\alpha|} U^\alpha \det((g_\alpha, l_{\alpha^*})) \in \mathbb{R}.$$

Proof. We consider $D(\lambda)^T$ instead of $D(\lambda)$ and the formula for the determinant directly comes from the linearity of the determinant with the columns. Then,

$$\begin{aligned}
\det(D(\lambda)) &= \sum_{\alpha + \alpha^* \in M_1,\ |\alpha + \alpha^*| = d_N} \lambda^{|\alpha|} U^\alpha \det((g_\alpha, l_{\alpha^*})) \\
&= \sum_{\alpha + \alpha^* \in M_1,\ |\alpha + \alpha^*| = d_N} \lambda^{|\alpha|} \det((g_\alpha, l_{\alpha^*})) \prod_{i \in J^N,\ j \in J^{2N}} U_j^{\alpha_{ij}} \\
&= \sum_{\alpha + \alpha^* \in M_1,\ |\alpha + \alpha^*| = d_N} \lambda^{|\alpha|} \det((g_\alpha, l_{\alpha^*})) \prod_{i \in J^N,\ j \in J^{2N}} \overline{U_j}^{\alpha_{-i,-j}}
\end{aligned}$$

Now we use the change of variable β defined as $\beta_{i,j} = \alpha_{-i,-j}$ and we notice that $|\beta| = |\alpha|$ and $(g_\alpha, l_{\alpha *}) = R\overline{(g_\beta, l_\beta)}$ where R is the reflection operator $R : (i,j) \mapsto (-i,-j)$. However we can go from $R\overline{(g_\beta, l_\beta)}$ to $\overline{(g_\beta, l_\beta)}$ by $\frac{d_N - 1}{2}$ interchanges of rows. But as $d_N - 1$ is divisible by 4 by assumption, the operator R does not change the determinant. As a consequence it gives

$$\begin{aligned}\det(D(\lambda)) &= \sum_{\beta+\beta^*\in M_1,\ |\beta+\beta^*|=d_N} \lambda^{|\beta|}\det(R\overline{(g_\beta, l_{\beta^*})}) \prod_{i\in J^N,\ j\in J^{2N}} \overline{U_j}^{\beta ij} \\ &= \sum_{\beta+\beta^*\in M_1,\ |\beta+\beta^*|=d_N} \lambda^{|\beta|}\overline{\det((g_\beta, l_{\beta^*}))} \prod_{i\in J^N,\ j\in J^{2N}} \overline{U_j}^{\beta ij}.\end{aligned}$$

This implies that $\det(D(\lambda))$ is equal to its conjugate, as $\lambda \in \mathbb{R}$, which means that $\det(D(\lambda))$ is real. □

We are now in a position to prove Proposition 4.2.1

Proof of Proposition 4.2.1. By Proposition 4.2.2 we know that the determinant of $D(\lambda)$ is a real number. Moreover we assume that $\det(G(U)) < 0$. Now $D(0) = L^N$ and by assumption A(3) we obtain $\det(D(0)) \neq 0$. We assume that $\det(D(0)) > 0$.

As $\det(G(U)) < 0$, we use the continuity of the determinant to conclude that there exists $\lambda > 0$ big enough such that $\det(D(\lambda)) < 0$.

To conclude we use again the continuity of the determinant and the intermediate value Theorem to prove that there exists $\lambda^* \neq 0$ such that $\det(D(\lambda^*)) = 0$.

The same reasoning holds if $\det(G(U)) > 0$ but we need to take $\lambda < 0$ small enough instead. Indeed as d is always odd (of the form $2p + 1$), we have $\lambda^d < 0$ whenever $\lambda < 0$. The same argument can be used if the determinant of L^N is negative. □

4.3 The hypothesis (H_3)

In this section we want to prove that H_3 is dense in H_2. The proof we use is very similar to the one used in Section 4.1. This time we place ourselves in the set of matrices of non-trivial kernel and we prove a similar result in a lower dimension thanks to isomorphism.

Proposition 4.3.1. H_3 is dense in H_2.

Proof. Clearly $H_3 \subset H_2$ by definition. Let $N \in \mathbb{N}^m$, from Proposition 4.1.1 we know that $K_1 := \{U \in \mathbb{C}^{d_N},\ \det(L^N + 2\lambda^* G^N(\tilde{U})) \neq 0\}$ is of dimension $d_N - 1$. Therefore, K_1 is isomorphic to a subspace of $\mathbb{C}^{d_N-1}$ that we call $\hat{K}_1$. For each $U \in K_1$, we call $\hat{U}$ its equivalent in $\hat{K}_1$. Similarly, the set of matrices $\{L^N + 2\lambda^* G^N(\tilde{U}), U \in K_1\}$ is isomorphic to a subspace of the square matrices of size $d_N - 1$. For $\hat{U} \in \hat{K}_1$, we call $M(\hat{U})$ the square matrix of size $d_N - 1$ corresponding to $L^N + 2\lambda^* G^N(\tilde{U})$. Then, we define $K_2 := \{\hat{U} \in \hat{K}_1,\ \det(M(\hat{U})) = 0\}$ and we obtain that K_2 is of dimension $d_N - 2$ with the same reasoning as in Proposition 4.1.1. Then, by isomorphism, we obtain that $\{U \in \mathbb{C}^{d_N},\ dim\ Ker(L^N + 2\lambda^* G^N(\tilde{U})) = 1\}$ is dense in K_1. Finally, we use the reasoning of Proposition 4.1.1 to prove that H_3 is dense in H_2. □

Remark 7. *This type of result was generalized by Sard, as he developed a theory on Hausdorff measure. We invite the interested reader to refer to [42] for more details.*

4.4 Extension from $\mathbb{U}_0$ to $X^{s+\theta} \bigcap H_4$

Following the results of sections 4.1-3, we can extend Theorem 2.4.1 to $X^{s+\theta} \bigcap H_4$.

Theorem 4.4.1. *H_3 is dense in $X^{s+\theta}$ and Theorem 2.4.1 can be extended to $X^{s+\theta} \bigcap H_4$. In particular, if $U_0 = 0$ then the branch given by the Theorem turns into the singleton $\{0\}$.*

Proof. We consider a vector $N \in \mathbb{N}_*^m$ such that d_N is divisible by 4. Under this condition, Proposition 4.2.1 gives that (H_5) implies (H_2). However, as H_5 is dense in $X^{s+\theta}$, it implies that H_2 is dense in $X^{s+\theta}$ as well. Then we just need to use Proposition 4.3.1 in order to obtain that $H_3 \subset H_2$ is dense in $X^{s+\theta}$.

To prove the extension of the Theorem 2.4.1, we use the continuity of the operators L and G in $X^{s+\theta}$ and the density of $H_3 \bigcap H_4$ in $X^{s+\theta} \bigcap H_4$. Then the fact that $H_3 \subset H_2 \subset H_1$ allows to extend the validity of the Theorem.

For the case $U_0 = 0$ we use the invertibility of L to conclude that the only solution to $LU = 0$ is $0 \in X^{s+\theta}$. □

Remark 8. *It is not obvious how to deal with the hypothesis (H_4). It seems that this condition has to be handled on a case to case basis. [15] gives the classical eigenvalue example where this hypothesis is easily verified.*

Moreover, our reasoning shows that the hypotheses (H_{1-3}) do not depend on N as it can be chosen as big as we want, as far as d_N is divisible by 4. Therefore we just have to find one example of that N big enough such that the conditions (3.1-4) are satisfied. Then we hope that (H_4) will be satisfied for such an N. As a consequence there is some remaining work to pursue on H_4 in order to extend our result to a wider range of vectors.

CHAPTER 5
Applications

The goal of this section is to present two kinds of problems that can be solved thanks to Theorem 2.4.1. Moreover we will give concrete examples by expliciting some equations.

5.1 Special case of time periodic solutions

Time periodic solutions are a recurrent subject in PDEs when one talks about periodic solutions. To study this kind of solutions, we specify our class of PDEs

$$\frac{\partial u}{\partial t} + \mathcal{L}_s u + \mathcal{G}(u,u) = f(x_1, ..., x_{m-1}). \tag{5.1}$$

with $\mathcal{L}_s$ being a linear differential operator and $\mathcal{G}$ is a bilinear differential operator. The variable t is defined as the *time variable* and $x_1, ..., x_{m-1}$ are the *space variables.* In order to be in adequation with our formulation, we denote $\mathcal{L} := \frac{\partial u}{\partial t} + \mathcal{L}_s$ and we suppose that the hypotheses $A(1-3)$ are satisfied. Moreover we suppose that $\mathcal{L}_s$ and $\mathcal{G}$ are spatial differential operators such that they do not depend on time differentiation. Then, the forcing f is supposed to be time independent which will lead us to the apparition of Spontaneous Periodic Orbits (SPO).

As $\mathcal{L}_s$ is a spatial differential operator, we can only consider its Fourier series transform in space. From that point, we call $\hat{L}_s$ its Fourier series transform in space and we remove the "hat" when we include the time direction. Similarly, we call $\hat{G}$ the operator G for the space directions. Finally, we call $\hat{X}^s$ the spatial coefficients in X^s and for $U \in X^s$, we call $U_n \in \hat{X}^s$ the space coefficients of U for the time index $n \in \mathbb{Z}$.

In this section we consider a subspace of X^s which will be our space of stationary parts

$$X_0^s := \{U \in X^s | \ U_n = 0 \text{ for all } n \neq 0\}.$$

Therefore $U_0 \in X_0^s$ (U_0 is still our central solution, we will make sure to distinguish it with our new notation) corresponds to a time-independent function in $C^s_{Fou}(\Omega)$. Moreover, in all the future work, δ and r are the parameters defined in Theorem 2.4.1.

To underline the correspondence between $\hat{X}^s$ and X^s, we give the following Proposition.

Proposition 5.1.1. Let $U_0 \in X_0^{s+\theta}$, then the two following statements are equivalent :

(1) : $Ker(L + G(U_0)) = span(U)$ for some non zero $U \in X^{s+\theta}$.

(2) : There exists a unique $n^* \in \mathbb{Z}$ and a non zero $\hat{U} \in \hat{X}^{s+\theta}$ such that $Ker(\hat{L}_s + \hat{G}(U_0) + \mathbf{i}n^*\hat{I}_d) = span(\hat{U})$.

where $\hat{I}_d$ is the identity operator in $\hat{X}^{s+\theta}$.

Proof. (1) $\implies$ (2)
Suppose (1), then by assumption we know that there exists $U \in X^{s+\theta}$ such that $Ker(L + G(U_0)) = span(U)$.

Moreover, as we explained, we can write $U := (..., U_n, ...)$ with $U_n \in \hat{X}^{s+\theta}$ and $n \in \mathbb{Z}$. Here n represents the time index. Therefore, as $U_0 \in X_0^s$, for each n we have

$$\mathbf{i}nU_n + \hat{L}_s U_n + \hat{G}(U_0)U_n = 0$$

As a consequence, for each $U_n \neq 0$ we obtain that U_n is an eigenvector of $\hat{L}_s + \hat{G}(U_0)$ for the eigenvalue $-\mathbf{i}n$. We need to show that there is only one $n^* \in \mathbb{Z}$ such that $U_{n^*} \neq 0$.

Indeed, suppose that there exist $n, m \in \mathbb{Z}$ such that $n \neq m$ and $U_n, U_m \neq 0$. In particular it implies that $(0, ..., 0, U_n, 0, ..., 0)$ and $(0, ..., 0, U_m, 0, ..., 0)$ are in the kernel of $L + G(U_0)$, which is in contradiction with our hypotheses. Therefore, there exists $n^* \in \mathbb{Z}$ such that $U_{n^*} \neq 0$ (as $U \neq 0$) is an eigenvector of $\hat{L}_s + \hat{G}(U_0)$ for the eigenvalue $-\mathbf{i}n^*$ and $-\mathbf{i}n$ is not an eigenvalue of $\hat{L}_s + \hat{G}(U_0)$ if $n \neq n^*$.

This proves the first implication

$(2) \implies (1)$

Suppose (2) and that $Ker(L + G(U_0))$ is at least of dimension 2. Then, following the previous proof, we can show that there would exit at least two indices $n, m \in \mathbb{Z}$ such that the eigenspaces of $-\mathbf{i}m$ and $-\mathbf{i}n$ for the operator $\hat{L}_s + \hat{G}(U_0)$ would be of dimension 1 (and 2 if $n = m$). But this is in contradiction with (2).
Now suppose that $Ker(L + G(U_0))$ is the singleton $\{0\}$. Then again we use the previous proof and (2) to build a non zero vector in the kernel of $L + G(U_0)$. $\square$

Remark 9. *If we were to use the full set of indices J instead of I in the previous proof, we would find that $-\boldsymbol{i}n^*$ and $\boldsymbol{i}n^*$ are eigenvalues because of the Fourier symmetry "$u_n = \overline{u_{-n}}$".*

5.1.1 Spontaneous Periodic Orbits

We study a special application of Theorem 2.4.1 which is the existence of Spontaneous Periodic Orbit (SPO). We call SPO a strictly (not constant in time) periodic solution arising from a time-independent forcing term, in the set up of the previous part.

In this section we consider a subspace of $X_0^{s+\theta}$ which will be our space of stationary parts

$$\mathbb{U}_0^{sta} := \{U_0 \in \mathbb{U}_0 |\ u_n = 0 \text{ for all } n \text{ such that } n_m \neq 0\}.$$

Therefore $U_0 \in \mathbb{U}_0^{sta}$ corresponds to a time-independent function in $C^{s+\theta}_{Fou}(\Omega)$. We can now give our existence result of SPO.

Theorem 5.1.1. *Let $U_0 \in \mathbb{U}_0^t$ and be u_0 the corresponding element in $C^{s+\theta}_{Fou}(\Omega)$, then if $n^* \neq 0$, defined in Proposition 5.1.1, there exists $\delta > 0$ and a non-trivial continuous branch l_u of 2π-spontaneous periodic solutions in $C^{s+\theta}_{Fou}(\Omega)$ to (5.1)*

$$l_u := \{u(r) + \lambda(r)u_0, r \in (-\delta, \delta)\}.$$

This branch corresponds to a continuous branch l_f of forcing terms in $C^s_{Fou}(\Omega)$ given by

$$l_f := \{\lambda(r)\mathcal{L}_1 u_0 + \lambda(r)^2 \mathcal{G}(u_0, u_0), r \in (-\delta, \delta)\}.$$

Proof. The proof is a direct consequence of Theorem 2.4.1. Moreover, by construction u_0 is time-independent, so the branch l_f is a branch of time-independent forcing terms. Then Proposition 5.1.1 assures that the non-trivial branch is a branch of (strictly) time dependent solutions. This justifies the title spontaneous. □

5.1.2 Example 1 : 2D incompressible Navier-Stokes equation (vorticity formulation)

$u := (u_1, u_2)$ and $w := \nabla \times u$ will solve the 2D Navier-Stokes vorticity equation if

$$\partial_t w + (u \cdot \nabla) w - \nu \Delta w = \nabla \times f$$
$$\nabla \cdot u = 0$$

where w, u_1 and u_2 are real valued and $\nu > 0$ is called the viscosity.

The standard way to study this equation is to introduce the stream function ψ. Indeed, the divergence free velocity in the Navier-Stokes equations enables us to defined ψ as follows

$$u_1 = \partial_y \psi$$
$$u_2 = -\partial_x \psi.$$

Then clearly $\nabla \cdot u = 0$ is satisfied by construction of u and we have that $w = -\Delta\psi$, therefore we can rewrite the system under the form

$$-\partial_t \Delta\psi + \nu\Delta^2\psi - \partial_y\psi\partial_x\Delta\psi + \partial_x\psi\partial_y\Delta\psi = \nabla \times f. \tag{5.2}$$

Therefore we can construct our operators L and G by defining the following quantities

$$\forall n, k \in I,$$
$$L_n = (n_1^2 + n_2^2)(in_3 + \nu(n_1^2 + n_2^2))$$
$$g(k, n) = \frac{1}{2}(k_2 n_1 - k_1 n_2)(k_1^2 + k_2^2 - (n_1 - k_1)^2 - (n_2 - k_2)^2).$$

Clearly $A(1)$ is satisfied and the higher order of L is $(n_1^2 + n_2)^2$. We call $\mathcal{L}_1 := \Delta^2$ as in the definition of 6.1.

Then G_1 and G_2 are of order 3 at most in n_1, n_2, whereas L is of order 4 as we saw. Moreover L is of order 1 in n_3 and G is constant in n_3, therefore $A(2)$ is validated as well.

Finally, by definition of w, the components corresponding to $(n_1^2 + n_2^2) = 0$ will be equal to 0. Therefore the set of definition for L will be the vectors in $U \in X^s$ such that $U_n = 0$ if $(n_1^2 + n_2^2) = 0$. Moreover, $g(k, n) = 0$ for all k and all n such that $n_1 = n_2 = 0$. Under that restriction, we can eliminate these terms from L and obtain a linear term invertible, which implies that $A(3)$ is satisfied.

Theorem 5.1.2 (SPO in 2D-Navier-Stokes)**.** *Let* $U_0 \in \mathbb{U}_0^{sta}$ *and be* u_0 *the corresponding element in* $C_{Fou}^{s+\theta}(\Omega)$*. Let* $\nu > 0$*, then if* $n^* \neq 0$ *(in Proposition 5.1.1), there exists* $\delta > 0$ *and a non-trivial continuous branch* l_u *of* 2π*-spontaneous periodic solutions in* $C_{Fou}^{s+\theta}(\Omega)$ *to (5.2)*

$$l_u := \{u(r) + \lambda(r)u_0, r \in (-\delta, \delta)\}.$$

This branch corresponds to a continuous branch l_f *of forcing terms in in* $C_{Fou}^{s}(\Omega)$ *given by*

$$l_f := \{\nu\lambda(r)\mathcal{L}_1 u_0 + \lambda(r)^2 \mathcal{G}(u_0, u_0), r \in (-\delta, \delta)\}.$$

Proof. The proof is a direct consequence of Theorem 5.1.1. □

5.1.3 Example 2 : 2D Kuramoto-Sivashinsky equation

u will solve the 2D Kuramoto-Sivashinsky equation if it satisfies

$$\partial_t u + \nu\Delta^2 u + \Delta u + \frac{1}{2}|\nabla u|^2 = f \tag{5.3}$$

where u is real valued and $\nu > 0$.

We consider the case where $\nu > 1$. Under that condition, we can easily check that the assumptions $A(1-3)$ are satisfied. In particular, $A(3)$ will be satisfied if we impose the mean value to be zero. We can then transform the problem, as in the Navier-Stokes case, to obtain L invertible.

Theorem 5.1.3 (SPO in 2D-Kuramoto-Sivashinsky)**.** *Let* $U_0 \in \mathbb{U}_0^{sta}$ *and be* u_0 *the corresponding element in* $C_{Fou}^{s+\theta}(\Omega)$*. Let* $\nu > 1$*, then if* $n^* \neq 0$ *(in Proposition 5.1.1), there exists* $\delta > 0$ *and a non-trivial continuous branch* l_u *of* 2π*-spontaneous periodic* *solutions in $C^{s+\theta}_{Fou}(\Omega)$ to (6.3)*

$$l_u := \{u(r) + \lambda(r)u_0, r \in (-\delta, \delta)\}.$$

This branch corresponds to a continuous branch l_f of forcing terms in $C^s_{Fou}(\Omega)$ given by

$$l_f := \{\lambda(r)(\Delta^2 + \nu\Delta)u_0 + \lambda(r)^2\mathcal{G}(u_0, u_0), r \in (-\delta, \delta)\}.$$

Proof. The proof is a direct consequence of Theorem 5.1.1. □

5.2 Internally Excited Equation

This new application puts in place a linear operator that may be non-invertible. In fact, we decompose the linear term between two invertible linear parts with different signs. The part with the higher degree of differentiation is modulated by some parameter ν. Therefore, if these linear operators are elliptic and if ν is small enough, then the full linear operator will have eigenvalues with positive real parts. Therefore if we add to this construction a time derivative, the system might have growing modes : the eigenvalues with positive real parts might engender instability. Moreover, as we will see in the Section 5.2.2, this kind of equation does not need a forcing term to give rise to a bifurcation. This explains why we denote them *Internally Excited Equations* (IEE).

5.2.1 Non-homogeneous case

We consider the following kind of equations

$$\nu\mathcal{L}_1 u - \mathcal{L}_2 u + \mathcal{G}(u, u) = f \tag{5.4}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are continuous and linear differential operators and $\mathcal{G}$ is a continuous bilinear differential operator. $\nu > 0$ is a positive constant.

Now still assume $A(2)$ and we consider $A(3)$ to be satisfied for both $\mathcal{L}_1$ and $\mathcal{L}_2$ separately but not necessarily to the sum. Moreover we turn $A(1)$ into the following assumption $A(4)$.

Assumption 4. *$\mathcal{L}_1 u := \sum\limits_{\alpha \leq \theta_1} a_\alpha \partial^\alpha u$ and $\mathcal{L}_2 u := \sum\limits_{\alpha \leq \theta_2} b_\alpha \partial^\alpha u$ where $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ are sequences of constant coefficients.*

Moreover we suppose that there exists $C_g > 0$ and $\gamma \geq (1, ..., 1)$ that gives $\left|\frac{\sum\limits_{\alpha \leq \theta_2} b_\alpha \xi^\alpha}{\sum\limits_{\alpha \leq \theta_1} a_{\alpha_i} \xi^\alpha}\right| \leq \frac{C_g}{w_\xi^\gamma}$ for all $i \in \{1, ..., m\}$ and $|\xi|$ big enough.

We also need to transform the set $\mathbb{U}_0$ to match our formulation. We say that $U_0 \in \mathbb{U}_0^I$ if it satisfies the following hypotheses :

$(H_1) : U_0 \in X^{s+\theta_1},\ U_0 \neq 0.$

(H_2) : There exists $\nu^* \in \mathbb{R}^*$ and $N \in \mathbb{N}^m$ such that $\nu^* L_1^N - L_2^N + 2G^N(U_0)$ is not invertible.

$(H_3) : \mathrm{Ker}(\nu^* L_1^N - L_2^N + 2G^N(U_0)) = \mathrm{span}\ (U^*).$

$(H_4) : L_1^N U^* \notin \mathrm{R}(\nu^* L_1^N - L_2^N + 2G^N(U_0)).$

Then again we can give an existence Theorem of periodic solutions in (5.4).

Theorem 5.2.1. *Let $U_0 \in \mathbb{U}_0^I$ and be u_0 the corresponding element in $C_{Fou}^{s+\theta_1}(\Omega)$, then there exists $\delta > 0$ and a non-trivial continuous branch l_u of 2π-periodic solutions in $C_{Fou}^{s+\theta_1}(\Omega)$ to (5.4) with $\nu := \nu(r)$*

$$l_u := \{(u(r), \nu(r)), r \in (-\delta, \delta)\}.$$

This branch corresponds to a continuous branch l_f of forcing terms in $C_{Fou}^s(\Omega)$ given by

$$l_f := \{\nu(r)\mathcal{L}_1 u_0 - \mathcal{L}_2 u_0 + \mathcal{G}(u_0, u_0), r \in (-\delta, \delta)\}.$$

Proof. We just need to notice that the proof of Theorem 2.4.1 can be applied if we replace the role of $2\lambda G(U_0)$ by νL_1 each time it appears. $\square$

5.2.2 Homogeneous case

In this section we show that the forcing is not necessary in the rise of periodic solutions in the case of IEE. In fact, the excited part of the equation enables the system to create periodic solutions by itself.

We consider the following class of equations

$$\nu\mathcal{L}_1 u - \mathcal{L}_2 u + \mathcal{G}(u,u) = 0 \tag{5.5}$$

where we also assume the same assumptions as in Section 5.2.1. This kind of equation is a perfect example for Crandall-Rabinowotz theory and the Theorem in [15] gives the existence of bifurcations from the trivial line. Therefore, we can take $U_0 = 0$ and apply our theory to obtain the same result as in Crandall-Rabinowitz theory.

Theorem 5.2.2. *Suppose that there exists $\nu^* \in \mathbb{R}$ such that ν^* is an eigenvalue of $L_1^N(L_2^N)^{-1}$, for $N \in \mathbb{N}^m$ big enough. Then, there exists $\delta > 0$ and a non-trivial continuous branch l_u of 2π-periodic solutions in $C^{s+\theta_1}_{Fou}(\Omega)$ to (5.5) with $\nu := \nu(r)$*

$$l_u := \{(u(r), \nu(r)), r \in (-\delta, \delta)/(u(0), \nu(0)) = (0, \nu^*)\}.$$

Proof. Let $N \in \mathbb{N}^m$ be as supposed, then by assumption L_1^N and L_2^N are invertible diagonal matrices, therefore the eigenvectors of $L_1^N(L_2^N)^{-1}$ will be part of the standard basis. From this point, it is obvious that the finite-dimensional hypotheses of Crandall-Rabinowitz Theorem will be satisfied, and by A(1-4) we can use Theorem 2.4.1 to conclude. $\square$

CHAPTER 6
Computational solutions

6.1 Energetic minimization

Even if we know that there exists spontaneous periodic orbits in (2.1) (thanks to Theorem 5.1.1), it is not obvious how to actually compute such solutions. Our goal in this section is to develop a numerical method to approximate them. The idea is to introduce an energy such that its minima will be solutions of (2.2) in the case of the Section 5.1. In this chapter we consider the full set of indices J^N and $d_N = |J^N|$.

We naturally want to involve the ℓ^2 norm of Fourier coefficients in our energy. Therefore, for a fixed $N \in \mathbb{N}^m$, we define E^N for all $U \in \mathbb{C}^{d_N}$ as

$$E^N(U) := ||(L^N U + G^N(U)U)_{n_m \neq 0}||_2^2.$$

For simplicity in this chapter, we write $\tilde{U}$ as U. We will use the tilde symbol when there is ambiguity. Then, if we find U such that $E(U) = 0$, then it means that the equivalent function of $L^N U + G^N(U)U$ in $C^\infty(\Omega)$ will be time-independent.

Remark 10. *As $U \in \mathbb{C}^{d_N}$, the complete energy should be $L^{2N}U^{2N} + G^{2N}(\tilde{U})U^{2N}$, where U^{2N} is a padding with zeros of U to match the dimension $2N$. In fact, having $E(U) = 0$ does not imply $(L\tilde{U} + G(\tilde{U})\tilde{U})_{n_m \neq 0} = 0$ because the convolution term makes "double frequencies" appear. However, this issue can be passed if we can show that the coefficients of U decrease quick enough.*

The problem of the energy E^N lies in the fact that it is not complex differentiable because of the norm operator. Therefore the idea is to decompose $U = R_U + iC_U$ with R_U and C_U respectively the real and imaginary parts of U. We call $P(U)U :=$

$(L^N U + G^N(U)U)_{n_m \neq 0}$ to simplify, then we define

$$\begin{aligned}\hat{E} \colon \mathbb{R}^{d_N} \times \mathbb{R}^{d_N} &\to \mathbb{R}^+ \\ (R_U, I_U) &\mapsto ||P(R_U + iI_U)(R_U + iI_U)||_2^2\end{aligned}$$

which is infinitely differentiable on $\mathbb{R}^{d_N} \times \mathbb{R}^{d_N}$.

6.2 Complexification

In this section, we introduce a simple trick that shows the equivalence of the minimization with $\hat{E}$ or E^N. What we call complexification is basically the identification between of the Taylor expansion of $\hat{E}$ and the one of E^N.

Lemma 6.2.1 (Complexification)**.** *For all $U, h \in \mathbb{C}^{d_N}$ respecting the symmetry $u_n = \overline{u_{-n}}$ for all n, such that $h = R_h + iC_h$ and $U = R_U + iR_U$, we have that*

$$\begin{aligned}
&E^N(U+h) = E^N(U) + E_1(U,h) + \frac{E_2(U,h)}{2} + \frac{E_3(U,h)}{6} + \frac{E_4(h)}{24} \quad \textit{with} \\
&E_1(U,h) = D\hat{E}(R_U, C_U)(R_h, C_h) \\
&E_2(U,h) = D^2\hat{E}(R_U, C_U)(R_h, C_h) \\
&E_3(U,h) = D^3\hat{E}(R_U, C_U)(R_h, C_h) \\
&E_4(h) = D^4\hat{E}(R_h, C_h)
\end{aligned}$$

where E_i is a polynomial of order i in h and $D^i\hat{E}$ is the i-th derivative of $\hat{E}$.

Proof. Let $U, h \in \mathbb{C}^{d_N}$ respecting the symmetry $u_n = \overline{u_{-n}}$ for all n, such that $h = R_h + iC_h$ and $U = R_U + iR_U$ and let $t \in \mathbb{R}$. Then $U + th$ also respects the symmetry of Fourier coefficients. Now $\hat{E}$ is a polynomial of order 4 so we can use its Taylor expansion (which will in fact be exact) :

$$\hat{E}(R_U + tR_h, C_U + tC_h)$$
$$= \hat{E}(R_U,C_U) + tD\hat{E}(R_U,C_U)(R_h,C_h) + \frac{t^2}{2}D^2\hat{E}(R_U,C_U)(R_h,C_h) + \frac{t^3}{6}D^3\hat{E}(R_U,C_U)(R_h,C_h) + \frac{t^4}{24}D^4\hat{E}(R_h,C_h).$$

Then by definition of $\hat{E}$ we have $\hat{E}(R_U + tR_h, C_U + tC_h) = E^N(U + th)$ and as E^N is a polynomial of order 4 we can expand $E^N(U + th)$ in h and obtain $E^N(U + th) = E^N(U) + tE_1(U, h) + t^2\frac{E_2(U,h)}{2} + t^3\frac{E_3(U,h)}{6} + t^4\frac{E_4(h)}{24}$ where E_i is a polynomial of order i in h.

Therefore we have two polynomials (in t) which are the same. As a consequence we can identify their coefficients and obtain the statement of the Lemma. □

Remark 11. *Therefore we see ,thanks to Lemma 6.2.1, that the computations of the derivatives of $\hat{E}$ are the same as the expansion coefficients of E^N. The interest of the complexification is to maintain the symmetries due to Fourier series and the complex space.*
Furthermore, $U - \nabla_c E^N(U)$ is equivalent to $(R_U, C_U) - \nabla\hat{E}(R_U, C_U)$. This is really useful for the implementation of a gradient descent minimization.

In particular we have $E_1(U, h) := \overline{h}^T\nabla_c E^N(U) = (R_h^T \;\; C_h^T)\nabla\hat{E}(R_U, C_U)$ and $E_2(U, h) := \overline{h}^T H_c E^N(U)h = (R_h^T \;\; C_h^T)H\hat{E}(R_U, C_U)(R_h^T \;\; C_h^T)^T$ respectively the complexification of the gradient and the Hessian of $\hat{E}$. All this quantities are real as we proved in Proposition 2.2.1.

Lemma 6.2.2 (Gradient and Hessian)**.** *For all $U \in \mathbb{C}^{d_N}$ such that $u_n = \overline{u_{-n}}$,*

$$\nabla_c E^N(U) = 2\overline{P(2U)}^T P(U)U$$
$$H_c E^N(U) = 2\overline{P(2U)}^T P(2U) + 4\sum_{k\in I_N}(P(U)U)_k\hat{G}_k$$

where $(\hat{G}_k)$ are derived from (G_k).

Proof. Let $U, h \in \mathbb{C}^{d_N}$ such that $u_n = \overline{u_{-n}}$ and similarly for h,

$$\begin{aligned}
&E^N(U+h) - E^N(U) \\
&= \overline{h}^T \overline{L}[P(U)U - F] + 2\overline{h}^T \begin{pmatrix} \overline{G_1 U} & . & . & . & \overline{G_N U} \end{pmatrix} [P(U)U - F] \\
&+ \overline{[P(U)U - F]}^T Lh + 2\overline{[P(U)U - F]}^T \begin{pmatrix} U^T G_1 \\ . \ . \ . \\ U^T G_N \end{pmatrix} h + o(||h||).
\end{aligned}$$

Therefore, from the previous Lemma we deduce that $\nabla_c E^N(U) = 2\overline{P(2U)}^T P(U)U$.

Now similarly, using Lemma 6.2.1 again,

$$\begin{aligned}
&E^N(U+h) - E^N(U) - \overline{h}^T \nabla_c E^N(U) \\
&= \overline{h}^T \overline{P(2U)}^T P(2U)h + 2 \begin{pmatrix} \overline{h}^T \overline{G_1 h} \\ . \ . \ . \\ \overline{h}^T \overline{G_N h} \end{pmatrix}^T P(U)U + o(||h||^2)
\end{aligned}$$

but $(\overline{G_k h})_i = \sum_{n \in J_N} \mathbb{1}_{i=k-n} \frac{g(-i,-k)+g(-n,-k)}{2} h_{-n} = \sum_{n \in J_N} \mathbb{1}_{i=k+n} \frac{g(-i,-k)+g(n,-k)}{2} h_n := (\hat{G}_k h)_i$.

As a consequence we obtained that $H_c E^N(U) = 2P(2\overline{U})^T P(2U) + 4 \sum_{k \in J_N} (P(U)U)_k \hat{G}_k$.

□

6.3 Minimization and Algorithm

Now that we introduced an energy that will lead to approximated solutions, we use a minimization method. The method we choose is the classical gradient descent. Therefore, for $N \in \mathbb{N}^m$ we can minimize E^N until machine precision.

Then the idea is to consider a strictly increasing sequence of vectors $(N_k)_k \subset \mathbb{N}^m$ ($N_1 < N_2 < ... < N_k < ...$) and minimize $E^k := E^{N_k}$ for all k. Therefore we create a sequence of minima (U_k) where $U_k \in \mathbb{C}^{d_{N_k}}$. Theorem 5.1.1 shows that we can find some solutions in $X^{s+\theta}$, which means that we might observe a decrease in the amplitude of the coefficients.

In order to find such solutions, we chose, for each k, each initial guess by padding the previous solution U_k by zeros to match the size in $\mathbb{C}^{d_{N_{k+1}}}$. If indeed the decrease of coefficients appears, the initial guess will be successively more accurate for each k.

We give the following algorithm to summarize our minimization process :

Algorithm 1: Minimization for Spontaneous Periodic Orbits

Initialisation :

- Take $N_0 \in \mathbb{N}^m$ and a random $\tilde{U}_0 \in \mathbb{C}^{d_0}$
- Find U_0, using gradient descent, that minimizes E^0

For $N = N_k > N_{k-1}$:

- We pad U_{k-1} with zeros, we call it $\tilde{U}_k$
- Find U_k, using gradient descent with $\tilde{U}_k$ as the initial condition, that minimizes E^k

---

6.4 Double frequencies

The convolution coming from $G^N(U)U$ is supposed to make “double frequencies” appear but we cut them in our formulation of the problem. The double frequencies are basically the frequencies between N strictly and $2N$.

We define the following operator associated to the double frequencies :

$$\begin{aligned} G_2^N : \mathbb{C}^{d_N} &\to \mathbb{C}^{d_{2N}} \\ U &\mapsto \mathbb{1}_{n \in J^{2N} - J^N} G^{2N}(\tilde{U}) \end{aligned}$$

where $\tilde{U} \in \mathbb{C}^{d_{2N}}$ is the padded version of U with zeros.

Lemma 6.4.1. *If we call* (U_k) *a sequence of minimizers generated by Algorithm 1 such that* (U_k) *converges to* $U \in X^{s+\theta}$*, then* $||G_2^{N_k}(U_k)U_k||_\infty$ *decreases to zero to the rate* s*. For* $U \in \mathbb{C}^{d_N}$*, we define the infinite norm* $||.||_\infty$ *as* $||U||_\infty := \max_{i \in I^N} |U_i|$*.*

Proof. Let $s > (1, ..., 1)$, then by definition of U_k, there exists k_1 such that

$$\forall |k| \geq |k_1|,$$
$$||LU^k - LU||_s \leq C w_{N_k}^s$$

for some $C > 0$ independent of k.

Then by Proposition 3.2.1, we know that $G(U)U \in X^s$, therefore there exists $C_1 > 0$ such that $||G_N(U)U||_s \leq C_1 w_N^s$ for all $N \in \mathbb{N}^m$.

As a consequence, we can find $|k| > |k_1|$ big enough such that

$$||G_2^{N_k}(U_k)U_k||_\infty \leq C w_{N_k}^s$$

with $C > 0$ independent of k. This proves our lemma. □

We are now ready to give the regularity result for our algorithm.

Theorem 6.4.2. *If the sequence* $(U_k)_k$ *generated by Algorithm 1 converges to some* U *in* $X^{s+\theta}$ *such that* $U \notin X_0^{s+\theta}$*, then the equivalent of* U *in* $C^{s+\theta}(\Omega)$ *is a Spontaneous Periodic Orbit.*

Proof. The Spontaneous character comes from the construction of the energy E^k and of Lemma 6.4.1. □

6.5 Numerical results for the 2D Navier-Stokes vorticity equation

6.5.1 An example of a solution of Algorithm 1

In practice the Algorithm 1 shows excellent results for the Navier-Stokes equations in a sense that we observe some exponential decay of the coefficients. This can be justified by our previous reasoning. Moreover, we can cite [20] in which is shown the Gevrey regularity of Navier-Stokes solutions and the exponential decay of Fourier coefficients for Gevrey forcing terms. Our algorithm converges to this smooth type of couple solution-forcing as we notice the exponential behavior after some critical vector of frequencies $N^* \in \mathbb{N}^m$.

We fix the viscosity $\nu = 1$ and we look for periodic solutions of period 2π in every directions thanks to Algorithm 1. We stop the Algorithm for $N^k = [10, 10, 10]$ and we build the forcing term by taking the time-independent part of $LU_k + G(U_k)U_k$. By doing so, we make an error of order $||G_2^{N_k}(U_k)U_k||_\infty$ as we saw in Lemma 6.4.1. However, if the coefficients of U_k decrease quickly enough, then this approximation will be numerically satisfying. We give the representation of the forcing in the following plot (which is actually the curl of some f for the vorticity equation).

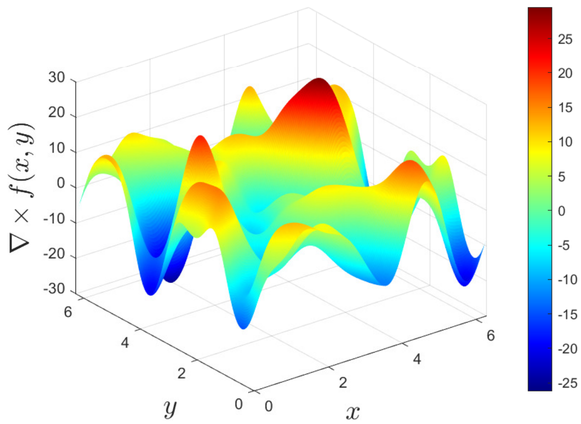

Figure 6–1: An example of a 2D Navier-Stokes forcing term generated by Algorithm 1

Now, to verify the numerical interest of our solution, we use a second order Adams–Bashforth solver and we solve for the previous forcing. Our initial condition is a small perturbation of the solution U^k we obtained in Algorithm 1. After some transient, the Algorithm converges to a periodic solution. We give the plots of the numerical solution at different times with a translation of the time to zero in order to skip the transition.

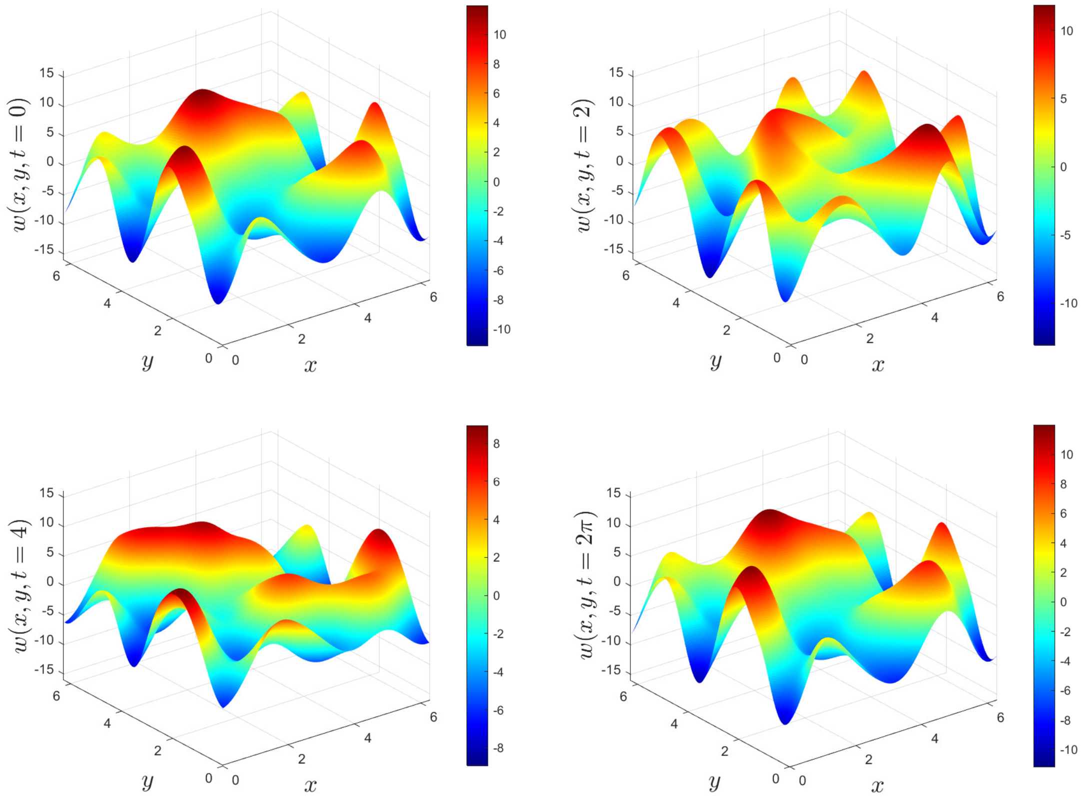

Figure 6–2: An example of the vorticity w at different times of a period in 2D Navier-Stokes

In addition to showing some stability against small perturbations, this verification also allows to highlight the fact that the couple solution-forcing is non trivial. Moreover, in practise we can change the values of ν, of the period and of the sizes of the domain, as we explained in Remark 4.

6.5.2 Approximation of the bifurcation

We now show that we can numerically follow the reasoning from Section 3. In fact, we want to develop a continuation method with respect to the amplitude of the steady part of the solution. In that way we will mimic the reasoning done with our parameter λ and try to numerically observe the bifurcation behavior of the system.

Therefore we fix a number of frequencies N_0 and we use Algorithm 1 to obtain a solution U^0. Then we multiply the part in $\mathbb{U}_0^{sta}$ (defined in Section 5.1) of U^0 by some $0.9 \leq \gamma < 1$. We call U_0^k the projection of U^k on the subspace $\mathbb{U}_0^{sta}$. Therefore, we obtain a vectors $\tilde{U}^1$. Finally we use again Algorithm 1 using $\tilde{U}^2$ as an initial condition and we repeat this procedure. We summarize the procedure in the following algorithm

---

Algorithm 2: Approximation of the amplitude bifurcation

Initialisation :

- Take $N_0 \in \mathbb{N}^m$ and a random $\tilde{U}^0 \in \mathbb{C}^{d_0}$
- Find U^0, thanks to gradient descent, that minimizes E^0

For $k \leq 1$:

- Find an adequate $0.9 \leq \gamma_k < 1$
- Multiply U_0^{k-1} by γ_k. We call $\tilde{U}^k$ this vector.
- Find U_k, thanks to gradient descent with $\tilde{U}_k$ as the initial condition, that minimizes E^0

Remark 12. *From Lemma 6.2.2, we notice that the second term of the Hessian will be negligible if γ_k is close to 1. Therefore, if the matrix $P(2U^k)$ is invertible, then the Hessian is positive definite in a neighborhood of U^k. As a consequence the gradient descent should converge. However, the Hessian will loose its invertibility out of a smaller and smaller neighborhood as we approach a possible bifurcation point. Therefore, γ_k has to be chosen closer and closer to 1 as we reach that point. We* *notice that this reasoning is very similar to the one we did in Section 3 for the fixed point operator T_r^N and the radius $R(r, N)$.*

We use Algorithm 2 and we build a collection of vectors of size d_0 solving the minimization problem associated with E^0. Then, for each vector, we separate the component in $\mathbb{U}_0^t$ from the rest. Then, using the previous notations, we plot for each solution, the ℓ^2 norm of the part in $\mathbb{U}_0^t$ ($||U_0||_2$) versus the time-dependent part of the solution ($||U_1||_2$).

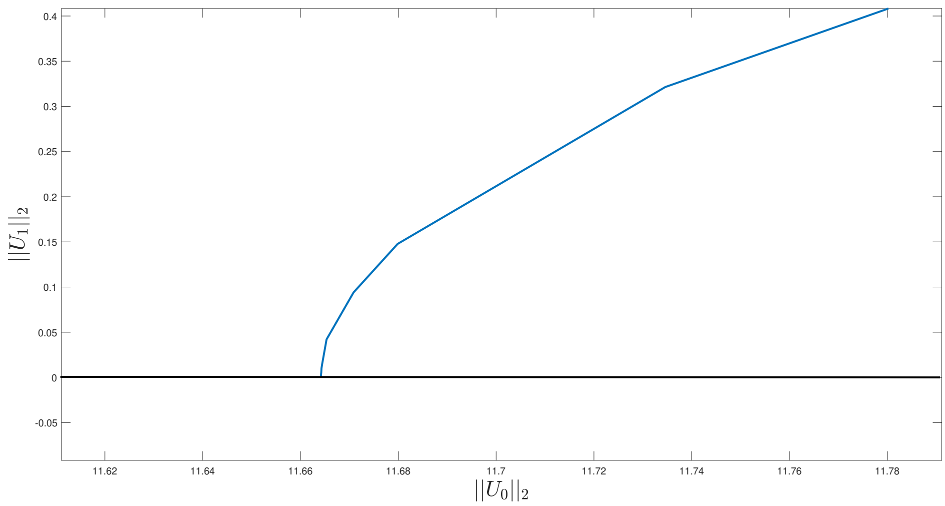

Figure 6–3: Evolution of the time-dependent part versus the stationary part

This is obviously not a rigorous proof of a bifurcation but it gives nonetheless a nice method to approximate them. In correlation with our theoretical results, we found a way to numerically approximate bifurcations of Section 3. In order to actually prove the bifurcation and all the solutions present on the branch, one could use a Newton-like operator and the technique presented in [48].

6.6 Numerical results for 1D Kuramoto-Sivashinsky equation

Similarly as the Navier-Stokes case, we want to use Algorithm 1 on the 1D Kuramoto-Sivashinsky equation in order to approximate spontaneous periodic orbits. We obtained the following time-independent forcing :

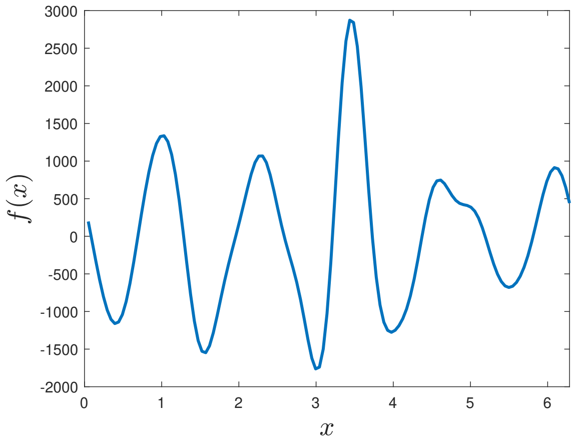

Figure 6–4: Example of a 1D Kuramoto-Sivashinsky forcing term generated by Algorithm 1

The solution we give is achieved for $\nu = 0.5$. Therefore we are in the range where the Kuramoto-Sivashinsky equation becomes internally excited, as the laplacian term is perturbating the system.

In the same manner as before, we use an Euler step solver using the previous forcing. Our initial velocity is a small perturbation of the solution we obtained in Algorithm 1. Now we give the results of the solver, after some transition time that we rescaled to zero.

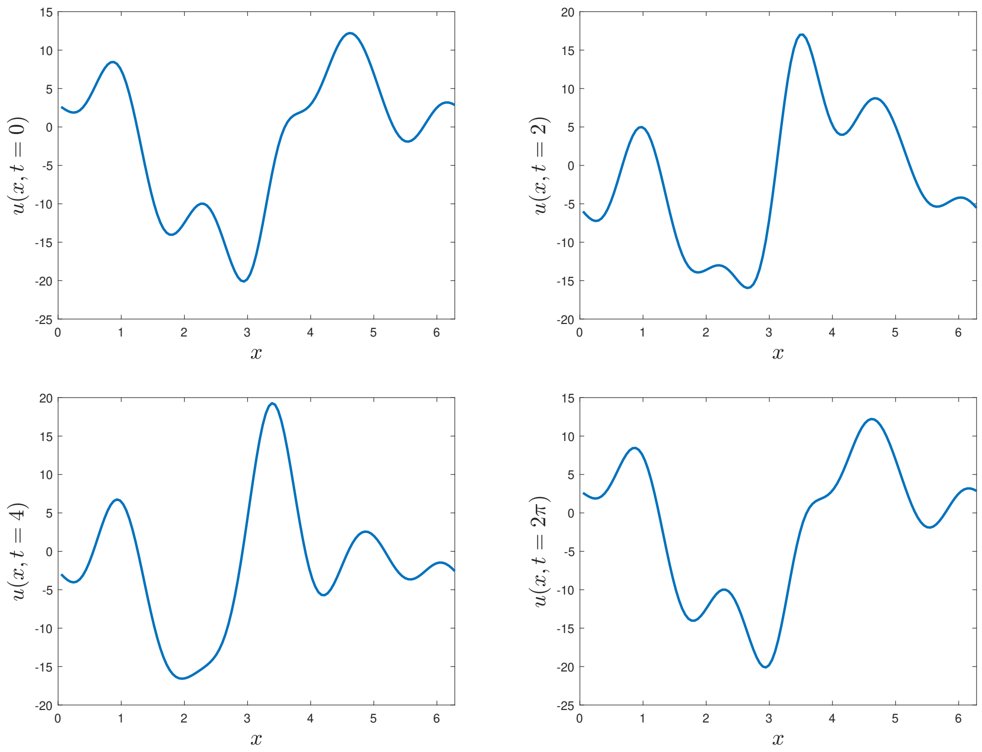

Figure 6–5: An example of solution u at different times of a period in 1D Kuramoto-Sivashinsky

Overall, it seems that the biharmonic part still ensures the stability of the system. Therefore we were able to give numerical results. It is also interesting so see that we can numerically approximate non trivial SPO in internally excited equations.

Conclusion

To conclude, we developed a method to prove the existence of periodic solutions in quadratic semilinear PDEs. We took advantage of the finite-dimensional approach in Fourier series to build up a fixed point technique. As explained, this result might be the entrance for a larger study such as the one of turbulence or chaos. This is especially true as a lot of physics phenomena are represented by equations of this type, we gave the examples of Navier-Stokes and Kuramoto-Sivashinsky equations.

Moreover, we tackled the problem of Spontaneous Periodic Orbits in Navier-Stokes equations. We gave the theoretical existence of such solutions and we also developed a numerical method to approximate them.

In addition to proving the answer of our specific Navier-Stokes problematic, we generalized it and built a detailed analysis of bifurcations towards periodic solutions in semilinear quadratic PDEs. We were also able to relax the hypotheses of our main Theorem in order to extend it. There is still some study to pursue on (H_4) to fully understand the emergence of bifurcations.

To push this study further, one might be interested in developing a Newton-like operator in order to use continuation. This could be particularly relevant in Navier-Stokes equations where a continuation with the viscosity is possible. In fact, as it is shown in [48], one can use the radii polynomial Theorem to actually prove the continuation branch, once it is computed. In addition to the numerical interest, this method could also extend our results to global ones. This would complete our study both from a theoretical and practical point of view.

REFERENCES

[1] R. A. Adams and J. J. Fournier. *Sobolev Spaces.* Elsevier, 2003.

[2] Z. Aliev and S. Hasanova. Global bifurcation of positive solutions of semilinear elliptic partial differential equations with indefinite weight. *Zeitschrift für Analysis und ihre Anwendungen*, 38:1–15, 01 2019.

[3] Z. S. Aliyev and S. M. Hasanova. Global bifurcation of positive solutions from zero in nonlinearizable elliptic problems with indefinite weight. *Journal of Mathematical Analysis and Applications*, page 124252, 2020.

[4] H. Amann. Fixed point equations and nonlinear eigenvalue problems in ordered banach spaces. *SIAM review*, 18(4):620–709, 1976.

[5] H. Amann. Periodic solutions of semilinear parabolic equations. In *Nonlinear analysis*, pages 1–29. Elsevier, 1978.

[6] D. M. Ambrose and A. L. Mazzucato. Global Existence and Analyticity for the 2D Kuramoto–Sivashinsky Equation. *Journal of Dynamics and Differential Equations*, 31(3):1525–1547, 2019.

[7] J. Argyris, G. Faust, and M. Haase. Routes to chaos and turbulence. A computational introduction. *Philosophical Transactions of the Royal Society of London. Series A: Physical and Engineering Sciences*, 344(1671):207–234, 1993.

[8] D. Bambusi and S. Paleari. Families of periodic solutions of resonant PDEs. *Journal of Nonlinear Science*, 11(1):69–87, 2001.

[9] T. Bartsch, N. Dancer, and Z.-Q. Wang. A liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system. *Calculus of Variations and Partial Differential Equations*, 37(3-4):345–361, 2010.

[10] A. Biswas and D. Swanson. Existence and generalized Gevrey regularity of solutions to the Kuramoto–Sivashinsky equation in $\mathbb{R}^n$. *Journal of Differential Equations*, 240(1):145–163, 2007.

[11] T. J. Bridges. The Hopf bifurcation with symmetry for the Navier-Stokes equations in $(L_p(\Omega))^n$, with application to plane Poiseuille flow. *Archive for Rational Mechanics and Analysis*, 106(4):335–376, 1989.

[12] G. Cerami, D. Fortunato, and M. Struwe. Bifurcation and multiplicity results for nonlinear elliptic problems involving critical sobolev exponents. In *Annales*

de l'Institut Henri Poincare (C) Non Linear Analysis, volume 1, pages 341–350. Elsevier, 1984.

[13] N. Chafee and E. F. Infante. A bifurcation problem for a nonlinear partial differential equation of parabolic type. *Applicable Analysis*, 4(1):17–37, 1974.

[14] K. Cliffe, T. Garratt, and A. Spence. Eigenvalues of the discretized Navier-Stokes equation with application to the detection of Hopf bifurcations. *Advances in Computational Mathematics*, 1(3):337–356, 1993.

[15] M. G. Crandall and P. H. Rabinowitz. Bifurcation from simple eigenvalues. *Journal of Functional Analysis*, 8(2):321–340, 1971.

[16] M. G. Crandall and P. H. Rabinowitz. *Bifurcation, perturbation of simple eigenvalues and linearized stability.* University of Wisconsin-Madison, Mathematics Research Center, 1973.

[17] M. G. Crandall and P. H. Rabinowitz. The Hopf bifurcation theorem in infinite dimensions. *Archive for Rational Mechanics and Analysis*, 67(1):53–72, 1977.

[18] Y. Ding. Computation of leading eigenvalues and eigenvectors in the linearized Navier-Stokes equations using Krylov subspace method. *International Journal of Computational Fluid Dynamics*, 17(4):327–337, 2003.

[19] L. C. Evans. *Partial Differential Equations*, volume 19. American Mathematical Soc., 2010.

[20] C. Foias, O. Manley, R. Rosa, and R. Temam. *Navier-Stokes equations and Turbulence*, volume 83. Cambridge University Press, 2001.

[21] J. M. Fraile, P. K. Medina, J. López-Gómez, and S. Merino. Elliptic eigenvalue problems and unbounded continua of positive solutions of a semilinear elliptic equation. *journal of differential equations*, 127(1):295–319, 1996.

[22] G. Galdi and M. Kyed. Time-periodic solutions to the Navier-Stokes equations. *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids*, pages 1–70, 2017.

[23] M. Gameiro and J.-P. Lessard. Analytic estimates and rigorous continuation for equilibria of higher-dimensional PDEs. *Journal of Differential Equations*, 249(9):2237–2268, 2010.

[24] M. Gameiro and J.-P. Lessard. A Posteriori Verification of Invariant Objects of Evolution Equations: Periodic Orbits in the Kuramoto–Sivashinsky PDE. *SIAM Journal on Applied Dynamical Systems*, 16(1):687–728, 2017.

[25] G. Gentile and M. Procesi. Periodic solutions for a class of nonlinear partial differential equations in higher dimension. *Communications in Mathematical Physics*, 289(3):863–906, 2009.

[26] G. Iooss and A. Mielke. Bifurcating time-periodic solutions of Navier-Stokes equations in infinite cylinders. *Journal of Nonlinear Science*, 1(1):107–146, 1991.

[27] D. Joseph and D. Sattinger. Bifurcating time periodic solutions and their stability. *Archive for Rational Mechanics and Analysis*, 45(2):79–109, 1972.

[28] S. Kaniel and M. Shinbrot. A reproductive property of the Navier-Stokes equations. *Archive for Rational Mechanics and Analysis*, 24(5):363–369, 1967.

[29] S. Kida, M. Yamada, and K. Ohkitani. A route to chaos and turbulence. *Physica D: Nonlinear Phenomena*, 37(1-3):116–125, 1989.

[30] H. Kielhöfer. Interaction of periodic and stationary bifurcation from multiple eigenvalues. *Mathematische Zeitschrift*, 192(1):159–166, 1986.

[31] H. Kielhöfer. *Bifurcation Theory: An Introduction with applications to partial differential equations*, volume 156. Springer Science & Business Media, 2011.

[32] M. Kyed and J. Sauer. On time-periodic solutions to parabolic boundary value problems. *Mathematische Annalen*, 374(1-2):37–65, 2019.

[33] L. Landau and E. Lifshitz. *Fluid Mechanics: Volume 6*. Number vol. 6. Elsevier Science, 2013.

[34] G. M. Lieberman. Time-periodic solutions of linear parabolic differential equations. *Communications in partial differential equations*, 24(3-4):631–663, 1999.

[35] A. Melcher, G. Schneider, and H. Uecker. A Hopf-Bifurcation Theorem for the Vorticity Formulation of the Navier-Stokes Equations in $\mathbb{R}^3$. *Communications in Partial Differential Equations*, 33(5):772–783, 2008.

[36] M. Nakao and H. Kato. On periodic solutions of a semilinear hyperbolic parabolic equation. *Bulletin of the Australian Mathematical Society*, 46(2):245–261, 1992.

[37] L. Nirenberg. *Topics in nonlinear functional analysis*, volume 6. American Mathematical Soc., 1974.

[38] M. Oliver and E. S. Titi. Remark on the rate of decay of higher order derivatives for solutions to the Navier-Stokes equations in $\mathbb{R}^n$. *Journal of Functional Analysis*, 172(1):1–18, 2000.

[39] V. Otto. *Partial Differential Equations: Time-periodic Solutions*. Martinus Nijhoff Publishers, 1982.

[40] K. Pflüger. Periodic Solutions of Non-linear Anisotropic Partial Differential Equations. *Mathematical methods in the applied sciences*, 19(5):363–374, 1996.

[41] P. H. Rabinowitz. Some aspects of nonlinear eigenvalue problems. *The Rocky Mountain Journal of Mathematics*, 3(2):161–202, 1973.

[42] A. Sard. Hausdorff measure of critical images on Banach manifolds. *American Journal of Mathematics*, 87(1):158–174, 1965.

[43] J. Serrin. A note on the exstencie of periodic solutions of the Navier-Stokes equations. *Archive for Rational Mechanics and Analysis*, 3(1):120–122, 1959.

[44] J. Serrin. On the stability of viscous fluid motions. *Archive for Rational Mechanics and Analysis*, 3(1):1–13, 1959.

[45] J. Shi and X. Wang. On global bifurcation for quasilinear elliptic systems on bounded domains. *Journal of Differential Equations*, 246(7):2788–2812, 2009.

[46] M. Shinbrot and S. Kaniel. The initial value problem for the Navier-Stokes equations. *Archive for Rational Mechanics and Analysis*, 21(4):270–285, 1966.

[47] Y. Teramoto et al. On the stability of periodic solutions of the Navier-Stokes equations in a noncylindrical domain. *Hiroshima Mathematical Journal*, 13(3):607–625, 1983.

[48] J. B. van den Berg, M. Breden, J.-P. Lessard, and L. van Veen. Spontaneous periodic orbits in the Navier-Stokes flow. *arXiv preprint arXiv:1902.00384*, 2019.