Cubical dimension and obstructions to actions on CAT(0) cube complexes

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Contents

Abstract

We study obstructions to group actions on CAT(0) cube complexes. Firstly, we are interested in the obstruction to proper actions on CAT(0) cube complexes of a fixed dimension. This leads to the notion of cubical dimension of a group G which is the infimum n such that G acts properly on an n-dimensional CAT(0) cube complex. We obstruct the actions of small cancellation groups in the following sense. For a fixed n, we construct a $C'(1/6)$ small cancellation group with cubical dimension bounded below by n. Another instance of finding a bound on the cubical dimension is an example of cubulated group G with virtual cubical dimension bounded below by 1. In fact, we show that G does not virtually split. This construction is based on our result on (cocompact) cubulation of a small cancellation quotient of free product of (cocompactly) cubulated groups. The second direction we pursue is to obstruct proper and cocompact actions. We show that *most* 2-dimensional or three-generator Artin groups do not act properly and cocompactly on CAT(0) cube complexes, even virtually. We give a classification of cocompactly cubulated 2-dimensional or three-generator groups in terms of their defining graphs.

Abrégé

Nous étudions des obstructions d'actions de groupes sur les complexes cubiques CAT(0). Nous nous intéressons d'abord aux obstructions d'actions propres sur les complexes cubiques CAT(0) de dimension fixe. Cela nous amène à définir une notion de dimension cubique d'un groupe G qui est l'infimum n tel que G agit proprement sur un complexe cubique CAT(0) de dimension n. Nous obstruons les actions des groupes à petite simplification comme suit: pour un n fixe, nous construisons un groupe à petite simplification $C'(1/6)$ de dimension cubique minorée par n. Un autre cas où nous donnons une borne pour la dimension cubique est un exemple de groupe G cubulé de dimension cubique virtuelle minorée par 1. En fait, nous montrons que G ne se scinde pas virtuellement. Cette construction est tirée de notre résultat sur la cubulation (cocompacte) de quotient à petite simplification d'un produit libre de groupes cubulé (cocompactement). La deuxième direction que nous poursuivons est d'obstruer les actions propres et cocompactes. Nous montrons que la plupart des groupes d'Artin de dimension 2 ou engendré par trois éléments n'agissent pas proprement et cocompactement sur les complexes cubiques CAT(0), même virtuellement. Nous donnons une classification des groupes de dimension 2 ou engendré par trois éléments qui sont cubulé cocompactement en fonction de leurs graphs définissants.

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Preface

The manuscript consists of the expository part and a collection of three papers with original research contribution to the theory of group actions on CAT(0) cube complexes. Here is an outline of this manuscript.

Chapter 1 contains definitions and basic properties, features and constructions of CAT(0) cube complexes and group actions on such. Chapter 2 surveys families of groups that have been shown to act nicely on CAT(0) cube complexes. In Chapter 3 we discuss a few instances of families of groups known to not admit nice actions on CAT(0) cube complexes. All the three first chapters are entirely expository and contain no original contribution.

Chapter 4 addresses the question of cubical dimension of small cancellation groups. All $C'(1/6)$ groups are known to be cocompactly cubulated, so in particular they are known to have finite cubical dimension. For each $n \geq 1$ we construct examples of $C'(1/6)$ groups with cubical dimension bounded below by n. The main result is Theorem 4.1.2. This chapter is based on the preprint *Lower bounds on cubical dimension of* $C'(1/6)$ *groups* [**Jan17b**].

In Chapter 5 we provide a simple proof of cubulating small cancellation quotients of free products of cubulated groups which was previously addressed by Martin-Steenbock [**MS16**]. The main result of Chapter 5 is Theorem 5.6.2. We use that result to construct a cocompactly cubulated group G that does not virtually split, i.e. in the terms of cubical dimension, the virtual cubical dimension of G is greater than 1. This chapter is based on the preprint *Cubulating small cancellation free products* [**JW17a**] which is joint work with Daniel Wise. Both authors contributed equally.

Chapter 6 concerns cocompact cubulations of Artin groups. The main new contributions are Theorem 6.1.1 and Theorem 6.1.2 which give complete classifications of cocompactly cubulated Artin groups of dimension 2 and on three generators respectively, in terms of the defining graph. This chapter is based on joint work with Jingyin Huang and Piotr Przytycki *Cocompactly cubulated* 2*-dimensional Artin groups* [**HJP16**] published in Comment. Math. Helv., 91(3):519-542, 2016. The three authors contributed equally.

CHAPTER 1

CAT(0) cube complexes

1.1. Cube complexes

A *cube of dimension* d is a copy of $[0,1]^d$. A *face* of a cube is the restriction of some of the coordinates to 0 or 1. A *cube complex* is a cell complex obtained from a disjoint union of cubes where the attaching maps restricted to single faces are isometries with respect to the Euclidean metric. The *dimension* of a cube complex is the supremum of the dimensions of its cubes. The n-skeleton of a cube complex X is denoted by X^n.

The *link* of a 0-cube v in a cube complex X is a complex $\mathrm{link}(v)$ defined as follows. The vertices of $\mathrm{link}(v)$ correspond to oriented edges incident to v. There is an $(n-1)$-simplex spanned on a collection of vertices in $\mathrm{link}(v)$ if corresponding edges in X are contained in an n-cube. It can be thought as an intersection of a small radius sphere around the vertex v in X. See Figure 1. A *flag complex* is a simplicial complex, such that each set of pairwise

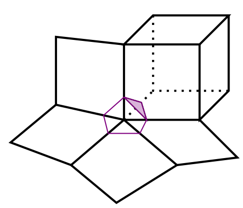

FIGURE 1. Vertex link.

adjacent vertices spans a simplex. A cube complex X is *non-positively curved* if all its vertex links are flag. Intuitively, that means that if there is a corner of a cube in X the whole cube is there. A CAT(0) *cube complex* is a simply connected, non-positively curved cube complex.

1.2. Metric

A map between metric spaces $\phi : (X, \mathsf{d}_X) \to (Y, \mathsf{d}_Y)$ is an *isometry* if for all $x, x' \in X$ we have $\mathsf{d}_X(x,x') = \mathsf{d}_Y(\phi(x), \phi(x'))$. A *geodesic metric space* is a metric space (X, d) such that for every x, x' there exists an isometry $\phi : [0, \mathsf{d}(x,x')] \to (X, \mathsf{d})$ with $\phi(0) = x$ and $\phi(\mathsf{d}(x,x')) = x'$ where the interval $[0, \mathsf{d}(x,x')]$ is equipped with the standard metric. The image of ϕ is denoted by $[x,y]$. A *geodesic triangle* Δ on a triple of point x,y,z is the union $[x,y]\cup[y,z]\cup[z,x]$. A *comparison triangle* $\bar\Delta$ is a triangle in the Euclidean space $\mathbb{E}^2$ with vertices $\bar x,\bar y,\bar z$ such that $\mathsf{d}_{\mathbb{E}^2}(\bar x,\bar y)=\mathsf{d}(x,y)$, $\mathsf{d}_{\mathbb{E}^2}(\bar y,\bar z)=\mathsf{d}(y,z)$, and $\mathsf{d}_{\mathbb{E}^2}(\bar z,\bar x)=\mathsf{d}(z,x)$. For any $z\in[x,y]$ there exists a unique $\bar z\in\bar\Delta$ such that $\mathsf{d}_{\mathbb{E}^2}(\bar z,\bar x)=\mathsf{d}(z,x)$ and $\mathsf{d}_{\mathbb{E}^2}(\bar z,\bar y)=\mathsf{d}(z,y)$. Similarly, we define $\bar z$ for $z\in[y,z]\cup[z,x]$. We say that X is a *CAT(0) space* if for any geodesic triangle Δ and any $z,z'\in\Delta$ we have $\mathsf{d}(z,z')\le\mathsf{d}_{\mathbb{E}^2}(\bar z,\bar z')$.

There is a natural path metric d on a cube complex X that is induced from the Euclidean metric where each cube is isometric to $[0,1]^d$. It is defined as $\inf\{\sum\mathsf{d}_{\mathbb{E}^2}(x_i,x_{i+1})$ where the infimum is taken over n and the choice on $x_0,\dots x_n$ where $x_0=x, x_n=y$, and each pair x_i,x_{i+1} is contained in a single cube. One can show that d is indeed a metric and (X,d) is a complete, geodesic metric space [**Bri91, Lea13, Mou88**].

By the theorem of Gromov [**Gro87**] a finite dimensional cube complex X with the metric d is locally a CAT(0) space in the sense above (i.e. each point has an open neighbourhood that is a CAT(0) space) if and only if it is non-positively curved in the sense of the flag link condition as in Section 6.3, and X is a CAT(0) space in the above sense if it is additionally simply connected.

1.3. Hyperplanes

Let X be a cube complex. A *midcube* of a cube $[0,1]^n$ is the $(n-1)$-dimensional cube that is obtained by restricting one coordinate to $\frac{1}{2}$. A midcube of a 1-cube is a *midpoint*. Let H be a cube complex whose cubes are all the midcubes of X and attaching maps are restrictions of attaching maps in X to midcubes. A connected component $\boldsymbol{\hbar}$ of H is a *hyperplane*. See Figure 2. The set of all hyperplanes $\mathscr{H}(X)$ in X can be identified with the equivalence classes of oriented 1-cubes where the equivalence relation is generated by parallelism, i.e. two 1-cubes are equivalent of they are opposite 1-cubes of a 2-cube. We say that $\boldsymbol{\hbar}$ is *dual* to the 1-cubes contained in the equivalence class.

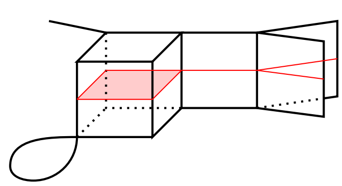

FIGURE 2. Hyperplane.

By the theorem of Sageev [**Sag95**] if X is a CAT(0) cube complex, then every hyperplane $\boldsymbol{h}$ is embedded, $\boldsymbol{h}$ separates X into two subspaces, called *halfspaces* and denoted by h^-, h^+, $\boldsymbol{h}$ is itself a CAT(0) cube complex and every collection of pairwise intersecting hyperplanes has a nonempty intersection. If $x \in h^+$ and $y \in h^-$ then we say that $\boldsymbol{h}$ *separates* x and y.

The *carrier* $N(\boldsymbol{h})$ of a hyperplane $\boldsymbol{h}$ in X is the union of all cubes in X that have non-empty intersection with $\boldsymbol{h}$. The carrier $N(\boldsymbol{h}) \simeq \boldsymbol{h} \times [0,1]$ and is a convex subcomplex of X [**Wis12**].

1.4. L^1-metric

There is a natural L^1-metric on X defined as

$$\mathsf{d}_1(x,y) \;=\; \inf\{\sum \mathsf{d}_{(\mathbb{E}^2,L^1)}(x_i,x_{i+1})\}$$

where the infimum is taken over n and the choice on $x_0,\dots x_n$ where $x_0 = x, x_n = y$, and each pair x_i, x_{i+1} is contained in a cube. If x, y lie in the 0-skeleton X^0 of a CAT(0) cube complex X then

$$\mathsf{d}_1(x,y) \;=\; \#\{\text{hyperplanes separating } x \text{ from } y\}.$$

1.5. Classification of isometries with respect to a hyperplane

Let X be a CAT(0) cube complex and a and isometry of X. We say that a acts *without hyperplane inversions* on X if $ah^+ \neq h^-$ for all $h \in \mathscr{H}(X)$. Every isometry of X acts without hyperplane inversions on the cubical subdivision of X. The isometry a is *elliptic* if a fixes a point in X. If a acts without hyperplane inversions, then a fixes a 0-cube. The isometry a is *hyperbolic* if a stabilizes a combinatorial geodesic (i.e. a geodesic in X^1 with respect to d_1), that is called a *combinatorial axis*. We will refer to it as just axis. Every isometry of X is either elliptic or hyperbolic [**Hag07**]. The *combinatorial translation length* $\delta(a)$ of an isometry a is defined as $\inf_{x\in X^0} \mathsf{d}_1(x,ax)$. If a acts without hyperplane inversions then the infimum is realized and $\delta(a^k) = k\delta(a)$ [**Hag07**] (see also [**Woo16a**]). In particular, any axis of a is also an axis of a^k.

Let a be a hyperbolic isometry of X and let $\boldsymbol{h}$ be a hyperplane. We recall the classification of isometries of a CAT(0) cube complex [**CS11**].

- a *skewers* $\boldsymbol{h}$ if $a^k h^+ \subsetneq h^+$ for one of the halfspaces h^+ of $\boldsymbol{h}$ and some $k > 0$. Equivalently, if some (equivalently, any) axis of a intersects $\boldsymbol{h}$ exactly once.

- a is *parallel* to $\boldsymbol{\hbar}$ if some finite neighbourhood of $\boldsymbol{\hbar}$ contains an axis of a.
- a is *peripheral* to $\boldsymbol{\hbar}$ if a does not skewer $\boldsymbol{\hbar}$ and is not parallel to $\boldsymbol{\hbar}$. Equivalently, $a^k h^+ \subsetneq h^-$ for some $k > 0$.

The type of behaviour of a with respect to $\boldsymbol{\hbar}$ is commensurability invariant, i.e. a^i has the same type as a with respect to $\boldsymbol{\hbar}$. The set of all hyperplanes in X skewered by a is denoted by $\mathrm{sk}(a)$.

1.6. Group actions on CAT(0) cube complexes

Let G act on a metric space X. We say that G acts

- *freely*, if for every $x \in X$ and every $g \in G$ we have $gx \neq x$.
- *properly*, if for every n we have $|\{g \mid gB_n(x) \cap B_n(x) \neq \emptyset\}| < \infty$.
- *cocompactly*, if there exists a compact set $K \subset X$ such that $GK = X$.
- *with a global fixed point*, if there exists $x \in X$ such that for all $g \in G$ we have $gx = x$.

We say that a group G is *(cocompactly) cubulated* if G acts isometrically properly (and cocompactly) on a CAT(0) cube complex.

1.7. Sageev's construction

There is a standard construction of a CAT(0) cube complex with an action on a group G obtained from an action of G on a *wallspace*. The construction is due to Sageev and can be found in [**Sag95**] where it was described in terms of codimension-1 subgroups, and a formulation in terms of wallspaces can be found in [**CN05, Nic04**].

A *wall* in a set X is a partition of X into two subsets which we call *halfspaces*. A wall *separates* $x, y \in X$ if x, y lie in different halfspaces. A *discrete wallspace* is a set X together with a collection $\mathcal{W}$ of walls such that for any two points $x, y \in X$ there are only finitely many elements of $\mathcal{W}$ that separate x from y.

An example of a discrete wallspace structure comes form hyperplanes in a CAT(0) cube complex. For each hyperplane $\boldsymbol{\hbar}$ choose one of the halfspaces h^+. That determines a wall partitioning X into the closed halfspace h^+ and an open halfspace h^-. Thus $(X, \mathcal{H}(X))$ form a discrete wallspace.

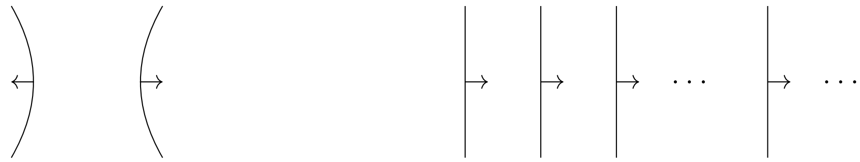

FIGURE 3. Choices of halfspaces for each hyperplane that do not define an orientation.

There is a natural pseudometric on the discrete wallspace $(X, \mathcal{W})$ defined as

$$\mathsf{d}_{\mathcal{W}}(x,y) \;=\; \#\{W \in \mathcal{W} \mid W \text{ separates } x \text{ and } y\}.$$

An *orientation* on $\mathcal{W}$ is a choice $o(W)$ of a halfspace for each wall W with the following properties:

- for $W, W' \in \mathcal{W}$, we have $o(W) \cap o(W') \neq \emptyset$,
- for every $x \in X$ there are only finitely many $W \in \mathcal{W}$ such that $x \notin o(W)$.

See Figure 3.

The CAT(0) cube complex *dual* to a discrete wallspace $(X, \mathcal{W})$ has orientations on $\mathcal{W}$ as its 0-skeleton. Two 0-cubes o, o' are joined by a 1-cube if and only if $|\{W \in \mathcal{W} \mid o(W) \neq o'(W)\}| = 1$. Finally, higher dimensional cubes are added whenever the 1-skeleton of a cube is there. For more details and to see that the resulting cube complex is CAT(0) see [**Sag95**] or [**Sag14**].

We say that G acts on a wallspace $(X, \mathcal{W})$ if $gW \in \mathcal{W}$ for every $g \in G$ and $W \in \mathcal{W}$. That action of G an $(X, \mathcal{W})$ induces an action of G on the dual CAT(0) cube complex.

1.8. Codimension-1 subgroups

Let X be a topological space such that $X = \bigcup_{i=1}^{\infty} K_i$ where all K_i are compact subsets and they form an ascending system

$$K_1 \hookrightarrow K_2 \hookrightarrow \dots \hookrightarrow K_i \hookrightarrow K_{i+1} \hookrightarrow \dots$$

There is an induced inverse system

$$\pi_0(X - K_1) \leftarrow \pi_0(X - K_2) \leftarrow \dots \leftarrow \pi_0(X - K_i) \leftarrow \pi_0(X - K_{i+1}) \leftarrow \dots$$

The *set of ends* $\mathrm{Ends}(X)$ of X is the inverse limit of this system. Equivalently, consider proper rays in X, i.e. $r : [0, \infty] \to X$ such that for every compact $K \subset X$ the preimage $r^{-1}(K)$ is compact. Two rays r_1, r_2 are equivalent if for every compact $K \subset X$ there exists

$t > 0$ such that the images $r_1([t, \infty])$ and $r_2([t, \infty])$ are contained in the same path component of $X - K$. The set $\mathrm{Ends}(X)$ can be identified with the equivalence classes of rays in X.

The *set of ends* of a finitely generated group G is defined as the set of ends of the Cayley graph of G with respect to some (any) finite generating set. It is a classical result, known as Stallings theorem, that every group has either 0, 1, 2 or ∞ ends [**Hop44**]. Moreover, it has 1 end if and only if G is finite, it has 2 ends if and only if G is virtually $\mathbb{Z}$, and it has ∞ ends if and only if G splits as a free product with amalgamation or an HNN extension over a finite group [**Sta68, Sta71**].

The *relative set of ends of G relative to H* $\mathrm{Ends}(G, H)$ where $H < G$ is the set of ends of the Schreier graph of G/H. A subgroup H in G is a *codimension-1 subgroup* if G has at least two ends relative to H.

Alternatively, we can define a codimension-1 subgroup H of G if the r-neighbourhood $N_r(H)$ of H in the Cayley graph Γ_G is separating in the following sense. A *deep* component of $\Gamma_G - N_r(H)$ is a connected component that is not contained in $N_s(H)$ for any $s > 0$. The subgroup H is codimension-1 in G if $\Gamma_G - N_r(H)$ has more than one H-orbit of deep components. Note that $\mathrm{Ends}(G, H) = 0$ implies $[G : H] < \infty$. If G splits non-trivially over a subgroup commensurable with H, then $\mathrm{Ends}(G, H) > 1$.

Let W be a deep component of $\Gamma_G - N_r(H)$. Then W and its complement W^c form a wall in the Cayley graph Γ_G. All the translates of $\{W, W^c\}$ by the elements of G form a discrete wallspace $(\Gamma_G, \mathcal{W})$ that G acts on [**HW14**]. Moreover the action of G on the dual cube complex is without a global fixed point [**Sag95**].

Conversely, if G acts on a finite dimensional CAT(0) cube complex without a global fixed point then there exists a hyperplane $\boldsymbol{\hbar}$ such that the stabilizer $\mathrm{Stab}(\boldsymbol{\hbar})$ is a codimension-1 subgroup of G [**Sag95**]. In particular, the stabilizer of a hyperplane that corresponds to $g\{W, W^c\}$ where W is a deep component of $\Gamma_G - N_r(H)$ as above is commensurable with gHg^{-1}.

1.9. Properties of the action on the dual cube complex

1.9.1. Cocompactness. Sageev proved that if G is hyperbolic and G has a quasiconvex codimension-1 subgroup, then the resulting action of G on the dual cube complex is cocompact [**Sag97**]. Hruska-Wise introduced the notion of relative cocompactness and generalized

Sageev's result [**HW14**]. Let G be a group and $\{P_1, \dots, P_k\}$ be a collection of subgroups of G. Then G acts *relatively cocompactly* on a CAT(0) cube complex X, if there exists a compact subcomplex K and P_i-invariant subcomplexes X_i such that

- $X = GK \cup \bigcup_i GX_i$, and
- $gX_i \cap hX_j \subset GK$ unless $i = j$ and $gX_i = hX_i$.

Hruska-Wise gave a criterion for relative cocompact actions of relatively hyperbolic groups [**HW14**].

1.9.2. Properness. Having a codimension-1 subgroup does not always lead to a proper action on a CAT(0) cube complex. Indeed, the Baumslag-Solitar group $BS(p,q) = \langle a, t \mid t^{-1}a^p t = a^q \rangle$ has a codimension-1 subgroup $\langle a \rangle$. If $p \neq \pm q$ then the subgroup $\langle a \rangle$ is distorted in $BS(p,q)$ and so $BS(p,q)$ does not act properly on a CAT(0) cube complex [**Hag07**]. If G acts properly (with respect to the pseudometric) on the discrete wallspace, then the induced action of G on the dual CAT(0) cube complex is proper. The properness of the action on the discrete wallspace can be interpreted in the following way: for any sequence $g_n \in G$ such that $|g_n|_S \to \infty$ we have $\mathsf{d}_{\mathcal{W}}(g_n x, x) \to \infty$ for any $x \in X$.

Often, the properness of the action on the wallspace follows from the slightly stronger property of the action that we describe now. Let $(X, \mathcal{W})$ be a wallspace where X is also equipped with a metric d. Then $(X, \mathcal{W})$ has the *linear separation property*, if there exist $K > 0, C \geq 0$ such that

$$\mathsf{d}_{\mathcal{W}}(x,y) \geq K\mathsf{d}(x,y) - C.$$

If (X, d) is metrically proper, has linear separation, and the action of G on (X, d) is proper, then G acts properly on the dual cube complex [**HW14**].

Bergeron-Wise gave the following boundary criterion for the properness and cocompactness of the action on the dual cube complex [**BW12**]. Let G be a hyperbolic group. Suppose that for every $u, v \in \partial G$ there exists a quasiconvex codimension-1 subgroup H such that u, v lie in the components of $\partial G - \partial H$ corresponding to distinct halfspaces of the wall defined by H. Then there exists a finite collection of quasiconvex codimension-1 subgroups such that the action of G on the dual CAT(0) cube complex is proper and cocompact.

1.9.3. Local finiteness. Hruska-Wise also gave a criterion for local finiteness of the CAT(0) cube complex dual to a metric pace (X, d) with a discrete wallspace structure $(X, \mathcal{W})$

[**HW14**]. If for every compact set $K \subset X$ there exists a constant $f(K)$ such that whenever $\mathsf{d}(K, W) \geq f(K)$ then there exists a wall W' separating K and W, then the dual cube complex is locally finite.

1.10. Special cube complexes

A map between cube complexes $\phi : X \to Y$ is *combinatorial* if every n-cube is mapped homeomorphically onto an n-cube. A combinatorial map $\phi : X \to Y$ is a *local isometry* if for each $x \in X^0$ the induced map $\mathrm{link}(x) \to \mathrm{link}(\phi(x))$ is an embedding as a *full* subcomplex, i.e. if vertices $u, v \in \mathrm{link}(\phi(x))$ are adjacent then u, v are adjacent in $\mathrm{link}(x)$.

A nonpositively curved cube complex X is *special* if it admits a local isometry to the Salvetti complex S_Γ of some right-angled Artin group A_Γ (for the definition of a right-angled Artin group and its Salvetti complex, see Section 2.1). Equivalently, specialness can be defined in terms of well-behaved hyperplanes. We say that a hyperplane $\boldsymbol{\hbar}$ is *one-sided* if $\boldsymbol{\hbar}$ is dual to one 1-cube with both orientations. The hyperplane $\boldsymbol{\hbar}$ *self-intersects* if $\boldsymbol{\hbar}$ is dual to two distinct non-parallel 1-cubes contained in a common 2-cube. The hyperplane $\boldsymbol{\hbar}$ *self-osculates* if it is dual to two 1-cubes that intersect at either their initial or terminal vertices. Two hyperplanes $\boldsymbol{\hbar}, \boldsymbol{k}$ *inter-osculate* if they cross and they also have dual 1-cubes that intersect but do not lie in a common 2-cube. The alternative definition is the following: X is special, if no hyperplane is one-sided, self-intersects or self-osculates and no two hyperplanes inter-osculate. See Figure 4. Every CAT(0) cube complex is special . It is not hard to see

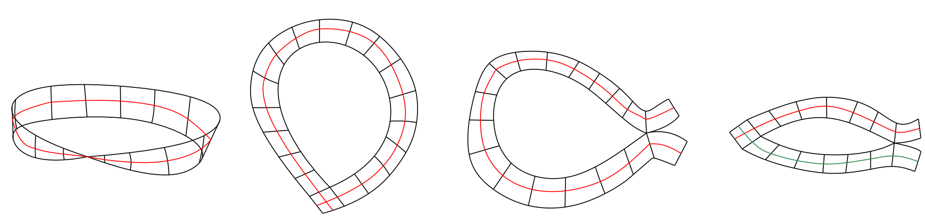

FIGURE 4. One-sided hyperplane. Self-intersecting hyperplane. Self-osculating hyperplane. Two inter-osculating hyperplanes.

that the Salvetti complex of a right-angled Artin group has well-behaved hyperplanes in the above sense. To prove the equivalence of the two definitions one defines a local isometry from a special cube complex X to a Salvetti complex of a right-angled Artin group whose generators are in 1-to-1 correspondence with $\mathscr{H}(X)$ and they commute if and only if the corresponding hyperplanes intersect in X. The properties of the hyperplanes of X ensure that such a local isometry exists.

We say that a group G is *(compact) special* if G is the fundamental group of a (compact) special cube complex. Since right-angled Artin groups are linear [**DJ00, HW99**], every compact special group is also linear. If G is virtually compact special and hyperbolic, then every quasiconvex subgroup of G is separable [**HW08**], in particular G is residually finite.

1.11. Nice properties of cubulated groups

1.11.1. CAT(0)-ness. A *CAT(0) group* is a a group that acts properly and cocompactly by isometries on a CAT(0) space (defined in Section 1.2). The nice geometry of CAT(0) spaces have strong algebraic consequences, e.g. central abelian subgroups of groups acting properly on CAT(0) spaces are virtually direct factors [**BH99**]. Moreover, CAT(0) groups are bicombable (see below). It is an important open problem whether all hyperbolic groups are CAT(0).

1.11.2. Biautomaticity. Niblo-Reeves proved that cocompactly cubulated groups are biautomatic [**NR98**]. See also [**Ś06**]. We recall the definition of biautomaticity below.

Let G be a group with a finite generating set S. In this section d denotes the distance in the Cayley graph $\Gamma(G, S)$. There is a natural monoid homomorphism $\mu : (S \cup S^{-1})^* \to G$ where $(S \cup S^{-1})^*$ is the free monoid on $(S \cup S^{-1})$. Any element w of S^* can be viewed as a path $[0, \infty] \to \Gamma(G, S)$ such that $w(0) = e$ and $w(t) = \mu(w)$ for $t \geq |w|$. A section $\sigma : G \to (S \cup S^{-1})^*$ of μ is a *combing* of G if there exists a constant $K > 0$ such that for every $g \in G$ and $s \in S$ the paths $\sigma(g)$ and $\sigma(gs)$ *K-fellow travel* on $\Gamma(G, S)$, i.e.

$$\mathsf{d}(\sigma(g)(t), \sigma(gs)(t)) \leq K$$

for all $t \geq 0$. A combing is a *bicombing* if additionally

$$\mathsf{d}(s(\sigma(g)(t)), \sigma(sg)(t)) \leq K$$

The group G is *(bi-)combable* if G has a (bi-)combing. The group G is *(bi-)automatic* if it has a (bi-)combing $\sigma : G \to (S \cup S^{-1})^*$ such that $\sigma(G)$ is a regular language. For more details, see [**ECH$^+$92**]. All these properties are independent of the finite generating set [**Sho90**]. Combable groups are finitely presented and have solvable word problem, and bicombable groups have solvable conjugacy problem [**Sho90**]. There are examples of groups that are combable but not bicombable or automatic, as well as examples of groups that are bicombable but not automatic [**Bri03**]. All CAT(0) groups are bicombable [**AB95**] but it is an open problem whether there are examples of CAT(0) groups that are not automatic or biautomatic. There are examples of groups that are automatic but not CAT(0), e.g. mapping class groups [**Mos95, Bri10**]. Also the fundamental group of a unit tangent bundle on a hyperbolic surface is bicombable but not CAT(0) [**AB95**].

1.11.3. The Haagerup Property. A locally compact topological group G has the *Haagerup Property* (or is *a-T-menable*), if G admits a proper, continuous, isometric action on a real Hilbert space. Niblo-Reeves implicitly showed that groups that act properly on CAT(0) cube complexes have Haagerup Property [**NR03**] (see also [**CCJ$^+$01**]). The Haagerup Property has important consequences in representation theory and operator K-theory: groups with Haagerup Property are uniformly embeddable into Hilbert spaces and satisfy the Baum-Connes conjecture and the Novikov conjecture. We do not discuss these problems here as they require too many definitions. The Haagerup property is a strong negation of Kazhdan's Property (T) that we briefly discuss in Section 3.1.

1.11.4. The Tits alternative. The theorem of Tits [**Tit72**] states that if G is a finitely generated linear group over a field then one of the following holds:

- either G is virtually solvable,
- or G contains F_2.

We say that a group G *satisfies the Tits alternative* if for every finitely generated subgroup $H < G$ either H is virtually solvable, or H contains a copy of F_2. Among groups satisfying the Tits alternative there are hyperbolic groups [**Gro87, GdlH90**], mapping class groups [**Iva84, McC85**] and $\mathrm{Out}(F_n)$ [**BFH00**]. Sageev-Wise proved that a group G that acts properly on a finite dimensional CAT(0) cube complex and that has an upper bound on the size of torsion subgroup satisfies the Tits alternative, in the following stronger form [**SW05**]: either G is virtually abelian, or it contains a copy of F_2. In particular, any torsion-free group that acts properly on a finite dimensional CAT(0) cube complex satisfies the above version of the Tits alternative. Note that the Tits alternative is not known for general CAT(0) groups.

1.12. Cubical dimension

The *(virtual) cubical dimension* of a group G is the infimum n such that (a finite index subgroup of) G acts properly on an n-dimensional CAT(0) cube complex. See Definition 4.1.1.

The fundamental theorem of Bass-Serre theory states that a finitely generated group G splits if and only if G acts on a tree without a global fixed point [**Ser77, Ser80**]. Thus the cubical dimension of G equal 1 in particular implies that G splits.

Wright showed that the asymptotic dimension of G is bounded above by the cubical dimension of G [**Wri12**]. We recall the definition of the asymptotic dimension. Let X be a metric space and n be an integer. The *asymptotic dimension* $\operatorname{asdim} X \leq n$ if for every $r \geq 1$ there exists a cover $\mathcal{U}$ of X such that

- $\sup_{U \in \mathcal{U}} \operatorname{diam} U < \infty$, and
- every ball of radius r in X intersects at most $n+1$ subsets from $\mathcal{U}$.

Since the definition of the cubical dimension is in terms of proper and not necessarily cocompact action there is no immediate connection between the cubical dimension and the geometric or cohomological dimension of G.

Often a naturally occuring discrete wallspaces structure for G leads to an action of G on a CAT(0) cube complex whose dimension is not optimal. For example, the presentation complex of the fundamental group $\pi_1\Sigma_g$ of a closed orientable surface of genus $g \geq 2$ has a discrete wallspace structure where the walls are the equivalence classes of edges where the equivalence relation is generated by being opposite in a 2-cell. The dimension of the cube complex X dual to this wallspaces has dimension $2g$. Indeed each 2-cell has $4g$ edges that belong to $2g$ distinct walls so $\dim X \geq 2g$. By the convexity of walls $\dim X = 2g$. On the other hand, $\pi_1\Sigma_g$ acts by deck transformation on $\mathbb{H}^2$ tessellated by $4g$-gons with $4g$ of them meeting at each vertex. By subdividing each $4g$-gon into $4g$ triangles with a common vertex, and then pairing two triangles that meet along an edge an $4g$-gon we get a tessellation of $\mathbb{H}^2$ by squares, with $4g$ squares meeting at each vertex. The group $\pi_1\Sigma_g$ preserves this tessellation, and so $\pi_1\Sigma_g$ acts properly and cocompactly on a CAT(0) cube complex of dimension 2. Since $\pi_1\Sigma_g$ does not split over a finite group, we conclude that the cubical dimension of $\pi_1\Sigma_g$ is exactly 2.

In Section 4 for each $n \geq 1$ we construct examples of $C'(1/6)$ groups (see Section 4.3) with cubical dimension bounded below by n. This is based on preprint [**Jan17b**].

In Section 5 we construct cocompactly cubulated groups with virtual cubical dimension greater than 1. This is joint work with Daniel Wise [**JW17a**].

CHAPTER 2

Survey on groups acting on CAT(0) cube complexes

2.1. Right-angled Artin groups

Let Γ be a simplicial graph with the vertex set V and the edge set E. The *right-angled Artin group* associated to Γ is given by the presentation

$$A_\Gamma = \langle v \in V \mid vw = wv \text{ for } (v,w) \in E\rangle.$$

There is a cube complex S_Γ, called the *Salvetti complex*, associated to a right-angled Artin group which is defined as follows. The complex S_Γ has a unique 0-cube x and it has $|V|$ 1-cubes labelled by s_v for $v \in V$ with both endpoints attached to x. For each $(v,w) \in E$ there is a 2-torus obtained from a 2-cube attached to the 1-skeleton of S_Γ along the path $s_v s_w s_v^{-1} s_w^{-1}$. We continue by attaching an n-torus for every n-clique in Γ to the n_1-dimensional faces. Since the 2-skeleton of S_Γ is the presentation complex for the standard presentation of A_Γ it is clear that $\pi_1(S_\Gamma) = A_\Gamma$. This complex was studied by Charney-Davis in [**CD95a**] where they proved that the universal cover $\widetilde{S}_\Gamma$ of S_Γ is a CAT(0) cube complex, and in particular S_Γ is a $K(\pi,1)$-space for A_Γ. Indeed, by construction, the link at x is the flag complex on Γ.

2.2. Coxeter groups

A *Coxeter group* is given by the presentation

$$W = \langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} = 1\rangle$$

where $m_{ij} = m_{ji} \in \{2,3,\dots\} \cup \{\infty\}$ and $m_{ii} = 2$ for all i. The relation $(s_i s_j)^\infty$ denotes no relation. A right-angled Coxeter groups is a Coxeter group where all $m_{ij} \in \{2,\infty\}$. All the data of a right angled Coxeter group can be encoded in a graph Γ, i.e. a right angled Coxeter group W_Γ can be defined as the quotient of the right-angled Artin group A_Γ where each generator has order 2, i.e. W_Γ has the following presentation

$$W_\Gamma = \langle v \in V \mid v^2 = 1 \text{ for all } v \in V, \text{and } vw = wv \text{ for } (v,w) \in E\rangle.$$

In the case of the right angled Coxeter groups the Davis complex has a structure of a cube complex that is CAT(0) [**Gro87, Dav08**] and so the right-angled Coxeter group are cocompactly cubulated. For other Coxeter groups the Davis complex is always a CAT(0) space (in general not a cube complex) with a proper and cocompact action of a Coxeter group, so Coxeter groups are CAT(0) groups [**Mou88, Dav08**].

Niblo-Reeves proved that every Coxeter groups W acts properly on a finite dimensional CAT(0) cube complex [**NR03**]. Moreover, they showed that if W is either hyperbolic or right-angled, then the action is cocompact. The construction of the wallspace for W is the following. Consider the Cayley complex for the above presentation of W and collapse each bigonal cell corresponding to the relator s_i^2 to a single undirected edge labelled by s_i, and then collapse $2m_{ij}$ copies of $2m_{ij}$-gons corresponding to the relator $(s_is_j)^{m_{ij}}$ to a single $2m_{ij}$-gon. See Figure 1. In the resulting complex the walls are defined as graphs joining the

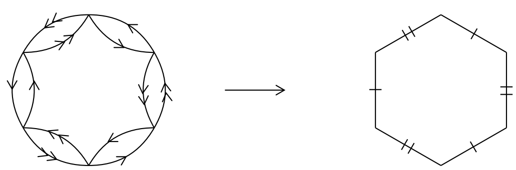

FIGURE 1. Collapsing bigonal cells to single edges.

opposite edges in each polygon.

Caprace-Mühlherr later showed that the action constructed by Niblo-Reeves is cocompact unless W contains a Euclidean triangle group, i.e. there exist s_i, s_j, s_k among the generators in the standard presentation of W such that $(m_{ij}, m_{jk}, m_{ki}) = (2,3,6), (2,4,4)$, or $(3,3,3)$ [**CM05**]. Finally, Haglund-Wise showed that the Niblo-Reeves cubulation is virtually special, i.e. W has a finite index subgroup F such that X_W/F is special where X_W is the CAT(0) cube complex constructed by Niblo-Reeves [**HW10**].

2.3. Small cancellation groups

We recall the basic notions of small cancellation theory following [**LS77**]. Let $G = \langle S \mid R \rangle$ be a group presentation where S is a finite generating set and R is a set of cyclically reduced relators. Let R^{cyc} be the set of all cyclic permutation of the words in R. A word w is *piece in* $r \in R$ if there exists $r' \neq r \in R^{cyc}$ such that

$$r = w \cdot u, \quad r' = w \cdot u'$$

are reduced spellings. Let $p \geq 1$. The presentation $G = \langle S \mid R\rangle$ satisfies

- the $C'(1/p)$ *small cancellation condition*, if $|w| < \frac{1}{p}|r|$,
- the $C(p)$ *small cancellation condition*, if r cannot be expressed as a concatenation of less than p pieces.

We say that G is a $C'(1/p)$ *(resp. $C(p)$) small cancellation group*, if there exists a presentation $G = \langle S \mid R\rangle$ that satisfies the $C'(1/p)$ (resp. $C(p)$) small cancellation condition.

The motivating examples are the fundamental groups of closed orientable surfaces of genus $g \geq 2$. The standard presentation is the following

$$\pi_1(\Sigma_g) = \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \cdots [x_g, y_g]\rangle.$$

Pieces are words of length ≤ 1. Thus $\pi_1(\Sigma_g)$ is $C'(1/(4g-1))$, in particular $C'(1/6)$.

The *word problem* is the following decision problem. For a given group with a generating set (G, S) does there exist an algorithm that takes as an input a word w in S and as an output gives *yes* if and only if w is an identity element in G.

The surface groups were shown to have solvable word problem by Dehn via what is now known as *Dehn's algorithm*. Greendlinger defined a class of groups that satisfy the $C'(1/6)$ small cancellation condition as groups where the Dehn's algorithm provides a solution to the word problem [**Gre60**]. Gromov defined *hyperbolic groups* as a common generalization of $C'(1/6)$ small cancellation groups and the fundamental groups of negatively curved manifolds [**Gro87**]. In the same paper, Gromov proved that in fact every hyperbolic group has a *Dehn presentation*, i.e. a presentation where the Dehn's algorithm solve the word problem.

An important feature of $C'(1/6)$ small cancellation groups is Greendlinger's Lemma which is as follows. Suppose $G = \langle S \mid R\rangle$ is a $C'(1/6)$ presentation and let w be a cyclically reduced word w in S that defines the identity in G. Then there exists a subword p of some cyclic permutation of w that is also a subword of a cyclic permutation of some $r \in R$ such that $|w| > \frac{1}{2}|r|$.

Wise showed that $C'(1/6)$ groups are cocompactly cubulated [**Wis04**]. The walls he constructed are obtained by joining the opposite edges (by possibly subdividing each edge) in the presentation complex and applying Sageev's construction.

The cube complex dual to the discrete wallspace as above construction does not necessarily give a CAT(0) cube complex of minimal dimension. For example for the usual presentation of $\pi_1(\Sigma_g)$ we get a cube complex of dimension $2g$, while there also exists a 2-dimensional CAT(0) cube complex that $\pi_1(\Sigma_g)$ acts on freely and cocompactly (see Section 1.12). In Chapter 4 for each $n \geq 1$ we construct example of $C'(1/6)$ groups with cubical dimension bounded below by n. Note that Wise's construction gives an upper bound on cubical dimension which is the maximum of the length of relators.

2.4. Random groups

The notion of random group in a density model was introduced and studied by Gromov [**Gro93**] and followed by Ollivier [**Oll04**][**Oll05**]. Fix an integer $m \geq 2$ and a real number $0 \leq d \leq 1$. A *random group at density* d with relators of length ℓ is given by the presentation

$$\langle a_1, \dots, a_m \mid r_1, \dots, r_{q(\ell)} \rangle$$

where $q(\ell) = \lfloor (2m-1)^{d\ell} \rfloor$ and r_i is chosen randomly independently and uniformly from the set of freely reduced words of length ℓ in the letters $a_1^{\pm 1}, \dots, a_m^{\pm 1}$. We say that a random group at density d has a property P *with overwhelming probability* if

$$\lim_{\ell \to \infty} \mathbb{P}\left(\langle a_1, \dots, a_m \mid r_1, \dots, r_{q(\ell)} \rangle \text{ has property } P \right) = 1.$$

Gromov and Ollivier proved that

- if $d > \frac{1}{2}$, then the random group at density d is a subgroup of $\mathbb{Z}/2\mathbb{Z}$ with overwhelming probability,
- if $d < \frac{1}{2}$, then the random group at density d is non-elementary hyperbolic, torsion-free and has cohomological dimension 2 with overwhelming probability.

Żuk proved that the random group at density $d > \frac{1}{3}$ has Property (T) with overwhelming probability [**Żuk03**] (completed by Kotowski-Kotowski [**KK13**]). In [**OW11**] Ollivier-Wise proved that a random group at density $d < \frac{1}{6}$ acts freely and cocompactly on a CAT(0) cube complex with overwhelming probability, and at density $d < \frac{1}{5}$ it acts without a global fixed point with overwhelming probability. The range where a random group is known to act non-trivially have been extended to $d < \frac{5}{24}$ by Mackay-Przytycki [**MP15**].

2.5. Groups with quasi-convex hierarchy

Let $\mathcal{C}$ be a family of groups. A group G has a *quasiconvex hierarchy terminating in* $\mathcal{C}$ if G either belongs to $\mathcal{C}$, or splits as an HNN-extension $A*_D$ or a free product with amalgamation $A *_D B$ where

- A and B have a quasiconvex hierarchy terminating in $\mathcal{C}$,
- D is finitely generated and quasi-isometrically embedded in $A*_D$ or $A *_D B$.

Note that it is not enough to assume that D is quasi-isometrically embedded in each A and B. For example, the Baumslag-Solitar group $BS(1,2) = \langle a, t \mid t^{-1}at = a^2\rangle$ is an HNN-extension $A*_D$ where $A = \langle a\rangle \simeq \mathbb{Z}$ and D is a copy of $\mathbb{Z}$ in A embedded as a subgroup of index 1 or 2, and in both cases it is quasi-isometrically embedded in A. The theorem of Wise states that if G is hyperbolic and has quasiconvex hierarchy terminating in trivial groups, then G is virtually special [**Wis11**]. We discuss more examples in Section 2.6 and Section 2.7.

2.6. One relator groups with torsion

A *one-relator group* is a group given by a presentation

$$G = \langle x_1, \dots, x_k \mid w^n\rangle$$

where $w \in F(x_1, \dots, x_k)$. We assume that w is not a proper power. A one-relator group G has torsion if and only if $n > 1$, in which case every torsion element in G is a conjugate of some power of w [**FKS72**]. Lauer-Wise showed that if $n \geq 4$, then G acts properly and cocompactly on a CAT(0) cube complex [**LW13**] by constructing walls in the Cayley complex of G which are quasiconvex and satisfy the linear separation condition. Later Wise proved that any one-relator group with torsion has a finite index subgroup that admits a quasiconvex hierarchy terminating in trivial groups and consequently, any such group is the fundamental group of a compact special cube complex [**Wis11**]. The Magnus-Moldavanskii hierarchy is a quasiconvex hierarchy of G terminating in finite groups [**LS77, Wis11**]. We describe it now. First, the *complexity* of a one-relator group is $\sum_i(\#x_i - 1)$ where $\#x_i$ is the number of appearances of $x_i^{\pm}$ in w. A *Magnus subgroup* of G is a subgroup generated by a subset of $x_1, \dots, x_k$ that omits at least one x_i for which $\#x_i > 0$. A Magnus subgroup is a free group [**LS77**].

If $\#x_i = 0$ for some i, then G splits as $\langle x_1, \ldots, \hat{x}_i, \ldots x_k \mid w^n \rangle * \mathbb{Z}$. If $\#x_i = 1$ for some i, then w is a primitive element in F_k and so $G = \langle x_1, \ldots, \hat{x}_i, \ldots x_k, w \mid w^n \rangle = F_{k-1} * \mathbb{Z}_n$. If $\#x_i > 1$ for all i, then $G' = G * \mathbb{Z}$ splits over its Magnus subgroup as an HNN-extension of a one-relator group of lower complexity. We choose new generators $y_i = x_i t^{p_i}$ where p_i are integers which are chosen so that the total exponent of w written in $y_1, \ldots, y_k, t$ is 0. Finally, we add new relators of the form $y_{i,j} = t^{-j} y_i t^j$ for all the necessary j's so that w can be now written as a word in the generators $y_{i,j}$ exclusively. We also add relators of the form $y_{i,j} = t^{-j} y_i t^j$. Let K be a one relator group generated by $y_{i,j}$'s. We get a presentation of G' as an HNN-extension of K with the stable letter t conjugating one Magnus subgroup of K to another. The details of this construction can be found in [**LS77**] or [**Wis11**]. Wise showed that the induced hierarchy of a finite index torsion-free subgroup of G is a quasiconvex hierarchy terminating in trivial groups [**Wis11**].

2.7. Hyperbolic 3-manifold groups

A subgroup H of G is *separable* if H is the intersection of the finite index subgroups of G containing H. The group G is *subgroup separable* if every finitely generated subgroup is separable in G. If Σ is an immersed surface in a 3-manifold M, then the immersion $\Sigma \looparrowright M$ induces the embedding $\pi_1\Sigma \hookrightarrow \pi_1 M$. If $\pi_1\Sigma$ is separable in $\pi_1 M$ then by a theorem of Scott Σ lifts to an embedding in a finite cover of M [**Sco78**].

Every hyperbolic 3-manifold can be realized as a quotient of the hyperbolic 3-space $\mathbb{H}^3$ by a Kleinian group, i.e. a discrete subgroup of the group $\mathrm{Isom}_+ \mathbb{H}^3 \simeq PSL(2, \mathbb{C})$ of orientation preserving isometries of $\mathbb{H}^3$. The *limit set* $\Lambda(\Gamma)$ of Γ is the set of accumulation points of the action of $\Gamma \curvearrowright \mathbb{H}^3$ on the boundary $\partial \mathbb{H}^3 \simeq S^2$. The group Γ is *quasi-fuchsian* if $\Lambda(\Gamma)$ is a quasi-circle. Quasi-fuchsian groups are conjugate to fuchsian groups by quasiconformal transformations.

A number of long standing conjecture about 3-manifolds have been resolved in the last 10 years. The surface subgroup conjecture of Waldhausen states that the fundamental group of every closed, irreducible 3-manifold is either finite, or contains a surface subgroup. The Waldhausen's virtual Haken conjecture states that every compact, orientable, irreducible 3-manifold with infinite fundamental group has a finite index cover which is *Haken*, i.e. contains a properly embedded two-sided incompressible surface. The Thurston's virtual fibered conjecture states that every closed, irreducible, atoroidal 3-manifold with infinite fundamental group has a finite cover which is a surface bundle over the circle, i.e. a finite cover of M is diffeomorphic to a mapping torus $M_\phi := \Sigma \times [0,1]/(x,0) \sim (\phi(x),1)$ for some diffeomorphism $\phi : \Sigma \to \Sigma$. See Figure 2.

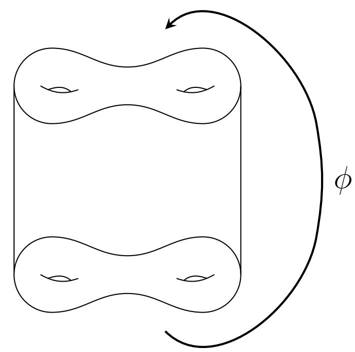

FIGURE 2. Surface bundle over a circle.

The proof of the geometrization conjecture by Perelman reduced all the above conjectures to the case of closed hyperbolic 3-manifolds. Cubulating hyperbolic 3-manifold groups played a crucial role in establishing the virtual Haken conjecture and the virtual fibered conjecture.

The surface subgroup conjecture was resolved in 2009 by Kahn-Markovic [**KM12**] in the following stronger form: for any great circle C in $\partial\mathbb{H}^3 \simeq S^2$ there exists a sequence of immersed quasi-fuchsian surfaces $\Sigma_i \hookrightarrow M$ such that $\partial\Sigma_i$ pointwise converges to C. In particular, any two points $p, q \in \partial\mathbb{H}^3$ are separated by some $\partial\Sigma_i$. The result of Kahn-Markovic reduced the virtual Haken conjecture to the question of separability of the quasi-fuchsian surface subgroups of $\pi_1 M$.

Bergeron-Wise used the Kahn-Markovic surfaces as walls in $\mathbb{H}^3$ and obtained a wallspace with an action of $\pi_1 M$ and proved that the resulting action on the dual cube complex is proper and cocompact [**KM12**]. Wise proved that if M contains a incompressible geometrically finite surface, then $\pi_1 M$ is virtually special [**Wis11**]. Using the criterion of Agol which states that if $\pi_1 M$ is virtually special, then M virtually fibers [**Ago08**], that also reduced the virtual fibered conjecture to the conjecture on separability of the quasi-fuchsian surface subgroups.

Finally, Agol proved that every hyperbolic cocompactly cubulated group is virtually special [**Ago13**]. That established the separability of quasiconvex subgroups, in particular the quasi-fuchsian surface subgroups and therefore completed the proof of the virtual Haken conjecture and virtual fibered conjecture for all closed hyperbolic 3-manifolds. Moreover, the result of Agol also gives the subgroup separability of $\pi_1 M$. Indeed, the tameness theorem [**Ago04, CG06**] states that every complete hyperbolic 3-manifold with finitely generated fundamental group is *topologically tame*, i.e. it is homeomorphic to the interior of a compact 3-manifold. Thus every finitely generated subgroup of $\pi_1 M$ is either geometrically finite and then quasiconvex, or is virtually the fundamental group of a fiber of a fibration over the circle, in which case it is virtually a normal subgroup with quotient $\mathbb{Z}$ and therefore it is also separable.

For more discussion on cubulated 3-manifold groups, see Section 3.2.2.

2.8. Hyperbolic free-by-cyclic groups

A *free-by-cyclic group* G is a group extension of a free group F_k by $\mathbb{Z}$ where $k \geq 2$, i.e. there is a short exact sequence

$$1 \to F_k \to G \to \mathbb{Z} \to 1.$$

Every such short exact sequence splits, so G is a semidirect product $F_k \rtimes_\phi \mathbb{Z}$ for some $\phi \in \operatorname{Aut} F_k$. Since composing ϕ on the right with some inner automorphism of F_k does not change the semidirect product we often write $\phi \in \operatorname{Out}(F_n)$ to determine $F_k \rtimes_\phi \mathbb{Z}$ where $\operatorname{Out}(F_k) := \operatorname{Aut}(F_k)/\operatorname{Inn}(F_k)$. This group has the following presentation

$$F_k \rtimes_\phi \mathbb{Z} = \langle x_1, \ldots, x_k, t \mid t^{-1} x_i t = \phi(x_i) \rangle$$

The group G is hyperbolic if and only if ϕ is *atoroidal*, i.e. no power of ϕ stabilizes a non-trivial conjugacy class in F_k [**BH92**].

There are many connections between free-by-cyclic groups and the fundamental group of 3-manifolds that fiber over S^1. The latter can be expressed as the fundamental groups of *mapping tori*

$$M_f = \Sigma \times [0,1]/(x,0) \sim (f(x),1)$$

where $f : \Sigma \to \Sigma$ is a homeomorphism. The mapping torus structure induces the short exact sequence of groups

$$1 \to \pi_1(\Sigma) \to \pi_1(M_f) \to \mathbb{Z} \to 1$$

which again splits, so $\pi_1(M_f) = \pi_1(\Sigma) \rtimes_\phi \mathbb{Z}$ where $\phi = [f_*]$ is the outer automorphism of $\pi_1(\Sigma)$ induced by f. Similarly, G can be represented as the fundamental group of the mapping torus of $f : V \to V$ where V is a finite graph with the fundamental group F_k. Hagen-Wise constructed quasiconvex walls in the mapping torus and using Bergeron-Wise criterion proved that the resulting action on a CAT(0) cube complexes is free and cocompact [**HW13**][**HW15**]. The proof is hard and technical so we do not discuss it here.

CHAPTER 3

Obstructions to group actions on CAT(0) cube complexes

3.1. Kazhdan's Property (T)

Let G be a locally compact topological group. There are many equivalent definitions of Property (T) (also known as Property (FH)) and we are going to use the one that has the closest relation to our interest. For other equivalent definitions, see [**BdlHV08**]. We say that G has *Kazhdan's Property (T)* if every continuous isometric action of G on a real Hilbert space has a fixed point. Niblo-Reeves [**NR97**] proved that if G has Property (T) and G acts isometrically on a finite dimensional CAT(0) then the action has a global fixed point. In particular, G has no codimension-1 subgroups. Property (T) is a strong negation of the Haagerup Property discussed in Section 1.11.3. Indeed, the only groups that have both Property (T) and the Haagerup Property are compact. Here are some other examples of groups with Property (T):

(1) Simple real Lie groups of rank at least 2 e.g.
   - special linear groups $SL(n, \mathbb{R}) = \{M \in M_n(\mathbb{R}) \mid \det M = 1\}$ for $n \geq 3$,
   - indefinite special orthogonal groups $SO(p, q) = \{M \in M_{p+q}(\mathbb{R}) \mid M^T I_{p,q} M = I_{p,q}\}$ for $p > q \geq 2$ or $p = q \geq 3$ where $I_{p,q} = diag(\underbrace{1, \dots, 1}_{p}, \underbrace{-1, \dots, -1}_{q})$.

(2) Indefinite spin groups $Sp(n, 1) = \{M \in M_{n+1}(\mathbb{H}) \mid M^* I_{n,1} M\}$ for $n \geq 2$ where $\mathbb{H}$ denotes the quaternions and M^* is the quaternion conjugate transpose.

(3) A rich family of examples of discrete groups with Property (T) is a consequence of Kazhdan's Theorem: A lattice Γ in a locally compact group G has Property (T) if and only if G has Property (T).

3.2. Obstructions to cocompact actions

3.2.1. Artin groups. An Artin group A is given by presentation

$$\langle a_1, \dots, a_n \mid \underbrace{a_i a_j a_i \dots}_{m_{ij}} = \underbrace{a_j a_i a_j \dots}_{m_{ij}} \rangle$$

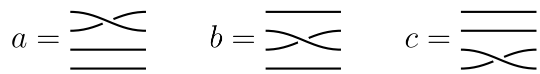

FIGURE 1. Generators of the braid group B_4.

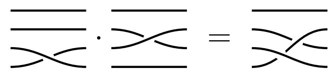

FIGURE 2. Group operation in the braid group B_4.

where $m_{ij} = m_{ji} \in \{2, 3, \ldots, \infty\}$ where $m_{ij} = \infty$ means no relation. The *defining graph* Γ_A has the vertices corresponding to the generators of A, and vertices a_i, a_j are joined by an edge labelled by m_{ij} if and only if $m_{ij} < \infty$.

An Artin group A is 2-*dimensional* if for every full subgraph $\Gamma \subset \Gamma_A$ on three vertices the corresponding Coxeter group is finite, i.e. if for any generators a_i, a_j, a_k the triple (m_{ij}, m_{jk}, m_{ki}) is not one of the following: $(2,3,5)$, $(2,3,4)$, $(2,3,3)$, or $(2,2,n)$ for some $n > 1$.

Together with Jingyin Huang and Piotr Przytycki [**HJP16**] we examine proper and cocompact actions of Artin groups and prove that many Artin groups fail to be cocompactly cubulated. The article is included in Section 6. We give a complete characterization of 2-dimensional or three generator Artin groups that are cocompactly cubulated, in terms of the defining graph. See Theorem 6.1.1 and Theorem 6.1.2.

The *braid group on four strands* B_4 is the Artin group with the following presentation

$$B_4 = \langle a, b, c \mid aba = bab, bcb = cbc, ac = ca \rangle.$$

The elements of B_4 can be intuitively represented by equivalence classes of four intertwined strands where the group operation is the concatenation of strands. See Figure 1 and Figure 2.

Note that our result in particular implies that the braid group B_4 on 4 strands is not virtually cocompactly cubulated. The group B_4 is known to be a CAT(0) group [**BM10**].

More recently, Haettel obtained the full classification of (virtually) cocompactly cubulated Artin groups [**Hae17**]. Haettel proves that A is virtually cocompactly cubulated if and only if the following conditions hold:

- for each i, j, k such that m_{ij} is odd, either $m_{jk} = m_{ki} = \infty$, or $m_{jk} = m_{ki} = 2$, and
- there exist an orientation of edges of Γ such that for each i, j, k such that m_{ij} is even and $\neq 2$, and (i, j) has positive orientation, we have one of the following:
  - $m_{ki} = m_{jk} = 2$,
  - $m_{ki} = m_{jk} = \infty$,
  - $m_{ki} = 2$ and $m_{jk} = \infty$, or
  - m_{ki} is even and $\neq 2$, (i, k) has positive orientation and $m_{jk} = \infty$.

In particular, it follows from result of Haettel that no braid group on at least 4 strands is cocompactly cubulated. The assumption on the cocompactness of the action is essential in both approaches [**HJP16**] and [**Hae17**].

3.2.2. Graph manifold groups. Let M be a compact oriented 3-manifold. We say that M is *irreducible*, if any embedded S^2 in M bounds an embedded ball B^3 in M. Let $\Sigma \subset M$ be a compact surface such that $\Sigma \neq S^2, P^2$. The surface Σ is *compressible* if there exists a disc D embedded in M such that $D \cap \Sigma = \partial D$ and the curve ∂D does not bound a disc in Σ. Otherwise, Σ is *incompressible* if there does not exists a disc D embedded in M that is essential in Σ such that $D \cap \Sigma = \partial D$. An irreducible manifold M is *atoroidal* if for every incompressible torus $T \subset M$ there exists a connected component of $M - T$ that is homeomorphic to $T \times I$.

A *model Seifert fibering of* $S^1 \times D^2$ is the decomposition of $S^1 \times D^2$ into *fibers* S^1 obtained from decomposing $[0, 1] \times D^2$ into intervals $[0, 1] \times \{x\}$ and then identifying $\{0\} \times D^2$ with $\{1\} \times D^2$ via the rotation by $\frac{p}{q}2\pi$ where p, q are relatively prime. A *Seifert fibered manifold* is a manifold with (possibly empty) boundary, together with a decomposition into S^1 such that each fiber has a neighbourhood that is fiber-preserving diffeomorphic to a neighbourhood of a fiber in a model Seifer fibering of $S^1 \times D^2$.

Suppose M is irreducible. Then there exists an upper bound on the size of the maximal collection $\{T_i\}_i$ of disjoint incompressible tori in M such that no connected component of $M - \bigcup_i T_i$ is homeomorphic to $T \times I$. Every connected component of $M - \bigcup_i T_i$, called a *block*, is either atoroidal or Seifert fibered. Such a decomposition obtained by cutting along $\{T_i\}_i$ is called a *JSJ decomposition* of M.

A *graph manifold* M is a compact oriented aspherical 3-manifold that has only Seifert fibered blocks in its JSJ decomposition. Liu showed that $\pi_1 M$ is virtually cubulated if and only if $\pi_1 M$ is virtually special if and only if M admits a nonpositively curved Riemannian metric [**Liu13**]. Hagen-Przytycki proved that $\pi_1 M$ is virtually cocompactly cubulated if and only if $\pi_1 M$ is virtually compact special if and only if M is *chargeless*, i.e. in each block there is a horizontal surface whose boundary circles are vertical in adjacent blocks. [**HP15**].

To summarize let us give a complete classification of closed irreducible 3-manifold groups that are virtually (cocompact) special. For the simplicity we assume that M is orientable (as we can always take an orientable double cover of M). Let us first look at geometric manifolds. It is easy to see that for M with one of the geometries: $S^3, \mathbb{R}^3, \mathbb{H}^2 \times \mathbb{R}, S^2 \times \mathbb{R}$, $\pi_1 M$ is virtually compact special. The fundamental groups of M with one of geometries Nil, Sol or $\widetilde{PSL_2}\mathbb{R}$ is not virtually cubulated. For the first two geometries $\pi_1 M$ contains distorted abelian subgroups so it is not a CAT(0) group by algebraic flat torus theorem [**BH99**] and it cannot act properly on CAT(0) cube complexes [**Woo17**]. If M has $\widetilde{PSL_2}\mathbb{R}$ geometry, then it contains a central subgroup isomorphic to $\mathbb{Z}$ that is not virtually a direct factor. If M is a hyperbolic 3-manifolds with boundary then $\pi_1 M$ is virtually compact special by [**Wis11**]. If M is a closed hyperbolic 3-manifolds then $\pi_1 M$ is virtually compact special by [**Ago13**] (see Section 2.7). It is known that Seifert fibered 3-manifolds are geometric (with one of the geometries $S^3, \mathbb{E}^3, \mathbb{H}^2 \times \mathbb{R}, S^2 \times \mathbb{R}, Nil, \widetilde{PSL_2}\mathbb{R}$). If M is not hyperbolic or Seifert fibered then M admits a nontrivial JSJ decomposition. The classification of graph manifolds have been discussed. If not all the blocks in JSJ decomposition are Seifert fibered, then $\pi_1 M$ is a mixed manifold and is virtually special by [**PW17**]. Moreover, $\pi_1 M$ is virtually compact special if and only if $\pi_1 M$ is virtually cocompactly cubulated if and only if M is chargeless [**Tid17**].

3.3. Tubular groups

A *tubular group* G is a group that splits as a graph of groups Γ with $G_v \simeq \mathbb{Z}^2$ for all $v \in V(\Gamma)$ and $G_e \simeq \mathbb{Z}$ for all $e \in E(\Gamma)$. The group G is the fundamental group of a graph of spaces X, called *tubular space* whose vertex spaces are tori and edges spaces are cylinders. The *intersection number* $\#[c,s]$ where $c, s \in \mathbb{Z}^2$ is defined as 0 if $\langle c, s\rangle \simeq \mathbb{Z}$ and as $[\mathbb{Z}^2 : \langle c, s\rangle]$ otherwise. The intersection number is the same as the minimal number of intersection of loops representing c, s as elements of $\pi_1 T^2$. For a finite set $S \subset \mathbb{Z}^2$ $\#[c,S]$ is defined as $\sum_{s\in S} \#[c,s]$. An *equitable set* for G is a collection of finite subsets $\{S_v \subset G_v \mid v \in V(\Gamma)\}$

such that $[G_v : \langle S_v \rangle] < \infty$, and $\#[c_{\overleftarrow{e}}, S_{\overleftarrow{e}}] = \#[c_{\overrightarrow{e}}, S_{\overrightarrow{e}}]$ where $\overleftarrow{e}, \overrightarrow{e}$ are the images of a fixed generator c of G_e in the vertex groups $G_{\overleftarrow{e}}$ and $G_{\overrightarrow{e}}$ respectively. Wise gave a characterization of tubular groups that act freely on CAT(0) cube complexes as precisely those that admit equitable sets [**Wis14**]. Woodhouse determined which tubular groups act freely on finite dimensional CAT(0) cube complexes [**Woo16b**] and characterized virtually special tubular groups as those that act freely on locally finite CAT(0) cube complexes [**Woo16c**].

CHAPTER 4

Lower bounds on cubical dimension of $C'(1/6)$ group

For each n we construct examples of finitely presented $C'(1/6)$ small cancellation groups that do not act properly on any n-dimensional CAT(0) cube complex. This section is based on [**Jan17b**].

4.1. Introduction

Groups that satisfy the $C'(1/6)$ small cancellation condition were shown to act properly and cocompactly on CAT(0) cube complexes by Wise in [**Wis04**]. In this chapter we are interested in the minimal dimension of a CAT(0) cube complex that such groups act properly on.

Definition 4.1.1. The *cubical dimension* of G is the infimum of the values n such that G acts properly on an n-dimensional CAT(0) cube complex.

Wise's complex is obtained from Sageev's construction [**Sag95**] with walls joining the opposite sides in each relator (after subdividing each edge into two if necessary). However, its dimension is not in general optimal. For example, the dimension of the CAT(0) cube complex associated to the usual presentation for the fundamental group of the surface of genus $g \geq 2$ is g, while its cubical dimension equals 2 as it acts on the hyperbolic plane with a CAT(0) square complex structure.

We prove the following:

Theorem 4.1.2. *For each $n \geq 1$ and each $p \geq 6$ there exists a finitely presented $C'(1/p)$ small cancellation group G such that the cubical dimension of G is greater than n.*

For $n = 1$, the stronger form of Theorem 6.1.1 was proved by Pride in [**Pri83**]. He gives an explicit example of an infinite $C'(1/6)$ group with property FA. Pride's construction has been revisited in [**JW17b**]. We observe that the case $n = 2$ can be deduced from the work of Kar and Sageev who study uniform exponential growth of groups acting freely on CAT(0) square complexes [**KS16**]. See Remark 4.4.1. As a consequence, the Kar–Sageev examples have finite cubical dimension that is strictly larger than the geometric dimension.

This chapter is organized as follows. In Section 4.2 we recall the classification of isometries of a CAT(0) cube complex with respect to hyperplanes. We refer to [**LS77**] for the background on small cancellation theory. In Section 3 we describe how to build a $C'(1/p)$ presentations where relators are positive products of given words. This technical result is applied in Section 4, which is the heart of the paper and contains the proof of Theorem 6.1.1. The argument heavily utilizes hyperplanes to create a dichotomy between free subsemigroups and subgroups having polynomial growth. The main ingredient of the proof of Theorem 6.1.1 is Lemma 4.4.2 which states that for any two hyperbolic isometries a, b of an n-dimensional CAT(0) cube complex one of the following holds: $\langle a, b\rangle$ is virtually abelian, or there is a hyperplane stabilized by certain conjugates of some powers of a or b, or there is a pair of words in a, b of uniformly bounded length that generates a free semigroup.

4.2. Isometries and hyperplanes in CAT(0) cube complexes

In this section we recall relevant facts about isometries of CAT(0) cube complexes and collect some lemmas that will be used in the proof of Theorem 6.1.1. For general background on CAT(0) cube complexes and groups acting on them we refer the reader to [**Sag14**].

Throughout the paper X will be a finite dimensional CAT(0) cube complex. The set of all hyperplanes of X is denoted by $\mathscr{H}(X)$ and a cube complex dual to a collection $\mathscr{H}$ of hyperplanes is denoted by $X(\mathscr{H})$. We use letters h^+, h^- to denote the halfspaces of a hyperplane $\boldsymbol{h}$, and $N(\boldsymbol{h})$ to denote the closed carrier of $\boldsymbol{h}$, i.e. the convex subcomplex of X that is the union of all the cubes intersecting $\boldsymbol{h}$. We say that a hyperplane $\boldsymbol{h}$ *separates* subsets $A, B \subset X$, if $A \subset h^+$ and $B \subset h^-$. The metric d is the ℓ_1-metric on X. All the paths we consider are combinatorial (i.e. concatenations of edges), all the geodesics are with respect to d, and all axes of hyperbolic isometries are combinatorial axes. The combinatorial translation length $\delta(a)$ of an isometry a is defined as $\inf_{x\in X^0} \mathsf{d}(x, ax)$. If a acts without hyperplane inversions then the infimum is realized and $\delta(a^k) = k\delta(a)$ [**Hag07**] (see also [**Woo16a**]). In particular, a has a combinatorial axis and any axis of a is also an axis of a^k. The *combinatorial minset* of x is

$$\mathrm{Min}^0(x) = \{p \in X^0 : \mathsf{d}(p, xp) = \delta(x)\}$$

where X^0 is the 0-skeleton of X. Every 0-cube p of $\mathrm{Min}^0(x)$ lies on an axis of x (any geodesic joining $\{x^i p\}_i$).

Let $n = \dim X$. Let a be a hyperbolic isometry of X and let $\boldsymbol{\hbar}$ be a hyperplane. We recall the classification of isometries of a CAT(0) cube complex. More details can be found in [**CS11**, Sec 2.4 and 4.2].

- a *skewers* $\boldsymbol{\hbar}$ if $a^k h^+ \subsetneq h^+$ for one of the halfspaces h^+ of $\boldsymbol{\hbar}$ and some $k > 0$. Equivalently, if some (equivalently, any) axis of a intersects $\boldsymbol{\hbar}$ exactly once.
- a is *parallel* to $\boldsymbol{\hbar}$ if some (equivalently, any) axis of a is in a finite neighbourhood of $\boldsymbol{\hbar}$.
- a is *peripheral* to $\boldsymbol{\hbar}$ if a does not skewer $\boldsymbol{\hbar}$ and is not parallel to $\boldsymbol{\hbar}$. Equivalently, $a^k h^+ \subsetneq h^-$ for some $k > 0$.

Note that the type of behaviour of a with respect to $\boldsymbol{\hbar}$ is commensurability invariant, i.e. a^i has the same type as a with respect to $\boldsymbol{\hbar}$. The set of all hyperplanes in X skewered by a is denoted by $\mathrm{sk}(a)$. The constant k in the above definitions can be chosen to be lat most n. Indeed, $\{\boldsymbol{\hbar}, a\boldsymbol{\hbar}, \dots, a^n\boldsymbol{\hbar}\}$ cannot all intersect in X with $\dim X = n$. In particular, if $\boldsymbol{\hbar} \in \mathrm{sk}(a)$ then $a^{n!}h^+ \subset a^{(\frac{n!}{k}-1)k}h^+ \subset \dots a^k h^+ \subset h^+$ for one of the halfspaces $h^+ \in \boldsymbol{\hbar}$ and for an appropriate $k < n$. Similarly, we have the following:

LEMMA 4.2.1. There exists a constant $K_3 = K_3(n)$ such that for each hyperplane $\boldsymbol{\hbar}$ in X and an isometry a there exist $k < k' \le K_3$ such that the hyperplanes $\{\boldsymbol{\hbar}, a^k\boldsymbol{\hbar}, a^{k'}\boldsymbol{\hbar}\}$ pairwise are disjoint or equal.

PROOF. Consider the graph Γ whose vertices correspond to integers, and two integers r, q are joined by an edge if and only if $a^r\boldsymbol{\hbar}$ and $a^q\boldsymbol{\hbar}$ are distinct and intersect. Cliques in Γ correspond to collections of distinct pairwise intersecting hyperplanes. Let K_3 be the Ramsey constant for numbers $(n+1)$ and 3. Since X is n-dimensional, there are no $(n+1)$-cliques in Γ. The induced subgraph of Γ on vertices $[0, K_3 - 1]$ must contain a 3-anticlique. This corresponds to a triple of hyperplanes $\{a^p\boldsymbol{\hbar}, a^q\boldsymbol{\hbar}, a^r\boldsymbol{\hbar}\}$ that pairwise are disjoint or equal where $p < q < r$. Hence the hyperplanes $\{\boldsymbol{\hbar}, a^{q-p}\boldsymbol{\hbar}, a^{r-p}\boldsymbol{\hbar}\}$ are pairwise disjoint or equal. $\square$

In the above Lemma the hyperplanes $\boldsymbol{\hbar}, a^k\boldsymbol{\hbar}, a^{k'}\boldsymbol{\hbar}$ are pairwise disjoint, or $a^{K_3!}$ stabilizes $\boldsymbol{\hbar}$ (and the two cases are not mutually exclusive).

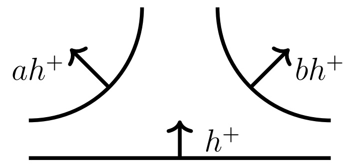

FIGURE 1. A ping-pong triple.

LEMMA 4.2.2. [**KS16**, Lem 12] Suppose a and b are hyperbolic isometries of X and there exists a hyperplane $\boldsymbol{\hbar} = (h^+, h^-)$ such that $ah^+ \subset h^+$, $bh^+ \subset h^+$ and $ah^+ \subset bh^-$. Then a, b freely generate a free semigroup. See Figure 1.

The triple $\{h^+, ah^+, bh^+\}$ as in Lemma 4.2.2 is called a *ping pong triple.* The following Lemma is a higher dimensional version of the All-Or-Nothing Lemma [**KS16**, Lem 13]. Our proof is based on the proof of Kar–Sageev but it differs slightly.

LEMMA 4.2.3. Let a and b be hyperbolic isometries and let $\boldsymbol{\hbar} \in \mathrm{sk}(a)$. Then one of the following holds

- b skewers all $a^{in!}\boldsymbol{\hbar}$ for $i \in \mathbb{Z}$, or
- b skewers none of $a^{in!}\boldsymbol{\hbar}$ for $i \in \mathbb{Z}$, or
- one of the following pairs of words freely generate a free semigroup for some $1 \leq k \leq n$:

$$\begin{aligned} &(a^{n!}, b^{kn!}a^{n!}), \\ &(a^{n!}, b^{-kn!}a^{n!}), \\ &(a^{-n!}, b^{kn!}a^{-n!}), \\ &(a^{-n!}, b^{-kn!}a^{-n!}). \end{aligned} \tag{$\star$}$$

PROOF. Let h^+ be the halfspace of $\boldsymbol{\hbar}$ such that $a^{n!}h^+ \subsetneq h^+$. Suppose that b skewers some hyperplane in P but not all of them. Without loss of generality we can assume that b skewers exactly one of $\boldsymbol{\hbar}, a^{n!}\boldsymbol{\hbar}$. First suppose b skewers $\boldsymbol{\hbar}$ but not $a^{n!}\boldsymbol{\hbar}$ i.e. the axis $\gamma_b \subset a^{n!}h^-$. Since γ_b goes arbitrarily deep in h^- we have that b is peripheral to $a^{n!}\boldsymbol{\hbar}$. We either have $b^{n!}h^+ \subset h^+$ or $b^{-n!}h^+ \subset h^+$. Let k be such that $b^{kn!}a^{n!}\boldsymbol{\hbar}$ and $a^{n!}\boldsymbol{\hbar}$ are disjoint. Either $b^{kn!}a^{n!}h^+ \subset b^{kn!}h^+ \subset h^+$ or $b^{-kn!}a^{n!}h^+ \subset b^{-kn!}h^+ \subset h^+$ and thus $\{h^+, a^{n!}h^+, b^{kn!}a^{n!}h^+\}$ or $\{h^+, a^{n!}h^+, b^{-kn!}a^{n!}h^+\}$ is a ping-pong triple. Similarly, if b skewers $a^{n!}\boldsymbol{\hbar}$ but not $\boldsymbol{\hbar}$, then one of $\{a^{n!}h^-, h^-, b^{kn!}h^-\}$ or $\{a^{n!}h^-, h^-, b^{-kn!}h^-\}$ is a ping-pong triple. $\square$

The *combinatorial convex hull* of a subset $A \subset X$ is the smallest convex cube complex containing A.

LEMMA 4.2.4.

(1) The combinatorial convex hull $\mathrm{Hull}(\gamma_x)$ of an axis γ_x of x isometrically embeds in $\mathbb{E}^k$ for some $k \geq 1$.
(2) The 0-skeleton of $\mathrm{Hull}(\mathrm{Min}^0(x))$ is contained in $\mathrm{Min}^0(x^{n!})$.

PROOF. Let p be some 0-cube of γ_x. Let $\boldsymbol{h}_1, \dots, \boldsymbol{h}_k$ denote all the hyperplanes separating p and $x^{n!}p$ (in particular, $k = n!\delta(x)$). Since $x^{n!}h_i^+ \subset h_i^+$ for all i and appropriate choice of halfspace h_i^+ of $\boldsymbol{h}_i$, the partition of the set of all hyperplanes skewered by x into $\{x^{in!}\boldsymbol{h}_1\}_{i\in\mathbb{Z}}, \dots, \{x^{in!}\boldsymbol{h}_k\}_{i\in\mathbb{Z}}$ gives an isometric embedding of $\mathrm{Hull}(\gamma_x)$ into a product of k trees by [**CH13**]. Since all the hyperplanes are intersected by a single bi-infinite geodesic (an axis of x), all the trees are in fact lines, i.e. $\mathrm{Hull}(\gamma_x)$ isometrically embeds in $\mathbb{E}^k$ with the standard cubical structure. The action of $x^{n!}$ extends to the action to $\mathbb{E}^k$ as a translation by the vector $[1, \dots, 1]$. Thus every 0-cube of the combinatorial convex hull $\mathrm{Hull}(\gamma_x)$ is translated by $k = n!\delta(x) = \delta(x^{n!})$ and therefore the 0-skeleton of $\mathrm{Hull}(\gamma_x)$ is contained in $\mathrm{Min}^0(x^{n!})$.

The subcomplex $\mathrm{Hull}(\mathrm{Min}^0(x))$ is the maximal subcomplex of the $\bigcap\{h^+ : \mathrm{Min}^0(x) \subset h^+\}$, i.e. $\mathrm{Hull}(\mathrm{Min}^0(x))$ is dual to $\mathscr{H}_x = \{\boldsymbol{h} : \mathrm{Min}^0(x) \cap h^+ \neq \emptyset \text{ and } \mathrm{Min}^0(x) \cap h^- \neq \emptyset\}$. If $p, p' \in \mathrm{Min}^0(x)$, $p \in h^+$ and $p' \in h^-$, then x is parallel to $\boldsymbol{h}$. Indeed, $x^i p \in h^+$ and $x^i p' \in h^-$ for all i and since $\mathsf{d}(x^i p, x^i p') = \mathsf{d}(p, p')$ the axis γ_x through p is contained in $N_d(\boldsymbol{h})$ where $d \leq \mathsf{d}(p, p')$. Thus the set $\mathscr{H}_x$ consists of hyperplanes skewered by x or parallel to x. It follows that $\mathrm{Hull}(\mathrm{Min}^0(x))$ decomposes as a product $Y \times Y^\perp$ where Y is dual to $\mathrm{sk}(x)$ and $Y^\perp$ is dual to the set of all the hyperplanes of $\mathscr{H}_x$ that are parallel to x. For each $p \in Y^\perp$ the complex $Y \times \{p\}$ is the combinatorial convex hull of an axis of x. It follows that $\mathrm{Hull}(\mathrm{Min}^0(x))$ is the union of the complexes of the form $\mathrm{Hull}(\gamma_x)$ and so the 0-skeleton of $\mathrm{Hull}(\mathrm{Min}^0(x))$ is contained in $\mathrm{Min}^0(x^{n!})$.

$\square$

LEMMA 4.2.5. Let X be a CAT(0) cube complex that is a subcomplex of a CAT(0) cube complex that is quasi-isometric to $\mathbb{E}^d$. Then any group G acting properly on X does not contain a copy of F_2. Moreover, if G is torsion-free, then G is virtually abelian.

PROOF. The growth of X^0 is a polynomial of degree at most d and so is the growth of G. Hence G cannot contain a copy of F_2. The second part follows from the Tits alternative for groups acting properly on CAT(0) cube complexes [**SW05**] which states that any such group with a bound on the size of finite subgroups either contains a copy of F_2, or is virtually abelian. $\square$

4.3. Constructing small cancellation presentations

The main goal of this section is the following.

PROPOSITION 4.3.1. Let $\mathcal{U} = \{(u_i, v_i)\}_{i=1}^m$ be a finite collection of pairs where for each i the elements $u_i, v_i \in F(a,b)$ are not powers of the same element. There exists a $C'(1/6)$ small cancellation presentation

$$\langle a, b \mid r_1, \dots, r_m \rangle$$

where r_i is a positive word in u_i, v_i that is not a proper power for $i = 1, \dots, m$.

By $F(a,b)$ in the above Lemma and throughout the section we denote the free group on generators a and b. The length of a word u with respect to a, b is denoted by $|u|$. A *spelling* of a nontrivial element $u \in F(a,b)$ is a concatenation $u_1 \cdots u_m = u$ where each *syllable* u_i is a nontrivial element of $F(a,b)$. The *cancellation* in the spelling uv is the value $\mathrm{canc}(u,v) = \frac{1}{2}(|u| + |v| - |uv|)$, i.e. the length of the common prefix of the reduced words representing u^{-1} and v. A spelling is *reduced* if $\mathrm{canc}(u_i, u_{i+1}) = 0$ for $i = 1, \dots, m-1$; in other words $|u| = \sum_i |u_i|$. For $u, v \in F(a,b)$ we say u, v are *virtually conjugate* and write $u \sim v$ if some powers of u and v are conjugate. We denote a free semigroup on u, v by $\{u, v\}^+$. Let u^* denote an element u^k for some $k \geq 0$.

LEMMA 4.3.2. Let H be a finitely generated subgroup of F_k. There exists a constant $C = C(H < F_k)$ such that the map between the conjugacy classes of maximal $\mathbb{Z}$-subgroups induced by the the inclusion $H \hookrightarrow F_k$ is at most C-to-1.

PROOF. Let $A \looparrowright B$ be an immersion of graphs where B is a wedge of k circles, whose induced map on the fundamental groups is the inclusion $H \hookrightarrow F_k$.

For any graph $\Gamma = A, B$, the conjugacy class of a $\mathbb{Z}$-subgroup in G can be represented by an immersion $L \looparrowright \Gamma$ of a line that factors as $L \looparrowright S \looparrowright \Gamma$ where S is a circle, taken modulo the orientation. Thus different conjugacy classes of $\mathbb{Z}$-subgroups in H that map into the same conjugacy class in F_k are some different lifts

$$\begin{tikzcd} & A \arrow[d] \\ L \arrow[ur, dotted] \arrow[r] & B \end{tikzcd}$$

The number of such lifts is bounded by the number of vertices in A. $\square$

Lemma 4.3.3. Let $u, v \in F(a,b)$ be such that u and v are not powers of the same element. There are infinitely many pairwise non virtually conjugate elements of the form $u^k v^k$.

Proof. Two elements of $F(x,y)$ are virtually conjugate if and only if they have the following reduced spellings

$$gw^i g^{-1}$$

$$h\bar{w}^j h^{-1}$$

where g, h, w are reduced words in x, y and $\bar{w}$ is a cyclic permutation of w. In particular the elements of the set $\{x^k y^k : n \in \mathbb{Z}\}$ are not virtually conjugate, i.e. they are contained in distinct conjugacy classes of maximal $\mathbb{Z}$-subgroups. Since u, v are not powers of the same element, the group $\langle u, v\rangle$ is a rank 2 free group. By Lemma 4.3.2 there exists a constant C such that the map between the conjugacy classes of maximal $\mathbb{Z}$-subgroups induced by the inclusion $\langle u, v\rangle \hookrightarrow F(a,b)$ is at most C-to-1. The lemma follows. $\square$

We say that elements $s, t \in F(a,b)$ are *non-cancellable*, if for any $w_1, w_2 \in \{s,t\}^+$

$$\mathrm{canc}(w_1, w_2) < \frac{1}{2}\min\{|s|, |t|\}.$$

In particular, we have $|w_1 w_2| \geq \max\{|w_1|, |w_2|\}$. Equivalently, it suffices that $\mathrm{canc}(s,t) < \frac{1}{2}\min\{|s|,|t|\}$ and $\mathrm{canc}(t,s) < \frac{1}{2}\min\{|s|,|t|\}$ for s, t to be non-cancellable. If s, t are non-cancellable then so are any two elements in $\{s,t\}^+$.

Lemma 4.3.4. Let $u, v \in F(a,b)$ not be powers of the same element. Then there exists elements $s, t \in \{u,v\}^+$ that are non-cancellable and are not powers of the same element.

Proof. If $\mathrm{canc}(u,v) > \frac{1}{2}\min\{|u|,|v|\}$ replace the pair (u,v) with (u, uv) if $|u| \leq |v|$, and with (v, uv) otherwise. If $\mathrm{canc}(v,u) > \frac{1}{2}\min\{|u|,|v|\}$ replace the pair (u,v) with (u, vu) if $|u| \leq |v|$, and with (v, vu) otherwise. Repeat these steps until $\mathrm{canc}(u,v), \mathrm{canc}(v,u) \leq$

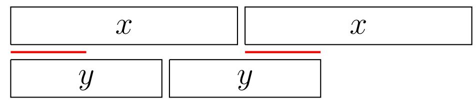

FIGURE 2. Long overlap between x^* and y^*. The red path is w.

$\frac{1}{2}\min\{|u|,|v|\}$. Since at each step the value $|u|+|v|$ strictly decreases, the procedure terminates in finitely many steps. Note that for any nontrivial element $x \in F(a,b)$ we have $\operatorname{canc}(x,x) < \frac{1}{2}|x|$, i.e. $|x^2| > |x|$. Let $s = u^2$ and $t = v^2$. We have $\operatorname{canc}(s,t) = \operatorname{canc}(u,v) \leq \frac{1}{2}\min\{|u|,|v|\}) < \frac{1}{2}\min\{|s|,|t|\}$ as wanted. Similarly, $\operatorname{canc}(t,s) < \frac{1}{2}\min\{|s|,|t|\}$. It follows that $\operatorname{canc}(w_1,w_2) < \frac{1}{2}\min\{|s|,|t|\}$ for every $w_1,w_2 \in \{s,t\}^+$. □

LEMMA 4.3.5. Let x,y be two cyclically reduced elements in $F(a,b)$ such that $|x| \geq |y| > 0$ such that x^2 is a prefix of y^*. Then x,y are powers of the same element.

PROOF. Suppose that x and y are not powers of the same element. In particular, x is not a power of y, so there exists a nonempty prefix w of x that is both some prefix of y and some suffix of y. See Figure 2. If $|w| \leq \frac{1}{2}|y|$, then y is of the form wuw for some u, and $x = (wuw)^k wu$ for some $k \geq 1$. Then x^2 has a prefix $(wuw)^k wu \cdot wuww = (wuw)^{k+1}uww$ which thus must coincide with $y^{k+1} = (wuw)^{k+2}$. In particular, $uw = wu$, which means that w,u are powers of the same element. That is a contradiction.

If $|w| > \frac{1}{2}|y|$, then $y = uw = wu'$ for some u,u' such that $|u| = |u'| < |w|$, and $x = (uw)^k u$ for some $k \geq 1$. The prefix $(uw)^k uuw$ of x^2 must coincide with the prefix $(uw)^{k+1}u$ of y^{k+2}. In particular $uw = wu$, which again is a contradiction. □

LEMMA 4.3.6. Let $u_i, v_i \in F(a,b)$ for $i = 1,2$ where for each $i = 1,2$ the elements u_i, v_i are non-cancellable and are not powers of the same element. Then for each $i = 1,2$ there exist $s_i, t_i \in \{u_i, v_i\}^+$ such that

- s_i, t_i are non-cancellable and are not virtually conjuagate,
- $\operatorname{canc}(s_i,t_i) = \operatorname{canc}(t_i,s_i) = \operatorname{canc}(s_i,s_i) = \operatorname{canc}(t_i,t_i)$, i.e. there exists g such that $s_i = g\bar{s}_i g^{-1}$ and $t_i = g\bar{t}_i g^{-1}$ are reduced spellings where $\bar{s}_i$, $\bar{t}_i$ are cyclically reduced and have no cancellation,
- every piece r between a word in $\{s_1,t_1\}^+$ and a word in $\{s_2,t_2\}^+$ we have $|r| < \min\{|s_i|,|t_i|\}$ for $i = 1,2$.

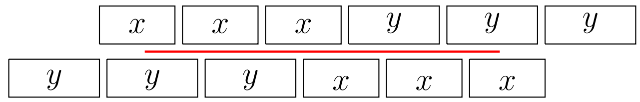

Figure 3. The red line is a maximal piece in r.

Proof. Since u_i, v_i are non-cancellable, the consecutive cancellations between syllables in any word $w \in \{u_i, v_i\}$ are separated from each other. For $i = 1, 2$ set $x_i = u_i^{n_1^i} v_i^{n_1^i}$ and $y_i = u_i^{n_2^i} v_i^{n_2^i}$ where $n_1^1, n_1^2, n_2^1, n_2^2$ are chosen so that x_1, y_1, x_2, y_2 are pairwise non virtually conjugate. This can be done by Lemma 4.3.3. Note that for $i = 1, 2$ we have $\operatorname{canc}(x_i, y_i) = \operatorname{canc}(x_i, x_i) = \operatorname{canc}(y_i, y_i) = \operatorname{canc}(v_i, u_i)$. Let $\bar{x}_i$ denote the cyclically reduced word representing an element conjugate to x_i such that the spelling $x_i = g\bar{x}_i g^{-1}$ is reduced where $|g| = \operatorname{canc}(v_i, u_i) < \frac{1}{2}\min\{|u_i|, |v_i|\}$. We have $y_i = g\bar{y}_i g^{-1}$ where $\bar{y}_i$ is cyclically reduced, and thus any positive word $w(x_i, y_i)$ in x_i, y_i has the reduced spelling $gw(\bar{x}_i, \bar{y}_i)g^{-1}$.

Let $N = 8\max\{|\bar{x}_1|, |\bar{y}_1|, |\bar{x}_2|, |\bar{y}_2|\}$ and set $s_i = x_i^N$ and $t_i = y_i^N$. Let r be a piece between a word in $\{s_1, t_1\}^+$ and a word in $\{s_2, t_2\}^+$ and suppose that $|r| \geq N$. There exists a subword r' of r of length $\geq \frac{1}{2}N$ that is a subword of $\bar{x}_1^*$ or of $\bar{y}_1^*$. There exists an even shorter subword r'' of r' of length $\geq \frac{1}{4}N$ that is also a subword of either $\bar{x}_2^*$ or of $\bar{y}_2^*$. Thus one of $\bar{x}_1^*, \bar{y}_1^*$ and one of $\bar{x}_2^*, \bar{y}_2^*$ have a common subword of length $\geq 2\max\{|\bar{x}_1|, |\bar{y}_1|, |\bar{x}_2|, |\bar{y}_2|\}$ and by Lemma 4.3.5 they are virtually conjugate. This is a contradiction. Thus $|r| < N$. We clearly also have $|x_i^N|, |y_i^N| \geq N$ for $i = 1, 2$, and thus we get $|r| < \min\{|s_i|, |t_i|\}$. $\square$

Lemma 4.3.7. Let x, y be cyclically reduced elements that are not proper powers in $F(a, b)$ such that x, y are not virtually conjugate. Let $r = x^{\alpha_1} y^{\beta_1} \cdots x^{\alpha_{2p}} y^{\beta_{2p}}$ for some p and w be a piece in r. If α_j, β_j are all different and greater than $2\max\{|x|, |y|\} + 1$, then for every piece w in r we have $|w| \leq (\max\{\alpha_j\} + 2)\,|x| + (\max\{\beta_j\} + 2)\,|y|$.

Proof. Let w be a piece in r and consider two subwords of r: $\eta_0\eta_1 \cdots \eta_k\eta_{k+1}$ and $\mu_0\mu_1 \cdots \mu_\ell\mu_{\ell+1}$ where $\eta_i, \mu_j \in \{x, y\}$ such that $\eta_1 \cdots \eta_k$ and $\mu_1 \cdots \mu_\ell$ are maximal words in syllables x, y entirely contained in w. We say that two syllables η_i and μ_j are *aligned* if $\eta_i = \mu_j$ and they entirely overlap in w.

Suppose two syllables η_i, μ_j overlap in w and $\eta_i = \mu_j = x$. If they are not aligned, say a proper suffix of η_i equals a proper prefix of μ_j then $\eta_{i+1} = y$ and $\mu_{j-1} = y$ (since x is not equal to any of its conjugates by Lemma 4.3.5). See Figure 3. Since x, y are not conjugate by Lemma 4.3.5 we get that $j \geq \ell - 1$ and $i \leq 2$. Thus $|w| < 6\max\{|x|, |y|\} < 6(|x| + |y|)$. From now on, assume that any two copies of x or y that overlap are aligned.

Suppose $\eta_i = x$ and $\mu_j = x$ are aligned where $1 \leq i \leq k$ and $1 \leq j \leq \ell$. If $\eta_{i+1} = x, \mu_{j+1} = y$, then $i + 2 \geq k + 1$. Indeed, consider three cases:

- $|x| = |y|$: Then necessarily $i = k$ and $j = \ell$.
- $|x| < |y|$: If $\eta_{i+2} = x$, then $i + 2 \geq k + 1$ because otherwise $\eta_{i+1}\eta_{i+2} = x^2$ was a subword of y^* (more specifically a subword of $\mu_{j+1}\mu_{j+2} = y^2$). If $\eta_{i+2} = y$ then η_{i+2} and μ_{j+1} are two overlapping not aligned copies of y so $i + 2 \geq k + 1$.
- $|x| > |y|$: If $\eta_{i+2} = x$, then $i + 2 \geq k + 1$ because otherwise $\mu_{j+1}\mu_{j+2} = y^2$ was a subword of x^*. If $\eta_{i+2} = y$, then η_{i+2} and μ_{j+2} are two overlapping not aligned copies of y so $i+2 \geq k+1$ (μ_{j+2} overlaps with η_{i+2} because otherwise $\mu_{j+1}\mu_{j+2} = y^2$ was a subword of x^*).

Similarly, if instead $\eta_{i-1} = x, \mu_{j-1} = y$, then $i - 2 \leq 0$. Similarly we can switch x and y. We are looking for an upper bound of $|w|$. If w contains a whole syllable x^{α_n} as a subword for some n and $\eta_{i+1} = \cdots = \eta_{i+\alpha_n} = x$ for $0 \leq i \leq k - \alpha_n$. In particular $\eta_i = y$ and $\eta_{i+\alpha_n+1} = y$. Since $\alpha_n \geq 2|y| + 1$ there must be a syllable μ_j contained in the subword spelled by $\eta_{i+1} \cdots \eta_{i+\alpha_n}$ because otherwise y and x were virtually conjugate. By the previous consideration μ_j and $\eta_{i'}$ are aligned for some $i + 1 \leq i' \leq i + \alpha_n$. Since $\alpha_1, \beta_1, \ldots, \alpha_{2p}, \beta_{2p}$ are all different, we can find i, j such that $\eta_i = x$ and μ_j are aligned and either η_{i+1}, μ_{j+1} or η_{i-1}, μ_{j-1} are different syllables (i.e. one of them is x and the other is y). By the consideration above, the subword x^{α_n} is contained less than two syllables from to the beginning of w or from the end of w. The same happens with a syllable y^{β_n} contained in w. We conclude that $|w| \leq (\max\{\alpha_i\} + 2)\,|x| + (\max\{\beta_i\} + 2)\,|y|$. □

Proof of Proposition 4.3.1. First by Lemma 4.3.4 we can assume that for $i = 1, \ldots, m$ the elements u_i, v_i are non-cancellable. Replace the pair (u_1, v_1) and (u_2, v_2) by (s_1, t_1) and (s_2, t_2) respectively as in Lemma 4.3.6, and continue replacing for each pair of indices $i < j \leq m$. After $\binom{m}{2}$ steps we have a collection $\{(s_i, t_i)\}_{i=1}^m$ where for every piece r between a word in (s_i, t_i) and a word in (s_j, t_j) where $i \neq j$ we have $|r| < \max\{|s_i|, |t_i|\}$ and where for any i the elements s_i and t_i are not virtually conjugate. Let $r_i(s_i, t_i) = s_i^{\alpha_1^i} t_i^{\beta_1^i} \cdots s_i^{\alpha_{2p}^i} t_i^{\beta_{2p}^i}$ where $\alpha_1^i, \beta_1^i, \ldots, \alpha_{2p}^i, \beta_{2p}^i$ are all distinct. Then for each piece

w between r_i and r_j where $i \neq j$ we clearly have $|w| < \max\{|s_i|, |t_i|\} < \frac{1}{p}|r_i|$. Moreover, if $\min\{\alpha_1^i, \beta_1^i, \ldots, \alpha_{2p}^i, \beta_{2p}^i\} > \frac{1}{2}\max\{\alpha_1^i, \beta_1^i, \ldots, \alpha_{2p}^i, \beta_{2p}^i\} + 1$ then also for any piece w that lies in r_i in two different ways we also have $|w| < \frac{1}{p}|r_i|$. Indeed, by Lemma 4.3.6 r_i has the reduced form $gr_i(\bar{s}_i, \bar{t}_i)g^{-1}$ where $g\bar{s}_ig^{-1}, g\bar{t}_ig^{-1}$ are reduced spellings of s_i, t_i respectively with $\bar{s}_i, \bar{t}_i$ cyclically reduced. Let $\bar{x}_i, \bar{y}_i$ be the words that are not proper powers such that $\bar{s}_i = \bar{x}_i^{n_x}$ and $\bar{t}_i = \bar{y}_i^{n_y}$, i.e. neither $\bar{x}_i$ or $\bar{y}_i$ is equal to any of its nontrivial cyclic permutations. Also, by Lemma 4.3.6 $\bar{x}_i^{\pm}, \bar{y}_i^{\pm}$ are not conjugate. Suppose r is disjoint from g, g^{-1}.

Then r is a word in $\bar{x}_i, \bar{y}_i$ and by Lemma 4.3.7 $|w| \leq \left(\max_j\{n_x\alpha_j^i\} + 2\right)|\bar{x}_i| + \left(\max_j\{n_y\beta_j^i\} + 2\right)|\bar{y}_i|$ and so

$$|w| \leq (\max_j\{\alpha_j^i, \beta_j^i\} + 2)(|\bar{s}_i| + |\bar{t}_i|) < 2\min_j\{\alpha_j^i, \beta_j^i\}(|\bar{s}_i| + |\bar{t}_i|) =$$
$$= \frac{1}{p}\left(2p\min_j\{\alpha_j^i, \beta_j^i\}(|\bar{s}_i| + |\bar{t}_i|)\right) < \frac{1}{p}|R_i|.$$

Finally if r overlaps with the prefix g or suffix g^{-1} then r is a subword of $g\bar{s}_i^{\alpha_1^i}\bar{t}_i^{\beta_1^i}$ or $\bar{s}_i^{\alpha_{2p}^i}\bar{t}_i^{\beta_{2p}^i}g^{-1}$. If we choose $\alpha_1^i, \beta_1^i, \ldots, \alpha_{2p}^i, \beta_{2p}^i$ sufficiently large so $\min_j\{\alpha_j^i, \beta_j^i\} > \frac{1}{2}\left(\max_j\{\alpha_j^i, \beta_j^i\} + |g| + 2\right)$ then we have

$$|r| \leq |g| + \max_j\{\alpha_j^i, \beta_j^i\}(|\bar{s}_i| + |\bar{t}_i|) < 2\min_j\{\alpha_j^i, \beta_j^i\}(|\bar{s}_i| + |\bar{t}_i|) < \frac{1}{p}|R_i|.$$

□

4.4. Proof of the main theorem

REMARK 4.4.1. The case $n = 2$ of Theorem 4.1.2 can be deduced from the work of Kar and Sageev who study uniform exponential growth of groups acting freely on CAT(0) square complexes [**KS16**]. They prove that for any two elements a, b there exists a pair of words of length at most 10 in a, b that freely generates a free semigroup, unless $\langle a, b\rangle$ is virtually abelian. One can construct a small cancellation presentation by applying Proposition 4.3.1 to $\mathcal{U} = \{(u, v) \mid |u|, |v| \leq 10 \text{ and } u, v \text{ are not powers of the same element}\}$. The resulting group cannot act properly on a CAT(0) square complex, since for each pair u, v there is a relator which is a positive word in u, v.

Let $R_n(x,y)$ be the union of the following pairs for all $k < n$ and $\ell < \ell' \leq K_3$

$$\begin{aligned}
&(x^{n!}, y^{kn!}x^{n!}), && (x^{-n!}y^{-k}x^{n!}y^{k\ell}, x^{-n!}y^{-k}x^{n!}y^{k\ell'}),\\
&(x^{n!}, y^{-kn!}x^{n!}), && (y^{-k}x^{-n!}y^{k\ell}x^{n!}, y^{-k}x^{-n!}y^{k\ell'}x^{n!}),\\
&(x^{-n!}, y^{kn!}x^{-n!}), && (x^{-n!}, y^{k\ell}x^{-n!}),\\
&(x^{-n!}, y^{-kn!}x^{-n!}), && (x^{n!}, y^{k\ell}x^{n!}).
\end{aligned}$$

Let $\mathcal{R}_1(x,y) = R_1(x,y) \cup R_1(y,x)$. Let

$$\mathcal{R}_n(x,y) = R_n(x,y) \cup R_n(y,x) \cup \mathcal{R}_{n-1}(y^N, x^{-n!}y^N x^{n!}) \cup \mathcal{R}_{n-1}(x^N, y^{-n!}x^N y^{n!})$$

where $N = n!K_3!$.

Lemma 4.4.2 (The Main Lemma). Suppose x and y are hyperbolic isometries of an n-dimensional CAT(0) cube complex. Then one of the following holds:

- one of the pairs in $R_n(x,y)$ freely generates a free semigroup, or
- either y^N and $x^{-n!}y^N x^{n!}$, or x^N and $y^{-n!}x^N y^{n!}$ stabilize a hyperplane, or
- the group $\langle x^N, y^N \rangle$ is virtually abelian.

Proof. Without loss of generality we may assume that the action of $\langle x, y\rangle$ is without hyperplane inversions, as we can always subdivide X to have this property of the action. Let γ_x, γ_y be axes of x, y respectively such that $\mathsf{d}(\gamma_x, \gamma_y)$ is minimal.

Suppose there exists a hyperplane $\boldsymbol{\hbar} \in \mathrm{sk}(x) - \mathrm{sk}(y)$. By Lemma 4.2.3, y does not skewer $x^{in!}\boldsymbol{\hbar}$ for any $i \in \mathbb{Z}$ unless one of the pairs in $\mathcal{R}_n(x,y)$ freely generates a free semigroup. Without loss of generality (by possibly renaming some $x^{in!}\boldsymbol{\hbar}$ as $\boldsymbol{\hbar}$) we can assume that $\gamma_y \subset h^+ \cap x^{n!}h^-$.

If $y^N\boldsymbol{\hbar} = \boldsymbol{\hbar}$ and $y^N x^{n!}\boldsymbol{\hbar} = x^{n!}\boldsymbol{\hbar}$ then the subgroup $\langle y^N, x^{-n!}y^N x^{n!}\rangle$ preserves $\boldsymbol{\hbar}$. We are now assuming that this is not the case, i.e. at least one of $\boldsymbol{\hbar}$ and $x^{n!}\boldsymbol{\hbar}$ is not preserved by y^N.

Suppose that y^N does not stabilize $\boldsymbol{\hbar}$. Let $k \leq n$ be minimal such that $y^k x^{n!}\boldsymbol{\hbar}$ and $x^{n!}\boldsymbol{\hbar}$ are disjoint or equal and let $\ell < \ell' \leq K_3$ such that $\{\boldsymbol{\hbar}, y^{k\ell}\boldsymbol{\hbar}, y^{k\ell'}\boldsymbol{\hbar}\}$ are pairwise disjoint (no two can be equal since y^N does not stabilize $\boldsymbol{\hbar}$). If $y^k x^{n!}\boldsymbol{\hbar} \neq x^{n!}\boldsymbol{\hbar}$, then we have $y^k x^{n!}h^+ \subset x^{n!}h^-$, and thus also $x^{n!}y^k x^{n!}h^+ \subset h^-$. Since $y^{k\ell}h^- \subset h^+$ and $y^{k\ell'}h^- \subset h^+$ there is a ping-pong triple $\{x^{-n!}y^k x^{n!}h^-, y^{k\ell}h^-, y^{k\ell'}h^-\}$. See Figure 4. Now suppose $y^k x^{n!}\boldsymbol{\hbar} = x^{n!}\boldsymbol{\hbar}$. We have $y^{k\ell}h^- \subset x^{n!}h^-$ because $h^- \subset x^{n!}h^-$, and thus $\{x^{n!}h^-, h^-, y^{k\ell}h^-\}$ is a ping-pong triple.

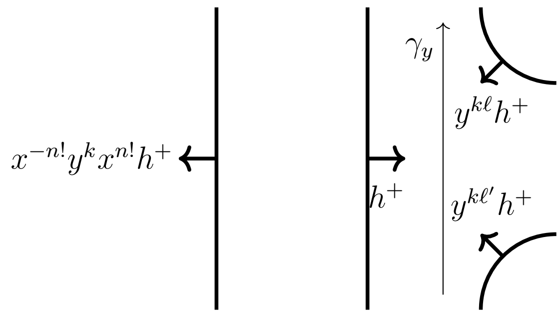

FIGURE 4. The case where $y^N\boldsymbol{h} \neq \boldsymbol{h}$ and $y^kx^{n!}\boldsymbol{h} \neq x^{n!}\boldsymbol{h}$.

Analogously, if y^N does not stabilize $x^{n!}\boldsymbol{h}$ then one of $\{x^{n!}y^kh^+, y^{k\ell}x^{n!}h^+, y^{k\ell'}x^{n!}h^+\}$ and $\{h^+, x^{n!}h^+, y^{k\ell}x^{n!}h^+\}$ is a ping-pong triple for some $k \leq n$ and $\ell < \ell' \leq K_3$.

Similarly, if there exists a hyperplane $\boldsymbol{h} \in \mathrm{sk}(y) - \mathrm{sk}(x)$, then one of the pairs in $\mathcal{R}_n(x, y)$ freely generates a free semigroup or $\langle x^N, y^{-n!}x^Ny^{n!}\rangle$ stabilizes a hyperplane. Otherwise $\mathrm{sk}(x) = \mathrm{sk}(y)$, which we now assume is the case.

Suppose there exists a hyperplane $\boldsymbol{h}$ separating γ_x, γ_y that is not stabilized by either $x^{K_3!}$ or $y^{K_3!}$. Let $k \leq n$ be minimal such that $x^kh \subset h^*$ for appropriate choice of halfspace h of $\boldsymbol{h}$. Let $\ell, \ell' \leq K_3$ such that $\{\boldsymbol{h}, y^{k\ell}\boldsymbol{h}, y^{k\ell'}\boldsymbol{h}\}$ are pairwise disjoint. The triple $\{x^kh, y^{k\ell}h, y^{k\ell'}h\}$ is a ping-pong triple.

We can now assume that every hyperplane separating any two axes of x and y is stabilized by $x^{K_3!}$ or $y^{K_3!}$. If a hyperplane $\boldsymbol{h}$ is stabilized by $x^{K_3!}$ then there are axes of $x^{K_3!}$ in both halfspaces h, h^*. In particular, no hyperplane separates $\mathrm{Min}^0(x^{K_3!})$ and $\mathrm{Min}^0(y^{K_3!})$, hence $\mathrm{Hull}(\mathrm{Min}^0(x^{K_3!})) \cap \mathrm{Hull}(\mathrm{Min}^0(y^{K_3!})) \neq \emptyset$. Let p be a 0-cube in the intersection $\mathrm{Hull}(\mathrm{Min}^0(x^{K_3!})) \cap \mathrm{Hull}(\mathrm{Min}^0(y^{K_3!}))$. By Lemma 4.2.4, p lies on axes of both x^N and y^N. The complex $\mathrm{Hull}(\gamma)$ where γ is an axis of x^N through p is a minimal convex subcomplex containing the $\langle x^N, y^N\rangle$-orbit of p, $\mathrm{Hull}(\gamma)$ is dual to $\mathrm{sk}(x) = \mathrm{sk}(y)$, and $\langle x^N, y^N\rangle$ acts properly on $\mathrm{Hull}(\gamma)$. By Lemma 4.2.4 $\mathrm{Hull}(\gamma)$ embeds in $\mathbb{E}^k$ and by Lemma 4.2.5 the group $\langle x^N, y^N\rangle$ is virtually abelian. □

In the following proof $|w|_*$ denotes the minimal number of syllables of the form $a^{\pm *}, b^{\pm *}$ in a spelling of w.

PROOF OF THEOREM 6.1.1. Let G be a group given by the $C'(1/p')$ presentation from Proposition 4.3.1 with $\mathcal{U} = \mathcal{R}_n(a, b)$ where $p' = \max\{p, 8 \cdot 3^n\}$. In particular, G is an infinite, torsion-free, non-elementary hyperbolic group. Since $p' \geq p$ the group G is $C'(1/p)$. Suppose that G acts properly on an n-dimensional CAT(0) cube complex.

By definition of G none of the pairs in $\mathcal{R}_n(a,b)$ can freely generate a free semigroup since there is a relator in the presentation of G associated to each pair. Also the subgroup $\langle a^N, b^N\rangle$ is not virtually abelian since the presentation of G is $C'(1/6)$, so by Lemma 4.4.2 one of the pairs $b^N, a^{-n!}b^N a^{n!}$ or $a^N, b^{-n!}a^N b^{n!}$ stabilizes a hyperplane and thus these two elements act on an $(n-1)$-dimensional CAT(0) cube complex. Since $R_{n-1}(b^N, a^{-n!}b^N a^{n!}) \subset \mathcal{R}_n(a,b)$ and $R_{n-1}(a^N, b^{-n!}a^N b^{n!}) \subset \mathcal{R}_n(a,b)$ we can apply Lemma 4.4.2 again and we conclude that either one of $\langle b^N, a^{-n!}b^N a^{n!}\rangle$ and $\langle a^N, b^{-n!}a^N b^{n!}\rangle$ is virtually abelian, or an appropriate pair of elements stabilizes a hyperplane. We can keep applying Lemma 4.4.2. As long as the pair of elements x, y stabilizes a hyperplanes, then by Lemma 4.4.2 one of the pairs $y^{N^2}, x^{-n!}y^{N^2}x^{n!}$ or $x^{N^2}, y^{-n!}x^{N^2}y^{n!}$ generates a virtually abelian subgroup or stabilizes a hyperplane. By construction, x and y at each step are some conjugates of one of the the original generators a and b, so $|x^k|_* = |x|_*$ and $|y^k|_* = |y_*|$ for any $k > 0$. Also,

$$|y^{-n!}x^{N^2}y^{n!}|_* \leq |y^{-n!}|_* + |x^{N^2}|_* + |y^{n!}|_* =$$

$$= |y|_* + |x|_* + |y|_* \leq 3\max\{|x|_*, |y|_*\},$$

and similarly $|x^{-n!}y^{N^2}x^{n!}|_* \leq 3\max\{|x|_*, |y|_*\}$. By repeating the argument up to n times, we eventually get a pair of elements x_0, y_0 that generates a virtually abelian subgroup and we have $|x_0|_*, |y_0|_* \leq 3^n$. Since all elements of G have infinite order and G contains no abelian groups of rank 2, we conclude that $\langle x_0, y_0\rangle$ is (virtually) $\mathbb{Z}$. In particular, $x_0^k = y_0^{k'}$ for some $k, k' \neq 0$ and we have $|x_0^k y_0^{-k'}|_* \leq 2\cdot 3^n$. By Greendlinger's Lemma [**LS77**] some subword w of $x_0^k y_0^{-k'}$ must be also a subword of some relator r with $|w| \geq \frac{1}{2}|r|$. On one hand $|w|_* \leq 2\cdot 3^n$. On the other hand, the length of each syllable of the form $a^{\pm *}$ or $b^{\pm *}$ in r is at most $1 + \frac{1}{p'}|r| < \frac{2}{p'}|r|$ because if a^k is a subword of r then a^{k-1} is a piece in r and the same for b. Thus for any subword w' of r of length at most $\frac{|r|}{2}$ we have $|w'|_* > \frac{p'}{4}$. Since $\frac{p'}{4} \geq 2\cdot 3^n$ we get a contradiction. $\square$

CHAPTER 5

Cubulating small cancellation free products

We give a simplified approach to the (cocompact) cubulation of small-cancellation quotients of free products of (cocompactly) cubulated groups. We construct fundamental groups of compact nonpositively curved cube complexes that do not virtually split. This section is based on joint work with Daniel Wise [**JW17a**]. The authors contributed equally.

5.1. Introduction

Martin and Steenbock recently showed that a small-cancellation quotient of a free product of cubulated groups is cubulated [**MS16**]. In this paper we revisit their theorem in a slightly weaker form, and reprove it in a manner that capitalizes on the available technology. Combined with an idea of Pride's about small-cancellation groups that do not split, we answer a question posed to us by Indira Chatterji by constructing an example of a compact nonpositively curved cube complex X such that $\pi_1 X$ is nontrivial but does not virtually split.

Section 5.2 recalls the definitions and theorems that we will use from cubical small-cancellation theory. Section 5.3 recalls properties of the dual cube complex in the relatively hyperbolic setting. Section 5.4 recalls the definition of small-cancellation over free products, and describe associated cubical presentations. Section 5.5 reproves Pride's result about small-cancellation groups that do not split. Section 5.6, relates small-cancellation over free products to cubical small-cancellation theory, and proves our main result which is Theorem 5.6.2. Finally, Section 5.7 combines Pride's method with Theorem 5.6.2 to provide cubulated groups that do not virtually split in Example 5.7.1.

5.2. Background on Cubical Small Cancellation

5.2.1. Nonpositively curved cube complexes. We shall assume that the reader is familiar with *CAT(0) cube complexes* which are CAT(0) spaces having cell structures, where each cell is isometric to a cube. We refer the reader to [**BH99, Sag95, Lea, Wis11**]. A *nonpositively curved cube complex* is a cell-complex X whose universal cover $\widetilde{X}$ is a CAT(0) cube complex. A *hyperplane* $\widetilde{U}$ in $\widetilde{X}$ is a subspace whose intersection with each n-cube $[0,1]^n$ is either empty or consists of the subspace where exactly one coordinate is restricted to $\frac{1}{2}$. For a hyperplane $\widetilde{U}$ of $\widetilde{X}$, we let $N(\widetilde{U})$ denote its *carrier*, which is the union of all closed cubes intersecting $\widetilde{U}$. The hyperplanes $\widetilde{U}$ and $\widetilde{V}$ *osculate* if $N(\widetilde{U}) \cap N(\widetilde{V}) \neq \emptyset$ but $\widetilde{U} \cap \widetilde{V} = \emptyset$. We will use the *combinatorial metric* on a nonpositively curved cube complex X, so the distance between two points is the length of the shortest combinatorial path connecting them. The *systole* $\|X\|$ is the infimal length of an essential combinatorial closed path in X. A map $\phi : Y \to X$ between nonpositively curved cube complexes is a *local isometry* if ϕ is locally injective, ϕ maps open cubes homeomorphically to open cubes, and whenever a, b are concatenatable edges of Y, if $\phi(a)\phi(b)$ is a subpath of the attaching map of a 2-cube of X, then ab is a subpath of a 2-cube in Y.

5.2.2. Cubical presentations and Pieces.

DEFINITION 5.2.1. A *cubical presentation* $\langle X \mid Y_1, \ldots, Y_m \rangle$ consists of a nonpositively curved cube complex X, and a set of local isometries $Y_i \looparrowright X$ of nonpositively curved cube complexes. We use the notation X^* for the cubical presentation above. As a topological space, X^* consists of X with a cone on Y_i attached to X for each i.

DEFINITION 5.2.2. A *cone-piece* of X^* in Y_i is a component of $\widetilde{Y}_i \cap g\widetilde{Y}_j$, where $\widetilde{Y}_i$ is a lift of Y_i to the universal cover $\widetilde{X}^*$ and $g \in \pi_1 X$, excluding the case where $i = j$ and $g \in \operatorname{Stab}(\widetilde{Y}_i)$. A *wall-piece* of X^* in Y_i is a component of $\widetilde{Y}_i \cap N(\widetilde{U})$, where $\widetilde{U}$ is a hyperplane that is disjoint from $\widetilde{Y}_i$. For a constant $\alpha > 0$, we say X^* satisfies the $C'(\alpha)$ *small-cancellation* condition if $\operatorname{diam}(P) < \alpha \|Y_i\|$ for every cone-piece or wall-piece involving Y_i.

When α is small, the quotient $\pi_1 X^*$ has good behaviour. For instance, when X^* is $C'(\frac{1}{12})$ then each immersion $Y_i \looparrowright X$ lifts to an embedding $\widetilde{Y}_i \hookrightarrow \widetilde{X}^*$. This is proven in [**Wis11**, Thm 4.1], and we also refer to [**Jan17a**] for analogous results at $\alpha = \frac{1}{9}$.

5.2.3. The $B(8)$ condition. We now describe a special case of the B(8) condition within the context of $C'(\alpha)$ metric small-cancellation. A *piece-path* in Y is a path in a piece of Y.

DEFINITION 5.2.3. The $B(8)$ *condition* assigns a wallspace structure to each Y_i as follows:

(1) The collection of hyperplanes of each Y_i are partitioned into classes such that no two hyperplanes in the same class cross or osculate, and the union $U = \cup U_i$ of the hyperplanes in a class forms a *wall* in the sense that $Y_i - U$ is the disjoint union of a left and right halfspace.

(2) If P is a path that is the concatenation of at most 8 piece-paths and P starts and ends on the carrier $N(U)$ of a wall then P is path-homotopic into $N(U)$.

(3) The wallspace structure is preserved by the group $\mathrm{Aut}(Y_i \to X)$ which consists of automorphisms $\phi : Y_i \to Y_i$ such that
$$\begin{tikzcd} Y_i \arrow[rr] \arrow[dr] & & Y_i \arrow[dl] \\ & X & \end{tikzcd}$$
commutes.

5.2.4. Properness Criterion. A *closed geodesic* $w \to Y$ in a nonpositively curved cube complex, is a combinatorial immersion of a circle whose universal cover $\tilde{w}$ lifts to a combinatorial geodesic $\tilde{w} \to \widetilde{Y}$ in the universal cover of Y.

A hyperplane U in Y is *piecefully convex* if the following holds: for any path $\xi\rho \to Y$ with endpoints on $N(U)$, if ξ is a geodesic and ρ is trivial or lies in a piece of Y containing an edge dual to U, then $\xi\rho$ is path-homotopic in Y to a path $\mu \to N(U)$. If U is piecefully convex then $N(U) \to Y$ is convex.

We quote the following criterion from [**FW16**]. Each Y_i deformation retracts to a closed combinatorial geodesic w_i. The wallspace that is assigned to each Y_i has a wall for hyperplanes dual to pairs of antipodal edges in w_i. (The complex X is subdivided to ensure that each $|w_i|$ is even.)

Theorem 5.2.4. Let $X^* = \langle X \mid Y_1, \ldots, Y_k \rangle$ be a cubical presentation. Suppose each Y_i deformation retracts to a closed combinatorial geodesic w_i, and each hyperplane U of Y_i is piecefully convex, U intersects w_i and U has an embedded carrier with $\operatorname{diam} N(U) < \frac{1}{20}\|Y_i\|$. If X^* is $C'(\frac{1}{20})$ then X^* is $B(8)$ and $\pi_1 X^*$ acts properly and cocompactly on the CAT(0) cube complex dual to the wallspace on $\widetilde{X^*}$.

Moreover, if each $\langle w_i \rangle \subset \pi_1 X$ is a maximal cyclic subgroup, then $\pi_1 X^*$ acts freely and cocompactly on the associated dual CAT(0) cube complex.

5.2.5. The wallspace structure.

DEFINITION 5.2.5 (The walls). When X^* satisfies the $B(8)$ condition, $\widetilde{X}^*$ has a *wallspace* structure which we now briefly describe: Two hyperplanes H_1, H_2 of $\widetilde{X}^*$ are *cone-equivalent* if $H_1 \cap Y_i$ and $H_2 \cap Y_i$ lie in the same wall of Y_i for some lift $Y_i \hookrightarrow \widetilde{X}^*$. Cone-equivalence generates an equivalence relation on the collection of hyperplanes of $\widetilde{X}^*$. A *wall* of $\widetilde{X}^*$ is the union of all hyperplanes in an equivalence class. When X^* is $B(8)$, the hyperplanes in an equivalence class are disjoint, and a wall w can be regarded as a wall in the sense that $\widetilde{X}^*$ is the union of two halfspaces meeting along w.

LEMMA 5.2.6. Let W be a wall of $\widetilde{X}^*$. let $Y \subset \widetilde{X}^*$ be a lift of some cone Y_i of X^*. Then either $W \cap Y = \emptyset$ or $W \cap Y$ consists of a single wall of Y.

The *carrier* $N(W)$ of a wall W of $\widetilde{X}^*$ consists of the union of all carriers of hyperplanes of W together with all cones intersected by hyperplanes of W. The following appears as [**Wis11**, Cor 5.27]:

LEMMA 5.2.7 (Walls quasi-isometrically embed). Let X^* be $B(8)$. Suppose that pieces in cones have uniformly bounded diameter. Then for each wall W, the map $N(W) \to \widetilde{X}^*$ is a quasi-isometric embedding.

We will need the following result of Hruska which is proven in [**Hru10**, Thm 1.5]:

THEOREM 5.2.8. Let G be a f.g. group that is hyperbolic relative to $\{G_i\}$. Let $H \subset G$ be a f.g. subgroup that is quasi-isometrically embedded. Then $H \subset G$ is relatively quasiconvex.

5.3. Relative Cocompactness

The following is a simplified restatement of [**HW14**, Thm 7.12] in the case $\heartsuit = \star$:

We use the notation $\mathcal{N}_d(S)$ for the closed d-neighbourhood of S.

THEOREM 5.3.1. Consider the wallspace $(\widetilde{X}^*, \mathcal{W})$. Suppose G acts properly and cocompactly on X preserving both its metric and wallspace structures, and the action on $\mathcal{W}$ has only finitely many G–orbits of walls. Suppose $\text{Stab}(W)$ is relatively quasiconvex and acts cocompactly on W for each wall $W \in \mathcal{W}$. Suppose G is hyperbolic relative to $\{G_1, \dots, G_r\}$. For each G_i let $\widetilde{X}_i \subset \widetilde{X}^*$ be a nonempty G_i–invariant G_i–cocompact subspace. Let $C(X)$ be the cube complex dual to $(\widetilde{X}^*, \mathcal{W})$ and for each i let $C_\star(\widetilde{X}_i)$ be the cube complex dual to

$(\widetilde{X}^*, \mathcal{W}_i)$ where $\mathcal{W}_i$ consists of all walls W with the property that $\operatorname{diam}\big(W \cap \mathcal{N}_d(\widetilde{X}_i)\big) = \infty$ for some $d = d(W)$.

Then there exists a compact subcomplex K such that $C(X) = GK \cup \bigcup_i GC_\star(\widetilde{X}_i)$. Hence G acts cocompactly on $C(X)$ provided that each $C_\star(\widetilde{X}_i)$ is G_i-compact.

In our application of Theorem 5.3.1, X is a long wedge of cube complexes $X_1, \ldots, X_r$ (see Construction 5.4.3) and $\widetilde{X}_i$ is a lift of the universal cover of X_i to $\widetilde{X}^*$. The wallspace structure of X^* is described in Section 5.2.5 (see also Lemma 5.4.4). We will be able to apply Theorem 5.3.1 because the cube complex $C_\star(\widetilde{X}_i)$ will be G_i-cocompact for the following reason:

Lemma 5.3.2. *Let W be a wall of $\widetilde{X}^*$. Suppose $\operatorname{diam}(W \cap \mathcal{N}_d(\widetilde{X}_i)) = \infty$ for some i. Then W contains a hyperplane of $\widetilde{X}_i$. Hence $C_\star(\widetilde{X}_i) = \widetilde{X}_i$ for each i.*

Proof. Each $\widetilde{X}_i$ has the property for each $d \geq 0, \lambda \geq 1, \epsilon \geq 0$ there exists d' such that if α is a (λ, ϵ)-quasigeodesic that starts and ends at points in $\mathcal{N}_d(\widetilde{X}_i)$ then $\alpha \subset \mathcal{N}_{d'}(\widetilde{X}_i)$. This follows from the analogous statement for parabolic subgroups of a relatively hyperbolic group which can be deduced from [**DS05**, Thm 1.12.(1)]. The quasigeodesics we consider are contained in $N(W)$ which is quasi-isometrically embedded by Lemma 5.2.7, so there is a uniform value of d' for all such quasigeodesics.

By the fellow-travelling property above, we see that if $\operatorname{diam}(N(W) \cap \mathcal{N}_d(\widetilde{X}_i)) = \infty$ then by cocompactness, there exists an infinite order element g stabilizing both W and $\widetilde{X}_i$. Each $\widetilde{X}_i \subset \widetilde{X}^*$ is convex by [**Wis11**, Lem 3.70], and we may therefore choose a geodesic $\tilde{\gamma}$ be in $\widetilde{X}_i$ that is stabilized by g, and let $\tilde{\lambda}$ be a path in $N(W)$ that is stabilized by g. We thus obtain an annular diagram A between closed paths γ and λ which are the quotients of $\tilde{\gamma}$ and $\tilde{\lambda}$ by $\langle g \rangle$. Suppose moreover that A has minimal complexity among all such choices (A, γ, λ) where $\gamma \to X_i$ has the property that $\widetilde{\gamma}$ is a geodesic, and $\lambda \to N(W)$ is a closed path. By [**Wis11**, Thm 5.53], A is a square annular diagram, and we may assume it is has no spur. (The hypothesis of Thm 5.53 requires "tight innerpaths" which holds at $C'(\frac{1}{16})$ by Lem 3.65.)

Observe that if s is a square with an edge in $\widetilde{X}_i$, then $s \subset \widetilde{X}_i$. Consequently, the minimality of A ensures that A has no square, and so $\gamma = A = \lambda$.

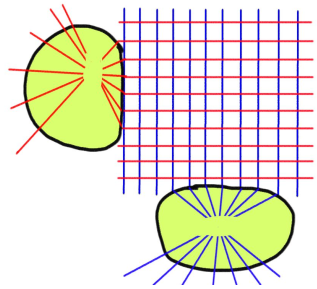

FIGURE 1. The walls associated to a 13-cube in the cubulation of a flat.

There are now two cases to consider: Either $\tilde{\lambda} \subset N(U)$ for some hyperplane U of W, or $\tilde{\lambda}$ has a subpath $u_1 y_j u_2$ traveling along $N(U_1), Y_j, N(U_2)$, where U_1, U_2 are distinct hyperplanes of W, and U_1, U_2 intersect the cone Y_j in antipodal hyperplanes.

In the latter possibility contradicts the $B(8)$ condition for Y_j, since $\widetilde{X}_i \cap Y_j$ contains the single piece-path y_j which starts and ends on carriers of distinct hyperplanes of the same wall of Y_j.

In the former possibility, $N(U) \subset \widetilde{X}_i$, and so the above square observation ensures that $N(U) \subset \widetilde{X}_i$. Hence W cuts $\widetilde{X}_i$ as claimed. □

EXAMPLE 5.3.3. Consider the quotient: $G = \mathbb{Z}^2 * \mathbb{Z}^2 / \langle\langle w_1, w_2 \rangle\rangle$, with the following presentation for some number $m > 0$:

$$\Big\langle \ \langle a, b \mid aba^{-1}b^{-1} \rangle \ * \ \langle c, d \mid cdc^{-1}d^{-1} \rangle \ \Big| \ a^1c^1a^2c^2 \cdots a^m c^m,\ b^1d^1b^2d^2 \cdots b^m d^m \Big\rangle$$

Note that each piece consists of at most 2 syllables, whereas the syllabic length of each relator is $2m$. Hence the $C'_*(\frac{1}{m-1})$ small-cancellation condition over free products is satisfied. See Definition 5.4.1.

So X is the long wedge of two tori X_1, X_2 corresponding to $\langle a, b \rangle$ and $\langle c, d \rangle$. And Y_1 is a bunch of rectangles glued together along arcs.

The cube complex dual to $\widetilde{X}^*$ has $\frac{m(m+1)}{4}$-dimensional cubes arising from the cone-cells Y_1 and Y_2. More interestingly, the cube complex dual to $(\widetilde{X}^*, \mathcal{W}_1)$ where $\mathcal{W}_1$ consists of the walls intersecting a copy of $\widetilde{X}_1$, has dimension $2m$. This is because all hyperplanes dual to the path a^m cross each other because of Y_1 and likewise all hyperplanes dual to the path b^m cross each other because of Y_2, and every hyperplane dual to the path a^m crosses every hyperplane dual to the path b^m because $\widetilde{X}_1$ is a 2-flat.

5.4. Small cancellation over free products

DEFINITION 5.4.1. [Small cancellation over a free product] Every element R in the free product $G_1 * \cdots * G_r$ has a unique *normal form* which is a word $h_1 \cdots h_n$ where each h_i lies in a factor of the free product and h_i and h_{i+1} lie in different factors for $i = 1, \ldots, n-1$. The number n, which we denote by $|R|_*$, is the *syllable length* of R. We say R is *cyclically reduced* if h_1 and h_n also lie in different factors. We say that R is *weakly cyclically reduced* if $h_n^{-1} \neq h_1$ or if $|R|_* \leq 1$. We refer to each h_i as a *syllable*. There is a *cancellation* in the concatenation $P \cdot U$ of two normal forms if the last syllable of P is the inverse of the first syllable of U.

Consider a *presentation over a free product* $\langle G_1 * \cdots * G_r \mid R_1, \ldots, R_s\rangle$ where each R_i is a cyclically reduced word in the free product. A word P is a *piece* of R_i, R_j if they have weakly cyclically reduced conjugates R_i', R_j' that can be written as concatenations $P \cdot U_i$ and $P \cdot U_j$ respectively with no cancellations. The presentation is $C'_*(\frac{1}{n})$ if $|P|_* < \frac{1}{n}|R_i'|_*$ whenever P is a piece.

Each factor G_i embeds in a $C'_*(\frac{1}{6})$ small-cancellation presentation G over a free product $G_1 * \cdots * G_r$ [**LS77**, Cor. 9.4], and thus G is nontrivial if some G_i is nontrivial. We quote the following result from [**Osi06**]:

LEMMA 5.4.2. Let G be a quotient of $G_1 * \cdots * G_r$ arising as a $C'_*(\frac{1}{6})$ small-cancellation presentation over a free product. Then G is hyperbolic relative to $\{G_1, \ldots, G_r\}$.

5.4.1. Cubical presentation associated to a presentation over a free product.

CONSTRUCTION 5.4.3. Let T_r be the union of directed edges $e_1, \ldots, e_r$ identified at their initial vertices. The *long wedge* of a collection of spaces $X_1, \ldots, X_r$ is obtained from T_r by gluing the basepoint of each X_j to the terminal vertex of e_j . We will later subdivide the edges of T_r. Given groups $G_1, \ldots G_r$ such that for each $1 \leq j \leq r$, let $G_j = \pi_1 X_j$ where X_j is a nonpositively curved cube complex, the long wedge X of the various X_j is a cube complex with $\pi_1 X = G_1 * \cdots * G_r$.

Given an element $R \in G_1 * \cdots * G_r$ with $|R|_* > 1$, there exists a local isometry $Y \to X$ where Y is a compact nonpositively curved cube complex with $\pi_1 Y = \langle R\rangle$. Indeed, let

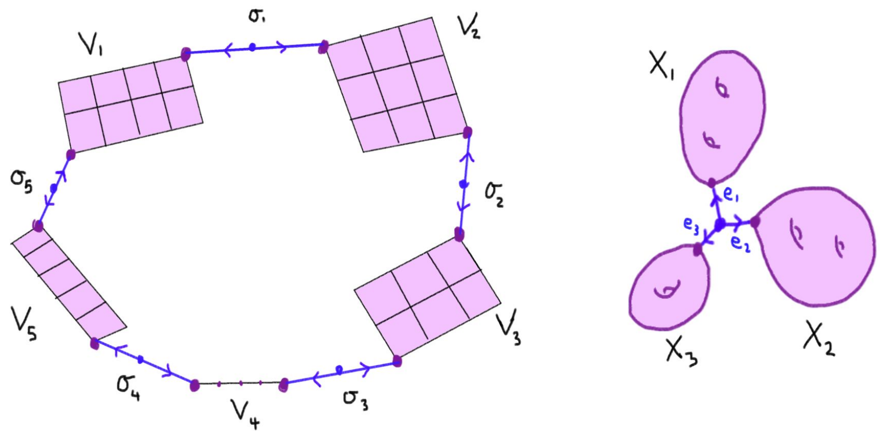

FIGURE 2. Y is depicted on the left and X on the right.

$R = h_1h_2 \cdots h_t$ where each h_k is an element of some $G_{m(k)}$. For each k let V_k be the compact cube complex that is the combinatorial convex hull of the basepoint p and its translate h_kp in the universal cover $\widetilde{X}_{m(k)}$. We call p the *initial vertex* of V_k and h_kp the *terminal vertex* of V_k. For each $1 \le k \le t$ let σ_k be a copy of $e_{m(k)}^{-1}e_{m(k+1)}$ where $m(t+1) = m(1)$. Finally we form Y from $\bigsqcup_{k=1}^t V_k$ and $\bigsqcup_{k=1}^t \sigma_k$ by gluing the terminal vertex of V_k to the initial vertex of σ_k and the terminal vertex of σ_k to the initial vertex of V_{k+1}. Note that there is an induced map $Y \to X$ which is a local isometry.

Given a presentation $\langle G_1, \ldots, G_r \mid R_1, \ldots, R_s\rangle$ over a free product there is an associated cubical presentation $X^* = \langle X \mid Y_1, \ldots, Y_s\rangle$ where each $Y_i \to X$ is a local isometry associated to R_i as above. Finally, any subdivision of the edges $e_1, \ldots, e_r$ induces a subdivision of X, and accordingly a subdivision of each Y_i. We thus obtain a new cubical presentation that we continue to denote by X^*.

LEMMA 5.4.4. Suppose $\langle X \mid Y_1, \ldots, Y_s\rangle$ is $B(8)$ (after subdividing). And let $\widetilde{X}_k$ be the universal cover of X_k with the wallspace structure such that each hyperplane is a wall. Then $\langle X \mid Y_1, \ldots, Y_s, \widetilde{X}_1, \ldots, \widetilde{X}_r\rangle$ is $B(8)$. Moreover, the walls of $\widetilde{X}^*$ with respect to the two structures are identical.

PROOF. The pieces between $\widetilde{X}_i$ and Y_j are copies of the V_k associated to X_i that appear in Y_j, and hence the $B(8)$ properties hold for each Y_j as before. For each $\widetilde{X}_i$, Conditions 5.2.3.(1) and 5.2.3.(3) hold automatically by our choice of wallspace structure, and Condition 5.2.3.(2) holds since $\widetilde{X}_i$ is contractible. $\square$

PROOF. This follows by combining Lemma 5.4.4 and Lemma 5.2.6. □

5.5. Construction of Pride

The following result was proven by Pride in [**Pri83**]. We give a slightly more geometric version of his proof, which was originally stated for a $C(n)$ presentation instead of a classical $C'(\frac{1}{n})$ presentation [**LS77**].

LEMMA 5.5.1. Let $G = \langle x, y \mid R_1, R_2, R_3, R_4, R_5, R_6 \rangle$ where the relators R_i are specified below for associated positive integers $\alpha_i, \beta_i, \gamma_i, \delta_i, \rho_i, \sigma_i, \tau_i, \theta_i$ for each $1 \leq i \leq k$, and $k \geq 1$. Then G does not split as an amalgamated product or HNN extension.

$$R_1(x,y) = xy^{\alpha_1}xy^{\alpha_2}\cdots xy^{\alpha_k}$$

$$R_2(x,y) = yx^{\beta_1}yx^{\beta_2}\cdots yx^{\beta_k}$$

$$R_3(x,y) = x^{\gamma_1}y^{-\delta_1}x^{\gamma_2}y^{-\delta_2}\cdots x^{\gamma_k}y^{-\delta_k}$$

$$R_4(x,y) = xy^{\rho_1}xy^{-\rho_1}xy^{\rho_2}xy^{-\rho_2}\cdots xy^{\rho_k}xy^{-\rho_k}$$

$$R_5(x,y) = yx^{\sigma_1}yx^{-\sigma_1}yx^{\sigma_2}yx^{-\sigma_2}\cdots yx^{\sigma_k}yx^{-\sigma_k}$$

$$R_6(x,y) = (xy)^{\tau_1}(x^{-1}y^{-1})^{\theta_1}(xy)^{\tau_2}(x^{-1}y^{-1})^{\theta_2}\cdots(xy)^{\tau_k}(x^{-1}y^{-1})^{\theta_k}$$

PROOF. Suppose $G = A *_C B$ or $G = A*_C$ and let T be the associated Bass-Serre tree. Without loss of generality, assume that the translation length of y is at least as large as the translation length of x. Choose a vertex $v \in \mathrm{Min}(x)$ for which $\mathsf{d}_T(y{\cdot}v, v)$ is minimal.

For use in the argument below, given a decomposition of $w \in G$ as a product $w = w_1w_2\cdots w_\ell$, the path $[v, w_1{\cdot}v][w_1{\cdot}v, w_1w_2{\cdot}v]\cdots[w_1w_2\cdots w_{\ell-1}{\cdot}v, w{\cdot}v]$ is said to *read* w.

We now show that $v \in \mathrm{Min}(y)$. First suppose that x, and hence y, is a hyperbolic isometry. If $v \notin \mathrm{Min}(y)$, then by the choice of v we have $[v, y{\cdot}v] \cap \mathrm{Min}(x) = \{v\}$, hence the concatenation of two nontrivial geodesics $[x^{-1}{\cdot}v, v][v, y{\cdot}v]$ would be a geodesic. See Figure 3. Similarly $[x{\cdot}v, v][v, y{\cdot}v]$, $[x^{-1}{\cdot}v, v][v, y^{-1}{\cdot}v]$ and $[x{\cdot}v, v][v, y^{-1}{\cdot}v]$ would be geodesics. Consequently, regarding R_6 as a product of elements $\{x^{\pm1}, y^{\pm1}\}$, the path reading R_6 would be a geodesic, which contradicts that $R_6 =_G 1$. Now, suppose that x is elliptic and so $x{\cdot}v = v$. Let e denote the initial edge of $[v, y{\cdot}v]$ and note that e is also the initial edge of $[v, y^{-1}{\cdot}v]$ since $v \notin \mathrm{Min}(y)$. The choice of v implies $x{\cdot}e \neq e$, and so the concatenation of the nontrivial geodesics $[y^{-1}{\cdot}v, v][v, xy{\cdot}v]$ is a geodesic, and similarly for $[y^{-1}{\cdot}v, v][v, x^{-1}y^{-1}v]$,

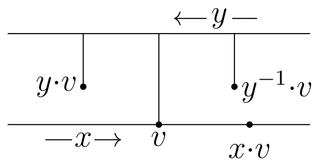

FIGURE 3. The case where $\mathrm{Min}(x) \cap \mathrm{Min}(y) = \emptyset$.

$[y{\cdot}v, v][v, xy{\cdot}v]$ and $[y{\cdot}v, v][v, x^{-1}y^{-1}v]$. It follows that regarding R_6 as a product of elements $\{xy, x^{-1}y^{-1}\}$, the path reading R_6 is a geodesic, which contradicts that $R_6 =_G 1$.

Since $v \in \mathrm{Min}(x) \cap \mathrm{Min}(y)$, the element y is a hyperbolic isometry, because otherwise x, y are elliptic and so v is be a global fixed point. Suppose x is also a hyperbolic isometry. At least one of $[y^{-1}{\cdot}v, v][v, x{\cdot}v]$ or $[x^{-1}{\cdot}v, v][v, y{\cdot}v]$ is not a geodesic, because otherwise the path reading R_1 regarded as a product of $\{x^{\pm 1}, y^{\pm 1}\}$ would be a geodesic. Consequently, both $[x{\cdot}v, v][v, y{\cdot}v]$ and $[x^{-1}{\cdot}v, v][v, y^{-1}{\cdot}v]$ are geodesics, and hence regarding R_3 as a product of elements $\{x^{\pm 1}, y^{\pm 1}\}$, the path reading R_3 must be a geodesic, which is a contradiction. Thus, x is an elliptic isometry.

Let e_+ and e_- denote the initial edges of $[v, y{\cdot}v]$ and $[v, y^{-1}{\cdot}v]$ respectively. See Figure 4. We must have $x{\cdot}e_+ = e_-$ because otherwise $[y^{-1}{\cdot}v, v][v, xy{\cdot}v]$ would be a geodesic since the last edge of $[y^{-1}{\cdot}v, v]$ is e_- and the first edge of $[v, xy{\cdot}v]$ is e_+. Likewise, for $n, m > 0$ the path $[y^{-n}{\cdot}v, v][v, xy^m{\cdot}v]$ would be a geodesic, and so too would be its translate $[v, xy^n{\cdot}v][xy^n{\cdot}v, xy^nxy^m{\cdot}v]$. Finally, regarding R_1 as a product $(xy^{\alpha_1})(xy^{\alpha_2})\cdots(xy^{\alpha_k})$, the path reading R_1 would be a geodesic, contradicting $R_1 =_G 1$.

Neither e_- nor e_+ is fixed by x. For any $n, m > 0$ the last edge of $[y^n{\cdot}v, v]$ is e_+ and the first edge of $[v, xy^m{\cdot}v]$ is $x{\cdot}e_+ = e_- \neq e_+$, and so the path $[y^n{\cdot}v, v][v, xy^m{\cdot}v]$ is a geodesic, and so is $[v, y^{-n}{\cdot}v][y^{-n}{\cdot}v, y^{-n}xy^m{\cdot}v]$. Similarly, the last edge of $[y^{-n}{\cdot}v, v]$ is e_- and the first edge of $[v, xy^{-m}{\cdot}v]$ is $x{\cdot}e_- \neq e_-$, and so the path $[y^{-n}{\cdot}v, v][v, xy^{-m}{\cdot}v]$ is a geodesic as is $[v, xy^n{\cdot}v][xy^n{\cdot}v, xy^n{\cdot}xy^{-m}{\cdot}v]$. Regarding R_4 as a product $(xy^{\rho_1})(xy^{-\rho_1})\cdots(xy^{\rho_k})(xy^{-\rho_k})$, the path reading R_4 is a geodesic, contradicting $R_4 =_G 1$. $\square$

REMARK 5.5.2. In the context of Lemma 5.5.1, for each n there are choices of k and $\{\alpha_i, \beta_i, \gamma_i, \delta_i, \rho_i, \sigma_i, \tau_i, \theta_i \ : \ 1 \leq i \leq k\}$, such that the presentation is $C'(\frac{1}{n})$.

Given $n > 1$, let $k = 3n$ and choose $8k$ numbers $\alpha_i, \beta_i, \gamma_i, \delta_i, \rho_i, \sigma_i, \tau_i, \theta_i$ that are all different and lie between $50n$ and $75n$. Then any piece P in R_i where $i \neq 6$ is of the form

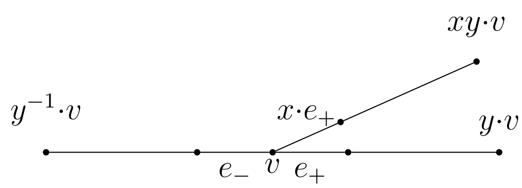

FIGURE 4. If $x{\cdot}e_+ \neq e_-$ then $[y^{-1}{\cdot}v, v][v, xy{\cdot}v]$ is a geodesic.

$x^l y x^m$ or $y^l x y^m$ for some l, m (possibly 0). Thus $|P| \leq l + m + 1 \leq 150n + 1$. We also have $|R_i| \geq (k+1)50n = (3n+1)50n$ and so $|P| \leq \frac{1}{n}(150n+1)n \leq \frac{1}{n}|R_i|$. If P is a piece in R_6, then P is of the from $(xy)^l(x^{-1}y^{-1})^m$ and so $|P| \leq 2(l+m) \leq 300n$. We also have $|R_6| = 2(\tau_1 + \theta_1 + \tau_2 + \cdots + \theta_k) \geq 2(2k)50n = 600n^2$. Hence $|P| \leq \frac{1}{n}|R_6|$.

COROLLARY 5.5.3. Let $G_1, \ldots, G_r$ be nontrivial groups generated by finite sets of infinite order elements, and suppose $r > 1$. For each $n > 0$ there is a finitely related $C'_*(\frac{1}{n})$ quotient G of $G_1 * \cdots * G_r$ that does not split.

PROOF. Let S_p be the given generating set of G_p for each p, and assume no proper subset of S_p generates G_p. The desired quotient G arises from a presentation $\langle G_1 * \cdots * G_r \mid \mathcal{R}\rangle$, where following Lemma 5.5.1, the set of relators is:

$$\mathcal{R} \;=\; \{\; R_\ell(x,y) \;:\; 1 \leq \ell \leq 6,\; (x,y) \in S_p \times S_q,\; \text{where}\;\; 1 \leq p < q \leq r\}$$

where $k(x,y) = 3n$ for each (x,y) and where the constants $\alpha_i(x,y)$, $\beta_i(x,y)$, $\gamma_i(x,y)$, $\delta_i(x,y)$, $\rho_i(x,y)$, $\sigma_i(x,y)$, $\tau_i(x,y)$, $\theta_i(x,y)$ will be described below. For each (x,y) let $\alpha_i(x,y)$, $\delta_i(x,y)$ and $\rho_i(x,y)$ be distinct integers > 1 and such that $y^m \notin \langle z\rangle$ for $m \in \{\alpha_i(x,y), \delta_i(x,y), \rho_i(x,y)\}$ and $z \in S_q - \{y\}$. This is possible because y has infinite order and $y \notin \langle z\rangle$. Similarly, let $\beta_i(x,y)$, $\gamma_i(x,y)$ and $\sigma_i(x,y)$ be distinct integers > 1 such that $x^m \notin \langle z\rangle$ for $m \in \{\beta_i(x,y), \gamma_i(x,y), \sigma_i(x,y)\}$ and $z \in S_p - \{x\}$. Finally, let $\tau_i(x,y)$ and $\theta_i(x,y)$ be distinct integers between $10n$ and $20n$.

Having chosen the above constants for each (x,y) we now show that the presentation for G is $C'_*(\frac{1}{n})$. We begin by observing that each $|R_\ell(x,y)|_* \geq 6n$. Let P be a piece in $R^1 = R_{\ell_1}(x_1,y_1)$ and $R^2 = R_{\ell_2}(x_2,y_2)$ where $x_1 \in S_{p_1}$, $y_1 \in S_{q_1}$, $x_2 \in S_{p_2}$, and $y_2 \in S_{q_2}$. If $\{p_1,q_1\} \neq \{p_2,q_2\}$ then $|P|_* \leq 1$. Assume that $\{p_1,q_1\} = \{p_2,q_2\}$. First suppose that $\ell_1 \neq 6$, then $|P|_* \leq 3$. Indeed, if $|P|_* \geq 4$ then two consecutive syllables would appear in distinct cyclically reduced forms of relators, which contradicts our choice of the constants. If $\ell_1 = 6$, then $|P|_* \leq \max\{\tau_i(x,y)\} + \max\{\theta_i(x,y)\} \leq 80n$. We also have $|R_6(x,y)|_* = 2\left(\tau_1(x,y) + \theta_1(x,y) + \cdots + \tau_k(x,y) + \theta_k(x,y)\right) \geq 2(2k)10n = 120n^2$, so $|P|_* \leq \frac{1}{n}|R_6(x,y)|_*$.

We now show that G does not split as an amalgamated product. For each $x \in S_p, y \in S_q$ with $p \leq q$ we let $H(x,y) = \langle x, y \mid R_\ell(x,y) : 1 \leq \ell \leq 6\rangle$. By Lemma 5.5.1, we see that $H(x,y)$ does not split. As there is a homomorphism $H(x,y) \to G$, we deduce that for any splitting of G as an amalgamated free product $G = A *_C B$, the elements x, y are either both in A or both in B. Considering all such pairs (x,y), we find that the generators of G are either all in A or all in B. Moreover G cannot split as an HNN extension, since the the relators $R_4(x,y)$ and $R_5(x,y)$ show that all generators have finite order in the abelianization of G. $\square$

5.6. Main theorem

LEMMA 5.6.1. If $\langle G_1, \ldots, G_r \mid R_1, \ldots, R_s\rangle$ is $C'_*(\frac{1}{n})$ then for a sufficient subdivision of $e_1, \ldots, e_r$ the cubical presentation X^* is $C'(\frac{1}{n})$.

PROOF. Let X' be a subdivision of X induced by a q-fold subdivision of each e_j. We accordingly let Y_i' be the induced subdivision of Y_i, so $Y_i' = \bigsqcup V_k \cup \bigsqcup \sigma_k$ as in Construction 5.4.3 and with each σ-edge subdivided q times. We thus obtain a new cubical presentation $\langle X' \mid Y_1', \ldots, Y_s'\rangle$. We have $\|Y_i'\| = \|Y_i\| + 2|R_i|_*(q-1)$. Note that $\|Y_i'\| > \sum_{i=1}^{|R_i|_*} |\sigma_i| = 2q|R_i|_*$ and so $\|Y_i'\| > 2(1+\epsilon)q|R_i|_*$ for sufficiently small $\epsilon > 0$. Let $M_i = \max_k \{\mathrm{diam}(V_k)\}$. For a wall-piece P we have $\mathrm{diam}(P) < M_i$. Consider a maximal cone-piece P in Y_i', and suppose it intersects ℓ different V_k's and contains ℓ' different e_k edges. Note that $2\ell \geq \ell'$ since if P starts or ends with an entire σ_k arc, then it intersects an additional V_k (possibly trivially). We have $\mathrm{diam}(P) \leq \ell M_i + q\ell'$. When $\ell' > 0$, for any $\epsilon > 0$ we can choose $q \gg 0$ so that $\mathrm{diam}(P) < (1+\epsilon)q\ell'$. Since P corresponds to a length ℓ syllable piece, the $C'_*(\frac{1}{n})$ hypothesis implies that $\ell < \frac{1}{n}|R_i|_*$, and so $\mathrm{diam}(P) < (1+\epsilon)q\ell' < 2(1+\epsilon)q(\frac{1}{n}|R_i|_*) < \frac{1}{n}\|Y_i'\|$. When $\ell' = 0$, then assuming $q > nM_i$ we have $\mathrm{diam}(P) \leq M_i < 2\frac{q}{n}|R_i|_* < \frac{1}{n}\|Y_i'\|$. $\square$

THEOREM 5.6.2. Suppose $G = \langle G_1, \ldots, G_r \mid R_1, \ldots, R_s\rangle$ satisfies $C'(\frac{1}{20})$. If each G_i is the fundamental group of a [compact] nonpositively curved cube complex, then G acts properly [and cocompactly] on a CAT(0) cube complex. Moreover, if no R_i is a proper power, then G is the fundamental group of a [compact] nonpositively curved cube complex.

PROOF. Let X^* be the associated cubical presentation. Lemma 5.6.1 asserts that X^* is $C'(\frac{1}{20})$ after a sufficient subdivision. For each hyperplane U in Y_i we have $\operatorname{diam} N(U) < \frac{1}{20}\|Y_i\|$ if the subdivision is sufficient. Since a 1-piece neighborhood of $N(U)$ has diameter much smaller than $\frac{1}{2}\|Y_i\|$, every path contained in such neighborhood with the endpoints in $N(U)$ is homotopic to a path in $N(U)$. Therefore every hyperplane U is Y_i is piecefully convex. Theorem 5.2.4 asserts that $\pi_1 X^*$ acts freely (or with finite stabilizers if relators are proper powers) on a CAT(0) cube complex C dual to $\widetilde{X}^*$

Let X'^* be the cubical presentation $\langle X \mid \{Y_i\}, \{\widetilde{X}_j\}\rangle$. By Lemma 5.4.4, X'^* satisfies $B(8)$ with our previously chosen wallspace structure on each Y_i and the hyperplane wallspace structure on each $\widetilde{X}_j$. Thus by Lemma 5.2.6 each $\widetilde{X}_j$ in $\widetilde{X}^* = \widetilde{X}'^*$ intersects the walls of $\widetilde{X}^*$ in hyperplanes of $\widetilde{X}_j$.

Lemma 5.4.2 asserts that $\pi_1 X^*$ is hyperbolic relative to $\{G_1, \ldots, G_r\}$.

The pieces in $X^* = \langle X \mid \{Y_i\}\rangle$ are uniformly bounded since $\operatorname{diam}(Y_i)$ is uniformly bounded. Thus $N(W) \to \widetilde{X}^*$ is quasi-isometrically embedded by Lemma 5.2.7. Hence $\operatorname{Stab}(N(W))$ is relatively quasiconvex with respect to $\{\pi_1 X_j\}$ by Theorem 5.2.8.

Theorem 5.3.1 asserts that $\pi_1 X^*$ acts relatively cocompactly on C. Lemma 5.3.2 asserts that each $C_\star(\widetilde{X}_i) = \widetilde{X}_i$. Hence if each X_i is compact, we see that C is compact. $\square$

5.7. A cubulated group that does not virtually split

Examples were given in [**Wis11**] of a compact nonpositively curved cube complex X such that X has no finite cover with an embedded hyperplane. It is conceivable that those groups have no (virtual) splitting, but this was not confirmed there.

EXAMPLE 5.7.1. There exists a nontrivial group G with the following two properties:

(1) $G = \pi_1 X$ where X is a compact nonpositively curved cube complex.
(2) G does not have a finite index subgroup that splits as an amalgamated product or HNN extension.

Let G_1 be the fundamental group of X_1 which is a compact nonpositively curved cube complex with a nontrivial fundamental group but no nontrivial finite cover. For instance, such complexes were constructed in [**Wis96**] or [**BM97**].

By Corollary 5.5.3 there exists a $C'_*(\frac{1}{20})$ quotient G of the free product $G_1 * \cdots * G_1$ of r copies of G_1, such that G does not split. The group G has no finite index subgroups since $G_1 * \cdots * G_1$ has none.

Since $G_1 = \pi_1 X_1$, by Theorem 6.1.1, G is the fundamental group of a compact nonpositively curved cube complex.

CHAPTER 6

Cocompactly cubulated 2-dimensional Artin groups

We give a necessary and sufficient condition for a 2-dimensional or a three-generator Artin group A to be (virtually) cocompactly cubulated, in terms of the defining graph of A. This section is based on joint work with Jingyin Huang and Piotr Przytycki [**HJP16**]. The authors contributed equally.

6.1. Introduction

We say that a group is (*cocompactly*) *cubulated* if it acts properly (and compactly) by combinatorial automorphisms on a CAT(0) cube complex. We say that a group is *virtually cocompactly cubulated*, if it has a finite index subgroup that is cocompactly cubulated. Such groups either fail to have Kazhdan's property (T) or are finite [**NR97**], are bi-automatic [**Ś06**], satisfy the Tits Alternative [**SW05**] and, if cocompactly cubulated, they satisfy rank-rigidity [**CS11**]. For more background on CAT(0) cube complexes, see the survey article of Sageev [**Sag14**].

The *Artin group* with generators s_i and exponents $m_{ij} = m_{ji} \geq 2$, where $i \neq j$, is presented by relations $\underbrace{s_is_js_i\cdots}_{m_{ij}} = \underbrace{s_js_is_j\cdots}_{m_{ij}}$. Here $\underbrace{s_is_js_i\cdots}_{m_{ij}}$ denotes the first half of the word $(s_is_j)^{m_{ij}}$. The *defining graph* of an Artin group has vertices corresponding to s_i and edges labeled m_{ij} between s_i and s_j whenever $m_{ij} < \infty$.

Artin groups that are *right-angled* (i.e. the ones with $m_{ij} \in \{2, \infty\}$) are cocompactly cubulated, and they play a prominent role in theory of special cube complexes of Haglund and Wise [**HW08**]. However, much less is known about other Artin groups, in particular about braid groups. In [**Wis11**] Wise suggested an approach to cubulating Artin groups using cubical small cancellation. However, we failed to execute this approach: we were not able to establish the B(6) condition.

In this chapter we consider Artin groups that have three generators, or are 2-*dimensional*, that is, their corresponding Coxeter groups have finite special subgroups of maximal rank

2 (or, equivalently, 2-dimensional Davis complex). We characterize when such a group is virtually cocompactly cubulated. This happens only for very rare defining graphs. An *interior* edge of a graph is an edge that is not a leaf.

THEOREM 6.1.1. Let A be a 2-dimensional Artin group. Then the following are equivalent.

(i) A is cocompactly cubulated,
(ii) A is virtually cocompactly cubulated,
(iii) each connected component of the defining graph of A is either
 - a vertex, or an edge, or else
 - all its interior edges are labeled by 2 and all its leaves are labelled by even numbers.

Moreover, if A is an arbitrary Artin group, then (iii) implies (i).

THEOREM 6.1.2. Let A be a three-generator Artin group. Then the following are equivalent.

(i) A is cocompactly cubulated,
(ii) A is virtually cocompactly cubulated,
(iii) the defining graph of A is as in Theorem 6.1.1(iii) or has two edges labelled by 2.

6.1.1. Remarks. From Theorem 6.1.2 it follows that the 4-strand braid group is not virtually cocompactly cubulated.

Note that, independently, Thomas Haettel [**Hae17**] has obtained a full classification of cocompactly cubulated Artin groups.

The equivalence of (i) and (ii) has no counterpart for Coxeter groups, where the group $\widetilde{A}_2$ generated by reflections in the sides of an equilateral triangle in $\mathbb{R}^2$ is virtually cocompactly cubulated, but not cocompactly cubulated.

There are Artin groups that do not satisfy the equivalent conditions from Theorem 6.1.1, but are cubulated. Namely, it follows from [**Bru92, HM99**] that if the defining graph of A is a tree, then A is the fundamental group of a link complement that is a graph manifold with boundary. Hence by the work of Liu [**Liu13**] or Przytycki and Wise [**PW14**] the Artin group A is cubulated.

Artin groups of *large type*, that is, with all $m_{ij} \geq 3$ are 2-dimensional. For many of them Brady and McCammond constructed 2-dimensional CAT(0) complexes with proper and cocompact action [**BM00**]. However, these complexes are built of triangles, not squares.

6.1.2. Some historical background. Sageev invented a way of *cubulating* groups (i.e. showing that they are cubulated) using codimension 1-subgroups [**Sag95**], which was later also explained in the language of *walls* in the Cayley complex of the group [**CN05, Nic04**]. Here we give a brief account on some cubulation results, for a more complete one see [**HW14**].

Using the technology of walls, Niblo and Reeves cubulated Coxeter groups [**NR03**], then Williams [**Wil99**] and Caprace and Mühlherr [**CM05**] analyzed when this cubulation is cocompact. It is not known if all Coxeter groups are virtually cocompactly cubulated. Wise cocompactly cubulated small cancellation groups [**Wis04**], and Ollivier and Wise cocompactly cubulated random groups at density $< \frac{1}{6}$ [**OW11**].

Furthermore, using the surfaces of Kahn and Markovic, Bergeron and Wise cocompactly cubulated the fundamental groups of closed hyperbolic 3-manifolds [**KM12, BW12**], and later Wise cocompactly cubulated the fundamental groups of compact hyperbolic 3-manifolds with boundary [**Wis11**]. Hagen and Wise cocompactly cubulated hyperbolic free-by-cyclic groups [**HW15**].

Groups that are not (relatively) hyperbolic are harder to cubulate cocompactly. Przytycki and Wise cubulated the fundamental groups of all compact 3-dimensional manifolds that are not graph manifolds, as well as graph manifolds with boundary [**PW14, PW17**]. In [**Liu13**] Liu gave a criterion for a graph manifold fundamental group to be virtually cubulated *specially* (meaning that the quotient of the action admits a local isometry into the Salvetti complex of a right-angled Artin group), but we do not know if this is equivalent to just being cubulated. Hagen and Przytycki gave a criterion for a graph manifold fundamental group to be cocompactly cubulated [**HP15**]. In general, it is difficult to find obstructions for groups to be cubulated. Another result of this type is Wise's characterization of tubular groups that are cocompactly cubulated [**Wis14**].

6.1.3. Proof outline for (i)⇒(iii) in Theorem 6.1.1. Given a 2-dimensional Artin group acting properly and cocompactly on a CAT(0) cube complex, we show that its two-generator special subgroups are convex cocompact. More precisely, each of them acts cocompactly on a convex subcomplex which naturally decomposes as a product of a vertical factor and a horizontal factor. Geometrically, the intersection of two such subgroups is either vertical or horizontal. However, if Theorem 6.1.1(iii) is not satisfied, then this intersection is neither vertical nor horizontal by algebraic considerations.

One of the ingredients of the proof is Theorem 6.3.8, which asserts that a top rank product of hyperbolic groups acting on a CAT(0) cube complex is always convex cocompact.

6.1.4. Organization. In Section 6.2 we give some background on CAT(0) spaces and CAT(0) cube complexes. Section 6.3 is devoted to the proof of Theorem 6.3.8. In Section 6.4 we give some background on Artin groups and discuss some algebraic properties of two-generator Artin groups. Finally, in Section 6.5 we prove Theorem 6.1.1 and in Section 6.6 we prove Theorem 6.1.2.

6.2. Preliminaries

A group is a CAT(0) *group* if it acts properly and cocompactly on a CAT(0) space. We assume the reader is familiar with the basics of CAT(0) spaces and groups. For background, see [**BH99**]. In this section we collect some less classical results.

6.2.1. Asymptotic rank. The following definition was introduced in [**Kle99**].

Definition 6.2.1. Let X be a CAT(κ) space. For $x \in X$ we denote by $\Sigma_x X$ the CAT(1) space that is the completion of the space of directions at x [**BH99**, Definition II.3.18]. The *geometric dimension* of X, denoted $\mathrm{GeomDim}(X)$ is defined inductively as follows.

- $\mathrm{GeomDim}(X) = 0$ if X is discrete,
- $\mathrm{GeomDim}(X) \leq n$ if $\mathrm{GeomDim}(\Sigma_x X) \leq n-1$ for any $x \in X$.

Definition 6.2.2. Let X be a CAT(0) space. Then its *asymptotic rank*, denoted by $\mathrm{asrk}(X)$, is the supremum of the geometric dimension of the asymptotic cones of X.

Theorem 6.2.3. Let X and Y be CAT(0) spaces. Then

(1) $\mathrm{asrk}(X \times Y) \geq \mathrm{asrk}(X) + \mathrm{asrk}(Y)$,

(2) if $\operatorname{asrk}(X) \leq 1$, then X is hyperbolic.

The first assertion follows from Theorem A of [**Kle99**] and the second assertion follows from Corollary 1.3 of [**Wen11**].

Definition 6.2.4. If G is a CAT(0) group acting properly and cocompactly on a CAT(0) space X, then the *asymptotic rank* of G is the asymptotic rank of X. By [**Kle99**, Theorem C] this is the maximal n for which there is a quasi-isometric embedding $\mathbb{R}^n \to X$. Hence it does not depend on the choice of the CAT(0) space X.

Lemma 6.2.5. Suppose that G is a CAT(0) group, and that G acts properly and cocompactly on a contractible n-dimensional cell complex X (not necessarily CAT(0)). Then the asymptotic rank of G is $\leq n$.

Proof. Choose any G-equivariant length metric on X. We will prove that there does not exist a quasi-isometric embedding $f : \mathbb{R}^k \to X$ for $k > n$. Otherwise, since X is contractible and admits a cocompact action of G, we can assume that f is a continuous quasi-isometry: such f can be defined by induction on consecutive skeleta of the standard cubical subdivision of $\mathbb{R}^k$.

Let $Y \subseteq X$ be the smallest subcomplex containing $f(\mathbb{R}^k)$. Then $f : \mathbb{R}^k \to Y$ is a quasi-isometry. Let $g : Y \to \mathbb{R}^k$ be a quasi-isometry inverse to f, we can again assume that g is continuous. For any $x \in \mathbb{R}^k$ the distance $d(g \circ f(x), x)$ is uniformly bounded and consequently there is a proper geodesic homotopy between $g \circ f$ and the identity map.

Recall that for a topological space X we can consider *locally finite chains* in X, which are formal sums $\Sigma_{\lambda \in \Lambda} a_\lambda \sigma_\lambda$ where a_λ are integers, σ_λ are singular simplices, and any compact set in X intersects the images of only finitely many σ_λ with $a_\lambda \neq 0$. This gives rise to *locally finite homology* of X, denoted by $H_*^{\mathrm{lf}}(X)$. Moreover, proper maps induce homomorphisms on locally finite homology. See [**BKS16**, Section 2.2] for more discussion.

Since there is a proper geodesic homotopy between $g \circ f$ and the identity map, $g \circ f$ induces the identity on $H_*^{\mathrm{lf}}(\mathbb{R}^k)$, and consequently $f_*\colon H_k^{\mathrm{lf}}(\mathbb{R}^k) \to H_k^{\mathrm{lf}}(Y)$ is injective. This leads to a contradiction, since $H_k^{\mathrm{lf}}(\mathbb{R}^k)$ contains the fundamental class $[\mathbb{R}^k]$ which is a nontrivial element, while $H_k^{\mathrm{lf}}(Y) = 0$ since $\dim(Y) < k$. $\square$

6.2.2. Gate and parallel set. All CAT(0) cube complexes in this chapter are finite-dimensional. Throughout this paper the only metric that we consider on a CAT(0) cube complex X is the CAT(0) metric d. The *convex hull* of a subspace $Y \subseteq X$ is the smallest convex subspace containing Y, and is not necessarily a subcomplex, while the *combinatorial convex hull* of Y is the smallest convex subcomplex of X containing Y. For a complete convex subspace $Y \subseteq X$ we denote by $\pi_Y : X \to Y$ the closest point projection onto Y.

The following lemma was proved in slightly different contexts by various authors [**BHS17, Hua17b, BKS08, AB08**]:

Lemma 6.2.6. [**Hua17b**, Lemma 2.10] Let X be a CAT(0) cube complex of dimension n, and let Y_1, Y_2 be convex subcomplexes. Let $\Delta = d(Y_1, Y_2)$, $V_1 = \{y \in Y_1 \mid d(y, Y_2) = \Delta\}$ and $V_2 = \{y \in Y_2 \mid d(y, Y_1) = \Delta\}$. Then:

(1) V_1 and V_2 are nonempty convex subcomplexes.
(2) π_{Y_1} maps V_2 isometrically onto V_1 and π_{Y_2} maps V_1 isometrically onto V_2. Moreover, the convex hull of $V_1 \cup V_2$ is isometric to $V_1 \times [0, \Delta]$.
(3) for every $\epsilon > 0$ there exists $\delta = \delta(\Delta, n, \epsilon) > 0$ such that if $y_1 \in Y_1$, $y_2 \in Y_2$ and $d(y_1, V_1) \geq \epsilon$, $d(y_2, V_2) \geq \epsilon$, then
$$d(y_1, Y_2) \geq \Delta + \delta d(y_1, V_1)\,,\ d(y_2, Y_1) \geq \Delta + \delta d(y_2, V_2)\,.$$

We call $V_1 \subseteq Y_1$ the *gate with respect to* Y_2, and $V_2 \subseteq Y_2$ the *gate with respect to* Y_1. We write $\mathcal{G}(Y_1, Y_2) = (V_1, V_2)$. We say that Y_1, Y_2 are *parallel* if $\mathcal{G}(Y_1, Y_2) = (Y_1, Y_2)$.

Lemma 6.2.7 ([**Hua17a**, Lemma 2.9]). Let X be a CAT(0) cube complex, and let $(V_1, V_2) = \mathcal{G}(Y_1, Y_2)$ for some convex subcomplexes $Y_1, Y_2 \subseteq X$. Let e be an edge in V_1 and let $\boldsymbol{h}$ be the hyperplane dual to e. Then $\boldsymbol{h} \cap V_2 \neq \emptyset$.

Lemma 6.2.8 ([**CS11**, Lemma 2.5]). A decomposition of a CAT(0) cube complex as a product of CAT(0) cube complexes corresponds to a partition $\mathcal{H}_1 \sqcup \mathcal{H}_2$ of the collection of hyperplanes of X such that every hyperplane in $\mathcal{H}_1$ intersects every hyperplane in $\mathcal{H}_2$.

The following lemma was also proved in [**BHS17**, Lemma 2.4].

Lemma 6.2.9. Let X be a CAT(0) cube complex and let $Y \subseteq X$ be a convex subcomplex. Let $\{Y_\lambda\}_{\lambda \in \Lambda}$ be the collection of all convex subcomplexes that are parallel to Y. Then the combinatorial convex hull P_Y of $\bigcup_{\lambda\in\Lambda} Y_\lambda$ admits a natural product decomposition $P_Y = Y \times Y^\perp$.

P_Y is called the *combinatorial parallel set* of Y.

Proof. Let $\mathscr{H}$ be the collection of hyperplanes in X that separate some points in $\bigcup_{\lambda\in\Lambda} Y_\lambda$ and let $\boldsymbol{h} \in \mathscr{H}$. We claim that either h intersects all Y_λ or it is disjoint from all Y_λ. Indeed, we have $\mathcal{G}(Y, Y_\lambda) = (Y, Y_\lambda)$ for all $\lambda \in \Lambda$. It follows from Lemma 6.2.7 that if h intersects some Y_λ, then it intersects Y, and hence it intersects all Y_λ.

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be the collections of hyperplanes satisfying the first assertion and the second assertion in the claim, respectively. For any $\boldsymbol{h} \in \mathscr{H}_2$, there exist $\lambda, \lambda' \in \Lambda$ such that h separates Y_λ from $Y_{\lambda'}$. Thus $\boldsymbol{h}$ intersects every hyperplane in $\mathscr{H}_1$. Note that $\mathscr{H}$ is the collection of hyperplanes that intersect P_Y and $\mathscr{H}_1$ is the collection of hyperplanes that intersect Y. Thus by Lemma 6.2.8, P_Y admits a product decomposition $P_Y = Y \times Y^\perp$. $\square$

6.3. Cocompact cores

The main goal of this section is to prove Theorem 6.3.8 on existence of cocompact cores for top rank products of hyperbolic groups. The first step towards it is to study flats in a CAT(0) cube complex, which we do in Section 6.3.1. A hurried reader can proceed directly to Section 6.3.2 and use [**WW17**, Theorem 2.6] instead. However, our Theorem 6.3.4 is of independent interest.

6.3.1. Combinatorial convex hull of a flat. Throughout this paper a *flat* is a CAT(0) flat, i.e. an isometrically embedded copy of $\mathbb{R}^n$, not necessarily combinatorial. A *half-flat* is an isometrically embedded copy of $\mathbb{R}^{n-1} \times [0, \infty)$.

Lemma 6.3.1. Let X be a CAT(0) cube complex and let $F \subseteq X$ be a flat. Let $\boldsymbol{h}$ be a hyperplane in X intersecting F, and let h^+ and h^- be the halfspaces of $\boldsymbol{h}$. Then either $F \subseteq \boldsymbol{h}$, or $\boldsymbol{h} \cap F$ is a codimension-1 flat in F. In the latter case, both $h^+ \cap F$ and $h^- \cap F$ are half-flats.

Proof. The carrier $N(\boldsymbol{h})$ of $\boldsymbol{h}$, which is its neighbourhood, has the form $N(\boldsymbol{h}) = \boldsymbol{h} \times [0, 1]$. Thus if $F \nsubseteq \boldsymbol{h}$, then $\boldsymbol{h} \cap F$ is a codimension-1 submanifold of F. Moreover, the intersections $\boldsymbol{h} \cap F$, $h^+ \cap F$, and $h^- \cap F$ are convex, thus the lemma follows. $\square$

LEMMA 6.3.2. Let $\mathfrak{h}$ be a hyperplane in a CAT(0) cube complex X. Suppose that l is a geodesic ray in X starting in $\mathfrak{h}$. If $l \nsubseteq \mathfrak{h}$, then there exists another hyperplane $\mathfrak{h}'$ in X intersecting l and disjoint from $\mathfrak{h}$.

PROOF. Let $N(\mathfrak{h})$ be the carrier of $\mathfrak{h}$. Let B be the first cube outside $N(\mathfrak{h})$ whose interior is intersected by l. We claim that there is a hyperplane $\mathfrak{h}'$ intersecting B and disjoint from $\mathfrak{h}$. Indeed, pick a vertex $v \in N(\mathfrak{h}) \cap B$ and let e be an edge of B containing v. If the hyperplane dual to e intersects $\mathfrak{h}$, then $e \subset N(\mathfrak{h})$. If this holds for any e, then $B \subset N(\mathfrak{h})$ by the convexity of $N(\mathfrak{h})$, which yields a contradiction. This justifies the claim.

By the claim, there a hyperplane $\mathfrak{h}'$ intersecting B and disjoint from $\mathfrak{h}$. It remains to prove that l intersects $\mathfrak{h}'$. Otherwise, since l intersects the interior of the carrier $N(\mathfrak{h}')$, we have that l is contained in $N(\mathfrak{h}')$. Since l starts at $\mathfrak{h}$, we have that h intersects $N(\mathfrak{h}')$ and hence it also intersects $\mathfrak{h}'$, which is a contradiction. $\square$

We will also use a consequence of a result of Haglund [**Hag08**, Theorem 2.28].

THEOREM 6.3.3. Let X be a hyperbolic CAT(0) cube complex. Then any quasi-isometrically embedded subspace of X is at finite Hausdorff distance from its combinatorial convex hull.

In the following theorem we generalise our results from [**HP15**, Section 3]. Here d_{Haus} denotes the Hausdorff distance.

THEOREM 6.3.4. Let X be a CAT(0) cube complex of asymptotic rank n and let $F \subseteq X$ be an n-flat. Let Y be the combinatorial convex hull of F. Then $d_{\mathrm{Haus}}(F, Y) < \infty$.

PROOF. If F is contained in the carrier $N(\mathfrak{h}) = \mathfrak{h} \times [0,1]$ of a hyperplane $\mathfrak{h}$, then we can replace X by $\mathfrak{h}$ and F by its projection to $\mathfrak{h}$. The combinatorial convex hull Y of F equals $Y' \times [0,1], Y' \times \{0\}$, or $Y' \times \{1\}$, where Y' is the combinatorial convex hull of the projection of F to $\mathfrak{h}$. Henceforth we can and will assume that F is not contained in the carrier of any hyperplane.

Let $\mathscr{H}$ be the collection of hyperplanes intersecting F. We define a *pencil of hyperplanes* to be an infinite collection of mutually disjoint hyperplanes $\{\mathfrak{h}_i\}_{i=-\infty}^{\infty}$ such that for each i, $\{\mathfrak{h}_j\}_{j=-\infty}^{i-1}$ and $\{\mathfrak{h}_j\}_{j=i+1}^{\infty}$ are in different halfspaces of $\mathfrak{h}_i$. It follows from Lemma 6.3.1 that every pencil of hyperplanes in $\mathscr{H}$ intersects F in a collection of parallel family of codimension-1 flats. A collection of pencils of hyperplanes in $\mathscr{H}$ is *independent* if their corresponding normal vectors are linearly independent in $F = \mathbb{R}^n$.

Let $\{P_i\}_{i=1}^m$ be a maximal collection of pairwise independent pencils in $\mathscr{H}$. We claim that $m = n$ and that $\{P_i\}$ is independent. Suppose first $m > n$. Note that if two pencils $P, P' \subseteq \boldsymbol{\hbar}$ are independent, then every hyperplane in P intersects every hyperplane in P'. This gives rise to a quasi-isometric embedding of $\mathbb{R}^m$ into X, contradicting the bound on the asymptotic rank of X. If $m < n$ or if $m = n$ but $\{P_i\}$ is dependent, then there is a geodesic line l in F parallel to $\boldsymbol{\hbar} \cap F$ for all hyperplanes $\boldsymbol{\hbar}$ in all P_i. Using Lemma 6.3.2, we can then produce a new pencil P formed of some hyperplanes intersecting l. Since P is independent from each P_i, this contradicts the maximality of m. This justifies the claim that $m = n$ and $\{P_i\}$ is independent.

For $1 \leq i \leq n$, choose $h_i \in P_i$ and let $F_i = h_i \cap F$. We will prove that for any hyperplane $\boldsymbol{\hbar} \in \mathscr{H}$, there exists F_i such that $h \cap F$ is parallel (possibly equal) to F_i. Otherwise, choose a geodesic line l in F transverse to $h \cap F$. By Lemma 6.3.2, $\boldsymbol{\hbar}$ is contained in a pencil $P_{\boldsymbol{\hbar}}$ of hyperplanes intersecting l. Note that $P_{\boldsymbol{\hbar}}$ is independent from each P_i, contradicting the maximality of m.

Let $\mathscr{H}_i \subseteq \mathscr{H}$ be the collection of hyperplanes whose intersection with F is parallel to F_i. The above discussion implies $\mathscr{H} = \bigsqcup_{i=1}^n \mathscr{H}_i$. Moreover, for $i \neq j$, every hyperplane in $\mathscr{H}_i$ intersects every hyperplane in $\mathscr{H}_j$. Let Y be the combinatorial convex hull of F. Since we assumed that F is not contained in the carrier of any hyperplane, the hyperplanes in Y are exactly the intersections with Y of the hyperplanes in $\mathscr{H}$. Two hyperplanes of Y intersect if and only if the corresponding hyperplanes in $\mathscr{H}$ intersect. Hence by Lemma 6.2.8, we have a product decomposition $Y = Y_1 \times \cdots \times Y_n$.

Let $\pi_i : Y \to Y_i$ be the coordinate projections. Let $l_i = \bigcap_{j \neq i} F_j$, which is a geodesic line in F. Note that for $j \neq i$ we have $l_i \subseteq F_j \subseteq h_j$ and hence the projection $\pi_j(l_i)$ is a single point. Thus the restriction of π_i to l_i is an isometric embedding. It follows that $F = \pi_1(l_1) \times \cdots \times \pi_1(l_n)$. Moreover, since $\pi_i(l_i) = \pi_i(F)$, each Y_i is the combinatorial convex hull of $\pi_i(l_i)$, since otherwise we could pass to a smaller convex subcomplex containing F.

Since each of Y_i contains a line and their product has asymptotic rank $\leq n$, by Theorem 6.2.3(1) each Y_i has asymptotic rank 1. By Theorem 6.2.3(2) each Y_i is hyperbolic. Thus by Theorem 6.3.3, we have $d_{\text{Haus}}(\pi_i(l_i), Y_i) < \infty$, and consequently $d_{\text{Haus}}(F, Y) < \infty$. □

While we will not need it in the remaining part of the paper, from the proof above we can deduce the following interesting result which concerns flats that are not necessarily of top rank.

Corollary 6.3.5. Let X be a CAT(0) cube complex and let $F \subseteq X$ be a flat. Let $Y \subseteq X$ be the combinatorial convex hull of F. Then Y has a natural decomposition $Y = Y_1 \times \cdots \times Y_n \times K$ such that:

(1) $n \geq \dim(F)$ and K is a cube.
(2) each Y_i contains an isometrically embedded copy of $\mathbb{R}$ that is the projection of a geodesic line in F.
(3) no Y_i contains a facing triple of hyperplanes, that is, a collection of three disjoint hyperplanes such that none of them separates the other two.

Roughly speaking, (3) means that Y_i do not "branch".

6.3.2. Product of hyperbolic groups.

Definition 6.3.6. Let X be a CAT(0) cube complex. A group $H \leq Aut(X)$ is *convex cocompact* if there is a convex subcomplex $Y \subseteq X$ that is H-*cocompact*, meaning that H preserves Y and acts on it cocompactly.

Lemma 6.3.7. Let X be a CAT(0) cube complex and let $H \leq Aut(X)$ be convex cocompact. Then there exists a minimal H-invariant convex subcomplex. Moreover, any minimal H-invariant convex subcomplex is H-cocompact and any two minimal H-invariant convex subcomplexes are parallel.

Proof. Let $Y \subseteq X$ be an H-cocompact convex subcomplex. Let $\mathscr{P}$ be the poset of H-invariant convex subcomplexes in Y. For the first assertion, by the Kuratowski–Zorn Lemma, it suffices to show that every descending chain of elements $\{Y_\lambda\}_\lambda \subseteq \mathscr{P}$ has a lower bound, or equivalently that their intersection is nonempty. Let $K \subseteq Y$ be compact such that $HK = Y$. Then each $K \cap Y_\lambda$ is nonempty, and by compactness of K so is their intersection.

For the second and third assertion, let $Y_{\min} \subseteq Y$ be a minimal element of $\mathscr{P}$ and let Y' be any other minimal H-invariant convex subcomplex. Let $(V, V') = \mathcal{G}(Y_{\min}, Y')$. Then both V and V' are H-invariant. By Lemma 6.2.6(1) both V and V' are convex subcomplexes, hence from minimality of $Y_{\min}$ and Y' we have $V = Y_{\min}$ and $V' = Y'$. Moreover, by Lemma 6.2.6(2) we have that Y' is H-equivariantly isometric to $Y_{\min}$ and thus it is H-cocompact. □

THEOREM 6.3.8. Let X be a locally finite CAT(0) cube complex of asymptotic rank n. Let $H \leq Aut(X)$ be a subgroup satisfying

(1) $H = H_1 \times \cdots \times H_n$, where each H_i is an infinite hyperbolic group, and
(2) for some (hence any) point $x \in X$ the orbit map $h \to h \cdot x$ from H to X is a quasi-isometric embedding.

Then H is convex cocompact. More precisely, if among H_i exactly $\{H_i\}_{i=1}^m$ are not virtually $\mathbb{Z}$, then there is a convex subcomplex $Y \subseteq X$ with a cubical product decomposition $Y = Y_0 \times \prod_{i=1}^m Y_i$ such that

(i) Y is H-cocompact, and the action $H \curvearrowright Y$ respects the product decomposition, and
(ii) the induced action of $\prod_{i=m+1}^n H_i$ on Y_0 is proper and cocompact, in particular Y_0 is quasi-isometric to $\mathbb{R}^{n-m}$, and
(iii) for any pair $i \neq j$ with $1 \leq j \leq m$ and $1 \leq i \leq n$, the induced action $H_i \curvearrowright Y_j$ is *almost trivial*, i.e. by isometries at uniformly bounded distance from the identity.

In the proof we need the notion of coarse intersection. Let X be a metric space and let $N_R(Y)$ be the R-neighbourhood of a subspace $Y \subseteq X$. A subspace $V \subseteq X$ is the *coarse intersection* of Y_1 and Y_2 if V is at finite Hausdorff distance from $N_R(Y_1) \cap N_R(Y_2)$ for all sufficiently large R. For example, in Lemma 6.2.6, in view of its part (3), the gates V_1, V_2 are the coarse intersections of Y_1 and Y_2. However, in general the coarse intersection of two subsets might not exist.

LEMMA 6.3.9 ([**MSW11**, Lemma 2.2]). Let X be a finitely generated group with word metric. Then the intersection of a pair of subgroups is their coarse intersection.

See [**MSW11**, Chapter 2] for more discussion on coarse intersection.

PROOF OF THEOREM 6.3.8. We first prove that H is convex cocompact, which we do by induction on m. Consider first the case $m = 0$. Recall that all CAT(0) cube complexes in the chapter were assumed to be finite-dimensional. Thus by [**Bri99**], H acts on X be semi-simple isometries. By the Flat Torus Theorem [**BH99**, Chapter II.7], H acts cocompactly on an n-flat $F \subseteq X$. By Theorem 6.3.4, the combinatorial convex hull Y of F is at finite Hausdorff distance from F. Since X is locally finite, Y is H-cocompact, as desired.

Suppose now that $m \geq 1$. Let $H' = \prod_{i \neq m} H_i$. We first prove that the group H' is convex cocompact. Choose a subgroup $Z \leq H_m$ isomorphic to $\mathbb{Z}$ and choose $h \in H_m$ such that the coarse intersection of hZ and Z is bounded. Let $G = H' \times Z \subset H$. By induction assumption, there exists a G-cocompact convex subcomplex $U \subset X$. Let $V \subset U$ be the gate with respect to $h \cdot U$. Note that both U and $h \cdot U$ are H'-invariant, so V is H'-invariant. By Lemma 6.2.6(3), V is the coarse intersection of U and $h \cdot U$. Hence, by Lemma 6.3.9 applied to G and hGh^{-1}, the action $H' \curvearrowright V$ is cocompact.

By Lemma 6.3.7, there exists a minimal H'-cocompact convex subcomplex, for which we keep the notation V. Then for any $h \in H_m$, the translate $h \cdot V$ is minimal H'-invariant, hence parallel to V by Lemma 6.3.7. Let $P_V = V \times V^{\perp}$ be the combinatorial parallel set of V (see Lemma 6.2.9). We have that P_V is H-invariant. Moreover, since V is H'-invariant, there are induced actions $H \curvearrowright V^{\perp}$ and $H_m \curvearrowright V^{\perp}$.

Choose a point $v \in V$. Let $\psi : H_m \to V^{\perp}$ be the composition of the orbit map $h \to h \cdot v$ with the coordinate projection. We claim that ψ is a quasi-isometric embedding. This follows from assumption (2) and the estimates below, where $\sim$ means equality up to a uniform multiplicative and additive constant. Namely, for any $h_1, h_2 \in H_m$ we have:

$$d_{H_m}(h_1, h_2) \sim d_H(h_1 H', h_2 H') \sim d_X(h_1 \cdot V, h_2 \cdot V) = d_{V^{\perp}}(\psi(h_1), \psi(h_2))$$

By Theorem 6.2.3, since V contains an isometrically embedded copy of $\mathbb{R}^{n-1}$, the asymptotic rank of $V^{\perp}$ is ≤ 1, and hence $V^{\perp}$ is hyperbolic. Let $V_m \subseteq V^{\perp}$ be the combinatorial convex hull of $\psi(H_m)$. Then $d_{\text{Haus}}(V_m, \psi(H_m)) < \infty$ by Theorem 6.3.3. Moreover, V_m is H-invariant under the action $H \curvearrowright V^{\perp}$ since $\psi(H_m)$ is invariant under H. Thus H acts cocompactly on the convex subcomplex $V \times V_m \subseteq P_V$. Notice that since $H' \curvearrowright \psi(H_m)$ is trivial, the action $H' \curvearrowright V_m$ is almost trivial.

By now we already know that H is convex cocompact. As for properties (i)—(iii), if $m = 1$, then it suffices to take $Y_0 = V$ and $Y_1 = V_1$. If $m \geq 2$, to obtain the required decomposition, we consider $X' = V \times V_m$, $H'' = \prod_{i \neq (m-1)} H_i$ and we repeat the previous argument. This gives rise to an H-cocompact convex subcomplex $V' \times V_{m-1} \subseteq V \times V_m$, where V' is a minimal H''-cocompact convex subcomplex. Since V_m is contained in some R-neighbourhood of a V', the intersection $V_{m-1} \cap V_m$ is compact. Moreover, V' and V_{m-1} admit cubical product decompositions $V' = (V' \cap V) \times (V' \cap V_m)$ and $V_{m-1} = (V_{m-1} \cap V) \times (V_{m-1} \cap V_m)$, thus $V' \times V_{m-1} = (V' \cap V) \times (V' \cap V_m) \times (V_{m-1} \cap V) \times (V_{m-1} \cap V_m)$. The H-action respects the above decomposition. Moreover, the induced action $H' \curvearrowright (V' \cap V_m)$ is almost trivial and the induced action $H'' \curvearrowright (V_{m-1} \cap V)$ is almost trivial. If $m = 2$, then we take $Y_1 = V_1 \cap V$, $Y_2 = V' \cap V_2$, and $Y_0 = (V \cap V') \cup (V_1 \cap V_2)$. If $m \geq 3$, then we let $X'' = V' \times V_{m-1}, H''' = \prod_{i \neq (m-2)} H_i$ and we repeat the previous process to obtain further product decomposition. We run this process m times, obtaining the required decomposition as the result of the last step. In each step, we possibly get nontrivial compact factors similar to $V_{m-1} \cap V_m$. We absorb all these compact factors into the factor Y_0 (we can also discard them). $\square$

6.4. Artin groups

6.4.1. Background on Artin groups. Let A be an Artin group with defining graph Γ, and generators S. Let W be the Coxeter group defined by Γ. For any $T \subseteq S$ let W_T (respectively A_T) be the *special subgroup* of W (respectively A) generated by T. The special subgroup W_T is naturally isomorphic to the Coxeter group defined by the subgraph Γ_T induced on T [**Bou68**]. Similarly, by [**H.83**] the special subgroup A_T of A is naturally isomorphic to the Artin group defined by Γ_T.

Lemma 6.4.1 ([**CP14**, Theorem 1.1]). Special subgroups of Artin groups are convex with respect to the word metric defined by standard generators.

A subset $T \subseteq S$ is *spherical* if the special subgroup W_T is finite. The *dimension* of the Artin group A is the maximal cardinality of a spherical subset of S.

The following is a consequence of [**CD95b**] and [**CD95a**, Corollary 1.4.2].

Theorem 6.4.2. Let A be an Artin group of dimension n. Suppose that

(A) $n \leq 2$, or

(B) every clique T in Γ is spherical.

Then there is a finite n-dimensional cell complex that is a $K(A, 1)$.

6.4.2. Two-generator Artin groups. We start with the description of most two-generator Artin groups as virtually $F_k \times \mathbb{Z}$, where F_k is the free group with k generators.

LEMMA 6.4.3. Let A be an Artin group with defining graph Γ a single edge labelled by $n > 2$. Then

(1) A has a finite index subgroup of form $F_k \times \mathbb{Z}$ with $k \geq 2$, and
(2) no power of one of the two standard generators lies in the $\mathbb{Z}$ factor.

PROOF. By [**BM00**] (or by our proof of Theorem 6.5.1) A acts freely and cocompactly on a product of a tree and a line, where a central element acts as a translation in the line factor. By [**BH99**, Theorem II.6.12] A virtually decomposes as $A' \times \mathbb{Z}$. The induced action of A' on the tree factor has finite vertex stabilizers so by Bass–Serre theory A' is a graph of finite groups, in particular A' is virtually free, justifying (1). Part (2) follows from the fact that standard generators act hyperbolically on the tree factor. $\square$

Throughout this section by $\bar{x}$ we denote the inverse of x. By x^z we denote the conjugate $\bar{z}xz$.

Let $A_n = \langle a, b \mid \underbrace{aba\dots}_{n} = \underbrace{bab\dots}_{n} \rangle$. Denote $\Delta = \underbrace{aba\dots}_{n} = \underbrace{bab\dots}_{n}$. Let A'_n be the kernel of the homomorphism sending each generator to the generator of $\mathbb{Z}/2$ i.e. the subgroup consisting of all words of even length. The group A'_n is generated by the elements: $r = ab, s = a\bar{b}, t = \bar{a}b$. If ϕ is a word in an alphabet Λ, and $x \in \Lambda$, then we denote by $\mathrm{Exp}_x(\phi)$ the sum of all the exponents at x in ϕ.

By direct computation we immediately establish the following:

LEMMA 6.4.4. If n is odd, then the conjugation by Δ is an order two automorphism sending $s \mapsto \bar{s}, t \mapsto \bar{t}, r \mapsto q$, where $q = ba = \bar{s}r\bar{t}$. In particular, Δ^2 is a central element.

If n is even, then Δ is a central element.

Let z be the element Δ^2 for n odd and the element Δ for n even.

LEMMA 6.4.5. If n is odd, then we have

$$b^n = \phi(s, t, r)\Delta,$$

where $\mathrm{Exp}_r(\phi) = 0$.

PROOF. Consider the following word ϕ expressed as a product of terms indexed by decreasing i:

$$\phi(s,t,r) = \bar{s} \prod_{i=\frac{n-3}{2}}^{0} \bar{t}^{r^i}$$

Since r^i appear in the expression defining ϕ only as elements that we conjugate by, we have $\mathrm{Exp}_r(\phi) = 0$.

To verify that $b^n = \phi\Delta$, note that

$$\phi = \bar{s} \prod_{i=\frac{n-3}{2}}^{0} \bar{r}^i \bar{t} r^i = \bar{s}(\bar{r}^{\frac{n-3}{2}} \bar{t} r^{\frac{n-3}{2}})(\bar{r}^{\frac{n-3}{2}-1} \bar{t} r^{\frac{n-3}{2}-1}) \ldots (\bar{r}\bar{t}r)\bar{t} = \bar{s}\bar{r}^{\frac{n-1}{2}}(r\bar{t})^{\frac{n-1}{2}}.$$

Since $\bar{s}\bar{r}^{\frac{n-1}{2}} = b\bar{a}(\bar{b}\bar{a})^{\frac{n-1}{2}} = b\bar{\Delta}$ and $r\bar{t}\Delta = \Delta q t = \Delta b^2$, we have

$$\phi(s,t,r)\Delta = \bar{s}\bar{r}^{\frac{n-1}{2}} \Delta b^{n-1} = b\bar{\Delta}\Delta b^{n-1} = b^n.$$

□

COROLLARY 6.4.6. If n is odd, we have

$$b^{2n}\bar{z} \in [A'_n, A'_n].$$

PROOF. We have

$$b^{2n} = \phi(s,t,r)\Delta\phi(s,t,r)\Delta = \phi(s,t,r)\phi(\bar{s},\bar{t},q)z.$$

Denote the word $\phi(s,t,r)\phi(\bar{s},\bar{t},q)$ by $\psi(s,t,r,q)$. By Lemma 6.4.5, we have $\mathrm{Exp}_r(\psi) = \mathrm{Exp}_q(\psi) = 0$. We also have $\mathrm{Exp}_s(\psi) = \mathrm{Exp}_t(\psi) = 0$ since the total exponents of s and t in $\phi(s,t,r)$ are equal to the total exponents of $\bar{s}$ and $\bar{t}$ in $\phi(\bar{s},\bar{t},q)$, respectively. Thus $\psi \in [A'_n, A'_n]$. □

6.4.3. Surface lemma. The following lemma will allow us to utilize the preceding result when discussing finite index subgroups of A_n.

LEMMA 6.4.7. Let G be a finitely generated group and let $z \in G$ be central. Let H be a finite index normal subgroup of G, and let $h \in H \cap z[G,G]$. Then for any homomorphism $\rho : H \to \mathbb{Z}$ such that $\rho(\langle z \rangle \cap H) \neq \{0\}$, there exist a positive integer m and $g \in G$ with $\rho((h^m)^g) \neq 0$.

Proof. Let X be a presentation complex for G. Let S be an oriented surface with connected ∂S and basepoint $s \in \partial S$, mapping to X, such that on the level of fundamental groups $\partial S \mapsto h\bar{z}$. Let $\widehat{X}$ be the finite cover of X corresponding to H and let $\widehat{S}$ be a finite cover of S such that $\widehat{S} \to S \to X$ lifts to $\widehat{S} \to \widehat{X}$. Choose a system Σ of nonintersecting arcs that join the basepoint of $\widehat{S}$ to the other preimages of s, one for each of the boundary components of $\widehat{S}$. Consider the surface S' obtained from $\widehat{S}$ by cutting along the arcs of Σ, and the mapping $S' \to \widehat{X}$ that factors through $\widehat{S}$. Then, as the boundary of a surface, $\partial S'$ is mapped to an element $f \in H = \pi_1(\widehat{X})$ contained in $[H, H]$. The arcs of Σ map to paths in $\widehat{X}$ that project to closed paths in X corresponding to some $g_i \in G$. Thus we have $f = \prod_{i=1}^{q} (h^{m_i})^{g_i} \bar{z}^M$, where $m_i \geq 1$ with $M = \sum m_i$.

Since H is normal, each $(h^{m_i})^{g_i}$ lies in H. We have $\rho(\prod_{i=1}^{q}(h^{m_i})^{g_i}) = \rho(z^M) \neq 0$. That means that there is at least one element $(h^{m_i})^{g_i}$ such that $\rho((h^{m_i})^{g_i}) \neq 0$. $\square$

Corollary 6.4.8. Let n be odd and let H be a finite index normal subgroup of A'_n. Then for any homomorphism $\rho : H \to \mathbb{Z}$ such that $\rho(\langle z \rangle \cap H) \neq \{0\}$, there exist a positive integer m and $g \in A'_n$ such that $b^m \in H$ and $\rho((b^m)^g) \neq 0$.

Proof. Let k be large enough so that $b^{2nk} \in H$. By Corollary 6.4.6, we can apply Lemma 6.4.7 with $G = A'_n, h = b^{2nk}$, and z^k in the role of z. $\square$

Corollary 6.4.9. Let n be even and let H be a finite index normal subgroup of A_n. Then for any homomorphism $\rho : H \to \mathbb{Z}$ such that $\rho(\langle z \rangle \cap H) \neq \{0\}$, there exist a positive integer m and $g \in A_n$ such that at least one of $(a^m)^g$ and $(b^m)^g$ lies in H and is not mapped to 0 under ρ.

Proof. Let $k = \frac{n}{2}k'$ be a nonzero integer such that $a^k, b^k \in H$. Since $z^{k'} = (ab)^k$, we have

$$a^k b^k \in z^{k'}[A_n, A_n].$$

By Lemma 6.4.7, we have $m > 0$ and $g \in A_n$ such that $\rho\big(((a^k b^k)^m)^g\big) \neq 0$. Let $f = (a^k)^g$ and $h = (b^k)^g$. We have $(fh)^m \in f^m h^m [H, H]$. Thus $\rho(f^m h^m) \neq 0$ and so at least one of $f^m = (a^{km})^g$ and $h^m = (b^{km})^g$ is not mapped to 0 under ρ. $\square$

6.5. The main theorem

In this section we prove Theorem 6.1.1. The implication (i)$\Rightarrow$(ii) is obvious.

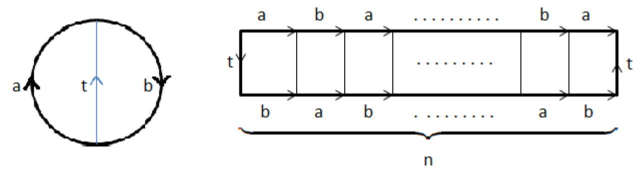

FIGURE 1. The complex K_n.

6.5.1. Implication (iii)⇒(i).

THEOREM 6.5.1. Let A be an Artin group with each connected component of the defining graph:

- a vertex, or an edge, or else
- all interior edges labeled by 2 and all leaves labelled by even numbers.

Then A is the fundamental group of a nonpositively curved cube complex.

PROOF. We assume without loss of generality that Γ is connected, since if Γ has more connected components, then A is the fundamental group of the wedge of the complexes obtained for its connected components.

If Γ is a single vertex, then A is the fundamental group of a circle.

If Γ is a single edge labelled by an odd n, then let K_n be the cube complex described in Figure 1. On the left side we see part of the 1-skeleton of K_n consisting of three edges labelled by a, b, t, and the right side indicates how to attach a rectangle (subdivided into n squares) along its boundary path $\underbrace{ab\ldots a}_{n}\bar{t}\underbrace{\bar{b}\bar{a}\ldots\bar{b}}_{n}\bar{t}$. It is easy to check that the link of each of the two vertices in K_n is isomorphic to the spherical join of two points with n points, hence K_n is nonpositively curved. By collapsing the t-edge we obtain the presentation complex for the standard presentation of A, so $\pi_1(K_n) = A$. We learned this construction from Daniel Wise.

If Γ is a single edge labelled by an even n, let $x = ab$. The group A is then presented as $\langle a, x \mid ax^{n/2} = x^{n/2}a\rangle$. Let $K_{n,a}$ be the cube complex described in Figure 2. One can check that the link of the unique vertex in $K_{n,a}$ is isomorphic to the spherical join of two points with n points, hence $K_{n,a}$ is nonpositively curved. It is clear that $\pi_1(K_{n,a}) = A$.

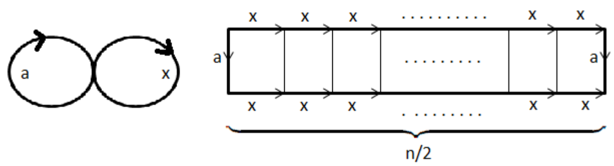

FIGURE 2. The complex $K_{n,a}$.

Similarly if we let $y = ba$, then A can be presented as $\langle b, y \mid by^{n/2} = y^{n/2}b \rangle$. We define $K_{n,b}$ in a similar way. Note that the a-circle in $K_{n,a}$ is a locally convex subcomplex, so is the b-circle in $K_{n,b}$.

If Γ contains more than one edge, then let $\Gamma' \subseteq \Gamma$ be the nonempty subgraph induced on all the vertices that have at least two neighbours. Thus the edges of Γ' are precisely the interior edges and by the hypothesis they are labelled by 2. Hence $A_{\Gamma'}$ is a right-angled Artin group. The *Salvetti complex* $S(\Gamma')$ is the nonpositively curved cube complex obtained from the presentation complex of $A_{\Gamma'}$ by adding the missing cubes of higher dimension (see [**Cha07**]). Let $\{(s_i, t_i)\}_{i=1}^k$ be the collection of leaves of Γ with $s_i \in \Gamma'$. Let n_i be the label of the edge (s_i, t_i), which is even. Let K be the amalgamation of $\{K_{n_i,s_i}\}_{i=1}^k$ and $S(\Gamma')$ along the s_i-circles. Then $\pi_1(K) = A$ and it follows from [**BH99**, Proposition II.11.6] that K is nonpositively curved. □

6.5.2. Implication (ii)⇒(iii).

THEOREM 6.5.2. Let A be a 2-dimensional Artin group. If A is virtually cocompactly cubulated, then each connected component of the defining graph of A is either

- a vertex, or an edge, or else
- all its interior edges are labeled by 2 and all its leaves are labelled by even numbers.

PROOF. Suppose that there exists a finite index subgroup $\hat{A} \leq A$ that acts properly and cocompactly by combinatorial automorphisms on a CAT(0) cube complex X. Without loss of generality, we assume that $\hat{A}$ is normal in A. It suffices to prove:

(1) no edge of Γ has an odd label, unless it is an entire connected component, and
(2) no interior edge of Γ has an even label ≥ 4.

Let us first prove (1). Suppose to the contrary that Γ has an edge (a, b) with odd label and another edge (b, c). Let A_{ab} be the special subgroup generated by a and b. By A'_{ab}

we denote its index-two subgroup that is the kernel of the homomorphism to $\mathbb{Z}/2$ sending both a and b to 1. Let $\hat{A}_{ab} = F_k \times \mathbb{Z}$ be a finite index subgroup of $A'_{ab} \cap \hat{A}$ guaranteed by Lemma 6.4.3(1). We can also assume that $\hat{A}_{ab}$ is normal in A'_{ab}. Similarly, let A_{bc} be the special subgroup generated by b and c, and let $\hat{A}_{bc} = F_l \times \mathbb{Z}$ be a finite index subgroup of $A_{bc} \cap \hat{A}$. Note that the edge (b, c) might be labelled by 2 and then $l = 1$.

Since $\hat{A}$ is a CAT(0) group, we can speak of its asymptotic rank. By Theorem 6.4.2(A), there exists a finite 2-dimensional cell complex that is a $K(A, 1)$. Thus by Lemma 6.2.5, the asymptotic rank of $\hat{A}$ is ≤ 2 and so is the asymptotic rank of X. The subgroup A_{ab} is convex with respect to the standard generators of A by Lemma 6.4.1 and so $\hat{A}_{ab}$ is quasi-isometrically embedded in $\hat{A}$. We can thus apply Theorem 6.3.8 to find a convex subcomplex Y_{ab} that is $\hat{A}_{ab}$-cocompact. Moreover, there is a cubical product decomposition $Y_{ab} = V_{ab} \times H_{ab}$ such that the action of $\hat{A}_{ab}$ respects this decomposition, the vertical factor V_{ab} is quasi-isometric to $\mathbb{R}$, and the $\mathbb{Z}$ factor Z of $\hat{A}_{ab}$ acts almost trivially on H_{ab}.

Consider $\mathrm{Min}(Z) = \mathbb{R} \times V_0 \subseteq V_{ab}$ for the induced action of Z, where $\mathbb{R}$ is an axis of Z. Since Z is contained in the centre of $\hat{A}_{ab}$, we have an induced action of $\hat{A}_{ab}$ on $\mathbb{R} \times V_0$ respecting this decomposition. The factor V_0 is bounded, so V_0 contains a fixed-point of the action of $\hat{A}_{ab}$. Thus $\mathbb{R} \times V_0$ contains an $\hat{A}_{ab}$–invariant line l. Let $\rho : \hat{A}_{ab} \to \mathrm{Isom}(l)$ be the induced map. Note that $\rho(\hat{A}_{ab})$ does not flip the ends of l. Moreover, since V_{ab} is a cube complex, the translation lengths on l are discrete. This gives rise to a homomorphism $\rho : \hat{A}_{ab} \to \mathbb{Z}$ assigning to each element of $\hat{A}_{ab}$ its translation length on l. Note that $\rho(Z) \neq 0$. By Corollary 6.4.8 applied to $H = \hat{A}_{ab}$, there exists a nonzero integer m and $g \in A'_{ab}$ such that $\rho((b^m)^g) \neq 0$.

By normality of $\hat{A}$, we have $(\hat{A}_{bc})^g \leq \hat{A}$. Let Y_{bc} be a convex $(\hat{A}_{bc})^g$-cocompact subcomplex guaranteed again by Theorem 6.3.8. By [**H.83**] we have $A_{ab} \cap A_{bc} = \langle b \rangle$, and hence the groups $\langle b^m \rangle^g$ and $\hat{A}_{ab} \cap (\hat{A}_{bc})^g$ have a common finite index subgroup B. Let $Y \subset Y_{ab}$ be the gate with respect to Y_{bc}. Then Y is the coarse intersection of Y_{ab} and Y_{bc} by Lemma 6.2.6(3). By Lemma 6.3.9, Y is B-cocompact.

Since Y is a convex subcomplex, it has a product structure $Y = Y_V \times Y_H$ where $Y_V \subseteq V_{ab}$ and $Y_H \subseteq H_{ab}$. We have $\rho(B) \neq 0$, so Y_V is unbounded. Since Y is quasi-isometric to $\mathbb{R}$, the factor Y_H is bounded. Since Z acts almost trivially on H_{ab}, any of its orbits in Y_{ab} is at a finite Hausdorff distance from Y. Hence Z is commensurable with B. Thus there exists an integer $j \neq 0$ such that $(b^g)^j \in Z$, and hence $b^j \in Z$, contradicting Lemma 6.4.3(2).

Let us now prove (2). Suppose that Γ has edges $(a,b),(b,c)$, and (c',a) (here c and c' are possibly the same), where (a,b) has an even label ≥ 4. Let $\hat{A}_{ab}, \hat{A}_{bc}, \hat{A}_{c'a}$ be finite index subgroups of $A_{ab} \cap \hat{A}, A_{bc} \cap \hat{A}, A_{c'a} \cap \hat{A}$, respectively, that are isomorphic to a product of a free group and $\mathbb{Z}$. Assume moreover that $\hat{A}_{ab}$ is normal in A_{ab}. Let $Y_{ab} = V_{ab} \times H_{ab}$ be a convex $\hat{A}_{ab}$-cocompact subcomplex, and let $\rho : \hat{A}_{ab} \to \mathbb{Z}$ be defined as before. By Corollary 6.4.9, there exist a nonzero integer m and $g \in A_{ab}$ such that at least one of $(a^m)^g$ and $(b^m)^g$ lies in $\hat{A}_{ab}$ and is not mapped to 0 under ρ. Without loss of generality we can assume $\rho((b^m)^g) \neq 0$. The rest of the argument is identical as in the proof of (1). $\square$

6.6. 3-generator Artin groups

This section is devoted to the proof of Theorem 6.1.2. Let A be the three-generator Artin group with $m_{ab} = 3, m_{bc} = 2$, and $m_{ac} = 3, 4$, or 5, and let W be the Coxeter group with the same defining graph. Consider a longest word in a, b, c which is a minimal length representative of the element it represents in W. This word represents also an element of A, which we call Δ.

Lemma 6.6.1. (i) The centre Z of A is generated by Δ^2 for $m_{ac} = 3$ and by Δ for $m_{ac} = 4$ or 5.

(ii) The intersections of A_{ab} and A_{bc} with Z are trivial.

(iii) In A we have $A_{ab} \times Z \cap A_{bc} \times Z = A_b \times Z$.

Proof. Assertion (i) follows from [**Del72**, Theorem 4.21].

For (ii), let $\Delta_{ab} = aba$. By [**Del72**, Proposition 4.17], each element of A_{ab} is represented by $\Delta_{ab}^{-k}\phi(a,b)$, where ϕ is a positive word in a, b, and $k \geq 0$. If we had $\phi(a,b) = \Delta_{ab}^k \Delta^l$ for some $l > 0, k \geq 0$, then by [**Del72**, Theorem 4.14] this equality would also hold in the Artin semigroup, contradicting the fact that Δ is expressed as a positive word involving all a, b, c. The same argument works for A_{bc}.

For (iii) we need to show $A_{ab} \times Z \cap A_{bc} \times Z \subseteq A_b \times Z$. Since b and c commute, it suffices to show that for each $m \neq 0$ we have $c^m \notin A_{ab} \times Z$. If $m_{ac} = 3$, then this follows from a well known fact that A/Z is the mapping class group of the four punctured disc, where A_{ab}

fixes a curve around the first three punctures and c is a half-Dehn twist in a curve around the third and the fourth.

If $m_{ac} = 4$ or 5, assume for contradiction that $c^m = gz$, for some $z \in Z$ and $g \in A_{ab}$. Thus $gc^m = g^2z = gzg = c^m g$. Let $g = \Delta_{ab}^{-k}\phi(a, b)$, where ϕ is a positive word in a, b, and $k \geq 0$ is even. Thus $\phi(a, b)c^m\Delta_{ab}^k = \Delta_{ab}^k c^m \phi(a, b)$.

By [**Del72**, Theorem 4.14] this equality also holds in the Artin semigroup. The relation $acac = caca$ or $acaca = cacac$ involves on each side 2 occurrences of c separated by an occurrence of a. The word $\phi(a, b)c^m\Delta_{ab}^k$ does not contain such a subword, and this property is invariant under the replacements $bc = cb$, $aba = bab$. Thus to pass from $\phi(a, b)c^m\Delta_{ab}^k$ to $\Delta_{ab}^k c^m \phi(a, b)$ one can only use $bc = cb$, and $aba = bab$, which is the relation defining A_{ab}. Thus there is l such that in A_{ab} we have $\phi(a, b)b^l = \Delta_{ab}^k$. Hence $g = b^{-l}$. Thus $c^m = b^{-l}z$, contradicting assertion (ii). $\square$

We also need the following consequence of rank-rigidity [**CS11**].

Lemma 6.6.2. Let G be a cocompactly cubulated group with centre containing $Z \cong \mathbb{Z}$. Then G has a finite index subgroup $G_0 \times Z$ with G_0 cocompactly cubulated.

Proof. Suppose that G acts properly and cocompactly by cubical automorphisms on a CAT(0) cube complex X. By [**CS11**, Corollary 6.4(iii)], if we replace X with its essential core, and G with a finite-index subgroup, we obtain a cubical product decomposition of X respected by G, such that for each factor there is an element $g \in G$ acting on it as a rank one isometry. Let X_V be a factor on which Z acts freely, and combine all other factors into X_H, so that $X = X_H \times X_V$. Let $g \in G$ act on X_V as a rank one isometry.

Note that the generator z of Z acts on X_V as a rank one isometry. Otherwise an axis of g would not be parallel to an axis of z. Hence g and z would generate $\mathbb{Z}^2$ acting properly on X_V, contradicting the fact that g has rank one. Consider $\mathrm{Min}(Z) = \mathbb{R} \times Y \subseteq X_V$, where $\mathbb{R}$ is an axis of Z. Since Z is contained in the centre of G, we have an induced action of G on $\mathbb{R} \times Y$ respecting this decomposition. Since z has rank one, we have that Y does not contain a geodesic ray, and hence is bounded. Consequently, Y contains a fixed-point of the action of G. Thus X_V contains a G–invariant line l.

Let $\rho : G \to \mathrm{Isom}(l)$ be the induced map. Note that $\rho(G)$ does not flip the ends of l. Moreover, since X_V is a cube complex, the translation lengths on l are discrete. Thus the image of ρ can be identified with $\mathbb{Z}$, which contains $\rho(Z)$ as a finite index subgroup. Let $G_0 = \ker(\rho)$. Thus $Z \times G_0$ is a finite index subgroup of G. Moreover, G_0 acts properly by cubical automorphisms on $X_H \subset X$. Since the action of Z on X_V is proper, the action of G_0 on X_H is cocompact. □

We complement Lemma 6.6.2 with the following:

Lemma 6.6.3. Let $G = G_0 \times Z$ be finitely generated, with $Z \cong \mathbb{Z}$. Let $H < G$ be a finite product of finitely generated free groups of rank ≥ 2 that is quasi-isometrically embedded.

(i) The map $H \to G/Z$ is a quasi-isometric embedding.
(ii) Let G be cocompactly cubulated. If we require that $H \cap Z$ is trivial, then assertion (i) holds also if in the product we allow free groups of rank 1.

Proof. If H is a free group of rank ≥ 2, then we choose in H a free generating set $S^{\pm}$. In Z we consider the generating set $\{\pm 1\}$ and in G_0 any symmetric generating set. Let $|\cdot|_H, |\cdot|_Z, |\cdot|_{G_0}$ denote the corresponding word-lengths. Let π_{G_0}, π_Z be the coordinate projections from G to G_0, Z, respectively. By assumption, there exists a constant c such that for any $h \in H$, we have $|h|_H \leq c\big(|\pi_{G_0}(h)|_{G_0} + |\pi_Z(h)|_Z\big)$. Viewing h as a reduced word over $S^{\pm}$, choose $s \in S^{\pm}$ such that the word $w = hsh^{-1}s^{-1}$ is reduced. Then $|\pi_Z(w)|_Z = 0$, and applying the above inequality with w in place of h we obtain $2|h|_H + 2 \leq c|\pi_{G_0}(w)|_{G_0} \leq 2c\big(|\pi_{G_0}(h)|_{G_0} + |\pi_{G_0}(s)|_{G_0}\big)$. Consequently $|h|_H \leq c|\pi_{G_0}(h)|_{G_0} + a$ for some uniform constant a, and thus the restriction of π_{G_0} to H is a quasi-isometric embedding, as desired.

Similarly, if H is a product of free groups H_i of rank ≥ 2, then we choose generating sets $S_i^{\pm}$ in H_i. Let $h = \prod h_i$ with $h_i \in H_i$. To get an estimate on $|h|_H$, it suffices to use a product of reduced words $w = \prod h_i s_i h_i^{-1} s_i^{-1}$, with $s_i \in S_i^{\pm}$. This proves assertion (i).

If G is cocompactly cubulated, then by Lemma 6.6.2, after passing to a finite index subgroup, the quotient G/Z acts properly and cocompactly on a CAT(0) cube complex X. Let $H = \mathbb{Z}^n \times H_0 \leq G$, where H_0 is a finite product of finitely generated free groups of rank ≥ 2. We keep the notation H for the isomorphic image of H in G/Z. Then H preserves $\mathrm{Min}(\mathbb{Z}^n) = \mathbb{R}^n \times Y \subseteq X$ and respects its product structure. We fix $v \in \mathbb{R}^n$ and $y \in Y$. From assertion (i), the orbit map $h_0 \to (h_0 \cdot v, h_0 \cdot y)$ from H_0 to $\mathbb{R}^n \times Y$ is a quasi-isometric embedding. Since the commutator of H_0 acts trivially on the $\mathbb{R}^n$ factor, using the same argument as for assertion (i), we obtain c satisfying $|h_0|_{H_0} \leq c d_Y(y, h_0 \cdot y)$. On the other hand, there is c' such that for $f \in \mathbb{Z}^n$ we have $|f|_{\mathbb{Z}^n} \leq c' d_{\mathbb{R}^n}(v, f \cdot v)$. Let d be the maximum of the displacements $d_{\mathbb{R}^n}(v, s \cdot v)$ over the generators s of H_0. For $fh_0 \in H$ consider the maximum norm $\|fh_0\| = \max\{|f|_{\mathbb{Z}^n}, 2c'd|h_0|_{H_0}\}$. If $|f|_{\mathbb{Z}^n} \geq 2c'd|h_0|_{H_0}$, then

$$c' d_{\mathbb{R}^n}(v, fh_0 \cdot v) \geq |f|_{\mathbb{Z}^n} - c'd|h_0|_{H_0} \geq \frac{1}{2}|f|_{\mathbb{Z}^n} \geq \frac{1}{2}\|fh_0\|.$$

Otherwise, if $|f|_{\mathbb{Z}^n} < 2c'd|h_0|_{H_0}$, then

$$cd_Y(y, fh_0 \cdot y) = cd_Y(y, h_0 \cdot y) \geq |h_0|_{H_0} > \frac{1}{2c'd}\|fh_0\|.$$

This proves assertion (ii). □

Proof of Theorem 6.1.2. The implication (i)⇒(ii) is obvious. The implication (iii)⇒(i) follows from Theorem 6.5.1 unless the defining graph Γ of A has two edges $(a, c), (b, c)$ with label 2. By Theorem 6.5.1, A_{ab} is the fundamental group of a nonpositively curved cube complex K. Then $K \times S^1$ is a nonpositively curved cube complex with fundamental group A.

The implication (ii)⇒(iii) follows from Theorem 6.5.2 if A is 2-dimensional. Suppose now that A is not 2-dimensional. Then, by the classification of finite Coxeter groups, the labels of Γ are $m_{ab} = 3, m_{bc} = 2$, and $m_{ac} = 3, 4$, or 5. Let Z be the centre of A described in Lemma 6.6.1(i).

Suppose that there exists a normal finite index subgroup $\hat{A} \leq A$ that is cocompactly cubulated. Let $\hat{Z} = \hat{A} \cap Z$. By Lemma 6.6.2, up to replacing $\hat{A}$ with a further finite index subgroup, we have $\hat{A} = \hat{A}_0 \times \hat{Z}$, where $\hat{A}_0$ is cocompactly cubulated. We keep the notation $\hat{A}_0$ for its isomorphic image in the quotient A/Z. Note that $\hat{A}_0 \leq A/Z$ is a normal finite index subgroup.

By Theorem 6.4.2(B), the Artin group A is the fundamental group of a 3-dimensional cell complex which is a $K(A, 1)$. Thus, by Lemma 6.2.5, the asymptotic rank of $\hat{A}$ is ≤ 3. Hence the asymptotic rank of $\hat{A}_0$ is ≤ 2.

By Lemma 6.6.1(ii), the intersections of A_{ab} and A_{bc} with Z are trivial. Thus A_{ab} and A_{bc} embed into A/Z under the quotient map, and we keep the notation A_{ab} and A_{bc} for their images in A/Z. By Lemma 6.6.1(iii) in A/Z we have $A_{ab} \cap A_{bc} = A_b$.

Let $\hat{A}_{ab} = F_k \times \mathbb{Z}$ be a finite index subgroup of $A'_{ab} \cap \hat{A}_0$ guaranteed by Lemma 6.4.3(1). We can assume that $\hat{A}_{ab}$ is normal in A'_{ab}. Let $\hat{A}_{bc} = A_{bc} \cap \hat{A}_0 = \mathbb{Z}^2$. By Lemmas 6.4.1 and 6.6.3(ii), $\hat{A}_{ab}, \hat{A}_{bc} < A/Z$ are quasi-isometric embeddings.

From this point we argue to reach a contradiction exactly as in part (1) of the proof of Theorem 6.5.2. $\square$

Summary

We have examined certain obstruction to group actions on CAT(0) cube complexes, either proper actions on a complex of a fixed dimension in the context of small cancellation groups, or proper and cocompact actions of Artin groups. Here are a few further questions we hope to address in the future.

Question 6.6.4. Are all $C(6)$ groups (cocompactly) cubulated?

Question 6.6.5. Is there a uniform bound on the virtual cubical dimension of $C'(1/6)$?

Question 6.6.6. Which Artin groups act properly on CAT(0) cube complexes? In particular, are braid groups cubulated?

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