Laws of large numbers for sequences and arrays of random variables

by: Gelila Tilahun

Department of Mathematics and Statistics,
McGill University, Montreal

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Abstract

This thesis presents an up-to-date survey of results concerning laws of large numbers for sequences and arrays of random variables. We begin with Kolmogorov's pioneering result, the strong law of large numbers, and preceed through to Hu *et al.'s,* and Gut's recent result for weakly dominated random variables, for which we provide a simpler proof. We insist in particular on the techniques of proof of Etemadi and Jamison *et al.*. Furthermore, analogues to the Marcinkiewicz-Zygmund theorem are given. This thesis illustrates the trade-off between the existence of higher moments and non i.i.d sequences and arrays of random variables to obtain the strong law of large numbers.

Résumé

Ce mémoire présente une revue des récents résultats sur la loi des grands nombres pour des suites et tableaux de variables aléatoires. Nous commonçons par les travaux précurseurs de kolmogorov sur la loi forte des grands nombres pour ensuite aboutir aux récents résultats de Hu *et al.* et de Gut sur les variables faiblement dominées, dont nous donnons une preuve plus simple. Nous insistons en particulier sur les techniques de preuves utilisées par Etemadi et Jamison *et al.*. Nous présentons aussi des résultats analogues aux théorèmes de Marcinkiewicz-Zygmund. Ce mémoire illustre en fait les liens entre l'existence de moments d'ordre élévé et les suites et tableaux de variables non necessairement i.i.d. pour obtenir la loi forte de grands nombres.

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Contents

1 Introduction **3**
1.1 Outline of Thesis . . . 3
1.2 Background Results . . . 6

2 Strong Laws of Large Numbers **9**
2.1 An Application . . . 9
2.2 The Strong Law for Pairwise Independent Random Variables 11
2.3 Marcinkiewicz-Zygmund's Strong Law of Large Numbers . . . 16
2.3.1 More on the Marcinkiewicz-Zygmund Theorem . . . 19
2.4 Orthogonal Random Variables . . . 22

3 Generalizations **24**
3.1 Weighted Sums of Random Variables . . . 24
3.2 Stability of Weighted Sums of Random Variables . . . 25
3.2.1 Arbitrarily Heavy Tails and the Strong Law . . . 38
3.2.2 Arbitrarily Heavy Tails and the Weak Law . . . 42
3.3 Arrays of Weights . . . 46

4 Arrays of Random Variables **56**
4.1 Arrays of i.i.d. Random Variables . . . 56
4.2 Rowwise Independent Random Variables . . . 60

5 Random Elements **72**
5.1 Banach Space Valued Random Elements . . . 72
5.2 Hilbert Space Valued Arrays of Random Elements . . . 78

6 Conclusion **81**

Chapter 1

Introduction

1.1 Outline of Thesis

In 1930, Kolmogorov proved what is now known as his classical strong law of large numbers–that for a sequence $\{X_n; n \geq 1\}$ of independent, identically distributed (i.i.d.) random variables with $E|X_1|$ finite, the average of the sequence converges to the mean with probability one; in symbols,

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_k}{n} = EX_1 \text{ a.s.}$$

A great deal of work has been done since in transporting this important result to more general settings, and in exploring variations on the theorem in which weaker conditions on the sequence of random variables implies weaker modes of convergence for the sequence of averages. This thesis gives an overview of some of these generalizations and variations. We begin by giving an outline of the thesis; for definitions please refer to the next section.

The second chapter of this thesis begins with one proof of the classical law, with the X_i's pairwise independent and identically distributed, but not necessarily i.i.d. We then discuss the Marcinkiewicz-Zygmund strong law, which examines the necessary and sufficient conditions for the strong convergence of $S_n/n^{1/p}$ where

$$S_n = \sum_{k=1}^{n} X_k$$

and $\{X_k\}$ is a sequence of i.i.d. random variables with $EX_1 = 0$ and $0 < p < 2$.

In the third chapter, a more general problem is examined; namely the almost surely and in probability convergence to a constant of the weighted sums

$$T_n = \sum_{k=1}^{n} a_{nk} X_k$$

(the a_{nk}'s are the weights). The purpose of this chapter is to provide more general results valid for a whole class of coefficient matrices $A = (a_{nk})$. This is in contrast to the previous chapter in which (a_{nk}) has a specific structure–for example, $a_{nk} = n^{-1}$ for $k \leq n$ and $a_{nk} = 0$ for $k > n$. Although the case where the X_i's are i.i.d. is studied primarily, we also examine the situation when the X_i are pairwise independent, dominated by a random variable, or orthogonal, and the situation when no first moment exists.

The fourth chapter studies results concerning *complete convergence* of arrays of random variables $\{X_{nk};\ 1 \leq k \leq n;\ n \geq 1\}$. Necessary and sufficient conditions, inspired by the Marcinkiewicz-Zygmund result, for the complete convergence of

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{nk} \ \text{ for } 0 < p < 2$$

with $\{X_{nk}\}$ an i.i.d. array are presented. Next, we examine the case in where no assumption of independence between the rows of the array is made; we present a new proof, based on Rosenthal's inequality, which eliminates some of the tedious technicalities of the original proof and extends the theorem to the case where $0 < p < 1$.

Finally, we conclude by examining some of the results which have been obtained in the previous chapters in a more general setting–that of random variables which take values in Banach spaces.

1.2 Background Results

The basic setting throughout this thesis is a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is called the sample space, $\mathcal{F}$ is a family of subsets of Ω a ($\sigma-$ field) whose elements are called events, and $P : \mathcal{F} \rightarrow [0,1]$ is a probability measure. For $A \in \mathcal{F}$, $P(A)$ is called the probability of the event A. The purpose of this section is to give the necessary background definitions and results which will be used in the remainder of the thesis. We begin with the Borel-Cantelli lemma which will play a vital role.

Definition 1 *If $\{A_n, n \geq 1\}$ is a sequence of elements of $\mathcal{F}$ then the elements $\omega \in \Omega$ which occur infinitely often (i.o.) in $\{A_n\}$ are those elements which are in A_n for infinitely many values of n. The set of elements of Ω which occur infinitely often in $\{A_n\}$ is thus given by*

$$\{A_n \text{ i.o. }\} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = \limsup A_n.$$

A useful fact to know about such events is

$$P(\limsup A_n) = \lim_{k \to \infty} P(\bigcup_{n=k}^{\infty} A_n).$$

Lemma 1 (Borel-Cantelli) *If $\{A_n, n \geq 1\}$ is a sequence of events for which*

$$\sum_{n=1}^{\infty} P\{A_n\} < \infty$$

then $P\{A_n \text{ i.o. }\} = 0$. *If the* A_n *are pairwise independent, then a partial converse holds–namely, if*

$$\sum_{n=1}^{\infty} P\{A_n\} = \infty$$

then $P\{A_n \text{ i.o. }\} = 1$.

Proof: See [2].

Let $\{X_n,\ n \geq 1\}$ and X be random variables.

Definition 2 *The sequence* $\{X_n\}$ *is said to converge* almost surely *(a.s.) to* X *if*

$$P\{\omega : X_n(\omega) \rightarrow X(\omega) \text{ as } n \rightarrow \infty\} = 1$$

or equivalently, if for all $\epsilon > 0$,

$$P\{\omega : |X_n(\omega) - X(\omega)| > \epsilon \ \text{ i.o. }\} = 0.$$

This will be denoted by $X_n \rightarrow X$ *a.s as* $n \rightarrow \infty$. *If* X *is degenerate (almost surely a constant) then* $\{X_n\}$ *is said to converge almost surely to a constant (a.s.c)* X.

Definition 3 *The sequence* $\{X_n\}$ *is said to converge in probability to* X *if for every* $\epsilon > 0$

$$\lim_{n \rightarrow \infty} P\{\omega : |X_n(\omega) - X(\omega)| > \epsilon\} = 0.$$

In this case, we will also say that $X_n \rightarrow X$ *in probability as* $n \rightarrow \infty$. *If* $X_n \rightarrow X$ *in probability and* X *is degenerate, then* $\{X_n\}$ *is said to converge in probability to a constant (i.p.c)* X.

We note that the almost sure convergence of X_n to X implies convergence in probability (to X), but the converse is not true in general.

Lemma 2 ([4] p.90) *For any $r > 0$ and any random variable X,*

$$\sum_{n=1}^{\infty} P\{|X| \geq n^{1/r}\} \leq E|X|^r \leq \sum_{n=0}^{\infty} P\{|X| > n^{1/r}\}.$$

Lemma 3 *Let $\{X_n\}$ be a sequence of random variables. Suppose that*

$$\sum_{n=1}^{\infty} P\{|X_n| > \epsilon\} < \infty \text{ for all } \epsilon > 0. \tag{1.1}$$

Then

$$X_n \to 0 \text{ a.s.}, \tag{1.2}$$

and the converse holds if the X_n's are also pairwise independent.

Proof. If (1.1) holds, then for any $\epsilon > 0$, by the Borel-Cantelli lemma with $A_n = \{|X_n| > \epsilon\}$ we have

$$P\{|X_n| > \epsilon \text{ i.o.}\} = 0.$$

This is equivalent to the statement $X_n \to 0$ a.s. Conversely, if the X_n are pairwise independent and (1.2) holds, that is, $X_n \to 0$ almost surely, then for any $\epsilon > 0$

$$P\{|X_n| > \epsilon \ i.o.\} = 0.$$

By the partial converse of the Borel-Cantelli lemma,

$$\sum_{n=1}^{\infty} P\{|X_n| > \epsilon\} < \infty$$

and so (1.1) holds. $\square$

Chapter 2

Strong Laws of Large Numbers

2.1 An Application

We begin with a simple and beautiful application of the strong law of large numbers due to Borel (1909). Let $\Omega = [0,1]$. Consider a decimal expansion $\omega = 0.x_1x_2,...$ for each $\omega \in \Omega$. (Some ω's have two decimal expansions, however, since the Lebesgue measure of such ω's is zero, our discussion will not be affected if we take the decimal expansion of ω to be either of the two possible). For $k = 0,1,2,...,9$ let $N_n^{(k)}(\omega)$ denote the number of times k appears among the first n x_i's of the decimal expansion of ω. We say $\omega \in \Omega$ is normal to base 10 if

$$N_n^{(k)}(\omega)/n \to 0.1 \quad \text{as } n \to \infty \quad \text{for } k = 0,1,2,...,9.$$

In what follows we will prove that almost every ω chosen randomly in $[0,1]$ is normal. That is, we will prove that for almost all ω, the frequency in the limit with which k appears among the first n x_i's of the decimal expansion of ω is the same for every k, namely 1 out of 10 times.

Let $\mathcal{F}$ be the Borel subsets of $\Omega = [0,1]$ and P the Lebesque measure. Let $X_n(\omega)$ be the nth number in the decimal expansion of ω for every ω chosen randomly in $[0,1]$. We can easily verify that the sequence of random variables $\{X_n;\ n \geq 1\}$ is i.i.d. with $P\{X_1 = k\} = 0.1$. (It is clear that $P(X_n^{-1}(i)) = 0.1$ for all $n \geq 1,\ 0 \leq i \leq 9$. To see the independence, for example, $P(X_1^{-1}(i) \cap X_2^{-1}(j)) = P(\omega \mid \omega = 0.ij\ldots) = 0.01$. In general,

$$P(X_n^{-1}(i) \cap X_m^{-1}(j)) = 0.01 = (0.1)^2 = P(X_n^{-1}(i))P(X_m^{-1}(j))\ .)$$

Define $f(x) = 1$ if $x = k$ and $f(x) = 0$ if $x \neq k$. Then $\{f(X_i); i \geq 1\}$ is a sequence of i.i.d random variables and $Ef(X_1) = 0.1$. Now by the strong law,

$$\sum_{i=1}^{n} f(X_i)/n \to 0.1 \quad \text{a.s.}$$

Since $\sum_{i=1}^{n} f(X_i(\omega))/n = N_n^{(k)}(\omega)/n$, we have that almost every number with respect to the Lebesgue measure is base 10 normal. Of course the same argument can be modified to obtain the normality of almost all real numbers for any base.

There are other interesting and useful applications of the strong law in areas such as statistics, classical real analysis and Monte Carlo simulation. For examples see Stout [28] p.123-125.

2.2 The Strong Law for Pairwise Independent Random Variables

We will now present Etemadi's proof of the strong law of large numbers. The proof is more direct than Kolmogorov's because it uses neither Kolmogorov's inequality nor results on convergence of series of random variables. Etemadi's proof involves results on the subsequence of the random variables and moreover, the sequence of random variables needs only to be pairwise independent. Nevertheless, Kolmogorov's proof is still important since the ingredients used in the proof provide information on the rate at which $|S_n/n| \to 0$

Theorem 1 (Etemadi [7]) *Let $\{X_n\}$ be a sequence of pairwise independent, identically distributed random variables. Let $S_n = \sum_{i=1}^{n} X_i$. Then for some finite constant c*

$$E|X_1| < \infty \quad \textit{if and only if} \quad \frac{S_n}{n} \to c \ \textit{ a.s. as } \quad n \to \infty,$$

and if so, $c = EX_1$.

Proof. If $S_n/n \to c$ then

$$\frac{X_n}{n} = \frac{S_n - nc}{n} - \left(\frac{n-1}{n}\right) \frac{S_{n-1} - nc}{n-1} \to 0 \text{ a.s.} \tag{2.1}$$

Hence, $P\{|X_n| > n \ i.o.\} = 0$ (taking $\epsilon = 1$), and by the partial converse of the Borel-Cantelli lemma,

$$\sum_{n=0}^{\infty} P\{|X_1| > n\} = \sum_{n=0}^{\infty} P\{|X_n| > n\} < \infty.$$

Thus $E|X_1| < \infty$ by lemma 2.

On the other hand, suppose $E|X_1| < \infty$. Write $X_n = X_n^+ - X_n^-$ where

$$X_n^+ = \max(0, X_n) \quad \text{and} \quad X_n^- = \max(0, -X_n).$$

Clearly, $\{X_n^+\}$ and $\{X_n^-\}$ satisfy the assumption of the theorem, and therefore without loss of generality, we can assume that $X_n \geq 0$. The basic idea of the proof is as follows: First, we truncate X_i at the level i, by putting $Y_i = X_i I\{X_i \leq i\}$ where I is the indicator function. Let $S'_n = \sum_{i=1}^n Y_i$ and $k(n) = \lfloor \alpha^n \rfloor$ where $\alpha > 1$. We will prove that for the sequence $\{k(n)\}$, $\frac{S'_{k(n)}}{k(n)} \to EX_1$ a.s. as $n \to \infty$. Once we prove that the sequence $\{Y_n\}$ and $\{X_n\}$ are asymptotically equivalent, that is

$$\sum_{n=1}^{\infty} P\{X_n \neq Y_n \ i.o\} < \infty,$$

using the monotonicity of S_n we will conclude the proof.

Given any $\epsilon > 0$, using Chebychev's inequality, we obtain from the pairwise independence of $\{Y_n\}$,

$$\sum_{n=1}^{\infty} P\left\{\left|\frac{S'_{k(n)} - ES'_{k(n)}}{k(n)}\right| > \epsilon\right\} \leq \frac{1}{\epsilon^2}\sum_{n=1}^{\infty} \frac{Var\ S'_{k(n)}}{k(n)^2} = \frac{1}{\epsilon^2}\sum_{n=1}^{\infty} \frac{1}{k(n)^2}\sum_{i=1}^{k(n)} Var\ Y_i.$$

Using Fubini's theorem to interchange the order of summation we have

$$\epsilon^{-2}\sum_{n=1}^{\infty} \frac{1}{k(n)^2}\sum_{i=1}^{k(n)} Var\ Y_i = \epsilon^{-2}\sum_{i=1}^{\infty} Var\ Y_i \sum_{n:k(n)\geq i} k(n)^{-2}.$$

Since $k(n) = \lfloor \alpha^n \rfloor$ and $\lfloor \alpha^n \rfloor \geq \alpha^n/2$, (because $\alpha > 1$) for $n \geq 1$,

$$\sum_{n:k(n)\geq i} \lfloor \alpha^n \rfloor^{-2} \leq 4 \sum_{n:k(n)\geq i} \alpha^{-2n} \leq 4\left(1 - \alpha^{-2}\right)^{-1} i^{-2}.$$

(The last inequality holds since if n_i were to denote the smallest integer n such that $k(n) \geq i$, then from summing a geometric series

$$\sum_{n:k(n)\geq i} \alpha^{-2n} = \alpha^{-2n_i} 1 - \alpha^{-2^{-1}}) .$$

It is precisely for this convergence reason that we had defined $k(n) = \lfloor \alpha^n \rfloor$ where $\alpha > 1$. Now, noting that $Y_i \geq 0$,

$$\sum_{n=1}^{\infty} P\left\{\left|\frac{S'_{k(n)} - ES'_{k(n)}}{k(n)}\right| > \epsilon\right\} \leq 4\,(1-\alpha^{-2})^{-1}\epsilon^{-2}\sum_{i=1}^{\infty}\frac{E\,(Y_i^2)}{i^2}$$

Letting $c = 4\,(1-\alpha^{-2})^{-1}\epsilon^{-2}$ and $F(x) = P\{X_1 \leq x\}$,

$$\begin{aligned}\sum_{n=1}^{\infty} P\left\{\left|\frac{S'_{k(n)} - ES'_{k(n)}}{k(n)}\right| > \epsilon\right\} &\leq c\sum_{i=1}^{\infty}\frac{EY_i^2}{i^2} = c\sum_{i=1}^{\infty}\frac{1}{i^2}\int_0^i x^2 dF(x) \\ &= c\sum_{i=1}^{\infty}\frac{1}{i^2}\left(\sum_{k=0}^{i-1}\int_k^{k+1} x^2 dF(x)\right).\end{aligned}$$

Using Fubini's theorem to interchange the order of summation in the above equation, and noting that $\sum_{i=k+1}^{\infty} i^{-2} \leq 1/(k+1)$, we obtain

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{i^2}\sum_{k=0}^{i-1}\int_k^{k+1} x^2 dF(x) &\leq \sum_{k=0}^{\infty}\frac{1}{k+1}\int_k^{k+1} x^2 dF(x) \\ &\leq \sum_{k=0}^{\infty}\int_k^{k+1} x dF(x) \\ &= EX_1 < \infty.\end{aligned}$$

Hence by lemma 3,

$$\frac{S'_{k(n)} - ES'_{k(n)}}{k(n)} \to 0 \quad \text{a.s} \quad (n \to \infty). \tag{2.2}$$

We also have

$$EX_1 = \lim_{n\to\infty}\int_0^n x dF(x) = \lim_{n\to\infty} EY_n = \lim_{n\to\infty}\frac{ES'_{k(n)}}{k(n)}. \tag{2.3}$$

The justification for the last equality is provided by the following argument: Let $\mu = EX_1$ and $\mu_i = EY_i$. Then

$$\left|\frac{ES'_{k(n)}}{k(n)} - EX_1\right| = \left|\frac{\sum_{i=1}^{k(n)} \mu_i}{k(n)} - \mu\right| \leq \frac{1}{k(n)} \sum_{i=1}^{k(n)} |\mu_i - \mu| \, .$$

Since $|\mu_i - \mu| \to 0$, so does its arithematic mean by Cesàro's summation theorem. Now, using results (2.2) and (2.3), it follows

$$\lim_{n\to\infty} \frac{S'_{k(n)}}{k(n)} = EX_1 \qquad \text{a.s.} \tag{2.4}$$

Using Fubini's theorem, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} P\{Y_n \neq X_n\} &= \sum_{n=1}^{\infty} P\{X_n > n\} = \sum_{n=1}^{\infty} \int_n^{\infty} xdF(x) = \sum_{n=1}^{\infty}\sum_{i=n}^{\infty} \int_i^{i+1} xdF(x) \\
&= \sum_{n=1}^{\infty} i \int_i^{i+1} xdF(x) \leq \sum_{n=1}^{\infty} \int_i^{i+1} xdF(x) \\
&\leq EX_1 < \infty.
\end{aligned}$$

By lemma 3 $X_n - Y_n \to 0$ a.s. Hence,

$$\frac{1}{k(n)} \sum_{i=1}^{k(n)} (X_i - Y_i) \to 0 \ \text{ a.s } \ (n \to \infty)$$

by Cesàro's summation theorem. (We note that Cesàro's summation theorem also holds in the case of almost sure convergence, but not necessarily for the case of convergence in probability).

By equation (2.4), it follows that

$$\lim_{n\to\infty} \frac{S_{k(n)}}{k(n)} = EX_1 \qquad \text{a.s.} \tag{2.5}$$

Noting that $Y_i \geq 0$, we now observe that if $k(n) \leq m < k(n+1)$ then

$$\frac{S_{k(n)}}{k(n+1)} \leq \frac{S_m}{m} \leq \frac{S_{k(n+1)}}{k(n)}. \tag{2.6}$$

We shall now note the following two points.

(a) From the above equation it is clear that

$$\liminf_{j\to\infty} {}_{k(n)\geq j} \frac{S_{k(n)}}{k(n+1)} \leq \liminf_{j\to\infty} {}_{m\geq j} \frac{S_m}{m} \tag{2.7}$$

(b) Recalling that $k(n) = \lfloor \alpha^n \rfloor$, $k(n) \leq k(n+1)$ implies that $\frac{k(n)}{k(n)+1} < \frac{k(n)}{\alpha^n} \leq \frac{k(n)}{k(n)}$, and therefore,

$$\lim_{n\to\infty} \frac{k(n)}{\alpha^n} = 1$$

and similarly,

$$\limsup_{j\to\infty} {}_{k(n)\geq j} \frac{k(n+1)}{k(n)} \leq \limsup_{j\to\infty} {}_{k(n)\geq j} \frac{\alpha^n}{k(n)}\alpha = \alpha. \tag{2.8}$$

We also have that

$$\frac{1}{\alpha}\frac{S_{k(n)}}{\alpha^n} \leq \frac{S_{k(n)}}{k(n+1)}.$$

This implies that

$$\frac{1}{\alpha} \inf_{k(n)\geq j} \frac{S_k(n)}{k(n)} \inf_{k(n)\geq j} \frac{k(n)}{\alpha^n} \leq \inf_{k(n)\geq j} \frac{S_k(n)}{k(n+1)} \quad a.s.$$

Upon taking the limits as $j \to \infty$ on both sides of the above equation and taking point (a), (2.5) and (2.6) into account,

$$\frac{1}{\alpha}EX_1 \leq \liminf_{j\to\infty} {}_{k(n)\geq j} \frac{S_{k(n)}}{k(n+1)} \leq \liminf_{j\to\infty} {}_{m\geq j} \frac{S_m}{m}. \tag{2.9}$$

On the other hand, by (2.5)

$$\begin{aligned}\limsup_{j\to\infty}\sup_{m\geq j}\frac{S_m}{m} &\leq \limsup_{j\to\infty}\sup_{k(n)\geq j}\left[\frac{S_{k(n+1)}}{k(n+1)}\,\frac{k(n+1)}{k(n)}\right]\\ &\leq \alpha \limsup_{j\to\infty}\sup_{k(n)\geq j}\frac{S_{k(n+1)}}{k(n+1)} = \alpha EX_1.\end{aligned}$$

This, together with (2.9) implies that

$$\frac{1}{\alpha}EX_1 \leq \liminf_{n\to\infty}\frac{S_m}{m} \leq \limsup_{n\to\infty}\frac{S_m}{m} \leq \alpha EX_1 \qquad \text{a.s.}$$

Since the above result is true for all $\alpha > 1$, the proof is complete. $\square$

2.3 Marcinkiewicz-Zygmund's Strong Law of Large Numbers

Kolmogorov's strong law of large numbers states that for a sequence of independent and identically distributed random variables the first moment exists if and only if S_n is of order smaller than n, that is $S_n = o(n)$ a.s. In general, for $p > 0$, what can we say about the asymptotic fluctuations of $\{S_n\}$ when $E|X|^p$ exists? The answer is provided by the following 1938 result of Marcinkiewicz and Zygmund which essentially says that $S_n = o(n^{1/p})$ a.s for $0 < p < 2$.

Theorem 2 *If X_n is a sequence of i.i.d random variables where*

$$S_n = \sum_{i=1}^{n} X_i$$

then,

i) For $0 < p < 1$, $E|X_1|^p < \infty$ *if and only if*

$$\frac{S_n}{n^{1/p}} \to 0 \quad a.s.$$

ii) For $1 \le p < 2$

$$\frac{S_n - nc}{n^{1/p}} \to 0 \quad a.s$$

if and only if $E|X_1|^p < \infty$, *and if so,* $c = EX_1$.

Proof. For a complete proof of this theorem, refer to Chow and Teicher [4] p.125 or to Zygmund and Marcinkiewicz [18] where this theorem was first proved. We will only present here an outline of the proof as in [4].

We need to employ a result from the Khintchine-Kolmogorov Convergence Theorem (see [4] p.113) which states:

Let $\{X_n;\ n \ge 1\}$ *be a sequence of independent random variables with finite variances and* $EX_n = 0$, $n \ge 1$. *If* $\sum_{j=1}^{\infty} EX_j^2 < \infty$, *then* $\sum_{j=1}^{\infty} X_j$ *converges to some random variable almost surely.*

As usual, we begin by truncating the random variable X_n. Let

$$Y_n = \frac{X_n I_{[\,|X_n| \le n^{1/p}]}}{n^{1/p}}.$$

Then, for $0 < p < \alpha$ we find that

$$\sum_{n=1}^{\infty} E|Y_n|^{\alpha} \le \frac{\alpha}{\alpha - p} E|X_1|^p < \infty. \tag{2.10}$$

Thus taking $\alpha = 2$ in (2.10), $\sum_{n=1}^{\infty}(Y_n - EY_n)$ converges almost surely by the Khintchine-Kolmogorov Convergence Theorem applied to $Y_n - EY_n$. Also,

$$\sum_{n=1}^{\infty} P\{X_n/n^{1/p} \neq Y_n\} = \sum_{n=1}^{\infty} P\{|X_1| > n^{1/p}\} \leq E|X_1|^p < \infty$$

so by the Borel-Cantelli lemma $P\{X_n/n^{1/p} \neq Y_n \ i.o\} = 0$. Therefore

$$P\{\sum_{n=1}^{\infty}(X_n/n^{1/p} - EY_n) \text{ converges } \} = 1$$

if and only if

$$P\{\sum_{n=1}^{\infty} Y_n - EY_n \text{ converges } \} = 1.$$

This implies that $\sum_{n=1}^{\infty}((X_n/n^{1/p}) - EY_n) < \infty$ almost surely.

For case (i) where $0 < p < 1$, taking $\alpha = 1$ in equation (2.10), we have that

$$\sum_{n=1}^{\infty} |EY_n| < \infty.$$

For case (ii) where $1 < p < 2$, assuming without loss of generality that $EX_1 = 0$, we find that the above equation also holds. Hence, the following series converge almost surely:

$$\sum_{n=1}^{\infty} \frac{X_n - EX_n}{n^{1/p}} \quad \text{for } 1 < p < 2, \text{ and } \quad \sum_{n=1}^{\infty} \frac{X_n}{n^{1/p}} \quad \text{for } 0 < p < 1.$$

Applying to the series Kronecker's lemma which states:

If $\{a_n\}$ and $\{b_n\}$ are sequence of real numbers with $0 < b_n \uparrow \infty$, $\sum_{j=1}^{\infty}(a_j/b_j)$ converging, then

$$\frac{1}{b_n}\sum_{j=1}^{n} a_j \to 0,$$

we conclude the proof of the theorem. $\square$

2.3.1 More on the Marcinkiewicz-Zygmund Theorem

Let $\{X_n; n \geq 1\}$ be a sequence of independent and identically distributed random variables and let $S_n = \sum_{i=1}^{n} X_i$. Marcinkiewicz-Zygmund's theorem states that for $1/2 < \alpha < \infty$, S_n is of order smaller than n^α if and only if the $1/\alpha$ th moment exists. It is natural then that we ask whether an analogous result holds if we assume that $0 < \alpha \leq 1/2$. The answer is no, as shown in the following theorem.

Theorem 3 *Suppose X_1 is non-degenerate and $p \geq 2$. Then*

$$\limsup |S_n - b_n|/n^{1/p} \;=\; \infty \quad a.s$$

for every choice of sequence of real constants $\{b_n;\ n \geq 1\}$.

In order to prove this result, the central limit theorem is crucial. The central limit theorem states:

Theorem 4 *If X_1 is nondegenerate with $var\, X_1 < \infty$, then*

$$P\left\{\frac{S_n - nEX_1}{(n\ var\ X_1)^{1/2}} \leq x\right\} \to \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} \exp\left(-y^2/2\right) dy$$

as $n \to \infty$ for each real x.

Proof. (Many introductory probability text books have the proof. See for example Chow &Teicher [4] p.299).

To prove theorem (3), the following definition and lemma are most useful.

Definition 4 *If $\{Y_i,\ i \geq 1\}$ is a sequence of random variables and $\{Y_i',\ i \geq 1\}$ is a sequence of random variables independent of $\{Y_i,\ i \geq 1\}$ with $\{Y_i',\ i \geq 1\}$ having same distributions as $\{Y_i,\ i \geq 1\}$, then*

$$\{Y_i^s,\ i \geq 1\} = \{Y_i - Y_i',\ i \geq 1\}$$

is called the symmetrized version of $\{Y_i,\ i \geq 1\}$.

We note by Kolmogorov's extension theorem, the existence of the symmetrized version $\{Y_i^s,\ i \geq 1\}$ is guaranteed.

Proof of theorem (3) (Stout [28] p.135). Suppose X_1 is nondegenerate. Assume that there exists a $K < \infty$ such that $P\{\limsup\ |S_n - b_n|/n^{1/p} < K\} > 0$. We will show that this assumption produces a contradiction.

Since the set $\{\limsup\ |S_n - b_n|/n^{1/p} < \infty\}$ is a tail event, by Kolmogorov's 0-1 law, $P\{\limsup |S_n - b_n|/n^{1/p} < K\} = 1$. Letting S_n^s be the symmetrized version of S_n, $|S_n - b_n|/n^{1/p} \geq |S_n^s|/n^{1/p} - |S_n' - b_n|/n^{1/p}$, and it follows that

$$P\{\ \limsup(\ |S_n^s|/n^{1/p} \leq 2K)\ \} = 1.$$

Noting that $S_n^s = \sum_{i=1}^n X_i^s$ we have

$$P\{\ \limsup(\ |\sum_{i=1}^n X_i^s|/n^{1/p} \leq 2K)\ \} = 1. \tag{2.11}$$

Since

$$P\{\ \lim_{j\to\infty} \sup_{n\geq j} |\sum_{i=1}^n X_i^s|/n^{1/p} > 2K\ \} \geq P\{\ \cap_{j=1}^{\infty} \cup_{n=j}^{\infty} |\sum_{i=1}^n X_i^s|/n^{1/p} > 2K\ \},$$

using (2.11)

$$
\begin{aligned}
0 &= \lim_{j\to\infty} P\{ \cup_{n=j}^{\infty} |\sum_{i=1}^{n} X_i^s|/n^{1/p} > 2K \} \\
&\geq \lim_{j\to\infty} P\{ |\sum_{i=1}^{j} X_i^s|/j^{1/p} > 2K \}. \qquad (2.12)
\end{aligned}
$$

From the equation

$$\frac{X_n^s}{n^{1/p}} = \frac{\sum_{i=1}^{n} X_i^s}{n^{1/p}} - \frac{\sum_{i=1}^{n-1} X_i^s}{(n-1)^{1/p}} \frac{(n-1)^{1/p}}{n^{1/p}},$$

it follows by (2.11) that

$$P\{ \limsup \frac{|X_n^s|}{n^{1/p}} \leq 4K \} = 1.$$

Therefore, $P\{ |X_i^s| > (4K)i^{1/p} \ i.o \} = 0$ and applying to this the Borel-Cantelli lemma for independent events, we have that $\sum_{i=1}^{\infty} P\{ |X_i^s|^p > (4K)i \} < \infty$. Now by lemma 2 we obtain $E|X_1^s|^p < \infty$, and since $p \geq 2$, it follows that $E|X_1^s|^2 < \infty$. Since the nondegeneracy of X_1 implies the nondegeneracy of X_1^s, it follows by the central limit theorem that

$$\lim_{n\to 0} P\{ \sum_{i=1}^{n} X_i^s/n^{1/2} > 2K\} = 1 - \int_{-\infty}^{(2K)/\sigma} e^{-u^2/2} du > 0;$$

but this contradicts (2.12), thus establishing the theorem. $\square$

We will now explore some variations of this theorem. For $\{X_i\}$ a sequence of pairwise independent and identically distributed random variables, the Marcinkiewicz-Zygmund strong law of large numbers holds for $0 < p \leq 1$. For $1 < p < 2$ however, thus far it's been shown that the condition

$E|X|^p(\log^+|X|)^p < \infty$ where $\log^+ x = \log(2 \vee x)$ is sufficient for the relation $(S_n - ES_n)/n^{1/p} \to 0$ a.s to hold (see Li [16]). Martikaine [19] has slightly improved the sufficiency condition. Namely, we only need that for $\gamma > 0$ and $\gamma > 4p-6$, $E|X_1|^p(\log^+|X_1|)^\gamma < \infty$ in order that $(S_n - ES_n)/n^{1/p} \to 0$ a.s.

2.4 Orthogonal Random Variables

We now state some results for the strong law of large numbers for the situation in which the sequence of random variables $\{X_k\}$ are no longer pairwise independent and identically distributed, but rather orthogonal. Doob's version of the strong law of large numbers states:

If $\{X_k : k = 1, 2, \ldots\}$ is a sequence of random variables with

$$E(X_k) = 0 \;\; \textit{and} \;\; E(X_k^2) = {\sigma_k}^2 < \infty \;\; (k = 1, 2, \ldots), \tag{2.13}$$

$$\sum_{k=1}^{\infty} \frac{{\sigma_k}^2}{k^2} (\log k)^2 < \infty, \tag{2.14}$$

$$E(X_k X_l) = 0 \;\; (k \neq l;\; k, l = 1, 2, \ldots), \tag{2.15}$$

then

$$\frac{1}{n}(X_1 + \ldots + X_n) \to 0 \;\; \textit{a.s.} \;\; (n \to \infty). \tag{2.16}$$

Mòricz [20] has shown that (2.16) remains valid when (2.15) is weakened as follows:

$$E(X_k X_l) = 0 \;\; (2^{p-1} < k < l \leq 2^p; p, k, l = 1, 2, \ldots).$$

We are however limited by how far we can weaken (2.15). Le Gac [13] has recently proved a conjecture of Mòricz which states:

For every $\alpha > 1$, there exists a sequence of random variables $\{X_k\}$ such that (2.13) *and* (2.14) *hold, and*

$$E(X_k X_l) = 0 \;\; (p^\alpha < k < l \leq (p+1)^\alpha; p, k, l = 1, 2, \ldots)$$

are satisfied, but

$$\limsup \frac{1}{n} |X_1 + \ldots + X_n| = \infty \;\; a.s. \;\; (n \to \infty).$$

Chapter 3

Generalizations

3.1 Weighted Sums of Random Variables

Let $\{X_n,\ n \geq 1\}$ be a sequence of i.i.d random variables and let S_n be the nth partial sum of the sequence. Let μ denote the mean of X_1 and assume the first moment of X_1 exists. Then, Kolmogorov's strong law tells us that $(S_n - EX_1)/n$ converges almost surely to μ. In this chapter we study the convergence properties of

$$\tilde{T}_n = \sum_{k=1}^{n} a_{nk} X_k$$

where $\{a_{nk};\ n \geq 1,\ 1 \leq k \leq n\}$ denotes a triangular array of real numbers. Our purpose is to find sufficient and/ or necessary conditions on $\{a_{nk};\ n \geq 1,\ 1 \leq k \leq n\}$ and $\{X_k;\ k \geq 1\}$ such that we obtain convergence almost surely and in probability to a constant for the sequence $\{\tilde{T}_n\}$. Following Stout's definition [28], we will say that $\tilde{T}_n$ is *stable* if $\tilde{T}_n \to c$ almost surely for some constant c. In this chapter, the results of sections 3.2 and 3.2.2 are from Jamison *et al.* [12], section 3.2.1 is from Wright *et al.* [30] and section

3.3 is from Pruitt [23].

3.2 Stability of Weighted Sums of Random Variables

Let $\{X_k;\ k \geq 1\}$ be a sequence of i.i.d. random variables and $\{\omega_k;\ k \geq 1\}$ be a sequence of positive real numbers. Let X be a random variable with the same distribution as the X_k's. Define $T_n = \sum_{k=1}^{n} \omega_k X_k$ and $W_n = \sum_{k=1}^{n} \omega_k$ (so in the notation introduced above, $\tilde{T}_n = T_n/W_n$ $a_{nk} \equiv \frac{\omega_k}{W_n}$ for all $k \leq n$). In studying the a.s.c convergence properties of T_n/W_n, we need to omit the following two trivial cases: when X is degenerate (almost surely a constant) and when $\sum_{k=1}^{\infty} \omega_k < \infty$.

1. Suppose X is degenerate, say $X = m$ (a.s) for some constant m. Then, $T_n/W_n \equiv \sum_{k=1}^{n} \omega_k X_k / W_n = \mathrm{m}$ (a.s) and consequently, $T_n/W_n \rightarrow m$ (a.s) as $n \rightarrow \infty$.

2. Suppose $\sum_{k=1}^{\infty} \omega_k = c < \infty$. Then, $W_n \rightarrow c$ for some constant c. Therefore, the convergence of T_n/W_n and the convergence of $\sum_{k=1}^{\infty} \omega_k X_k$ are equivalent and so, by Kolmogorov's $0-1$ law, either T_n/W_n fails to converge in probability or else it converges almost surely to a non-degenerate limit. Hence, T_n/W_n can not even converge in probability.

Therefore, it is throughout assumed that X is non-degenerate and $\sum \omega_k = \infty$. Such a sequence of weights will be called a divergent sequence of weights.

We will in addition assume that $\omega_n/W_n \to 0$ as $n \to \infty$. As proved in the proposition below, without this assumption, T_n/W_n need not be stable. We will also note (and it is not difficult to show) that

$$\sum_{k=1}^{\infty} \omega_k = \infty \quad \text{and} \quad \omega_n/W_n \to 0$$

if and only if

$$\max_{1 \leq k \leq n} \{\omega_k/W_n\} \to 0 \text{ as } n \to \infty.$$

The condition $\max_{1 \leq k \leq n}\{\omega_k/W_n\} \to 0$ simply says that as n gets large, the contribution of X_n to T_n is significantly reduced.

Remark: The Marcinkiewicz-Zygmund theorem is not generalized in this chapter for under the assumption that $W_n = \sum_{k=1}^{n} \omega_k$, we cannot find an array $\{a_{nk};\ n \geq 1,\ 1 \leq k \leq n\}$ such that $a_{nk} = n^{-p}$, $(k = 1, 2, \ldots, n)$, $0 < p < 2$, $p \neq 1$. However, by letting $\omega_k \equiv 1$ for $k \leq n$, the results of this chapter extend that of Kolmogorov's strong law.

In order for T_n/W_n to be stable, the growth rate of W_n relative to ω_n is crucial.

Proposition 1 *T_n/W_n is stable implies $\omega_n/W_n \to 0$ as $n \to \infty$.*
(In fact, it is necessary that $\omega_n/W_n \to 0$ as $n \to \infty$, in order for convergence i.p.c to hold for $\{T_n/W_n\}$).

Proof. Consider the identity

$$\frac{T_n}{W_n} - \frac{T_{n-1}}{W_{n-1}} = \frac{\omega_n}{W_n}\left(X_n - \frac{T_{n-1}}{W_{n-1}}\right) \qquad (n \geq 2)$$

This identity makes it evident that even the weak law for T_n/W_n fails unless $\omega_n/W_n \to 0$. If we suppose that $T_n/W_n \to c$ in probability for some number c, and that there exists a constant M such that $|W_n/\omega_n| \leq M$ for all n, then using the above identity, for any $\epsilon > 0$

$$0 = \lim_{n\to\infty} P\left(\left|\frac{\omega_n}{W_n}\left(X_n - \frac{T_{n-1}}{W_{n-1}}\right)\right| > \epsilon\right).$$

This implies that

$$0 = \lim_{n\to\infty} P\left(\left|X_n - \frac{T_{n-1}}{W_{n-1}}\right| > M\epsilon\right).$$

Therefore, $X_n - \frac{T_{n-1}}{W_{n-1}} \to 0$ in probability, and since $T_{n-1}/W_{n-1} \to c$ in probability, it follows that $X_n \to c$ in probability. Since convergence in probability implies convergence in distribution, the fact that the X_n's are identically distributed implies that the distribution of X is almost surely equal to c. This contradicts the assumption that X is a non-degenerate random variable. $\square$

Definition 5 *For $x > 0$, let $N(x)$ be the number of subscripts n such that $W_n/\omega_n \leq x$.*

The following corollary establishes the connection between the function $N(x)$ and the stability of T_n/W_n.

Corollary 1 *If $N(x) = \infty$ for some $x > 0$, then the stability of T_n/W_n fails.*

Proof. If T_n/W_n is stable, then by proposition 1, for $x > 0$ there exists a natural number N_x such that $|\omega_n/W_n| < 1/x$ for all $n \geq N_x$. Therefore, $N(x) < N_x < \infty$. $\square$

The following strong law theorem gives a sufficient condition involving the function $N(x)$ so that T_n/W_n is stable.

Theorem 5 *If $E|X| < \infty$, $EN(|X|) < \infty$ and*

$$\int x^2 \int_{y \geq |x|} \frac{N(y)}{y^3} dy dF(x) < \infty, \tag{3.1}$$

then $T_n/W_n \to EX$ almost surely as $n \to \infty$.

(The condition $EN(|X|) < \infty$ is stated only as a convenience. It can be omitted since it is a consequence of equation (3.1): note that

$$\int x^2 \int_{y \geq |x|} \frac{N(y)}{y^3} dy dF(x) \geq \int x^2 N(|x|) \int_{y \geq |x|} \frac{1}{y^3} dy dF(x) = 1/2\ EN(|X|)\ ).$$

The proof of this theorem uses the following two lemmas. In each of these lemmas we will assume that the conditions of theorem (5) hold.

For given positive weights and for each $x > 0$, define

$$N_k(x) = \begin{cases} 1 & \text{if } W_k/\omega_k \leq x \\ 0 & \text{otherwise} \end{cases} .$$

Clearly, $N(x) \equiv \sum_{k=1}^{\infty} N_k(x)$. Also, define a sequence of random variables $\{Y_k;\ k \geq 1\}$ such that

$$Y_k = \begin{cases} X_k & \text{if } |X_k| < W_k/\omega_k \\ 0 & \text{otherwise} \end{cases}$$

and let $Z_n = \sum_{k=1}^{n} \omega_k Y_k$.

Using the usual truncation technique, instead of 'working' with the random variables X_k's, we will use the bounded random variable Y_k's. We will show that Z_n/W_n is stable, and that its stability in turn implies the stability of T_n/W_n.

Lemma 4 $\frac{T_n}{W_n} - \frac{Z_n}{W_n} \to 0$ *almost surely as $n \to \infty$.*

Proof.

$$\sum_{k=1}^{\infty} P(X_k \neq Y_k) = \sum_{k=1}^{\infty} \int_{|x| \geq W_k/\omega_k} dF(x) = \sum_{k=1}^{\infty} \int N_k(|x|)dF(x)$$

$$= \int \sum_{k=1}^{\infty} N_k(|x|)dF(x) = \int N(|x|)dF(x) = EN(|X|) < \infty.$$

Hence, by the Borel-Cantelli lemma, $P(\{X_k \neq Y_k\}\ i.o) = 0$, that is to say if we let $E_k = \{X_k \neq Y_k\}$, then $P(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k) = 0$. Now,

$$\left\{\frac{T_n}{W_n} - \frac{Z_n}{W_n} \not\to 0\right\} \subset \{\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k\}.$$

To see this , suppose $t \notin \{\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k\}$. Then, there exists a number A such that $\forall k \geq A, X_k(t) = Y_k(t)$. This implies that $\sum_{K=A}^{\infty} \omega_k (X_k(t) - Y_k) = 0$. Therefore, for any positive integer n,

$$\sum_{k=1}^{n} \omega_k (X_k(t) - Y_k(t)) \leq \sum_{k=1}^{A-1} \omega_k (X_k(t) - Y_k(t)) \leq C < \infty.$$

Now, since $W_n \to \infty$ as $n \to \infty$, it follows that

$$\frac{1}{W_n} \sum_{k=1}^{n} \omega_k (X_k(t) - Y_k(t)) \leq \frac{C}{W_n} \to 0 \text{ as } n \to \infty.$$

Therefore, $P\left(\frac{T_n}{W_n} - \frac{Z_n}{W_n} \not\to 0\right) \le P\left(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k\right) = 0$. Hence,

$$\frac{T_n}{W_n} - \frac{Z_n}{W_n} \to 0 \quad \text{a.s .}$$

Lemma 5 $\frac{Z_n}{W_n} - E\left(\frac{Z_n}{W_n}\right) \to 0$ *almost surely.*

Proof. Loève [17] p.238 has shown that

If $\{X_n\}_{n=1}^{\infty}$, *are independent and* $\sum_{n=1}^{\infty} Var\left(\frac{X_n}{b_n}\right) < \infty$, $b_n \uparrow \infty$ *and* $S_n = \sum_{i=1}^{n} X_i$, *then* $\frac{S_n}{b_n} - \frac{E(S_n)}{b_n} \to 0$ *almost surely.*

In light of the above result, we will prove the lemma by showing that $\sum_{k=1}^{\infty} \frac{Var(\omega_k Y_k)}{W_k^2} < \infty$.

$$\sum_{k=1}^{\infty} \frac{Var(\omega_k Y_k)}{W_k^2} = \sum_{k=1}^{\infty} \frac{\omega_k^2}{W_k^2} Var(Y_k) = \sum_{k=1}^{\infty} \frac{\omega_k^2}{W_k^2} \int (Y_k^2 - (EY_k)^2) dP$$

$$\le \sum_{k=1}^{\infty} \frac{\omega_k^2}{W_k^2} \int Y_k^2 dP = \sum_{k=1}^{\infty} \frac{\omega_k^2}{W_k^2} \int_{|x| < \frac{W_k}{\omega_k}} x^2 dF(x) = \int x^2 \sum_{\{k:\ \frac{W_k}{\omega_k} > |x|\}} \frac{\omega_k^2}{W_k^2}\, dF(x).$$

In order to estimate the latter equation, using integration by parts observe that

$$\begin{aligned} \sum_{\{k:\ |x| < \frac{W_k}{\omega_k} \le z\}} \frac{{\omega_k}^2}{W_k^2} &= \int_{|x| < y \le z} \frac{dN(y)}{y^2} \\ &= \frac{N(z)}{z^2} - \frac{N(|x|)}{x^2} + 2\int_{|x| < y \le z} \frac{N(y)}{y^3} dy. \end{aligned} \tag{3.2}$$

Using integration by parts again we have that

$$\int_{z < y \le a} \frac{dN(y)}{y^2} = \frac{N(a)}{a^2} - \frac{N(z)}{z^2} + 2\int_{z < y \le a} \frac{N(y)}{y^3} dy,$$

and $\int_{z<y\le a}\frac{dN(y)}{y^2} \ge \frac{1}{a^2}(\int_{z<y\le a} dN(y)) = \frac{1}{a^2}(N(a)-N(z))$. This implies

$$\frac{N(z)}{z^2} - \frac{N(z)}{a^2} \le 2\int_{z<y\le a} \frac{N(y)}{y^3}dy.$$

Letting $a \to \infty$, we obtain that

$$\frac{N(z)}{z^2} \le 2\int_z^\infty \frac{N(y)}{y^3}dy < \infty \tag{3.3}$$

where the integral converges as a result of (3.1). Now, using (3.1), (3.2) and (3.3) we obtain

$$\int x^2 \sum_{\{k:\, \frac{W_k}{\omega_k}>|x|\}} \frac{\omega_k^2}{W_k^2}\, dF(x) \le 2\int x^2 \left(\int_{|x|}^z \frac{N(y)}{y^3}dy + \int_{|x|}^\infty \frac{N(y)}{y^3}dy\right) dF(x)$$

$$= 2\int x^2 \int_{|x|<y} \frac{N(y)}{y^3}dy\, dF(x) < \infty.$$

Hence, $\sum_{k=1}^\infty \frac{Var(\omega_k Y_k)}{W_k^2} < \infty$ and $W_k \uparrow \infty$ thus completing the proof of the lemma. $\square$

Proof of theorem 5. Let $\mu_k = EY_k$ and $\mu = EX$. Then, letting I to be the indicator function,

$$\mu_k = \int_{|x|<\frac{W_k}{\omega_k}} x dF(x) = \int x I_{|x|<\frac{W_k}{\omega_k}} dF(x).$$

Since the random variable $XI_{\{|X|<\frac{W_k}{\omega_k}\}}$ converges to X (recall that X is a random variable that has the same distribution as the X_k's) and $E|X| < \infty$, by the dominated convergence theorem

$$\mu_k = \int_{|x|<\frac{W_k}{\omega_k}} x\, dF(x) \to \mu \text{ as } k \to \infty.$$

Since $W_n \to \infty$ and $\omega_k/W_n \to 0$ as $n \to \infty$ (k fixed), using Toeplitz's lemma,

$$\left|E\left(\frac{Z_n}{W_n}\right) - \mu\right| = \left|\frac{1}{W_n}\sum_{k=1}^{n}\omega_k(\mu_k - \mu)\right| \to 0 \quad (n \to \infty)$$

Equivalently, $E\left(\frac{Z_n}{W_n}\right) \to \mu$ as $n \to \infty$. Therefore by lemma (4) and lemma (5), $T_n/W_n \to \mu$ thus proving theorem (5). $\square$

Our objective now is to find a class of weights $\{\omega_k\}$ such that T_n/W_n is stable with $|X|$ having a finite moment. The result which we're seeking will not directly involve the function $N(x)$ for computing this function could be difficult. In order to achieve our goal however we need to study further the role which the function $N(\cdot)$ plays in the stability of T_n/W_n.

Lemma 6 *T_n/W_n is stable implies $\omega_n X_n/W_n \to 0$ almost surely, and the latter condition is equivalent to $EN(\,c|X|\,) < \infty$ for every $c > 0$.*

Proof. Recalling the identity in proposition (1), we see that if T_n/W_n is stable then $\omega_n X_n/W_n \to 0$ (a.s). Now, by lemma 3 (and its converse as $\{X_n\}$ is an independent sequence of r.v's),

$$\frac{\omega_n X_n}{W_n} \to 0 \,(\text{a.s}) \;\text{ if and only if }\; \sum_{n=1}^{\infty} P\left(\left|\frac{\omega_n X_n}{W_n}\right| \geq \epsilon\right) < \infty$$

for any $\epsilon > 0$. Since

$$\int N\left(\frac{|x|}{\epsilon}\right)\, dF = \sum_{n=1}^{\infty}\int_{|x|\geq\epsilon\frac{W_n}{\omega_n}} dF = \sum_{n=1}^{\infty} P\left(\left|\frac{\omega_n X_n}{W_n}\right| \geq \epsilon\right),$$

it follows that $EN(\,c|X|\,) < \infty$ for every $c > 0$ if and only if $\omega_n X_n/W_n \to 0$ almost surely. $\square$

Proposition 2 *For a given sequence of weights* $\{\omega_k\}$, $T_n/W_n \to EX$ *almost surely with* $E|X| < \infty$, *if and only if* $\limsup N(x)/x < \infty$ *as* $x \to \infty$.

Remark: $\limsup N(x)/x < \infty$ as $x \to \infty$, is equivalent to the existence of a constant $K < \infty$ such that

$$N(n)/n \leq K \tag{3.4}$$

for all $n \geq 1$.

Example 1 (Stout[28] p.220) This example helps in interpreting equation (3.4). Clearly, if $\omega_n = 1$ for all $k \geq 1$, equation (3.4) holds and so Kolmogorov's strong law is included in the statement of the proposition. However, if we let $\omega_1 = \omega_2 = 1$ and $\omega_k = W_{k-1}/(-1 + \log k)$ for $k \geq 3$, (hence $W_k/\omega_k = \log k$), then equation (3.4) fails. To see this, note from the definition of $N(n)$

$$\begin{aligned} N(n) &= |\{k : W_k/\omega_k \leq n\}| = |\{k : \log k \leq n\}| \\ &= |\{k : \log k \leq n\}| = |\{k : k \leq \exp(n)\}| \\ &= |\{k : k \leq \exp(n)\}| = \lfloor \exp(n) \rfloor . \end{aligned}$$

Hence, $N(n)/n \geq (\exp(n) - 1)/n \to \infty$ as $n \to \infty$. Roughly, equation (3.4) implies there cannot be too many ω_k's whose magnitude relative to W_k is too large. In the example above, the 'largeness' of the ω_k's allowed $N(\cdot)$ to be an exponential function.

Proof of proposition 2. Suppose $\limsup N(x)/x < \infty$ as $x \to \infty$. Then, for some $M < \infty$, $N(x) < Mx$ for all $x > 0$. Hence, $E|X| < \infty$ implies

$$\int x^2 \int_{y \geq |x|} \frac{N(y)}{y^3} dy \, dF(x) \leq \int x^2 \frac{M}{|x|} dF(X) = ME|X| < \infty.$$

By theorem 5, $T_n/W_n \to EX$ a.s as $n \to \infty$.

On the other hand, if $\limsup N(x)/x = \infty$, then there exists a sequence $\{x_k\}$ such that $N(x_k)/x_k > k$, $k \geq 1$. By choosing $f_k = 1/(ck^2 n_k)$ where $c = \sum_{k \geq 1}(1/k^2 x_k)$, we have that

$$\sum_{k \geq 1} f_k = 1 \quad \text{and} \quad f_k x_k = x_k/(ck^2 x_k) = 1/ck^2$$

and so $\sum_{k \geq 1} f_k x_k < \infty$. Since $f_k N(x_k) > k f_k x_k = 1/ck$, it follows that $\sum_{k \geq 1} f_k N(x_k) = \infty$. The sequence $\{f_k\}$ defines a distribution such that $E|X| < \infty$, but $EN(|X|) = \infty$. Hence, T_n/W_n is not stable by lemma 6. $\square$

Let us return to our main objective, that is, finding a class of weights $\{\omega_k\}$ such that T_n/W_n is stable and X has a finite expectation. From proposition 1 we know that if T_n/W_n is stable, then $\omega_n/W_n \to 0$. The converse of proposition 1 in general is not true. We can see this from example 1 where $\omega_n/W_n = 1/\log n \to 0$ as $n \to \infty$ but, $N(x)$ grew arbitrarily large and in light of proposition 2, T_n/W_n failed to be stable.

The constraints on the weights $\{\omega_k\}$ Jamison *et al.* consider in [12] in order for the converse of proposition 1 to hold involve the uniform bounding of the ω_k's. Without loss of generality, we can assume this uniform bound of the ω_k's to be one so that $0 < \omega_k \leq 1$ and $W_n \to \infty$.

Lemma 7 *For a divergent sequence of positive weights bounded by one ,*

$$\limsup N(x)/x \log x \ \le 2 \qquad as \quad x \to \infty.$$

Proof. Fix $x > 0$. Let $B_n = \{k : \ n < W_k \le n+1\}$ and V_n be the number of k's in B_n such that $x \ge \frac{W_k}{\omega_k}$. Let

$$B'_n = \{k : \ n < W_k \le n+1, \text{ and } \frac{W_k}{\omega_k} \le x\}.$$

Since W_k diverges, there can only be finite number of k's in B_n, and consequently, B'_n also contains a finite number of k's.

Let $k_1, \dots, k_r$ be all the k's in $B\prime_n$. Then, $| \{k_1, \dots, k_r\} | = \ V_n$ and

$$\frac{W_{k_1} + \cdots + W_{k_r}}{x} \le \sum_{k \in B\prime_n} \omega_k \le \sum_{k \in B_n} \omega_k.$$

Now,

$$\frac{W_{k_1} + \cdots + W_{k_r}}{x} > \frac{nV_n}{x} \quad \text{as each } W_{k_i} > n.$$

Therefore,

$$\frac{nV_n}{x} \le \sum_{k \in B_n} \omega_k.$$

Now let $k_{r_1}, \dots k_{r_j}$, in an increasing order, be the elements of B_n. Then, the fact that $n < W_{k_{r_1}-1} + \omega_{k_{r_1}} = W_{k_{r_1}} \ \le n+1$ implies that $n - 1 < W_{k_{r_1}-1}$. Since $W_{k_{r_1}-1} + \sum_{i \in B_n} \omega_i \le n+1$, it follows that

$$\sum_{i \in B_n} \omega_i \le (n+1) - (n-1) = 2.$$

We now have that $\frac{nV_n}{x} \leq \sum_{k \in B_n} \omega_k \leq 2$ and so, $V_n \leq \frac{2x}{n}$. Recalling the definition of the function $N(x)$, it follows that

$$N(x) = \sum_{n=0}^{\lfloor x \rfloor} V_n \leq V_0 + 2x \sum_{n=1}^{\lfloor x \rfloor} \frac{1}{n} \leq V_0 + 2x \log x \ .$$

Since V_0 is a finite number, $\limsup_{x \to \infty} N(x)/x \log x \leq 2$. $\square$

It is interesting to note that Jamison *et al.* [12](p.43, ex.1) have constructed a sequence of weights $\{\omega_k \ k \geq 1\}$ such that $|\omega_k| < \infty$ for all $k \geq 1$ and yet $\limsup_{x \to \infty} N(x)/x = \infty$. This suggest that additional conditions, perhaps on the random variables $\{X_k;\ k \geq 1\}$, are needed if T_n/W_n is to be stable. The next theorem tells us what happens when we consider a condition slightly stronger than the existence of $E|X|$.

Theorem 6 *: Let $\{\omega_k;\ k \geq 1\}$ be any bounded sequence of weights. If $E|X| \log^+ |X| < \infty$, then $T_n/W_n \to EX$ a.s as $n \to \infty$.*

Proof. Without loss of generality, assume the weights are bounded by one. In this proof, we will use the results of lemma 7 and theorem 5. By lemma 7, there exists a a number $R < \infty$ such that $N(y) \leq R\, y \log y$ for all $y > 0$. Since $N(y) = 0$ for $y < 1$, (because $W_n/\omega_n = [(\omega_1 + \dots + \omega_{n-1})/\omega_n] + 1$), we have that

$$\int_{y>|x|} \frac{N(y)}{y^3} dy = \int_{y>|x|\geq 1} \frac{N(y)}{y^3} dy \ \leq R \int_{|x|}^{\infty} \frac{\log^+(y)}{y^2} dy$$

$$= R \left(\frac{\log^+ |x|}{|x|} + \frac{1}{|x|} \right) .$$

Therefore,

$$\int x^2 \int_{y>|x|} \frac{N(y)}{y^3} dy\, dF(x) \leq (2+R) \int x^2 \left(\frac{\log^+ |x|}{|x|} + \frac{1}{|x|} \right) dF(x)$$

$$= R \int (\, |x| \log^+ |x| + |x| \,) \, dF(x)$$

$$= R \left[\int |x| \log^+ |x| \, dF(x) + \int |x| \, dF(x) \right]$$

Since $E|X| \log^+ |X| < \infty$ implies $E|X| < \infty$ it follows that

$$\int x^2 \int_{y>|x|} \frac{N(y)}{y^3} dy\, dF(x) < \infty.$$

By theorem 5, $T_n/W_n \to EX$ as $n \to \infty$.

Remark: Etemadi [8] has shown that theorem 5 remains true even when the sequence of the random variables $\{X_n\}$ are pairwise independent. The proof involves the usage of the subsequence technique similar to the one employed in the proof of Etemadi's version of the strong law (see chapter 1).

Remark: It is interesting to compare the classical Kolmogorov's strong law of large number with theorem 6. In the first instance,

$$\sum_{k=1}^{n} X_k/n \to c \quad \text{a.s}$$

for a constant c if and only if $E|X| < \infty$, while in the second instance,

$$\frac{T_n}{W_n} = \frac{\sum_{k=1}^{n} \omega_k X_k}{W_n} \to c \quad \text{a.s}$$

for a constant c if $E|X| \log^+ |X| < \infty$. This latter condition is not necessary and in fact, the next section illustrates the stability of T_n/W_n for some admissible sequence $\{\omega_k\}$ even when $E|X| = \infty$.

3.2.1 Arbitrarily Heavy Tails and the Strong Law

Without any moment assumption, in proposition 1 we saw that in order for convergence i.p.c for the sequence $\{T_n/W_n\}$ it is necessary that

$$W_n \to \infty \quad \text{and} \quad \omega_n/W_n \to 0 \quad \text{as} \quad n \to \infty. \tag{3.5}$$

However, restricting our attention to those X's for which $E|X| < \infty$, we were able to construct positive weights $\{\omega_k\}$ which satisfied (3.5) although T_n/W_n failed to be stable (see example 1). Eventhough for $\omega_k \equiv 1$ the stability of T_n/W_n is a moment result (this is Kolmogorov's strong law of large numbers), with example 1 in mind Wright *et al.*, [30], pose the following question: does a sequence of positive weights $\{\omega_k\}$ and a sequence of random variables $\{X_k\}$ exist such that (3.5) hold and T_n/W_n is stable, but $E|X| = \infty$? The answer to this question is provided by letting $g(x) = |x|$ in the next proposition.

Proposition 3 *Let g be a nonnegative function defined for nonnegative real numbers with $g(x) \to \infty$ as $x \to \infty$. Then there exist a sequence of i.i.d random varibales $\{X_k\}$, a sequence of positive weights $\{\omega_k\}$ satisfying* (3.5) *and a constant c for which $T_n/W_n \to c$ almost surely and $Eg(X^+) = Eg(X^-) = \infty$.*

In addition, Wright *et al.* extend theorem (5) of Jamison *et al.*. In this new result, theorem (5) of Jamison *et al.* has been slightly modified in order to include random variables which do not have a first moment.

Theorem 7 *Suppose $\{\omega_k\}$ is a sequence of weights satisfying* (3.5). *Let μ be a constant number. If*

$$\int_{|x|<T} x dF(x) \to \mu \quad as \quad T \to \infty \tag{3.6}$$

and if

$$\int x^2 \int_{y \geq |x|} \frac{N(y)}{y^3} dy \, dF(x) < \infty, \tag{3.7}$$

then $T_n/W_n \to \mu$ almost surely.

Truncating X at a number T and using the dominated convergence theorem, if $E|X| < \infty$ then $\int_{|x|<T} x dF(x) \to \mu = EX$ as $T \to \infty$. On the other hand, if $E|X| = \infty$ then the mean does not exist.

Proof of theorem 7. The proof is exactly the same as the proof of theorem 5 of section 3.2 except in showing that $E(Z_n)/W_n \to \mu$. In this case, we are given that $EY_k = \int_{|x|<\frac{W_k}{\omega_k}} x \, dF(x) \to \mu$. Using (3.5) and (3.6), it follows that $E(Z_n)/W_n \to \mu$. $\square$

Wright *et al.* also points out a result of Chow & Teicher [3] in which they show that for any random variable X for which

$$\liminf_{x \to \infty} xP\{|X| > x\} > 0, \tag{3.8}$$

the stability of T_n/W_n fails for any choice of of positive weights $\{\omega_k\}$ satisfying (3.5). For a proper interpretation of (3.8) note that

$$\liminf_{x \to \infty} xP\{|X| > x\} > 0 \Longrightarrow E|X| = \infty. \tag{3.9}$$

The reason for this implication is as follows. If $E|X| < \infty$ then $|X| < \infty$ a.s. Hence, given an $x > 0$, $|X|I_{|X|>x} \to 0$ and as $x \to \infty$. By the dominated convergence theorem

$$0 = \lim_{x\to\infty} \int_{|X|>x} |X| dP \geq \lim_{x\to\infty} xP(|X| > x).$$

Therefore the class of i.i.d random variables considered in theorem 7 are those with

$$\liminf_{x\to\infty} xP\{|X| > x\} = 0.$$

For examples which (3.8) is satisfied, consider the St.Petersburg paradox ($P\{X = 2^k\} = 2^{-k}$ for $k \geq 1$) and the Cauchy distribution. For further detail see Durrett [5] (p.32 example 5.6.)

Wright *et al.* also extends theorem 2 of Jamison *et al.*. They show that theorem 2 is a special case when $r = 1$ of the following corollary of theorem 7.

Corollary 2 *Let* $1 \leq r < 2$ *and let* $\{\omega_k\}$ *be a sequence of weights which satisfy equation* (3.5). *The stability of* T_n/W_n *holds for all* X *with* $E|X|^r < \infty$ *if and only if* $\limsup_{x\to\infty} N(x)/x^r < \infty$.

Wright *et al.* also makes an interesting point: if $0 < r < 1$, then there can not exist weights which satisfy $\limsup_{x\to\infty} N(x)/x^r < \infty$ due to the following result which Wright *et al.* prove as proposition 1.

Let $\{\omega_k\}$ *be a sequence of positive numbers satisfying* $\omega_n/W_n \to 0$. *Then* $\sum_k \omega_k = \infty$ *if and only if* $I = \int_1^\infty N(x)x^{-2}dx$ *is infinite.*

If such a sequence $\{\omega_k\}$ were to exist then for all $x \geq 1$, there exists a numbre k such that

$$\frac{N(x)}{x^2} \leq \frac{k}{x^{2-r}} .$$

Thus the correspodning integral I is finite contradicting the proposition.

3.2.2 Arbitrarily Heavy Tails and the Weak Law

In this section, we will examine the necessary and sufficient conditions for convergence i.p.c to hold for the sequence $\{T_n/W_n\}$ when $E|X|$ does not necessarily exist. Let us for a moment return to Chow & Teicher's result which says that condition

$$\liminf_{x\to\infty} P\{|X| > x\} = 0$$

is necessary for a.s.c convergence of $\{T_n/W_n\}$ to hold. By adding an additional condition, namely

$$\lim_{c\to\infty} \int_{|x|<c} x dF(x) < \infty,$$

we obtain the following theorem. (As in the previous sections, let $\{\omega_n\}$ be a sequence of non-negative real numbers, and put $W_n = \sum_{k=1}^{n} \omega_k$).

Theorem 8 *$\{T_n/W_n\}$ converges i.p.c for all divergent sequences $\{\omega_k\}$ such that $\omega_k/W_n \to 0$ (or equivalently $\max_{1\le k\le n} \omega_k/W_n \to 0$) if and only if*

$$\lim_{c\to\infty} cP(|X| > c) = 0 \quad \textit{and} \quad \lim_{c\to\infty} \int_{|x|<c} x dF(x) \textit{ exists.} \tag{3.10}$$

Remark: Although condition (3.10) is weaker than $E|X| < \infty$, it 'almost' says that $E|X| < \infty$. This is due to the following result of Rohatgi [25] which states:

Let X be a random variable with a distribution satisfying $n^\alpha P\{|X| > n\} \to 0$ as $n \to \infty$ for some $\alpha > 0$. Then $E|X|^\beta < \infty$ for $0 < \beta < \alpha$.

Proof of theorem 8. Suppose (3.10) is true. Let X_{nk} be X_k truncated at W_n/ω_k as follows:

$$X_{nk} = \begin{cases} X_k & \text{if } |X_k| < W_n/\omega_k \\ 0 & \text{otherwise} \ . \end{cases}$$

Define $S_n = \sum_{k=1}^{n} \omega_k X_k$ and $S_{nn} = \sum_{k=1}^{n} \omega_k X_{nk}$. The proof involves the evaluation of the probability limit S_{nn}/W_n, and then showing that this probability limit equals the probability limit of S_n/W_n. We have that

$$P\{S_{nn} \neq S_n\} = P\{ \sum_{k=1}^{n} \omega_k(X_k - X_{nk}) \neq 0 \} = P\{ \bigcup_{k=1}^{n} ((X_k - X_{nk}) \neq 0) \}$$

$$\leq \sum_{k=1}^{n} P\{ (X_k - X_{nk}) \neq 0\} = \sum_{k=1}^{n} P\{X_k \neq X_{nk}\} = \sum_{k=1}^{n} P\{ |X_k| \geq W_n/\omega_k\}$$

Since $\max_{1 \leq k \leq n} \omega_k/W_n \to 0$, we have that for all $k \leq n$, $W_n/\omega_k \to \infty$ as $n \to \infty$. Therefore, from hypothesis (3.10), given $\epsilon > 0$, there exists a natural number N such that for all $n \geq N$, $P\{|X_k| \geq W_n/\omega_k\} < \epsilon\,(\omega_k/W_n)$. This implies

$$\sum_{k=1}^{n} P\{|X_k| \geq W_n/\omega_k\} < \epsilon \sum_{k=1}^{n} \omega_k/W_n = \epsilon.$$

Hence,

$$P\{S_{nn} \neq S_n\} \to 0 \text{ as } n \to \infty. \tag{3.11}$$

Using integration by parts we have,

$$\begin{aligned} \frac{1}{T}\int_{|x|<T} x^2 dF(x) &= \frac{1}{T}\left(-T^2 P\{|X| > T\} + 2\int_{0\leq x\leq T} xP\{|x| > x\}dx\right) \\ &= -TP\{|X| > T\} + \frac{2}{T}\int_{0\leq x\leq T} xP\{|x| > x\}dx. \quad (3.12) \end{aligned}$$

By the first condition of (3.10), for any $\epsilon > 0$ there exists an N such that for all $x > N$ $xP\{|X| > x\} < \epsilon$. Assuming, without loss of generality, that $T > N$, for the second part of equation (3.12) we have that

$$\begin{aligned}\frac{2}{T}\int_{0\le x\le T} xP\{|X|\ge x\}\,dx &= \frac{2}{T}\left[\int_0^N xP\{|X|\ge x\}\,dx + \int_N^T xP\{|X|>x\}\,dx\right]\\ &\le \frac{2}{T}\left[\,N^2+\epsilon(T-N)\,\right] \le \frac{2N^2}{T}+\epsilon.\end{aligned}$$

Therefore

$$\frac{1}{T}\int_{|x|\le T} x^2\,dF(x) \le -TP\{|X|\ge T\} + \frac{2N^2}{T} + \epsilon \to \epsilon \text{ as } T\to\infty. \quad (3.13)$$

Since ϵ is arbitrary,

$$\frac{1}{T}\int_{|x|\le T} x^2\,dF(x) \to 0 \text{ as } T\to\infty. \quad (3.14)$$

Employing (3.14) it now follows that for a sufficiently large n,

$$\begin{aligned}\mathrm{Var}\left(\frac{S_{nn}}{W_n}\right) &= \frac{1}{W_n^2}\sum_{k=1}^n \omega_k^2\ \mathrm{Var}\,(X_{nk}) \le \frac{1}{W_n^2}\sum_{k=1}^n\left(\omega_k^2\int_{|x|<W_n/\omega_k} x^2dF(x)\right)\\ &= \frac{1}{W_n^2}\sum_{k=1}^n\left(\omega_k^2\,\frac{W_n}{\omega_k}\,\frac{\omega_k}{W_n}\int_{|x|<W_n/\omega_k} x^2dF(x)\right) \le \frac{1}{W_n^2}\sum_{k=1}^n \omega_k^2\,\frac{W_n}{\omega_k}\,\epsilon = \epsilon\end{aligned}$$

which is equivalent to saying that

$$\mathrm{Var}\left(\frac{S_{nn}}{W_n}\right) \to 0 \text{ as } n\to\infty.$$

Using Chebyshev's inequality, we will proceed to show that for any $a > 0$

$$P\left(\left|\frac{S_n}{W_n}-\mu\right|\ge a\right)\to 0 \text{ as } n\to\infty.$$

We first note that

$$E(S_{nn}/W_n) = \frac{1}{W_n}\sum_{k=1}^{n}\omega_k E(X_{nk}) = \frac{1}{W_n}\sum_{k=1}^{n}\omega_k \int_{|x|<W_n/\omega_k} x\,dF(x) \to \mu \quad (3.15)$$

as $n \to \infty$. Letting $\mu_n = E(S_{nn}/W_n)$, by Chebyshev's inequality

$$P\left(\left|\frac{S_{nn}}{W_n} - \mu_n\right| \geq a\right) \leq \frac{\mathrm{Var}\left(\frac{S_{nn}}{W_n}\right)}{a^2} \to 0.$$

Since $\mu_n \to \mu$, by (3.15) and the fact that $\frac{S_{nn}}{W_n} - \mu_n \to 0$ in probability, it follows that $\frac{S_{nn}}{W_n} \to \mu$ in probability . Combining this with (3.11) we obtain

$$P\left\{\left|\frac{S_n}{W_n} - \mu\right| \geq a\right\}$$

$$= P\left\{S_{nn} = S_n\ ,\ \left|\frac{S_n}{W_n} - \mu\right| \geq a\right\} + P\left\{S_{nn} \neq S_n\ ,\ \left|\frac{S_n}{W_n} - \mu\right| \geq a\right\}$$

$$\leq P\left\{\left|\frac{S_{nn}}{W_n} - \mu\right| \geq a\right\} + P\left\{S_{nn} \neq S_n\right\} \to 0.$$

Hence $S_n/W_n \to \mu$ in probability. This ends the sufficiency part of the proof.

For necessity, suppose T_n/W_n converges i.p.c. Then applying the classical degenerate convergence criterion, (see Loève [17] p.278), with $\omega_k \equiv 1$, (3.10) is obtained. $\square$

3.3 Arrays of Weights

As before, let $\{X_k\}$ be a sequence of independent, identically distributed random variables with $E|X_1| < \infty$ and $EX_1 = \mu$. Let X be a random variable with the same distribution as the X_k's. Proposition 1 in section 3.2 says that the a.s.c convergence of $\{T_n/W_n\}$ implies $\omega_n/W_n \to 0$ as $n \to \infty$ where $\sum_1^n \omega_n = W_n$ and $W_n \to \infty$ as $n \to \infty$. However, the converse of the implication does not hold in general, as was shown by example 1. In this section, we will prove that when $E|X| < \infty$ the convergence of $\{T_n/W_n\}$ i.p.c holds if and only if $\omega_n/W_n \to 0$ and $W_n \to \infty$. More generally, we will prove the analogous result for arrays of weights $A = (a_{nk})$ where A is a Toeplitz matrix. Furthermore, a moment condition on the random variable X will be established in order for $Y_n = \sum_{k=1}^{\infty} a_{nk}X_k \to \mu$ almost surely as $n \to \infty$.

We say $A = (a_{nk})$ is a Toeplitz matrix if:

$$\lim_{n\to\infty} a_{nk} = 0 \text{ for every } k, \tag{3.16}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1, \text{ and} \tag{3.17}$$

$$\sum_{k=1}^{\infty} |a_{nk}| \leq M \text{ for all } n. \tag{3.18}$$

We should note that the structure of the matrix A defined by a sequence of positive numbers $\{\omega_k\}$ such that

$$a_{nk} = \begin{cases} \omega_k / W_n & \text{if } 1 \leq k \leq n, \\ 0 & \text{if } k > n \end{cases}$$

where $W_n = \sum_{k=1}^{n} \omega_k$ is a special case of the Toeplitz matrix- it is for this type of Toeplitz matrix on which the results in the previous sections are based.

Since for each $n \geq 1$ $E \sum_{k=1}^{\infty} |a_{nk} X_k| = E|X| \sum_{k=1}^{\infty} |a_{nk}| \leq ME|X| < \infty$, the random sum $\sum_{k=1}^{\infty} |a_{nk} X_k|$ converges absloutely with probability one, and so, the sequence of random variables $\{Y_n;\ n \geq 1\}$ is well defined (since $E|Y_n| < \infty$, $Y_n < \infty$ a.s.).

In the first part of this section, we will provide a necessary and sufficient condition for the convergence of Y_n i.p.c to hold. (The trivial case when X is almost surely equal to μ will be omitted).

Theorem 9 *A necessary and sufficient condition for* $Y_n \to \mu = EX_1$ *in probability is that* $\max_{1 \leq k \leq n} |a_{nk}| \to 0$ *as* $n \to \infty$.

Proof. Suppose $\max_{1 \leq k \leq n} |a_{nk}| \to 0$ as $n \to \infty$. Let

$$X_{nk} = \begin{cases} a_{nk} X_k & \text{if } |a_{nk} X_k| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and let $Z_n = \sum_{k=1}^{\infty} X_{nk}$. We will first prove that $Z_n - Y_n \to 0$ in probability. Then, it will suffice to show that $Z_n \to \mu$ in probability in order to complete the sufficiency part of the proof. In order to show this however, we will beforehand need to prove that $\mathrm{Var}(Z_n) \to 0$ as $n \to \infty$; for then, an application of the Chebyshev's inequality will enable us to conclude that $Z_n \to \mu$ in probability.

First recall that (by 3.9) $E|X| < \infty$ implies

$$\lim_{T\to\infty} TP\{|X| > T]\} = 0. \tag{3.19}$$

We will now show that $Z_n - Y_n \to 0$ in probability. To begin with, we have

$$\begin{aligned} P\{Z_n \neq Y_n\} &= P\left\{\sum_{k=1}^{\infty} X_{nk} \neq \sum_{k=1}^{\infty} a_{nk}X_k\right\} = P\left\{\cup_{k=1}^{\infty}\{X_k - X_{nk} \neq 0\}\right\} \\ &\leq \sum_{k=1}^{\infty} P\{|X_k| > |a_{nk}|^{-1}\} = \sum_{k=1}^{\infty} P\{|X| > |a_{nk}|^{-1}\}. \end{aligned} \tag{3.20}$$

Since $|a_{nk}|^{-1} \to \infty$ as $n \to \infty$ $(1 \leq k \leq n)$, for a given $\epsilon > 0$ by (3.19) there exists a natural number N such that for all $n \geq N$,

$$|a_{nk}|^{-1}P\{|X| \geq |a_{nk}|^{-1}\} < \epsilon/M.$$

Since $\sum_{k=1}^{\infty} |a_{nk}| \leq M$ for all n,

$$\sum_{k=1}^{\infty} P\{|X| \geq |a_{nk}|^{-1}\} \leq \sum_{k=1}^{\infty} |a_{nk}|\epsilon/M \leq \epsilon.$$

Hence, returning to (3.20),

$$P\{Z_n \neq Y_n\} \to 0 \quad \text{as} \quad n \to \infty.$$

that is, $Z_n - Y_n \to 0$ in probability. To prove $\mathrm{Var}(Z_n) \to 0$, by using integration by parts we have the following equality:

$$\frac{1}{T}\int_{|x|\leq T} x^2\, dF(x) = -TP\{|X| \geq T\} + \frac{2}{T}\int_{0\leq x\leq T} xP\{|X| \geq x\}\, dx. \tag{3.21}$$

By (3.19), given any $\epsilon > 0$ there exists a natural number N such that for all $x \geq N$, $xP[|X| \geq x] < \epsilon$. Assuming without loss of generality that $T > N$, for the second part of the sum in equation (3.21) we have that

$$\begin{aligned}
\frac{2}{T}\int_{0\leq x\leq T} xP[|X|\geq x]\,dx &= \frac{2}{T}\left[\int_0^N xP[|X|\geq x]\,dx + \int_N^T xP[|X|>x]\,dx\right] \\
&\leq \frac{2}{T}\left[\,N^2+\epsilon(T-N)\,\right] \\
&\leq \frac{2N^2}{T}+\epsilon.
\end{aligned}$$

Therefore,

$$\frac{1}{T}\int_{|x|\leq T} x^2\,dF(x) \leq -TP\{|X|\geq T\} + \frac{2N^2}{T} + \epsilon \to \epsilon \text{ as } T\to\infty.$$

Since ϵ is arbitrary,

$$\frac{1}{T}\int_{|x|\leq T} x^2\,dF(x) \to 0 \text{ as } T\to\infty. \tag{3.22}$$

Now, since $\sum_{k=1}^{\infty}|X_{nk}| \leq \sum_{k=1}^{\infty}|a_{nk}X_k|$, the random sum $\sum_{k=1}^{\infty}|X_{nk}|$ is finite almost surely. Using the monotone convergence theorem,

$$E(|Z_n|^2) = \int \lim_{m\to\infty}(\sum_{k=1}^{m}|X_{nk}|)^2\,dP = \lim_{m\to\infty}\int(\sum_{k=1}^{m}|X_{nk}|)^2\,dP$$

$$= \lim_{m\to\infty}\int\left(\sum_{k=1}^{m}|X_{nk}|^2dP + 2\sum_{1\leq i<j\leq m}|X_{ni}||X_{nj}|\right)dP$$

$$= \lim_{m\to\infty}\left[\sum_{k=1}^{m}\int_{|a_{nk}X_k|\leq 1} a_{nk}^2X_k^2\,dP + 2\lim_{m\to\infty}(E|X|)^2\sum_{1\leq i<j\leq m}|a_{ni}||a_{nj}|\right] \tag{3.23}$$

Using (3.22) and assumption (3.18) for the first sum of the latter equation, given $\epsilon > 0$ there exists an N_1 such that for all $n \geq N_1$

$$\lim_{m\to\infty}\sum_{k=1}^{m}\int_{|a_{nk}X_k|\leq 1} a_{nk}^2X_k^2\,dP = \lim_{m\to\infty}\sum_{k=1}^{m}|a_{nk}||a_{nk}|\int_{|x|\leq|a_{nk}|^{-1}} x^2\,dF(x)$$

$$\leq \lim_{m\to\infty} \sum_{k=1}^{m} |a_{nk}| (\epsilon/2M)$$
$$\leq \epsilon/2.$$

For the second sum of equation (3.23), since $\lim_{n\to\infty} a_{nj} = 0$ for a every j, there exists an N_2 such that for all $j \leq m$ and for all $n \geq N_2$,

$$|a_{nj}| < \epsilon / [4(E|X|)^2 M].$$

Therefore,

$$\begin{aligned} &\lim_{m\to\infty} 2(E|X|)^2 \sum_{1\leq i<j\leq m} |a_{ni}||a_{nj}| \\ \leq &\lim_{m\to\infty} 2(E|X|)^2 \epsilon / [4(E|X|)^2 M] \sum_{i=1}^{m} |a_{ni}| \\ \leq &\; 2(E|X|)^2 \epsilon/[4(E|X|)^2 M] M = \epsilon/2 \end{aligned}$$

Letting $N = \max\{N_1, N_2\}$, we now have that for all $n \geq N$

$$\lim_{m\to\infty} \sum_{k=1}^{m} \int_{|a_{nk}X_k|\geq 1} a_{nk}^2 X_k^2 \, dP + 2 \lim_{m\to\infty} (E|X|)^2 \sum_{1\leq i<j\leq m} |a_{ni}||a_{nj}|$$
$$\leq \epsilon/2 + \epsilon/2 = \epsilon$$

and therefore, by equation (3.23), it follows that $E|Z_n|^2 \to 0$ as $n \to \infty$. We now need one last result to conclude the proof for sufficiency. Letting $\mu_n = E(Z_n)$, we have

$$\mu_n - \mu = \sum_{k=1}^{\infty} a_{nk} \left[\int_{|x|<|a_{nk}|^{-1}} x dF - \mu \right] + \mu[\sum_{k=1}^{\infty} a_{nk} - 1] \to 0$$

as $n \to \infty$. Using Chebyshev's inequality, for any $c > 0$

$$P\{ |Z_n - \mu| \geq c \} \leq \frac{1}{c^2} E((Z_n - \mu)^2) = \frac{1}{c^2} \left[E({Z_n}^2) - \mu\mu_n + \mu^2 \right] \to 0$$

as $n \to \infty$. Hence, $Z_n \to \mu$ in probability.

For the necessity, suppose $Y_n \to \mu$ in probability. Let

$$U_m^{(n)} = \sum_{k=1}^{m} a_{nk} \left(X_k - \mu\right) \text{ and } U^{(n)} = \sum_{k=1}^{\infty} a_{nk} \left(X_k - \mu\right).$$

Also let

$$g(u) = E(\, e^{iu(X_k - \mu)})$$

be the characteristic function of $X_k - \mu$. By the continuity of the exponential function, $\lim_{m\to\infty} e^{iuU_m^{(n)}} = e^{iuU^{(n)}}$. Since $E|\, e^{iuU^{(n)}}\, | \le 1$, by the dominated convergence theorem and the fact that $\{X_k\}$ is a sequence of i.i.d random variables, we have

$$\prod_{k=1}^{\infty} g(ua_{nk}) = E(\, e^{iuU^{(n)}}\,).$$

Since $U_n \to 0$ in probability (and hence in distribution),

$$\lim_{n\to\infty} \prod_{k=1}^{\infty} g(ua_{nk}) = \lim_{n\to\infty} E(\, e^{iuU^{(n)}}\,) = 1.$$

But,

$$|\prod_{k=1}^{\infty} g(ua_{nk})| \le |g(ua_{nm})| \le 1 \text{ for any } m. \tag{3.24}$$

Therefore, for any sequence k_n,

$$|g(ua_{nk_n})| \to 1 \tag{3.25}$$

We now use corollary 2 of Chow &Teicher [4]p.280 which states

A characteristic function $g(u)$ satisfies either (i) $|g(u)| < 1$ for all $u \neq 0$, (ii) $|g(u)| \equiv 1$, or (iii) $|g(u)| = 1$ for countably many isolated values of u.

Case (ii) can be eliminated since X_k is non-degenerate. Therefore there exists u_0 such that $|g(u)| < 1$ for $0 < |u| < u_0$. Letting $u = u_0/2M$, it follows that

$$|a_{nk_n}u| = |a_{nk_n}||u| \leq M|u| < \frac{u_0}{2} < u_0.$$

Hence, $|g(a_{nk_n}u)| < 1$. This implies that $a_{nk_n} \to 0$ for otherwise, by the continuity of $g(u)$, $g(a_{nk_n}u) \not\to g(0) = 1$ and this contradicts (3.25).
Now choosing k_n to satisfy $|a_{nk_n}| = \max_{1\leq k\leq n} |a_{nk_n}|$ the proof is complete.
$\square$

In light of theorem 9, the condition $\max_{1\leq k\leq n} |a_{nk}| \to 0$ as $n \to \infty$ is not sufficient to guarantee the a.s.c convergence of $\{Y_n\}$ although it is necessary. However, by strengthening the growth rate of $\max_{1\leq k\leq n} a_{nk}$ and by considering a moment condition on the random variable X, Pruitt was able to show that $\{Y_n\}$ converges a.s.c.

Theorem 10 . *If* $\max_{1\leq k\leq n} |a_{nk}| = O(n^{-\gamma})$, $\gamma > 0$, *then* $E|X_k|^{1+1/\gamma} < \infty$ *implies that* $Y_n \to \mu$ *almost surely.*

In light of this theorem, if the matrix $A = (a_{nk})$ has a specific structure satisfying the condition

$$a_{nk} = \begin{cases} \omega_k/W_n & \text{if } 1 \leq k \leq n, \\ 0 & \text{if } k > n \end{cases}$$

where $\{\omega_k\}$ a sequence of positive numbers and $W_n = \sum_{k=1}^{n} \omega_k$, then the result of theorem 6 is sharper; in theorem 6 we only need that

$$\max_{1\leq k\leq n} a_{nk} \to 0 \text{ as } n \to \infty \quad \text{and} \quad E|X| \log^+ |X| < \infty$$

in order that $Y_n \to \mu$ almost surely.

Pruitt also shows that theorem 10 is sharp in the sense that for every $\gamma > 0$, one can construct a Toeplitz matrix $A = (a_{nk})$ with $\max_{1\le k\le n} |a_{nk}| = O(n^{-\gamma})$ such that if $Y_n \to \mu$ a.s, then $E|X_k|^{1+1/\gamma} < \infty$.

We will now present an outline for the proof of theorem 10, deriving it from the following three lemmas which we will not prove.

Lemma 8 *If $E|X|^{1+1/\gamma} < \infty$ and $\max_{1\le k\le n} |a_{nk}| \le Bn^{-\gamma}$, then for every $\epsilon > 0$, $\sum_{n=1}^{\infty} P[|a_{nk}X_k| \ge \epsilon$ for some $k] < \infty$.*

Lemma 9 *If $E|X|^{1+1/\gamma} < \infty$ and $\max_{1\le k\le n} |a_{nk}| \le Bn^{-\gamma}$, for $\alpha < \gamma/2(\gamma+1)$, $\sum_{n=1}^{\infty} P[|a_{nk}X_k| \ge n^{-\alpha}$ for at least two values of $k] < \infty$.*

Lemma 10 *If $EX = 0$, $E|X|^{1+1/\gamma} < \infty$, and $\max_{1\le k\le n} |a_{nk}| \le Bn^{-\gamma}$, then for every $\epsilon > 0$, $\sum_{n=1}^{\infty} P[|\sum_k{}' a_{nk}X_k| \ge \epsilon] < \infty$, where*

$$\sum_k{}' a_{nk}X_k = \sum_{\{k:\, |a_{nk}X_k| < n^{-\alpha}\}} a_{nk}X_k,$$

and $0 < \alpha < \gamma$.

Proof of theorem 10. First observe that

$$Y_n = \sum_k a_{nk}X_k = \sum_k a_{nk}(X_k - \mu) + \mu \sum_k a_{nk}$$

and that the last term converges to μ since $\sum_k a_{nk} \to 1$ as $n \to \infty$. Therefore, to prove the theorem, we will need to show

$$\sum_k a_{nk}(X_k - \mu) \to 0 \text{ as } n \to \infty.$$

Without loss of generality, assume $\mu = 0$. By the Borel-Cantelli Lemma, it will suffice to show that for every $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P\left\{\left|\sum_{k=1}^{\infty} a_{nk}X_k\right| \geq \epsilon\right\} < \infty. \tag{3.26}$$

First we will show that

$$\left\{ \left|\sum_k a_{nk}X_k\right| \geq \epsilon \right\} \subset \left\{ \left|\sum_k{}' a_{nk}X_k\right| \geq \frac{\epsilon}{2} \right\} \bigcup \left\{ |a_{nk}X_k| \geq \frac{\epsilon}{2} \text{ for some k} \right\}$$
$$\bigcup \quad \{ |a_{nk}X_k| \geq n^{-\alpha} \text{ for at least two values of k } \}. \tag{3.27}$$

Suppose $\omega \in \{ |\sum_k{}' a_{nk}X_k| < \frac{\epsilon}{2} \} \bigcap \{ |a_{nk}X_k| < \frac{\epsilon}{2} \text{ for all } k \}$ $\bigcap \{ |a_{nk}X_k| \geq n^{-\alpha}$ for at most one value of $k\}$. If in the case that for all k $\omega \in \{|a_{nk}X_k| < n^{-\alpha}\}$, then $\omega \in \{ |\sum_k a_{nk}X_k| < \epsilon \}$. Otherwise, since there can be at most one value of k, say k', such that $|a_{nk'}X_{k'}(\omega)| \geq n^{-\alpha}$, and since $|a_{nk}X_k(\omega)| < \frac{\epsilon}{2}$ for all k ,

$$|\sum_k a_{nk}X_k(\omega)| = |\sum_k{}' a_{nk}X_k(\omega)| + |a_{nk'}X_{k'}(\omega)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, $\omega \in \{ |\sum_k a_{nk}X_k| < \epsilon\}$ thereby showing (3.27). Now, if $0 < \alpha < \gamma/2$ then $\alpha < \gamma$. As a consequence of the three lemmas and (3.27), (3.26) holds. This completes the proof. $\square$

Remark: Suppose $\{X_n : n \geq 1\}$ is a sequence of independent but not necessarily identically distributed random variables. If the random variables $\{X_n\}$ are uniformly dominated by a random variable X in the sense that

$$P\{|X_n| \geq x\} \leq P\{|X| \geq x\} \quad \text{for all } x > 0,$$

and $A = (a_{nk})$ is a Toeplitz matrix, then Pruitt's results, theorems 9 and 10, hold. This was proved by Rohatgi [26]. Note that if $\{X_n\}$ is identically distributed, then the random variables $\{X_n\}$ are uniformly dominated by X_1, so Rohatgi's result contains Pruitt's theorems 9 and 10.

Chapter 4

Arrays of Random Variables

4.1 Arrays of i.i.d. Random Variables

According to the Marcinkiewicz-Zygmund 1937 result, if $\{X_n; n \geq 1\}$ is a sequence of i.i.d random variables and $S_n = \sum_{k=1}^{n} X_k$ with $EX_1 = 0$, then for any p, $0 < p < 2$,

$$\frac{S_n}{n^{1/p}} \to 0 \quad \text{a.s.} \quad (n \to \infty) \tag{4.1}$$

if and only if

$$E|X_1|^p < \infty.$$

(Note, the case when p=1 was already proved by Kolmogorov).

We will now explore the possibility of extending the Marcinkiewicz-Zygmund result to arrays of random variables. If $\{X_{nk};\ 1 \leq k \leq n, n \geq 1\}$ is an array of i.i.d random variables, does a moment condition on X_{11} exist which is necessary and sufficient for the a.s.c convergence of $S_n/n^{1/p}$ where $EX_1 = 0$ and $0 < p < 2$? It is interesting to note that Zaman and Zaman

[31] provide an example where $\{X_{nk};\ 1 \le k \le n, n \ge 1\}$ is an array of i.i.d random variables with $EX_{11} = 0$ and $E|X_{11}|^p < \infty$ for $1 \le p < 2$, but for which

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{nk} \not\to 0 \quad \text{almost surely.}$$

This suggests that a stricter moment condition on X_{11} is needed. In order to tackle this problem, we will need the following definition of Hsu and Robbins [10].

Definition 6 *(Hsu and Robbins) A sequence of random variables* $\{X_n : n = 1, 2, \cdots\}$ *is said to converge to 0* completely *if for every* $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P\{|X_n| > \epsilon\} < \infty.$$

Applying the Borel-Cantelli lemma, complete convergence implies almost sure convergence, and the converse is not necessarily true unless the sequence of random variables $\{X_n\}$ are independent. In 1949, Erdös [6] showed that complete convergence of $S_n/n^{1/p}$ holds in (4.1) for a sequence of i.i.d. random variables $\{X_k; k \ge 1\}$ if and only if $E|X_1|^{2p} < \infty$ $(1 \le p < 2)$. Based on this result, we obtain a simple proof for the following result.

Proposition 4 *(Erdös). Let* $\{X_{nk};\ 1 \le k \le n, n \ge 1\}$ *be an array of i.i.d random variables such that* $EX_{11} = 0$ *and* $1 \le p < 2$. *Then*

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{nk} \to 0 \quad \textit{completely} \quad (n \to \infty)$$

if and only if $E|X_{11}|^{2p} < \infty$.

Proof. (Hu *et al.*,[11]) By considering the rows for a fixed column of the array $\{X_{nk}\}$, $\{X_{n1}\}$ is a sequence of i.i.d random variables. For each $n \geq 1$ and $\epsilon > 0$,

$$P\left\{\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_{nk}\right| > \epsilon\right\} = P\left\{\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_{k1}\right| > \epsilon\right\}$$

Hence, by Erdös' result we have

$$\sum_{n=1}^{\infty} P\left\{\frac{1}{n^{1/p}}\sum_{k=1}^{n} |X_{nk}| > \epsilon\}\right\} = \sum_{n=1}^{\infty} P\left\{\frac{1}{n^{1/p}}\sum_{k=1}^{n} |X_{k1}| > \epsilon\}\right\} < \infty$$

if and only if $E|X_{11}|^{2p} < \infty$. $\square$

Qi [24] has recently extended the above proposition for the case $0 < p < 1$.

Theorem 11 *(Qi) Let $\{X_{nk}; 1 \leq k \leq n, n = 1, 2, \ldots\}$ be an array of i.i.d random variables with $0 < p < 2$ and let $S_n = \sum_{k=1}^{n} X_{nk}$. Then*

$$\frac{S_n - n\mu}{n^{1/p}} \to 0 \quad \textit{completely} \quad (n \to \infty) \tag{4.2}$$

if and only if $E|X_{11}|^{2p} < \infty$ where $\mu = EX_{11}$ when $1 \leq p < 2$, and $\mu = 0$ when $0 < p < 1$.

To prove the theorem, we will need to use the following two results of Baum and Katz [1]. In these results, the sequence of random variables $\{X_k; k \geq 1\}$ are i.i.d and $T_n = \sum_{k=1}^{n} X_k$.

(i) *Let $t > 0$, $r > 1$, $r/t > 1$. Then*

$$E|X_k|^t < \infty \textit{ if and only if } \sum_{n=1}^{\infty} n^{r-2} P\{|T_n| > n^{r/t}\epsilon\} < \infty.$$

(ii) *Let* $t > 0,\ r > 1,$ *and* $1/2 < r/t \leq 1$. *Also let* $EX_k = \mu$. *Then,*

$$E|X_k|^t < \infty \text{ if and only if } \sum_{n=1}^{\infty} n^{r-2} P\{|T_n - n\mu| > n^{r/t}\epsilon\} < \infty.$$

Proof of theorem 11. By (4.2), for any $\epsilon > 0$

$$\sum_{n=1}^{\infty} P\{|S_n - n\mu| > \epsilon n^{1/p}\} < \infty. \tag{4.3}$$

Substituting $r = 2$ and $t = 2p$ in the above theorems, for $p < 1$ case (i) applies and for $1 \leq p < 2$ case (ii) applies. Hence for $0 < p < 1$, (4.3) is equivalent to $E|X_{11}|^{2p} < \infty$ with $\mu = 0$, and for $1 \leq p < 2$ (4.3) is equivalent to $E|X_{11}|^{2p} < \infty$ with $\mu = EX_{11}$. $\square$

Remark: By the Borel-Cantelli lemma, if a sequence of random variables converge completely then it will also converge almost surely. The converse does not hold in general. For example, letting $\{X_k;\ k \geq 1\}$ to be a sequence of i.i.d random variables and $S_n = \sum_{k=1}^{n} X_k,\ n \geq 1$, by Erdös' theorem the complete convergence of $\{S_n/n,\ n \geq 1\}$ holds if and only if $\mathrm{Var}X_1$ is finite, whereas by the strong law of large numbers, the a.s.c convergence of $\{S_n/n,\ n \geq 1\}$ holds if and only if the mean is finite. For the case of i.i.d arrays, the almost sure convergence and the complete convergence of (4.2) are equivalent. The reason for this is as follows. Let $\{X_{nk}; 1 \leq k \leq n, n \geq 1\}$ be an array of i.i.d random variables and let

$$T_n = \sum_{k=1}^{n} X_{nk}.$$

Since the sequence $\{X_{nk}; 1 \leq k \leq n, n \geq 1\}$ is rowwise independent,

$\{T_n/n^{1/p}\}$ is an i.i.d. sequence. In this case, lemma 3 states that

$$\sum_{n=1}^{\infty} P\left\{|T_n/n^{1/p}| > \epsilon\right\} < \infty$$

for any $\epsilon > 0$ if and only if

$$T_n/n^{1/p} \to 0 \text{ a.s..}$$

4.2 Rowwise Independent Random Variables

Let $\{X_{nk};\ 1 \le k \le n, n \ge 1\}$ be an array of rowwise independent random variables, that is, no assumptions of independence between the rows are assumed. Also assume (without loss of generality) that $EX_{nk} = 0$ for $\{1 \le k \le n, n \ge 1\}$. In this section we will examine the sufficient conditions that are needed in order for

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{nk} \to 0 \quad \text{completely} \qquad (n \to \infty) \tag{4.4}$$

to hold. Hu *et al.*, [11] 1989 have obtained the following as a main result.

Theorem 12 *Let $\{X_{nk} : 1 \le k \le n, n \ge 1\}$ be an array of rowwise independent r.v's such that $EX_{nk} = 0$ for all $\{1 \le k \le n, n \ge 1\}$. Also, assume that there exists a random variable X such that for all $t > 0$ and all $n, k \ge 1$,*

$$P\{|X_{nk}| > t\} \le P\{|X| > t\} \tag{4.5}$$

and

$$E|X|^{2p} < \infty$$

where $1 \leq p < 2$. *Then* (4.4) *holds.*

Motivated by Hu *et al.*'s result, Gut [9] has extended theorem 12 to include the case $0 < p < 1$. In addition, he has weakened assumption (4.5). We will prove Gut's version of theorem 12, however, the proof we provide is different from that of Gut. The new proof, based on Rosenthal's inequality [27], uses some of the lemmas and techniques of Hu *et al.*. At the same time, for the case $1 \leq p < 2$, the new proof is much shorter than that of Hu *et al.* since Rosenthal's inequality allows us to it avoids some of their fairly long and technical details of the proof.

Theorem 13 *Let* $\{X_{nk}; 1 \leq k \leq n, n \geq 1\}$ *be an array of rowwise independent r.v's such that* $EX_{nk} = 0$ *for all* $n, k = 1, 2, \ldots$ *Also, assume that there exists a random variable* X *and* $\alpha > 0$ *such that for all* $t > 0$ *and* $n \geq 1$,

$$\frac{1}{n}\sum_{k=1}^{n} P\{|X_{nk}| > t\} \leq \alpha P\{|X| > t\} \tag{4.6}$$

and

$$E|X|^{2p} < \infty \tag{4.7}$$

where $0 < p < 2$. *Then* (4.4) *holds.*

Before we prove theorem 13, let us first examine equation (4.6).

Definition 7 *We say that the array* $\{X_{nk};\ 1 \leq k \leq n, n \geq 1\}$ *is uniformly dominated by the random variable* X *if* (4.5) *is satisfied and the array is weakly dominated by a random variable* X *if* (4.6) *is satisfied.*

The definitions of uniform domination and weak domination were introduced in order to overcome the lack of identical distribution between the rows of the arrays of random variables. Uniform domination clearly implies weak domination (take $\alpha = 1$), but the reverse is not true in general as shown in the following example.

Example 2 (Gut) Suppose $P\{X_{nk} = 1\} = P\{X_{nk} = -1\} = 1/2$ for $k = 1, 2, \ldots, n-1$ and that $P\{X_{nn} = n^{1/4}\} = P\{X_{nn} = -n^{1/4}\} = 1/2, n \geq 1$. Then there clearly is no uniformly dominating random variable X, however since

$$\frac{1}{n}\sum_{k=1}^{n} P\{|X_{nk}| > t\} = \begin{cases} \frac{1}{n} & \text{for } 1 < t \leq \sqrt{n} \\ 0 & \text{for } t > \sqrt{n}, \end{cases}$$

for the random variable X such that $P\{|X| \geq \sqrt{k}\} = 2/k$ for $k \geq 2$, the condition of weak domination is satisfied.

In order to prove theorem 13, we will need to employ the following lemmas.

Lemma 11 *For any $r \geq 1, E|X|^r < \infty$ if and only if*

$$\sum_{n=1}^{\infty} n^{r-1} P\{|X| > n\} < \infty.$$

In fact,

$$r2^{-r}\sum_{n=1}^{\infty} n^{r-1} P\{|X| > n\} \leq E|X|^r \leq 1 + r2^r \sum_{n=1}^{\infty} P\{|X| > n\}.$$

Lemma 12 *If $r \geq 1$ and $p > 0$, then*

$$E\left(|X|^r I_{[|X| \leq n^{1/p}]}\right) \leq r\int_0^{n^{1/p}} t^{r-1} P[|X| > t]\, dt$$

and

$$E\left(|X|^r I_{[|X|>n^{1/p}]}\right) = n^{1/p} P\left\{|X| > n^{1/p}\right\} + \int_{n^{1/p}}^{\infty} P\{|X| > t\}\, dt.$$

For the proofs of these lemmas refer to [11].

Lemma 13 *Suppose $\{\xi_n\}$ and $\{\eta_n\}$ are sequences of random variables such that $|\xi_n - \eta_n| \to 0$ completely as $n \to \infty$. If also $\eta_n \to 0$ completely, then $\xi_n \to 0$ completely as $n \to \infty$.*

Proof of lemma 13: For $\epsilon > 0$ we have that

$$P\{\ |\xi_n| > \epsilon\ \} \leq P\{\ |\xi_n - \eta_n| + |\eta_n| > \epsilon\ \}$$

$$\leq P(\{\ |\xi_n - \eta_n| > \epsilon/2\} \cup \{|\eta_n| > \epsilon/2\ \}).$$

Since $\{\xi_n - \eta_n\}$ and $\{\eta_n\}$ converge to 0 completely,

$$\sum_{n=1}^{\infty} P\{|\xi_n| > \epsilon\} \ \leq \sum_{n=1}^{\infty} P\{|\xi_n - \eta_n| > \epsilon/2\} \ + \ \sum_{n=1}^{\infty} P\{|\eta_n| > \epsilon/2\} < \infty. \ \square$$

We will now present the basic outline for the proof of theorem 13 before we embark onto the formal proof. For the case $1 \leq p < 2$, we proceed by first truncating X_{nk} at $n^{1/p}$ and then letting Y_{nk} be the truncated part of X_{nk}. Using the moment condition on X, we will show that

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} (X_{nk} - Y_{nk}) \to 0 \quad \text{completely} \quad (n \to \infty).$$

To complete the proof of the theorem, by lemma 13 it will suffice to prove that

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} Y_{nk} \to 0 \quad \text{completely} \quad (n \to \infty). \tag{4.8}$$

In order to accomplish this, we first center the mean of Y_{nk} at 0 by letting

$$Z_{nk} = Y_{nk} - EY_{nk} \quad (k = 1, 2, \dots, n : \ n = 1, 2, \dots).$$

We then use Rosenthal's inequality (i) which states

Suppose $X_1, \dots, X_n$ *are independent random variables and* $EX_k = 0$ *for* $k = 1, \dots, n$. *Furthermore, suppose* $l \geq 2$ *and let* $S_n = \sum_{k=1}^{n} X_k$. *Then*

$$E|S_n|^l \leq c(l) \left[\sum_{k=1}^{n} E|X_k|^l + \left(\sum_{k=1}^{n} EX_k^2 \right)^{l/2} \right]$$

where $c(l)$ *is a positive constant depending on* l *only.*

We then show that

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} Z_{nk} \to 0 \quad \text{completely} \quad (n \to \infty),$$

and that this implies (4.8).

For the case $0 < p < 1/2$ and $1/2 \leq p < 1$, we will use Rosenthal's inequality (ii). It states,

Suppose $X_1, \dots, X_n$ *are independent random variables and* $l > 1$. *Let* $S_n = \sum_{k=1}^{n} X_k$. *Then,*

$$E|S_n|^l \leq c(l) \left[\sum_{k=1}^{n} E|X_k|^l + \left(\sum_{k=1}^{n} E|X_k| \right)^l \right]$$

where $c(l)$ *is a positive constant depending on* l *only.*

(Notice the absence of the condition $EX_k = 0$ in Rosenthal's inequality (ii). Also, see [14] for a Banach version of of Rosenthal's inequalities.)

Proof of theorem 13. Define

$$Y_{nk} = X_{nk} I_{[|X_{nk}| \leq n^{1/p}]} \quad (k = 1, 2, \dots; \ n = 1, 2, \dots).$$

Applying Lemma 11 with $r = 2$, we have that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} P\{X_{nk} \neq Y_{nk}\} = \sum_{n=1}^{\infty}\sum_{k=1}^{n} P\left\{|X_{nk}| > n^{1/p}\right\}$$
$$\leq \sum_{n=1}^{\infty} n\alpha P\left\{|X| > n^{1/p}\right\} \leq 2\alpha E|X|^{2p} < \infty.$$

Next, note that for any $\epsilon > 0$ and $n \geq 1$,

$$\left\{\omega \middle| \left|\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_{nk}(\omega) - \frac{1}{n^{1/p}}\sum_{k=1}^{n} Y_{nk}(\omega)\right| > \epsilon\right\} \subseteq \bigcup_{k=1}^{n} \{\omega | X_{nk}(\omega) \neq Y_{nk}(\omega)\}.$$

Hence,

$$\sum_{n=1}^{\infty} P\left\{\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_{nk} - \frac{1}{n^{1/p}}\sum_{k=1}^{n} Y_{nk}\right| > \epsilon\right\}$$
$$\leq \sum_{n=1}^{\infty} P\left\{\bigcup_{k=1}^{n}(X_{nk} \neq Y_{nk})\right\} \leq \sum_{n=1}^{\infty}\sum_{k=1}^{n} P\{X_{nk} \neq Y_{nk}\} < \infty.$$

Therefore,

$$\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_{nk} - \frac{1}{n^{1/p}}\sum_{k=1}^{n} Y_{nk}\right| \to 0 \text{ completely } (n \to \infty).$$

By lemma 13, it now suffices to prove that

$$\frac{1}{n^{1/p}}\sum_{k=1}^{n} Y_{nk} \to 0 \text{ completely } (n \to \infty). \tag{4.9}$$

For the case $1 \leq p < 2$.

Let

$$Z_{nk} = Y_{nk} - EY_{nk} \quad (k = 1, 2, \ldots, n).$$

Then for $1 \leq q \leq 2p$, using Holder's and Lyaponov's inequality, and the moment condition on X, it follows that

$$(E|Z_{nk}|^q)^{1/q} \leq 2(E|Y_{nk}|^q)^{1/q} \leq 2\left(E|Y_{nk}|^{2p}\right)^{1/(2p)} \leq 2(E|X|^{2p})^{1/(2p)}.$$

Since $E|X|^{2p} < \infty$ by (4.7)

$$E|Z_{nk}|^q < \infty. \tag{4.10}$$

We now let ν denote the least integer such that

$$\frac{2\nu}{3}\left(\frac{2}{p} - 1\right) > 1, \tag{4.11}$$

and we note that $2\nu \geq 2$. Applying Rosenthal's inequality (ii) to the random variables Z_{nk}; $k = 1, \ldots, n$, we have that

$$E\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n} Z_{nk}\right|^{2\nu} \leq c(\nu)\left[\sum_{k=1}^{n}\frac{1}{n^{2\nu/p}}E\left|Z_{nk}\right|^{2\nu} + \frac{1}{n^{2\nu/p}}\left(\sum_{k=1}^{n} E|Z_{nk}|^2\right)^{\nu}\right].$$

where $c(\nu)$ is a constant depending on ν only. Since $E|Z_{nk}|^2 < \infty$ by (4.10), for some number r, $E|Z_{nk}|^2 \leq r$. Hence

$$\sum_{n=1}^{\infty} E\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n} Z_{nk}\right|^{2\nu} \leq c(\nu)\left(\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{n^{2\nu/p}}E\left|Z_{nk}\right|^{2\nu}\right) + c(\nu)r^{\nu}\sum_{n=1}^{\infty}\frac{n^{\nu}}{n^{2\nu/p}}.$$

We will now show the finiteness of the sum for the right hand side in the above equation. By the definition of ν, we have that $\frac{2\nu}{p} - \nu > \frac{3}{2} > 1$. Hence

$$c(\nu)r^{\nu}\sum_{n=1}^{\infty}\frac{n^{\nu}}{n^{2\nu/p}} < \infty. \tag{4.12}$$

Now, using the second result of lemma 12 and the assumption that the array of random variable $\{X_{nk}\}$ are weakly uniformly bounded by a random variable X, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{n^{2\nu/p}}\sum_{k=1}^{n} EZ_{nk}^{2\nu} \;\leq\; 2^{2\nu}\sum_{n=1}^{\infty}\frac{1}{n^{2\nu/p}}\sum_{k=1}^{n} EY_{nk}^{2\nu}$$

$$
\begin{aligned}
&\leq 2^{2\nu} \sum_{n=1}^{\infty} \frac{1}{n^{2\nu/p}} \sum_{k=1}^{n} 2\nu \int_0^{n^{1/p}} t^{2\nu-1} P\{|X_{nk}| > t\}\, dt \\
&\leq 2^{2\nu} \sum_{n=1}^{\infty} \frac{1}{n^{2\nu/p}} 2\nu \int_0^{n^{1/p}} t^{2\nu-1} \sum_{k=1}^{n} P\{|X_{nk}| > t\}\, dt \\
&\leq 2^{2\nu} \alpha \sum_{n=1}^{\infty} \frac{1}{n^{2\nu/p}} 2\nu n \int_0^{n^{1/p}} t^{2\nu-1} P\{|X| > t\}\, dt.
\end{aligned}
$$

Letting $t = n^{1/p}s^{1/2\nu}$, and applying lemma 11 and the moment condition on the random variable X, (4.7), we have that

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^{2\nu/p}} \sum_{k=1}^{n} E_{nk}^{2\nu} &\leq 2^{2\nu} \alpha \sum_{n=1}^{\infty} n \int_0^1 P\left\{|X| > n^{1/p}s^{1/2\nu}\right\} ds \\
&= 2^{2\nu} \alpha \int_0^1 \sum_{n=1}^{\infty} nP\left\{\left|s^{-1/2\nu}X\right|^p > n\right\} ds \\
&\leq 2^{2\nu+1} \alpha \int_0^1 s^{-p/\nu} E|X|^{2p} ds \\
&= 2^{2\nu+1} \alpha \frac{\nu}{\nu - p} E|X|^{2p} < \infty. \qquad (4.13)
\end{aligned}
$$

This result along with (4.12) shows that

$$
E\left|\frac{1}{n^{1/p}} \sum_{k=1}^{n} Z_{nk}\right|^{2\nu} < \infty.
$$

By Chebychev inequality, for $\epsilon > 0$,

$$
\sum_{n=1}^{\infty} P\left\{\left|\frac{1}{n^{1/p}} \sum_{k=1}^{n} Z_{nk}\right| > \epsilon\right\} \leq \frac{1}{\epsilon^{2\nu}} \sum_{n=1}^{\infty} E\left|\frac{1}{n^{1/p}} \sum_{k=1}^{n} Z_{nk}\right|^{2\nu} < \infty
$$

and so,

$$
\frac{1}{n^{1/p}} \sum_{k=1}^{n} Z_{nk} \to 0 \text{ completely } \ (n \to \infty). \qquad (4.14)
$$

We now refer to a simple fact, namely, if $\{\eta_n\}$ is a sequence of random variables and a_n a numerical sequence such that $\eta_n \to 0$ completely and

$a_n \to 0$ then $\eta_n + a_n \to 0$ completely. Hence, to prove (4.9) we need to only show that

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} EY_{nk} \to 0 \quad (n \to \infty).$$

To accomplish our goal, we will prove that

$$\sum = \sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \sum_{k=1}^{n} |EY_{nk}| < \infty. \tag{4.15}$$

By the definition of Y_{nk},

$$Y_{nk} = X_{nk} I_{[|X_{nk}| \leq n^{1/p}]} = X_{nk} - X_{nk} I_{[|X_{nk}| > n^{1/p}]}.$$

Since $EX_{nk} = 0$,

$$|EY_{nk}| \leq E(|X_{nk}| I_{[|X_{nk}| > n^{1/p}]}).$$

Thus, by the second part of lemma 12

$$\sum \leq \sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \sum_{k=1}^{n} E\left(|X_{nk}| I_{[|X_{nk}| > n^{1/p}]}\right) =$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \sum_{k=1}^{n} \left[n^{1/p} P\{|X_{nk}| > n^{1/p}\} + \int_{n^{1/p}}^{\infty} P\{|X_{nk}| > t\} dt \right] =$$

$$\alpha \sum_{n=1}^{\infty} \left[nP\{|X| > n^{1/p}\} + \frac{n}{n^{1/p}} \int_{n^{1/p}}^{\infty} P\{|X| > t\} dt \right].$$

Letting $t = n^{1/p}s$ and applying lemma 11, we conclude that

$$\sum \leq \alpha \sum_{n=1}^{\infty} nP\{|X|^p > n\} + \alpha \sum_{n=1}^{\infty} n \int_{1}^{\infty} P\{|X| > n^{1/p}s\} ds$$

$$\leq 2\alpha E|X|^{2p} + \alpha \int_{1}^{\infty} \sum_{n=1}^{\infty} nP\{|s^{-1}X|^p > n\} ds \leq 2\alpha E|X|^{2p} + 2\alpha \int_{1}^{\infty} s^{-2p} EX|^{2p} ds$$

$$= \frac{4p}{2p-1} E|X|^{2p} < \infty.$$

This proves (4.9), thereby concluding the proof of theorem 13 for the case $1 \leq p < 2$.

For the case $0 < p < 1/2$.

By Rosenthal's inequality (ii) we have that

$$\sum_{n=1}^{\infty} E\left|\frac{1}{n^{1/p}} \sum_{k=1}^{n} Y_{nk}\right|^2 \leq c(2) \sum_{n=1}^{\infty} \frac{1}{n^{2/p}} \left[\sum_{k=1}^{n} E|Y_{nk}|^2 + \left(\sum_{k=1}^{n} E|Y_{nk}|\right)^2\right]$$

where $c(2)$ is a positive constant depending on the number 2 only. The finitness of $\sum_{n=1}^{\infty} \frac{1}{n^{2/p}} \sum_{k=1}^{n} E|Y_{nk}|^2$ can be obtained by imitating the derivation of equation (4.13) where, in place of ν we have the number 1. Thus,

$$2^2 \sum_{n=1}^{\infty} \frac{1}{n^{2/p}} \sum_{k=1}^{n} EY_{nk}^2 \leq 2^3 \alpha \frac{1}{1-p} E|X|^{2p} < \infty.)$$

We will now show the finitness of

$$\sum_{n=1}^{\infty} \frac{1}{n^{2/p}} \left(\sum_{k=1}^{n} E|Y_{nk}|\right)^2.$$

Applying lemma 12 we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^{2/p}} \left(\sum_{k=1}^{n} E|Y_{nk}|\right)^2 &\leq \sum_{n=1}^{\infty} \frac{1}{n^{2/p}} \left(\sum_{k=1}^{n} \int_0^{n^{1/p}} P[|X_{nk}| > t]dt\right)^2 \\ &\leq \alpha \sum_{n=1}^{\infty} \left(\frac{n}{n^{1/p}} \int_0^{n^{1/p}} P[|X| > t]dt\right)^2 \qquad (4.16) \end{aligned}$$

Letting $t = n^{1/p}s$ and applying lemma 11,

$$\sum_{n=1}^{\infty} \left(\frac{n}{n^{1/p}} \int_0^{n^{1/p}} P[|X| > t]dt\right)^2 \leq \left(\sum_{n=1}^{\infty} \frac{n}{n^{1/p}} \int_0^{n^{1/p}} P[|X| > t]dt\right)^2$$

$$\begin{aligned}
&\le \left(\sum_{n=1}^{\infty} \frac{n}{n^{1/p}} \int_0^1 P[|X| > n^{1/p}s]\, n^{1/p} ds\right)^2 \\
&\le \left(\sum_{n=1}^{\infty} n \int_0^1 P[|Xs^{-1}|^p > n] ds\right)^2 \\
&= \left(\int_0^1 \sum_{n=1}^{\infty} nP[|Xs^{-1}|^p > n] ds\right)^2 \\
&\le 2^2 \left(\int_0^1 E|Xs^{-1}|^{2p} ds\right)^2 = 2^2 \left(E|X|^{2p}\right)^2 \frac{1}{(1-2p)^2} < \infty.
\end{aligned}$$

Therefore, by Chebychev's inequality, for $\epsilon > 0$

$$\sum_{n=1}^{\infty} P\left\{\left|\frac{1}{n^{1/p}} \sum_{k=1}^{n} Y_{nk}\right| > \epsilon\right\} \le \frac{1}{\epsilon^2} \sum_{n=1}^{\infty} E\left|\frac{1}{n^{1/p}} \sum_{k=1}^{n} Z_{nk}\right|^2 < \infty.$$

This proves equation (4.9) thereby concluding the proof of theorem 13 for the case $0 < p < 1/2$.

For the case $1/2 \le p < 1$.

Let $\tilde{\nu}$ denote the greatest least integer such that

$$2\tilde{\nu}\left(\frac{1}{p} - 1\right) > 1.$$

By Rosenthal's inequality (ii) we have

$$\sum_{n=1}^{\infty} E\left|\frac{1}{n^{1/p}} \sum_{k=1}^{n} Y_{nk}\right|^{2\tilde{\nu}} \le c(\tilde{\nu}) \sum_{n=1}^{\infty} \frac{1}{n^{2\tilde{\nu}/p}} \left[\sum_{k=1}^{n} E|Y_{nk}|^{2\tilde{\nu}} + \left(\sum_{k=1}^{n} E|Y_{nk}|\right)^{2\tilde{\nu}}\right].$$

The finitness of $\sum_{n=1}^{\infty} \frac{1}{n^{2\tilde{\nu}/p}} \sum_{k=1}^{n} E|Y_{nk}|^{2\tilde{\nu}}$ can be obtained by imitating the derivation of equation (4.13) where instead of ν we we have $\tilde{\nu}$. Now, since $E|X|^{2p} < \infty$ for $1/2 \le p < 1$, by (4.7), $E|X| < \infty$. Hence, by lemma 12 and the definition of $\tilde{\nu}$ it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2\tilde{\nu}/p}} \left(\sum_{k=1}^{n} E|Y_{nk}|\right)^{2\tilde{\nu}} \le \sum_{n=1}^{\infty} \frac{1}{n^{2\tilde{\nu}/p}} \left(\sum_{k=1}^{n} \int_0^{n^{1/p}} P\{|X_{nk}| > t\} dt\right)^{2\tilde{\nu}}$$

$$\leq \alpha \sum_{n=1}^{\infty} \frac{n^{2\tilde{\nu}}}{n^{2\tilde{\nu}/p}} \int_0^{n^{1/p}} P\{|X| > t\}dt \leq E|X| \sum_{n=1}^{\infty} \frac{n^{2\tilde{\nu}}}{n^{2\tilde{\nu}/p}} < \infty.$$

By Chebychev's inequality,

$$\sum_{n=1}^{\infty} P\left\{\left|\frac{1}{n^{1/p}} \sum_{k=1}^{n} Y_{nk}\right| > \epsilon\right\} \leq \frac{1}{\epsilon^{2\tilde{\nu}}} \sum_{n=1}^{\infty} E\left|\frac{1}{n^{1/p}} \sum_{k=1}^{n} Y_{nk}\right|^{2\tilde{\nu}} < \infty$$

This proves equation (4.9) thereby concluding the proof of theorem 13 for the case $1/2 \leq p < 1$. $\square$

Remark: Hu *et al.*, [11] point out that the assumption $p < 2$ is essential in theorem 11 and theorem 13. Relation (4.4) cannot hold for $p = 2$ even in the case of weakly bounded r.v's. Using the law of iterated logarithm, the Rademacher functions serve as a counter example.

Chapter 5

Random Elements

5.1 Banach Space Valued Random Elements

Let $(\Omega,\mathcal{F},P)$ be a probability space. In this section we will briefly discuss some of results of the previous chapters that can be generalized when a function takes value in a general topological space, in particular when the topological space is Banach. A Banach space is defined to be a complete normed linear space where the real-valued function $||\cdot||$ denotes the norm on the space.

Let $(\Omega,\mathcal{F},P)$ be a probability space. Let $\mathcal{X}$ denote a topological space (for our purposes, this space is Banach) and let $\mathcal{B}(\mathcal{X})$ denote the Borel subsets of $\mathcal{X}$, that is the smallest σ-algebra containing all the open subsets of $\mathcal{X}$.

Definition 8 *A function $V: \Omega \to \mathcal{X}$ is said to be a random element in $\mathcal{X}$ if $\{\omega \in \Omega : V(\omega) \in B\} \in \mathcal{F}$ for each $B \in \mathcal{B}(\mathcal{X})$.*

As we can see from this definition, a random element is a generalization of random variables since the σ-algebra generated by all the intervals of the form $[b, \infty]$ is the class of Borel subsets of the real numbers $\mathcal{R}$. However, it is not possible to extend all the properties of random variables to random elements. For example, sums of two random variables is a random variable, but sums of two random elements may not be defined. This poses a problem for our purpose since we are interested in examining the results of the previous chapters where instead of random variables, we have random elements. One way to overcome this obstacle is to assume that the topological space $\mathcal{X}$ is also separable, for then, a function $V : \Omega \to \mathcal{X}$ is a random element if and only if $f(V)$ is a random variable for each $f \in \mathcal{X}^*$ where $\mathcal{X}^*$ denotes the dual space of $\mathcal{X}$.

Analogous to the case of random variables, independence and distribution for Banach-valued random elements are defined in the usual way (simply replace absolute values with $\|\cdot\|$). Moreover, probability modes of convergence are defined as follow.

Definition 9 *A sequence of Banach-valued random elements $\{V_n\}$ converges to V in probability if for any $\epsilon > 0$,*

$$\lim_{n\to\infty} P\{\|V_n - V\| > \epsilon\} = 0$$

and $\{V_n\}$ converges to V almost surely if

$$P\{\lim_{n\to\infty} V_n = V\} = 1.$$

We define the expected value of a random element via the Pettis integral as follow:

Definition 10 *A random element V in a linear topological space $\mathcal{X}$ is said to have expected value EV if there exists an element $EV \in \mathcal{X}$ such that $E(f(V)) = f(EV)$ for each $f \in \mathcal{X}^*$ (the dual of $\mathcal{X}$).*

(For general discussions regarding the properties of the expected value of a random element, see [29] (p. 38-43) and [22].)

We will now state some useful results concerning random elements.

Proposition 5 *Let $\mathcal{X}$ be a separable Banach space. The random elements V and G are identically distributed (independent) if and only if $f(V)$ and $f(G)$ are identically distributed (independent) random variables for each $f \in \mathcal{X}^*$*

Proposition 6 *Let $\mathcal{X}$ be a separable Banach space and V a random element. If $E\|V\| < \infty$ then $E(V)$ exists.*

The Marcinkiewicz-Zygmund's strong law of large numbers can be generalized to the case where $\{X_i; i \geq 1\}$ is a sequence of i.i.d random elements with values in a separable Banach space. Let $S_n = \sum_{i=1}^{n} X_i$. For $0 < p \leq 1$ we have the result that

$$\frac{S_n}{n^{1/p}} \to 0 \text{ almost surely if and only if } E\|X_1\|^p < \infty$$

(and $EX_1 = 0$ for $p = 1$). For the case $1 < p < 2$ however, we would require an equality such as

$$E\|\sum_{i=1} Y_i\|^p \leq C \sum_{i=1} E\|Y_i\|^p \tag{5.1}$$

for every finite sequence $\{Y_i; i \geq 1\}$ of independent centered random elements where C depends on p only. Such an inequality does not hold in a general separable Banach space and those with (5.1) as an additional property are said to be of *type p*. Clearly, every separable Banach space is of type 1 and every separable Hilbert space is of type 2 . Actually, separable Hilbert spaces are the 'best' possible type 2 spaces for if $\{Y_n\}_{n\geq 1}$ is an orthogonal set, then equality holds in (5.1) with $C = 1$. If $1 \leq p < 2$ where $\{X_i; i \geq 1\}$ is a sequence of random elements with values in a separable type p Banach space, then

$$\frac{S_n}{n^{1/p}} \to 0 \text{ if and only if } E\|X\|^p < \infty \text{ and } EX = 0. \tag{5.2}$$

In fact (5.2) hold if and only if the separable Banach space is of type p (see [15] p.259).

We can also extend the results of Pruitt, theorem 9 and theorem 10 of chapter 4, for i.i.d random elements with values in a separable Banach space. Let $A = (a_{nk})$ be a Toeplitz matrix.

Theorem 14 *(Taylor [29] p.110) Let $\{V_n; n \geq 1\}$ be a sequence of identically distributed random elements in a separable Banach space $\mathcal{X}$ and let V be a random element with the same distribution as the V_n's. Suppose*

$E\|V\| < \infty$. *Then, for each* $f \in \mathcal{X}^*$

$$\sum_{k=1}^{n} a_{nk}\ f(V_k - EV_1) \to 0 \tag{5.3}$$

in probability if and only if

$$\|\sum_{k=1}^{n} a_{nk}\ (V_k - EV_1)\| \to 0 \tag{5.4}$$

in probability.

If in addition the random elements $\{V_n\}$ are independent, then

$$\max_{1\le k\le n} |a_{nk}| \to 0 \text{ as } n \to \infty$$

yields the convergence in (5.4) by Pruitt's result (theorem 9 in chapter 4) and theorem 14. Regarding Pruitt's second result (theorem 10 in chapter 4), we have the following extension.

Theorem 15 *Let* $\{V_n\}$ *be a sequence of independent and identically distributed random elements in a seperable Banach space* $\mathcal{X}$ *with* $EV_1 = 0$, *and let* $A = (a_{nk})$ *be a Toeplitz matrix. Assume that* $\max_{1\le k\le n} |a_{nk}| = O(n^{-\gamma})$ *for some* $\gamma > 0$. *If* $E\|V_1\|^{1+1/\gamma} < \infty$, *then*

$$\sum_{k=1}^{n} a_{nk} V_k \to 0$$

almost surely.

For theorem 5 and theorem 6 of chapter 3, there so far is no extension when $\{X_i; i \geq 1\}$ is an i.i.d sequence of random elements with values in a separable Banach space. Since the real valued function $N(\cdot)$ (see definition 5) plays a vital role in the proofs of the theorems, it would seem an analogous function that plays the role of $N(\cdot)$ is required if the space we're dealing with is no longer the special case $\mathcal{R}$ but a general separable Banach space.

On the other hand, it seems that it is possible to extend the results of chapter 4 for the case $\{X_{nk} : 1 \leq k \leq n; n = 1, 2, \ldots\}$ is an array of random elements with values in a seperable Banach space.

Definition 11 *A sequence of Banach-valued random elements $\{X_n; n = 1, 2, \ldots\}$ is said to converge to 0 completely ($X_n \to 0$ completely) if for every $\epsilon > 0$,*

$$\sum_{n=1}^{\infty} P\{\|X_n\| > \epsilon\} < \infty.$$

If the array of the random elements $\{X_n; n = 1, 2, \ldots\}$ is also weakly dominated, that is, there exists a random element X with values in a separable Banach space such that for all $t > 0$ and $\alpha > 0$

$$\frac{1}{n} \sum_{k=1}^{n} P\{\|X_{nk}\| > t\} \leq \alpha P\{\|X\| > t\} \text{ for all } n \geq 1,$$

then for $0 < p < 2$

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{nk} \to 0 \text{ completely } (n \to \infty).$$

The proof would follow the same steps as in the proof of theorem 13 where the absolute values are replaced with $\|\cdot\|$ in the lemmas and the Rosenthal's inequalities are replaced by Ledoux's Banach space versions as in [14].

5.2 Hilbert Space Valued Arrays of Random Elements

Let $\mathcal{H}$ be a separable Hilbert space with inner product denoted by $(\cdot,\cdot)$. We say $\{X_{nk} : 1 \leq k \leq n; n = 1, 2, \ldots\}$ is a sequence of *rowwise orthogonal array* of random elements with values in $\mathcal{H}$ if

$$\sigma_{nk}{}^2 := E\|X_{nk}\|^2 < \infty \tag{5.5}$$

and

$$E(X_{nk}, X_{nj}) = 0 \quad (k \neq j; k, j = 1, 2, \ldots) \tag{5.6}$$

(The norm in (5.5) is induced by the inner product $(\cdot,\cdot)$).

In this section we provide a simple sufficient conditions to ensure the complete convergence of

$$\xi_n := \frac{1}{n^\alpha} \sum_{k=1}^{n} X_{nk} \quad (n = 1, 2, \ldots)$$

Theorem 16 *(Mòricz and Taylor [21]) Let X_{nk} be a rowwise orthogonal array in a separable Hilbert space $\mathcal{H}$. If*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \sum_{k=1}^{n} \sigma_{nk}{}^2 < \infty \tag{5.7}$$

for some $\alpha > 0$, then

$$\xi_n \to 0 \quad \textit{completely} \quad (n \to \infty).$$

Proof. By (5.5), (5.6) and the properties of the inner product,

$$E\|\xi_n{}^2\| = \frac{1}{n^{2\alpha}} E\left[\left(\sum_{k=1}^{n} X_{nk}, \sum_{j=1}^{n} X_{nj}\right)\right] = \frac{1}{n^{2\alpha}} \sum_{k=1}^{n}\sum_{j=1}^{n} E(X_{nk}, X_{nj})$$
$$= \frac{1}{n^{2\alpha}} \sum_{k=1}^{n} \sigma_{nk}{}^2.$$

By Chebyshev's inequality and (5.7), it follows that

$$\sum_{n=1}^{\infty} P\{\|\xi_n\| > \epsilon\} \le \frac{1}{\epsilon^2} \sum_{n=1}^{\infty} E\|\xi_n\|^2$$
$$= \frac{1}{\epsilon^2} \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \sum_{k=1}^{n} \sigma_{nk}{}^2 < \infty.$$

Mòricz and Taylor furthermore construct an example to show that theorem 5.7 is the best possible even

1. for real valued ($\mathcal{H} = \mathcal{R}$) random variables; and
2. if orthogonality is required not only within each row , but between any two rows in the array $\{X_{nk}\}$.

In short, we have the following theorem.

Theorem 17 *Let $\{\sigma_{nk}\}$ be an array of nonnegative numbers such that*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \sum_{k=1}^{n} \sigma_{nk}{}^2 = \infty$$

for some $\alpha > 0$. Then there exists an array $\{X_{nk}\}$ of random variables such that for $n \neq m$ or $k \neq j; k = 1, 2, \ldots, n; j = 1, 2, \ldots, m; n, m = 1, 2, \ldots$

$$\begin{aligned} EX_{nk} &= 0, \\ EX_{nk}{}^2 &= \sigma_{nk}{}^2, \\ EX_{nk}X_{mj} &= 0 \end{aligned} \tag{5.8}$$

and

$$\limsup_{n\to} |\xi_n| = \infty.$$

Chapter 6

Conclusion

In this thesis we have surveyed results which link moment conditions of sequences of random variables to the almost sure and probability convergence of the average of the sequence. We have outlined this connection for sequences of random variables that are i.i.d., pairwise independent and identically distributed and weighted with weights that satisfy the Toeplitz matrix. We have also stated some of the results which can be preserved when the random variables are Banach valued.

We now conclude by summarizing the results in this thesis. Let $\{X_k; k \geq 1\}$ be a sequence of i.i.d. random variables. Let $\{\omega_k; k \geq 1\}$ be a sequence of positive weights as in chapter 3 and $W_n = \sum_1^n \omega_k$. Let $A = (a_{nk})$ be a Toeplitz matrix and let c be a constant. Also let

$$S_n^{(1)} = \frac{\sum_{k=1}^n X_k}{n}, \quad S_n^{(2)} = \frac{\sum_{k=1}^n \omega_k X_k}{W_n} \quad \text{and} \quad S_n^{(3)} = \sum_{k=1}^n a_{nk} X_k.$$

Conditions involving the existence of a moment

i) Kolmogorov's strong law of large numbers says

$$S_n^{(1)} \to c \text{ (a.s)} \iff E|X_1| < \infty \text{ in which case } c = EX_1.$$

(Etemadi has generalized this result to $\{X_k\}$ pairwise independent and identically distributed).

ii)Marcinkiewicz and Zygmund have generalized Kolmogorov's result when $p = 1$. Assuming, without loss of generality, $EX_1 = 0$ then

$$\frac{\sum_{k=1}^{n} X_k}{n^{1/p}} \to 0 \text{ (a.s)} \iff E|X_1|^p < \infty \text{ for } 0 < p < 2.$$

iii) Hu *et al.* and Qi have proved an analogue to Marcinkiewicz's and Zygmund's result for $\{X_{nk}; 1 \le k \le n, n = 1, 2, \ldots\}$ an array of i.i.d. sequence of random variables. Again, assuming without loss of generality $EX_{11} = 0$, then

$$S_n^{(4)} := \frac{\sum_{k=1}^{n} X_{nk}}{n^{1/p}} \to 0 \text{ (a.s)} \iff E|X_{11}|^{2p} < \infty \text{ for } 0 < p < 2.$$

Furthermore, if $\{X_{nk}\}$ are rowwise independent and weakly dominated by a random variable X and $EX_{nk} = 0$ for all n and k, then

$$E|X|^{2p} < \infty \Longrightarrow S_n^{(4)} \to 0 \text{ (completely)} \Longrightarrow S_n^{(4)} \to 0 \text{ (a.s)}.$$

iv) For weighted sequences of i.i.d. random variables, Jamison *et al.* have shown that

$$S_n^{(2)} \to EX_1 \text{ (a.s)} \iff \frac{N(x)}{x} < \infty \text{ as } x \to \infty.$$

Also for a bounded sequence of weights $\{\omega_k\}$,

$$E|X|\log^+|X| < \infty \Longrightarrow S_n^{(2)} \to EX_1.$$

v) Pruitt has shown that

$$S_n^{(3)} \to EX_1 \text{ (in probability)} \Longleftrightarrow \max_{1\le k\le n} |a_{nk}| \to 0 \text{ as } n \to \infty$$

and

$$S_n^{(3)} \to EX_1 \text{ (a.s)} \Longleftarrow \max_{1\le k\le n} |a_{nk}| = O(n^{-\gamma}) \text{ and } E|X_1|^{1+\frac{1}{\gamma}} < \infty$$

for $\gamma > 0$.

<u>When the first moment does not exist</u>

vi) When $E|X|$ is not necessarily finite Jamison *et al.* have shown

$$S_n^{(2)} \to c \text{ (in probability)} \Longleftrightarrow$$

$$\lim_{c\to\infty} cP\{|X_1| > c\} = 0 \text{ and } \lim_{c\to\infty} \int_{|x|<c} x dF(x) \text{ exists.}$$

vii) Wright *et al.* have shown that if $\lim_{c\to\infty} cP\{|X_1| > c\} = 0$ holds and a certain integral involving $N(\cdot)$ is finite, then $S_n^{(2)} \to c$ (a.s).

The most general result is that of Gut theorem 13 and I was happy to have provided a different proof.

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