Hodge Decompositions and Computational Electromagnetics

by

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ABSTRACT

The handling of topological aspects in boundary value problems of engineering electromagnetics is often considered to be an engineer's art and not a science. This thesis is an attempt to show that the opposite is true. Through the use of differential forms and rudimentary concepts from homology theory a paradigm variational boundary value problem is formulated and investigated. It is seen that reasoning in terms of the Tonti diagram for this problem may lead to false conclusions if cohomology groups are ignored. As a prelude to this investigation, a suitable orthogonal decomposition of differential forms is derived and the roles played by the long exact homology sequence and topological duality theorems for compact orientable manifolds with boundary are considered in detail.

RÉSUMÉ

L'utilisation correcte des aspects topologiques des problèmes avec conditions aux frontières tels que rencontrés en électromagnétisme est souvent considérée comme un art plutôt qu'une science. Cette thèse tente de démontrer le contraire. En utilisant des formes différentiables et des concepts rudimentaires de la théorie de l'homologie, un modèle de problème variationnel avec conditions aux frontières est formulé, puis analysé. Il est démontré que, pour ce problème, les raisonnements utilisant les diagrammes de Tonti peuvent conduirent à des résultats erronnés si les groupes de cohomologie sont ignorés. Comme prélude à cette investigation, un théorème sur la décomposition orthogonale des formes différentiables est présenté et les rôles joués par la longue séquence exacte d'homologie et les théorèmes de dualité pour des variétés différentiables, orientables, compactes et avec frontières sont considérés en détail.

ACKNOWLEDGEMENTS

I would like to thank my thesis supervisor Professor P. Silvester for listening to my half baked ideas and asking the type of questions which helped this piece of work become an engineering thesis and not a summary of my favorite library books. When it came time to put the thesis together my colleagues Sylvia Márquez, Denis Martin, and Nevine Nassif played such a great role that I doubt that I would ever have submitted this thesis without their help. I will always be grateful to them.

I am also grateful for the opportunity to interact wih Professor G. Cambrell. Our discussions had a definite influence on my thesis. The patience of Professor P. Russell, who consistently gave clear answers to my muddled questions about algebraic topology will always be remembered and it is a pleasure to say that my interaction with Professor P. Caines has been most enjoyable and rewarding.

I would like to thank certain people who helped me open certain doors in the literature. In particular I would like to mention that Professor L. Nirenberg informed me about the papers by Sibner and Sibner while Professor C. Herz told me about Maxwell's periphractic and cyclomatic numbers.

Finally, I would like to express my gratitude for the financial support given to me by NSERC during the last two years and I am sure that many booksellers who know me would like to express the same gratitude.

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CHAPTER 1

Introduction and Prerequisite ideas from homology theory

"It could be said if all the text that concerned the application of boundary conditions in electromagnetic problems, and all the topological arguments, were removed from this book, there would be little left. To some extent both topics could be said to be more of an art than a science".

E.R. Laithwaite,
Induction Machines for Special Purposes
[1966] *p.* 326.

"It seems probable to the author that many of the objectively important problems in mathematical physics, geometry, and analysis cannot be solved without radical additions to the methods of what is now strictly regarded as pure analysis. Any problem which is non-linear in character, which involves more than one coordinate system or more than one variable, or whose structure is initially defined in the large, is likely to require considerations of topology and group theory in order to arrive at its meaning and its solution. In the solution of such problems classical analysis will frequently appear as an instrument in the small, intregrated over the whole problem with the aid of group theory or topology."

Marston Morse,
The Calculus of Variations in the Large
[1934] *p. iii*

"We are here led to considerations belonging to the Geometry of Position, a subject which, though its importance was pointed out by Leibnitz and illustrated by Gauss, has been little studied."

James Clerk Maxwell,
A Treatise on Electricity and Magnetism
[1891] *Art.* 17

1.1 Introduction

The purpose of this thesis is to show how homology groups play a key role in deriving orthogonal decompositions of differential forms which enable a variety of questions associated with variational boundary value problems of electromagnetics to be formulated and answered in a general way. More precisely, an orthogonal decomposition and a paradigm problem will be developed and it will be seen that questions regarding existence of potential, gauge transformations, and existence of solution can only be answered in a complete way if information concerning certain relative homology groups is used. The classical theorem of Helmholtz is an orthogonal decomposition which can be used to give complete answers to the questions which will be adressed in the case where the region of interest is $\mathbb{R}^3$ and the vector fields vanish at infinity.Unfortunately, boundary value problems of computational electromagnetics are usually set in compact regions which may have topological intricacies not encountered in $\mathbb{R}^3$. Thus, there is a need for variants of the Helmholtz Theorem for finite regions of arbitrary topology. In classical electromagnetics the need for such theorems was made obvious at an early stage in the area of cavity resonators throug the early papers of Teichmann and Wigner [1953], Kurokawa [1958] and Van Bladel [1960], [1962]. For bounded regions Ω the variants of the Helmholtz theorem decompose a vector field into the gradient of a scalar function, the curl of a vector field, and a harmonic vector field which is both irrotational and solenoidal. In addition, each of these three subspaces is subjected to certain boundary conditions. There have been many excellent papers on this subject, for example, Weyl [1940], Bykhovski and Smirnov [1960], Werner [1963], [1983] Foias and Teman [1978], and Saranen [1982, 1983]. Although most authors have stressed the interplay between the harmonic vector fields and the topology of the region Ω, to the best of the author's knowledge, the electrical engineer has no simple account of how homology groups are related to orthogonal decompositions and other aspects of boundary value problems.

The interplay between orthogonal decompositions and homology groups of a manifold has been understood for a long time. In the seminal paper by de Rham [1931] an isomorphism between the homology groups of a manifold and the cohomology groups defined in terms of differential forms is established. This work prompted a series of investigations by Hodge [1952] which culminated in the Hodge decomposition theorem — an orthogonal decomposition theorem for differential forms on closed manifolds which is a generalisation of the Helmholtz Theorem. A nice exposition of the Hodge decomposition is given in the paper by Bidel and de Rham [1946] while the basic idea is explained by Eckmann [1944]. For manifolds with boundary the analogues of de Rham's theorems were developed by Duff [1952] while the analogues of the Hodge decomposition are worked out by Duff and Spencer [1952]. The papers by Duff, Connor, Friedrichs, Gaffney, Milgram and Rosenbloom, Morrey and Eells should also be consulted. The book by de Rham [1955] is often considered to mark the end of this classical period. Modern extensions and applications of the Hodge decomposition in the context of continuum mechanics are given in the book by Marsden [1974] and in the paper by Sibner and Sibner [1970]. All of the fundamental work in this area has been done in the formalism of differential forms and unfortunately, if the author's experience is any indication, practical people in computational electromagnetics often regard the formalism of differential forms as a plot devised by a group of mathematicians intent on undermining the greatness of Hamilton, Gibbs and Heaviside. Welcome exceptions to this view are the book by Balasubramanian et al. [1970] and the paper by Nedelec [1978]. Along another route, the paper by Milani and Negro [1982] show how the Hodge decomposition theorem can be used when prescribing boundary conditions on a vector potential.

It should be noted that there is along history of singular homology theory in numerical analysis — one merely has to recall the work of Kron [1959] and Roth [1955].

More recently, the work of Bossavit [1982], [1983] recognises the essential role played by homology groups in boundary value problems for eddy currents, while Brown [1984] has gone a step further and used standard techniques in the homology theory of simplicial complexes to show how topological intricacies associated with the so called $T - \Omega$ method can be handled by computer. Considering the work of Brown [1984], Mantyla [1983], Eastman and Preiss [1984] it is obvious that singular homology theory will play an essential role in the automatic construction of finite element models. Similarly, through the work of Kron [1959], Kondo [1955], and Whitney [1957] one can see that the practicioners of the finite element method have done very little to keep in touch with useful techniques developed in algebraic topology. This view is clearly reinforced by the work of Baker [1982], Komorowski [1975], and Dodziuk [1976, 1977, 1981 and 1982]. These recent developments relieve much of the author's guilt about using the words "homology group" or "differential form" in an engineering thesis.

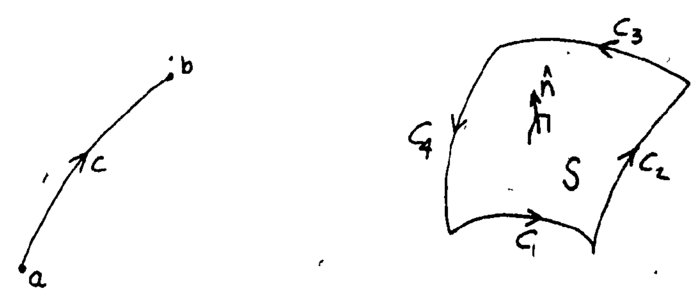
Certain accomodations have been made for the audience of this thesis. Firstly the calculus of differential forms ($d, \wedge, *$, etc) is avoided as much as possible in Chapter 1. This enables the reader who has little inclination to learn about differential forms to get an intuitive appreciation of the formalism without being forced into any calculations. Secondly, almost all questions of analysis have been ignored in this thesis. In particular, facts concerning topological spaces, and discussions of direct and inverse limits in the definitions of various cohomology theories, have been avoided while orthogonal decompositions are to be understood in the pre-Hilbert space sense. Thirdly, in order to avoid boredom on the part of the uninitiated reader, the development of homology groups is entirely heuristic and all schemes for their computation have been ignored. This enables one to appreciate the usefulness of homology theory in electromagnetics at the earliest possible stage. It is felt that making these accomodations does not deter from the message of the thesis since every effort has been made to ensure that the bibliography contains the material which fills in the gaps. Furthermore, it is felt that concentrating on the intuitive consequences of homology theory without regard to any particular scheme of computation is justified since the Steenrod axioms for a homology theory (see for example Hu [1966]) enable one to talk about the unique consequences of a homology theory without regard to how homology is computed.

The objective of this introductory chapter is to give a heuristic appreciation of the formalism of homology theory in the context of boundary value problems of classical electromagnetics. The handling of topological aspects in these problems is often considered to be an engineer's art and not a science, however, the author hopes to indicate how homology groups totally characterise the relevant topological aspects. The first step in this task is to formalise the geometric intuition gained through using Maxwell's equations in integral form. Hence the only prerequisite knowledge assumed for this introductory chapter is familiarity with Maxwell's equations in integral form, simple vector analysis, and an aquaintance with linear space jargon. The basic reference for linear algebra is taken to be Halmos [1958].

1.2 Chains, Cochains and Integration

Homology theory characterises certain problems which arise in the use of the integral theorems of Gauss, Green, Newton-Leibnitz, Stokes and their generalistion. Stokes' Theorem on manifolds which may be considered to be the fundamental theorem of multivariable calculus is the generalisation of these classical integral theorems. In order to appreciate how these problems arise, the process of integration must be reinterpreted in an algebraic manner.

Consider a n-dimensional region Ω and let the set of all possible p-dimensional regions, over which a p-fold integration can be performed, be denoted by $C_p(\Omega)$†. Here it is understood that $0 \leq p \leq n$ and a 0-fold integration is the sum of values of a function evaluated at some finite set of points. The elements of $C_p(\Omega)$ will be called p-chains. In order to serve their intended function, the elements of $C_p(\Omega)$ must be more than p-dimensional surfaces, for in evaluating integrals it is essential to associate an orientation to a chain. The idea of an orientation is crucial if one is to consider the oriented boundary of a chain; for example



$\partial c = b - a$ $\qquad\qquad$ $\partial S = c_1 + c_2 + c_3 + c_4$

The set of integrands of p-fold integrals is called the set of p-forms (or p-cochains) and is denoted by $C^p(\Omega)$. Thus for $c \in C_p(\Omega)$ and $\omega \in C^p(\Omega)$ the integral of ω over c is denoted by

$$\int_c \omega$$

and hence integration can be regarded as a mapping:

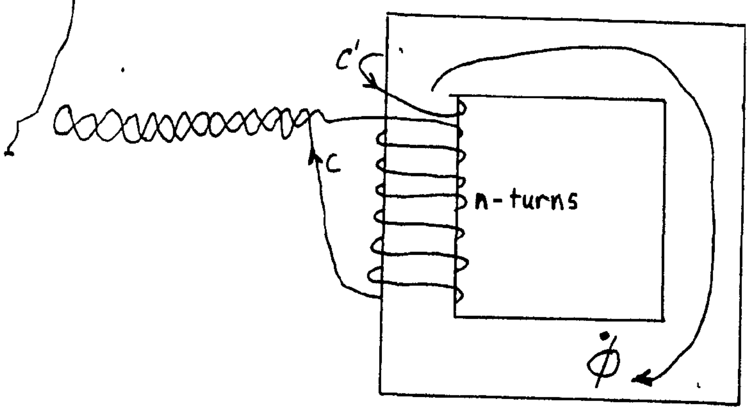
$$\int : C_p(\Omega) \times C^p(\Omega) \to \mathbb{R}, \qquad 0 \leq p \leq n$$

† An informal approach is required if one is to avoid a barrage of formalism.

where $\mathbb{R}$ is the set of real numbers. Integration, with respect to p-forms, is a linear operation: that is, given $a^1, a^2 \in \mathbb{R}, \omega_1, \omega_2 \in C^p(\Omega)$ and $c \in C_p(\Omega)$:

$$\int_c a_1\omega_1 + a_2\omega_2 = a_1 \int_c \omega_1 + a_2 \int_c \omega_2.$$

Thus $C^p(\Omega)$ may be regarded as a vector space and as such it can be denoted as $C^p(\Omega, \mathbb{R})$. Similarly, it is convenient to regard $C_p(\Omega)$ as having some algebraic structure. For example it is often convenient to consider it as an abelian group as the following example will show.



for $c, c' \in C_1(\mathbb{R}^3)$, $\qquad c = nc'$

Fig. 1

When analysing an ideal transformer, a current carrying coil with n turns which can be considered as a 1-chain is modelled by n copies of another 1-chain as shown in Fig. 1. Calculating the voltage induced in loop c in terms of the voltage of loop c' yields

$$V_c = \int_c \mathbf{E} \cdot \mathbf{t}\, dl = \int_{nc'} \mathbf{E} \cdot \mathbf{t}\, dl = n \int_{c'} \mathbf{E} \cdot \mathbf{t}\, dl = nV_{c'}.$$

Thus in this case it is convenient to regard linear combinations of 1-chains with integer coefficients as 1-chains. In this way $C_1(\mathbb{R}^3)$ becomes an abelian group written additively. That is, for all $c, c', c'' \in C_1(\mathbb{R}^3)$

$$c - c' \in C_1(\mathbb{R}^3)$$

$$c + (-c) = 0$$

$$c + 0 = c$$

$$c - c' = c' + c$$

$$c + (c' + c'') = (c + c') + c''$$

Note that the inverse operation in the group reverses the orientation of the chain, that is

$$\int_{-c} \omega = -\int_c \omega.$$

Similarly, given any n-dimensional region Ω, the set of p-chains $C_p(\Omega)$ can be considered to be an abelian group by taking linear combiantions of p-chains with coefficients in $\mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. This group will be denoted by

$$C_p(\Omega, \mathbb{Z})$$

and called: "the group of p-chains with coefficients in $\mathbb{Z}$".

If in the above construction, linear combinations of p-chains with coefficients in the field $\mathbb{R}$ are taken, the set of p-chains can be regarded as a vector space. This vector space will be used extensively in this thesis and can be denoted by

$$C_p(\Omega, \mathbb{R})$$

and called "the group of p-chains with coefficients in $\mathbb{R}$". In this case the following is true. For $a_1, a_2 \in \mathbb{R}, c_1, c_2 \in C_p(\Omega, \mathbb{R}), \omega \in C^p(\Omega, \mathbb{R})$,

$$\int_{a_1 c_1 + a_2 c_2} \omega = a_1 \int_{c_1} \omega + a_2 \int_{c_2} \omega$$

In a similar fashion. taking a ring R and forming linear combinations of p-chains with coefficients in R, an R-module[†] is obtained:

$$C_p(\Omega, R)$$

which is called: "the group of p-chains on Ω with coefficients in R". This construction has as special cases the previous two. Also, for p-cochains it is possible to construct the analogous p-cochain groups; there will be no need to do so in this thesis.

In order to resolve topological problems arising in multiple integration, it is sufficient to regard the set of p-chains as a vector space. However, for the purposes of numerical computation it may be advantageous to consider p-chains with coefficients in $\mathbb{Z}$. For this reason only the coefficients in $\mathbb{R}$ or $\mathbb{Z}$ are considered in this thesis. Furthermore, for simplicity, the following notation will be adopted

$$C_p(\Omega) = C_p(\Omega, \mathbb{R})$$
$$C^p(\Omega) = C^p(\Omega, \mathbb{R}).$$

For coefficients in $\mathbb{R}$, it is apparent that the operation of integration can be regarded as a bilinear pairing between p-chains and p-forms. Furthermore, for reasonable p-chains and p-forms this bilinear pairing is nondegenerate. That is

$$\text{If } \int_c \omega = 0 \text{ for all } c \in C_p(\Omega), \text{ then } \omega = 0$$

and

$$\text{If } \int_c \omega = 0 \text{ for all } \omega \in C^p(\Omega), \text{ then } c = 0.$$

Although this statement requires some sophisticated discretisation procedure and limiting argument for its justification. it is simple to understand and is quite plausible.

† Knowledge of rings and modules is irrelevant at this point, the construction of $C_p(\Omega, R)$ is intended to illustrate how the notation is developed.

In conclusion, it is important to regard $C_p(\Omega)$ and $C^p(\Omega)$ as vector spaces and to consider integration as a bilinear pairing between them. In order to reinforce this point of view, the process of integration will be written using the linear space notation:

$$\int_c \omega = [c, \omega]$$

that is, $C^p(\Omega)$ is to be considered the dual space of $C_p(\Omega)$.

1.3 Integral Laws and Homology

Consider the fundamental theorem of calculus

$$\int_c \frac{\partial f}{\partial x} dx = f(b) - f(a), \qquad c \in C_1(\mathbb{R}^1),\ c = [a, b]$$

its analogues for two dimensional surfaces Ω,

$$\begin{aligned} \int_c \operatorname{grad}\phi \cdot \mathbf{t}\, dl &= \phi(p_2) - \phi(p_1), & c \in C_1(\Omega),\ \partial c = p_2 - p_1 \\ \int_S \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\, dS &= \int_{\partial S} \mathbf{F} \cdot \mathbf{t}\, dl, & S \in C_2(\Omega) \end{aligned}$$

and brethren from three dimensional vector analysis, $(\Omega \subset \mathbb{R}^3)$

$$\begin{aligned} \int_c \operatorname{grad}\phi \cdot \mathbf{t}\, dl &= \phi(p_2) - \phi(p_1), & c \in C_1(\Omega),\ \partial c = p_2 - p_1 \\ \int_S \operatorname{curl}\mathbf{F} \cdot \mathbf{n}\, dS &= \int_{\partial S} \mathbf{F} \cdot \mathbf{t}\, dl, & S \in C_2(\Omega) \\ \int_D \operatorname{div}\mathbf{F}\, dV &= \int_{\partial D} \mathbf{F} \cdot \mathbf{n}\, dS, & D \in C_3(\Omega). \end{aligned}$$

† When dealing with a p-chain c_i its algebraic properties may be ignored and the symbol c_i may be used to denote a point set when no confusion may arise.

These integral theorems along with four dimensional versions which arise in covariant formulations of electromagnetics, are special instances of the general version called Stokes' Theorem on manifolds. The general theorem, which will be developed in the next chapter, takes the form

$$\int_c d\omega = \int_{\partial c} \omega$$

where the linear operator

$$\partial : \bigoplus_p C_p(\Omega) \to \bigoplus_p C_{p-1}(\Omega)$$

$$d : \bigoplus_p C^{p-1} \to \bigoplus_p C^p(\Omega)$$

defined in terms of direct sums are called the boundary and the exterior derivative respectively (when p-forms are called p-cochains, d is called the coboundary operator). For an n-dimensional region Ω the following definition is made:

$$\left.\begin{aligned} C^p(\Omega) &= 0 \\ C_p(\Omega) &= 0 \end{aligned}\right\} \qquad p < 0, \quad p > n.$$

In this way, the boundary operator on p-chains has an intuitive meaning which carries over from vector analysis. On the other hand, the exterior derivative must be regarded as the operator which makes Stokes' theorem true. When a formal definition of the exterior derivative is given in the next chapter, it will be a simple computation to verify the special cases listed above.

For the time being, let the restriction of the boundary operator to p-chains be denoted by ∂_p and the restriction of the exterior derivative to p-forms be denoted by d^p. Thus

$$\partial_p : C_p(\Omega) \to C_{p-1}(\Omega)$$

$$d^p : C^p(\Omega) \to C^{p+1}(\Omega).$$

Considering various n-dimensional regions Ω and p-chains for various values of p, it is apparent that

$$(\partial_p \partial_{p+1})c = 0 \quad for\ all\ c \in C_{p+1}(\Omega) \tag{1}$$

or, in words, "the boundary of a boundary is zero". The interesting question which arises is the converse: "If the boundary of a $p-1$ chain is zero, then is the chain the boundary of some other p-chain in $C_{p+1}(\Omega)$?" The answer to this is, of course, no in general, since otherwise the question would not be interesting. However, in order to give a serious answer to the question and to see its implications for vector analysis more formalism is required.

Rewriting Equation (1) as

$$\text{Image } \partial_{p+1} \subset \text{Kernel } \partial_p \tag{2}$$

the question reduces to asking if the above inclusion is an equality. In order to regain the geometric flavor of the question, define

$$B_p(\Omega) = \text{Image } \partial_{p+1}$$

$$Z_p(\Omega) = \text{Kernel } \partial_p$$

where elements of $B_p(\Omega)$ are called p-boundaries and elements of $Z_p(\Omega)$ are called p-cycles. Thus the inclusion (2) can be rewritten as $B_p(\Omega) \subset Z_p(\Omega)$ and the question at hand is really an inquiry into the size of the quotient group†

$$H_p(\Omega) = Z_p(\Omega)/B_p(\Omega)$$

which is called the pth homology group of Ω. In order to simplify the language used when talking about the cosets of $H_p(\Omega)$, the following equivalence relation is introduced: Given $z_1, z_2 \in Z_p(\Omega)$,

$$z_1 \sim z_2 \text{ (read } z_1 \text{ is homologous to } z_2\text{) if } z_1 - z_2 = b \text{ for some } b \in B_p(\Omega)$$

† This construction can be performed with any coefficient group. In the present case $Z_p(\Omega)$, $B_p(\Omega)$ are vector spaces and $H_p(\Omega)$ is a quotient space.

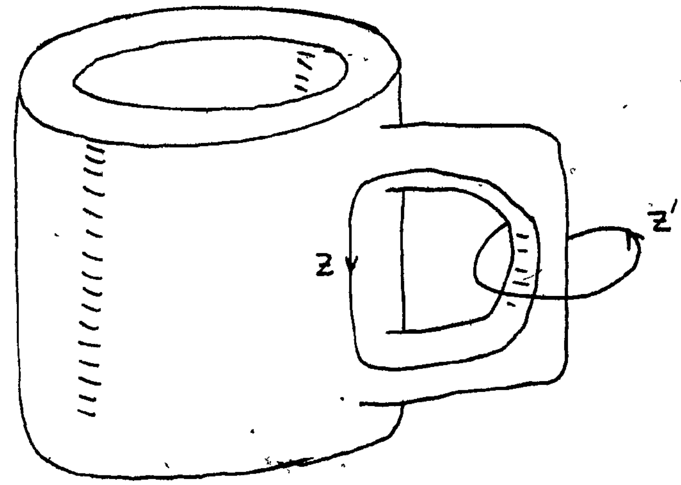


Fig. 2

Hence z_1 is homologous to z_2 if z_1 and z_2 lie in the same coset of $H_p(\Omega)$. In the present case $H_p(\Omega)$ is a vector space and the dimension of the pth homology "group" is called the pth Betti number. i.e.

$$\beta_p(\Omega) = \dim(H_p(\Omega)).$$

In order to see what the cosets of $H_p(\Omega)$ mean and to get an answer to "the interesting question", consider a few examples.

Example 1 (A three dimensional example, $\beta_2 \neq 0$)

Consider three concentric spheres and let Ω be the three-dimensional shell whose boundary is formed by the innermost and outermost spheres. Next, let $z \in Z_2(\Omega)$ be the sphere between the innermost and outermost spheres, oriented by the unit outward normal. Since z is a closed surface, $\partial_2 z = 0$ however z is not the boundary of any three dimensional chain in Ω, that is $z \neq \partial_3 c$ for any $c \in C_3(\Omega)$. Hence z represents a nonzero coset in $H_2(\Omega)$. In this case $\beta_1(\Omega) = 1$ and $H_2(\Omega)$ is generated by cosets of the form

$$az + B_2(\Omega) \qquad \text{where } a \in \mathbb{R}.$$

Example 2 (Another three-dimensional example, $\beta_1 \neq 0$)

Suppose $\Omega \subseteq \mathbb{R}$ is the region occupied by a coffee cup. Let $z \in Z_1(\Omega)$ be a closed curve going around the inside of the handle while $z' \in Z_1(\mathbb{R}^3 - \Omega)$ is a closed curve "linking" the handle (see Fig. 2). A little reflection shows that $z \notin B_1(\Omega)$ and that $\beta_1(\Omega) = 1$, that is, the cosets of $H_1(\Omega)$ look like $az + B_1(\Omega)$ where $a \in \mathbb{R}$. Dually, $z^1 \in B_1(\mathbb{R}^3 - \Omega)$ and $\beta_1(\mathbb{R}^3 - \Omega) = 1$, hence the cosets of $H_1(\mathbb{R}^3 - \Omega)$ look like

$$a'z' + B_1(\mathbb{R}^3 - \Omega)$$

where $a' \in \mathbb{R}$.

Example 3 ($\Omega \subset \mathbb{R}^3$, looking for $H_2(\Omega)$, $H_0(\mathbb{R}^3 - \Omega)$)

Suppose Ω is a compact† connected subset of $\mathbb{R}^3$ and‡ $\partial_3\Omega = S_0 \cup S_1 \cup S_2 \cup \ldots \cup S_n$ where $S_i \in Z_2(\Omega), 0 \leq i \leq n$ are the connected components of $\partial_3\Omega$. (Think of Ω as a piece of swiss cheese). Furthermore, let S_0 be the connected component of $\partial_3\Omega$ which, when taken with the opposite orientation, becomes the boundary of the unbounded component of $\mathbb{R}^3 - \Omega$. Given that Ω is connected, it is possible to find $n+1$ components Ω_i' of $\mathbb{R}^3 - \Omega$ such that

$$\partial_3\Omega_i' = -S_i \qquad 0 \leq i \leq n$$

It is obvious that $S_i, 0 \leq i \leq n$, cannot possibly represent independent generators of $H_2(\Omega)$ since their sum (as chains) is homologous to zero, that is

$$\sum_{i=0}^{n} S_i = \partial_3\Omega$$

or

$$\sum_{i=0}^{n} S_i \sim 0.$$

† A compact set in this case, means a closed and bounded set.

‡ By an abuse of language, we assume $\Omega \in C_3(\Omega)$ where, when considered as a chain, $\partial\Omega$ has the usual orientation. That is Ω is considered as both a chain and set.

However, it is quite plausible that $H_2(\Omega)$ is generated by cosets of the form:

$$\sum_{i=1}^{n} a_i S_i + B_2(\Omega).$$

The justification of this statement proceeds as follows. Choose 0-cycles p_i (points), $0 \leq i \leq n$, such that $p_i \in Z_0(\Omega_i')$ and define 1-chains (curves) $c_i \in C_1(\mathbb{R}^3)$, $1 \leq i \leq n$, by the following:

$$\partial c_i = p_i - p_0.$$

That is, p_i, are $n+1$ generators of $H_0(\mathbb{R}^3 - \Omega)$ while the c_i connect the components of $\mathbb{R}^3 - \Omega$. It is apparent that for $1 \leq i,\ j \leq n$, c_i can be arranged to intersect S_i once and not intersect S_j if $i \neq j$. This being possible it is clear that if the c_i are regarded as point sets,

$$\beta_2\left(\Omega - \left(\bigcup_{i=1}^{n} c_i\right)\right) = 0$$

$$\beta_0\left(\left(\overline{\mathbb{R}^3 - \Omega}\right) \cup \left(\bigcup_{i=1}^{n} c_i\right)\right) = 1$$

where in the latter case multiples of the 0-cycle p_0 can be taken to generate the 0th homology group. It is apparent that this property cannot be achieved by taking fewer than n such c_i. That is, for every c_i which goes through Ω there corresponds one and only one generator of $H_2(\Omega)$. Hence in summary

$$\beta_2(\Omega) = n = \beta_0(\overline{\mathbb{R}^3 - \Omega}) - 1$$

where the n-independent cosets of the form

$$\sum_{i=1}^{n} a_i S_i + B_2(\Omega) \qquad a_i \in \mathbb{R}$$

$$\sum_{i=0}^{n} a_i' P_i + B_0(\overline{\mathbb{R}^3 - \Omega}) \qquad a_i' \in \mathbb{R}$$

can be used to generate $H_2(\Omega)$, $H_0(\overline{\mathbb{R}^3 - \Omega})$ respectively†.

The general case where Ω may be disconnected is handled by applying the above argument to each connected component of Ω and choosing the same p_0 for every component. In this case it will also be true that

$$\beta_0(\overline{\mathbb{R}^3 - \Omega}) = \beta_2(\Omega) + 1.$$

End of Example 3

Example 4 (Ω an orientable surface. $H_1(\Omega)$ of interest)

It is a well known fact that any orientable 2-dimensional surface is homeomorphic to "a sphere with n handles and k holes". That is, for some integers n and k, any orientable 2-dimensional surface can be mapped in a 1-1 continuous fashion into some surface like the one pictured in Fig. 3 (see Massey [1977] Chapter 1 or Cairns [1961] Chapter 2 for more pictures and explanations).

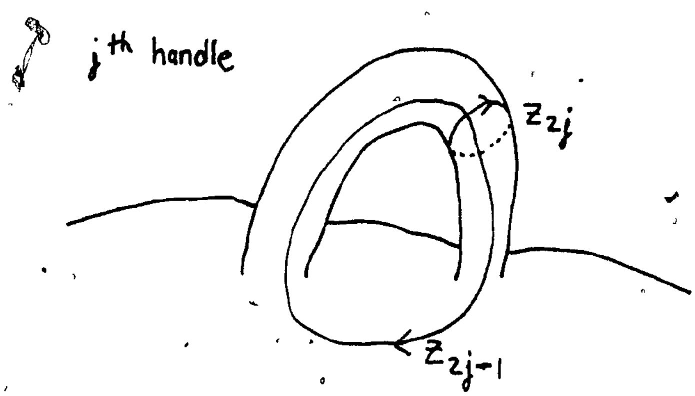


Fig. 4

† The arguments presented here are essentially those of Maxwell (1891) Article 22. In his terminology the periphractic number of a region Ω is $\beta_2(\Omega)$.

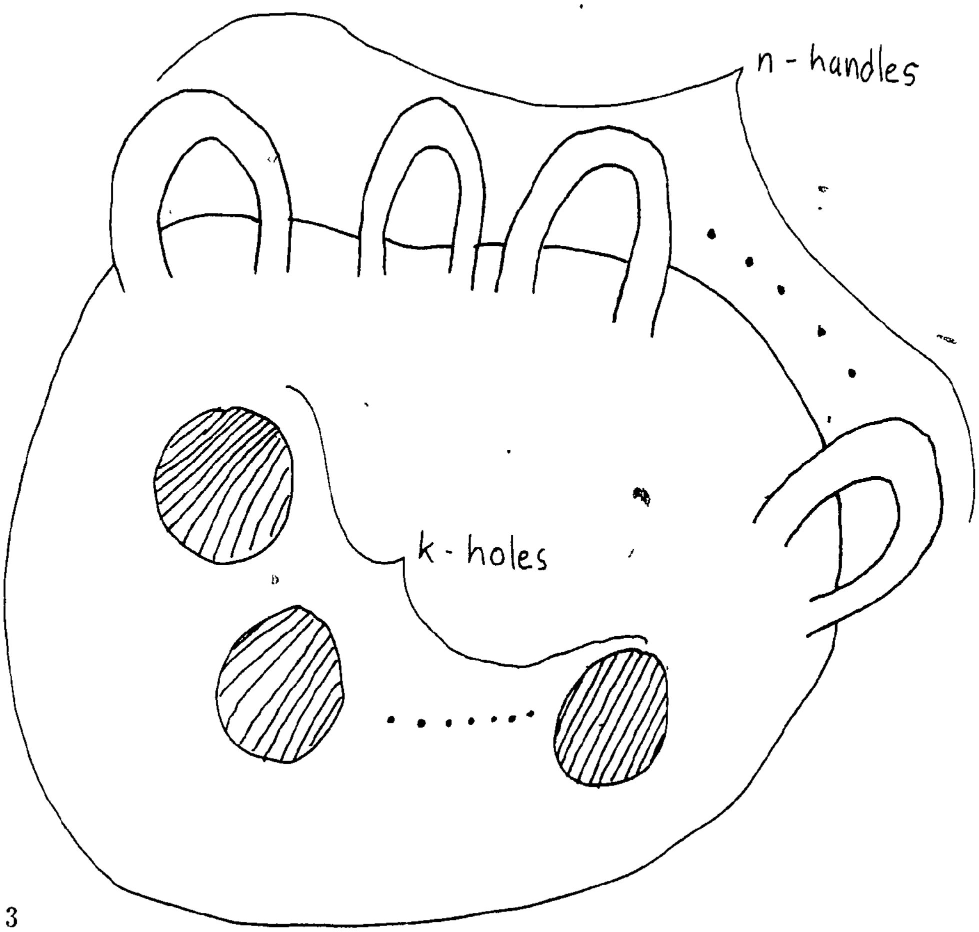


Fig 3

Let Ω be the surface described above and let $\beta_1 = 2n + k - 1$. Consider 1-cycles $z_i \in Z_1(\Omega), 1 \le i \le \beta_1$ where z_{2j-1}, z_{2j}, $1 \le j \le n$ is a pair of cycles which are related to the jth handle in the way shown in Fig. 4 while z_{2n+j}, $1 \le j \le k-1$ is related to the jth hole in the way shown in Fig. 5. Note that the kth hole is ignored as far as the z_i are concerned.

It is obvious that $z_i \in Z_1(\Omega)$ and $z_i \notin B_1(\Omega)$ for $1 \le i \le \beta_1$. What is less obvious is that $H_1(\Omega)$ can be generated by β_1 linearly independent cosets of the form

$$\sum_{i=1}^{\beta_1} a_i z_i + B_1(\Omega) \qquad a_i \in \mathbb{R}.$$

That is, no linear combination with non-zero coefficients of the z_i is homologous to zero and any 1-cycle in $Z_1(\Omega)$ is homologous to a linear combination of the z_i. In order to

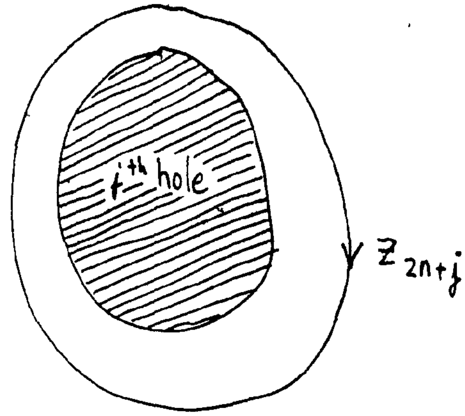


Fig. 5

justify this statement, consider 0-cycles (points) $p_j, 1 \leq j \leq k$, such that p_j is on the boundary of the jth hole that is, the p_j can be regarded as generators of $H_0(\partial_2 \Omega)$. Next define β_1 1-chains $c_i \in C_1(\Omega)$ such that

$$\left.\begin{aligned} z_{2j} &= c_{2j-1} \\ z_{2j-1} &= c_{2j} \end{aligned}\right\} \quad 1 \leq j \leq n$$

$$\partial c_{2n+j} = p_j - p_k \qquad 1 \leq j \leq k-1.$$

Furthermore, c_i intersects z_i once, $1 \leq i \leq \beta_1$ and does not intersect z_l if $i \neq l$. Hence pictorially as seen in Fig. 6.

Notice that if the surface could be cut along the c_i it would become simply connected while remaining connected. Furthermore it is not possible to make Ω simply connected with fewer than β_1 cuts. Hence, regarding the c_i as sets, one can write

$$H_1\left(\Omega - \left(\bigcup_{i=1}^{\beta_1} c_i\right)\right) = 0$$

and the c_i act like branch-cuts in complex analysis. Since removing the c_i successively introduces a new generator $H_1(\Omega)$ at each step, it is clear that

$$\beta_1(\Omega) = \beta_1 = 2n + k - 1$$

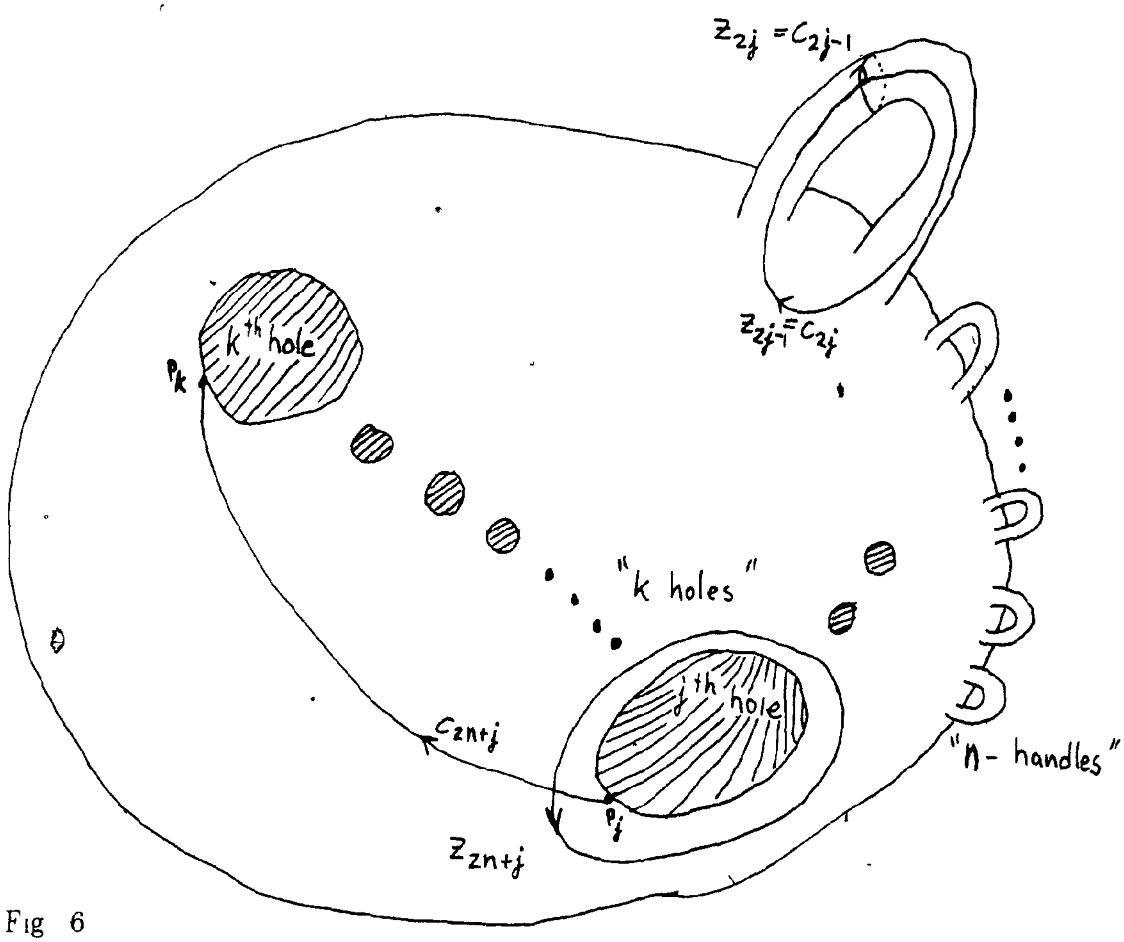


Fig 6

and that the z_i, $1 \leq i \leq \beta_1(\Omega)$ are indeed generators of $H_1(\Omega)$.

Throughout the above construction the reader may have wondered about the special status of the kth hole. It should be clear that

$$0 \sim \partial_2\Omega \sim \sum_{i=1}^{k-1} z_{2n-i} + \partial_2(kth\ hole)$$

hence associating a z_{2n+k} to the kth hole as z_{2n+j} is associated with the jth hole would not introduce a new generator into $H_1(\Omega)$. Finally, if Ω is not connected, then the above considerations can be applied to each connected component of Ω.

End of Example 4

Intuitively the ranks of $H_p(\Omega)$ were, respectively, $1, 1, n, 2n + k - 1$, in Examples 1,2,3,4. In order to prove this fact, it is necessary to have a way of computing homology.

From the definiton $H_p(\Omega) = Z_p(\Omega)/B_p(\Omega)$ involving the quotient of two infinite groups (vector spaces) it is not apparent that the homology groups should have finite rank! In general, compact manifolds have homology groups of finite rank. It is not worthwhile to pursue this point since no method of computing homology has been introduced so far. Instead the relation between homology and vector analysis will now be explored in order to show the importance of homology theory in the context of electromagnetics.

1.4 Cohomology and Vector Analysis

To relate homology groups to vector analysis consider Stokes' Theorem

$$\int_c d\omega = \int_{\partial c} \omega$$

rewritten for the case of p-chains on Ω

$$[c, d^{p-1}\omega] = [\partial_p c, \omega].$$

Stokes' Theorem shows that d^{p-1} and ∂_p act like adjoint operators. Furthermore, since $\partial_p \partial_{p+1} = 0$ we have

$$\begin{aligned} [c, d^p d^{p-1}\omega] &= [\partial_{p+1} c, d^{p-1}\omega] \\ &= [\partial_p \partial_{p+1} c, \omega] \\ &= 0 \qquad \textit{for all } c \in C_p(\Omega), \omega \in C^p(\Omega). \end{aligned}$$

Thus, assuming integration to be a non-degenerate bilinear pairing gives the operator equation

$$d^p d^{p-1} = 0 \qquad \text{for all } p.$$

Hence, surveying the classical versions of Stokes' Theorem it is apparent that the following vector identities

$$\operatorname{div}(\operatorname{curl}) = 0$$

$$\operatorname{curl}(\operatorname{grad}) = 0$$

follow as special cases.

In analogy with the case of the boundary operator, the identity $d^p d^{p-1} = 0$ does not imply that $\omega = d^{p-1}\eta$ for some $\eta \in C^{p-1}(\Omega)$ whenever $d^p\omega = 0$ and it is useful to define subgroups of $C^p(\Omega)$ as follows:

$$Z^p(\Omega) = \text{Kernel}\,(d^p),$$

the group of p-cocycles (or closed forms) on Ω and

$$B^p(\Omega) = \text{Image}\,(d^{p-1}),$$

the group of p-coboundaries (or exact forms) on Ω. The equation

$$d^p d^{p-1} = 0$$

can thus be rewritten as

$$B^p(\Omega) \subset Z^p(\Omega)$$

and

$$H^p(\Omega) = Z^p(\Omega)/B^p(\Omega),$$

the pth cohomology group of Ω, is defined as a measure of the extent by which the inclusion misses being an equality†. In order not to become tongue-tied when talking about the cosets of $H^p(\Omega)$, the following equivalence relation is introduced. Given $z^1, z^2 \in Z^p(\Omega)$,

$$z^1 \sim z^2 \text{ (read } z^1 \text{ is cohomologous to } z_2)$$

† The groups $B^p(\Omega)$, $Z^p(\Omega)$ are vector spaces while $H^p(\Omega)$ is a quotient space in the present case since the coefficient group is $\mathbb{R}$.

$$\iota f\ z^1 - z^2 \in b \text{ for some } b \in B^p(\Omega).$$

That is, z^1 is cohomologous to z^2 if z^1 and z^2 lie in the same coset of $H^p(\Omega)$.

The topological problems in vector analysis can now be reformulated in a neat way Consider a uniformly n-dimensional region Ω which is a bounded subset[‡] of $\mathbb{R}$ and consider the following questions:

- Given a vector field $\mathbf{D}$ such that $\operatorname{div}\mathbf{D} = 0$ on Ω, is it possible to find a continuous vector field $\mathbf{C}$ such that $\mathbf{D} = \operatorname{curl}\mathbf{C}$?

- Given a vector field $\mathbf{H}$ such that $\operatorname{curl}\mathbf{H} = 0$ in Ω, is it possible to find a continuous single-valued function ψ such that $\mathbf{H} = \operatorname{grad}\psi$?

- Given a scalar function ϕ such that $\operatorname{grad}\phi = 0$ in Ω. is $\phi = 0$ in Ω?

It is apparent that for $p = 2, 1, 0$ respectively the above questions have the common form: Given a $\omega \in Z^p(\Omega)$ is $\omega \in B^p(\Omega)$? Alternatively, this question can be rephrased as: Given $\omega \in Z^p(\Omega)$, is ω cohomologous to zero?

Given an n-dimensional Ω, suppose for a moment that, for all p, $C^p(\Omega)$ and $C_p(\Omega)$ are both *finite* dimensional. In this case, the fact that ∂_p and d^{p-1} are adjoint operators gives an instant solution to the above questions since, the identity

$$\text{Annihilator}(\text{Image } d^{p-1}) = \text{Kernel}\,(\partial_p)$$

that is,

$$\text{Annihilator}\,(B^p(\Omega)) = Z_p(\Omega)$$

‡ Technically speaking Ω is a compact 3-dimensional manifold with boundary - the appropriate definitions will appear in the next chapter.

can be rewritten in a more intuitive way:

$$\omega \in B^p(\Omega) \text{ iff } \int_z \omega = 0 \qquad \text{for all } z \in Z_p(\Omega). \tag{3}$$

Next, suppose $\omega \in Z^p(\Omega)$ and consider the integral of ω over the coset

$$z + B_p(\Omega) \in H_p(\Omega).$$

Letting $b = \partial_{p+1}c'$ ($c' \in C_{p+1}(\Omega)$) be an arbitrary element of $B_p(\Omega)$ gives

$$\begin{aligned} \int_{z+b} \omega &= \int_z \omega + \int_{\partial_{p+1}c'} \omega && \text{by linearity} \\ &= \int_z \omega + \int_{c'} d^p\omega && \text{by Stokes' Theorem} \\ &= \int_z \omega && \text{since } \omega \in Z^p(\Omega). \end{aligned}$$

Hence, when $\omega \in Z^p(\Omega)$, the compatibility condition (3) depends only on the coset of z in $H_p(\Omega)$. Thus condition (3) can be rewritten as:

$$\omega \in B^p(\Omega) \qquad \text{iff} \omega \in Z^p(\Omega) \qquad \text{and} \qquad \int_{Z_i} \omega = 0,\ 1 \le i \le \beta_p(\Omega)$$

where $H_p(\Omega)$ is generated by cosets of the form

$$\sum_{i=1}^{\beta_p(\Omega)} a_i z_i + B_p(\Omega).$$

It turns out that the result of this simple† investigation is true under very general conditions. The result of de Rham which is stated in the next section amounts to saying

$$H^p(\Omega) \simeq H_p(\Omega)$$

† Simple since $C^p(\Omega), C_p(\Omega)$ are seldomly finite dimensional.

— an isomorphism obtained through integration — and

$$\beta^p(\Omega) = \beta_p(\Omega) < \infty$$

where

$$\beta_p(\Omega) = \dim H_p(\Omega), \qquad \beta^p(\Omega) = \dim H^p(\Omega).$$

Hence, for an n-dimensional region Ω the answer to the question: "Given $\omega \in Z^p(\Omega)$ is $z \in B^p(\Omega)$? is "yes" provided that

$$\int_z \omega = 0$$

over $\beta_p(\Omega)$ independent p-cycles whose cosets in $H_p(\Omega)$ are capable of generating $H_p(\Omega)$. To the uninitiated, this point of view may seem unintuitive and excessively algebraic. For this reason several examples illustrating de Rham's theorem will be considered next along with the original statement of de Rham's Theorem.

1.5 19th Century Problems Which Illustrate the Work of George de Rham

In order to state the theorems of de Rham in their original form the notion of a period is required. Consider a n-dimensional region Ω. Define the period of $\omega \in Z^p(\Omega)$ on $z \in Z_p(\Omega)$ to be the value of the integral

$$\int_z \omega.$$

Note that by Stokes' Theorem, the period of ω on z depends only on the coset of z in $H_p(\Omega)$ and the coset of ω in $H^p(\Omega)$. That is,

$$\begin{aligned}\int_{z+\partial_{p+1}c'} \omega + d^{p-1}\omega' &= \int_z \omega + \int_{\partial_{p+1}c'} \left(\omega + d^{p-1}\omega'\right) + \int_z d^{p-1}\omega' \\ &= \int_z \omega + \int_{c'} d^p\omega + \int_{\partial_p z} \omega' \qquad \text{by Stokes' Theorem} \\ &= \int_z \omega \qquad \text{since } \omega \in Z^p(\Omega), \quad z \in Z_p(\Omega).\end{aligned}$$

Postponing technicalities pertaining to differentiable manifolds, de Rham's original two theorems can be stated as follows. Let $\tilde{z}_i$, $1 \leq i \leq \beta_p(\Omega)$ be homology classes (cosets in $H_p(\Omega)$) which generate $H_p(\Omega)$..Then:

1) A closed form whose periods on the $\tilde{z}_i$ vanish is an exact form. That is, $\omega \in B^p(\Omega)$ if $\omega \in Z^p(\Omega)$ and

$$\int_{\tilde{z}_i} \omega = 0, \qquad 1 \leq i \leq \beta_p(\Omega);$$

2) Given numbers $a_i, 1 \leq i \leq \beta_p(\Omega)$, there exist a closed form ω such that the period of ω on $\tilde{z}_i$ is a_i, $1 \leq i \leq \beta_p(\Omega)$. That is, given a_i, $1 \leq i \leq \beta_p(\Omega)$, there exists a $\omega \in Z^p(\Omega)$ such that

$$\int_{\tilde{z}_i} \omega = a_i \qquad 1 \leq i \leq \beta_p(\Omega).$$

The two above theorems are an explicit way of saying that $H^p(\Omega)$ and $H_p(\Omega)$ are isomorphic.

The following examples will illustrate how the isomorphism between homology and cohomology groups occurs in vector analysis and, whenever possible, the approach will mimic the nineteenth century reasoning.

Example 5 ($\Omega \subset \mathbb{R}^3$ $H^2(\Omega)$ is of concern)

Let Ω be a three-dimensional subset of $\mathbb{R}^3$ and consider a continuous vector field $\mathbf{D}$ such that $\operatorname{div} \mathbf{D} = 0$ in Ω. When is it possible to find a vector field $\mathbf{C}$ such that $\mathbf{D} = \operatorname{curl} \mathbf{C}$?

If Ω has no cavities, that is if $\mathbb{R}^3 - \Omega$ is connected, then it is safe to say that such a vector field exists. (i.e. $0 = H_2(\Omega) \Rightarrow H^2(\Omega) = 0$.) In order to see that there may be no such vector field $\mathbf{C}$ if $H_2(\Omega) \neq 0$ consider the following situation.

Suppose a conducting ball of radius 1, centred about the origin in $\mathbb{R}^3$, supports a non zero net charge Q. Suppose this ball is centred in a spherical metallic shell of radius 3 and let Ω be the space interior to the shell but exterior to the ball. It is obvious that a sphere of radius 2, centered about the origin and oriented by its unit outward normal is not homologous to zero and that $\beta_2(\Omega) = 1$. Interpreting $\mathbf{D}$ as the electric flux density vector and assuming that it is related to an electric vector potential by the relation

$$\mathbf{D} = \operatorname{curl} \mathbf{C}$$

leads to a contradiction, because calculating the period of the field $\mathbf{D}$ over the nontrivial homology class yields

$$\begin{aligned} Q &= \int_S \mathbf{D} \cdot \mathbf{n} = \int_S \operatorname{curl} \mathbf{C} \cdot \mathbf{n}\, dS \\ &= \int_{\partial S} \mathbf{C} \cdot \mathbf{t}\, dl = 0 \qquad \text{since } \partial S = 0 \end{aligned}$$

in other words $0 \neq Q = 0$ – a contradiction.

More generally, the "intuitive" condition for ensuring that such a vector field $\mathbf{C}$ exists if $\operatorname{div} \mathbf{D} = 0$ can be given as follows (see also Stevenson (1954), Maxwell (1891)[†], Article 22).

Consider again the region Ω of Example 3 where the boundary of Ω had $n+1$ connected components S_i, $0 \leq i \leq n$, S_0 being the boundary of the unbounded component of $\mathbb{R}^3 - \Omega$. In this case $H_2(\Omega)$ is generated by linear combinations of the S_i, $1 \leq i \leq n$, and the conditions for ensuring that $\mathbf{D} = \operatorname{curl} \mathbf{C}$ in Ω if $\operatorname{div} \mathbf{D} = 0$ in Ω are:

$$\int_{S_i} \mathbf{D} \cdot \mathbf{n}\, dS = 0 \qquad 1 \leq i \leq n = \beta_2(\Omega).$$

[†] When reading Maxwell the following terminology is useful to know:
Ω is a periphractic region - $H_2(\Omega) \neq 0$ the periphractic number of Ω – $\beta_2(\Omega)$.

This is also the answer to be expected by de Rham's Theorem. If is interesting to note that the above integral condition is satisfied identically on S_0 in this case since

$$0 = \int_\Omega \operatorname{div} \mathbf{D}\, dV = \int_{\partial\Omega} \mathbf{D} \cdot \mathbf{n}\, dS = \sum_{i=0}^{n} \int_{S_i} \mathbf{D} \cdot \mathbf{n}\, dS = \int_{S_0} \mathbf{D} \cdot \mathbf{n}\, dS$$

this reaffirms that

$$\sum_{i=0}^{n} S_i \sim 0.$$

The case where Ω may not be connected is easily handled by applying the above considerations to each connected component of Ω.

End of Example 5

Example 6 ($\Omega' \subset \mathbb{R}^3, H^0(\Omega')$ is of concern).

Let Ω' be a three dimensional subset of $\mathbb{R}^3$ and consider a function ϕ such that $\operatorname{grad}\phi = 0$ in Ω. When is it possible to say that $\phi \in B^{-1}(\Omega)$, that is $\phi = 0$? If Ω' is connected then ϕ is determined to within a constant (i.e. $\beta_0(\Omega) = 1 \Leftrightarrow \beta^0(\Omega) = 1$). In order to see that ϕ is not necesarily a constant if $\beta_0(\Omega) > 1$, consider the situation of electrostatics:

Suppose that there are n connected bodies Ω'_i each carrying a charge Q_i, $1 \leq i \leq n$ inside a conducting shell Ω'_0 which supports a charge Q_0. Let

$$\Omega' = \bigcup_{i=0}^{n} \Omega'_i.$$

Inside each conducting body the electric field vector $\mathbf{E} = -\operatorname{grad}\phi$ vanishes. However, depending on the charges Q_i and hence on the charge

$$-\sum_{i=0}^{n} Q_i$$

somewhere exterior to the problem, it is well known that

$$\phi\Big|_{\Omega'_i} = \phi_i \text{ (constants)}, \qquad 1 \le i \le n$$

can be assigned arbitrarily. In general, the scalar potential ϕ vanishes only if the above constants all vanish, hence

$$\beta_0(\Omega') = n + 1.$$

This trivial example can be used to illustrate an additional point. In electrostatics it is customary to let

$$\phi\Big|_{\Omega'_0} = 0 \qquad \text{(datum)}$$

$$Q_0 = \sum_{i=1}^{n} Q_i \qquad \text{(conservation of charge)}$$

Let $\Omega' = \mathbb{R}^3 - \Omega$ where Ω'_i, $0 \le i \le n$ are the connected components of Ω' while Ω'_0 is the unbounded component of Ω'. Using the final equation of Example 3, it is clear that

$$\begin{aligned} n = \beta_2(\Omega) &= \beta_0(\mathbb{R}^3 - \Omega) - 1 \\ &= \beta_0(\Omega') - 1 \end{aligned}$$

Interpreting $\beta_2(\Omega)$ as the number of independent charges in the problem and $\beta_0(\Omega') - 1$ as the number of independent potential differences the above equation says that the number of independent charges equals the number of independent potential differences.

End of Example 6

Example 7 (Ω a 2 dimensional surface, $H^1(\Omega)$ of interest) Let Ω be a two dimensional orientable surface and consider the conjugate versions of the usual integral theorems[†]:

$$\int_c \overline{\text{curl}}\,\phi \cdot \mathbf{n}\, dl = \phi(p_2) - \phi(p_1) \qquad c \in C_1(\Omega)$$
$$\partial c = p_2 - p_1$$

$$\int_c \text{div}\,\mathbf{J}\, dS = \int_{\partial c} \mathbf{J} \cdot \mathbf{n}\, dl \qquad c \in C_2(\Omega).$$

where $\mathbf{n}$ is the unit vector normal to the curve c. In this case, the operator identity

$$\text{div}\,(\overline{\text{curl}}) = 0$$

shows that it is natural to ask the following question. Consider a vector field $\mathbf{J}$ such that $\text{div}\,\mathbf{J} = 0$ on Ω. When is it possible to write $\mathbf{J} = \overline{\text{curl}}\,\phi$ for some single valued stream function ϕ?

If Ω is simply connected[†], then it is well known that $\mathbf{J} = \overline{\text{curl}}\,\phi$, i.e. $\beta_1(\Omega) = 0 \Rightarrow \beta^1(\Omega) = 0$. In order to see that it may not be possible to find such a ϕ if Ω is not simply connected, consider the following example.

Suppose Ω is homeomorphic to an annulus. On Ω let $\mathbf{J}$ flow outward in the radial direction, and let $z \in Z_1(\Omega)$ be a 1-cycle which encircles the hole (see Fig. 7). Interpreting this situation as a steady current flow on a surface of finite thickness, the period of $\mathbf{J}$ on the cycle z will be called "the current per unit of thickness through z"

† $\overline{\text{curl}}\,\phi$ is defined as $\mathbf{n}' \times \text{grad}\,\phi$ where $\mathbf{n}'$ is the unit normal vector to the two dimensional orientable surface. The notation $\overline{\text{curl}}$ is taken from Nedelec [1978], p.582.

† Given a space X, "X is simply connected" means that the first homotopy group $\pi(X)$ is trivial which in turn implies that $H_1(X) = 0$ by an old theorem of Poincaré (see Greenberg (1981) Chapter 12). Generally speaking, higher homotopy groups, $\pi_n(X)$, can be defined, but they are useless for computing homology (see Bott (1982) p.225).

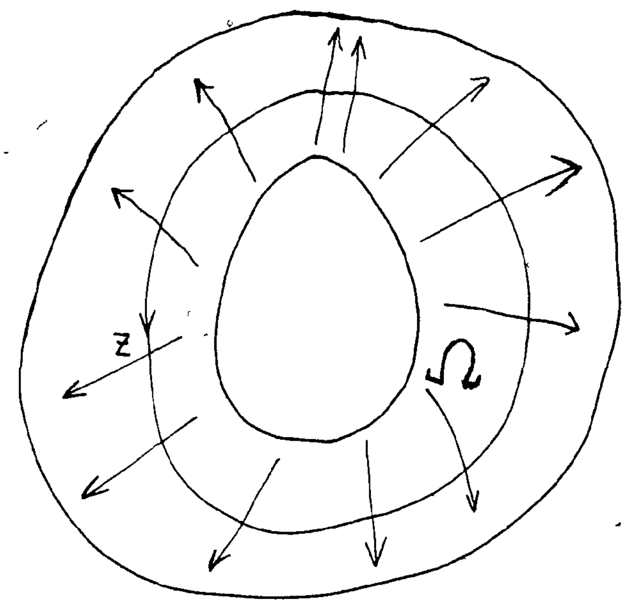


Fig. 7

and denoted by I. In this case, relating $\mathbf{J}$ to a single valued stream function ϕ leads to a contradiction since

$$0 \neq I = \int_z \mathbf{J} \cdot \mathbf{n}\, dl = \int_z \overline{\mathrm{curl}}\, \phi \cdot \mathbf{n}\, dl = \int_{\partial z} \phi = 0 \quad \text{since } \partial z = 0.$$

Hence $0 \neq I = 0$ – a contradiction.

More generally, (see Klein [1893] for more pictures, interpretations and references to the nineteenth century literature) consider the surface Ω of Example 4 where there are z_i, $1 \leq i \leq \beta_1(\Omega) = 2n + k - 1$ generators of $H_1(\Omega)$ and cuts c_i, $1 \leq i \leq \beta_1(\Omega)$ such that

$$\Omega^- = \Omega - \left(\bigcup_{i=1}^{\beta_1(\Omega)} c_i \right)$$

was connected and simply connected. Since Ω^- is simply connected it is possible to define a stream function ϕ^- on Ω^- such that

$$\mathbf{J} = \overline{\mathrm{curl}}\, \phi^- \qquad on\ \Omega^-.$$

Letting the current flowing through z_i be I_i

$$\int_{z_i} \mathbf{J} \cdot \mathbf{n}\, dl = I_i,$$

it is apparent from the integral laws that

$$I_i = \int_{z_i \cap \Omega^-} \overline{\text{curl}}\, \phi^- = (\text{jump in } \phi^- \text{ across } c_i).$$

That is ϕ^- is in general multivalued and it is single valued if and only if all the periods of $\mathbf{J}$ on the z_i vanish, that is each I_i must vanish. Hence $\mathbf{J} = \overline{\text{curl}}\, \phi$ on Ω for some single valued ϕ if and only if $\text{div}\, \mathbf{J} = 0$ and

$$\int_{z_i} \mathbf{J} \cdot \mathbf{n}\, dl = 0 \qquad 1 \leq i \leq \beta_1(\Omega),$$

End of Example 7

Example 8 ($\Omega \subset \mathbb{R}^3$, $H^1(\Omega)$ is of concern)

Let Ω be a three dimensional subset of $\mathbb{R}^3$ and consider a vector field $\mathbf{H}$ such that $\text{curl}\, \mathbf{H} = 0$ in Ω. Is there a single valued function ψ such that $\mathbf{H} = \text{grad}\, \psi$?

If Ω is simply connected, that is, if every closed curve in Ω can be shrunk to a point in a continuous fashion, then it is possible to find such a single valued function ψ. In other words, simple connectivity, $H_1(\Omega) = 0$, and $H^1(\Omega) = 0$ are equivalent statements in this case.

In order to see that there may be no such function ψ if Ω is not simply connected, consider the region Ω to be the region exterior to a thick resistive wire connected across a battery and $\Omega' = \mathbb{R}^3 - \Omega$ as shown in Fig. 8. Here, $\beta_1(\Omega) = \beta_1(\Omega') = 1$, $z \in Z_1(\Omega)$, and $z' \in Z_1(\Omega')$ represent nontrivial homology classes of $H_1(\Omega)$ and $H_1(\Omega')$ respectively. Let

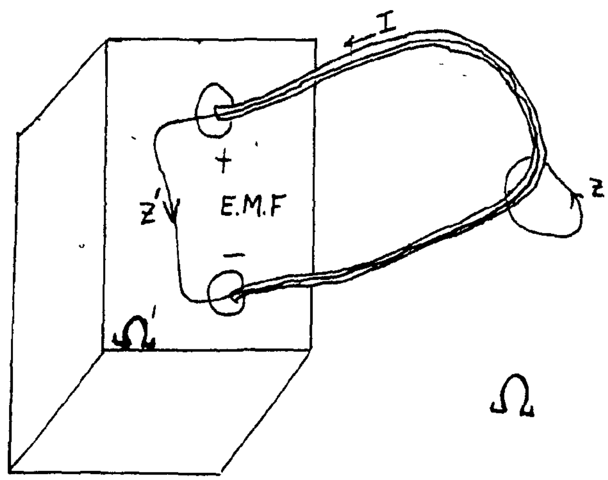


Fig. 8

$S, S' \in C_2(\mathbb{R}^3)$ be a pair of 2-chains which, when considered as sets, are homeomorphic to discs and

$$\partial S' = z'$$

$$\partial S = z.$$

Since the problem is assumed to be static, it is obvious that

$$\operatorname{curl} \mathbf{H} = 0 \qquad in\ \Omega$$

$$\operatorname{curl} \mathbf{E} = 0 \qquad in\ \Omega'$$

and that the periods

$$\int_z \mathbf{H} \cdot \mathbf{t}\, dl = I, \qquad \int_{z'} \mathbf{E} \cdot \mathbf{t}\, dl = E.M.F.$$

are nonzero. However, assuming that $\mathbf{E}$ and $\mathbf{H}$ can be represented as gradients of single valued scalar potentials ψ' and ψ respectively leads to contradictions since

$$0 \neq I = \int_z \mathbf{H} \cdot \mathbf{t}\, dl = \int_z \operatorname{curl} \psi \cdot \mathbf{t}\, dl = \int_{\partial z} \psi = 0$$

since $\partial z = 0$ and

$$0 \neq E.M.F. = \int_{z'} \mathbf{E} \cdot \mathbf{t}\, dl = \int_{z'} \operatorname{curl} \psi' \cdot \mathbf{t}\, dl = \int_{\partial z'} \psi' = 0$$

since $\partial z' = 0$. In this case, note that

$$H_1(\Omega - S') = 0$$

$$H_1(\Omega' - S) = 0$$

hence, the magnetic field can be represented as the gradient of a scalar ψ in $\Omega - S'$ where the scalar has a jump of value I whenever S' is traversed in the direction of its normal. Similarly, the electric field can be represented as the gradient of a scalar ψ' in $\Omega' - S$ where the scalar has a jump of value $E.M.F$ whenever S is traversed in the direction of its normal. Note that ψ, ψ' are continuous and single valued on Ω, Ω' respectively if and only if

$$I = 0, \qquad E.M.F. = 0.$$

Thus it is seen that the irrotational fields $\mathbf{H}$ in Ω and $\mathbf{E}$ in Ω can be expressed in terms of single-valued scalar functions once the cuts S and S' are introduced.

The general intuitive conditions for representing an irrotational vector field $\mathbf{H}$ as the gradient of a scalar potential have been studied for a long time. See Kelvin [1867], Maxwell [1891] articles† 18–20, 421 and Lamb [1932] articles 47–55, 132–134, and 139–141 are also of interest. A formal justification for introducing cuts into a space involves duality theorems for homology groups of orientable manifolds. These theorems will be

† When reading articles 18–20 in Maxwell, the following correspondences are useful to remember. Ω is acyclic means Ω is simply connected. Cyclosis means multiple connectivity, cyclic constants are periods on generators of $H_1(\Omega)$. "Cyclic constants" were usually called "Kelvin's constants of circulation" in the nineteenth century literature.

considered in Section 1.10 For the time being the general procedure for introducing cuts will be illustrated by trying to generalize the above case involving a battery and a wire

Let Ω be a connected subset of $\mathbb{R}^3$. The first thing to do is to find 2-chains $S_i' \in C_2(\mathbb{R}^3)$, $1 \leq i \leq n$ which when considered as surfaces satisfy the following:

1) $\Omega - (\bigcup_{i=1}^{n} S_i')$ is connected and simply connected, and $\partial S_i' \in Z_1(\mathbb{R}^3 - \Omega), i \leq i \leq n$.

2) $H_1(\mathbb{R}^3 - \Omega)$ is generated by cosets of the form

$$\sum_{i=1}^{n} a_i' \partial S_i' + B_1(\mathbb{R}^3 - \Omega), \qquad a_i \in \mathbb{R}$$

and n is chosen such that $n = \beta_1(\mathbb{R}^3 - \Omega)$. Note that $\partial S_i' \notin B_1(\mathbb{R}^3 - \Omega), 1 \leq i \leq \beta_1(\mathbb{R}^3 - \Omega)$.

It turns out that one can also do the reverse, that is find 2-chains $S_i \in C_2(\mathbb{R}^3)$, $1 \leq i \leq n$, which when considered as surfaces and satisfy the following:

3) $(\mathbb{R}^3 - \Omega) - (\bigcup_{i=1}^{n} S_i)$ is connected and simply connected, and $\partial S_i \in Z_1(\Omega), 1 \leq i \leq n$.

4) $H_1(\Omega)$ is generated by cosets of the form

$$\sum_{i=1}^{n} a_i \partial S_i + B_1(\Omega) \qquad a_i \in \mathbb{R}$$

and n is chosen such than $n = \beta_1(\Omega)$. Note $\partial S_i \neq B_1(\Omega), 1 \leq i \leq \beta_1(\Omega)$.

If one is lucky, the ∂S_i intersect S_j' very few times and likewise for $\partial S_i'$ and S_j. The result†

$$\beta_1(\Omega) = \beta_1(\mathbb{R}^3 - \Omega)$$

† This result was known to Maxwell (1891) Article 18.

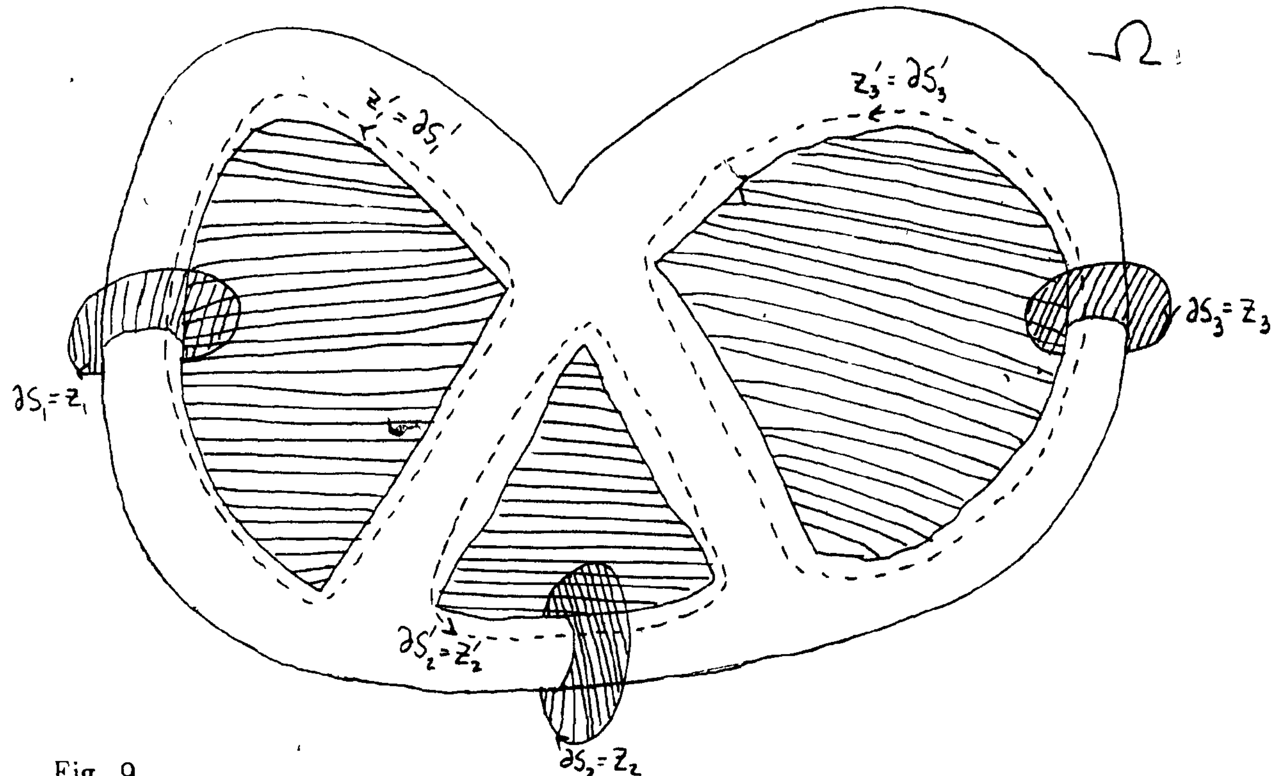


Fig. 9

is apparent at this stage.

If $\operatorname{curl}\mathbf{H} = 0$ in Ω then by the above construction, there exist a

$$\psi \in C^0\left(\Omega - \left(\bigcup_{i=1}^{\beta^1(\Omega)} S_i'\right)\right)$$

such that

$$\mathbf{H} = \operatorname{grad}\psi \qquad on\ \Omega - \left(\bigcup_{i=1}^{\beta_1(\Omega)} S_i'\right).$$

Furthermore the jump in ψ over the surface S_i' can be deduced from the periods

$$\int_{\partial S_l} \mathbf{H}\cdot\mathbf{t}\,dl = I_l, \qquad 1 \le l \le \beta_1(\Omega)$$

by solving a set of linear equations which have trivial solutions if and only if all of the periods vanish.

As a simple instance of this procedure, consider a current carrying pretzel $\mathbb{R}^3 - \Omega$ and its complement Ω as shown in Fig. 9. Here

$$I_i = \int_{\partial S_i} \mathbf{H} \cdot \mathbf{t}\, dl = (\text{current flowing through } S_i)$$

$$\beta_1(\Omega) = \beta_1(\mathbb{R}^3 - \Omega) = 3$$

$$\mathbf{H} = \operatorname{grad} \psi \qquad on\ \Omega - \left(\bigcup_{i=1}^{3} S_i' \right)$$

and the jumps in ψ across S_i' are given by I_i. It is clear that the scalar potential will be continuous and single valued in Ω if and only if $I_i = 0$, $1 \le i \le 3$.

Although this illustration makes the general procedure look like a silly interpretation of mesh analysis in network theory, problems where the $\partial S_i'$ are necessarily not in the same plane may be harder to tackle as are problems where $\beta_0(\Omega) > 1$. The case where Ω may be disconnected is handled by reasoning in terms of the connected components of Ω, separately. As a non trivial mental exercise the reader may convince himself that

$$\beta_1(\Omega) = \beta_1(\mathbb{R}^3 - \Omega) = 2n + k - 1$$

when Ω is the two-dimensional region of Example 4. This is actually quite simple when one realizes that generators of $H_1(\Omega)$ can be taken to be the boundaries of cuts in $\mathbb{R}^3 - \Omega$ and generators of $H_1(\mathbb{R}^3 - \Omega)$ can be considered to be boundaries of surfaces which intersect Ω along the cuts $\mathbf{c}_i$, $1 \le i \le \beta_1(\Omega)$.

1.6 Chain and Cochain Complexes

Chain and cochain complexes are the setting for homology theory. Algebraically speaking, a chain complex $C. = \{C_p, \partial_p\}$ is a sequence of modules C_i over a ring R and a sequence of homomorphisms

$$\partial_p : C_p \to C_{p-1}$$

such that

$$\partial_{p-1}\partial_p = 0.$$

For the purposes of this thesis, the ring R will be $\mathbb{R}$ or $\mathbb{Z}$ in which case the modules C_p are vector spaces or abelian groups respectively. A familiar example is the chain complex

$$C_\cdot(\Omega; \mathbb{R}) = \{C_p(\Omega; \mathbb{R}), \partial_p\}$$

considered up to now. Similarly, one has the chain complex

$$C_\cdot(\Omega; \mathbb{Z}) = \{C_p(\Omega; \mathbb{Z}), \partial_p\}$$

when the coefficient group is $\mathbb{Z}$.

Cochain complexes are defined in a similar fashion except that "the arrows are reversed". That is, a cochain complex $C^\cdot = \{C^p, d^p\}$ is a sequence of modules C^p and homomorphisms

$$d^p : C^p \to C^{p+1}$$

such that

$$d^{p+1}d^p = 0.$$

An example of a the cochain complex is

$$C^\cdot(\Omega; \mathbb{R}) = \{C^p(\Omega; \mathbb{R}), d^p\}$$

which has been considered in the context of integration†. From the definition of chain

† When the coefficient group is not mentioned, it is understood to be $\mathbb{R}$.

and cochain complexes, it is obvious how homology and cohomology are defined:

$$\begin{array}{ll} B_p = \text{Image } \partial_{p+1} & B^p = \text{Image } d^{p-1} \\ Z_p = \text{Kernel } \partial_p & Z^p = \text{Kernel } d^p \\ B_p \subset Z_p & B^p \subset Z^p \\ H_p = Z_p/B_p & H^p = Z^p/B^p \\ \beta_p = \text{Rank } H_p & \beta^p = \text{Rank } H^p \\ \text{for homology} & \text{for cohomology} \end{array}$$

When dealing with chain and cochain complexes, it is often convenient to supress the subscript on ∂_p and the superscript on d^p and let ∂ and d be the boundary and coboundary operators in the complex where their interpretation is clear from context.

The reader should realise that the introduction has thus far aimed to motivate the idea of chain and cochain complexes and the resulting homology and cohomology. Explicit methods for setting up complexes and computing homology from triangulations or cell decompositions can be found in many texts (see Massey [1980], Giblin [1981], Wallace [1957], or Greenberg and Harper [1980], for example) while computer programs to compute Betti numbers and other topological invariants have been around for almost two decades (see Pinkerton [1966].) In contrast to the vast amount of literature on homology theory, there seems to be no systematic exposition on its rôle in boundary value problems of electromagnetics — the papers by Bossavit [1981], [1982], Bossavit and Verité [1982], [1983], Milani and Negro [1982], Brown [1984], Nedelec [1978] and Post [1978], [1984] are valuable first steps.

It is important to realise that the notion of complex is actually a basic idea in network theory where, if A is the usual incidence matrix and B is the loop matrix of a network, there is a chain complex

$$0 \rightarrow \{meshes\} \xrightarrow{B^T} \{branches\} \xrightarrow{A} \{nodes\} \rightarrow 0$$

(since $AB^T = 0$) in which 2-chains are linear combinations of mesh currents and 1-chains are linear combinations of branch currents. Taking the transpose of this complex, a cochain complex is obtained

$$0 \leftarrow \{meshes\} \overset{B}{\longleftarrow} \{branches\} \overset{A^T}{\longleftarrow} \{nodes\} \leftarrow 0$$

(since $BA^T = 0$) where 0-cochains are linear combinations of node potentials and 1-cochains are linear combinations of branch voltages (see Balabanian and Bickart [1969] Sect 2.2) Furthermore, if the network is planar and A is the reduced incidence matrix obtained by ignoring one node in each connected component of the network, then the homology of the complex is trivial. Kirchhoff's laws can be expressed as

The Kirchhoff Voltage Law: $v \in$ Image A^T (or v is a 1-coboundary)

The Kirchhoff Current Law: $\imath \in$ Kernel A (or $\imath$ is a 1-cycle)

so that if $v = A^T e$ for some set of nodal potentials e, then Tellegen's Theorem is easily deduced:

$$0 = (e, A\imath) = (A^T e, \imath) = (v, \imath).$$

Thus Tellegen's Theorem is an example of orthogonality between cycles and coboundaries. This view of electrical network theory is usually attributed to Weyl [1923], (See also Slepian [1968], Flanders [1971] and Smale, [1972]). Systematic use of homology theory in electrical network theory can be found in the work of J. P. Roth (see bibliography) and Ching [1968]. Kron [1959] generalises electrical network theory by introducing branch relations associated with k-dimensional ones. Unfortunately, by calling complexes "multidimensional space filters", Kron manages to confuse much of his audience — engineer and mathematician alike. An explanation of Kron's method as well as references to additional papers by Kron can be found in Balasubramanian et. al [1970].

The interplay between continuum and network models through the use of complexes is developed by Branin [1966] and Tonti [1977], and scattered throughout Kondo

1955 . Examples of cochain complexes for differential operators encountered in the work of Tonti and Branin are

$$0 \rightarrow \left\{ \begin{array}{c} scalar \\ functions \end{array} \right\} \xrightarrow{\text{grad}} \left\{ \begin{array}{c} fields \\ vectors \end{array} \right\} \xrightarrow{\text{curl}} \left\{ \begin{array}{c} flux \\ vectors \end{array} \right\} \xrightarrow{\text{div}} \left\{ \begin{array}{c} volume \\ densities \end{array} \right\} \rightarrow 0$$

$$\text{curl}\,(\text{grad}\,) = 0$$

$$\text{div}\,(\text{curl}\,) = 0$$

for vector analysis in three dimensions, and

$$0 \rightarrow \left\{ \begin{array}{c} scalar \\ functions \end{array} \right\} \xrightarrow{\text{grad}} \left\{ \begin{array}{c} field \\ vectors \end{array} \right\} \xrightarrow{\text{curl}} \{densities\} \rightarrow 0$$

in two dimensions. Note that the above two complexes are special cases of the complex $C^{\cdot}(\Omega)$ considered thus far and that when there is no mention of the domain Ω over which functions are defined, it is impossible to say anything about the homology of the complex. Hence, unless an explicit dependence on the domain Ω is recognised in the definition of the complex, it is virtually impossible to say anything concrete about global aspects of solvability conditions, gauge transformations or complementary variational principles, since these aspects depend on the cohomology groups of the complex which in turn depend on the topology of the domain Ω. Furthermore, imposing boundary conditions on some subset $S \subset \partial\Omega$ necessitates the consideration of relative cohomology groups to resolve questions of solvability, gauge ambiguity, etc., and again the situation becomes hopelessly complicated unless a complex which depends explicitly on Ω and S is defined. The cohomology groups of this complex, which are called the relative cohomology groups of Ω modulo S are the ones required to describe the global aspects of the given problem. Relative homology and cohomology groups will be considered in the next section, and as a prelude, it is necessary to introduce the idea of chain and cochain homomorphisms.

Just as groups, fields and vector spaces are examples of algebraic structures, complexes are a type of algebraic structure and as such it is useful to consider mappings between complexes. In the case of a chain complex the useful mappings to consider are the ones which have nice properties when it comes to homology. Such mappings, called chain homomorphisms, are defined as follows. Given two complexes

$$C_* = \{C_p, \partial_p\}, \qquad C'_* = \{C'_p, \partial'_p\}$$

a chain homomorphism

$$f_* : C_* \to C'_*$$

is a sequence of homomorphisms $\{f_p\}$ such that

$$f_p : C_p \to C'_p$$

and

$$\partial'_p f_p = f_{p-1} \partial_p.$$

That is, for each p, the following diagram is commutative

$$\begin{array}{ccc} \downarrow & & \downarrow \\ C_p & \xrightarrow{f_p} & C'_p \\ \downarrow \partial_p & & \downarrow \partial'_p \\ C_{p-1} & \xrightarrow{f_{p-1}} & C'_{p-1} \\ \downarrow & & \downarrow \end{array}$$

In the case of cochain complexes, cochain homomorphisms are defined analogously.

In order to illustrate chain and cochain homomorphisms, consider a region Ω and a closed and bounded subset S. Since

$$C_p(S) \subset C_p(\Omega) \qquad for\ all\ p$$

and the boundary operator ∂' in the complex $C_\cdot(S)$ is the restriction of the boundary operator ∂ in $C_\cdot(\Omega)$, $C_\cdot(S)$ is a subcomplex of $C_\cdot(\Omega)$ and there is a chain homomorphism

$$\imath_\cdot : C_\cdot(S) \to C_\cdot(\Omega).$$

where

$$\imath_p : C_p(S) \to C_p(\Omega)$$

is an inclusion. Obviously

$$i_{p-1}\partial'_p c = \partial_p i_p c \qquad for\ all\ c \in C_p(S)$$

as required. Similarly, considering the restriction of a p-form on Ω to one on S for all values of p. there is a cochain homomorphism

$$r^\cdot : C^\cdot(\Omega) \to C^\cdot(S)$$

where

$$r^p : C^p(\Omega) \to C^p(S).$$

If the coboundary operator (exterior derivative) in $C^\cdot(\Omega)$ is d and the corresponding coboundary operator in $C^\cdot(S)$ is d' then

$$d'^p r^p \omega = r^{p-1} d^p \omega \qquad \text{for all } \omega \in C^p(\Omega)$$

as required.

1.7 Relative Homology Groups

Relative chain, cycle, boundary and homology groups of a region Ω modulo a subset S will now be considered. It turns out that relative homology groups provide the generalisation of ordinary homology groups which is necessary in order to describe the topological aspects of cochains (forms) subject to boundary conditions.

Consider a region Ω and the chain complex $C_*(\Omega) = \{C_p(\Omega), \partial_p\}$ associated with it. Let S be a compact subset of Ω and $C_*(S) = \{C_p(S), \partial'_p\}$ be the chain complex associated with S. Note that the boundary operator of $C_*(S)$ is the one in $C_*(\Omega)$ with a restricted domain. Furthermore,

$$C_p(S) \subset C_p(\Omega) \qquad \text{for all } p$$

It is useful to define the quotient group

$$C_p(\Omega, S) = C_p(\Omega)/C_p(S)$$

— the group of p-chains on Ω modulo S — when one wants to consider p-chains on Ω while disregarding what happens on some subset S. In this way, the elements of $C_p(\Omega, S)$ are cosets of the form

$$c + C_p(S) \qquad \textit{where } c \in C_p(\Omega).$$

Although this definition makes sense with any coefficient ring R, when the coefficients are in $\mathbb{R}$ the definition of $C_p(\Omega, S)$ is made intuitive if one defines† $C^p(\Omega, S)$ to be the subset of $C^p(\Omega)$ where the support of $\omega \in C^p(\Omega, S)$ lies in $\Omega - S$. In this case it is possible to salvage the idea that integration should be a bilinear pairing between $C_p(\Omega, S)$ and $C^p(\Omega, S)$. That is

$$\int : C^p(\Omega, S) \times C_p(\Omega, S) \to \mathbb{R}$$

† This argument is intended to be entirely heuristic.

should satisfy:

$$\int_c \omega = 0 \qquad for\ all\ \omega \in C^p(\Omega, S) \Rightarrow c \in C_p(S)$$

$$\int_c \omega = 0 \qquad for\ all\ c \in C_p(\Omega, S) \Rightarrow \omega = 0$$

Note that when $S = \phi$ (the null set) the definitions of relative chain and cochain groups reduce to those of their absolute counterparts.

Returning to the general case where the chains could be considered with coefficients in any ring such as $\mathbb{R}$ or $\mathbb{Z}$, the induced boundary operator

$$\partial_p'' : C_p(\Omega)/C_p(S) \rightarrow C_{p-1}(\Omega)/C_{p-1}(S)$$

makes the following definitions appropriate:

$$Z_p(\Omega, S) = \text{Kernel}\left(C_p(\Omega, S) \xrightarrow{\partial_p''} C_{p-1}(\Omega, S) \right),$$

the group of relative p-cycles of Ω modulo S and

$$B_p(\Omega, S) = \text{Image}\left(C_{p+1}(\Omega, S) \xrightarrow{\partial_{p+1}''} C_p(\Omega, S) \right),$$

the group of relative p-boundaries of Ω mod S. Intuitively, relative cycles and boundaries can be interpreted as follows. Given $z, b \in C_p(\Omega)$

$$z + C_p(S) \in Z_p(\Omega, S) \quad \text{if} \quad \partial_p z \in \imath_{p-1}(C_{p-1}(S))$$

$$b + C_p(S) \in B_p(\Omega, S) \quad \text{if} \quad \partial_{p+1} c - b \in \imath_p(C_p(S))$$

for some $c \in C_{p+1}(\Omega)$. Hence, z is a relative p-cycle if its boundary lies in the subset S while b is a relative p-boundary if it is homologous to some p-chain on S.

From the definition of ∂_p'', it is apparent that

$$\partial_p'' \partial_{p+1}'' = 0.$$

Hence

$$B_p(\Omega, S) \subset Z_p(\Omega, S)$$

and the pth relative homology group of Ω modulo S and the relative pth Betti number of Ω modulo S can be defined as follows.

$$H_p(\Omega, S) = Z_p(\Omega, S)/B_p(\Omega, S)$$

$$\beta_p(\Omega, S) = \text{Rank}\, H_p(\Omega, S).$$

By defining $C_p(\Omega, S) = 0$ for $p < 0$ and $p > n$, the above definitions make it apparent that

$$C_\cdot(\Omega, S) = \{C_p(\Omega, S), \partial_p''\}$$

is a complex Futhermore, if

$$j_p \,.\, C_p(\Omega) \to C_p(\Omega, S)$$

is the homomorphism which takes a $c \in C_p(\Omega)$ into a coset of $C_p(\Omega, S)$ according to the rule

$$j_p(c) = c + C_p(S)$$

then the collection of homomorphisms $j_* = \{j_p\}$ is a chain homomorphism

$$j_* : C_\cdot(\Omega) \to C_\cdot(\Omega, S)$$

since

$$\partial_p'' j_p(c) = j_{p-1}\partial_p(c) \qquad \textit{for all } c \in C_p(\Omega).$$

Though the definitions leading to relative homology groups seem formidable at first sight, they are actually quite a bit of fun as the following example shows.

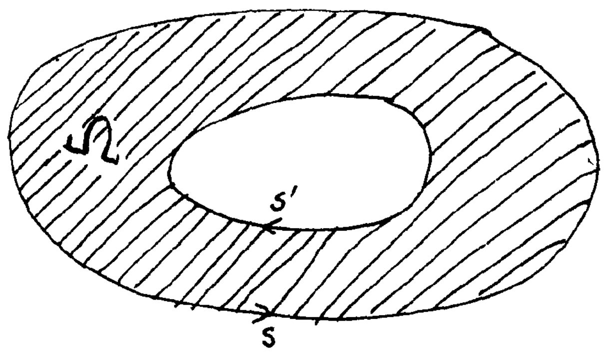


Fig. 10

Example 9 (a 2-D example)

In this example the relative homology groups associated with the cross-section of a coaxial cable are considered. The usefulness of relative homology groups will become apparent in later sections once relative cohomology groups have been introduced. Consider a piece of coaxial cable of elliptic cross section and let Ω be the "insulator" as shown in Fig 10 and consider 1-chains z, z', z'' and 2-chains c, c' as shown in Fig. 11. From the picture it is apparent that z, z', z'' represent nontrivial cosets in $Z_1(\Omega, \partial\Omega)$ but

$$j_1(z) \sim 0 \text{ in } H_1(\Omega, \partial\Omega) \text{ since } \partial c - z \in i_1\left(C_1(\partial\Omega)\right)$$

$$j_1(z') \sim j_1(z'') \text{ in } H_1(\Omega, \partial\Omega) \text{ since } \partial c' - z' + z'' \in i_1\left(C_1(\partial\Omega)\right).$$

However, it is apparent that $j_1(z')$ is not homologous to zero in $H_1(\Omega, \partial\Omega)$ and that $\beta_1(\Omega, \partial\Omega) = 1$ so that the cosets of $H_1(\Omega, \partial\Omega)$ look like

$$az' + B_1(\Omega, \partial\Omega) \qquad a \in \mathbb{R}.$$

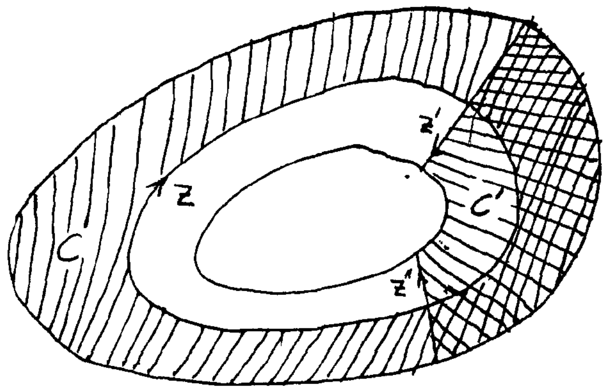


Fig 11

Next consider the other relative homology groups. If is obvious that $\beta_0(\Omega, \partial\Omega) = 0$ since any point in Ω can be joined to the boundary by a curve which lies in Ω. Furthermore, considering Ω as a 2-chain in $C_2(\Omega, \partial\Omega)$ it is apparent that

$$\Omega \in Z_2(\Omega, \partial\Omega) \text{ since } \partial_2(\Omega) \in \imath_1(C_1(\partial\Omega))$$

hence, since $B_2(\Omega, \partial\Omega) = 0$, Ω is a nontrivial generator of $H_2(\Omega, \partial\Omega)$ and since the region is planar, it is plausible that there are no other independent generators of $H_2(\Omega, \partial\Omega)$. Thus $\beta_2(\Omega, \partial\Omega)$ and the cosets of $H_2(\Omega, \partial\Omega)$ look like

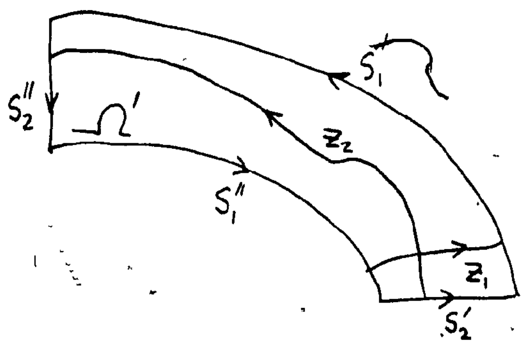
$$a\Omega \qquad a \in \mathbb{R}.$$

In the light of the previous examples the absolute homology groups of the region Ω are obvious once one notices that $\beta_0(\Omega) = 1$, the 1-cycle z is the only independent generator of $H_1(\Omega)$ hence $\beta_1(\Omega) = 1$, and $\beta_2(\Omega) = 0$ since $Z_2(\Omega) = 0$. Hence in summary

$$\beta_0(\Omega) = \beta_2(\Omega, \partial\Omega) = 1$$

$$\beta_1(\Omega) = \beta_1(\Omega, \partial\Omega) = 1$$

$$\beta_2(\Omega) = \beta_0(\Omega, \partial\Omega) = 0.$$



$$\partial\Omega' = S_1 + S_2$$

$$S_1 = S_1' + S_1''$$

$$S_2 = S_2' + S_2''$$

Fig. 12

Next, in order to exercise the newly acquired concepts, suppose that the capacitance of the cable was to be determined by a direct variational method. In this case it is convenient to exploit the inherent symmetry to reduce the problem to one a quarter of the original size. Thus consider the diagram shown in Fig. 12. It is apparent that for $a_1, a_2 \in \mathbb{R}$, the cosets of $H_1(\Omega', S_1)$ and $H_1(\Omega', S_2)$ look like

$$a_1 z_1 + B_1(\Omega', S_1)$$

and

$$a_2 z_2 + B_1(\Omega', S_2)$$

respectively and that

$$Z_2(\Omega', S_1) = 0 = Z_2(\Omega', S_2).$$

Hence, it is apparent that:

$$\beta_0(\Omega', S_1) = 0 = \beta_2(\Omega', S_2)$$

$$\beta_1(\Omega', S_1) = 1 = \beta_1(\Omega', S_2)$$

$$\beta_2(\Omega', S_1) = 0 = \beta_0(\Omega', S_2).$$

End of Example 9

Let the pth relative homology group of Ω modulo a subset S_1 with coefficient in $\mathbb{Z}$. be denoted by

$$H_p(\Omega, S, \mathbb{Z})$$

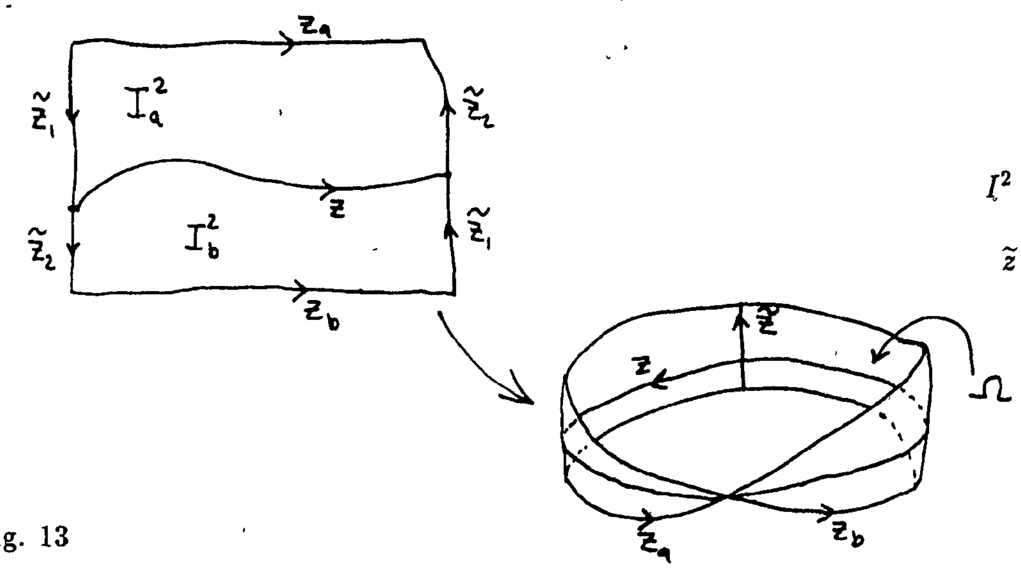
This is an abelian group by construction. By the structure theorem for finitely generated abelian groups (see Jacobson [1974] Theorem 3.13 or Giblin [1981] Theorem A.26 and Corollary A.27), this relative homology group is isomorphic to the direct sum of a free abelian group F on $\beta_p(\Omega, S)$ generators and a torsion group T on $\tau_p(\Omega, S)$ generators, where $\tau_p(\Omega, S)$ is called the pth torsion number of Ω modulo S When homology is computed with coefficients in $\mathbb{R}$ one obtains all the information associated with the free subgroup F and no information about the torsion subgroup T. If turns out that $\tau_p(\Omega, S) = 0$ if Ω is a subset of $\mathbb{R}^3$ and $S = \phi$. In other words, for subsets of $\mathbb{R}^3$ the torsion subgroups of the homology groups

$$H_p(\Omega; \mathbb{Z}) \qquad 0 \leq p \leq 3$$

are trivial (see, for example, Massey [1980] Chapter 9 exercise 6.6 for details). The relationship between $H_p(\Omega, S, \mathbb{R})$ and $H_p(\Omega, S, \mathbb{Z})$ is important since problems in vector analysis are resolved by knowing the structure of $H_p(\Omega, S, \mathbb{R})$ while it is convenient to use integer coefficients in numerical computations and determine $H_p(\Omega, S, \mathbb{Z})$. When $H_p(\Omega, S, \mathbb{Z})$ is found the absolute homology groups with coefficients in $\mathbb{R}$ are easily deduced and relative homology groups with coefficients in $\mathbb{R}$ are deduced by throwing away torsion information. The following example illustrates a relative homology group with non trivial torsion subgroup.

Example 10 (Torsion Phenomena Illustrated)

Consider a Möbius band which is obtained by identifying the sides of a square I^2 as shown in Fig. 13. Let Ω be the Möbius band and $S = z_a + z_b$ be a 1-chain which



$$I^2 = I_a^2 + I_b^2$$

$$\tilde{z} = \tilde{z}_1 + \tilde{z}_2$$

Fig. 13

is the edge of the band. Regarding S as a set, the following homology groups will be deduced:

$$H_1(\Omega;\mathbb{Z}), \qquad H_1(\Omega, S;\mathbb{Z}).$$

It is easy to see that the cosets of $H_1(\Omega;\mathbb{Z})$ look like

$$az + B_1(\Omega;\mathbb{Z}), \qquad a \in \mathbb{Z}$$

so that $\beta_1(\Omega) = 1$. In contrast, something really neat happens when the relative homology group is considered. Observe that $j_1(\tilde{z})$ is not homologous to zero in $H_1(\Omega, S;\mathbb{Z})$, that is $\partial\tilde{z} \in C_0(S;\mathbb{Z})$ but there is no $c \in C_2(\Omega;\mathbb{Z})$ such that $\partial c - \tilde{z} \in C_1(S;\mathbb{Z})$. Note however, that the square I^2 from which Ω was obtained has boundary

$$\partial_2(I^2) = 2\tilde{z} + z_b - z_a$$

hence

$$\partial_2(I^2) - 2\tilde{z} \in \imath_1\left(C_1(S;\mathbb{Z})\right)$$

or

$$j_1(2\tilde{z}) \sim 0 \ in \ H_1(\Omega, S;\mathbb{Z}).$$

Thus $\tilde{z}$ is an element of the torsion subgroup of the relative homology group since it is not homologous to zero, but a multiple of it is. Similarly:

$$j_1(z) \not\sim 0 \qquad in\ H_1(\Omega, S; \mathbb{Z})$$

$$j_1(2z) \sim 0 \qquad in\ H_1(\Omega, S; \mathbb{Z}).$$

The way to see this is to imagine the Mōbius band to be made out of paper which can be cut along the 1-cycle z to yield a surface Ω'. The surface Ω is orientable and

$$\partial_2(\Omega') = 2z - z_a - z_b$$

or

$$2z - \partial_2(\Omega') \in C_1(S; \mathbb{Z}).$$

Hence z and $\tilde{z}$ are nontrivial generators of $H_1(\Omega, S; \mathbb{Z})$. However $z \sim \tilde{z}$ since, referring back to the picture, it is apparent that

$$\partial_2(I_a^2) = \tilde{z}_1 + z + \tilde{z}_2 - z_a = \tilde{z} + z - z_a$$

$$\partial_2(I_b^2) = \tilde{z}_1 - z + \tilde{z}_2 + z_b = \tilde{z} - z + z_b$$

hence

$$\tilde{z} - (-z) - \partial_2(I_a^2) \in C_1(S; \mathbb{Z})$$

$$\tilde{z} - (z) - \partial_2(I_b^2) \in C_1(S; \mathbb{Z})$$

that is

$$j_1(\tilde{z}) \sim j_1(z) \qquad and \qquad j_1(\tilde{z}) \sim -j_1(z)\ in\ H_1(\Omega, S; \mathbb{Z}).$$

Thus, is quite plausible that

$$H_1(\Omega, S; \mathbb{Z}) \simeq \mathbb{Z}/2 \qquad \text{(the integers modulo 2)}$$

and hence in summary

$$\beta_1(\Omega, S) = 0 \qquad \beta_1(\Omega) = 1$$

$$\tau_1(\Omega, S) = 1 \qquad \tau_1(\Omega) = 0.$$

End of Example 10

Before considering the role of relative homology groups in resolving topological problems of vector analysis it is useful to consider *the long exact homology sequence* since it is the key to understanding relative homology.

1.8 The Long Exact Homology Sequence

For the purposes of this thesis the long exact homology sequence is a result which enables one to visualise a full set of generators for the homology of a region Ω modulo a closed subset S in situations where one's intuition can only be trusted with the absolute homology groups of Ω and S. To see how the long exact homology sequence comes about, consider the three complexes:

$$C_*(\Omega) = \{C_p(\Omega), \partial_p\}$$

$$C_*(S) = \{C_p(S), \partial'_p\}$$

$$C_*(\Omega, S) = \{C_p(\Omega, S), \partial''_p\}$$

and the two chain homomorphisms

$$i* = \{i_p\}, \qquad j_* = \{j_p\}$$

$$0 \to C_*(S) \xrightarrow{i_*} C_*(\Omega) \xrightarrow{j_*} C_*(\Omega, S) \to 0$$

where i_p, takes a p-chain on S and sends it to a p-chain which coincides with it on S and vanishes elsewhere on $\Omega - S$ while j_p takes a $c \in C_p(\Omega)$ and sends it into the coset $c + C_p(S)$ in $C_p(\Omega, S)$. It is clear that i_* is injective, j_* is surjective and that Image (i_*) = Kernel (j_*). Such a sequence of three complexes and chain homomorphisms

$\imath$, $\jmath$. is an example of a "short exact sequence of complexes". When written out in full, the situation looks like:

$$
\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{p+1}(S) & \xrightarrow{\imath_{p+1}} & C_{p+1}(\Omega) & \xrightarrow{\jmath_{p+1}} & C_{p+1}(\Omega, S) & \longrightarrow & 0 \\
 & & \downarrow \partial'_{p+1} & & \downarrow \partial_{p+1} & & \downarrow \partial''_{p-1} & & \\
0 & \longrightarrow & C_p(S) & \xrightarrow{\imath_p} & C_p(\Omega) & \xrightarrow{\jmath_p} & C_p(\Omega, S) & \longrightarrow & 0 \\
 & & \downarrow \partial'_p & & \downarrow \partial_p & & \downarrow \partial''_p & & \\
0 & \longrightarrow & C_{p-1}(S) & \xrightarrow{\imath_{p-1}} & C_{p-1}(\Omega) & \xrightarrow{\jmath_{p-1}} & C_{p-1}(\Omega, S) & \longrightarrow & 0 \\
 & & \downarrow \partial'_{p-1} & & \downarrow \partial_{p-1} & & \downarrow \partial''_{p-1} & &
\end{array}
$$

It is a fundamental and purely algebraic result (see Jacobson [1980] Vol. II Sect 6 3 Theorem 6.1) that given such a short exact sequence of complexes, there is a long exact sequence in homology. This means that if $\tilde{\imath}_p$ and $\tilde{\jmath}_p$ are the homomorphisms which $\imath_p$ and $\jmath_p$ induce on homology, and δ_p is a map on homology classes which takes $z \in Z_p(\Omega, S)$ into $z' \in Z_{p-1}(S)$ according to the rule

$$(z' + B_{p-1}(S)) = (\imath_{p-1})^{-1} \partial_p (\jmath_p)^{-1} (z + B_p(\Omega, S))$$

then the diagram

$$
\begin{array}{cccccccc}
0 & \longrightarrow & H_n(S) & \xrightarrow{\tilde{\imath}_n} & \cdots & & & \\
 & & & & & \longrightarrow & H_{p+1}(\Omega, S) & \longrightarrow \\
\xrightarrow{\delta_{p+1}} & & H_p(S) & \xrightarrow{\tilde{\imath}_p} & H_p(\Omega) & \xrightarrow{\tilde{\jmath}_p} & H_p(\Omega, S) & \longrightarrow \\
\xrightarrow{\delta_p} & & H_{p-1}(S) & \xrightarrow{\tilde{\imath}_{p-1}} & H_{p-1}(\Omega) & \xrightarrow{\tilde{\jmath}_{p-1}} & H_{p-1}(\Omega, S) & \xrightarrow{\delta_{p-1}} \\
 & & & & \cdots & \xrightarrow{\tilde{\jmath}_0} & H_0(\Omega, S) & \longrightarrow \; 0
\end{array}
$$

satisfies

$$\text{Kernel}\,(\tilde{\imath}_p) = \text{Image}\,(\delta_{p+1}) \qquad (i)$$

$$\text{Kernel}\,(\tilde{\jmath}_p) = \text{Image}\,(\tilde{\imath}_p) \qquad (ii)$$

$$\text{Kernel}\,(\delta_p) = \text{Image}\,(\tilde{\jmath}_p). \qquad (iii)$$

Although the above result is valid for the coefficient groups $\mathbb{Z}$ and $\mathbb{R}$, consider for now, the case where homology is computed with coefficients in $\mathbb{R}$ so that everything in sight is a vector space. Let

$$H_p(\Omega, S) \simeq \left(\frac{H_p(\Omega, S)}{\text{Kernel}\,(\delta_p)}\right) \oplus \text{Kernel}\ \delta_p$$

and let the two summands be interpreted as follows

$$\begin{aligned} \frac{H_p(\Omega, S)}{\text{Kernel}\,(\delta_p)} &\simeq \delta_p^{-1}\,(\text{Image}\,(\delta_p)) \\ &\simeq \delta_p^{-1}\,(\text{Kernel}\,(\tilde{\imath}_{p-1})) && \textit{by } (i) \end{aligned}$$

and

$$\begin{aligned} \text{Kernel}\,(\delta_p) &= \text{Image}\,(\tilde{\jmath}_p) && \textit{by } (iii) \\ &\simeq \tilde{\jmath}_p\left(\frac{H_p(\Omega)}{\text{Kernel}\,(\tilde{\jmath}_p)}\right) \\ &\simeq \tilde{\jmath}_p\left(\frac{H_p(\Omega)}{\tilde{\imath}_p\,(H_p(S))}\right). && \textit{by } (ii) \end{aligned}$$

Thus, combining the above three isomorphisms gives

$$H_p(\Omega, S) \simeq \delta_p^{-1}\,(\text{Kernel}\,(\tilde{\imath}_{p-1})) \oplus \tilde{\jmath}_p\left(\frac{H_p(\Omega)}{\tilde{\imath}_p\,(H_p(S))}\right).$$

Using the above identity it is usually easy to deduce a set of generators of $H_p(\Omega, S)$ if $H_p(S), H_p(\Omega), H_{p-1}(S)$ and $H_{p-1}(\Omega)$ are known. This is accomplished by using the following three step recipe:

Step 1 Find a basis for the vector space V_p where V_p is defined by the following equation:

$$H_p(\Omega) = (\text{Image}\,(\tilde{\imath}_p)) \oplus V_p$$

Hence, $\tilde{\jmath}_p(V_p)$ gives $(\beta_p(\Omega) - \dim\ \text{Image}\,(\tilde{\imath}_p))$ generators of $H_p(\Omega, S)$.

Step 2 Find a basis for Kernel$(\tilde{\imath}_{p-1})$ from the basis for $H_{p-1}(S)$ so that the

$$\dim \text{Kernel}\,(\tilde{\imath}_{p-1})$$

remaining generators of $H_p(\Omega, S)$ can be deduced from

$$\delta_p^{-1}\,(\text{Kernel}\,(\tilde{\imath}_{p-1}))\,.$$

This is done as follows: Let $\tilde{z}_i$ be a basis for Kernel $(\tilde{\imath}_{p-1})$ and find a set of z_i, $1 \leq i \leq$ dim Kernel $(\tilde{\imath}_{p-1})$ such that

$$z_i = \jmath_p(\partial_p)^{-1}\imath_{p-1}\tilde{z}_i.$$

Step 3 $H_p(\Omega, S) = (\jmath_p^{\cdot} V_p) \oplus \delta_p^{-1}\,(\text{Kernel}\,(\tilde{\imath}_{p-1}))$ where a basis is given in Steps 1 and 2. Furthermore

$$\beta_p(\Omega, S) = \beta_p(\Omega) \;-\dim\; \text{Image}\;(\tilde{\imath}_p) + \dim\; \text{Kernel}\;(\tilde{\imath}_{p-1}).$$

Although this recipe is quite algebraic, it enables one to proceed in a systematic but intuitive way in complicated problems. The folowing example will illustrate this procedure.

Example 11 (Ω is the surface of Example 4)

Recalling the 2-dimensional surface with n "handles" and k "holes" which was considered in Example 4, the following relative homology groups will now be deduced.

i) $H_1(\Omega, \partial\Omega)$

ii) $H_2(\mathbb{R}^3, \Omega)$

i) Consider the long exact homology sequence for the pair $(\Omega, \partial\Omega)$:

$$\begin{array}{ccccccc}
 & 0 & \xrightarrow{\tilde{\imath}_2} & H_2(\Omega) & \xrightarrow{\tilde{\jmath}_2} & H_2(\Omega,\partial\Omega) & \xrightarrow{\delta_2} \\
\xrightarrow{\delta_2} & H_1(\partial\Omega) & \xrightarrow{\tilde{\imath}_1} & H_1(\Omega) & \xrightarrow{\tilde{\jmath}_1} & H_1(\Omega,\partial\Omega) & \xrightarrow{\delta_1} \\
\xrightarrow{\delta_1} & H_0(\partial\Omega) & \xrightarrow{\tilde{\imath}_0} & H_0(\Omega) & \xrightarrow{\tilde{\jmath}_0} & H_0(\Omega,\partial\Omega) & \longrightarrow \;0
\end{array}$$

Following the three step recipe outlined above $H_1(\Omega, \partial\Omega)$ is obtained as follows.

Step 1 Image $(\tilde{\imath}_1)$ and V_1 are readily indentified to be of the form

$$\sum_{j=1}^{k-1} a_{2n+j} z_{2n+j} + B_1(\Omega)$$

and

$$\sum_{j=1}^{2n} a_j z_j + B_1(\Omega)$$

respectively. That is, $j_1(z_{2n+j}), 1 \le j \le k-1$, are homologous to zero in $H_1(\Omega, \partial\Omega)$ while $j_1(z_j), 1 \le j \le 2n$, are not homologous to zero in $H_1(\Omega, \partial\Omega)$.

Step 2 Kernel $(\tilde{\imath}_0)$ is seen to be of the form

$$\sum_{j=1}^{k-1} a_j(p_j - p_k) + B_0(\partial\Omega)$$

while the point p_k can be used to generate $H_0(\Omega)$. Thus the curves $j_1(c_{2n+j}), 1 \le j \le k-1$, which served as cuts in Example 4, can be used as $k-1$ additional generators in $H_1(\Omega, \partial\Omega)$ since

$$\partial c_{2n+j} = p_j - p_k \qquad 1 \le j \le k-1.$$

Step 3 Looking at the definitions of the $c_j, 1 \le j \le 2n+k-1$, it is clear that

$$\sum_{j=1}^{2n} a_j c_j + B_1(\Omega, \partial\Omega) = \text{Image } (\tilde{\jmath}_1)$$

$$\sum_{j=2n+1}^{2n+k-1} a_j c_j + B_1(\Omega, \partial\Omega) = \delta_1^{-1}\left(\text{Kernel } (\tilde{\imath}_0)\right).$$

Thus, the cosets of $H_1(\Omega, \partial\Omega)$ look like:

$$\sum_{j=1}^{2n+k+1} a_j c_j + B_1(\Omega, \partial\Omega)$$

that is to say, the chaing along which cuts were made to obtain a simply connected surface yield a set of generators for $H_1(\Omega, \partial\Omega)$ and

$$\beta_1(\Omega) = 2n - k - 1 = \beta_1(\Omega, \partial\Omega).$$

ii) Consider part of the long exact homology sequence for the pair $(\mathbb{R}^3, \Omega)$

$$\ldots \xrightarrow{\tilde{i}_2} H_2(\mathbb{R}^3) \xrightarrow{\tilde{j}_2} H_2(\mathbb{R}^3, \Omega) \xrightarrow{\delta_2} H_1(\Omega) \xrightarrow{\tilde{i}_1} H_1(\mathbb{R}^3) \xrightarrow{\tilde{j}_1} \ldots$$

Noticing that

$$H_p(\mathbb{R}^3) \simeq \begin{cases} \mathbb{R}, & \text{if } p = 0 \\ 0, & \text{if } p \neq 0 \end{cases}$$

the part of the long exact sequence displayed above reduces to:

$$0 \xrightarrow{\tilde{j}_2} H_2(\mathbb{R}^3, \Omega) \xrightarrow{\delta_2} H_1(\Omega) \xrightarrow{\tilde{i}_1} 0.$$

Hence δ_2 is an isomorphism since the sequence is exact. It is instructive to deduce $H_2(\mathbb{R}^3, \Omega)$ by using the three step recipe outlined above.

Step 1 Can be ignored since $H_2(\mathbb{R}^3) \simeq 0$ implies that Image $(\tilde{j}_2) = 0$.

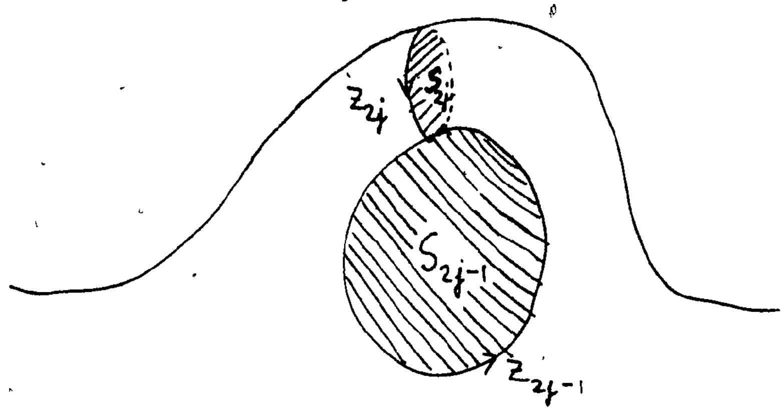
Step 2 Since $H_2(\mathbb{R}^3, \Omega) \simeq H_1(\Omega)$ take generators z_i, $1 \leq i \leq \beta_1(\Omega)$, of $H_1(\Omega)$ and see what $\delta_2^{-1} z_i$ looks like. In other words, relative 2-cycles $j_2(S_i) \in Z_2(\mathbb{R}^3, \Omega)$ must be found such that

$$j_2(S_i) = j_2(\partial_2^{-1}) i_1 z_i \qquad 1 \leq i \leq \beta_1(\Omega)$$

can be used to generate a basis vector of $H_1(\mathbb{R}^3, \Omega)$. By considering the "handles" and "holes" of Ω individually, it is obvious that such a set can be found as seen in Fig. 14 for the jth handle and Fig. 15 for the jth hole.

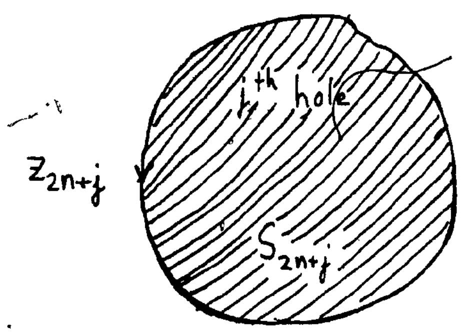
Step 3 There is nothing to do at this stage, the cosets of $H_2(\mathbb{R}^3, \Omega)$ look like

$$\sum_{i=0}^{2n+k-1} a_i S_i + B_2(\mathbb{R}^3, \Omega) \qquad a_i \in \mathbb{R}$$



$1 \leq j \leq n$

Fig. 14



$1 \leq j \leq k-1$

Fig. 15

and

$$\beta_2(\mathbb{R}^3, \Omega) = 2n + k - 1 = \beta_1(\Omega).$$

It is useful to realise that the exact same arguments holds if Ω is twisted up or has several connected components with the exception that the generators of basis vectors of $H_2(\mathbb{R}^3, \Omega)$ may not look like discs.

End of Example 11

Instead of considering more examples of relative homology groups, a heuristic argument will now be considered in order to illustrate the use of relative homology groups in vector analysis.

1.9 Relative Cohomology and Vector Analysis

Given a region Ω, one can form a vector space $C_c^p(\Omega)$ by considering linear combinations of p-cochains (p-forms) which have compact support in Ω. Since the coboundary operator (exterior derivative) applied to a p-cochain of compact support yields a $p+1$-cochain of compact support, one can define a complex

$$C_c^{\cdot}(\Omega) = \{C_c^p(\Omega), d^p\}$$

and by virtue of the fact that one has a complex, cocycle, coboundary, cohomology groups, as well as Betti numbers can be defined as usual:

$$\begin{aligned} Z_c^p(\Omega) &= \text{Kernel}\ \left(C_c^p(\Omega) \xrightarrow{d^p} C_c^{p+1}(\Omega)\right) \\ B_c^p(\Omega) &= \text{Image}\ \left(C_c^{p-1}(\Omega) \xrightarrow{d^{p-1}} C_c^p(\Omega)\right) \\ H_c^p(\Omega) &= Z^c(\Omega)/B_c^p(\Omega) \\ \beta_c^p(\Omega) &= \text{Rank}\ \left(H_c^p(\Omega)\right). \end{aligned}$$

In general, if Ω is a compact region then the cohomology of the complexes

$$C_c^{*}(\Omega), \quad C^{\cdot}(\Omega)$$

is identical. However, if Ω is an open set then the cohomology of the set of complexes will in general be different since the cochains with compact support have restrictions on the boundary of the set.

In order to formulate the idea of relative cohomology, let Ω be a compact region, S a compact subset, and consider the complexes

$$C_c^{\cdot}(\Omega) = \{C_c^p(\Omega), d^p\}$$

$$C_c^{\cdot}(S) = \{C_c^p(S), d^p\}.$$

It is understood that the coboundary operator in the latter complex is the coboundary operator of the first complex except that the domain is restricted. In the heuristic motivation for relative homology groups it was mentioned that, in order to regard integration as a bilinear pairing between $C^p(\Omega, S)$ and $C_p(\Omega, S)$, the definition of $C_p(\Omega, S)$ makes sense if $C^p(\Omega, S)$ is taken to be the set of linear combinations of p-forms whose support lies in $\Omega - S$. Hence in the present case where Ω and S are assumed to be compact, define the set of relative p-cochains to be

$$C_c^p(\Omega - S).$$

Hence there is a cochain complex

$$C_c^{\cdot}(\Omega - S) = \{C_c^p(\Omega - S), d^p\}$$

where it is understood that the coboundary operator is the restriction of the one in $C_c^{\cdot}(\Omega)$. In analogy with the case of homology, consider the following sequence of complexes and cochain homomorphisms,

$$0 \longrightarrow C_c^{\cdot}(\Omega - S) \xrightarrow{e^{\cdot}} C_c(\Omega) \xrightarrow{r^{\cdot}} C_c(S) \longrightarrow 0$$

$$e^{\cdot} = \{e^p\}, \quad r^{\cdot} = \{r^p\}$$

where e^p takes a p-cochain on $\Omega - S$ and it extends it by 0 to the rest of Ω, while r^p takes an p-cochain on Ω and gives its restriction to S. Although this sequence of complexes fails to be exact at $C_c^{\cdot}(\Omega)$, (that is Image $(e^{\cdot}) \neq$ Kernel $(r^{\cdot})$), a careful limiting argument shows that there is still a long exact sequence in cohomology (see Spivak [1979] p 589, Theorems 12.13) Furthermore for there are relative de Rham isomorphisms (see Duff 1952 for the basic constructions).

Instead of trying to develop the idea that the coboundary operator in the complex $C_c^{\cdot}(\Omega - S)$ is adjoint to the boundary operator in the complex $C_{\cdot}(\Omega, S)$, and trying to justify a relative de Rham isomorphism, familiar examples of the relative isomorphism will soon be considered. These examples will serve to solidify the notion of relative homology and cohomology groups and relative de Rham isomorphism so that an intuitive feel can be developed before a more concise formalism is given in the next chapter. When considering relative chains on $C_c^{\cdot}(\Omega - S)$ there are certain boundary conditions which cochains must satisfy when approaching S from within $\Omega - S$. Although these conditions are transparent in the formalism of differential forms, in the next few examples they will be stated often, without proof, since in specific instances they are easily, deduced by using the integral form of Maxwell's Equations.

In the upcoming examples. the relative de Rham isomorphism is understood to mean

$$H_p(\Omega, S) \simeq H_c^p(\Omega - S) \qquad \textit{for all } p.$$

Also, two forms $\omega_1, \omega_2 \in Z_c^p(\Omega - S)$ are said to be cohomologous in the relative sense if

$$\omega_1 - \omega_2 \in B_c^p(\Omega - S).$$

As usual, this forms an equivalence relation where the above is written as

$$\omega_1 \sim \omega_2.$$

The notion of a relative period is defined as follows. If $\omega \in Z_c^p(\Omega - S)$ and $z \in Z_p(\Omega, S)$ then the integral

$$\int_z \omega$$

is called the relative period of ω on z where, by Stokes Theorem, it is easily verified that the period depends only on the cohomology and homology classes of ω and z respectively. Thus the relative de Rham theorem should be interpreted as asserting that integration induces a nondegenerate bilinear pairing

$$\int \quad H_p(\Omega, S) \times H_c^p(\Omega - S) \rightarrow \mathbb{R}$$

where the values of this bilinear pairing can be deduced from evaluating the periods of basis vectors of $H_c^p(\Omega - S)$ on basis vectors of $H_p(\Omega, S)$. In most cases these periods have the interpretation of voltages currents, charges or fluxes

Example 12 (Three-Dimensional Electrostatics)

Consider a compact region Ω which contains no conducting bodies or free charges but whose boundary $\partial\Omega$ may contain a subset which is an interface with a conducting body. Let

$$\partial\Omega = S_1 \cup S_2 \qquad (S_1 \cap S_2 \text{ has no area})$$

where

$$\mathbf{n} \times \mathbf{E} = 0 \qquad \textit{on } S_1$$

$$\mathbf{D} \cdot \mathbf{n} = 0 \qquad \textit{on } S_2.$$

The boundary condition on S_1 is associated with the boundary of a conducting body or certain symmetry plane while the boundary condition on S_2 can be associated with a symmetry plane. Alternatively, the boundary conditions associated with S_1 and S_2 arise on $\partial\Omega$ if one has an interface where there is a sudden change in permittivity as one crosses $\partial\Omega$.

Elements of $C_c^1(\Omega - S_1)$ are associated with vector fields whose components tangent to S_1 vanish, hence the electric field intensity $\mathbf{E}$ can be associated with an element of $Z_c^1(\Omega - S_1)$ since

$$\operatorname{curl} \mathbf{E} = 0 \qquad in\ \Omega$$

$$\mathbf{n} \times \mathbf{E} = 0 \qquad on\ S_1.$$

Dually, elements of $C_c^2(\Omega - S_2)$ can be identified with vector fields whose component normal to S_2 vanishes. Hence the electric field flux density $\mathbf{D}$ can be identified with an element of $Z_c^2(\Omega - S_2)$ since

$$\operatorname{div} \mathbf{D} = 0 \qquad in\ \Omega$$

$$\mathbf{D} \cdot \mathbf{n} = 0 \qquad on\ S_2.$$

By considering a few concrete situations, the reader can easily convince himself that the periods of $\mathbf{E}$ on generators of $H_1(\Omega, S_1)$ are associated with prescribed $E.M.F.$'s while the periods of $\mathbf{D}$ on the generators of $H_2(\Omega, S_2)$ are associated with charges.

It is useful to illustrate how the various spaces associated with the cochain complexes $C_c^{\cdot}(\Omega - S_1)$ and $C_c^{\cdot}(\Omega - S_2)$ arise in variational principles. Assume that there is a tensor constitutive relation

$$\mathbf{D} = \mathcal{D}(\mathbf{E}, \mathbf{r})$$

and an inverse transformation

$$\mathbf{E} = \mathcal{E}(\mathbf{D}, \mathbf{r})$$

such that

$$\mathcal{D}\left(\mathcal{E}(\mathbf{D}, \mathbf{r}), \mathbf{r}\right) = \mathbf{D}.$$

Furthermore, assume that the matrix

$$\frac{\partial \mathcal{D}_i}{\partial E_j}$$

is symmetric positive definite. Consequently

$$\frac{\partial \mathcal{E}_i}{\partial D_k}$$

is symmetric positive definite. In this case the principles of stationary capacitive energy and coenergy can be stated as follows (see MacFarlane 1970 pages 332-333).

Stationary Capacitive Coenergy Principle

$$U'(\mathbf{E}) = \inf_{\mathbf{E}\in Z_c^1(\Omega-S_1)} \int_\Omega \left(\int_0^{\mathbf{E}} \mathcal{D}(\xi,\mathbf{r}) \cdot d\xi \right) dV$$

subject to the constraint that on generators of $H_1(\Omega, S_1)$ periods are prescribed as follows

$$V_i = \int_{c_i} \mathbf{E} \cdot d\mathbf{l} \qquad 1 \le i \le \beta_1(\Omega, S_1).$$

Stationary Capacitive Energy Principle

$$U'(\mathbf{D}) = \inf_{\mathbf{D}\in Z_c^2(\Omega-S_2)} \int_\Omega \left(\int_0^{\mathbf{D}} \mathcal{E}(\xi,\mathbf{r}) \; d\xi \right) dV$$

subject to the constraint that on generators of $H_2(\Omega, S_2)$ periods are prescribed as follows:

$$Q_i = \int_{\Sigma_i} \mathbf{D} \cdot \mathbf{n}\, dS \qquad 1 \le i \le \beta_2(\Omega, S_2).$$

Note that in both the variational principles the extremal is a relative cocycle and when the principal conditions are prescribed on the generators of a (co)homology group the variation of the extremal is constrained to be a relative coboundary. This is readily seen from the identities:

$$Z_c^1(\Omega - S_1) \simeq H_c^1(\Omega - S_1) \oplus B_c^1(\Omega - S_1)$$

$$Z_c^2(\Omega - S_2) \simeq H_c^2(\Omega - S_2) \oplus B_c^2(\Omega - S_2).$$

In the case where the constitutive relations are linear, the coenergy and energy principles can be used to obtain upper bounds on capacitance and elastance lumped parameters respectively. This is achieved by expressing the minimum of the functional as a quadratic form in the prescribed periods and making the identification

$$U'(\mathbf{E}) = \frac{1}{2} \sum_{i,j=1}^{\beta_1(\Omega,S_1)} V_i C_{ij} V_j$$

$$U'(\mathbf{D}) = \frac{1}{2} \sum_{i,j=1}^{\beta_2(\Omega,S_2)} Q_i p_{ij} Q_j .$$

From the upper bound on elastance, a lower bound on capacitance can be obtained in the usual way. The estimation of partial capacitance can be obtained by leaving certain periods free so that their values can be determined as a by product of the minimisation

Not only is the above statement of stationary capacitive energy and coenergy principles succinct but it also gives a direct correspondence with the lumped parameter versions of the same principles. The derivation of the corresponding variational principles in terms of scalar and vector potentials is instructive since insight is gained into why the coenergy principle is naturally formulated in terms of a scalar potential while the formulation of the energy principle in terms of a vector potential requires topological constraints on the model in order for the principle to be valid.† Let the coenergy principle in terms of a scalar potential be considered first.

Considering the long exact homology sequence for the pair (Ω, S_1), one has

$$H_1(\Omega, S_1) = \delta_1^{-1}(\text{Kernel } i_0) \oplus \tilde{j}_1 \left(\frac{H_1(\Omega)}{\tilde{i}_1\,(H_1(S_1))} \right)$$

† The author could not resist including a discussion of this point since it clearly shows the necessity of using homology groups.

where the relevant portion of the long exact homology sequence is

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{\partial_2} & H_1(S_1) & \xrightarrow{\tilde{\imath}_1} & H_1(\Omega) & \xrightarrow{\tilde{\jmath}_1} & H_1(\Omega, S_1) & \longrightarrow & \\ & \xrightarrow{\partial_1} & H_0(S_1) & \xrightarrow{\tilde{\imath}_0} & H_0(\Omega) & \longrightarrow & & & \ldots \end{array}$$

Let $c_i, 1 \leq i \leq \beta_1(\Omega, S)$ be a set of curves which are associated with the generators of $H_1(\Omega, S_1)$. These curves can be arranged into two groups according to the three step recipe.

Group 1 There are $\dim \operatorname{Image}(\jmath_1)$ generators of $H_1(\Omega, S_1)$ which are homologous in the absolute sense to generators of $H_1(\Omega)$. These generators can be associated with closed curves $c_i, 1 \leq i \leq \dim \operatorname{Image}(\tilde{\imath}_1)$. In this case, the period

$$\int_{c_i} \mathbf{E} \cdot d\mathbf{l} = V_i$$

is equal to the rate of change of magnetic flux which links c_i. Thus, although there is a static problem in Ω. the above periods are associated with magnetic circuits in $\mathbb{R}^3 - \Omega$ (it is usually wise to set these periods equal to zero if one has the chance).

Group 2 There are $\dim \operatorname{Kernel}(\tilde{\imath}_0)$ remaining generators of $H_1(\Omega, S_1)$ which can be associated with simple open curves whose end points lie in distinct points of S_1. In other words, if

$$c_{(\dim \operatorname{Image}(\tilde{\jmath}_1)+i)} \qquad 1 \leq i \leq \dim \operatorname{Kernel}(\tilde{\imath}_0)$$

are such a set of curves, then they can be defined (assuming Ω is connected) so that

$$\partial c_{(\dim \operatorname{Image}(\tilde{\jmath})+i)} = p_i - p_0$$

where p_0 is a datum node lying in some connected component of S_1 and each p_i lies in some distinct connected component of S_1. That is, there is one p_i in each connected component of S_1. In this case, the period

$$\int_{c_i} \mathbf{E} \cdot d\mathbf{l} = V_i$$

is associated with potential differences between connected components of S_1.

Now suppose that one can ignore the "magnetic circuits" in $\mathbb{R}^3 - \Omega$ so that the periods of $\mathbf{E}$ on generators of group 1 vanish. In this case it is seen (from the long exact sequence) that the period of $\mathbf{E}$ vanishes on all generators of $H_1(\Omega)$ since the tangential components of $\mathbf{E}$ vanish on S_1. Hence, $\mathbf{E}$ may be expressed as the gradient of a single valued scalar ϕ. Furthermore, the scalar is a constant on each connected component of S_1. That is

$$\mathbf{E} = \operatorname{grad}\phi \qquad \text{in } \Omega$$

$$\phi = \phi(p_i) \qquad \text{on the } i\text{th component of } S_1.$$

When $\mathbf{E}$ is expressed in this form, the periods of $\mathbf{E}$ on the generators of $H_1(\Omega, S_1)$ which lie in group 2 are easy to calculate:

$$V_{\left(\dim \operatorname{Image}(\tilde{j}_1)+i\right)} = \int_{c_{\left(\dim \operatorname{Image}(\tilde{j}_1)+i\right)}} \operatorname{grad}\phi \cdot d\mathbf{l} = \phi(p_i) - \phi(p_0)$$

since

$$\partial c_{\left(\dim \operatorname{Image}(\tilde{j}_1)+1\right)} = p_i - p_0.$$

In this way the coenergy principle can be restated as follows:

Stationary Capacitive Coenergy Principle ($E = grad\phi$)

$$U(\operatorname{grad}\phi) = \inf_{\operatorname{grad}\phi \in Z_c^1(\Omega - S_1)} \int_\Omega \left(\int_0^\phi \mathcal{D}(\operatorname{grad}\eta, \mathbf{r}) \cdot \operatorname{grad}(d\eta) \right) dV$$

subject to the constraints:

$$V_{\left(\dim \operatorname{Image}(\tilde{j}_1)+i\right)} = \phi(p_i) - \phi(p_0), \qquad 1 \le i \le \dim \operatorname{Kernel}(\tilde{i}_0).$$

Note that for this functional the space of admissible variations is still $B_c^1(\Omega - S_1)$, that is, ϕ can be varied by any scalar which vanishes on S_1.

Although the coenergy principle seems to be more natural when expressed in terms of a scalar potential, the only difference between the two principles is how they treat magnetic circuits in $\mathbb{R}^3 - \Omega$. The situation is quite different when one tries to express the energy principle in terms of a vector potential since, in general, the energy principle cannot be reformulated in terms of the vector potential alone. Considering the long exact homology sequence associated with the pair (Ω_1, S_2) one has

$$H_2(\Omega, S_2) = \delta_2^{-1}\left(\text{Kernel}\,(\tilde{\imath}_1)\right) \oplus \tilde{\jmath}_2 \left(\frac{H_2(\Omega)}{\tilde{\imath}_2\left(H_2(S_2)\right)}\right)$$

where the relevant portion of the long exact homology sequence is:

$$\begin{array}{ccccccc} \ldots \xrightarrow{\delta_3} & H_2(S_2) & \xrightarrow{\tilde{\imath}_2} & H_2(\Omega) & \xrightarrow{\tilde{\jmath}_2} & H_2(\Omega, S_2) & \longrightarrow \\ \xrightarrow{\delta_2} & H_1(S_2) & \xrightarrow{\tilde{\imath}_1} & H_1(\Omega) & \xrightarrow{\tilde{\jmath}_1} & \ldots & \end{array}$$

Let $\Sigma_i, 1 \leq i \leq \beta_2(\Omega, S_2)$ be a set of surfaces which are associated with the generators of $H_2(\Omega, S_2)$. These surfaces can be arranged into two groups according to the three step recipe:

Group 1 There are $\dim \text{Image}\,(\tilde{\jmath}_2)$ generators of $H_2(\Omega, S_2)$ which are homologous to generators of $H_2(\Omega)$. These generators can be associated with closed surfaces, that is $\Sigma_i, 1 \leq i \leq \dim \text{Image}\,(\tilde{\jmath}_2)$, can be associated with this group and so $\partial\Sigma_i = 0$ for each of these surfaces.

Group 2 There are $\dim \text{Kernel}\,(\tilde{\imath}_1)$ remaining generators of $H_2(\Omega, S_2)$ which can be identified with open surfaces

$$\Sigma_{\dim \text{Image}\,(\tilde{\jmath}_2)+i} \qquad 1 \leq i \leq \dim \text{Kernel}\,(\tilde{\imath}_1)$$

whose boundaries form, in $H_1(S_2)$ a basis for $\text{Kernel}\,(\tilde{\imath}_1)$.

If there are any generators of $H_2(\Omega, S_2)$ which belong to group 1 then the energy principle cannot be reformulated in terms of a vector potential because it is not possible to prescribe a nonzero period on generators of $H_2(\Omega, S_2)$ which have no boundary. In other words, if

$$\mathbf{D} = \operatorname{curl} \mathbf{C} \qquad in\ \Omega$$

then

$$Q_i = \int_{\Sigma_i} \mathbf{D} \cdot \mathbf{n}\, dS = \int_{\Sigma_i} \operatorname{curl} \mathbf{C} \cdot \mathbf{n}\, dS$$
$$= \int_{\partial\Sigma_i} \mathbf{C} \cdot d\mathbf{l} = 0$$

since

$$\partial\Sigma_i = 0 \qquad 1 \leq i \leq \dim \operatorname{Image}(j_2).$$

This reasoning is straight out of Example 5. Recalling the counter example in Example 5 it is apparent that it is not possible to estimate the capacitance of two concentric conducting ellipsoids if there is no symmetry to be exploited. However if a model of a problem can be constructed (by exploiting symmetries etc.) in which there are no nonzero periods of $\mathbf{D}$ on generators in group 1 then $\mathbf{D}$ can be expressed as the curl of a vector potential C and the periods of $\mathbf{D}$ can be prescribed on the generators of group 2 by prescribing

$$Q_j = \int_{\Sigma_j} \mathbf{D} \cdot \mathbf{n}\, dS = \int_{\partial\sigma_j} \mathbf{C} \cdot d\mathbf{l}$$
$$\dim \operatorname{Image}(\tilde{j}_2) - 1 \leq j \leq \beta_2(\Omega, S_2).$$

In this way, under the assumption that the periods of $\mathbf{D}$ vanish on all generators of $H_2(\Omega)$, the stationary capacitive energy principle can be stated in terms of a vector potential as follows:

Stationary Capacitive Energy Principle ($\mathbf{D} = \operatorname{curl} \mathbf{C}$)

$$U'(\operatorname{curl} \mathbf{C}) = \inf_{\operatorname{curl} \mathbf{C} \in Z_c^2(\Omega - S_2)} \int_\Omega \left(\int_0^{\mathbf{C}} \mathcal{E}\,(\operatorname{curl} \xi, \mathbf{r}) \cdot \operatorname{curl}(d\xi) \right) dV$$

subject to the constraint on generators of group 2:

$$Q_j^{\bullet} = \int_{\partial\Sigma_j} \mathbf{C} \cdot d\mathbf{l}, \qquad \dim \operatorname{Image}(\tilde{\jmath}_2) + 1 \leq j \leq \beta_2(\Omega, S_2).$$

Note that for this functional the space of admissible variations is still $B_c^2(\Omega - S_2)$, that is, the extremal $\mathbf{D}$ can be varied by the curl of any vector field, whose components tangent to S_2 vanish.

It is worthwhile considering how the tangential components of the vector potential are prescribed on S_2 in order to ensure that $\mathbf{D} \cdot \mathbf{n} = 0$ and the integral constraints can be satisfied. First, it is important to realise that one cannot ensure that

$$\operatorname{curl} \mathbf{C} \cdot \mathbf{n} = 0 \qquad on\ S_2$$

by imposing

$$\mathbf{n} \times \mathbf{C} = 0 \qquad on\ S_2$$

because this would imply

$$Q_i = \int_{\partial\Sigma_i} \mathbf{C} \cdot d\mathbf{l} = 0 \qquad since\ \partial\Sigma_i \subset S_2.$$

Instead, one has to let:

$$\mathbf{n} \times \mathbf{C} = \mathbf{n} \times \operatorname{grad} \psi$$

$$= \overline{\operatorname{curl}}\, \psi \qquad on\ S_2$$

where ψ is a function of the coordinates on S_2 which is not necessarily single valued and continuous. A single valued continuous function would set all the Q_i equal to zero. In order to get a better understanding of how the scalar function ψ is to be chosen, one can use the reasoning developed in the first part of Example 11.

Let $d_j, 1 \leq j \leq \beta_1(S_2, \partial S_2)$, be a set of generators for $H_1(S_2, \partial S_2)$ so that

$$H_1\left(S_2 - \bigcup_{j=1}^{\beta_1(S_2,\partial S_2)} d_j\right) \simeq 0.$$

In this way the scalar ψ can be made single valued on

$$S_2 - \bigcup_{j=1}^{\beta_1(S_2,\partial S_2)} d_j .$$

Hence, it remains to find a way of prescribing the periods Q_i in terms of the jumps in ψ as the "cuts" d_j are traversed. By considering the following portion of the long exact homology sequence for the pair (Ω, S_2):

$$\ldots \longrightarrow H_2(\Omega) \xrightarrow{\widetilde{j}_2} H_2(\Omega, S_2) \xrightarrow{\delta_2} H_1(S_2) \xrightarrow{\widetilde{i}_1} H_1(\Omega) \xrightarrow{\widetilde{j}_1} H_1(\Omega, S_2) \longrightarrow \ldots$$

the exact same argument which led to the famous three step recipe can be repeated in the context of $H_1(S_2)$ to yield

$$H_1(S_2) \simeq (\widetilde{i}_1)^{-1}\,(\text{Kernel}\,(\widetilde{j}_1)) \oplus \delta_2 \left(\frac{H_2(\Omega, S_2)}{\widetilde{j}_2\,(H_2(\Omega))} \right).$$

Thus the generators of $H_2(S_2)$ can be arranged into two groups to which cycles $z_i \in Z_1(S_2)$ can be associated as follows:

Group 1

$$\partial\Sigma_{\dim \text{Image}\,(\widetilde{j}_2)+i} = z_i \qquad 1 \le i \le \dim \text{Image}\,\delta_2$$

-this takes care of the part of $H_1(S_2)$ which is associated with Image (δ_2).

Group 2 There are $\dim \text{Kernel}\,(\widetilde{j}_1)$ remaining generators of $H_1(S_2)$

$$z_{\dim \text{Image}\,(\delta_2)+i} \qquad 1 \le i \le \dim \text{Kernel}\,(\widetilde{j}_1)$$

which are homologous to closed curves associated with generators of $H_1(\Omega)$. Now in order to prescribe the periods of the vector potential one merely has to find a way of prescribing the periods of ψ on the

$$z_i, \qquad 1 \le i \le \dim \text{Image}\,(\delta_2)$$

by specifying the jumps in ψ across the d_j. Let

$$[\psi]_{d_j}$$

denote the jump in ψ as d_j is traversed in a given direction, and m_{ij} is the number of oriented intersections of z_i with d_j. In this case

$$\begin{aligned} Q_{\dim \operatorname{Image}(\widetilde{j}_2)+i} &= \int_{\partial \Sigma_{\dim \operatorname{Image}(\widetilde{j}_2)+i}} \mathbf{C} \cdot d\mathbf{l} \\ &= \int_{z_i} \mathbf{C} \cdot d\mathbf{l} = \int_{z_i} \operatorname{grad} \psi \cdot d\mathbf{l} \\ &= \sum_{j=1}^{\beta_1(S_2, \partial S_2)} m_{ij} [\psi]_{d_j} \end{aligned}$$

for $1 \leq i \leq \dim \operatorname{Image}(\delta_2)$. This is a set of equations which expresses some of the periods of grad ψ in terms of the $\{[\psi]_{d_j}\}$. One can also choose arbitrarily constants

$$a_i, \qquad \dim \operatorname{Image}(\delta_2) + 1 \leq i \leq \beta_1(S_2)$$

and form the equations

$$a_i = \sum_{j=1}^{\beta_1(S_2, \partial S_2)} m_{ij} [\psi]_{d_j}$$

where

$$\dim \operatorname{Image}(\delta_2) + 1 \leq i \leq \beta_1(S_2).$$

The above two sets of equations, taken together form a set of $\beta_1(S_2)$ equations in $\beta_1(S_2, \partial S_2)$ unknowns. The Lefschetz duality theorem (see Greenberg [1980] page 242) states that

$$\beta_1(S_2) = \beta_c^1(S_2 - \partial S_2)$$

so that

$$\beta_1(S_2, \partial S_2) = \beta_1(S_2).$$

This result also follows from Example 11(i). Hence, the matrix whose entries are m_{ij} is square and from the Lefschetz Duality Theorem it can be shown that it is nonsingular. If M_{ij} are the entries of the matrix inverse to the matrix with entries m_{ij} then

$$[\psi]_{d_k} = \sum_{i=1}^{\dim \operatorname{Image}(\delta_2)} M_{ki} Q_{\dim \operatorname{Image}(\tilde{j}_2)+i} + \sum_{i=\dim \operatorname{Image}(\delta_2)+1}^{\beta_1(S_2)} M_{ki} a_i .$$

Thus the periods of the vector potential can be specified indirectly by the jumps $[\psi]_{d_k}$. Hence the stationary capacitive energy principle can be stated as follows.

Stationary Capacitive Energy Principle ($\mathbf{D} = \operatorname{curl} \mathbf{C}$)

If the representation $D = \operatorname{curl} \mathbf{C}$ is valid then

$$U'(\operatorname{curl} \mathbf{C}) = \inf_{\operatorname{curl} \mathbf{C} \in Z_c^2(\Omega - S_2)} \int_\Omega \left(\int_0^{\mathbf{C}} \mathcal{E}(\operatorname{curl} \xi, \mathbf{r}) \cdot \operatorname{curl} d\xi \right) dV$$

subject to the condition:

$$\mathbf{n} \times \mathbf{C} = \overline{\operatorname{curl} \psi} \qquad on\ S_2$$

where $[\psi]_{d_j}$ are prescribed on generators of $H_1(S_2, \partial S_2)$ but ψ is otherwise arbitrary.

Several remarks are in order. Firstly, the method of prescribing the tangential components of the vector potential in terms of a cocycle in S_2 was inspired by the paper of Milani and Negro [1982]. The approach given here differs from theirs in that S_2 need not be $\partial\Omega$ and ψ need not be a harmonic function. Secondly, when the energy principle is stated in terms of a vector potential, the solution is nonunique. The nonuniqueness consists of an element of $Z_c^1(\Omega - S_2)$, that is, an irrotational vector field whose components tangent to S_2 vanish. Since

$$Z_c^1(\Omega - S_2) = H_c^1(\Omega - S_2) \oplus B_c^1(\Omega - S_2)$$

the nonuniqueness of the vector potential can be eliminated by specifying the periods of C on generators of $H_1(\Omega, S_2)$ and constraining the part of the vector potential associated with $B_c^1(\Omega - S_2)$. The elements of $B_c^1(\Omega - S_2)$ look like

$$\operatorname{grad} \chi \qquad \text{in } \Omega$$

where

$$\chi = 0 \qquad \text{on } S_2$$

hence if

$$\begin{aligned} &\operatorname{div} \mathbf{C} && \text{specified in } \Omega \\ &\mathbf{C} \cdot \mathbf{n} && \text{specified on } S_1 \end{aligned}$$

then

$$\begin{aligned} \operatorname{div} \operatorname{grad} \chi & \qquad \text{specified in } \Omega \\ \chi = 0 & \qquad \text{on } S_2 \\ \frac{\partial \chi}{\partial n} & \qquad \text{specified on } S_1. \end{aligned}$$

In this case, the gradient of χ is uniquely defined, and the nonuniqueness of C associated with $B_c^1(\Omega - S_2)$ has been eliminated by specifying the divergence of $\mathbf{C}$ and its normal component on S_1.

In summary, the problem of electrostatics involves a region Ω where

$$\partial\Omega = S_1 \cup S_2, \qquad (S_1 \cap S_2 \text{ has no area})$$

$$\begin{aligned} \mathbf{n} \times \mathbf{E} = 0 & \qquad \text{on } S_1 \\ \mathbf{D} \cdot \mathbf{n} = 0 & \qquad \text{on } S_2. \end{aligned}$$

The electric field **E** is associated with an element of $Z_c^1(\Omega - S_1)$ while the electric flux density **D** is associated with an element of $Z_c^2(\Omega - S_2)$. The nondegenerate bilinear pairings which integration induces on homology and cohomology classes

$$\int : H_1(\Omega, S_1) \times H_c^1(\Omega - S_1) \rightarrow \mathbb{R}$$

$$\int : H_2(\Omega, S_2) \times H_c^2(\Omega - S_2) \rightarrow \mathbb{R}$$

are associated with potential differences and charges respectively. The fact that there are just as many independent potential differences as there are independent charges seems to indicate that

$$\beta_1(\Omega, S_1) = \beta_2(\Omega, S_2)$$

which will be shown to be true later. For variational principles where the electric field **E** is the independent variable and potential differences are prescribed, the variation of the extremal lies in the space $B_c^1(\Omega - S_1)$. Dually, for variational principles where the electric flux density **D** is the independent variable and charges are prescribed, the variation of the extremal takes place in the space $B_c^2(\Omega - S_2)$.

The long exact homology sequence is useful for showing the appropriateness of the variational principles involving scalar potentials and the limited usefulness of variational principles involving an electric vector potential. When the electric vector potential is used, the long exact homology sequence indicates how to prescribe the tangential components of the vector potential in terms of a scalar function defined on S_2. Finally, the vector potential is unique up to an element of $Z_c^1(\Omega - S_2)$ when its tangential components are prescribed on S_2.

Example 13 (3-D Magnetostatics)

Consider a connected compact three dimensional region Ω which contains no infinitely permeable or superconducting material, but whose boundary may contain an interface with an infinitely permeable or superconducting body. Let

$$\partial\Omega = S_1 \cup S_2 \qquad (S_1 \cap S_2 \text{ has no area})$$

where

$$\mathbf{B} \cdot \mathbf{n} = 0 \,, \quad on\ S_1$$

$$\mathbf{n} \times \mathbf{H} = 0 \quad on\ S_2.$$

The boundary condition on S_1 is associated with the boundary of a superconductor, a symmetry plane or, alternatively, when Ω contains a very permeable body part of whose boundary coincides with $\partial\Omega$ and it is known that no flux can escape through that part. This latter situation occurs if Ω is an ideal magnetic circuit. The boundary condition on S_2 is associated with boundaries of infinitely permeable bodies or symmetry planes.

Assume that no free currents flow in Ω. Since elements of $C_c^2(\Omega - S_1)$ can be identified with vector fields whose component normal to S_1 vanishes, the magnetic flux density $\mathbf{B}$ can be associated with an element of $Z_c^2(\Omega - S_1)$ because

$$\operatorname{div} \mathbf{B} = 0 \quad in\ \Omega$$

$$\mathbf{B} \cdot \mathbf{n} = 0 \quad on\ S_1.$$

Similarly, elements of $C_c^1(\Omega - S_2)$ can be associated with vector fields whose components tangent to S_2 vanish. Since it is assumed that no free currents can flow, the magnetic field intensity $\mathbf{H}$ can be associated with an element of $Z_c^1(\Omega - S_2)$ because

$$\operatorname{curl} \mathbf{H} = 0 \quad in\ \Omega$$

$$\mathbf{n} \times \mathbf{H} = 0 \quad on\ S_2.$$

As in the previous example lumped variables are associated with the generators of relative homology groups. That is, if

$$\Sigma_i, \qquad 1 \le i \le \beta_2(\Omega, S_1)$$

is a set of surfaces associated with a basis of $H_2(\Omega, S_1)$ and

$$c_i, \qquad 1 \le i \le \beta_1(\Omega, S_2)$$

is a set of curves associated with a basis of $H_1(\Omega, S_2)$ then the periods of $\mathbf{B}$ on the Σ_i

$$\Phi_i = \int_{\Sigma_i} \mathbf{B} \cdot \mathbf{n}\, dS, \qquad 1 \le i \le \beta_2(\Omega, S_1)$$

are associated with fluxes while the periods of $\mathbf{H}$ on the c_i

$$I_i = \int_{c_i} \mathbf{H} \cdot d\mathbf{l}, \qquad 1 \le i \le \beta_1(\Omega, S_2)$$

are associated with currents in $\mathbb{R}^3 - \Omega$.

In order to illustrate the role of cochain complexes in the statement of variational principles, a constitutive relation must be introduced. Let

$$\mathbf{H} = \mathcal{H}(\mathbf{B}, \mathbf{r})$$

be a a tensor constitutive relation and let

$$\mathbf{B} = \mathcal{B}(\mathbf{H}, \mathbf{r})$$

be the inverse transformation which satisfies

$$\mathcal{B}\left(\mathcal{H}(\mathbf{B}, \mathbf{r}), \mathbf{r}\right) = \mathbf{B}.$$

Furthermore, assume that the matrix with entries

$$\frac{\partial \mathcal{H}_i}{\partial B_j}$$

is symmetric positive definite. Consequently,

$$\frac{\partial \mathcal{B}_i}{\partial H_j}$$

is symmetric positive definite In this case the principles of stationary inductive coenergy and energy can be stated as follows (see Mac Farlane [1970] pages 330-332).

Stationary Inductive Coenergy Principle

$$T'(\mathrm{H}) = \inf_{\mathbf{H}\in Z_c^1(\Omega - S_2)} \int_\Omega \int_0^{\mathbf{H}} \mathcal{B}(\xi, \mathbf{r}) \cdot d\xi dV$$

subject to the constraints which prescribe periods of $\mathbf{H}$ on generators of $H_1(\Omega, S_2)$:

$$I_i = \int_{c_i} \mathbf{H} \cdot d\mathbf{l}, \qquad 1 \leq i \leq \beta_1(\Omega, S_2).$$

Stationary Inductive Energy Principle

$$T(\mathbf{B}) = \inf_{\mathbf{B}\in Z_c^2(\Omega - S_1)} \int_\Omega \int_0^{\mathbf{B}} \mathcal{H}(\xi, \mathbf{r}) \cdot d\xi dV$$

subject to the constraints which prescribe periods of $\mathbf{B}$ on generators of $H_2(\Omega, S_1)$

$$\Phi_j = \int_{\Sigma_i} \mathbf{B} \cdot \mathbf{n}\, dS \qquad 1 \leq i \leq \beta_2(\Omega, S_1).$$

As in the previous example, in both variational principles the extremal is constrained to be a relative cocycle and when principal conditions are prescribed on the generators of a (co)homology group the variation of the extremal is constrained to be a relative coboundary. This is readily seen from the identities

$$Z_c^1(\Omega - S_2) \simeq H_c^1(\Omega - S_2) \oplus B_c^1(\Omega - S_2)$$
$$Z_c^2(\Omega - S_1) \simeq H_c^2(\Omega - S_1) \oplus B_c^1(\Omega - S_1)$$

and the fact that the following relative de Rham isomorphisms have been assumed

$$H_c^1(\Omega - S_2) \simeq H_1(\Omega, S_2)$$
$$H_c^2(\Omega - S_1) \simeq H_2(\Omega, S_1).$$

For linear constitutive relations the coenergy principle gives an upper bound for inductance while the energy principle gives an upper bound for inverse inductance. This is done by expressing the minimum of the functional as a quadratic form in the prescribed periods and making the identification

$$T'(\mathbf{H}) = \sum_{i,j=1}^{\beta_1(\Omega,S_2)} I_i L_{ij} I_i$$

$$T(\mathbf{B}) = \sum_{i,j=1}^{\beta_2(\Omega,S_1)} \Phi_i \Gamma_{ij} \Phi_j .$$

From the upper bound on inverse inductance a lower bound on inductance can be found in the usual way. The estimation of partial inductances can be obtained by leaving some of the periods free in a given variational principle so that their values can be determined by the minimisation.

These variational principles are interesting since they provide a direct link with lumped parameters and show how the various subspaces of the complexes $C_c^{\cdot}(\Omega - S_1)$ and $C_c^{\cdot}(\Omega - S_2)$ play a role. As in the case of electrostatics it is useful to futher investigate the relative (co)homology groups of concern in order to see how the above variational principles can be rephrased in terms of vector and scalar potentials and to know the topological restrictions which may arise.

Considering the long exact homology sequence for the pair (Ω, S_1) one has

$$H_2(\Omega, S_1) = \delta_2^{-1}\left(\text{Kernel}\,(\tilde{\imath}_1)\right) \oplus \tilde{\jmath}_2 \left(\frac{H_2(\Omega)}{\tilde{\imath}_2\,(H_2(S_1))} \right)$$

where the relevant portion of the long exact homology sequence is

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{\delta_3} & H_2(S_1) & \xrightarrow{\tilde{\imath}_2} & H_2(\Omega) & \xrightarrow{\tilde{\jmath}_2} & H_2(\Omega, S_1) & \longrightarrow & \\ & \xrightarrow{\delta_2} & H_1(S_1) & \xrightarrow{\tilde{\imath}_1} & H_1(\Omega) & \xrightarrow{\tilde{\jmath}_1} & & \ldots & \end{array}$$

Thus, using the three step recipe. one can arrange the $\Sigma_i, 1 \leq i \leq \beta_2(\Omega, S_1)$ into two groups.

Group 1 There are dim Image $(\tilde{j}_2)$ generators of $H_2(\Omega, S_1)$ which are homologous in the absolute sense to generators of $H_2(\Omega)$. Thus let

$$\partial\Sigma_i = 0, \qquad 1 \leq i \leq \dim \mathrm{Image}\,(\tilde{j}_2)$$

and associated to these Σ_i is a basis of Image$(\tilde{j}_2)$ in $H_2(\Omega, S_1)$.

Group 2 There are dim Kernel $(\tilde{i}_1)$ remaining generators of $H_2(\Omega, S_1)$ whose image under δ_2 form in $H_1(S_1)$ a basis for Kernel$(\tilde{i}_1)$. Hence let

$$\partial\Sigma_{\dim \mathrm{Image}\,(\tilde{j}_2)+i} = z_i \qquad 1 \leq i \leq \dim \mathrm{Kernel}\,(\tilde{i}_1)$$

where the z_i are associated with Kernel$(\tilde{i}_1)$.

Considering the periods of $\mathbf{B}$ on the generators of $H_2(\Omega, S_1)$ which are in the first group, it is clear that if

$$\mathbf{B} = \mathrm{curl}\,\mathbf{A} \qquad in\ \Omega$$

then the periods must vanish because

$$\Phi_i = \int_{\Sigma_i} \mathbf{B} \cdot \mathbf{n} dS = \int_{\partial\Sigma_i} \mathbf{A} \cdot d\mathbf{l} = 0$$

since

$$\partial\Sigma_i = 0, \qquad 1 \leq i \leq \dim \mathrm{Image}\,(\tilde{j}_2).$$

Though this is a restriction, it is still natural to formulate the problem in terms of a vector potential since the nonzero periods of $\mathbf{B}$ on the generators of group 1 can only be associated with distributions of magnetic monopoles in $\mathbb{R}^3 - \Omega$. Assuming then that the periods of $\mathbf{B}$ on the generators of group 1 vanish, and $\mathbf{B}$ is related to a vector potential

$\mathbf{A}$, the periods of $\mathbf{B}$ on the generators of $H_2(\Omega, S_1)$ which lie in group 2, can easily be expressed in terms of the vector potential as follows.

$$\begin{aligned}\Phi_{\dim \text{Image}(\tilde{j}_2)+i} &= \int_{\Sigma_{\dim \text{Image}(\tilde{j}_2)+1}} \mathbf{B} \cdot \mathbf{n}\, dS \\ &= \int_{\partial\Sigma_{\dim \text{Image}(\tilde{j}_2)+i}} \mathbf{A} \cdot d\mathbf{l} \\ &= \int_{z_i} \mathbf{A} \cdot d\mathbf{l}\end{aligned}$$

since

$$\partial\Sigma_{\dim \text{Image}(\tilde{j}_2)+i} = z_i, \qquad 1 \leq i \leq \dim \text{Kernel}(\tilde{i}_1).$$

Next, it is worthwhile considering how the tangential components of $\mathbf{A}$ are to be prescribed on S_1 so that $\mathbf{B} \cdot \mathbf{n} = 0$ and the above periods can be prescribed. One cannot impose

$$\text{curl}\, \mathbf{A} \cdot \mathbf{n} = 0 \qquad on\ S_1$$

by forcing

$$\mathbf{n} \times \mathbf{A} = 0 \qquad on\ S_1$$

because this would imply

$$\Phi_{\dim \text{Image}(\tilde{j}_2)+i} = \int_{z_i} \mathbf{A} \cdot d\mathbf{l} = 0$$

since

$$z_i \in C_c^1(S_1).$$

Instead, as in the analogous case of electrostatics, one has to let

$$\begin{aligned}\mathbf{n} \times \mathbf{A} &= \mathbf{n} \times \text{grad}\, \psi \\ &\doteq \overline{\text{curl}}\, \psi \qquad on\ S_1\end{aligned}$$

where ψ is a multivalued function of the coordinates on S_1. Following the reasoning in Example 4, this function can be made single valued on

$$S_1 - \bigcup_{j=i}^{\beta_1(S_1,\partial S_1)} d_i$$

where the d_i are a set of curves associated with the generators of $H_1,(S_1,\partial S_1)$ and the periods of the multivalued function ψ are given by specifying the jumps, denoted by

$$\psi|_{d_j}, \qquad 1 \le j \le \beta_1(S_1,\partial S_1)$$

of ψ on the d_j. To see how this is done, consider the following portion of the long exact homology sequence for the pair (Ω, S_1)

$$\begin{array}{cccccccc} & \ldots & \longrightarrow & H_2(\Omega) & \xrightarrow{\tilde{j}_2} & H_2(\Omega,S_1) & \rightarrow \\ \xrightarrow{\delta_2} & H_1(S_1) & \xrightarrow{\tilde{i}_1} & H_1(\Omega) & \xrightarrow{\tilde{j}_1} & H_1(\Omega,S_1) & \xrightarrow{\delta_1} \end{array}$$

and using the same reasoning as in the "three step recipe" one has

$$H_1(S_1) \simeq (\tilde{i}_1)^{-1}(\text{Kernel}\,(\tilde{j}_1)) \oplus \delta_2\left(\frac{H_2(\Omega,S_1)}{\tilde{j}_2(H_2(\Omega))}\right).$$

Thus the generators of $H_2(S_1)$ can be arranged into two groups where one can choose $\beta_1(S_2)$ curves z_i and associate

$$z_i, \qquad 1 \le i \le \dim \text{Image}\,\delta_2$$

with boundaries of generators of $H_2(\Omega, S_1)$ and

$$z_{\dim \text{Image}\,(\delta_2)+i}, \qquad 1 \le i \le \dim \text{Kernel}\,(\tilde{j}_1)$$

which are homologous in $H_1(\Omega)$ to a set of generators of Image $(\tilde{i}_1)$.

In analogy with the previous example one can define a matrix whose elements m_{ij} count the number of oriented intersections of z_i with d_j, so that if p_j is the period of $\text{grad}\,\psi$ on z_j then

$$p_i = \int_{z_i} \mathbf{A} \cdot d\mathbf{l} = \int_{z_i} \text{grad}\,\psi \cdot d\mathbf{l}$$
$$= \sum_{j=1}^{\beta_1(S_1,\partial S_1)} m_{ij}[\psi]_{d_j}, \qquad 1 \le i \le \beta_1(S_1)$$

where

$$p_i = \Phi_{\dim \text{Image}\,(\tilde{\jmath}_2)+i}, \qquad 1 \le i \le \dim \text{Image}\,(\delta_2)$$

and the remaining p_i are prescribed arbitrarily. Assuming, as before, that the matrix with entries m_{ij} is nonsingular, the above system of linear equations can be inverted to give the jumps in the scalar ψ in terms of the $\dim \text{Image}\,(\delta_2)$ periods of the vector potential and $\dim \text{Kernel}\,(\tilde{\jmath}_1)$ ~~other~~ arbitrary constants. Note that this technique generalises and simplifies that of Milani and Negro [1982]. The assumption that the matrix with entries m_{ij} is square and nonsingular is a consequence of the Lefschetz Duality Theorem which will be considered soon.

It is now possible to restate the stationary inductive energy principle in terms of a vector potential.

Stationary Inductive Energy Principle ($\mathbf{B} = \text{curl}\,\mathbf{A}$)

$$T(\text{curl}\,\mathbf{A}) = \inf_{\mathbf{A}} \int_\Omega \int^{\mathbf{A}} \mathcal{H}(\text{curl}\,\xi, \mathbf{r}) \cdot \text{curl}\,(d\xi) dV$$

subject to the principal boundary condition

$$\mathbf{n} \times \mathbf{A} = \overline{\text{curl}\,\psi} \qquad \text{on } S_1$$

where $\psi|_{d_j}$ is prescribed on generators of $H_1(S_1, \partial S_1)$ but ψ is otherwise an arbitrary single valued function. Note that in this formulation of the energy principle the extremal $\mathbf{A}$ is unique to within an element of $Z_c^1(\Omega - S_1)$. Since

$$Z_c^1(\Omega - S_1) = H_c^1(\Omega - S_1) \oplus B_c^1(\Omega - S_1)$$

the nonuniqueness can be overcome by specifying the periods of $\mathbf{A}$ on the generators of $H_1(\Omega, S_1)$ and, in analogy with the uniqueness considerations of the electric vector potential, specifying the divergence of $\mathbf{A}$ and its normal component on S_2 eliminates the ambiguity in $B_c^1(\Omega - S_1)$. This statement refines the one in Kotiuga [1982], Theorem 5.1.

Returning to the case of the coenergy principle which will now be reformulated in terms of a scalar potential, consider the following portion of the long exact homology sequence associated with the pair (Ω, S_2)

$$\begin{array}{cccccccc} \ldots & \xrightarrow{\delta_2} & H_1(S_2) & \xrightarrow{\tilde{\imath}_1} & H_1(\Omega) & \xrightarrow{\tilde{\jmath}_1} & H_1(\Omega, S_2) & \longrightarrow \\ & \xrightarrow{\delta_1} & H_0(S_2) & \xrightarrow{\tilde{\imath}_0} & H_0(\Omega) & \xrightarrow{\tilde{\jmath}_0} & \ldots & \end{array}$$

In order to gain a better understanding of how the periods of $\mathbf{H}$ on the generators of $H_1(\Omega, S_2)$ are prescribed, write

$$H_1(\Omega, S_2) \simeq \delta_1^{-1}(\mathrm{Kernel}(\tilde{\imath}_0)) \oplus \tilde{\jmath}_1\left(\frac{H_1(\Omega)}{\tilde{\imath}_1(H_1(S_2))}\right)$$

and consider the three step recipe. Let

$$c_i, \qquad 1 \le i \le \beta_1(\Omega, S_2)$$

be a set of curves which are associated with the generators of $H_1(\Omega, S_2)$ which are arranged into two groups as follows.

Group 1 There are $\dim \text{Image}(\tilde{\jmath}_1)$ generators of $H_1(\Omega, S_2)$ which are homologous in the absolute sense to generators of $H_1(\Omega)$. These generators can be associated with closed curves $c_i, 1 \le i \le \dim \text{Image}(\tilde{\jmath}_1)$. In this case, the period

$$\int_{c_i} \mathbf{H} \cdot d\mathbf{l} = I_i$$

is equal to the current in $\mathbb{R}^3 - \Omega$ which links the generator of $H_1(\Omega, S_2)$ associated with c_i.

Group 2 There are $\dim \text{Kernel}(\tilde{\imath}_0)$ remaining generators of $H_1(\Omega, S_2)$ which can be associated with simple open curves whose end points lie in distinct connected components of S_2. That is, in each connected component of Ω, one can find curves

$$c_{\dim \text{Image}(\tilde{\jmath}_1) - i}, \qquad 1 \le i \le \dim \text{Kernel}(\tilde{\imath}_0)$$

such that

$$\partial c_{\dim \text{Image}(\tilde{\jmath}_1) + i} = p_i - p_0$$

where p_0 is a datum node lying in some connected component of S_2 and each p_i lies in some other distinct connected component of S_2. In this case, the period

$$\int_{c_j} \mathbf{H} \cdot d\mathbf{l} = I_j, \qquad \dim \text{Image}(\tilde{\jmath}_1) \le j \le \beta_1(\Omega, S_2)$$

is associated with a magnetomotice force.

It is often convenient to describe the magnetic field intensity $\mathbf{H}$ in terms of a scalar potential ς† since, from a practical point of view, it is much easier to work with a scalar function than three components of a vector field. However, when the periods of $\mathbf{H}$ do

† The usual symbol for the scalar potential ς is Ω which cannot be used here.

not vanish on the 1-cycles in group 1, then it is not possible to make ς continuous and single valued since this would imply

$$I_i = \int_{c_i} \mathbf{H} \cdot d\mathbf{l} = \int_{c_i} \operatorname{grad} \varsigma \cdot d\mathbf{l} = 0$$

since

$$\partial c_i = 0 \qquad 1 \leq i \leq \dim \operatorname{Image}(\tilde{j}_1).$$

In order to overcome this difficulty one can perform an analogue of the procedure used to prescribe the tangential components of the vector potential. In general, it is possible to find "barrier" surfaces

$$\Sigma_i', \qquad 1 \leq i \leq \dim \operatorname{Image}(\tilde{j}_1)$$

such that the Σ_i' are associated with dim Image $(\tilde{j}_1)$ dimensional subspace of $H_2(\Omega, S_1)$ and if n_{ij} is the number of oriented intersections of the 1-chain c_i with the relative 2-chain Σ_j' then the $\dim \operatorname{Image}(\tilde{j}_1) \times \dim \operatorname{Image}(\tilde{j}_1)$ matrix with entries n_{ij} is nonsingular. (The reason why this works will be apparent when duality theorems are considered.) It turns out that one can make the scalar potential single valued on

$$\Omega^- = \Omega - \bigcup_{i=1}^{\dim \operatorname{Image}(\tilde{j}_1)} \Sigma_i'.$$

There is another way of looking at the selection of the Σ_i'. Considering the following portion of the long exact homology sequence for the pair (Ω^-, S_2)

$$\begin{array}{llllllll} & & & & \longrightarrow & H_2(\Omega^-, S_2) & \longrightarrow \\ \xrightarrow{\delta_2} & H_1(S_2) & \xrightarrow{\tilde{i}_1} & H_1(\Omega^-) & \xrightarrow{\tilde{j}_1^-} & H_1(\Omega^-, S_2) & \longrightarrow \\ \xrightarrow{\delta_1} & H_0(S_2) & \longrightarrow & & & & \end{array}$$

one has

$$H_1(\Omega^-) = (\tilde{j}_1^-)^{-1} (\operatorname{Kernel}(\delta_1)) \oplus \tilde{i}_1 \left(\frac{H_1(S_2)}{\delta_2 (H_2(\Omega^-, S_2))} \right).$$

Since

$$\mathbf{n} \times \mathbf{H} = 0 \qquad on\ S_2$$

the periods of $\mathbf{H}$ vanish on the generators of $H_1(\Omega^-)$ associated with Image $(\widetilde{\imath}_1)$. Hence the field intensity $\mathbf{H}$ can be represented by the gradient of a single valued scalar ς on Ω^- if the barriers Σ'_i can be arranged so that $\widetilde{\jmath}_1^-$ becomes the trivial homomorphism. This is what happens when the n_{ij} form a non singular matrix.

Assuming that the Σ'_i have been chosen properly, the periods of $\mathbf{H}$ on the generators of $H_1(\Omega, S_2)$ which lie in group 1 are easily expressed in terms of the jumps in ς as the Σ'_i are traversed. That is if $[\varsigma]_{\Sigma'_i}$ are these jumps then for $1 \leq i \leq \dim \text{Image}(\widetilde{\jmath}_1)$ one has

$$I_i = \int_{c_i} \mathbf{H} \cdot d\mathbf{l} = \int_{c_i} \mathbf{grad}\,\varsigma \cdot d\mathbf{l}$$
$$= \sum_{j=1}^{\dim \text{Image}(\widetilde{\jmath}_1)} n_{ij} [\varsigma]_{\Sigma'_j}.$$

This forms a set of linear equations which can be inverted to yield

$$[\varsigma]_{\Sigma'_i} = \sum_{j=1}^{\dim \text{Image}(\widetilde{\jmath}_1)} (n_{ij})^{-1} I_j.$$

Thus, once the "barriers" have been selected, one has an explicit way of prescribing the first $\dim \text{Image}(\widetilde{\jmath}_1)$ periods of $\mathbf{H}$ in terms of the jumps in ς. In order to specify the remaining $\dim \text{Kernel}(\widetilde{\imath}_1)$ periods of $\mathbf{H}$ in terms of the scalar function ς, one defines a $\dim \text{Kernel}(\widetilde{\imath}_0) \times \dim \text{Image}(\widetilde{\jmath}_1)$ matrix whose entries n'_{ij} count the number of oriented intersections of

$$c_{\dim \text{Image}(\widetilde{\jmath}_1)+i} \qquad with\ \Sigma'_i.$$

Hence the periods $\mathbf{H}$ on the remaining generators can be expressed as follows:

$$
\begin{aligned}
I_{\dim \operatorname{Image}(\widetilde{j}_1)+i} &= \int_{c_{\dim \operatorname{Image}(\widetilde{j}_1)+i}} \mathbf{H}\cdot d\mathbf{l} \\
&= \int_{c_{\dim \operatorname{Image}(\widetilde{j}_1)+i}} \operatorname{grad}\varsigma \cdot d\mathbf{l} \\
&= \varsigma(p_i) - \varsigma(p_0) + \sum_{j=1}^{\dim \operatorname{Image}(\widetilde{j}_1)} n'_{ij}[\varsigma]_{\Sigma'_j}.
\end{aligned}
$$

Hence, if $\varsigma(p_0)$ is chosen arbitrarily then

$$
\begin{aligned}
\varsigma(p_i) &= \varsigma(p_0) + I_{\dim \operatorname{Image}(\widetilde{j}_1)+i} - \sum_{j=1}^{\dim \operatorname{Image}(\widetilde{j}_1)} n'_{ij}[\varsigma]_{\Sigma'_j} \\
&= \varsigma(p_0) + I_{\dim \operatorname{Image}(\widetilde{j}_1)+i} - \sum_{l,j=1}^{\dim \operatorname{Image}(\widetilde{j}_1)} n'_{ij}(n_{jl})^{-1} I_l
\end{aligned}
$$

which is an explicit formula giving the value of $\varsigma(p_i)$ in terms of the remaining periods to be prescribed.

Note that on the ith connected component of S_2

$$\varsigma = \varsigma(p_i), \qquad 0 \leq i \leq \dim \operatorname{Image}(\delta_1)$$

since the tangential components of $\mathbf{H}$ vanish on S_2. Having completed the description of $\mathbf{H}$ in terms of a scalar potential, the stationary inductive coenergy principle can be stated in terms of a scalar potential.

Stationary Inductive Coenergy Principle ($\mathbf{H} = \operatorname{grad}\varsigma$)

$$T'(\operatorname{grad}\varsigma) = \inf_{\varsigma} \int_{\Omega^-} \int^{\varsigma} \mathcal{B}(\operatorname{grad}\xi, \mathbf{r}) \cdot \operatorname{grad}(d\xi)\, dV$$

subject to the constraints:

$$[\varsigma]_{\Sigma'_i} \textit{ prescribed on barriers } \Sigma'_i, \qquad 1 \le i \le \dim \mathrm{Image}\,(\tilde{\jmath}_1)$$

and

$$\varsigma = \varsigma(p_i), \quad \text{a given constant on each connected component of } S_2.$$

In summary, the problem of magnetostatics involves a region Ω where

$$\partial\Omega = S_1 \cup S_2 \quad (S_1 \cap S_2 \text{ has no area})$$

$$\mathbf{B} \cdot \mathbf{n} = 0 \qquad on\ S_1$$

$$\mathbf{H} \times \mathbf{n} = 0 \qquad on\ S_2.$$

The magnetic flux density $\mathbf{B}$ is associated with an element of $Z_c^2(\Omega - S_1)$ while the magnetic field intensity $\mathbf{H}$ is associated with an element of $Z_c^1(\Omega - S_2)$. The nondegenerate bilinear pairings which integration induces on homology and cohomology classes

$$\int : H_2(\Omega, S_1) \times H_c^2(\Omega - S_1) \to \mathbb{R}$$

$$\int : H_1(\Omega, S_2) \times H_c^1(\Omega - S_2) \to \mathbb{R}$$

are associated with fluxes and magnetomotive forces respectively.

For variational principles involving the magnetic flux density $\mathbf{B}$ where fluxes are prescribed, the variation of the extremal lies in the space $B_c^2(\Omega - S_1)$. It is convenient to reformulate such variational principles in terms of a vector potential $\mathbf{A}$. When this is done the tangential components of $\mathbf{A}$ are prescribed on S_1 in order to specify fluxes corresponding to generators of $H_c^2(\Omega - S_1)$ and to ensure that the normal component of $\mathbf{B}$ vanishes on S_1. In such cases the vector potential which gives the functional its stationary value is unique to within an element of $Z_c^1(\Omega - S_1)$.

Dually, for variational principles where the magnetic field intensity **H** is the independent variable and magnetomotive forces are prescribed, the variation of the extremal takes place in the space $B_c^1(\Omega - S_2)$. Though it is not possible in general to reformulate such principles in terms of a continuous single valued scalar potential, it is possible to find a scalar potential formulation if one introduces suitable barriers into Ω, prescribing jumps to the scalar potential as these barriers are crossed and fixing the scalar potential to be a different fixed constant on each connected component of S_2.

As in the previous example, the long exact homology sequence played a crucial role in understanding the topological implications of formulating variational principles in terms of potentials and prescribing boundary conditions for the vector potential.

End of Example 13

Example 14 (Currents in Three Dimensional Conducting Bodies)

Consider a connected compact region Ω of finite, nonzero conductivity and whose boundary may contain interfaces with nonconducting or perfectly conducting bodies. Let

$$\partial\Omega = S_1 \cup S_2 \qquad (S_1 \cap S_2 \text{ has no area})$$

$$\operatorname{div} \mathbf{J} = 0 \qquad \text{in } \Omega$$

$$\mathbf{J} \cdot \mathbf{n} = 0 \qquad \text{on } S_1$$

$$\operatorname{curl} \mathbf{E} = 0 \qquad \text{in } \Omega$$

$$\mathbf{n} \times \mathbf{E} = 0 \qquad \text{on } S_2.$$

It is readily seen that under the transformation

$$\mathbf{J} \to \mathbf{B}$$

$$\mathbf{E} \to \mathbf{H}$$

this problem is formally equivalent to Example 13. It is clear that current density **J** can be associated with an element of $Z_c^2(\Omega - S_1)$ while the electric field intensity **E** can be associated with an element of $Z_c^1(\Omega - S_2)$. Note that the boundary condition on S_1 can be associated with a symmetry plane or interface with a nonconducting body while the boundary condition on S_2 can be associated with another type of symmetry plane or the interface of a perfectly conducting body. Furthermore, if

$$\Sigma_i, \qquad 1 \leq i \leq \beta_2(\Omega, S_1)$$

is a set of surfaces associated with a basis of $H_2(\Omega, S_1)$ and

$$c_i, \qquad 1 \leq i \leq \beta_1(\Omega, S_2)$$

is a set of curves associated with a basis of $H_1(\Omega, S_2)$, then the periods of **J** on the Σ_i

$$I_i = \int_{\Sigma_i} \mathbf{J} \cdot \mathbf{n}\, dS, \qquad 1 \leq i \leq \beta_2(\Omega, S_1)$$

are associated with currents and the periods of **E** on the c_i,

$$V_i = \int_{c_i} \mathbf{E} \cdot d\mathbf{l}, \qquad 1 \leq i \leq \beta_1(\Omega, S_2)$$

are associated with voltages, or electromotive forces..

In order to obtain a variational formulation of the problem, consider a constitutive relation

$$\mathbf{E} = \mathcal{E}(\mathbf{J}, \mathbf{r})$$

and an inverse constitutive relation

$$\mathbf{J} = \mathcal{J}(\mathbf{E}, \mathbf{r})$$

which satisfies

$$\mathcal{E}\left(J(\mathbf{E},\mathbf{r}),\mathbf{r}\right)=\mathbf{E}.$$

Furthermore, assume that the two matrices with elements

$$\frac{\partial \mathcal{E}_i}{\partial J_j}, \qquad \frac{\partial J_i}{\partial E_j}$$

are symmetric and positive definite. In this case the principles of stationary content and cocontent can be stated as follows (see MacFarlane [1970] pp 329-330).

Stationary Content Principle

$$G(\mathbf{J}) = \inf_{\mathbf{J}\in Z_c^2(\Omega - S_1)} \int_{\Omega_c} \left(\int_0^{\mathbf{J}} \mathcal{E}(\xi,\mathbf{r})\cdot d\xi \right) dV$$

subject to the constraints which prescribe the periods of $\mathbf{J}$ on generators of $H_2(\Omega, S_1)$

$$I_i = \int_{\Sigma_i} \mathbf{J}\cdot\mathbf{n}\, dS, \qquad 1 \le i \le \beta_2(\Omega, S_1)$$

Stationary Cocontent Principle

$$G'(\mathbf{E}) = \inf_{\mathbf{E}\in Z_c^1(\Omega - S_2)} \int_{\Omega} \left(\int_0^{\mathbf{E}} J(\xi,\mathbf{r})\cdot d\xi \right) dV$$

subject to the constraints which prescribe the periods of $\mathbf{E}$ on generators of $H_1(\Omega, S_2)$

$$V_i = \int_{c_i} \mathbf{E}\cdot dl, \qquad 1 \le i \le \beta_1(\Omega, S_2).$$

As in the previous two examples, the two variational principles stated above constrain the extremal to be a relative cocycle and when additional constraints are prescribed on the generators of a (co)homology group, the variation of the extremal is constrained to be a relative coboundary.

For linear constitutive relations the content principle gives an upper bound for resistance while the cocontent principle gives an upper bound on conductance. As usual the upper bounds are obtained by expressing the minimum of the functional as a quadratic form in the prescribed periods and making the identification

$$G(\mathbf{J}) = \sum_{i,j=1}^{\beta_2(\Omega,S_1)} I_i R_{ij} I_j$$

$$G(\mathbf{E}) = \sum_{i,j=1}^{\beta_1(\Omega,S_2)} V_i G_{ij} V_j.$$

From the upper bound on conductance, a lower bound on resistance can be found in the usual way.

By attempting to express the variational principles in terms of vector and scalar potentials one will find, as in previous examples, many topological subtleties. Noticing the mathematical equivalence between this example involving steady currents and the previous one involving magnetostatics, one may form a transformation of variables

$$\mathbf{J} \rightarrow \mathbf{B}$$

$$\mathbf{E} \rightarrow \mathbf{H}$$

$$\mathbf{T} \rightarrow \mathbf{A}$$

$$\phi \rightarrow \varsigma$$

as soon as one tries to define potentials ϕ and $\mathbf{T}$ such that

$$\mathbf{J} = \operatorname{curl} \mathbf{T}$$

$$\mathbf{E} = \operatorname{grad} \phi.$$

Summarising the results of exploiting the mathematical analogy, one can say the following about the cocontent principle in terms of a vector potential $\mathbf{T}$. As in the previous example, one can consider the long exact homology sequence for the pair (Ω, S_1) and obtain

$$H_2(\Omega, S_1) = \delta_2^{-1}(\text{Kernel}(\widetilde{\imath}_1)) \oplus \widetilde{\jmath}_2\left(\frac{H_2(\Omega)}{\widetilde{\imath}_2(H_2(S_1))}\right)$$

where the relevant portion of the long exact homology sequence is

$$\begin{array}{cccccccc} \ldots & \xrightarrow{\delta_3} & H_2(S_1) & \xrightarrow{\widetilde{\imath}_2} & H_2(\Omega) & \xrightarrow{\widetilde{\jmath}_2} & H_2(\Omega, S_1) & \longrightarrow \\ & \xrightarrow{\delta_2} & H_1(S_1) & \xrightarrow{\widetilde{\imath}_1} & H_1(\Omega) & \xrightarrow{\widetilde{\jmath}_1} & \ldots & \end{array}$$

As in the case of the vector potential $\mathbf{A}$ in Example 13, the surfaces $\Sigma_i, 1 \leq i \leq \beta_2(\Omega, S_1)$ can be split up into two groups where the first group is associated with a basis of the second term in the direct sum and the periods vanish on this first group. That is:

$$\Sigma_i, \qquad 1 \leq i \leq \dim \text{Image}(\widetilde{\jmath}_2)$$

is related to a basis for Image $(\widetilde{\jmath}_2)$ and

$$\begin{aligned} I_i &= \int_{\Sigma_i} \mathbf{J} \cdot \mathbf{n}\, dS \\ &= \int_{\partial \Sigma_i} \mathbf{T} \cdot d\mathbf{l} = 0 \end{aligned}$$

since

$$\partial \Sigma_i = 0, \qquad 1 \leq i \leq \dim \text{Image}(\widetilde{\jmath}_2).$$

The second group is associated with a basis for the first term in the direct sum. The periods of the current density $\mathbf{J}$ is easily expressed in terms of the vector potential in this case. Let

$$\partial \Sigma_{\dim \text{Image}(\widetilde{\jmath}_2)+i}, \qquad 1 \leq i \leq \dim \text{Image}(\delta_2)$$

be associated with a basis of Image (δ_2)

$$\begin{aligned} I_{\dim \text{Image}(\widetilde{\jmath}_2)+i} &= \int_{\Sigma_{\dim \text{Image}(\widetilde{\jmath}_2)+i}} \mathbf{J} \cdot \mathbf{n}\, dS \\ &= \int_{\partial \Sigma_{\dim \text{Image}(\widetilde{\jmath}_2)+i}} \mathbf{T} \cdot d\mathbf{l}. \end{aligned}$$

The fact that the periods of $\mathbf{J}$ vanish on the first $\dim \mathrm{Image}\,(\tilde{\jmath}_2)$ generators of $H_2(\Omega, S_1)$ in order for a vector potential to exist imposes no real constraint on the problem since these periods represent the rate of change of net charge in some connected component of $\mathbb{R}^3 - \Omega$. Since the problem is assumed to be static, these periods are taken to be zero. The next problem which arises is the prescription of the tangential components of the vector potential on S_1 so that the normal component of $\mathbf{J}$ vanishes there and the periods of $\mathbf{J}$ on the $\delta_2^{-1}\,(\mathrm{Kernel}\,(\tilde{\imath}_1))$ remaining generators of $H_2(\Omega, S_1)$ can be prescribed in terms of the vector potential. This problem can be overcome by using exactly the same technique as in Example 13. That is, let

$$\mathbf{n} \times \mathbf{T} = \overline{\mathrm{curl}}\,\psi \qquad on\ S_1$$

where the jumps

$$[\psi]_{d_j}, \qquad 1 \leq j \leq \beta_1(S_1, \partial S_1)$$

are prescribed on the curves d_i which are associated with a basis of $H_1(S_1, \partial S_1)$. As in that previous example, selecting a set of curves $z_i, 1 \leq i \leq \beta_1(S_1)$, associated with a basis of $H_1(S_1)$ where

$$\partial\Sigma_{\dim \mathrm{Image}\,(\tilde{\jmath}_2)+i} = z_i, \qquad 1 \leq i \leq \dim \mathrm{Image}\,(\delta_2)$$

one can explicitly describe the jumps $[\psi]_{d_j}$ in terms of the periods

$$I_{\dim \mathrm{Image}\,(\tilde{\jmath}_2)+i}, \qquad 1 \leq i \leq \dim \mathrm{Image}\,(\delta_2).$$

In this way it is possible to restate the stationary content principle as follows:

Stationary Content Principle ($\mathbf{J} = \mathrm{curl}\,\mathbf{T}$)

$$G(\mathrm{curl}\,\mathbf{T}) = \inf_{\mathbf{T}} \int_\Omega \int^{\mathbf{T}} \mathcal{E}\,(\mathrm{curl}\,\xi, \mathbf{r}) \cdot \mathrm{curl}\,(d\xi) dV$$

subject to the principal boundary condition

$$\mathbf{n} \times \mathbf{T} = \overline{\text{curl}}\,\psi \qquad on\ S_1$$

where $[\psi]_{d_j}$ are prescribed on curves representing generators of $H_1(S_1, \partial S_1)$ and ψ is otherwise an arbitrary single-valued function.

Turning to the other variational principle, and using the mathematical analogy between this example and the previous example, one sees that the stationary cocontent principle cannot in general be expressed in terms of a continuous single-valued scalar potential. To see why this is so, one considers the following portion of the long exact homology sequence of the pair (Ω, S_2).

$$\begin{array}{llllllll} \cdots & \xrightarrow{\delta_2} & H_1(S_2) & \xrightarrow{\tilde{\imath}_1} & H_1(\Omega) & \xrightarrow{\tilde{\jmath}_1} & H_1(\Omega, S_2) & \longrightarrow \\ & \xrightarrow{\delta_1} & H_0(S_2) & \xrightarrow{\tilde{\imath}_0} & H_0(\Omega) & \xrightarrow{\tilde{\jmath}_0} & & \cdots \end{array}$$

Let

$$c_i, \qquad 1 \leq i \leq \dim \operatorname{Image}(\tilde{\jmath}_1)$$

be a set of curves associated with a basis of Image $(\tilde{\jmath}_1)$ in $H_1(\Omega, S_2)$. The periods of the electric field intensity $\mathbf{E}$ on these curves are in general non zero, but would be zero if $\mathbf{E}$ is the gradient of a continuous single-valued scalar potential. As in Example 13, this problem can be overcome by letting

$$\Sigma_i', \qquad 1 \leq i \leq \dim \operatorname{Image}(\tilde{\jmath}_1)$$

be a set of surfaces associated with generators of $H_2(\Omega, S_1)$ which act like barriers that enable the scalar potential to be single valued on

$$\Omega^- = \Omega - \bigcup_{i=1}^{\dim \operatorname{Image}(\tilde{\jmath}_1)} \Sigma_i'.$$

Furthermore, the periods

$$V_i = \int_{c_i} \mathbf{E} \cdot d\mathbf{l}, \qquad 1 \leq i \leq \dim \text{Image}\,(\tilde{j}_1)$$

can be prescribed in terms of the jumps

$$\psi|_{\Sigma'_i}$$

and when this is done the remaining periods

$$V_{\dim \text{Image}\,(\tilde{j}_1)+i} = \int_{c_{\dim \text{Image}\,(\tilde{j}_1)+i}} \mathbf{E} \cdot d\mathbf{l}$$

can be expressed in terms of the scalar potential which is constant value on each connected component of S_2. That is if

$$\partial c_{\dim \text{Image}\,(\tilde{j}_1)-i} = p_i - p_0, \qquad 1 \leq i \leq \dim \text{Image}\,(\delta_1)$$

then

$$\phi = \phi(p_i)$$

on the ith connected component of S_2 and if p_0 is some datum node then the last $\dim \text{Image}\,(\delta_1)$ periods of $\mathbf{E}$ can be prescribed by specifying the potential differences

$$\phi(p_i) - \phi(p_0) \qquad 1 \leq i \leq \dim \text{Image}\,(\delta_1).$$

When this is done the stationary cocontent principle can be rephrased in terms of a scalar potential as follows.

Stationary Cocontent Principle ($\mathbf{E} = \text{grad}\,\phi$)

$$G'(\text{grad}\,\phi) = \inf_{\phi} \int_{\Omega} \int^{\phi} J(\text{grad}\,\xi, \mathbf{r}) \cdot \text{grad}\,(d\xi) dV$$

subject to the constraints

$$\phi|_{\Sigma_i}, \qquad \text{prescribed on barriers } \Sigma_i'$$

and

$$\phi = \phi(p_i), \qquad \text{on the } i\text{th connected component of } S_2.$$

In summary, the problem of calculating steady current distributions in conducting bodies and the problem of three dimensional magnetostatics are equivalent under the change of variables

$$\mathbf{A} \leftrightarrow \mathbf{T}, \qquad \mathbf{B} \leftrightarrow \mathbf{J}$$

$$\varsigma \leftrightarrow \phi, \qquad \mathbf{H} \leftrightarrow \mathbf{E}$$

hence the mathematical considerations in using vector or scalar potentials are the same in both problems. Thus it is necessary to summarise only the physical interpretations of the periods and potentials. In this example vector fields $\mathbf{J}$ and $\mathbf{E}$ were associated with elements of $Z_c^2(\Omega - S_1)$ and $Z_c^1(\Omega - S_2)$ respectively and the nondegenerate bilinear pairings which integration induces on homology and cohomology classes

$$\int : H_2(\Omega, S_1) \times H_c^2(\Omega - S_1) \to \mathbb{R}$$

$$\int : H_1(\Omega, S_2) \times H_c^1(\Omega - S_2) \to \mathbb{R}$$

are associated with currents and electromotive forces respectively. The formulas

$$Z_c^2(\Omega - S_1) \simeq H_c^2(\Omega - S_1) \oplus B_c^2(\Omega - S_1)$$

$$Z_c^1(\Omega - S_2) \simeq H_c^1(\Omega - S_2) \oplus B_c^1(\Omega - S_2)$$

show that when there is a variational principle where either $\mathbf{J}$ or $\mathbf{E}$ are independent variables, conditions fixing the periods of these relative cocycles restrict the variation of the extremal to be a relative coboundary. Alternatively, when the variational principles are formulated in terms of potentials, the potentials are unique to within an element of

$$Z_c^1(\Omega - S_1) \qquad \text{for } \mathbf{T}$$

$$Z_c^0(\Omega - S_2) \qquad \text{for } \phi$$

and techniques of the previous example show how to eliminate this nonuniquenes in the case of the vector potential.

End of Example 14

The previous examples show that homology groups arise naturally in boundary value problems of electromagnetics. It is beyond the intended scope of this thesis to give a heuristic account of axiomatic homology theory in the context of the boundary value problems being considered because additional mathematical machinery such as categories, functors, and homotopies are required to explain the axioms which underlie the theory (see Hocking and Young [1961] Sect 7.7 for an explanation of the axioms.) It suffices to say that the existence of a long exact homology sequence is only one of the seven axioms of a homology theory! The other six axioms of a homology theory, once understood, are "intuitively obvious" in the present context and have been used implicitly in many of the previous examples.

One of the virtues of axiomatic homology theory is that one can show that once a method of computing homology for a certain category of spaces, such as manifolds, has been devised, the resulting homology groups are unique up to an isomorphism. Thus, for example, the de Rham isomorphism can be regarded as a consequence of devising a method of computing cohomology with differential forms and simplicial complexes, and showing that both methods satisfy the requirements of the axiomatic theory in the case of differentiable manifolds.

1.10 Duality Theorems

The next useful topic from homology theory which sheds light on the topological aspects of boundary value problems are duality theorems. Duality theorems serve three functions:

1. They show a duality between certain sets of lumped parameters which are conjugate in the sense of the Legendre tranformation.
2. They show the relationship between the generators of the pth homology group of an n-dimensional space and the $n-p$ dimensional barriers which must be inserted into the space to make the pth homology group trivial.
3. They show a global duality between compatibility conditions on the sources in a boundary value problem and the gauge transformation or nonuniqueness of a potential.

In order to simplify ideas, let the discussion be restricted to manifolds where homology is calculated with coefficients in the field $\mathbb{R}$. In general duality theorems are formulated for orientable n-dimensional manifolds M and have the form

$$H_c^p(\text{something}) \simeq H_{n-p}(\text{something else}).$$

As will be seen in the next chapter, there is a way of multiplying an r-form and a q-form to get an $r+q$-form. Thus multiplying a p-form and an $n-p$-form one obtains an n-form. This multiplication, called the exterior product, leads to duality theorems as follows. For orientable manifolds one can construct an n-form so that integration over the manifold behaves like a nondegenerate bilinear pairing:

$$\int_M : C_c^p(M) \times C^{n-p}(M) \to \mathbb{R}$$

which induces a nondegenerate bilinear pairing on homology

$$\int_M : H_c^p(M) \times H^{n-p}(M) \to \mathbb{R}$$

where the multiplication on forms (exterior multiplication) induces a multiplication on homology classes which is called the cup product. Summarising then, one can say that duality theorems are a consequence of identifying a nondegenerate bilinear pairing associated with integration just as the de Rham theorem comes about as a result of a nondegenerate bilinear pairing between chains and cochains See Massey [1980] Chapter 9 or Greenberg and Harper [1981] Part 3 for derivations of the most useful duality theorems which do not depend on the formalism of differential forms.

The oldest form of these duality theorems is the Poincaré Duality theorem which says that for an orientable n-dimensional manifold M which has no boundary one has:

$$H_c^p(M) \simeq H_{n-p}(M)$$

where for compact closed manifolds the geometric picture is easily seen when one writes

$$H_p(M) \simeq H_{n-p}(M)$$

and verifies this for all of the 1 and 2 dimensional manifolds which one can think of.

For boundary value problems in electromagnetics one requires duality theorems which apply to manifolds with boundary. The classical prototype of this type of theorem is the Lefschetz Duality theorem which says that for a compact n-dimensional region Ω

$$H_c^{n-p}(\Omega) \simeq H_p(\Omega, \partial\Omega)$$

and so by de Rham's Theorem

$$H_{n-p}(\Omega) \simeq H_c^p(\Omega - \partial\Omega).$$

Hence

$$\beta_p(\Omega) = \beta_{n-p}(\Omega, \partial\Omega)$$

and by the de Rham Theorem

$$H_{n-p}(\Omega) \simeq H_p(\Omega, \partial\Omega).$$

In order to appreciate the intuitive content of this duality consider the following examples.

Example 15 (3-D Electrostatics, n=3,p=1)

Consider a dielectric region Ω whose boundary $\partial\Omega$ is an interface to conducting bodies. In this case each connected component of $\partial\Omega$ is associated with an equipotential and the generators of $H_1(\Omega, \partial\Omega)$ can be associated with curves $c_i, 1 \leq i \leq \beta_1(\Omega, \partial\Omega)$ whose end points can be used to specify the $\beta_1(\Omega, \partial\Omega)$ independent potential differences in the problem. Dually, the generators of $H_2(\Omega)$ can be associated with closed sufaces $\Sigma_j, 1 \leq j \leq \beta_2(\Omega)$ which can be used to specify the total flux of the $\beta_2(\Omega)$ independent charge distributions of the problem. That is,

$$\int_{c_i} \mathbf{E} \cdot d\mathbf{l} = V_i, \qquad 1 \leq j \leq \beta_1(\Omega, \partial\Omega)$$

$$\int_{\Sigma_j} \mathbf{D} \cdot \mathbf{n}\, dS = Q_j, \qquad 1 \leq j \leq \beta_2(\Omega).$$

Furthermore, the equation

$$\beta_1(\Omega, \partial\Omega) = \beta_2(\Omega)$$

expresses the fact that there are just as many indepentent potential differences as there are independent charges in the problem.

Another interpretation of the Lefschetz Duality theorem is obtained by constructing a matrix of intersection numbers (m_{ij}) where m_{ij} is the number of oriented intersections of c_i with Σ_j. The Lefschetz theorem then asserts that this matrix is nonsingular. Hence the c_i can be interpreted as a minimal set of curves which when considered as obstacles make

$$H_2\left(\Omega - \bigcup_{i=1}^{\beta_1(\Omega,\partial\Omega)} c_i\right) \simeq 0$$

or in Maxwell's terminology, the c_i eliminate the periphraxity of the region Ω. Thus if one replaces the c_i by a tubular neighbourhood of the c_i, one can always write

$$\mathbf{D} = \operatorname{curl} \mathbf{C} \qquad in \quad \Omega - \bigcup_{i=1}^{\beta_1(\Omega,\partial\Omega)} c_i$$

whenever

$$\operatorname{div} \mathbf{D} = 0 \qquad in\ \Omega$$

regardless of how charge is distributed in the exterior of the region and on the boundary.

End of Example 15

Example 16 (3-D Magnetostatics $p = 2, n = 3$)

Consider a nonconducting region Ω whose boundary $\partial\Omega$ is an interface to superconducting bodies. In this case the generators of $H_2(\Omega, \partial\Omega)$ can be associated with open surfaces $\Sigma_i, 1 \leq i \leq \beta_2(\Omega, \partial\Omega)$, which can be used to compute the $\beta_2(\Omega, \partial\Omega)$ independent fluxes in the problem. Dually, the generators of $H_1(\Omega)$ are associated with closed curves $c_j, 1 \leq j \leq \beta_1(\Omega)$ which can be used to specify the number of independent currents in the problem. That is, these generators are related to periods as follows:

$$\phi_i = \int_{\Sigma_i} \mathbf{B} \cdot \mathbf{n}\, dS = \int_{\partial \Sigma_i} \mathbf{A} \cdot d\mathbf{l}, \qquad 1 \leq i \leq \beta_2(\Omega, \partial\Omega)$$

$$I_j = \int_{c_j} \mathbf{H} \cdot d\mathbf{l}, \qquad 1 \leq j \leq \beta_1(\Omega).$$

Furthermore, the equation

$$\beta_2(\Omega, \partial\Omega) = \beta_1(\Omega)$$

expresses the fact that there are just as many independent fluxes as there are currents in the problem.

Another interpretation of the Lefschetz Duality theorem can be obtained by constructing a matrix of intersection numbers (m_{ij}) where m_{ij} is equal to the number of oriented intersections of Σ_i with c_j. The Lefschetz theorem asserts that this matrix is nonsingular. Hence the Σ_i can be interpreted as a set of barriers which make

$$H_1\left(\Omega - \bigcup_{i=1}^{\beta_2(\Omega,\partial\Omega)} \Sigma_1\right) \simeq 0$$

or in Maxwell's terminology, the Σ_i eliminate the cyclosis of the region Ω. Thus one can always write

$$\mathbf{H} = \operatorname{grad} \varsigma \qquad in \quad \Omega - \bigcup_{i=1}^{\beta_2(\Omega,\partial\Omega)} \Sigma_i \ .$$

whenever

$$\operatorname{curl} \mathbf{H} = 0 \qquad in\ \Omega$$

regardless of how currents flow in the exterior of the region and on the boundary.

End of Example 16

Example 17 (Steady currents in three dimensions $p = 2, n = 3$)

By making the substitutions

$$\mathbf{B} \to \mathbf{J} \qquad \mathbf{A} \to \mathbf{T} \qquad \Phi \to I$$

$$\mathbf{H} \to \mathbf{E}\,. \qquad \varsigma \to \phi \qquad I \to V$$

the problem of steady currents in a conducting region bounded by a nonconducting region becomes mathematically identical to Example 16 and hence the rôle of the Lefschetz Duality Theorem in problems involving steady currents is easily deduced by referring to the previous example. Thus the analogy between Examples 17 and 16 is the same as the analogy between Examples 14 and 13.

End of Example 17

Example 18 (Currents on orientable sheets $p = 1, n = 2$)

Consider a conducting sheet Ω which is homeomorphic to a sphere with n handles and k holes as in Example 4. Suppose that slowly varying magnetic fields are inducing currents on Ω and that the boundary of Ω does not touch any other conducting body. Thus

$$\operatorname{div} \mathbf{J} = 0 \qquad on\ \Omega$$

$$J_n = 0 \qquad on\ \partial\Omega.$$

Note that this problem is dual to the one considered in Example 7 in the sense that current flow normal to the boundary of the plate must vanish. Let us define a vector $\mathbf{K}$ by the relation†

$$\mathbf{J} \times \mathbf{n}' = \mathbf{K}$$

then, since locally,

$$\mathbf{J} = \overline{\operatorname{curl}}\, \psi = \mathbf{n}' \times \operatorname{grad} \psi$$

† Readers familiar with differential forms will note that the operation $\mathbf{n}'\times$ corresponds to the Hodge star operator on 1-forms in two dimensions.

one has

$$\mathbf{K} = (\mathbf{n}' \times \operatorname{grad} \psi) \times \mathbf{n}' = (\mathbf{n}' \cdot \mathbf{n}')\operatorname{grad} \psi - (\mathbf{n}' \cdot \operatorname{grad} \psi)\mathbf{n}' = \operatorname{grad} \psi.$$

Furthermore, since the current density $\mathbf{J}$ is tangent to the boundary, the vector field $\mathbf{K}$ has vanishing tangential components at the boundary and can be associated with an element of

$$Z_c^1(\Omega - \partial\Omega).$$

Thus if $\mathbf{K}$ can be described by a scalar potential ψ then

$$\psi = \text{ constant}$$

on each connected component of $\partial\Omega$.

Let $c_i, 1 \leq i \leq \beta_1(\Omega, \partial\Omega)$ be a set of curves which are associated with generators of $H_1(\Omega, \partial\Omega)$ and $z_i, 1 \leq i \leq \beta_1(\Omega)$ be a set of curves associated with generators of $H_1(\Omega)$ as in Example 7. Dual to the situation in Examples 7 and 11 the z_i act like cuts which enable ψ to be single valued on

$$\Omega - \bigcup_{i=1}^{\beta_1(\Omega)} z_i$$

while $\psi = 0$ on $\partial\Omega$. Furthermore, the jumps of ψ on the $z_i, [\psi]_{z_i}$, are given by calculating the periods

$$\begin{aligned} I_i &= \int_{c_i} \mathbf{J} \times \mathbf{n}' \cdot d\mathbf{l} = \int_{c_i} \mathbf{K} \cdot d\mathbf{l} \\ &= \int_{c_i} \operatorname{grad} \psi \cdot d\mathbf{l} = \sum_{j=1}^{\beta_1(\Omega)} m_{ij} [\psi]_{z_j} \end{aligned}$$

where m_{ij} is the number of oriented intersections of c_i with z_j. The matrix with entries m_{ij} is square since by the Lefschetz duality theorem

$$\beta_1(\Omega) = \beta_1(\Omega, \partial\Omega)$$

and nonsingular if the c_i and z_j actually correspond to bases of $H_1(\Omega, \partial\Omega)$ and $H_1(\Omega)$ respectively. The matrix can be inverted to yield

$$\psi]_{z_j} = \sum_{i=1}^{\beta_1(\Omega,\partial\Omega)} (m_{ij})^{-1} I_i.$$

Hence the duality between the homology groups i useful in prescribing periods of vector fields in terms of jumps in the scalar potential on curves associated with a dual group.

End of Example 18

Example 19 (Stream functions on orientable surfaces $p = 1, n = 2$)

When considering the current density vector $\mathbf{J}$ on a sheet as in Example 7 and when prescribing the components of a vector tangent to a surface as in Examples 12, 13, 14, the following situation occured. Given a two dimensional surface S suppose

$$(\operatorname{curl} \mathbf{C}) \cdot \mathbf{n}' = 0$$

$$\text{or} \qquad (\operatorname{curl} \mathbf{A}) \cdot \mathbf{n}' = 0$$

$$\text{or} \qquad (\operatorname{curl} \mathbf{T}) \cdot \mathbf{n}' = 0$$

on S or if $\mathbf{J}$ is a vector field defined in the surface, suppose

$$\operatorname{div} \mathbf{J} = 0 \qquad \textit{on } S$$

and no boundary conditions are prescribed on ∂S. In this case it is useful to set

$$\left.\begin{array}{r} \mathbf{n}' \times \mathbf{C} = \\ \mathbf{n}' \times \mathbf{A} = \\ \mathbf{n}' \times \mathbf{T} = \\ \mathbf{J} = \end{array}\right\} \overline{\operatorname{curl}}\, \psi \qquad \text{locally } \textit{on } S.$$

Next let $z_i, 1 \leq i \leq \beta_1(S)$, be a set of curves associated with a basis of $H_1(S)$ and $c_i, 1 \leq i \leq \beta_1(S, \partial S)$ be a set of curves associated with a basis of $H_1(S, \partial S)$. In this case

$$H_1\left(S - \bigcup_{i=1}^{\beta_1(S,\partial S)} c_i\right) \simeq 0$$

so that the stream function can be made continuous and single valued if one avoids the cuts c_i. Furthermore the jumps in ψ on the cuts c_i, denoted by $[\psi]_{c_i}$ can be used to prescribe the periods

$$p_i = \int_{z_i} \left\{ \begin{array}{c} -\mathbf{C} \\ -\mathbf{A} \\ -\mathbf{T} \\ \mathbf{J} \times \mathbf{n}' \end{array} \right\} \cdot d\mathbf{l}.$$

Since

$$p_i = \int_{z_i} (\overline{\mathrm{curl}}\, \psi \times \mathbf{n}') \cdot d\mathbf{l} = \int_{z_i} \mathbf{grad}\, \psi \cdot d\mathbf{l}$$

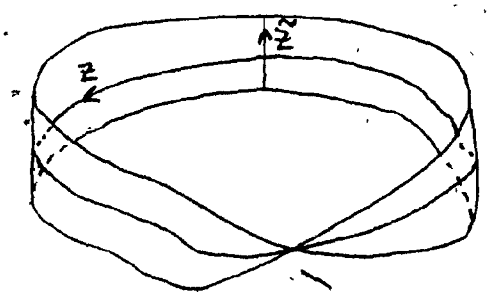
$$= \sum_{j=1}^{\beta_1(S,\partial S)} m_{ij} [\psi]_{c_j}$$

where m_{ij} is the number of oriented intersections of c_i with z_j and the Lefschetz duality theorem ensures that the above matrix equation is uniquely soluble for the $[\psi]_{c_i}$ if the c_i and z_i are actually associated with a full set of generators for $H_1(S, \partial S)$ and $H_1(S)$ respectively.

End of Example 19

Example 20 (A counterexample)

It has been mentioned that the duality theorems are true for orientable manifolds. In order to see that the Lefschetz duality theorem is not true in the case of a nonorientable manifold, consider the Möbius strip of Example 10 as shown in Fig. 16. Recall



Ω is the Mõbius band

S is its edge

Fig. 16

that

$$\beta_1(\Omega, S) = 0 \neq 1 = \beta_1(\Omega).$$

Hence the Lefschetz duality theorem does not apply in this case.

It is interesting to consider the Mõbius band in the light of Example 17. That is if

$$\operatorname{div} \mathbf{J} = 0 \qquad on\ \Omega$$

$$J_n = 0 \qquad on\ S$$

then one attempts, since S is connected, to set

$$J = \overline{\operatorname{curl}}\,\psi \qquad on\ \Omega$$

and

$$\psi = constant \qquad on\ S.$$

However, if the current flowing around the band is nonzero, then it is not possible to set the stream function ψ to a constant on S even if S is connected. That is, if I is the current which flows around the band then if ψ is single valued

$$I = \int_{\tilde{z}} \left((\overline{\operatorname{curl}}\,\psi) \times \mathbf{n}' \right) \cdot d\mathbf{l} = \psi_s - \psi_s = 0$$

where ψ_s is the value of ψ on S. Furthermore, since

$$\beta_1(\Omega, S) = 0$$

there is no way to take a curve associated with a generator of $H_1(\Omega, S)$ and use it to specify the current flowing around the loop. Thus it is seen that the method of considering curves associated with generators of $H_1(\Omega)$ and $H_1(\Omega, S)$ and using one set as cuts and the other to specify the periods of the vector field, is critically dependent on the Lefschetz duality theorem. In order to see how the current flow can be described in terms of a stream function, it is best to look at the problem in terms of $H_1(\Omega, S; \mathbb{Z})$ where $\tilde{z} \sim z$.

The reader may convince himself that this generator of the torsion subgroup of $H_1(\Omega, S; \mathbb{Z})$ can be used as a cut in Ω which enables one to describe the current density $\mathbf{J}$ in terms of a single valued stream function. Considering the diagram, there are two obvious ways of doing this.

Method # 1 Take z as the cut and impose the condition $\psi \to -\psi$ as one crosses the cut and set

$$\psi_s = \pm \frac{I}{2}$$

where the sign is chosen depending on the sense of the current.

Method # 2 Take $\tilde{z}$ as the cut and note that $S - \tilde{z}$ has two connected components which shall be called S' and S''. In order to describe the current flow in terms of the stream function, let $\psi \to -\psi$ as one crosses the cut and let

$$\psi_{s'} = -\psi_{s''} = \pm \frac{I}{2}$$

where again the sign is chosen depending on the sense of the current.

The moral of this example is that one should not expect techniques which make implicit use of duality theorems to work in situations where the hypotheses underlying the duality theorems are not satisfied.

End of Example 20

It is important to realise that, for an n-dimensional manifold, the interpretation of the Lefschetz Duality Theorem in terms of oriented intersections of p and $n-p$ dimensional submanifolds makes the duality intuitive when n is less than four. For a proper account of this interpretation see Greenberg and Harper [1981] Chapter 31 while for a leisurely but rigorous development of intersections see Guillemin and Pollack [1974] Chapters 2 and 3.

The boundary value problems considered in Examples 12, 13, and 14 show that the Lefschetz duality theorem is inadequate for dealing with complicated problems where different boundary conditions occur on different connected components of $\partial\Omega$ or when symmetry planes have been used to reduce a given problem to one a fraction of the original size. In other words the Lefschetz duality theorem is inadequate for many problems formulated for numerical computation. For cases like Examples 12, 13, and 14, the following duality theorems apply:

$$H_c^p(\Omega - S_1) \simeq H_{n-p}(\Omega, S_2)$$

$$H_c^p(\Omega - S_2) \simeq H_{n-p}(\Omega, S_1)$$

where

$$\partial\Omega = S_1 \cup S_2$$

$S_1 \cap S_2$ has no (n − 1) dimensional volume.

Here it is understood that the connected components of S_1 and S_2 correspond to intersections of symmetry planes with some original problem or to connected components of the boundary of the original problem which was reduced by identifying symmetry planes. The above duality was first observed (to the best of the author's knowledge) by Connor [1954] for the case where S_1 and S_2 are the union of connected components of $\partial\Omega$. The proof of the theorem in this case is outlined in Vick [1973] Sect 5.25. The more general version which is assumed in this thesis can be obtained from the version known to Connor by the usual method of doubling (see Duff [1952] or Friedrichs [1955]).

The above duality theorem implies that

$$\beta_p(\Omega, S_1) = \beta_{n-p}(\Omega, S_2)$$

and

$$H_p(\Omega, S_1) \simeq H_{n-p}(\Omega, S_2)$$

where the isomorphism between these two homology groups can be interpreted as asserting that there is a nondegenerate bilinear pairing between the two groups which can be represented by a square nonsingular matrix whose entries count the number of oriented intersections of p and $n-p$ dimensional submanifolds associated with the generators of both groups. The special cases of this theorem for the case $n = 3$ can be found in Examples 12, 13 and 14 (where $p = 1, 2, 2$ respectively). It is worthwhile to consider several others.

Example 21 ($p = 1, n = 2$, currents on conducting surfaces)

Consider again the two dimensional surface of Example 4 which is homeomorphic to a sphere with n handles and k holes and suppose that the component of the magnetic field normal to the surface is negligible and that the frequency of excitation is low enough to make displacement currents negligible. Hence let

$$\begin{aligned}
\partial\Omega &= S_1 \cup S_2 && (S_1 \cap S_2 \text{ has no length}) \\
\operatorname{div} \mathbf{J} &= 0 && \text{on } \Omega \\
J_n &= 0 && \text{on } S_1 \\
\operatorname{curl} \mathbf{E} &= 0 && \text{on } \Omega \\
E_t &= 0 && \text{on } S_2.
\end{aligned}$$

Thus S_1 is associated with the edge of the plate which does not touch any other conducting body and S_2 is associated with the interface of a perfect conductor. Alternatively S_1 or S_2 can be identified with symmetry planes. In this case the electric field $\mathbf{E}$ is associated with an element of $Z_c^1(\Omega - S_2)$ while

$$\mathbf{n}' \times \mathbf{J}$$

where $\mathbf{n}'$ is the vector normal to the sheet, can be associated with an element of $Z_c^1(\Omega - S_1)$. That is,

$$\begin{aligned}
\operatorname{curl}(\mathbf{n}' \times \mathbf{J}) \cdot \mathbf{n}' &= 0 && \text{on } \Omega \\
(\mathbf{n}' \times \mathbf{J})_t &= 0 && \text{on } S_1.
\end{aligned}$$

Let $c_i, 1 \le i \le \beta_1(\Omega, S_1)$ be a set of curves associated with generators of $H_1(\Omega, S_1)$ and $z_j, 1 \le j \le \beta_1(\Omega, S_2)$ be another set of curves which are associated with generators of $H_1(\Omega, S_2)$ and let these sets of curves be arranged in intersecting pairs as in Example 4. That is if m_{ij} is the number of oriented intersections of c_i with z_j then

$$m_{ij} = \delta_{ij} \qquad \text{(Kronecker delta)}.$$

Furthermore let the periods of the two cocycles on these sets of cycles be denoted by

$$I_i = \int_{c_i} (\mathbf{J} \times \mathbf{n}') \cdot d\mathbf{l}, \qquad 1 \le i \le \beta_1(\Omega, S_1)$$

$$\Phi_j = \int_{z_j} \mathbf{E} \cdot d\mathbf{l}, \qquad 1 \le j \le \beta_1(\Omega, S_2).$$

If d is the thickness of the plate and σ the conductivity of the material, the stationary content and cocontent principles can be restated as follows (note that $\mathbf{J}$ has units of current per length in this problem)

Stationary Content Principle

$$G(\mathbf{J}) = \inf_{\mathbf{J}\times\mathbf{n}'\in Z_c^1(\Omega-S_1)} \int_\Omega \frac{|\mathbf{J}|^2}{2\sigma d} dS$$

subject to the constraints which prescribe the periods of $\mathbf{J}\times\mathbf{n}'$ on generators of $H_1(\Omega, S_1)$

$$I_i = \int_{c_i} (\mathbf{J}\times\mathbf{n}')\cdot d\mathbf{l}, \qquad 1 \le i \le \beta_1(\Omega, S_1).$$

Stationary Cocontent Principle

$$G'(\mathbf{E}) = \inf_{\mathbf{E}\in Z_c^1(\Omega-S_2)} \int_\Omega \frac{\sigma d|\mathbf{E}|^2}{2} dS$$

subject to the constraint which prescribe the periods of $\mathbf{E}$ on generators of $H_1(\Omega, S_2)$

$$V_j = \int_{z_j} \mathbf{E}\cdot d\mathbf{l}, \qquad 1 \le j \le \beta_1(\Omega, S_2).$$

As in Examples 12, 13 and 14 the extremals are constrained to be relative cocycles and when the periods on generators of relative homology groups are specified, the space of admissible variations of the extremal is the space of coboundaries These considerations follow from the identities

$$Z_c^1(\Omega - S_1) \simeq H_c^1(\Omega - S_1) \oplus B_c^1(\Omega - S_1)$$
$$Z_c^1(\Omega - S_2) \simeq H_c^1(\Omega - S_2) \oplus B_c^1(\Omega - S_2).$$

The relationship of these variational principles to the lumped parameters of resistance and conductance is the same as in Example 14 and hence will not be discussed

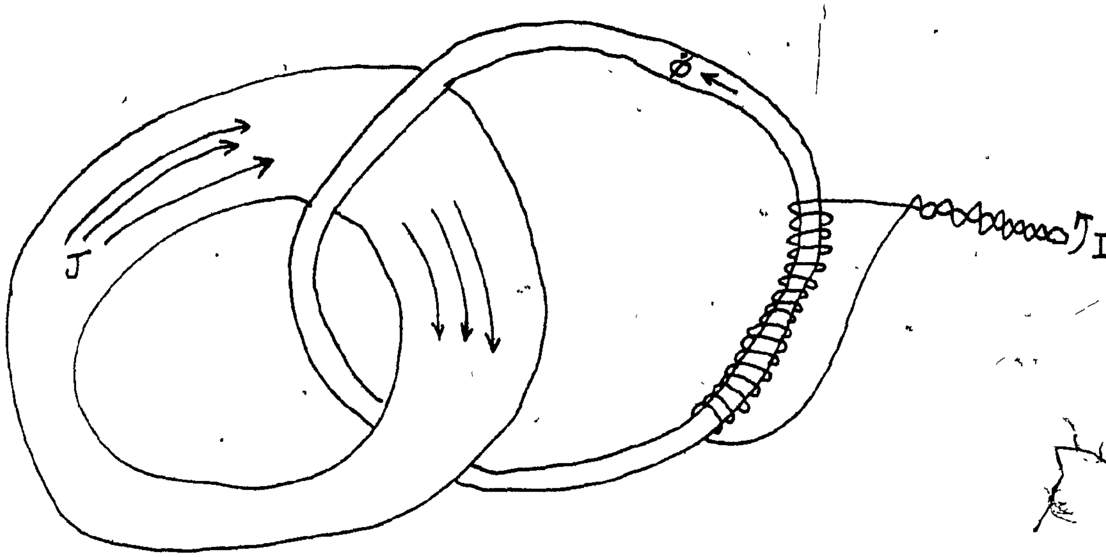


Fig. 17 $\dot{\mathbf{B}}$ is small outside of the magnetic circuit.

here. Instead, it is instructive to reformulate the above variational principles in terms of scalar potentials. By interpreting the c_i and the z_i as cuts, one can set

$$\mathbf{J} = \overline{\operatorname{curl}}\,\psi \qquad on\ \Omega - \bigcup_{i=1}^{\beta_1(\Omega,S_2)} z_i$$

$$\psi = 0 \qquad on\ S_1$$

and let $[\psi]_{z_i}$ denote the jump of ψ as z_i is traversed. Similarly one can set

$$\mathbf{E} = \operatorname{grad}\phi \qquad on\ \Omega - \bigcup_{i=1}^{\beta_1(\Omega,S_1)} c_i$$

$$\phi = 0 \qquad on\ S_2$$

where the jumps of ϕ on the c_i are denoted by $\phi|_{c_i}$. Note that it is quite natural to associate $\mathbf{J}$ with an element of $Z_c^1(\Omega - S_1)$ which has nonzero periods. These periods result naturally from a nonzero current being forced by time varying magnetic fields. On the other hand the assumption that the electric field is irrotational in the plate seems to preclude the possibility of the electric field having nonzero periods. The fact that this is not necessarily so is illustrated when the sheet links a magnetic circuit as shown in Fig. 17.

The periods of the cocycles in terms of the jumps in the corresponding potentials are easily calculated. Here the duality

$$H_1(\Omega, S_1) \simeq H_1(\Omega, S_2)$$

comes in nicely, for if

$$m_{ij} = \delta_{ij}$$

then

$$\begin{aligned} I_i &= \int_{c_i} (\mathbf{J} \times \mathbf{n}') \cdot d\mathbf{l} = \int_{c_i} \left((\overline{\operatorname{curl}}\, \psi) \times \mathbf{n}' \right) \cdot d\mathbf{l} = \int_{c_i} \operatorname{grad} \psi \cdot d\mathbf{l} \\ &= \sum_{i=1}^{\beta_1(\Omega, S_2)} m_{ij} [\psi]_{z_j} \qquad \text{since } \partial c_i \in C_0(S_1) \\ &= [\psi]_{z_i}, \qquad 1 \le i \le \beta_1(\Omega, S_1) \end{aligned}$$

and similarly

$$\begin{aligned} V_j &= \int_{z_j} \mathbf{E} \cdot d\mathbf{l} = \int_{z_j} \operatorname{grad} \phi \cdot d\mathbf{l} \\ &= \sum_{i=1}^{\beta_1(\Omega, S_1)} m_{ij} [\phi]_{c_i} \qquad \text{since } \partial z_i \in C_0(S_2) \\ &= [\phi]_{c_i}. \end{aligned}$$

Thus when the bases of $H_1(\Omega, S_1)$ and $H_1(\Omega, S_2)$ are arranged so that the intersection matrix is the unit matrix, the stationary content and cocontent principles can be restated as:

Stationary Content Principle ($\mathbf{J} = \overline{\operatorname{curl}}\, \psi$)

$$G(\overline{\operatorname{curl}}\, \psi) = \inf_{\psi} \int_{\Omega} \frac{|\overline{\operatorname{curl}}\, \psi|^2}{2\sigma d} dS$$

subject to

$$\psi = 0 \qquad on\ S_1$$

and the constraints which prescribe the periods of $\mathbf{J} \times \mathbf{n}'$ on generators of $H_1(\Omega, S_1)$:

$$I_i = [\psi]_{z_i} \qquad 1 \leq i \leq \beta_1(\Omega, S_1).$$

Stationary Cocontent Principle ($\mathbf{E} = \text{grad}\,\phi$)

$$G'(\text{grad}\,\phi) = \inf_{\phi} \int_{\Omega} \frac{\sigma d\,|\text{grad}\,\phi|^2}{2} dS$$

subject to $\phi = 0$ on S_2 and the constraints which prescribe the periods of $\mathbf{E}$ on generators of $H_1(\Omega, S_2)$:

$$V_i = [\phi]_{c_i}.$$

Thus, by playing down the role implicitly played by the long exact homology sequence, the role of duality theorems in handling topological aspects has become more apparent in this example.

End of Example 21

Example 22 (Magnetostatics with Current Sources)

The purpose of this example is to show another manifestation of duality theorems for orientable manifolds with boundary. Consider a magnetostatics problem in a compact region Ω where

$$\partial\Omega = S_1 \cup S_2 \qquad (S_1 \cap S_2 \text{ has no } 2-\text{dimensional area})$$

$$\text{div}\,\mathbf{B} = 0 \qquad in\ \Omega$$

$$\mathbf{B} \cdot \mathbf{n} = 0 \qquad on\ S_1$$

$$\text{curl}\,\mathbf{H} = \mathbf{J} \qquad in\ \Omega$$

$$\mathbf{H} \times \mathbf{n} = 0 \qquad on\ S_2.$$

As before S_1 is an the interface to a superconductor or a symmetry plane while S_2 is an interface to infinitely permeable bodies. As in Example 13, the magnetic flux density vector $\mathbf{B}$ can be associated with an element of $Z_c^2(\Omega - S_1)$ and in general it is not a relative coboundary. However, considering the long exact homology sequence for the pair (Ω, S_1) one has

$$H_2(\Omega, S_1) \simeq \delta_2^{-1}(\text{Kernel}(\tilde{\imath}_1)) \oplus \tilde{\jmath}_2\left(\frac{H_2(\Omega)}{\tilde{\imath}_2\,(H_2(S_1))}\right)$$

where the relevant portion of the long exact sequence is

$$\begin{array}{llllllll} \xrightarrow{\delta_3} & H_2(S_1) & \xrightarrow{\tilde{\imath}_2} & H_2(\Omega) & \xrightarrow{\tilde{\jmath}_2} & H_2(\Omega, S_1) & \longrightarrow \\ \xrightarrow{\delta_2} & H_1(S_1) & \xrightarrow{\tilde{\imath}_1} & H_1(\Omega) & \xrightarrow{\tilde{\jmath}_1} & \ldots & \end{array}$$

Since the periods of $\mathbf{B}$ on the basis of Image$(\tilde{\jmath}_2)$ correspond to distributions of magnetic poles in $\mathbb{R}^3 - \Omega$, it is natural to set these periods equal to zero. When this is done the periods of $\mathbf{B}$ on the generators of $H_2(\Omega)$ vanish and $\mathbf{B}$ can be written as

$$\mathbf{B} = \text{curl}\,\mathbf{A} \qquad in\ \Omega.$$

However, as in Example 13, one cannot insist that the components of the vector potential tangent to S_1 vanish unless the periods of $\mathbf{B}$ on the basis of $\delta_2^{-1}(\text{Kernel}(\tilde{\imath}_1))$ vanish. Since this is not true in general, one lets

$$\mathbf{n} \times \mathbf{A} = \overline{\text{curl}}\,\psi \qquad on\ S_1 - \bigcup_{i=1}^{\beta_1(S_1, \partial S_1)} d_i$$

as in Example 13, where $d_i, 1 \leq i \leq \beta_1(S_1, \partial S_1)$ form as basis for $H_1(S_1, \partial S_1)$. As in Example 13, one can express the periods of $\mathbf{B}$ on the basis of $\delta_2^{-1}(\text{Kernel}(\tilde{\imath}_1))$ by prescribing the jumps of ψ on the d_i.

By assuming a constitutive relation

$$\mathbf{H} = \mathcal{H}(\mathbf{B}, \mathbf{r})$$

which is defined in Example 13, one can rewrite the variational formultation for magnetostatics as follows (see Kotiuga [1982] Section 1.21).

Variational Principle ($\mathbf{B}=\text{curl}\,\mathbf{A}$)

$$F(\mathbf{A}) = \text{ext}_{\mathbf{A}} \int_{\Omega} \int^{\text{curl}\,\mathbf{A}} \mathcal{H}(\xi, \mathbf{r}) \cdot d\xi - \mathbf{A} \cdot \mathbf{J} dV$$

subject to the principal boundary condition

$$\mathbf{n} \times \mathbf{A} = \overline{\text{curl}}\,\psi \qquad on\ S_1 - \bigcup_{i=1}^{\beta_1(S_1, \partial S_1)} d_i$$

where

$$[\psi|_{d_i}, \qquad 1 \leq i \leq \beta_1(S_1, \partial S_1)$$

are chosen so that the periods of $\mathbf{B}$ on $\delta_2^{-1}(\text{Kernel}\,(\tilde{i}_1))$ have their desired values, and ψ is otherwise arbitrarily chosen.

The above functional has a nonunique extremal (whenever an extremal exists) and, as in the energy formulation of Example 13, this nonuniqueness corresponds to an element of $Z_c^1(\Omega - S_1)$. That is, if $\mathbf{A}$ and $\mathbf{A}'$ correspond to two vector potentials which give the functional its stationary value, then

$$\mathbf{A} - \mathbf{A}' \in Z_c^1(\Omega - S_1).$$

As noted in Example 13, one can write

$$Z_c^1(\Omega - S_1) \simeq H_c^1(\Omega - S_1) \oplus B_c^1(\Omega - S_1)$$

and the nonuniqueness of $\mathbf{A}$ can be eliminated by specifying the periods of $\mathbf{A}$ on generators of $H_1(\Omega, S_1)$ as well as the divergence of $\mathbf{A}$ in Ω and the normal component of $\mathbf{A}$ on S_2. This can be done by either making these conditions principal conditions on the above functional or, as in Kotiuga [1982], Chapter 5, by constucting another variational formulation for which these conditions are a consequence of extremising the functional. The question of alternate variational formulations for this problem is taken up in full generality in Chapter 3.

At this point one can expose the interplay between the nonuniqueness of **A** (the gauge transformation) and the conditions on the solvability of the associated boundary value problem (the conditions for the functional to have an extremum). As was noted in Kotiuga [1982] certain convexity conditions on the constitutive relation are sufficient to ensure that questions of solvability can be answered by a "Fredholm alternative" type of argument which implies that the problems has a solution if and only if

$$0 = F(\mathbf{A}) - F(\mathbf{A}') = \int_\Omega (\mathbf{A} - \mathbf{A}') \cdot \mathbf{J}\, dV \quad \text{for all } \mathbf{A} - \mathbf{A}' \in Z_c^1(\Omega - S_1).$$

By brute force calculation (see Kotiuga [1982] Chapter 4 and in particular Theorem 4.3) the above orthogonality condition can be restated entirely in terms of the current density vector **J**. In the present case of homogeneous boundary conditions on S_2, the conditions on the current density vector **J** can be restated as follows. If $\Sigma_i, 1 \leq i \leq \beta_2(\Omega, S_2)$ is a set of generators of $H_2(\Omega, S_2)$ then the conditions for the solvability of the equations for the extremum of the functional are:

locally:

$$\operatorname{div} \mathbf{J} = 0 \qquad \text{in } \Omega$$

$$\mathbf{J} \cdot \mathbf{n} \qquad \text{continuous across interfaces}$$

$$\mathbf{J} \cdot \mathbf{n} = 0 \qquad \text{on } S_2;$$

globally:

$$\int_{\Sigma_i} \mathbf{J} \cdot \mathbf{n}\, dS = 0 \qquad 1 \leq i \leq \beta_2(\Omega, S_2),$$

where the global constraints are verified by using the three step recipe and the long exact homology sequence for the pair (Ω, S_2) in the usual way. The local conditions in this set of solvability conditions merely state that **J** can be associated with an element of $Z_c^2(\Omega - S_2)$ while the global conditions ensure that the projection of this cocycle into $H_c^2(\Omega - S_2)$ is zero. Thus the solvability conditions merely state that **J** can be associated with a relative coboundary in $B_c^2(\Omega - S_2)$. This, however, is exactly what one requires of **J** in order to write

$$\mathbf{J} = \operatorname{curl} \mathbf{H}. \qquad in\ \Omega$$

with

$$\mathbf{n} \times \mathbf{H} = 0 \qquad on\ S_2.$$

Returning to the duality theorems, in Example 13 the duality between lumped variables was expressed by

$$H_2(\Omega, S_1) \simeq H_{3-2}(\Omega, S_2) = H_1(\Omega, S_2).$$

Here, in constrast, when sources are added the duality

$$H_1(\Omega, S_1) \simeq H_{3-1}(\Omega, S_2) = H_2(\Omega, S_2)$$

expresses a duality between global degrees of freedom in the gauge transformation and compatibility conditions on the prescription of the current density vector **J**.

One final remark is in order. The global ambiguity of the gauge of the vector potential is associated with unspecified fluxes through "handles" of Ω or unspecified time integrals of electromotive forces between connected components of S_1, while the compatibility conditions on **J** ensure that Ampère's law can be used without contradiction.

The intersection matrix for the generators of the groups $H_1(\Omega, S_1)$ and $H_2(\Omega, S_2)$ can be used to help see how a magnetostatics problem is improperly posed by checking to see which degrees of freedom in the gauge transformation do not leave the value of the functional invariant. To the best of the author's knowledge this idea was first exploited, in a rather pedestrian way, in Kotiuga [1982], Chapter 5.

End of Example 22

Finally, the Alexander Duality Theorem will be considered. Although the Alexander Duality Theorem theorem is not considered as visual or easy as the Lefschetz or Poincaré dualities, various special cases of this general theorem were known to Maxwell. In its most general form the Alexander Duality Theorem states that for an n-dimensional manifold M and a closed subset Ω,

$$H^p(\Omega) = H_{n-p}(M, M - \Omega).$$

There is a question of limits which has been ignored here (see Greenberg and Harper [1981], p 233 for an exact statement). Skipping over the details of limits is justified since one can safely say that the exceptions can truly be considered pathological (see Massey [1980] Chapter 9, Sect. 6 for an example). For most applications M is taken to be $\mathbb{R}^3$ in which case the theorem says

$$H^p(\Omega) = H_{n-p}(\mathbb{R}^3, \mathbb{R}^3 - \Omega).$$

There is a classical version of the Alexander Duality theorem which can be obtained as a corollary of the above one by the following simplified argument (see Greenberg and Harper [1981] Sect. 27.9 for a proof which follows on similar lines). Consider the long exact homology sequence for the pair $(\mathbb{R}^3, \mathbb{R}^3 - \Omega)$.

$$\begin{array}{cccccccc}
0 & \longrightarrow & H_3(\mathbb{R}^3 - \Omega) & \xrightarrow{\tilde{\imath}_3} & H_3(\mathbb{R}^3) & \xrightarrow{\tilde{\jmath}_3} & H_3(\mathbb{R}^3, \mathbb{R}^3 - \Omega) & \rightarrow \\
 & \xrightarrow{\delta_3} & H_2(\mathbb{R}^3 - \Omega) & \xrightarrow{\tilde{\imath}_2} & H_2(\mathbb{R}^3) & \xrightarrow{\tilde{\jmath}_2} & H_2(\mathbb{R}^3, \mathbb{R}^3 - \Omega) & \rightarrow \\
 & \xrightarrow{\delta_2} & H_1(\mathbb{R}^3 - \Omega) & \xrightarrow{\tilde{\imath}_1} & H_1(\mathbb{R}^3) & \xrightarrow{\tilde{\jmath}_1} & H_1(\mathbb{R}^3, \mathbb{R}^3 - \Omega) & \rightarrow \\
 & \xrightarrow{\delta_1} & H_0(\mathbb{R}^3 - \Omega) & \xrightarrow{\tilde{\imath}_0} & H_0(\mathbb{R}^3) & \xrightarrow{\tilde{\jmath}_0} & H_0(\mathbb{R}^3, \mathbb{R}^3 - \Omega) & \rightarrow 0.
\end{array}$$

Since

$$H_p(\mathbb{R}^3) \simeq \begin{cases} 0, & \text{if } p \neq 0; \\ \mathbb{R}, & \text{if } p = 0, \end{cases}$$

then if $\Omega \neq \phi$ the long exact sequence tells us that

$$H_{3-p}(\mathbb{R}^3, \mathbb{R}^3 - \Omega) \simeq H_{2-p}(\mathbb{R}^3 - \Omega) \qquad if\ p \neq 2$$

$$\mathbb{R} \oplus H_{3-p}(\mathbb{R}^3, \mathbb{R}^3 - \Omega) \simeq H_{2-p}(\mathbb{R}^3 - \Omega) \qquad if\ p = 2.$$

Combining this with the Alexander Duality theorem yields:

$$H^p(\Omega) \simeq H_{2-p}(\mathbb{R}^3 - \Omega) \qquad if\ p \neq 2$$

$$\mathbb{R} \oplus H^2(\Omega) \simeq H_0(\mathbb{R}^3 - \Omega)$$

or

$$\beta_p(\Omega) = \beta_{2-p}(\mathbb{R}^3 - \Omega) \qquad if\ p \neq 2$$

$$1 + \beta_2(\Omega) = \beta_0(\mathbb{R}^3 - \Omega).$$

These are the classical versions of the Alexander Duality Theorem. The case where $p = 1$ was known to Maxwell [1981], Section 18 in the following form:

"The space outside the region has the same cyclomatic number as the region itself."

and

"The cyclomatic number of a closed surface is twice that of either of the regions it bounds."

The reader may turn back to Example 6 to see how the classical version of the Alexander Duality Theorm was used in the case where $p = 2$, and Example 8 for the case $p = 1$. In general, the classical version of the Alexander duality is very useful when one wants to figure out how global aspects of gauge transformations, solvability conditions or potential formulations for a problem defined in a region Ω are a result of sources in $\mathbb{R}^3 - \Omega$.

In summary, there are three types of duality theorems which are invaluable when considering electromagnetic boundary value problems in complicated domains. They are

1. Lefschetz Duality Theorem (Ω n-dimensional)

$$H_p(\Omega) \simeq H_{n-p}(\Omega, \partial\Omega).$$

2. When Ω is n-dimensional and $\partial\Omega = S_1 \cup S_2$ where S_1 and S_2 are two regions whose intersection does not have any "$n-1$-dimensional volume" and are associated with dual boundary conditions on symmetry planes and interfaces then

$$H_p(\Omega, S_1) \simeq H_{n-p}(\Omega, S_2).$$

3. Alexander Duality Theorem

$$H_p(\Omega) \simeq H_{2-p}(\mathbb{R}^3 - \Omega) \qquad p \neq 2$$

$$\mathbb{R} \oplus H_2(\Omega) \simeq H_0(\mathbb{R}^3 - \Omega).$$

It is also important to remember that the first two duality theorems can be interpreted in terms of an intersection matrix. The classical version of Alexander Duality can be interpreted through the notion of a linking number (see Flanders [1963] section 6.4).

1.11 Outline of Thesis

The ultimate aim of this thesis is to develop orthogonal decompositions of differential forms which enable a variety of questions associated with variational boundary value problems of electromagnetics to be formulated and answered in a general way. In order to derive such orthogonal decompostions, rudimentary concepts from homology theory and the calculus of differential forms are required. Since it is not safe to assume that these mathematical concepts are standard parts of an electrical engineer's tool kit, the author has developed these subjects in Chapters 1 and 2 so that the main results of the thesis could be presented succinctly in Chapter 3.

The first chapter of this thesis may be regarded as an attempt to present certain key results from homology theory in the context of electromagnetics and to develop techniques for handling topological aspects of boundary value problems. To this end, Sections 1.2–1.5 present the process of integration as relating global information involving vector fields over a region (cohomology groups) to the "topology" of the region (homology groups). The interrelationship between homology and cohomology (the de Rham isomorphism) is presented in Section 1.4 and illustrated in Section 1.5 by means of several examples. Having established the de Rham isomorphism, Sections 1.6–1.8 present certain algebraic techniques which are required to handle boundary conditions. These algebraic concepts are then illustrated in Section 1.9 by means of several lengthy examples. More specifically, Section 1.6 introduces the notion of a complex which forms the backdrop for homology theory and Section 1.7 goes on to develop the notion of a relative homology group which is the key to formalising intuitions about boundary conditions. Section 1.8 presents the long exact homology sequence in order to develop a facility for deducing the structure of relative homology groups of a region Ω modulo a subset S in terms of absolute homology groups of Ω and S. The notion of relative cohomology and the relative de Rham isomorphism is developed in Section 1.9 and illustrated by several lengthy examples which demonstrate the role of relative homology groups in familiar boundary value problems. Finally, after the detailed analysis of the boundary value problems of Section 1.9, commonly used duality theorems for orientable manifolds with boundary are presented in Section 1.10 and their relevance in electromagnetic theory is illustrated by additional examples. The duality theorems are interpreted in terms of the matrices of intersection numbers between generators of two "dual" homology groups.

The first chapter of this thesis has been written in a heuristic manner which avoids the formalism of differential forms. The second chapter complements this point of view by developing the formalism of differential forms and exterior algebra, reorganising the material found in the first chapter, and going on to develop orthogonal decompositions in a Hilbert space setting. Specifically, Sections 2.2–2.4 develop the notions of manifold, differential form, and exterior algebra while Sections 2.5–2.8 present the cochain complex associated with differential forms and exterior derivative on a manifold, and the resulting cohomology of the cochain complex. Having this basic material, in Section 2.9, Stokes' theorem is presented, an integration by parts formula is derived, and much of the material considered in Chapter 1 is reconsidered. Specifically, the "five lemma" is used to derive a relative de Rham isomorphism from the de Rham isomorphism for absolute homology groups, and duality theorems are considered in the context of a bilinear pairing between cohomology groups which is induced by integration over the whole manifold. Starting in Section 2.10 a Hilbert space formalism is developed so that orthogonal decompositions can be derived. In Section 2.10 the Hodge star operator is introduced so that an inner product can be defined. Given an inner product on p-forms, Section 2.11 goes on to find an operator adjoint to the exterior derivative. This sets the stage for orthogonal decompositions. In Section 2.12, the Hodge decomposition is discussed and in Section 2.13 an orthogonal decomposition of p-forms on a manifold with boundary is derived and a proof of the duality theorems based on the Hodge star operator is given. It is seen that the orthogonal decomposition of p-forms on a manifold with boundary is a consequence of the complex associated with differential forms and a basic property concerning linear operators and their adjoints.

Having developed the relevant aspects of homology theory and derived an orthogonal decomposition theorem for differential forms which makes explicit the role played by homology groups, the third chapter goes on to formulate and investigate a paradigm situation encountered in classical electromagnetic theory. In Section 3.1, this paradigm variational boundary value problem is introduced and various familiar instances are listed. In Section 3.2 the conditions which must be imposed on the problem in order for a potential to exist are investigated and when such a potential exists, a variational formulation of the problem is given. The deep interrelationship between gauge transformations and compatibility conditions (conservation laws) is given in Section 3.3. In particular, it is seen how the duality theorems of homology theory give insight into the global interrelationship between gauge transformations and conservation laws. In Section 3.4 modified variational principles are considered for which there is always a unique extremal from which information concerning the solvability of the original problem can be deduced. More precisely, whenever the original problem is physically meaningful the unique extremal of the modified variational formulation corresponds to a solution of the original problem and the projection of the extremal into the space of "gauge transformations" vanishes. Alternatively, whenever the original problem is not posed in a physically meaningful way, the unique extremal of the modified variational formulation is a solution to the "nearest" physically meaningfull problem and the projection of the extremal into the space of "gauge transformations" can be used to obtain an a posteriori measure of how the problem was unphysical. Finally, the thesis culminates in Section 3.5 where the Tonti diagram for the paradigm problem is considered along with the complementary variational principle for the paradigm problem. The main observation made is that the Tonti diagram is essentially two differential complexes, one complex involving the operators adjoint to the operators of the other complex, and negelecting the (co)homology groups of these complexes may lead to false conclusions when using the Tonti diagram. In particular, the roles of the six different cohomology groups and three duality isomorphisms are explained in the context of defining potentials, investigating gauge invariance and determining compatibility conditions for both the original and complementary problems. The thesis ends with an outline for future work and a list of original contributions.

CHAPTER 2

A Short Course in Differential Forms

"The assemblage of points on a surface is a two manifoldness; the assemblage of points in tri-dimensional space is a three fold manifoldness; the values of a continuous function of n arguments an n-fold manifoldness."

G. Crystal,
Encyclopedia Britannica, 1892.

"The committee which was set up in Rome for the unification of vector notation did not have the slightest success, which was only to have been expected."

F. Klein,
Elementary Mathematics from an Advanced Standpoint, 1925.

"In the year 1844 two remarkable events occured, the publication by Hamilton [1844] of his discovery of quarternions, and the publication by Grassmann [1844] of his "Ausdehnungslehre". With the advantage of hindsight we can see that Grassmann's was the greater contribution to mathematics, containing the germ of many of the concepts of modern algebra, and including vector analysis as a special case. However, Grassmann was an obscure high-school teacher in Stettin, while Hamilton was the world famous mathematician whose official titles occupy six lines of print after his name at the beginning of his 1844 paper. So it is regrettable but not suprising, that quarternions were hailed as a great discovery while Grassmann had to wait 23 years before his work received any recognition at all from professional mathematicians."

F.J. Dyson,
Missed Opportunities,
*Bull.AMS.***78***, Sept.*(1972)*p.*644

2.1 Introduction

The systematic use of differential forms in electromagnetic theory started with the truly remarkable paper of Hargraves [1908] in which the space-time covariant form of

Maxwell's equations was deduced. Despite the efforts of great engineers such as Gabriel Kron (see the book by Balasubramanian et al. [1970] for a bibliography) the use of differential forms in electrical engineering is, unfortunately, still quite rare. The reader is referred to the paper by Deschamps [1981] for an introductory view of the subject.

The purpose of this chapter is to summarise the properties of differential forms which are needed to facilitate the presentation of the material in the next chapter. A detailed development is, of course, impossible. There are two reasons for this. Firstly, the body of results concerning applications of differential forms in mathematical physics is so much greater than what is required to understand this thesis that the uninitiated reader would not appreciate reading more than the absolute minimum. Secondly, there is little excuse to take up space with material which is so easily found elsewhere and presented from so many points of view. For example the book by Balasubramanian et al. [1970] does a marvellous job of presenting most of the topics in this chapter from the point of view of the numerical analyst interested in network models for Maxwell's equations. There are also the "Advanced Calculus" type of books by Loomis and Sternberg [1968], Flemming [1965], Lichnerowicz [1967]. Cartan [1967] and Spivak [1965]. The books by Flanders [1963] Bishop and Goldberg [1968], Schutz [1980], Arnold [1974], Abraham et al. [1983] and von Westenholz [1978] are aimed at the physics student. Alternatively. Hodge [1952] and Slebodzinski [1970] are excellent books which avoid all modern notation while de Rham [1955] is the classic work.

There are several books which the author found particularly invaluable. These are Warner [1971] Chapters 4 and 6 for a proof of Stokes' theorem and the Hodge decomposition for a manifold without boundary, Spivak [1979] Chapters 8 and 11 for integration theory and cohomology theory in terms of differential forms, Bott and Tu [1982] for a quick route into cohomology and Yano [1970] for results concerning manifolds with boundary. Finally, the papers by Duff, Spencer, Connor, and Fiedrichs (see bibliography) were always useful for basic intuitions about orthogonal decompositions on manifolds with boundary.

Though this chapter may seem more formal than the previous one, the basic intuitions about differential forms come from integration which was considered in the previous chapter. What remains to be developed is a systematic way of manipulating differential forms which involves only basic linear algebra and partial differentiation. Once the basic operations on differential forms have been defined, all of the properties of cohomology groups appear as in the previous chapter.

2.2 Differentiable Manifolds

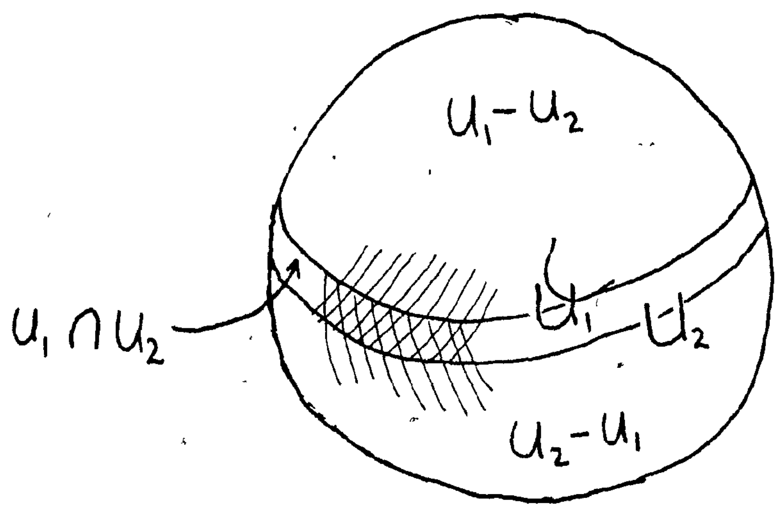


Fig. 18

In order to talk about differential forms, it is important first to have an aquaintance with the notion of a differentiable manifold. Roughly speaking, a differentiable manifold of dimension n can be described locally by n coordinates, that is, given any point p in an n-dimensional differentiable manifold M, one can find a neighborhood U of p which is homeomorphic to a subset of $\mathbb{R}^n$. More accurately, the 1-1 continuous mapping φ which takes U into a subset of $\mathbb{R}^n$ is differentiable a specified number of times. The reason why one is required to work in terms of open sets and not the whole manifold is because the simplest of n-dimensional manifolds are not homeomorphic to any subset of $\mathbb{R}^n$. Consider, for example, the two dimensional sphere shown in Fig. 18 which requires at least two such open sets to cover it.

More formally, in order to describe an "n-dimensional differentiable manifold M of class C^k" one forms what is called an atlas. An atlas $\mathcal{A}$ is a collection of pairs (U_i, φ_i) called charts where U_i is an open set of M and φ_i is a 1-1 bijective map, differentiable of class C^k, mapping U_i into an open set of $\mathbb{R}^n$. In addition the charts in the atlas are assumed to satisfy:

1) $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is a differentiable function of class C^k whenever $(U_i, \varphi_i), (U_j, \varphi_j) \in \mathcal{A}$ (see Fig. 19). The functions $\varphi_i \circ \varphi_j^{-1}$ are called transition functions.

2) $\bigcup U_i = M$.

Thus, referring back to the example of the sphere we see that it is a 2-dimensional differentiable manifold of class C^∞ which can be described by an atlas which contains two charts. The actual definition of a differentiable manifold involves not only an atlas but an equivalence class of atlases where if for a manifold M, $\mathcal{A}$ and $\mathcal{B}$ are atlases, then $\mathcal{A} \cup \mathcal{B}$ is also an atlas. That is if

$$(U_i, \varphi_i) \in \mathcal{A}$$

$$(W_j, \psi_j) \in \mathcal{B}$$

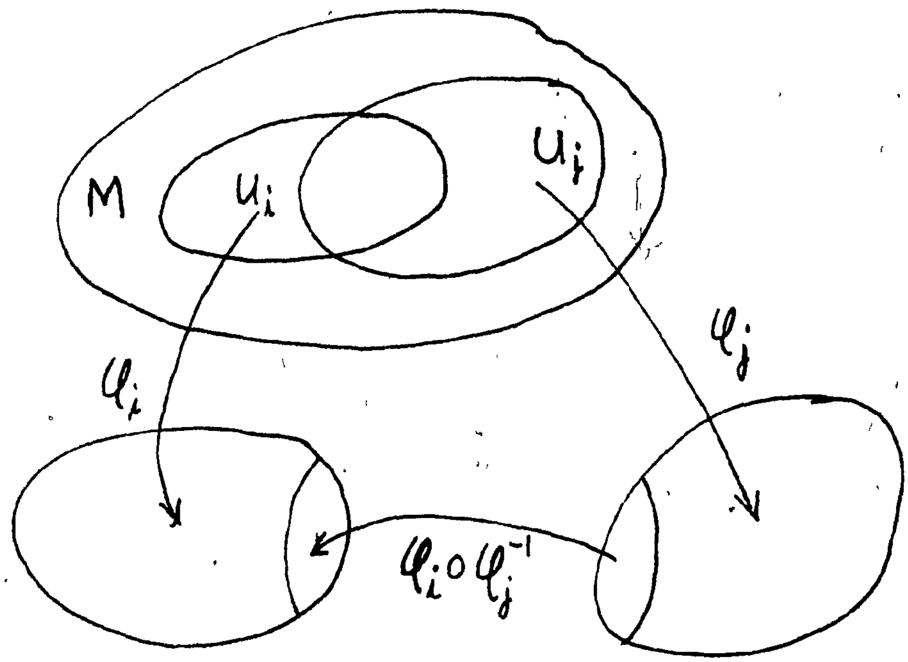


Fig. 19

then

$$\varphi_i \circ \psi_j^{-1} : \psi_j(U_i \cap W_j) \to \varphi_i(U_i \cap W_j)$$

is a continuous map which is just as smooth as φ_i or ψ_j. Thus a set M together with an equivalence class of atlases is called a differentiable manifold.

The local nature of the definition of a manifold is essential if one is not to constrain the global topology of the manifold. A fundamental property of differentiable manifolds is paracompactness which enables one to construct partitions of unity (see pages 5–10 in Warner [1971] for a decent explanation of what all this means). The existence of a partition of unity is what is required to specify smoothly a geometrical object such as a vector field, differential form or Riemannian structure globally on a differentiable manifold by specifying the geometrical object locally in terms of the coordinate charts. Throughout this chapter it will be assumed that such geometrical objects are defined globally and most computations will be performed in local coordinate charts without regard to how the charts fit together globally. Another almost immediate consequence of the definition of a manifold is that once a notion of distance (Riemannian structure) is defined, the cohomology of the manifold is easily computed in terms of differential forms (see Bott [1982] §5). Holding off on the questions of homology, cohomology etc. the exposition will now concentrate on the algebraic properties of differential forms.

2.3 Tangent Vectors and the Dual Space of 1-forms

Suppose that in a neighborhood of a point p in an n-dimensional manifold M there are local coordinates

$$x^i, \qquad 1 \leq i \leq n$$

The tangent space M_p at the point $p \in M$ is defined to be the linear span of all linear first order differential operators. That is, if $X \in M_p$ then X can be represented as

$$X = \sum_{i=1}^{n} X^i(p) \frac{\partial}{\partial x^i}.$$

It is obvious that

$$\frac{\partial}{\partial x^i} \qquad 1 \leq i \leq n$$

form a basis for M_p. The interpretation of the tangent space is obtained by considering

$$X(f)\Big|_p = \sum_{i=1}^{n} X^i(p) \frac{\partial f}{\partial x^i}\Big|_p .$$

Thus it is seen that the tangent vectors can be interpreted as giving directional derivatives of functions. The collection of all tangent spaces to a manifold is called the tangent bundle and is denoted by $T(M)$. Thus

$$T(M) = \bigcup_{p \in M} M_p.$$

A vector field on M is defined to be a smooth section of the tangent bundle, that is, if one writes a vector field X in terms of local coordinates in a neighborhood of p then

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$$

where the X^i are smooth functions of the local coordinates.

Since a vector space has been defined it is natural to inquire about its dual space. An element of the dual space to M_p is a first order differential form (or 1-form) evaluated at p. Such a beast looks like

$$\omega = \sum_{i=1}^{n} a_i(p) dx^i.$$

The dual space to the tangent space M will be denoted by M_p^*. In this scheme one identifies

$$dx^i, \qquad 1 \leq i \leq n$$

as a basis for M_p^* and the dual basis to

$$\frac{\partial}{\partial x^i} \qquad 1 \leq i \leq n.$$

Thus

$$dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta^i_j$$

and

$$\begin{aligned} \omega(X)\Big|_p &= \sum_{i=1}^{n} a_i(p) dx^i \left(\sum_{j=1}^{n} X^j(p) \frac{\partial}{\partial x^j} \right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i(p) X^j(p) dx^i \left(\frac{\partial}{\partial x^j} \right) \\ &= \sum_{i=1}^{n} a_i(p) X^i(p). \end{aligned}$$

Having defined the dual space to vectors as differential forms, one can also define the cotangent bundle to the manifold M as

$$T^*(M) = \bigcup_{p \in M} M_p^*.$$

In order to verify that $\omega(X)$ is really an invariant quantity, it is essential to know how ω and X behave under coordinate transformations. Suppose φ is a mapping between an m-dimensional manifold M' and an n-dimensional manifold M:

$$\varphi : M' \to M.$$

Choose points $p' \in M'$, $p \in M$ such that $\varphi(p') = p$. What is desired is the form of the induced transformations:

$$\varphi^{\#} : T^*(M) \to T^*(M')$$
$$\varphi_{\#} : T(M') \to T(M)$$

on 1-forms (called a pull back) and vector fields respectively, which have the following property:

If

$$\omega \in T^*(M), \qquad X \in T(M')$$

then

$$(\varphi^{\#}\omega)(X)\big|_{p'} = \omega(\varphi_{\#}X)\Big|_{p}.$$

Let $(y^1, \ldots, y^m), (x^1, \ldots, x^n)$ be local coordinates around $p' \in M'$ and $p \in M$ respectively. In terms of these local coordinates there is a functional relationship

$$x^i = x^i(y^1, \ldots, y^n), \qquad 1 \le i \le n$$

and the induced transformations $\varphi^{\#}, \varphi_{\#}$ transform basis vectors according to the rule

$$dx^j = \sum_{i=1}^{m} \frac{\partial x^j}{\partial y^i} dy^i$$

$$\frac{\partial}{\partial y^j} = \sum_{i=1}^{n} \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i},$$

Hence, if

$$\omega = \sum_{i=1}^{n} a_i(p) dx^i$$

$$X = \sum_{j=1}^{m} X^j \left(\varphi(p')\right) \frac{\partial}{\partial y^j}$$

then

$$\varphi^{\#}(\omega) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_i \left(\varphi(p')\right) \frac{\partial x^i}{\partial y^j} \right) dy^j$$

$$\varphi_{\#}(X) = \sum_{i=1}^{n} \left(\sum_{i=1}^{m} X^j(p) \frac{\partial x^i}{\partial y^j} \right) \frac{\partial}{\partial x^i}$$

and

$$\begin{aligned} (\varphi^{\#}\omega)(X)\Big|_{p'} &= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i \left(\varphi(p')\right) \frac{\partial x^i}{\partial y^j} X^j \left(\varphi(p')\right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i(p) \frac{\partial x^i}{\partial y^j} X^j(p) \\ &= \omega\left(\varphi_{\#} X\right)\Big|_p \end{aligned}$$

which is the desired transformation. Note that the transformation rule for the basis vectors

$$dx^i, \quad 1 \le i \le n, \qquad \frac{\partial}{\partial y^j}, \quad 1 \le j \le m$$

make sense if these basis vectors are regarded as infinitesimals and partial derivatives, and the usual rules of calculus are used. However the reader should avoid making any interpretation of the symbol d untill the exterior derivative is defined. That is,

$$d(\text{something})$$

should not be interpreted as an infinitesimal.

One more remark is in order. Suppose M'' is another manifold and there is a transformation

$$\psi : M'' \to M'$$

then there is a composite transformation $\varphi \circ \psi$ which makes the following diagram commutative

$$\begin{array}{ccc} M'' & \xrightarrow{\psi} & M' \\ & {\scriptstyle \varphi\circ\psi}\searrow & \downarrow{\scriptstyle \varphi} \\ & & M \end{array}$$

and the chain rule for partial derivatives shows that the induced transformations on vector fields and 1-forms make the following diagrams commutative

$$\begin{array}{ccc} T(M'') & \xrightarrow{\psi_\#} & T(M') \\ & {\scriptstyle (\varphi\psi)_\#}\searrow & \downarrow{\scriptstyle \varphi_\#} \\ & & T(M) \end{array} \qquad \begin{array}{ccc} T^*(M'') & \xleftarrow{\psi^\#} & T^*(M') \\ & {\scriptstyle (\varphi\psi)^\#}\nwarrow & \uparrow{\scriptstyle \varphi^\#} \\ & & T^*(M) \end{array}.$$

Hence

$$(\varphi\psi)_\# = \varphi_\#\psi_\#$$

$$(\varphi\psi)^\# = \psi^\#\varphi^\#.$$

Thus vector fields transform covariantly while 1-forms transform contravariantly and the whole scheme is invariant under transformations.

2.4 Higher Order Differential Forms and Exterior Algebra

The identification of 1-forms at a point p as elements of the dual space of M_p enables one to regard a differential 1-form at a point p as a linear functional on the tangent space M_p. Higher order k-forms are a generalisation of this idea. At a point $p \in M$, a differential k-form is defined to be an alternating k-linear functional on the tangent space M_p. That is if ω is a k-form then

$$\omega\Big|_p : \underbrace{M_p \times M_p \times \ldots \times M_p}_{k\ times} \to \mathbb{R}$$

which is linear in each argument and satisfies the following. If

$$X_1, X_2, \ldots, X_k \in M_p$$

then for any permutation π of k integers $(1, \ldots, k)$

$$\omega\left(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(k)}\right) = \text{sgn}(\pi)\omega(X_1, X_2, \ldots, X_k)$$

where

$$\text{sgn}(\pi) = \begin{cases} 1, & \text{if } \pi \text{ is an even permutation,} \\ -1, & \text{if } \pi \text{ is an odd permutation.} \end{cases}$$

Thus, the set of k-forms at a point p form a vector space. It is denoted by

$$\Lambda_k(M_p^*).$$

In addition, the following definitions are made:

$$\Lambda_k(M_p^*) = 0, \quad k < 0$$

$$\Lambda_1(M_p^*) = M_p^*$$

$$\Lambda_0(M_p^*) = \text{values of functions evaluated at } p.$$

When thinking of alternating multilinear mappings, it is useful to remember the *Alternation Mapping* which sends any multilinear mapping into an alternating one:

$$\text{Alt}\left(T(X_1, \ldots X_k)\right) = \sum_{\pi \in S_k} \frac{\text{sgn}(\pi)}{k!} T\left(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(k)}\right)$$

where S_k is the group of permutations of k objects. The alternation mapping has the exact same properties as the determinant function and from this fact one can deduce:

$$\Lambda_k(M_p^*) = 0 \qquad if\ k > n.$$

The exterior algebra of M_p^* is defined as

$$\Lambda(M_p^*) = \bigoplus_{k=0}^{n} \Lambda_k(M_p^*).$$

By forming

$$\Lambda_k^*(M) = \bigcup_{p\in M} \Lambda_k(M_p^*)$$

and considering the k-forms whose coefficients are differentiable functions of coordinates, one has the exterior k-bundle of the manifold M. The set of all differential forms on a manifold M form the exterior algebra bundle of M which is defined as

$$\Lambda^*(M) = \bigcup_{p\in M} \Lambda(M_p^*) = \bigoplus_{k=0}^{n} \Lambda_k^*(M).$$

In this swarm of definitions the reader has been short-changed. The word *exterior algebra* has been used several times without any mention of what this algebra is. There is a product

$$\wedge : \Lambda^*(M) \times \Lambda^*(M) \to \Lambda^*(M)$$

which is called the exterior (wedge, Grassmann) product which takes a k-form and an l-form and gives a $k+l$-form according to the following rule. If $\omega \in \Lambda_k^*(M), \eta \in \Lambda_l^*(M)$, and $(X_1, X_2, \ldots, X_{k+l}) \in T(M)$ then

$$(\omega \wedge \eta)(X_1, X_2, \ldots, X_{k-l})$$
$$= \frac{1}{(k+l)!} \sum_{\pi\in S_{k-l}} \operatorname{sgn}(\pi)\omega\left(X_{\pi(1)}, \ldots, X_{\pi(k)}\right)\eta\left(X_{\pi(k+l)}, \ldots, X_{\pi(k+l)}\right).$$

The above definition of wedge multiplication is not very useful for explicit calculations; its usefulness is like that of the formal definition of a determinant. For practical computations it is important to remember that wedge multiplication is:

1. Bilinear:

$$\left.\begin{aligned}\omega \wedge (a_1\eta_1 + a_2\eta_2) &= a_1(\omega \wedge \eta_1) + a_2(\omega \wedge \eta_2)\\ (a_1\eta_1 + a_2\eta_2) \wedge \omega &= a_1(\eta_1 \wedge \omega) + a_2(\eta_2 \wedge \omega)\end{aligned}\right\} \quad \text{for } a_1, a_2 \in \mathbb{R}.$$

2. Associative:

$$(\lambda \wedge \mu) \wedge \eta = \lambda \wedge (\mu \wedge \eta).$$

3. Graded commutative:

$$\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega \qquad \text{for } \omega \in \Lambda_k^*(M), \eta \in \Lambda_l^*(M).$$

Before considering some examples of wedge multiplication it is worth considering what differential forms look like at a point $p \in M$ where $(x^1, x^2, \ldots, x^n)$ are local coordinates. Let

$$dx^i, \qquad 1 \le i \le n$$

be a basis for M_p^*. By taking repeated wedge products in all possible ways, $\Lambda_k(M_p^*)$ is seen to be spanned by expressions of the form

$$dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}.$$

Furthermore for $\pi \in S_k$ one has

$$dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} = \operatorname{sgn}(\pi) dx^{i_{\pi(1)}} \wedge \ldots \wedge dx^{i_{\pi(k)}}$$

and since $dx^i, 1 \le i \le n$ span $\Lambda_1(M_p^*)$, one sees that $\Lambda_k(M_p^*)$ has a basis of the form

$$dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}, \qquad 1 \le i_1 < \ldots < i_k \le n.$$

Therefore, for $k > 0, \omega \in \Lambda_k(M_p^*)$ looks like:

$$\omega = \sum_{i_1 < i_2 < \ldots < i_k} a_{i_1, i_2, \ldots, i_k}(p) dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

and

$$\dim \Lambda_k(M_p^*) = \frac{n!}{(n-k)!k!} = \binom{n}{k}.$$

By the binomial theorem it is trivial to calculate the dimension of the exterior algebra of M_p^*

$$\begin{aligned} \dim\left(\Lambda(M_p^*)\right) &= \sum_{k=0}^{n} \dim\left(\Lambda_k(M_p^*)\right) = \sum_{k=0}^{n} \binom{n}{k} \\ &= (1+1)^n = 2^n. \end{aligned}$$

At this point it is useful to consider an example.

Example 23 (Wedge multiplication)

Consider the cotangent bundle to a three dimensional manifold embedded in $\mathbb{R}^m$ for some $m \geq 3$. Let $p \in M$ and dx^1, dx^2, dx^3 be a basis of $\Lambda_1(M_p^*) = M_p^*$. If

$$\omega_1, \omega_2, \omega_3 \in \Lambda_1(M_p^*), \qquad \eta \in \Lambda_2(M_p^*)$$

where

$$\begin{aligned} \omega_1 &= A_1 dx^1 + A_2 dx^2 + A_3 dx^3 \\ \omega_2 &= B_1 dx^1 + B_2 dx^2 + B_3 dx^3 \\ \omega_3 &= C_1 dx^1 - C_2 dx^2 + C_3 dx^3 \\ \eta &= P_1 dx^2 \wedge dx^3 + P_2 dx^3 \wedge dx^1 + P_3 dx^1 \wedge dx^2 \end{aligned}$$

then using the rules for wedge multiplication one obtains

$$\eta \wedge \omega_3 = (P_1 dx^2 \wedge dx^3 + P_2 dx^3 \wedge dx^1 + P_3 dx^1 \wedge dx^2) \wedge (C_1 dx^1 + C_2 dx^2 + C_3 dx^3)$$

$$= (P_1 C_1 + P_2 C_2 + P_3 C_3) dx^1 \wedge dx^2 \wedge dx^3$$

$$\omega_1 \wedge \omega_2 = (A_1 dx^1 \wedge A_2 dx^2 \wedge A_3 dx^3) \wedge (B_1 dx^1 + B_2 dx^2 + B_3 dx^3)$$

$$= (A_2 B_3 - A_3 B_2) dx_2 \wedge dx_3 + (A_3 B_1 - A_1 B_3) dx^3 \wedge dx^1 + (A_1 B_2 - A_2 B_1) dx^1 \wedge dx^2.$$

Identifying $\omega_1 \wedge \omega_2$ with η, the above two formulas give

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = [(A_2 B_3 - A_3 B_2) C_1 + (A_3 B_1 - A_1 B_3) C_2 + (A_1 B_2 - A_2 B_1) C_3] dx^1 \wedge dx^2 \wedge dx^3.$$

Hence, the scalar product, vector product and scalar triple product of vector calculus arise in the wedge multiplication of forms of various degrees on a three dimensional manifold.

End of Example 23

2.5 Behavior of Differential Forms Under Mappings

In the previous section many of the properties of differential forms were seen to be properties of alternating multilinear functionals over a vector space. The following fact is also a consequence of the definition of alternating multilinear functionals. Suppose there is a linear transformation on M_p then there is an induced exterior algebra homomorphism. That is, suppose there is an homorphism

$$f_\# : M'_p \to M_p$$

which induces

$$f^\# : \Lambda(M_p^*) \to \Lambda(M_p'^*).$$

If

$$\omega \to f^{\#}\omega$$

$$\eta \to f^{\#}\eta$$

$$\omega \wedge \eta \to f^{\#}(\omega \wedge \eta)$$

then

$$f^{\#}(\omega \wedge \eta) = (f^{\#}\omega) \wedge (f^{\#}\eta).$$

Earlier, when discussing covariance and contravariance, the pull back $\varphi^{\#}$ on 1-forms induced by a mapping φ was considered along with the induced transformation $\varphi_{\#}$ on vector fields. The above fact enables one to define a pull back on all differential forms:

Theorem. Given $\varphi : M' \to M$. there is an induced homomorphism

$$\varphi^{\#} : \Lambda^{\cdot}(M) \to \Lambda^{\cdot}(M')$$

such that

$$\varphi^{\#}(\omega \wedge \eta) = (\varphi^{\#}\omega) \wedge (\varphi^{\#}\eta)$$

for all $\omega, \eta \in \Lambda^{\cdot}(M)$.

The significance of this result is best appreciated in local coordinates since it dictates a "change of variables" formula for differential forms. Consider $p \in M$ with local coordinates $(x^1, \ldots, x^n)$ and $p' \in M'$ with local coordinates $(y^1, \ldots, y^m)$ along with

$$\varphi : M' \to M$$

such that $\varphi(p') = p$. This transformation induces (via $\varphi^{\#}$) a linear transformation on $\Lambda_1^{\cdot}(M)$ where

$$dx^i = \sum_{j=1}^{m} \frac{\partial x^i}{\partial y^j} dy^j.$$

The exterior algebra homomorphism says that given a k-form, $\omega \in \Lambda_k^*(M)$ where

$$\omega = \sum_{i_1 < i_2 < \cdots < i_k}^{n} a_{i_1 i_2 \ldots i_k}(p) dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

we have

$$\varphi^{\#}\omega = \sum_{i_1 < \cdots < i_k}^{n} a_{i_1 i_2 \ldots i_k}(\varphi(p')) \left(\sum_{j_1 = 1}^{m} \frac{\partial x^{i_1}}{\partial y^{j_1}} dy^{j_1} \right) \wedge \ldots \wedge \left(\sum_{j_k = 1}^{n} \frac{\partial x^{i_k}}{\partial y^{j_k}} dy^{j_k} \right).$$

This transformation is precisely the one required to leave

$$(\varphi^{\#}\omega)(X_1, \ldots, X_k) = \omega(\varphi_{\#} X_1, \ldots, \varphi_{\#} X_k).$$

Furthermore, the change of variables formula for multiple integrals takes on the following form

$$\int_{R'} \varphi^{\#}\omega = \int_{\varphi(R')} \omega.$$

This change of variable formula is most easily understood by means of a few examples.

Example 24 (Change of variables formula in two dimensions)

Suppose $R' \subset M'$ has local coordinates u, v while $\varphi(R') \subset M$ has local coordinates s, t. Let

$$I = \int_{\varphi(R')} f(s,t) ds \wedge dt$$

and consider the change of variables

$$s = s(u,v), \qquad t = t(u,v).$$

Since

$$ds = \frac{\partial s}{\partial u} du + \frac{\partial s}{\partial v} dv,$$
$$dt = \frac{\partial t}{\partial u} du + \frac{\partial t}{\partial v} dv$$

and

$$\begin{aligned} ds \wedge dt &= \left(\frac{\partial s}{\partial u}du + \frac{\partial s}{\partial v}dv\right) \wedge \left(\frac{\partial t}{\partial u}du + \frac{\partial t}{\partial v}dv\right) \\ &= \left(\frac{\partial s}{\partial u}\frac{\partial t}{\partial v} - \frac{\partial s}{\partial v}\frac{\partial t}{\partial u}\right) du \wedge dv \\ &= \frac{\partial(s,t)}{\partial(u,v)} du \wedge dv, \end{aligned}$$

one has

$$I = \int_{R'} f\left(s(u,v), t(u,v)\right) \frac{\partial(s,t)}{\partial(u,v)} du \wedge dv.$$

End of Example 24

Example 25 (A variant on Example 24)

Suppose $R' \subset M'$ has local coordinates u, v, while $\varphi(R') \subset M$ has local coordinates x, y, z. Let

$$J = \int\!\!\int_{\varphi(R')} B_z dx \wedge dy + B_y dz \wedge dx + B_x dy \wedge dz$$

where B_x, B_y, B_z, are functions of x, y, z. Consider a change of variables

$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v).$$

By the same type of calculation as in the previous example.

$$\begin{aligned} dx \wedge dy &= \frac{\partial(x,y)}{\partial(u,v)} du \wedge dv, \\ dz \wedge dx &= \dots \end{aligned}$$

one has

$$J = \int_{R'} \left[B_z \frac{\partial(x,y)}{\partial(u,v)} + B_y \frac{\partial(z,x)}{\partial(u,v)} + B_x \frac{\partial(y,z)}{\partial(u,v)}\right] du \wedge dv.$$

This is a generalisation of the usual change of variables formula.

(End of Example 25)

Example 26 (Change of variables formula in three dimensions)

Suppose $R' \subset M'$ has coordinates u, v, w while $\rho(R') \subset M$ has coordinates x, y, z. Let

$$I = \int_{\varphi(R')} p\, dx \wedge dy \wedge dz.$$

Tranforming coordintates,

$$x = x(u,v,w), \quad y = y(u,v,w), \quad z = z(u,v,w),$$

taking differentials, and using the triple product of Example 23 gives

$$I = \int_{R'} p \frac{\partial(x,y,z)}{\partial(u,v,w)} du \wedge dv \wedge dw.$$

End of Example 26

It is now time to consider the formal definition of the exterior derivative, which will enable us to define a complex associated with the exterior algebra bundle and the corresponding cohomology in terms of differential forms.

2.6 The Exterior Derivative

The exterior derivative will now be introduced in a formal way and illustrated in specific instances. As a preliminary to its definition it is useful to introduce certain vocabulary:

Definition: An endomorphism l of the exterior algebra bundle $\Lambda^{\cdot}(M)$ is

a) a derivation if for $\omega, \eta \in \Lambda^{\cdot}(M)$

$$l(\omega \wedge \eta) = l(\omega) \wedge \eta + \omega \wedge l(\eta);$$

b) an antiderivation if for $\omega \in \Lambda_k^{\cdot}(M), \eta \in \Lambda^{\cdot}(M)$

$$l(\omega \wedge \eta) = l(\omega) \wedge \eta + (-1)^k \omega \wedge l(\eta);$$

c) of degree k if $l : \Lambda_j^{\cdot}(M) \to \Lambda_{j+k}^{\cdot}(M)$ for all j.

The following theorem characterises the exterior derivative as a unique mapping which satisfies certain properties.

Theorem: There exists a unique antiderivation $d : \Lambda^{\cdot}(M) \to \Lambda^{\cdot}(M)$ of degree $+1$ such that

a) $d \circ d = 0$

b) for $f \in \Lambda_0^{\cdot}(M)$, $df(X) = X(f)$, that is, df is the differential of f.

For a globally valid construction of the exterior derivative, the reader is referred to Warner [1971]. Next it is advantageous to see what the exterior derivative does when a local coordinate system is introduced. To this end, an obvious corollary of the above theorem is considered in order to strip the discussion of algebraic terminology.

Corollary: Consider a chart about a point $p \in M$ where there is a local coordinate system with coordinates $(x^1, \ldots, x^n)$. In this chart there exist a unique mapping

$$d : \Lambda_i^{\cdot}(M) \to \Lambda_{i+1}^{\cdot}(M)$$

which satisfies

a) $d(d\omega) = 0, \qquad for\ \omega \in \Lambda^{\cdot}(M),$

b) $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$ for $f \in \Lambda_0^{\cdot}(M)$,

c) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ where $\omega \in \Lambda_k^{\cdot}(M), \eta \in \Lambda^{\cdot}(M)$.

From this corollary, it is easily verified that for a k-form

$$\omega = \sum_{i_1 < i_2 < \ldots < i_k}^{n} a_{i_1 i_2 \ldots i_k}(x^j) dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

the exterior derivative is given by:

$$d\omega = \sum_{i_1 < \ldots < i_k}^{n} \left(\sum_{j=1}^{n} \frac{\partial a_{i_1 \ldots i_k}}{\partial x^j}(x^k) dx^j \right) \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}.$$

Hence when local coordinates are introduced, the exterior derivative can be computed in a straightforward way. The properties which the exterior derivative satisfies according to the above corollary will now be examined. The property

$$d(d\omega) = 0 \quad for\ \omega \in \Lambda^{\cdot}(M)$$

should hold if Stokes' theorem is to hold and the exterior derivative is to be considered an operator adjoint to the boundary operator (recall Chapter 1). The property

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$$

is what is required to make bases of the dual space to the tangent space transform in a contravariant way. Finally the last condition of the corollary is what is required to make the following theorem true:

Theorem: Given $\varphi : M' \to M$ and $\omega \in \Lambda^{\cdot}(M)$

$$\varphi^{\#} d\omega = d(\varphi^{\#} \omega).$$

For a proof of this theorem and the previous one, see Warner [1971] pages 65-68. The next sensible thing to do is consider a series of examples which serve the dual purpose of illustrating exterior differentiation and introducing Stokes' Theorem.

Example 27 (An illustration of exterior differentiation in 1-dimension)

Consider a one dimensional manifold with local coordinate t and a 0-form.

$$\omega = f(t) \quad \text{implies} \quad d\omega = \frac{\partial f(t)}{\partial t} dt.$$

The fundamental theorem of calculus states that

$$\int_a^b \frac{\partial f(t)}{\partial t} dt = f \Big|_a^b$$

or rewritten in terms of differential forms

$$\int_{[a,b]} d\omega = \int_{\partial [a,b]} \omega.$$

End of Example 27

Example 28 (Complex Variables)

Let f be a function of a complex variable. That is,

$$f(z) = f(x + iy) = U(x,y) + iV(x,y).$$

Consider

$$\omega = f(z)dz = [U(x,y) + iV(x,y)]\,(dx + idy)$$

hence

$$\begin{aligned} d\omega &= \left[\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + i\frac{\partial V}{\partial x}dx + i\frac{\partial V}{\partial y}dy\right] \wedge (dx + idy) \\ &= \left[-\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) + i\left(\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}\right)\right] dx \wedge dy. \end{aligned}$$

In this case, "Green's theorem in the plane" states that

$$\int_{\partial R} \omega = \int_R d\omega$$

and the Cauchy-Riemann equations

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}, \quad \frac{\partial V}{\partial y} = \frac{\partial U}{\partial x} \qquad in\ R$$

are equivalent to the statement

$$d\omega = 0 \qquad in\ R.$$

Hence, if the Cauchy-Riemann equations hold in the region R, the

$$\int_{\partial R} f(z)dz = 0 = \int_{\partial R} \omega.$$

This is the Cauchy integral theorem. Furthermore, if one considers the above equation for arbitrary 1-cycles, partitions these cycles into homology classes and uses de Rham's Theorem, then one obtains the residue formula of complex analysis. Also, by partitioning the above expressions into real and imaginary parts one obtains the integral formulas associated with irrotational and solenoidal flows in two dimensions.

End of Example 28

Example 29 (The classical version of Stokes' theorem)

Let u, v, w be local coordinates in a three dimensional manifold and R a region in a two dimensional submanifold. Consider the 1-form:

$$\omega = A_u(u,v,w)du + A_v(u,v,w)dv + A_w(u,v,w)dw.$$

Using the rules for wedge multiplication and exterior differentiation one has

$$
\begin{aligned}
d\omega &= dA_u \wedge du + dA_v \wedge dv + dA_w \wedge dw \\
&= \left(\frac{\partial A_v}{\partial u} - \frac{\partial A_u}{\partial v}\right) du \wedge dv + \left(\frac{\partial A_w}{\partial v} - \frac{\partial A_v}{\partial w}\right) dv \wedge dw + \left(\frac{\partial A_u}{\partial w} - \frac{\partial A_w}{\partial u}\right) dw \wedge du
\end{aligned}
$$

and the classical version of Stokes' theorem becomes

$$\int_{\partial R} \omega = \int_R d\omega.$$

End of Example 29

Example 30 (The divergence theorem in three dimensions)

Next consider a 2-form on a three dimensional manifold with local coordinates u, v, w. Let

$$\omega = D_u(u,v,w)dv \wedge dw + D_v(u,v,w)dw \wedge du + D_w(u,v,w)du \wedge dv.$$

Then using the usual rules one has

$$
\begin{aligned}
d\omega &= dD_u \wedge dv \wedge dw + dD_v \wedge dw \wedge du + dD_w \wedge du \wedge dv \\
&= \left(\frac{\partial D_u}{\partial u} + \frac{\partial D_v}{\partial v} + \frac{\partial D_w}{\partial w}\right) du \wedge dv \wedge dw.
\end{aligned}
$$

In this case, Ostrogradskii's formula becomes

$$\int_{\partial R} \omega = \int_R d\omega.$$

End of Example 30

Example 31 (Electrodynamics)

Consider the four dimensional space-time continuum with local coordinates x, y, z, t. Let

$$\alpha = A_x dx + A_y dy + A_z dz - \phi dt$$

$$\beta = (E_x dx + E_y dy + E_z dz) \wedge dt + (B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy)$$

$$\eta = (H_x dx + H_y dy + H_z dz) \wedge dt - (D_x dy \wedge dz + D_y dz \wedge dx + D_z dx \wedge dy)$$

$$\lambda = (J_x dy \wedge dz + J_y dz \wedge dx + J_z dx \wedge dy) \wedge dt - \rho dx \wedge dy \wedge dz.$$

By a straightforward calculation it is easy to verify that Maxwell's equations can be written as

$$d\beta = 0$$

$$d\eta = \lambda.$$

Furthermore, if φ is a transformation of coordinates then the identity

$$\varphi^{\#} d = d\varphi^{\#}$$

is an expression of the principle of general covariance. Also, putting aside considerations of homology theory, the identity $d \circ d = 0$ enables one to write the field in terms of potentials

$$\beta = d\alpha.$$

The general covariance of Maxwell's equations is nicely expressed in the paper by Bateman [1909] and makes the study of electrodynamics in noninertial reference frames tractable. Following Hargraves [1908] one can rewrite Maxwells equations in integral form by using Stokes' theorem:

$$\int_{b_2 = \partial c_3'} \beta = \int_{c_3'} d\beta = 0$$

$$\int_{\partial c_3} \eta = \int_{c_3} d\eta = \int_{c_3} \lambda$$

where b_2 is any 2-boundary and c_3, c_3' are any 3-chains. Hence, putting aside considerations of homology

$$\int_{z_2} \beta = 0$$

$$\int_{\partial c_3} \eta = \int_{c_3} \lambda$$

where z_2 is any 2-cycle. For modern uses of these equations the reader is referred to Post [1978], [1984].

End of Example 31

2.7 Cohomology with Differential Forms

It is now possible to restate the ideas of Chapter 1 in a more formal way. Rewrite

$$d : \Lambda^*(M) \to \Lambda^*(M)$$

as

$$d : \bigoplus_p \Lambda_p^*(M) \to \bigoplus_p \Lambda_p^*(M)$$

and define the restriction of the exterior derivative to p-forms by

$$d^p : \Lambda_p^*(M) \to \Lambda_{p+1}^*(M).$$

As usual, one can define the set of p-cocycles (or in the language of differential forms, the space of closed p-forms) as

$$Z^p(M) = \text{Kernel}\,(\Lambda_p^*(M) \xrightarrow{d^p} \Lambda_{p+1}^*(M))$$

and the set of p-coboundaries (or in the language of differential forms, the space of exact p-forms) as

$$B^p(M) = \text{Image}\,\left(\Lambda_{p-1}^*(M) \xrightarrow{d^{p-1}} \Lambda_p^*(M)\right).$$

The equation

$$d_{p+1} \circ d_p = 0$$

shows that $\Lambda^{\cdot}(M)$ is a cochain complex and that

$$B^p(M) \subset Z^p(M).$$

Thus one defines

$$H^P_{deR}(M) = Z^p(M)/B^p(M)$$

to be the de Rham cohomology of the manifold. In order to relate the notation of this and the previous chapter, define

$$C^{\cdot}(M) = \Lambda^{\cdot}(M)$$

where

$$C^p(M) = \Lambda^{\cdot}_p(M).$$

It is important to note that one can define many other cohomology theories if the definition of cohomology is written out explicitly. That is if

$$H^p(M) = (Z^p(M) \cap C^p(M)) \,/\, ((d^{p-1}C^{p-1}(M)) \cap C^p(M)).$$

Thus if $C^p(M)$ were p-forms of compact support or p-forms with square integrable coefficients then one would obtain "cohomology with compact support" or "L^2 cohomology". Although these cohomology theories tend to agree on compact manifolds, they do not agree in general. The precise definition of compactly supported cohomology involves a limiting procedure which can be avoided in this thesis (see Greenberg and Harper [1980] Chapter 26). Thus for example

$$H^3_{deR}(\mathbb{R}^3) \simeq 0 \not\simeq \mathbb{R} \simeq H^3_c(\mathbb{R}^3).$$

Although this result has not been proven here, it is easily deduced from Spivak [1979] p. 371. In this thesis the regions of interest are bounded subsets of $\mathbb{R}^n$ and in this case cohomology with compact support is easily interpreted in terms of boundary conditions, an interpretation which will soon be given. As in the previous chapter, the complex associated with differential forms with compact support in a region Ω will be denoted by

$$C_c^{\cdot}(\Omega)$$

and the associated cocycle, coboundary, and cohomology spaces will be distinguished by the subscript c.

The de Rham cohomology vector spaces play a central role in this thesis as does the cohomology with compact support in the context of relative cohomology. The L^2 cohomology spaces, although important in the context of finite energy constraints on variational functionals, will not be considered in this thesis. There are two reasons for this. First, the properties of L^2 cohomology are harder to articulate mathematically and secondly, for bounded regions of $\mathbb{R}^3$ the de Rham cohomology and cohomology with compact support give the required insight into L^2 cohomology while unbounded domains in $\mathbb{R}^3$ can be handled by attaching a point at infinity and mapping $\mathbb{R}^3$ onto the unit sphere S^3 in $\mathbb{R}^4$ (this procedure is analogous to stereographic projections in complex variables).

2.8 Cochain Maps Induced by Mappings Between Manifolds

Having defined the cochain complexes associated with de Rham cohomology and cohomology with compact support, it is useful to consider how mappings between manifolds induce cochain maps between cochain complexes. Given a map

$$\varphi : M' \to M$$

there are covariant and contravariant transformations

$$\varphi_{\#} : T(M') \to T(M)$$

$$\varphi^{\#} : \Lambda^{\cdot}(M) \to \Lambda^{\cdot}(M')$$

on vector fields and differential forms respectively. For a given k and

$$\omega \in \Lambda_k^{\cdot}(M), \qquad X_i \in T(M'),\ 1 \leq i \leq k$$

one has

$$\omega\left(\varphi_{\#}(X_1), \varphi_{\#}(X_2), \ldots, \varphi_{\#}(X_k)\right) = (\varphi^{\#}\omega)(X_1, X_2, \ldots, X_k)$$

which express the invariance of the whole scheme. Having defined

$$C^{\cdot}(M) = \Lambda^{\cdot}(M)$$

$$C^{\cdot}(M') = \Lambda^{\cdot}(M')$$

the formula

$$\varphi^{\#} d_M = d_{M'} \varphi^{\#}$$

where the d on the right is the coboundary operator (exterior derivative) in the complex $C^{\cdot}(M')$ while the d on the left is the exterior derivative in the complex $C^{\cdot}(M)$, shows that $\varphi^{\#}$ is a cochain homomorphism. That is, if φ^p is the restriction of $\varphi^{\#}$ to p-forms then the following diagram is commutative for all k:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow d_M^{k-1} & & \downarrow d_{M'}^{k-1} \\
C^k(M) & \xrightarrow{\varphi^k} & C^k(M') \\
\downarrow d_M^{k} & & \downarrow d_{M'}^{k} \\
C^{k+1}(M) & \xrightarrow{\varphi^{k+1}} & C^{k+1}(M') \\
\downarrow d_M^{k+1} & & \downarrow d_{M'}^{k+1} \\
\vdots & & \vdots
\end{array}$$

A very important special case of this construction occurs when M' is a submanifold of M and φ is the injection mapping. That is if S is a submanifold of M and

$$i : S \rightarrow M$$

is the injection map then the pull back

$$i^{\#} : C^{*}(M) \rightarrow C^{*}(S)$$

is a cochain homomorphism. When this happens, it is possible to construct a long exact cohomology sequence in several ways (see Spivak [1979] p. 571-591 or Bott and Tu [1982] p. 78-79). This topic however, will not be pursued here since once the de Rham theorem is established, it is easier to think in terms of cycles and the long exact homology sequence.

2.9 Stokes' Theorem, de Rham's Theorems and Duality Theorems

As a prelude to Stokes' Theorem, the concepts of an orientation and regular domain are required. Since $\Lambda_n(M_p^{*})$ is one dimensional, it follows that $\Lambda_n(M_p^{*}) - \{0\}$ has two connected components. An orientation of M_p^{*} is a choice of connected component of $\Lambda_n(M_p^{*}) - \{0\}$. An n-dimensional manifold is said to be orientable if it is possible to make an unambiguous choice of orientation for M_p^{*} at each $p \in M$. If M is not connected then M is orientable if each of its connected components is. Thus, following the discussion in Warner [1971] p. 138-140, the following proposition clears up the intuitive picture about orientation.

Proposition: If M is a differentiable manifold of dimension n, the following are equivalent:

1) M is orientable,

2) there is an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ such that

$$\left(\frac{\partial(x^1, \ldots, x^n)}{\partial(y^1, \ldots, y^n)}\right) \geq 0 \qquad \text{on } U_i \cap U_j$$

whenever

$$\left(U_i, (x^1, \ldots, x^n)\right), \left(U_j, (y^1, \ldots, y^n)\right) \in \mathcal{A},$$

3) there is a nowhere vanishing n-form on M.

The proof of this proposition can be found in the above reference. An example of a nonorientable surface is the Möbius band of Example 10. The notion of a regular domain is given by the following definition:

Definition: A subset D of a manifold M is called a regular domain if for each $p \in M$ one of the following holds:

1) There is an open neighborhood of p which is contained in $M - D$.

2) There is an open neighborhood of p which is contained in D.

3) Given $\mathbb{R}^n$ with Gartesian coordinates $(r^1, \ldots, r^n)$ there is a centered coordinate system (U, φ) about p such that $\varphi(U \cap D) = \varphi(U) \cap H^n$ where H^n is the half-space of $\mathbb{R}^n$ defined by $r^n \geq 0$.

Thus if one puts the point p in a regular domain under an infinitely powerful microscope, one would see Fig. 20. A property of a regular domain is that its boundary is a manifold. In situations where this definition is too restrictive, for example if D is a square, one can use the notion of an almost regular domain (see Loomis and Sternberg [1968] p. 424-427).

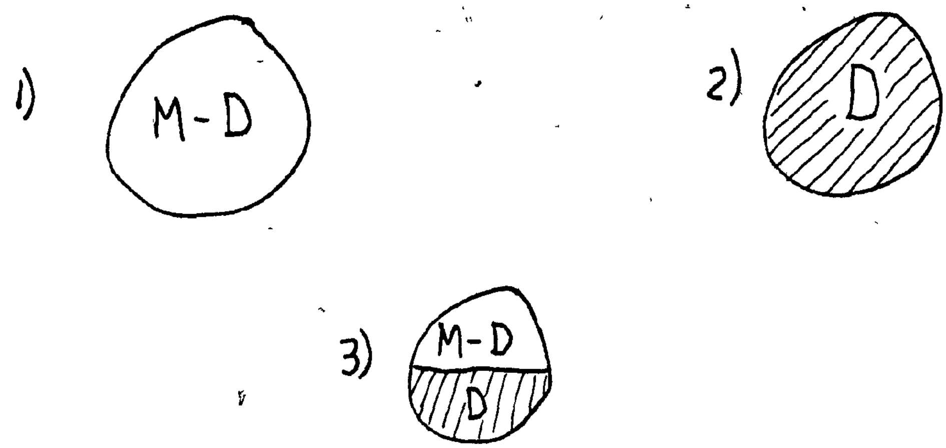


Fig. 20 .

The main result of this section is the following version of Stokes' Theorem.

Theorem: Let D be a regular domain in an oriented n-dimensional manifold M and let ω be a smooth $(n-1)$-form of compact support. Then

$$\int_D d\omega = \int_{\partial D} i^{\#}\omega$$

where $\iota : \partial D \to M$ induces the pull back

$$i^{\#} : C^{\cdot}(M) \to C^{*}(\partial D).$$

For a nice, simple proof of this theorem, see Warner [1971] pages 140-148.

At this point it is worthwhile interpreting integration as a bilinear pairing between differential forms and chains so that the de Rham isomorphism is easy to understand. It has been assumed all along that integration is a bilinear pairing between chains and cochains (forms). In the heuristic development of Chapter 1 this was emphasised by writing

$$\int : C_p(M) \times C^p(M) \to \mathbb{R}.$$

Furthermore, the reader was lead to believe that differential forms were linear functionals on differentiable chains. This was emphasised notationally by writing

$$\int_c \omega = [c, \omega].$$

Stokes' Theorem was then written as

$$[\partial_p c, \omega] = [c, d^{p-1}\omega] \qquad \textit{for all } p$$

and hence was interpreted as saying that the exterior derivative (coboundary) operator and the boundary operator were adjoint operators. This is the setting for de Rham's theorem, for if the domain of the bilinear pairing is restricted to cocycles (closed forms) and cycles, that is

$$\int : Z_p(M) \times Z^p(M) \to \mathbb{R}$$

then it is easy to show that the value of this bilinear pairing depends only on the homology class of the cycle and the cohomology class of the closed form (cocycle). This is easily verified by the following calculation. Let

$$z^p \in Z^p(M) \qquad c^{p-1} \in C^{p-1}(M)$$

$$z_p \in Z_p(M) \qquad c_{p+1} \in C_{p+1}(M)$$

then

$$[z_p + \partial_{p+1}c_{p+1}, z^p + d^{p-1}c^{p-1}] = [z_p, z^p] + [z_p, d^{p-1}c^{p-1}] + [\partial_{p+1}c_{p+1}, z^p + d^{p-1}c^{p-1}]$$

(by linearity)

$$= [z_p, z^p] + [\partial_p z_p, c^{p-1}] + [c_{p+1}, d^p z_p + d^p \circ d^{p-1}c^{p-1}]$$

(by Stokes' Theorem)

$$= [z_p, z^p] + [0, c^{p-1}] + [c_{p-1}, 0]$$

(by definition)

$$= [z_p, z^p].$$

Hence if is verified that when the domain of the bilinear pairing is restricted to chains and cochains one obtains a bilinear pairing between homology and cohomology. The theorems of de Rham assert that this induced bilinear pairing is non degenerate and hence there is an isomorphism

$$H_p(M) \simeq H^p_{deR}(M).$$

As noted in Chapter 1, this thesis is not the place to prove that such an isomorphism exists since no formal way of computing homology is considered. The reader will find down to earth proofs of the de Rham isomorphism in de Rham [1931] or Hodge [1952], Chapter 2. A sophisticated modern proof can be found in Warner [1971], Chapter 5 while less formal proofs can be found in Goldberg [1962], Appendix A or Massey [1980], Appendix.

Thoughout this thesis a de Rham type of isomorphism is required for relative homology and cohomology groups. Though this type of isomorphism is not readily found in books (if at all) there are two methods of obtaining such an isomorphism once the usual de Rham isomorphism is established. The first approach is to read the paper by Duff [1952] and refrain from sweating blood while following the arguments presented there. The second approach is to reduce the problem to a purely algebraic one and use the so called five lemma. Though this second approach is straightforward, the author is unable to find it in the literature and hence was forced to present it here.

Consider, for example, a manifold M with compact boundary ∂M. In this case (see Spivak [1979 Theorem 13 p. 589) there is the following long exact cohomology sequence.

$$\ldots \to H^k_c(M) \to H^k(\partial M) \to H^{k+1}_c(M - \partial M) \to H^{k+1}_c(M) \to H^{k+1}(\partial M) \to \ldots$$

Also there is a long exact homology sequence (see Greenberg and Harper [1981] Chapter 14 for example)

$$\ldots \leftarrow H_k(M) \leftarrow H_k(\partial M) \leftarrow H_{k+1}(M,\partial M) \leftarrow H_{k+1}(M) \leftarrow H_{k+1}(\partial M) \leftarrow \ldots$$

and the following de Rham isomorphisms are known to exist

$$H_k(M) \simeq H_c^k(M) \quad (M \text{ compact}) \text{ for all } k$$

$$H_k(\partial M) \simeq H^k(\partial M).$$

The above isomorphisms are induced by integration and there is also a bilinear pairing between $H_c^{k+1}(M-\partial M)$ and $H_{k+1}(M,\partial M)$ which is induced by integration. In this case the following diagram is commutative:

$$\begin{array}{ccccccccc}
H_{k+1}(\partial M) & \rightarrow & H_{k+1}(M) & \rightarrow & H_{k+1}(M,\partial M) & \rightarrow & H_k(\partial M) & \rightarrow & H_k(M) \\
\updownarrow\simeq & & \updownarrow\simeq & & \updownarrow & & \updownarrow\simeq & & \updownarrow\simeq \\
H^{k+1}(\partial M) & \leftarrow & H_c^{k+1}(M) & \leftarrow & H_c^{k+1}(M-\partial M) & \leftarrow & H^k(\partial M) & \leftarrow & H_c^k(M)
\end{array}$$

What is required is to show that the middle vertical arrow in this picture (and hence every third arrow in the long sequence of commutative squares) is an isomorphism. To do this one first considers the dual spaces:

$$\left(H^k(\partial M)\right)^*$$

$$\left(H_c^k(M)\right)^*$$

$$\left(H_c^k(M-\partial M)\right)^*$$

and notices that, by definition there is a commutative diagram

$$\begin{array}{ccccccccc}
H^{k+1}(\partial M) & \leftarrow & H_c^{k+1}(M) & \leftarrow & H_c^{k+1}(M-\partial M) & \leftarrow & H^k(\partial M) & \leftarrow & H_c^k(M) \\
\updownarrow\simeq & & \updownarrow\simeq & & \updownarrow\simeq & & \updownarrow\simeq & & \updownarrow\simeq \\
\left(H^{k+1}(\partial M)\right)^* & \rightarrow & \left(H_c^{k+1}(M)\right)^* & \rightarrow & \left(H_c^{k+1}(M-\partial M)\right)^* & \rightarrow & \left(H^k(\partial M)\right)^* & \rightarrow & \left(H_c^k(M)\right)^*
\end{array}$$

where the vertical arrows are all ismorphisms, the two rows are exact sequences and the mappings on the bottom row are the adjoints of the mappings directly above them. Combining the above two commutative diagrams one has the following commutative diagram:

$$
\begin{array}{ccccccccc}
H_{k+1}(\partial M) & \rightarrow & H_{k+1}(M) & \rightarrow & H_{k+1}(M,\partial M) & \rightarrow & H_k(\partial M) & \rightarrow & H_k(M) \\
\updownarrow \simeq & & \updownarrow \simeq & & \updownarrow & & \updownarrow \simeq & & \updownarrow \simeq \\
\left(H^{k+1}(\partial M)\right)^* & \rightarrow & \left(H_c^{k+1}(M)\right)^* & \rightarrow & \left(H_c^{k+1}(M-\partial M)\right)^* & \rightarrow & \left(H^k(\partial M)\right)^* & \rightarrow & \left(H_c^k(M)\right)^*
\end{array}
$$

where the rows are exact sequences and one wants to know whether the middle homomorphism is an isomorphism. To see that the answer is yes consider the following lemma (see Greenberg and Harper [1981] p. 77-78).

Five Lemma: Given a diagram of R-modules and homomorphisms with all rectangles commutative

$$
\begin{array}{ccccccccc}
A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & A_3 & \xrightarrow{a_3} & A_4 & \xrightarrow{a_4} & A_5 \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\
B_1 & \xrightarrow{b_1} & B_2 & \xrightarrow{b_2} & B_3 & \xrightarrow{b_3} & B_4 & \xrightarrow{b_4} & B_5
\end{array}
$$

such that the rows are exact (at joints 2, 3, 4) and the four outer homomorphisms $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is an isomorphism.

It is obvious that the five lemma applies in the above situation (since a vector space over $\mathbb{R}$ is an instance of an R-module) and hence

$$H_{k+1}(M,\partial M) \simeq \left(H_c^{k+1}(M-\partial M)\right)^* .$$

Thus

$$H_{k+1}(M,\partial M) \simeq H_c^{k+1}(M-\partial M)$$

and the relative de Rham isomorphism is proven. It is also obvious that the isomorphism would be true if ∂M were replaced by a collection of connected components of ∂M or parts of ∂M which arise from symmetry planes as in Chapter 1 all that is required is the existence of long exact (co)homology sequences and the usual de Rham isomorphism.

Having seen how the de Rham isomorphism can be understood with the help of Stokes' Theorem, a simple corollary of Stokes' Theorem will now be used to give a heuristic understanding of duality theorems. Suppose M is an oriented n-dimensional manifold, D is a regular domain in M and $\lambda \in C_c^k(M), \mu \in C^{n-k-1}(M)$. Since λ has compact support, if

$$\omega = \lambda \wedge \mu$$

then

$$\omega \in C_c^{n-1}(M).$$

Furthermore

$$d\omega = d(\lambda \wedge \mu) = (d\lambda) \wedge \mu + (-1)^k \lambda \wedge d\mu$$

and if $\imath$ is the injection of ∂D into M then, as usual, the pull back $\imath^\#$ satisfies

$$i^\#(\omega) = \imath^\#(\lambda \wedge \mu) = i^\#(\lambda) \wedge i^\#(\mu).$$

Substituting this expression for ω into Stokes' Theorem one obtains the following important corollary

Corollary: (Integration by parts)

If D is a regular domain in an oriented n-dimensional manifold M and

$$\lambda \in C_c^k(M), \quad \mu \in C^{n-k-1}(M)$$

then

$$(-1)^k \int_D \lambda \wedge d\mu = \int_{\partial D} (i^\# \lambda) \wedge (i^\# \mu) - \int_D d\lambda \wedge \mu$$

where

$$i : \partial D \to M$$

induces

$$i^{\#} : C^{*}(M) \to C^{*}(\partial D).$$

Just as Stokes' Theorem is often called the fundamental theorem of multivariable calculus since it generalises the usual fundamental theorem of integral calculus, the above corollary is the multivariable version of "integration by parts". This integration by parts formula is of fundamental importance in the calculus of variations and in obtaining an interpretation of duality theorems on manifolds. These duality theorems will be considered next.

Consider first the situation of a manifold without boundary M and the Poincaré duality theorem. In this case the integration by parts formula reduces to

$$(-1)^k \int_M \lambda \wedge d\mu = -\int_M (d\lambda) \wedge \mu$$

whenever $\lambda \in C_c^k(M), \mu \in C^{n-k-1}(M)$. Also there is a bilinear pairing

$$\int_M : C_c^p(M) \times C^{n-p}(M) \to \mathbb{R}$$

where a p-form of compact support is wedge multiplied with an $(n-p)$-form to yield an n-form of compact support which is then integrated over the entire manifold. The heart of the proof of the Poincaré duality theorem involves restricting the domain of this bilinear pairing from chains to cycles and noticing that one has a bilinear pairing on dual (p and $n-p$) cohomology groups. To see how happens. Though the Poincaré Duality Theorem will not be proved in this thesis, it is useful to see how this bilinear pairing on cohomology comes about. Consider

$$\int_M : Z_c^p(M) \times Z^{n-p}(M) \to \mathbb{R}$$

where for

$$z^p \in Z^p_c(M), \quad z^{n-p} \in Z^{n-p}(M)$$

one computes

$$\int_M z^p \wedge z^{n-p}.$$

To see that the value of this integral depends only on the cohomology classes of z^p and z^{n-p} one lets

$$c^{p-1} \in C^{p-1}_c(M), \quad c^{n-p-1} \in C^{n-p-1}(M)$$

and considers the following computation

$$\begin{aligned}
&\int_M \left(z^p + d^{p-1}c^{p-1}\right) \wedge \left(z^{n-p} + d^{n-p-1}c^{n-p-1}\right) \\
&= \int_M z^p \wedge z^{n-p} + \int_M \left(d^{p-1}c^{p-1}\right) \wedge z^{n-p} + \int_M \left(z^p + d^{p-1}c^{p-1}\right) \wedge d^{n-p-1}c^{n-p-1} \\
&= \int_M z^p \wedge z^{n-p} + (-1)^p \int_M c^{p-1} \wedge d^{n-p}z^{n-p} - (-1)^p \int_M \left(d^p z^p + d^p \circ d^{p-1}c^{p-1}\right) \wedge c^{n-p-1} \\
&\qquad \text{(integrating by parts)} \\
&= \int_M z^p \wedge z^{n-p} \qquad \text{(using the definition of cocycle).}
\end{aligned}$$

Thus it is seen that restricting the domain of the bilinear form from cochains to cocycles induces the following bilinear pairing on cohomology

$$\int_M : H^p_c(M) \times H^{n-p}_{deR}(M) \to \mathbb{R}.$$

The Poincaré Duality Theorem asserts that this bilinear pairing is nondegenerate. Thus

$$H^p_c(M) \simeq H^{n-p}_{deR}(M) \qquad \textit{for all } p$$

or if M is compact

$$H^p_{deR}(M) \simeq H^{n-p}_{deR}(M).$$

This statement of the Poincaré Duality theorem is not the most general version (see Bott and Tu [1982] pp. 44–47 for a proof and explanation of the subtleties encountered in generalising the above). The implicit assumption in the above argument is the finite dimensionality of the cohomology vector spaces. A nice discussion of this aspect is given in Spivak [1979] p. 600 and the pages leading up to page 600. As was mentioned in the previous chapter, Massey [1980] Chapter 9 and Greenberg and Harper [1981] Chapter 26 have proofs of the Poincaré Duality theorem which do not appeal to the formalism of differential forms.

When the manifold M is not compact, the Poincaré duality theorem may be used to show the difference between de Rham cohomology and cohomology with compact support. Take for example $\mathbb{R}^n$ where, by Poincaré duality and the arguments of the last chapter one has

$$H_c^{n-p}(\mathbb{R}^n) \simeq H_{deR}^p(\mathbb{R}^n) \simeq \begin{cases} \mathbb{R}, & \text{if } p = 0 \\ 0, & \text{if } p \neq 0. \end{cases}$$

Hence, in the case of $\mathbb{R}^3$

$$H_c^3(\mathbb{R}^3) \simeq \mathbb{R} \not\simeq 0 \simeq H_{deR}^3(\mathbb{R}^3)$$

$$H_{deR}^0(\mathbb{R}^3) \simeq \mathbb{R} \not\simeq 0 \simeq H_c^0(\mathbb{R}^3).$$

As stated in the previous chapter, the Poincaré duality theorem does not have many direct applications in boundary value problems of electromagnetics. For the purposes of this thesis, attention will be paid to compact manifolds with boundary and for these there is a variety of duality theorems. In this case, it is useful to get certain ideas established once and for all. First, if

$$\imath : \partial M \to M$$

$$i^{\#} : C^{\cdot}(M) \to C^{\cdot}(\partial M)$$

then

$$c \in C_c^{\cdot}(M - \partial M)$$

if

$$i^{\#} c = 0.$$

Thus

$$z^p \in Z_c^p(M - \partial M) \qquad \text{if } dz^p = 0,\ i^{\cdot} z^p = 0,$$

$$b^p \in B_c^p(M - \partial M) \qquad \text{if } b^p = dc^{p-1} \text{ for some } c^{p-1} \in C_c^{p-1}(M - \partial M).$$

In this case, it is customary to denote the symbol $i^{\#}$ by t, and avoid referring to the injection $\imath$. Thus $t\omega$ denotes the pull back of ω to ∂M and is the "tangential" part of ω. In this notation the integration by parts formula takes the form:

$$(-1)^k \int_M \lambda \wedge d\mu = \int_{\partial M} (t\lambda) \wedge (t\mu) - \int_M (d\lambda) \wedge \mu$$

where λ is a k-form, and μ is a $(n-k-1)$-form.

To see how the Lefschetz Duality Theorem comes about, consider an orientable compact n-dimensional manifold with boundary and the bilinear pairing

$$\int_M : C_c^p(M - \partial M) \times C^{n-p}(M) \to \mathbb{R}.$$

Note that if the boundary of the manifold is empty then the situation is identical to that of the Poincaré duality theorem. Restricting the domain of this bilinear pairing to cocycles (closed forms) one can easily show that there is an induced bilinear pairing on two cohomology groups. That is considering

$$\int_M : Z_c^p(M - \partial M) \times Z^{n-p}(M) \to \mathbb{R}$$

the integration by parts formula shows that the integral

$$\int_M z^p \wedge z^{n-p}$$

depends only on the cohomology classes of z^p and z^{n-p} whenever

$$z^p \in Z_C^p(M - \partial M), \quad z^{n-p} \in Z^{n-p}(M).$$

Hence there is a bilinear pairing

$$\int_M : H_c^p(M - \partial M) \times H_{deR}^{n-p}(M) \to \mathbb{R}$$

induced by integration. The Lefschetz Duality Theorem asserts that this bilinear pairing is non degenerate. Hence

$$H_c^p(M - \partial M) \simeq H_{deR}^{n-p}(M).$$

Again one can find the proof of this type of theorem in Massey [1980] Sect 9.7, or Greenberg and Harper [1981] Chapter 28. Connor [1954] has shown that there is a generalisation of the Lefschetz duality theorem. To see what this generalisation is, let

$$\partial M = \bigcup_{i=1}^{m} C_i$$

where each c_i is a connected manifold without boundary and let

$$S_1 = \bigcup_{i=1}^{r} C_i, \quad S_2 = \bigcup_{i=r+1}^{m} C_i$$

where M is a compact, orientable n-dimensional manifold with boundary. In this case the result of Connor states that

$$H_c^p(M - S_1) \simeq H_c^{n-p}(M - S_2):$$

This result can be interpreted, as before, by saying that the bilinear pairing

$$\int_M : C_c^p(M - S_1) \times C_c^{n-p}(M - S_2)$$

descends into a nondegenerate bilinear pairing on cohomology when the domain is restricted to cocycles. This is verified by using the integration by parts formula to show that the restricted bilinear pairing does indeed depend on cohomology classes only.

In this thesis, the situation is slightly more general in that S_1 and S_2 are not necessarily disjoint but at the intersection $S_1 \cap S_2$ a symmetry plane and a component of the boundary of some original problem meet at right angles. From the usual proofs of the Lefschetz duality theorem (which construct the double of a manifold) it is apparent that the duality theorem

$$H_c^p(M - S_1) \simeq H_c^{n-p}(M - S_2)$$

is still true. It is useful to note that the interpretation of the above duality theorems is in some sense dual to the approach taken in the previous chapter in that the homology point of view stresses intersection numbers while the cohomology point of view stresses the bilinear pairing induced by integration. It is important to keep this interplay in mind since topological problems in electromagnetics involve the bilinear pairing in cohomology and these problems can be resolved very conveniently by thinking in terms of intersection numbers. In the case of the Alexander duality theorem there is no nice intersection number or integral interpretation. This is apparent from the proof of the Alexander Duality Theorem (see Massey [1980] Sect 9.6 or Greenberg and Harper [1981] Chapter 27). For a different but down to earth exposition of intersection numbers from the point of view of differential forms and many other topics treated so far in this chapter the reader is referred to Hermann [1977] Part 5 Chapter 34.

2.10 Riemannian Structures, the Hodge Star Operator and an Inner Product for Differential Forms

So far in this chapter those aspects of differential forms which are independent of the notion of distance in the manifold have been considered. These include the complex structure associated with the exterior algebra bundle of the manifold, change of variables formulas for integrals, Stokes' theorem, de Rham's theorems, and duality theorems in homology and cohomology. The next thing to do is to put the whole framework in a Hilbert space setting.

The inner product on differential forms is a consequence of an inner product structure on the tangent bundle of the manifold or Riemannian structure. Since this idea is so important, it is worthy of a formal definition

Definition: A Riemannian structure on a differentiable manifold M is a smooth choice of a positive definite inner product $(\cdot,\cdot)$ on each tangent space M_p (recall M_p is the tangent space to M at p).

In the above definition, *smooth* means that if the functions in the charts of an atlas for M are differentiable of order C^k and if $X, Y \in T(M)$ have components which are C^k differentiable, then the function (X, Y) is a C^k differentiable function of the coordinates of M. It is a basic fact in Riemannian geometry that any manifold admits a Riemannian structure (see for example Warner 1971, p. 52 or Bott and Tu 1982 p. 42–43). A Riemannian manifold is, by definition, a differentiable manifold with a Riemannian structure, hence any differentiable manifold can be made into a Riemannian manifold.

In terms of local coordinates $(x^1, \ldots, x^n)$ about a point $p \in M$, if $X, Y \in M_p$ and

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}, \qquad Y = \sum_{i=1}^{n} Y^i \frac{\partial}{\partial x^i}$$

then there is a symmetric positive definite matrix (called the metric tensor) with entries

$$g_{ij} = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)_p$$

so that

$$(X, Y)_p = \sum_{i=1}^{n} \sum_{j=1}^{n} X^i g_{ij} Y^j.$$

Since 1-forms were defined to be elements of M_p^* (the dual space to M_p). the above inner product induces one on the dual space. That is, if $\omega, \eta \in M_p^*$ where in terms of local coordinates

$$\omega = \sum_{i=1}^{n} a_i dx^i, \quad \eta = \sum_{j=1}^{n} b_j dx^j$$

then

$$(\omega, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i g^{ij} b_j$$

where

$$g_{ij} g^{jk} = \delta_i^k \qquad \text{(Kronecker delta)}.$$

A Riemannian structure on a differentiable manifold induces an inner product on k-forms and the immediate objective at this point is to see how this inner product comes about. Given a Riemannian structure on the tangent bundle of a manifold it is always possible to do local computations in terms of an orthonormal basis obtained by the Gram-Schmidt procedure and patching together the results with a partition of unity. Hence, in order to define a pointwise inner product on k-forms it suffices to work in terms of local coordinates. Having made these observations, let $\omega_i, 1 \leq i \leq n$ be an orthonormal basis for M_p^*, (that is, $\Lambda_1(M_p^*)$ in some coordinate chart. This means:

$$(\omega_i, \omega_j) = \delta_{ij} \qquad \text{(Kronecker delta)}.$$

By taking all possible exterior products of these basis forms it is seen that $\Lambda_k(M_p^*)$ is spanned by $\binom{n}{k}$, k-forms which look like

$$\omega_{i_1} \wedge \omega_{i_2} \wedge \ldots \wedge \omega_{i_k}, \qquad 1 \leq i_1 < i_2 < \ldots < i_k \leq n$$

and in particular $\Lambda_n(M_p^*)$ is spanned by the one element

$$\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n.$$

This n-form is called the volume form. Next, by the symmetry of binomial coefficients it is seen that

$$\dim \Lambda_k(M_p^*) = \binom{n}{k} = \binom{n}{n-k} = \dim \Lambda_{n-k}(M_p^*).$$

Hence the two spaces are isomorphic. Consider an isomorphism (called the Hodge star operator)

$$* : \Lambda_k(M_p^*) \rightarrow \Lambda_{n-k}(M_p^*)$$

which acts on the above basis vectors in the following way. Let π be a permutation of n integers and let

$$\omega_{\pi(1)} \wedge \omega_{\pi(2)} \wedge \ldots \wedge \omega_{\pi(k)}$$

be a basis vector in $\Lambda_k(M_p^*)$ so that

$$\omega_{\pi(k+1)} \wedge \omega_{\pi(k+2)} \wedge \ldots \wedge \omega_{\pi(n)}$$

becomes a basis vector in $\Lambda_{n-k}(M_p^*)$. Define

$$*\left(\omega_{\pi(1)} \wedge_{\pi(2)} \wedge \ldots \omega_{\pi(k)}\right) = \text{sgn}(\pi)\left(\omega_{\pi(k+1)} \wedge \ldots \wedge \omega_{\pi(n)}\right)$$

and hence, since the linear transformation is defined on the basis vectors of $\Lambda_k(M_p^*)$ the linear transformation is completely defined. Alternatively one can define the operation of $*$ on basis vectors of $\Lambda_k(M_p^*)$ by the following

$$\left(\omega_{\pi(1)} \wedge \omega_{\pi(2)} \wedge \ldots \wedge \omega_{\pi(k)}\right) \wedge *\left(\omega_{\pi(1)} \wedge \ldots \wedge \omega_{\pi(k)}\right) = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n.$$

Using the usual abuse of notation, one defines the volume form

$$dV = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n$$

where it is understood that dV is not necessarily the exterior derivative of any $(n-1)$-form. If

$$\omega = \omega_{\pi(1)} \wedge \ldots \wedge \omega_{\pi(k)}$$

then the rules for wedge multiplication and the definition of the star operator show that

$$\begin{aligned} dV &= \omega \wedge (*\omega) \\ &= (-1)^{k(n-k)}(*\omega) \wedge \omega \end{aligned}$$

and

$$dV = (*\omega) \wedge (*(*\omega)).$$

hence

$$**\omega = (-1)^{k(n-k)}\omega, \quad 1 = *(dV).$$

By linearity, this is true for all k-forms. Furthermore if $\omega, \eta \in \Lambda_k(M_p^*)$ then

$$*(\omega \wedge *\eta) = *(\eta \wedge *\omega)$$

is a symmetric positive definite function (an inner product) on $\Lambda^{\cdot}(M_p^{\cdot})$ (see Flanders [1963] Chapter 2 for a discussion of this result). This completes the construction of a pointwise inner product on differential forms. At this point several remarks are in order:

1. Given an orientation on $\Lambda_n(M_p^{\cdot})$, the definition of the Hodge star operator is independent of the orthonormal basis chosen. That is, if the Hodge star operator is defined in terms of an orthonormal basis then the definition of the star operator is satisfied on any other orthonormal basis related to the first by an orthogonal matrix with positive determinant.

2. On an orientable manifold, it is possible to choose an orientation consistently over the whole manifold and hence the star operator can be defined smoothly as a mapping

$$* : \Lambda_k^{\cdot}(M) \to \Lambda_{n-k}^{\cdot}(M)$$

   or equivalently

$$* : C^k(M) \to C^{n-k}(M).$$

3. When there is a pseudo-Riemannian structure on the manifold, that is, a Riemannian structure which is nondegenerate but not positive definite then it is still possible to define a star operator, but it does not give rise to a positive definite bilinear pairing on k-forms. Such a star operator depends on the "signature" of the metric and occurs in four dimensional formulations of electrodynamics (see Flanders [1963] Sects 2.6, 2.7 and Balasubramanian et al. [1970] Sect 3.5).

The folowing examples shows how the operations $(d, \wedge, *)$ are related to the operators of vector analysis.

Example 32 (Three dimensional vector analysis)

Suppose x^1, x^2, x^3 are orthogonal curvilinear coordinates in a subset of $\mathbb{R}^3$, that is

$$g_{ij} = \begin{cases} h_i^2, & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

so that $\omega^i = h_i dx^i, 1 \leq i \leq 3$ is an orthonormal basis for 1-forms. In this case, if π is the permutation of three integers which sends 1, 2, 3 into i, j, k then

$$*1 = \omega_1 \wedge \omega_2 \wedge \omega_3 \qquad *(\omega_1 \wedge \omega_2 \wedge \omega_3) = 1$$

$$*\omega_k = \operatorname{sgn}(\pi)\omega_i \wedge \omega_j \qquad *(\omega_j \wedge \omega_k) = \operatorname{sgn}(\pi)\omega_i$$

hence

$$*1 = h_1 h_2 h_3 dx^1 \wedge dx^2 \wedge dx^3$$

$$*(dx^k) = \operatorname{sgn}(\pi) \left(\frac{h_i h_j}{h_k} \right) dx^i \wedge dx^j$$

$$*(dx^j \wedge dx^k) = \operatorname{sgn}(\pi) \left(\frac{h_i}{h_j h_k} \right) dx^i$$

$$*(dx^i \wedge dx^j \wedge dx^k) = \frac{1}{h_1 h_2 h_3} \operatorname{sgn}(\pi).$$

Furthermore, if

$$\omega = \sum_{i=1}^{3} F_i \omega_i = \sum_{i=1}^{3} F_i h_i dx^i$$

$$\eta = \sum_{i=1}^{3} G_i \omega_i = \sum_{i=1}^{3} G_i h_i dx^i$$

and f is a function, then it is a straightforward computation to show that

$$df = \sum_{i=1}^{3} \left(\frac{1}{h_i} \frac{\partial f}{\partial x^i} \right) \omega_i$$

$$*d\omega = \begin{vmatrix} \frac{\omega_1}{h_2 h_3} & \frac{\omega_2}{h_1 h_3} & \frac{\omega_3}{h_1 h_2} \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

$$*d*\omega = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x^1}(h_2 h_3 F_1) + \frac{\partial}{\partial x^2}(h_1 h_3 F_2) + \frac{\partial}{\partial x^3}(h_1 h_2 F_3) \right]$$

$$*(\omega \wedge \eta) = \begin{vmatrix} \frac{\omega_1}{h_2 h_3} & \frac{\omega_2}{h_1 h_3} & \frac{\omega_3}{h_1 h_2} \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{vmatrix}$$

$$*(\omega \wedge *\eta) = F_1 G_1 + F_2 G_2 - F_3 G_3.$$

Thus the operations grad, curl, div, $\times$ and $\cdot$ from vector analysis are easily constructed from operations on differential forms and the correspondence is made clear by making the following identifications:

$$d^0 f \leftrightarrow \text{grad}\, f$$

$$*d^1 \omega \leftrightarrow \text{curl}\, \mathbf{F}$$

$$*d^1 * \omega \leftrightarrow \text{div}\, \mathbf{F}$$

$$*(\omega \wedge \eta) \leftrightarrow \mathbf{F} \times \mathbf{G}$$

$$*(\omega \wedge *\eta) \leftrightarrow \mathbf{F} \cdot \mathbf{G}.$$

Note that in vector analysis it is customary to identify flux vector fields (arising from 2-forms) with vector fields arising from 1-forms by means of the Hodge star operator. Furthermore, one has

$$*d(df) = *(dd)f = 0 \Rightarrow \text{curl}\,\text{grad}\, f = 0$$

$$*d * (*d\omega) = *d(**)d\omega = *(dd)\omega = 0 \Rightarrow \text{div}\,\text{curl}\,\mathbf{F} = 0$$

as well as the following identities which are used when integrating by parts.

$$*d * (f\omega) = *(d(f * \omega)) = *(df \wedge *\omega) - f *d * \omega \Rightarrow \text{div}\,(f\mathbf{F}) = (\text{grad}\, f) \cdot \mathbf{F} + f \text{div}\, \mathbf{F}$$

$$*d(f\omega) = *(df \wedge \omega) + f * d\omega \Rightarrow \text{curl}\,(f\mathbf{F}) = (\text{grad}\, f) \times \mathbf{F} + f \text{curl}\, \mathbf{F}$$

$$
\begin{aligned}
d (*(\omega \wedge \eta)) &= *(d * *(\omega \wedge \eta)) = *(d(\omega \wedge \eta)) \\
&= *((d\omega) \wedge \eta) - *(\omega \wedge d\eta) \\
&= *(*(*d\omega) \wedge \eta) - *(\omega \wedge *(*d\eta)) \\
\Rightarrow \operatorname{div}(\mathbf{F} \times \mathbf{G}) &= (\operatorname{curl} \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\operatorname{curl} \mathbf{G})
\end{aligned}
$$

Thus, once the algebraic rules for manipulating differential forms are understood, commonly used vector identities can be derived systematically.

End of Example 32

Example 33 (Two dimensional vector analysis)

Suppose x^1, x^2 are orthogonal curvilinear coordinates in a 2-dimensional manifold, that is

$$
g_{ij} = \begin{cases} h_i^2, & i = j; \\ 0, & i \neq j. \end{cases}
$$

so that $\omega^i = h_i dx^i, 1 \leq i \leq 2$ is an orthonormal basis for 1-forms. In this case

$$
*1 = \omega_1 \wedge \omega_2, \quad *\omega_1 = \omega_2, \quad *\omega_2 = -\omega_1, \quad *(\omega_1 \wedge \omega_2) = 1
$$

hence

$$
\begin{aligned}
*(dx^1 \wedge dx^2) &= \frac{1}{h_1 h_2} & *1 &= h_1 h_2 dx^1 \wedge dx^2 \\
*(dx^1) &= \frac{h_2}{h_1} dx^2 & *dx^2 &= -\frac{h_1}{h_2} dx^1.
\end{aligned}
$$

Furthermore, if

$$
\omega = \sum_{i=1}^{2} F_i \omega_i, \quad \eta = \sum_{i=1}^{2} G_i \omega_i
$$

and f is any function, then it is a straightforward computation to show that

$$
\begin{aligned}
df &= \left(\frac{1}{h_1} \frac{\partial f}{\partial x^1} \right) \omega_1 + \left(\frac{1}{h_2} \frac{\partial f}{\partial x^2} \right) \omega_2 \\
*df &= \left(-\frac{1}{h_2} \frac{\partial f}{\partial x^2} \right) \omega^1 + \left(\frac{1}{h_1} \frac{\partial f}{\partial x^1} \right) \omega_2
\end{aligned}
$$

$$
\begin{aligned}
*d\omega &= \frac{1}{h_1 h_2}\left(\frac{\partial}{\partial x^1}(h_2 F_2) - \frac{\partial}{\partial x^2}(h_1 F_1)\right) \\
d\omega &= \frac{1}{h_1 h_2}\left(\frac{\partial}{\partial x^1}(h_2 F_1) - \frac{\partial}{\partial x^2}(h_1 F_2)\right) \\
*(\omega \wedge *\eta) &= F_1 G_1 + F_2 G_2.
\end{aligned}
$$

Thus the operators grad , $\overline{\text{curl}}$, curl , div and $\cdot$ are easily constructed from operations on differential forms and the correspondence is made explicit by making the following identifications

$$
\begin{aligned}
d^0 f &\leftrightarrow \operatorname{grad} f \\
*d^0 f &\leftrightarrow \overline{\text{curl}}\, f \\
*d^1 \omega &\leftrightarrow \operatorname{curl} \mathbf{F} \\
*d^1 * \omega &\leftrightarrow \operatorname{div} \mathbf{F} \\
*(\omega \wedge *\eta) &\leftrightarrow \mathbf{F} \cdot \mathbf{G}.
\end{aligned}
$$

In addition one sees that

$$
\begin{aligned}
*d(df) = *(ddf) = 0 &\Rightarrow \operatorname{curl} \operatorname{grad} f = 0 \\
*d * (*df) = *d(**)df = - * ddf = 0 &\Rightarrow \operatorname{div} \overline{\text{curl}}\, f = 0 \\
*d * (df) = *d(*df) &\Rightarrow \operatorname{div} \operatorname{grad} f = - \triangle f = \operatorname{curl} \overline{\text{curl}}\, f
\end{aligned}
$$

and the following commonly used identities used when integrating by parts:

$$
\begin{aligned}
*d * (f\omega) &= *d(f * \omega) = *(df \wedge *\omega) + f * d * \omega \\
&\Rightarrow \operatorname{div}(f\mathbf{F}) = (\operatorname{grad} f) \cdot \mathbf{F} + f \operatorname{div} \mathbf{F} \\
*d(f\omega) &= *(d(f\omega)) \\
&= *(df \wedge \omega) + f * d\omega \\
&= - * [(*(*df)) \wedge \omega] + f * d\omega \\
&\Rightarrow \operatorname{curl}(f\omega) = -(\overline{\text{curl}}\, f) \cdot \mathbf{F} + f \operatorname{curl} \mathbf{F}.
\end{aligned}
$$

These are the formulas used in Nedelec [1978]. Once again with the use of the formalism of differential forms commonly used vector identities can be derived systematically.

End of Example 33

Hopefully the reader has realised that the formalism of differential forms encompasses the types of computations encountered in vector analysis and more general computations in n-dimensional manifolds. For simple calculations involving Maxwell's equations in four dimensions, see Flanders [1963] Sects. 2.7, 4.6 and Balasubramanian et al. [1970] Chapter 4. Returning to the topic of inner products, recall that for an orientable Riemannian manifold, the expression

$$*(\omega \wedge *\eta) \qquad \omega, \eta \in C^k(M)$$

can be used to define a smooth symmetric positive definite bilinear form on $\Lambda_k(M_p^*)$ for all $p \in M$. Hence let

$$\langle \omega, \eta \rangle_k = \int_M [*(\omega \wedge *\eta)]\, dV$$

be an inner product on $C^k(M)$. This inner product will be of fundamental importance in deriving orthogonal decompositions. Before moving on, there are three fundamental properties of the star operator which should be remembered. They are

$$\left.\begin{aligned} **\omega &= (-1)^{k(n-k)}\omega \\ \omega \wedge *\eta &= \eta \wedge *\omega \end{aligned}\right\} \eta, \omega \in C^k(M)$$

$$*dV = 1 \qquad \text{where } dV \text{ is the volume n form.}$$

These expressions enable one to express the above inner product in four different ways. Note that

$$\begin{aligned} [*(\omega \wedge *\eta)]\, dV &= **\left([*(\omega \wedge *\eta)]\, dV\right) \\ &= *[*(\omega \wedge *\eta)] \\ &= (\omega \wedge *\eta). \end{aligned}$$

Hence using this expression and the symmetry of the inner product gives

$$\langle\omega,\eta\rangle_k = \int_M *(\omega \wedge *\eta)\, dV = \int_M \omega \wedge *\eta$$
$$= \int_M *(\eta \wedge *\omega)\cdot dV = \int_M \eta \wedge *\omega.$$

For simplicity, assume that M is compact. The above inner product makes $C^k(M)$ into a Hilbert space. This is the first step toward obtaining useful orthogonal decompositions.

2.11 The Operator Adjoint to the Exterior Derivative

Having an inner product on the exterior k-bundle of an orientable Riemannian manifold M (which will henceforth be assumed compact) and an operator

$$d^p : C^p(M) \to C^{p+1}(M),$$

one wants to know the form of the Hilbert space adjoint

$$\delta_{p+1} : C^{p+1}(M) \to C^p(M)$$

which satisfies the following equation:

$$\langle d^p\omega, \eta\rangle_{p+1} = \langle\omega, \delta_{p+1}\eta\rangle_p + \text{ boundary terms} \qquad \text{for } \omega \in C^p(M), \quad \eta \in C^{p-1}(M).$$

This type of formula will now be deduced from the integration by parts formula which was developed as a corollary to Stokes' Theorem. Let

$$\omega \in C^p(M), \quad \mu \in C^{n-p-1}(M)$$

then

$$\int_M d\omega \wedge \mu = \int_{\partial M} t\omega \wedge t\mu - (-1)^p \int_M \omega \wedge d\mu.$$

Next let $\mu = *\eta$ for some $\eta \in C^{p+1}(M)$ so that

$$\begin{aligned}\langle d\omega, \eta\rangle_{p+1} &= \int_M d\omega \wedge *\eta \\ &= \int_{\partial M} t\omega \wedge t(*\eta) - (-1)^p \int_M \omega \wedge d(*\eta).\end{aligned}$$

However, using the fact that $(-1)^{p(n-p)} * *\gamma = \gamma,\ \gamma \in C^p(M)$ and

$$-(-1)^p(-1)^{p(n-p)} = (-1)^{np+1-p(1-p)} = (-1)^{np-1}$$

one has

$$\begin{aligned}\langle d\omega, \eta\rangle_{p+1} &= -(-1)^p \int_M \omega \wedge d(*\eta) + \int_{\partial M} t\omega \wedge t(*\eta) \\ &= -(-1)^p(-1)^{p(n-p)} \int_M \omega \wedge *(*d * \eta) + \int_{\partial M} t\omega \wedge t(*\eta) \\ &= \int_M \omega \wedge * \left[(-1)^{np+1} * d * \eta\right] + \int_{\partial M} t\omega \wedge t(*\eta).\end{aligned}$$

Hence

$$\langle d^p\omega, \eta\rangle_{p+1} = \langle \omega, \delta_{p+1}\eta\rangle_p + \int_{\partial M} t\omega \wedge t(*\eta)$$

where

$$\delta_{p+1} = (-1)^{np+1} * d^{n-p-1} * \qquad \text{on } (p+1)\text{–forms.}$$

In order to gain an intuitive understanding of what is happening on the boundary, let us rework the boundary term. Up to now the operator t which gives the tangential components of a differential form was considered to be the pull back on differential forms induced by the map

$$i : \partial M \to M.$$

That is, $t = i^{\#}$. Given a Riemannian metric on M and if ∂M is smooth then given a point $p \in \partial M$ one can find a set of orthogonal curvilinear coordinates such that

p has coordinates $(0,\ldots,0)$, ∂M has local coordinates $(u^1,\ldots,u^{n-2},u^{n-1},0)$, and $(u^1,u^2,\ldots,u^n)$, $u^n \leq 0$ are a set of local coordinates in M. In terms of these local coordinates a k-form looks like:

$$\omega = \Sigma_{1\leq i_1<i_2<\ldots<i_k\leq n} a_{i_1 i_2\ldots i_k} du^{i_1} \wedge du^{i_2} \wedge \ldots \wedge du^{i_k}.$$

On ∂M the component of this form tangent to ∂M is given by

$$t\omega = \Sigma_{1\leq i_1<i_2<\ldots<i_k<n} a_{i_1 i_2\ldots i_k} du^{i_1} \wedge du^{i_2} \wedge \ldots \wedge du^{i_k}$$

while the normal component is given by

$$n\omega = \omega - t\omega.$$

It is apparent that each term in $n\omega$ involves dx^n. This definition of the normal component of a differential form seems to be due to Duff [1952] and is heavily used in subsequent literature (see for instance the papers by Duff, Spencer, Morrey, and Connor in the bibliography). By considering the k-form ω written as

$$\omega = t\omega + n\omega$$

in the above orthogonal coordinate system, it is apparent that $*\omega$ can be decomposed in the following two ways:

$$*(\omega) = *(t\omega) + *(n\omega)$$

$$(*\omega) = t(*\omega) + n(*\omega).$$

Thus subtracting the above two equations, one deduces that

$$*t\omega - n*\omega = *n\omega - t*\omega.$$

Noticing that each term in the right hand side of this equation involves dx^n and that no term in the left hand side involves dx^n one has:

$$t*\omega = *n\omega$$

$$n * \omega = *t\omega.$$

Furthermore, since exterior differentiation commutes with pull backs one has

$$dt\omega = td\omega$$

and

$$dt * \omega = td * \omega.$$

This formula as well as the above formulas relating normal and tangential components can be used to derive the following identity:

$$\begin{aligned} dt * \omega &= td * \omega \\ \Rightarrow *dt * \omega &= *td * \omega \\ \Rightarrow *d * n\omega &= n * d * \omega \\ \Rightarrow \delta n\omega &= n\delta\omega. \end{aligned}$$

Thus in summary

$$\begin{aligned} n\omega &= \omega - t\omega \\ n * \omega &= *t\omega \\ *n\omega &= t * \omega \\ dt\omega &= td\omega \\ \delta n\omega &= n\delta\omega. \end{aligned}$$

Finally, the above identities can be used to rewrite the integration by parts formula. From the above identities involving the normal components of a differential form, one has

$$\langle d^* \omega, \eta \rangle = \langle \omega, \delta_{k+1}\eta \rangle + \int_{\partial M} t\omega \wedge *n\eta.$$

Next, suppose $\partial M = S_1 \cup S_2$ where $S_1 \cap S_2$ is $(n-2)$-dimensional and where S_1 and S_2 are collections of connected components of ∂M or parts of M where symmetry planes exist. In this latter case S_1 and S_2 may not be disconnected but meet at right angles. The above integration by parts formula can then be reworked into the following form which will be essential in the derivation of orthogonal decompositions:

$$\langle d^k\omega, \eta\rangle_{k+1} - \int_{S_1} t\omega \wedge *n\eta = \langle \omega, \delta_{k+1}\eta\rangle_k + \int_{S_2} t\omega \wedge *n\eta.$$

2.12 The Hodge Decomposition

On a compact orientable Riemannian manifold, an inner product structure on $C^{\cdot}(M)$ and an operator adjoint to the exterior derivative enables one to define the Laplace-Beltrami operator

$$\triangle_p = d^{p-1}\delta_p + \delta_{p+1}d^p$$

(an elliptic operator on p-forms) and harmonic forms (solutions of the equation $\Delta\omega = 0$). Furthermore, when the manifold has no boundary, one has the Hodge decomposition theorem which generalises the Helmholtz Theorem of vector analysis. For compact orientable manifolds without boundary, the Hodge decomposition theorem asserts that

$$C^p(M) = \text{Image}\,(d^{p-1}) \oplus \text{Image}\,(\delta_{p+1}) \oplus \mathcal{H}^p(M),$$

where $\mathcal{H}^p(M)$ is the space of harmonic p-forms. Using the tools of elliptic operator theory and the de Rham isomorphism, one can show that

$$\dim \mathcal{H}^p(M) = \beta_p(M) < \infty$$

and that the basis vectors for the de Rham cohomology vector spaces may be represented by harmonic forms. A self contained proof of the Hodge decomposition as well as an explanation of the relevant machinery from elliptic operator theory can be found in Warner [1971] Chapter 6. Alternatively, a short and sweet account of the Hodge decomposition theorem along the lines of this chapter is given in Flanders [1963] Section 6.4 while a nice proof of the theorem in the case of 2-dimensional surfaces is usually given in any decent book on Riemann surfaces (see for example Springer [1957] or Schiffer and Spencer [1954]).

For orthogonal decompositions of p-forms on orientable Riemannian manifolds with boundary, the tools of elliptic operator theory are less sucessful in obtaining a nice orthogonal decomposition which relates harmonic forms to the relative cohomology groups of the manifold. The history of this problem starts with the papers of Kodaira [1948],Duff and Spencer [1952], and ends with the work of Friedrichs [1955], Morrey [1956], and Connor [1956]. A general reference for this problem is the book by Morrey [1966] Chapter 7. The basic problem encountered in the case of a manifold with boundary is that the space of harmonic p-forms is generally infinite dimensional and the questions of regularity at the boundary are quite thorny. There is a way of getting an orthogonal decomposition for p-forms on manifolds with boundary which completely avoids elliptic operator theory by defining harmonic p-fields (p-forms which satisfy $d^p\omega = 0, \delta_p\omega = 0$). Such a decomposition is called a Kodaira decomposition after Kunihiko Kodaira [1948] who introduced the notion of a harmonic field and the associated decompositions of p-forms. It turns out that for compact orientable Riemannian manifolds without boundary the proof of the Hodge decomposition theorem shows that harmonic fields and harmonic forms are equivalent concepts. Thus the decomposition of Kodaira in some way generalises the Hodge decomposition and variants of it will now be derived by using the differential cochain complexes already defined.

2.13 Orthogonal Decompositions of p-Forms and Duality Theorems

The immediate objective is to show that the structure of a complex with an inner product enables one to derive useful orthogonal decompositions of p-forms. As usual let M be a compact orientable n-dimensional Riemannian manifold with boundary where

$$\partial M = S_1 \cup S_2$$

and $S_1 \cap S_2$ is an $n-2$ dimensional manifold where a symmetry plane meets the boundary of some original problem at right angles. Consider the cochain complexes $C^*(M), C_c^*(M - S_1)$ and recall that

$$C_c^p(M - S_1) = \{\omega \mid \omega \in C^p(M), t\omega = 0 \text{ on } S_1\}$$

$$Z_c^p(M - S_1) = \{\omega \mid \omega \in C_c^p(M - S_1), d\omega = 0\}$$

$$B_c^p(M - S_1) = \{\omega \mid \omega = d\nu, \nu \in C_c^{p-1}(M - S_1)\}$$

$$H_c^p(M - S_1) = Z_c^p(M - S_1)/B_c^p(M - S_1).$$

Next define the complex $\widetilde{C}_*(M, S_2)$ where

$$\widetilde{C}_p(M, S_2) = \{\omega \mid \omega \in C^p(M), n\omega = 0 \text{ on } S_2\}$$

and the "boundary operator" in this complex is the Hilbert space formal adjoint δ of the exterior derivative d. Note that $\tau_p(M, S_2)$ is actually a complex since if η is a $p-1$-form in this complex then

$$n\eta = 0 \text{ on } S_2 \Rightarrow \delta_{p+1} n\eta = n(\delta_{p+1}\eta) = 0 \text{ on } S_2$$

and

$$(-1)^n \delta_p \delta_{p+1}\eta = *d^{n-p} * *d^{n-p-1} * \eta = (-1)^{p(n-p)} * d^{n-p} d^{n-p-1} * \eta = 0.$$

Hence define

$$\begin{aligned}\widetilde{Z}_p(M,S_2) &= \left\{\eta \mid \eta \in \widetilde{C}_p(M,S_2), \delta_p\eta = 0\right\}\\ \widetilde{B}_p(M,S_2) &= \left\{\eta \mid \eta = \delta_{p+1}\gamma; \gamma \in \widetilde{C}_{p+1}(M,S_2)\right\}\\ \widetilde{H}_p(M,S_2) &= \widetilde{Z}(M,S_2)/\widetilde{B}_p(M,S_2).\end{aligned}$$

The "cycles" of this complex are called coclosed differential forms while the "boundaries" are called coexact differential forms. The first step in deriving an orthogonal decomposition of p-forms on $C^p(M)$ is to recall the inner product version of the integration by parts formula:

$$\langle d^k\omega, \eta\rangle_{k+1} - \int_{S_1} t\omega \wedge *n\eta = \langle \omega, \delta_{k+1}\eta\rangle_k + \int_{S_2} t\omega \wedge *n\eta.$$

If $k = p$ and $\omega \in Z_c^p(M - S_1)$ then the left side of this formula vanishes and it is seen that closed p-forms are orthogonal to coexact p-forms in $\widetilde{C}_*(M,S_2)$. Alternatively, if $k+1 = p$ and $\eta \in \widetilde{Z}_p(M,S_2)$ then it is easily seen that coclosed p-forms are orthogonal to the exact p-forms in $C^a st_c(M - S_1)$. Actually, one has

$$\left.\begin{aligned}C^p(M) &= Z_c^p(M - S_1) \oplus \widetilde{B}_p(M,S_2)\\ C^p(M) &= \widetilde{Z}_p(M,S_2) \oplus B_c^p(M, M - S_1)\end{aligned}\right\} \qquad (1)$$

Since these identitities express the fact that if A is an operator between Hilbert spaces, then

$$(\text{Image}\,(A))^{\perp} = \text{Kernel}\,(A^{adj}).$$

Next by virtue of the fact that $C_c^p(M - S_1)$ and $\widetilde{C}_p(M,S_2)$ are complexes, one has

$$\left.\begin{aligned}\widetilde{B}_p(M,S_2) &\subset \widetilde{Z}_p(M,S_2)\\ B_c^p(M - S_1) &\subset B_c^p(M - S_1)\end{aligned}\right\} \qquad (2)$$

Finally, defining the relative harmonic p-fields as

$$\mathcal{H}^p(M,S_1) = \widetilde{Z}_p(M,S_2) \cap Z^p(M - S_1)$$

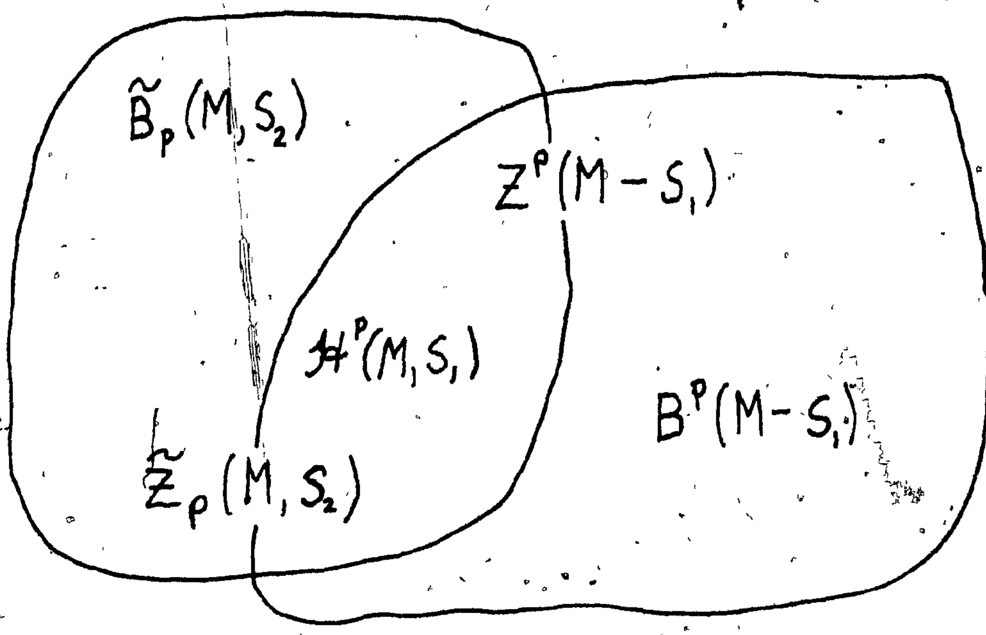


Fig. 21

the orthogonal decomposition is immediate once Equations (1) and (2) above are expressed in terms of a Venn diagram of orthogonal spaces as shown in Fig. 21. Thus

$$\widetilde{Z}_p(M,S_2) = \widetilde{B}_p(M,S_2) \oplus \mathcal{H}^p(M,S_1)$$

$$Z^p_c(M-S_1) = B^p_c(M-S_1) \oplus \mathcal{H}^p(M,S_1)$$

$$C^p(M) = \widetilde{B}_p(M,S_2) \oplus \mathcal{H}^p(M,S_1) \oplus B^p_c(M-S_1)$$

where the direct summands are mutually orthogonal with respect to the inner product $\langle\ ,\ \rangle_p$. In order to relate this orthogonal decomposition to the relative cohomology of the pair (M,S_1) consider the identities

$$Z^p_c(M-S_1) = B^p_c(M-S_1) \oplus \mathcal{H}^p(M,S_1)$$

$$H^p_c(M-S_1) = Z^p_c(M-S_1)/B^p_c(M-S_1)$$

and so, one has

$$H^p_c(M-S_1) \simeq \mathcal{H}^p(M,S_1)$$

that is, in each de Rham cohomology class there is exactly one harmonic field. A more concrete way of seeing this is to write the above orthogonal decomposition explicity in terms of differential forms and use the de Rham isomorphism. That is, if $\omega \in C^p(M)$ then ω can be decomposed into three unique, mutually orthogonal factors as follows

$$\omega = d\nu + \delta\gamma + \chi$$

where $\nu \in C_c^{p-1}(M - S_1), \gamma \in \widetilde{C}_{p+1}(M, S_2), \chi \in \mathcal{H}^p(M, S_2)$. Furthermore,

$$\omega \in Z_c^p(M - S_1) \Rightarrow \omega = d\nu + \chi$$

$$\omega \in \widetilde{Z}_p(M, S_1) \Rightarrow \omega = \delta\gamma + \chi.$$

Thus if $\omega \in Z_c^p(M - S_1)$ and $z_p \in Z_p(M, S_1)$ then

$$\int_{z_p} \omega = \int_{z_p} d\nu + \int_{z_p} \chi = \int_{z_p} \chi \quad \text{since} \int_{z_p} d\nu = 0.$$

Hence there is at least one harmonic field in each de Rham cohomology class. However it is easy to show that there cannot be more than one distinct harmonic field in each de Rham cohomology class. Suppose that

$$\omega_1 - \omega_2 = d\beta \qquad \beta \in C_c^{p-1}(M - S_1)$$

and

$$\omega_1 = d\nu_1 + h_1$$

$$\omega_2 = d\nu_2 + h_2$$

then

$$h_1 - h_2 = d(\beta - \nu_1 + \nu_2) \in B_c^p(M - S_1)$$

but by the orthogonal decomposition

$$h_1 - h_2 \in \mathcal{H}^p(M, S_1), \quad \mathcal{H}^p(M, S_1) \perp B_c^p(M - S_1)$$

so $h_1 = h_2$ and there is necessarily exactly one harmonic field in each de Rham cohomology class. Thus, explicity, it has been shown that

$$H^p_c(M - S_1) \simeq \mathcal{H}^p(M, S_1)$$

hence

$$\infty > \beta_p(M, S_1) = \beta^p_c(M - S_1) = \dim \mathcal{H}^p(M, S_1)$$

where the first equality results from the de Rham theorem. The above isomorphism shows that the projection of ω on $\mathcal{H}^p(M, S_1)$ is deduced from the periods of ω on a basis of $H_p(M, S_1)$.

At this point it is best to summarise the above discussion in the form of a theorem.

Theorem (Orthogonal Decomposition of p-forms). Given M, S_1, S_2 as usual and

$$\mathcal{H}^p(M, S_1) = Z^p_c(M - S_1) \cap \widetilde{Z}_p(M, S_2)$$

one has

1) direct sum decompositions:

$$C^p(M) = B^p_c(M - S_1) \oplus \mathcal{H}^p(M, S_1) \oplus \widetilde{B}_p(M, S_2)$$

$$Z^p_c(M - S_1) = B^p_c(M - S_1) \oplus \mathcal{H}^p(M, S_1)$$

$$\widetilde{Z}_p(M, S_2) = \widetilde{B}_p(M, S_2) \oplus \mathcal{H}^p(M, S_1)$$

where the direct summands are mutually orthogonal with respect to the inner product $\langle \; , \; \rangle_p$;

2) a unique harmonic field in each de Rham cohomology class, that is, an isomorphism

$$\mathcal{H}^p(M, S_1) \simeq H^p_c(M - S_1)$$

so that

$$\infty > \beta_p(M, S_1) = \beta_c^p(M - S_1) = \dim \mathcal{H}^p(M, S_1).$$

Having established the orthogonal decomposition theorem, it is useful to see how duality theorems come about as a result of the Hodge star operator. As a preliminary, several formulas must be derived. Recall that if $\omega \in C^p(M)$ then

$$* * \omega = (-1)^{p(n-p)}\omega, \quad \delta_p\omega = (-1)^{n(p+1)+1} * d * \omega.$$

Thus

$$*\delta_p = (-1)^{n(p+1)+1} * * d^{n-p} * = (-1)^{n(p+1)+1+(p+1)(n-p-1)} d^{n-p} *$$

$$*d^{n-p} = (-1)^{p(n-p)} * d^{n-p} * * = (-1)^{p(n-p)+n(p+1)+1} \delta_p * .$$

Cleaning up the exponents with modulo 2 arithmetic gives

$$*\delta_p = (-1)^p d^{n-p} *, \quad *d^{n-p} = -(-1)^{n-p} \delta_p *$$

and hence

$$*\delta_{n-p} = (-1)^{n-p} d_p *, \quad *d^p = -(-1)^p \delta_{n-p} * .$$

Next recall the identities

$$*t = n*, \quad *n = t * .$$

The above six formulas will now be used to make some useful observations. Let $\lambda \in C^k(M)$, and $\mu \in C^{n-k}(M)$ where $*\lambda = \mu$. In this case one has

$$\delta_k \lambda = 0 \Leftrightarrow 0 = *\delta_k \lambda = (-1)^k d^{n-k} * \lambda = (-1)^k d^{n-k} \mu$$

$$d^k \lambda = 0 \Leftrightarrow 0 = *d^k \lambda = -(-1)^k \delta_{n-k} * \lambda = -(-1)^k \delta_{n-p} \mu$$

$$t\lambda = 0 \Leftrightarrow 0 = *t\lambda = n * \lambda = n\mu$$

$$n\lambda = 0 \Leftrightarrow 0 = *n\lambda = t * \lambda = t\mu.$$

In other words, if $\lambda, \mu \in C^{\cdot}(M)$ and $*\lambda = \mu$ then

$$\delta\lambda = 0 \Leftrightarrow d\mu = 0, \qquad d\lambda = 0 \Leftrightarrow \delta\mu = 0$$

$$t\lambda = 0 \Leftrightarrow n\mu = 0, \qquad n\lambda = 0 \Leftrightarrow t\mu = 0.$$

With a little reflection, the above four equivalences show that the Hodge star operator induces the following isomorphisms, (here $1 \leq i \leq 2, 1 \leq k \leq n$)

$$C^p(M) \simeq C^{n-p}(M)$$

$$\widetilde{C}_k(M, S_i) \simeq C_c^{n-k}(M - S_i)$$

$$\widetilde{Z}_k(M, S_i) \simeq Z_c^{n-k}(M - S_i)$$

$$\widetilde{B}_k(M, S_i) \simeq B_c^{n-k}(M - S_i)$$

$$\widetilde{H}_k(M, S_i) \simeq H_c^{n-k}(M - S_i).$$

What is particularly interesting is the following computation: (here $1 \leq i, j \leq 2, i \neq j, 1 \leq l \leq n$)

$$\begin{aligned} *\mathcal{H}^l(M, S_i) &= *\left[(Z_c^l(M - S_i)) \cap \left(\widetilde{Z}_l(M, S_j)\right)\right] \\ &= \left[*(Z_c^l(M - S_i))\right] \cap \left[*\left(\widetilde{Z}_l(M, S_j)\right)\right] \\ &= \left(\widetilde{Z}_{n-l}(M, S_i)\right) \cap (Z_c^{n-l}(M - S_j)) \\ &= \mathcal{H}^{n-l}(M, S_j). \end{aligned}$$

In order to interpret this result, notice that the derivation of the orthogonal decomposition is still valid if S_1 and S_2 are interchanged everywhere. Hence juxtaposing the following two orthogonal decompositions,

$$C^p(M) = B_c^p(M - S_1) \oplus \mathcal{H}^p(M, S_1) \oplus \widetilde{B}_p(M, S_2)$$

$$C^{n-p}(M) = \widetilde{B}_{n-p}(M, S_1) \oplus \mathcal{H}^{n-p}(M, S_2) \oplus B_c^{n-p}(M - S_2)$$

it is seen that each term in the above decompositions is related to the one directly above or below it by the Hodge star operator. Also, the star operation performed twice maps $\mathcal{H}^p(M, S_1), \mathcal{H}^{n-p}(M, S_2)$ isomorphically back onto themselves, since in this case

$$(-1)^{p(n-p)} ** = \text{Identity}.$$

Hence

$$\mathcal{H}^p(M, S_1) \simeq \mathcal{H}^{n-p}(M, S_2).$$

At this point it is useful to summarise the isomorphisms in (co)homology derived in this chapter where coefficients are taken in $\mathbb{R}$ (of course) and M is an orientable, compact, n-dimensional Riemannian manifold with boundary where $\partial M = S_1 \cup S_2$ in the usual way. This is best expressed in the form of the following theorem

Theorem:

$$\begin{array}{ccccc} \mathcal{H}^k(M, S_1) & \simeq & H_c^k(M - S_1) & \simeq & H_k(M, S_1) \\ \wr\vert & & & & \\ \mathcal{H}^{n-k}(M, S_2) & \simeq & H_c^{n-k}(M - S_2) & \simeq & H_{n-k}(M, S_2) \end{array}$$

This theorem expresses the relative de Rham isomorphism (on the right), the representability of relative de Rham cohomology classes by harmonic fields (in the center), and the duality isomorphism induced on harmonic fields by the Hodge star (on the left). For the inspiration behind this theorem see Connor [1954].

In order to let the orthogonal decomposition sink in, it is useful to rewrite it in terms of differential forms and then consider several concrete examples. Thus consider the following theorem which restates the orthogonal decompostion theorem in a more palatable way:

Theorem: If M, S_1, and S_2 have their usual meaning and $\omega \in C^p(M)$ then ω has the unique representation

$$\omega = d\nu + \delta\gamma + \chi$$

where

$$t\nu = 0, \quad t\chi = 0 \qquad on\ S_1$$

$$n\gamma = 0, \quad n\chi = 0 \qquad on\ S_2$$

$$d\chi = 0, \quad \delta\chi = 0 \qquad in\ M\ and\ on\ \partial M.$$

Furthermore,

1) the three factors are mutually orthogonal with respect to the inner product $\langle\ ,\ \rangle_p$

2) if $d\omega = 0$ in M and $t\omega = 0$ on S_1 then one can take $\gamma = 0$

3) if $\delta\omega = 0$ in M and $n\omega = 0$ on S_2 then one can take $\nu = 0$.

Let the theorem be illustrated by a couple of examples.

Example 34 (Three dimensional vector analysis, $n = 3, p = 1$)

In vector analysis it is customary to identify flux vector fields (arising from 2-forms) with vector fields arising from 1-forms by means of the Hodge star operator. Keeping this in mind, the identifications established in Example 32 show that in the case of $n = 3, p = 1$ the above theorem can be rewritten as follows. If M is an orientable, compact three dimensional manifold with boundary embeded in $\mathbb{R}^3$ then any vector field $\mathbf{V}$ can be uniquely expressed as

$$\mathbf{V} = \operatorname{grad}\varphi + \operatorname{curl}\mathbf{F} + \mathbf{G}$$

where

$$\varphi = 0, \quad \mathbf{n}\times\mathbf{G} = 0 \qquad on\ S_1$$

$$\mathbf{F}\cdot\mathbf{n} = 0, \quad \mathbf{n}\cdot\mathbf{G} = 0 \qquad on\ S_2$$

$$\operatorname{curl}\mathbf{G} = 0, \quad \operatorname{div}\mathbf{G} = 0 \qquad in\ M\ and\ on\ \partial M.$$

Furthermore,

1) the three factors are mutually orthogonal with respect to the inner product:

$$\langle \mathbf{U}, \mathbf{V} \rangle_1 = \int_M \mathbf{U} \cdot \mathbf{V} \, dV;$$

2) if $\operatorname{curl} \mathbf{V} = 0$ in M and $\mathbf{n} \times \mathbf{V} = 0$ on S_1 then $\mathbf{F}$ may be set equal to zero;

3) if $\operatorname{div} \mathbf{V} = 0$ in M and $\mathbf{V} \cdot \mathbf{n} = 0$ on S_2 then φ may be set equal to zero.

Note that in practical problems it is customary to describe $\mathbf{G}$ by a (possibly multivalued) scalar potential and that the dimension of the space of harmonic vector fields which satisfy the conditions imposed on $\mathbf{G}$ is $\beta_1(M - S_1) = \beta_2(M - S_2)$.

End of Example 34

Example 35 (Two dimensional vector analysis $n = 2, p = 1$)

Let M be an orientable compact 2-dimensional Riemanian manifold with boundary where $\partial M = S_1 \cup S_2$ in the usual way. Using the identifications established in Example 33 one can rephrase the orthogonal decomposition theorem as follows. Any vector field $\mathbf{V}$ on M can be written as

$$\mathbf{V} = \operatorname{grad} \phi + \overline{curl \psi} + \mathbf{G}$$

where

$$\begin{aligned} \phi &= 0, & \mathbf{n} \times \mathbf{G} &= 0 & &\textit{on } S_1 \\ \psi &= 0, & \mathbf{G} \cdot \mathbf{n} &= 0 & &\textit{on } S_2 \\ \operatorname{curl} \mathbf{G} &= 0, & \operatorname{div} \mathbf{G} &= 0 & &\textit{in } M \textit{ and on } \partial M. \end{aligned}$$

Furthermore,

1) the three factors are mutually orthogonal with respect to the inner product given by the metric tensor in the usual way:

$$\langle \mathbf{U}, \mathbf{V} \rangle_1 = \int_M \mathbf{U} \cdot \mathbf{V} dV;$$

2) if $\text{curl}\,\mathbf{V} = 0$ in M and $\mathbf{n} \times \mathbf{V} = 0$ on S_1 then ψ may be set equal to zero;

3) if $\text{div}\,\mathbf{V} = 0$ in M and $\mathbf{V} \cdot \mathbf{n} = 0$ on S_2 then ϕ may be set equal to zero.

Note that in practical problems $\mathbf{G}$ is invariably described in terms of a (possibly multi-valued) scalar potential or stream function and that the dimension of the space of harmonic vector fields which satisfy the conditions imposed on $\mathbf{G}$ is $\beta_1(-S_1) = \beta_1(M - S_2)$.

End of Example 35

One final remark is appropriate. In the case of electrodynamics there is no positive definite inner product on p-forms since the metric tensor is not positive definite. One can, however, define all of the spaces found in the orthogonal decomposition and obtain a direct sum decomposition of $C^p(M)$ even though there is no positive definite inner product.

CHAPTER 3

A Paradigm Problem in Electromagnetic Theory

"The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact."

A.N. Whitehead
Science and the modern world, 1925

"It is important for him who wants to discover not to confine himself to one chapter of science, but to keep in touch with various others."

Jacques Hadamard
The psychology of invention in the mathematical field

"It has been said many times that Geometry is the art of correct reasoning supported by incorrect figures, but in order not to be misleading, these figures must satisfy certain conditions..."

Henri Poincaré
Analysis Situs Paper, 1895

3.1 Introduction to the Paradigm Problem

The purpose of this chapter is to show how the formalism of differential forms reduces various broad classes of problems in computational electromagnetics to a common form. For this class of problems, the differential complexes and orthogonal decompositions associated with differential forms make questions of existence and uniqueness of solution simple to answer in a complete way which exposes the role played by relative homology groups. When this class of problems is formulated variationally, the orthogonal decomposition theorem developed in the last chapter generalises certain well known interrelationships between gauge transformations and conservation laws (see Tonti [1968]) to include global conditions between dual cohomology groups. The orthogonal decomposition theorem can then be used to construct an alternate variational principle whose unique extremal always exists and can be used to obtain a posteriori measures of problem solvability, that is to verify if any conservation law was violated in the statement of the problem. Finally, a diagrammatic representation of the problem along the lines of Tonti [1972a] will be given and the role of homology groups will be reconsidered in this context. This of course will be of interest to people working in the area complementary variational principles. In addition to the usual literature cited in the bibliography, the work of Tonti [1968], [1969], [1972a], [1972b] and [1977], Sibner and Sibner [1970], [1979], [1981] and Kotiuga [1982] have been particularly useful in developing the ideas presented in this chapter.

The paradigm problem of this chapter will now be considered. Let M be a compact orientable n-dimensional Riemannian manifold with boundary. In the paradigm problem to be considered, the field is described by two differential forms

$$\beta \in C^p(M), \qquad \eta \in C^{n-p}(M)$$

which are related to another differential form

$$\lambda \in C^{n-p+1}(M)$$

which describes the sources in the problem. These differential forms are required to satisfy the following pair of equations:

$$\int_{\partial c_{p+1}} \beta = 0$$

$$\int_{\partial c_{n-p+1}} \eta = \int_{c_{n-p+1}} \lambda$$

for all $c_{p+1} \in C_{p+1}(M), c_{n-p+1} \in C_{n-p+1}(M)$. If S is a set of $n-1$ dimensional interface surfaces where β may be discontinuous, the first integral equation implies that:

$$d\beta = 0 \qquad on\ M - S.$$

Also one can define an orientation on S so that there is a plus side and a minus side and

$$t\beta^{+} = t\beta^{-} \qquad \text{as } S \text{ is traversed.}$$

It is natural to inquire whether there exists a potential

$$\alpha \in C^{p-1}(M)$$

such that

$$\beta = d\alpha.$$

In other words, the first integral equation shows that β is a closed form and one would like to know whether it is exact. The answer, of course, is given by the de Rham isomorphism, that is, β is exact if all of its periods vanish on a basis of the homology group $H_p(M)$. In addition to the above structure, the paradigm problem to be considered has a constitutive relation relating β and η. Further consideration will not be given to this constitutive relation until the next section.

Although various boundary conditions can be imposed on β and η so that a boundary value problem can be defined, to simplify the presentation it is assumed in accordance with the general philosophy of this thesis that

$$\partial M = S_1 \cup S_2$$

$$t\beta = 0 \qquad on\ S_1$$

$$t\eta = 0 \qquad on\ S_2$$

where $S_1 \cap S_2$ is an $(n-2)$-dimensional manifold whose connected components represent intersections between symmetry planes and connected components of the boundary of an original problem where symmetries were not exploited. Finally, before going on, it is important to list the specific problems which occur as special cases of this general problem. They are:

1) Electrodynamics in four dimensions.

2) Electrostatics in three dimensions.

3) Magnetostatics in three dimensions.

4) Currents in three dimensional conducting bodies where displacement currents are neglected.

5) Low frequency steady or eddy current problems where currents are confined to surfaces which are modelled as two dimensional manifolds. In this case, the local sources or "excitation" is the time variation of the magnetic field transverse to the surface.

6) Magnetostatics problems which are two dimensional in nature because of translational or rotational symmetry in a given three dimensional problem.

7) Electrostatics problems which are two dimensional in nature because of translational or rotational symmetry in a given three dimensional problem.

Note that the last two problems have not been discussed so far in this thesis because of their "topologically uninteresting" properties. They are included here for completeness, and a word of caution is in order. For two dimensional problems which arise from axially symmetric three dimensional problems it is important to remember that the metric tensor on M is not the one inherited from $\mathbb{R}^3$ but rather is a function of the distance from the axis of symmetry.

Tables 1 and 2 summarise the correspondence between the paradigm problem defined in terms of differential forms and the various cases listed above. Note that two dimensional problems are assumed to be embedded in a three dimensional space where $\mathbf{n}'$ is the unit normal to M. Also Table 2 lists examples considered so far in this thesis which are useful for sorting out topological or other details.

	n,p	α	β	η	λ	$t\beta = 0$ *on* S_1	$t\eta = 0$ *on* S_2
1	4,2	$\mathbf{A},\phi$	$\mathbf{E},\mathbf{B}$	$\mathbf{D},\mathbf{H}$	$\mathbf{J},\rho$	$\mathbf{n}\times\mathbf{E}=0$ $\mathbf{B}\cdot\mathbf{n}=0$	$\mathbf{n}\times\mathbf{H}=0$ $\mathbf{D}\cdot\mathbf{n}=0$
2	3,1	ϕ	$\mathbf{E}$	$\mathbf{D}$	ρ	$\mathbf{n}\times\mathbf{E}=0$	$\mathbf{D}\cdot\mathbf{n}=0$
3	3,2	$\mathbf{A}$	$\mathbf{B}$	$\mathbf{H}$	$\mathbf{J}$	$\mathbf{B}\cdot\mathbf{n}=0$	$\mathbf{n}\times\mathbf{H}=0$
4	3,2	$\mathbf{H}$	$\mathbf{J}$	$\mathbf{E}$	$-\frac{\partial\mathbf{B}}{\partial t}$	$\mathbf{J}\cdot\mathbf{n}=0$	$\mathbf{n}\times\mathbf{E}=0$
5	2,1	ψ	$\mathbf{n}'\times\mathbf{J}$	$\mathbf{E}$	$-\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{n}'$	$J_n=0$	$E_t=0$
6	2,1	A_n	$\mathbf{n}'\times\mathbf{B}$	$\mathbf{H}$	$\mathbf{J}\cdot\mathbf{n}'$	$B_n=0$	$H_t=0$
7	2,1	ϕ	$\mathbf{E}$	$\mathbf{n}'\times\mathbf{D}$	$\rho/length$	$E_t=0$	$D_n=0$

Table 1

Example of paradigm problem	Previous example relevant
1	31
2	5,6,12,15,19,23,32,34
3	8,13;16,19,22,23,32,34
4	8,14,17,19,23,32,34
5	4,7,11,18,19,33,35
6	7,33,35
7	7,33,35

Table 2

Thus, in summary, the paradigm problem considered in this chapter takes place on an oriented compact n-dimensional Riemannian manifold M where $S_1 \cup S_2 = \partial M$ in the usual way and

a) there is a $\beta \in Z_c^p(M - S_1)$ for which one would like to find a potential $\alpha \in C_c^{p-1}(M)$ such that

$$\beta = d\alpha.$$

In the language of Tonti [1972a] this equation is called the defining equation.

b) There is an $\eta \in C_c^{n-p}(M - S_2)$ and a prescribed $\lambda \in B_c^{n-p+1}(M - S_2)$ such that

$$d\eta = \lambda$$

in order to satisfy the integral equation relating η and λ. Thus when prescribing λ, care must be taken to ensure that λ is an exact form in the relative sense. In the language of Tonti [1972a] this equation is called a balance equation.

c) There is a constituitive relation relating β and η which will be discussed next.

3.2 The Constitutive Relation and Variational Formulation

In order to define a constitutive relation between β and η, consider a mapping

$$C : C_c^p(M) \to C_c^p(M)$$

which, when restricted to a point of M, becomes a transformation which maps one differential form into another. In addition, if

$$\omega, \omega_1, \omega_2 \in C_c^p(M)$$

but are otherwise arbitrary and there is a positive definite Riemannian structure on M which induces a positive definite inner product $\langle\ ,\ \rangle_p$ on $C_c^p(M)$ then the following two properties are required of the mapping C:

1) Strict monotonicity:

$$\langle C(\omega_1) - C(\omega_2), \omega_1 - \omega_2\rangle \geq 0$$

with equality if and only if $\omega_1 = \omega_2$.

2) Symmetry: defining the linear functional

$$f_\omega(\omega_1) = \langle C(\omega), \omega_1\rangle_p$$

and denoting its Gateaux variation by

$$f'_\omega(\omega_1, \omega_2) = \langle C'_\omega(\omega_2), \omega_1\rangle_p,$$

it is required that this function is a symmetric bilinear function of ω_1 and ω_2. That is

$$\langle C'_\omega(\omega_1), \omega_2\rangle_p = \langle C'_\omega(\omega_2), \omega_1\rangle_p.$$

The first of these two conditions ensures the invertibility of C (see Tonti [1972b] Theorem 10). When there is a pseudo Riemannian structure on the manifold the inner product $\langle\ ,\ \rangle_p$ is indefinite as is the case in four dimensional versions of electromagnetics and the appropriate reformulation of Condition 1 is found in Tonti [1972b] pp. 351–352. The second of the above two conditions will imply that there exists a variational principle for the problem at hand. See Vainberg [1964] or Tonti [1969] for a thorough discussion. This being said, let the constitutive relation between β and η be of the form

$$\eta = *C(\beta).$$

The next step in formulating the paradigm problem variationally, is to relate $\beta \in Z_c^p(M - S_1)$ to a potential α. In all of the special cases of the paradigm problem shown in Table 1, with the exception of Case 5 involving current flow on sheets, the physics of the problem shows that it is reasonable to assume

$$\beta = d\alpha$$

since $M \subset \mathbb{R}^n$ and the above equation is true for $\mathbb{R}^n$. In Case 5 involving currents on sheets, one can use the techniques developed in Example 21 to express the current density vector in terms of a stream function which has jump discontinuities on a set of curves representing generators of $H_1(M, S_2)$. The values of these jump discontinuities are related to the time rate of change of magnetic flux through "holes" and "handles" and are prescribed as a principal condition in any variational formulation. Keeping this in mind, it is assumed that

$$\beta = d\alpha \qquad \text{for some } \alpha \in C^{p-1}(M)$$

in the paradigm problem. The next thing to do in formulating a variational principle, where $\beta \in Z_c^p(M - S_1)$ is imposed as a principal condition, is to figure out a way of imposing the condition

$$t\beta = 0 \qquad on\ S_1$$

in terms of a vector potential α. In general the observation that

$$t\alpha = 0 \Rightarrow 0 = dt\alpha = td\alpha = t\beta \qquad on\ S_1$$

does not mean that it is advisable to make the pullback of α to S_1 vanish. To see why this is so. consider the following portion of the long exact homology sequence for the pair (M, S_1):

$$\cdots \xrightarrow{\tilde{\delta}_{p+1}} H_p(S_1) \xrightarrow{\tilde{\imath}_p} H_p(M) \xrightarrow{\tilde{\jmath}_p} H_p(M, S_1) \longrightarrow$$
$$\xrightarrow{\tilde{\delta}_p} H_{p-1}(S_1) \xrightarrow{\tilde{\imath}_{p-1}} H_{p-1}(M) \xrightarrow{\tilde{\jmath}_{p-1}} H_{p-1}(M, S_1) \longrightarrow \cdots$$

Following the three step recipe introduced in Chapter 1 gives

$$H_p(M, S_1) \simeq \delta_p^{-1}\left(Kernel(\widetilde{\imath}_{p-1})\right) \oplus \widetilde{\jmath}_p\left(\frac{H_p(M)}{\widetilde{\imath}_p\left(H_p(S_1)\right)}\right).$$

The above arguments concerning the existence of a potential α deal with the periods of β on generators of $H_p(M)$ and hence the generators of $H_p(M, S_1)$ corresponding to Image$(\widetilde{\jmath}_p)$. It remains to consider how the periods of β on generators of $H_p(M, S_1)$ corresponding to $\delta_p^{-1}\left(\text{Kernel}\left(\widetilde{\imath}_{p-1}\right)\right)$ depend on the tangential components of α on S_1. Let $z_p \in Z_p(M, S_1)$ represent a nonzero homology class in $\delta_p^{-1}\left(\text{Kernel}\left(\widetilde{\imath}_{p-1}\right)\right)$ and consider the calculation of the period of β on this homology class:

$$\int_{z_p} \beta = \int_{z_p} d\alpha = \int_{\partial z_p} t\alpha = 0 \qquad if\ t\alpha = 0\ on\ S_1.$$

Hence, unless the periods of β vanish on $\delta_p^{-1}\left(\text{Kernel}\left(\widetilde{\imath}_{p-1}\right)\right)$ there is no hope of making the tangential components of α vanish on S_1. Instead, one must find a way of prescribing $t\alpha$ on S_1 such that

$$dt\alpha = td\alpha = t\beta = 0 \qquad on\ S_1$$

and the periods of β on generators of $\delta_p^{-1}\left(\text{Kernel}\left(\widetilde{\imath}_{p-1}\right)\right)$ are prescribed. This, of course, is simple in the case where $p = 1$ since a scalar potential or stream function is forced to be a constant on each connected component of S_1 if its exterior derivative vanishes there. For vector potentials ($p = 2$), the problem is a little more tricky since the tangential components of the vector potential on S_1 are related to some scalar function which has jump discontinuities on curves representing generators of $H_1(S_2, \partial S_2)$. This situation should present no difficulties since it has been considered ad nauseam in Examples 12, 13, 14.

As a prelude to the variational formulation of the paradigm problem, one has

$$\eta = *C(\beta) \qquad in\ M$$

$$\beta = d\alpha \qquad in\ M$$

$$t\alpha \qquad \text{specified on } S_1.$$

The last two conditions are used to ensure that $\beta \in Z_c^p(M - S_1)$ and the periods of β on $\delta_p^{-1}\left(\text{Kernel}\left(\widetilde{i}_{p-1}\right)\right)$ are prescribed in some definite way. One is now required to find a variational principle which would have

$$d\eta = \lambda \qquad in\ M$$

$$t\eta = 0 \qquad on\ S_2$$

as Euler-Lagrange equation and natural boundary condition respectively. A variational principle for this problem is a functional

$$F : C^{p-1}(M) \to \mathbb{R}$$

which is stationary at the $p-1$ form α and satisfies the above requirements. In order to define a variational principle, consider a family of $(p-1)$-forms parametrized differentiably by s, that is, a curve in $C^{p-1}(M)$

$$\gamma : [0,1] \to C^{p-1}(M)$$

where

$$\gamma(0) = \alpha_0, \text{ some initial state,}$$

$$\gamma(1) = \alpha, \text{ an extremal}$$

and in order to respect the principal boundary condition on S_1,

$$t\gamma(s) = t\alpha \qquad \text{for all } s \in [0,1]$$

is fixed. In addition to the above, no other constraints are placed on γ so that

$$\left.\frac{\partial\gamma}{\partial s}\right|_{s=1} = \tilde{\alpha} \qquad \text{the variation of the extremal,}$$

can be any arbitrary element of $C_c^p(M - S_1)$, the space of admissible variations. Next, the symmetry condition which states that:

$$\langle C'_\omega(\omega_1), \omega_2\rangle_p = \langle C'_\omega(\omega_2), \omega_1\rangle_p$$

for all $\omega, \omega_1, \omega_2 \in C^p(M)$ ensures that the value of the functional F defined by:

$$F(\alpha) = F(\alpha_0) + \int_0^1 \left(\left\langle C\left(d\gamma(s)\right), d\left(\frac{\partial\gamma(s)}{\partial s}\right)\right\rangle_p + (-1)^r \left\langle *\lambda, \frac{\partial\gamma(s)}{\partial s}\right\rangle_{p-1}\right) ds$$

where $r = (n - p + 1)(p - 1) + (p - 1)$ is independent of the path in $C^{p-1}(M)$ joining α_0 and α. That is, the value of the right hand side of the above equation does not depend on the way in which $\gamma(s)$ goes from α_0 to α as s goes from zero to one (see Tonti [1972 b]). For a general view of this formulation of variational functionals the reader is referred to Tonti [1969] and Vainberg [1964].

In order to verify that the extremal of the above functional has the properties required of it, recall that an extremal of the functional and the variation of the extremal are assumed to be .

$$\gamma(1) = \alpha, \qquad \left.\frac{\partial\gamma(s)}{\partial s}\right|_{s=1} = \tilde{\alpha}.$$

This implies that variations of the extremal can be considered by looking at $\gamma(1 - \varepsilon)$ for ε sufficiently small, and that the condition for the functional to be stationary at α is:

$$\left.\frac{\partial F}{\partial \varepsilon}\left(\gamma(1 - \varepsilon)\right)\right|_{\varepsilon=0} = 0.$$

Using the definition of the inner product, one can rewrite the above functional as

$$F(\alpha) = F(\alpha_0) + \int_0^1 \left(\int_M d\left(\frac{\partial\gamma(s)}{\partial s}\right) \wedge *C(d\alpha) + (-1)^{p-1} \int_M \left(\frac{\partial\gamma(s)}{\partial s}\right) \wedge \lambda \right) ds.$$

Using this form of the functional it is seen that the functional is stationary at α when

$$0 = \frac{\partial}{\partial\varepsilon} \int_0^{1-\varepsilon} \left(\int_M d\left(\frac{\partial\gamma(s)}{\partial s}\right) \wedge *C\left(d\gamma(s)\right) + (-1)^{p-1} \int_M \left(\frac{\partial\gamma(s)}{\partial s}\right) \wedge \lambda \right) ds\bigg|_{\varepsilon=0}$$

$$= -\int_M d\left(\frac{\partial\gamma(1)}{\partial s}\right) \wedge *C(d\gamma(1)) - (-1)^{p-1}\int_M \left(\frac{\partial\gamma(1)}{\partial s}\right) \wedge \lambda$$

$$= -\int_M d\tilde{\alpha} \wedge *C(d\alpha) - (-1)^{p-1}\int_M \tilde{\alpha} \wedge \lambda$$

for all $\tilde{\alpha} \in C_c^{p-1}(M - S_1)$. The integration by parts formula which was obtained as a corollary to Stokes' theorem shows that:

$$\int_M d\tilde{\alpha} \wedge *C(d\alpha) = \int_{\partial M} t\tilde{\alpha} \wedge t\,(*C(d\alpha)) - (-1)^{p-1}\int_M \tilde{\alpha} \wedge d * C(d\alpha).$$

Combining the above two equations, it is seen that the functional is stationary at α if

$$0 = (-1)^{p-1}\int_M \tilde{\alpha} \wedge (d * C(d\alpha) - \lambda) - \int_{\partial M} t\tilde{\alpha} \wedge t\,(*C(d\alpha))$$

for all $\tilde{\alpha} \in C_c^{p-1}(M - S_1)$. This of course means that

$$d * C(d\alpha) = \lambda \qquad in\ M$$

$$t\,(*C(d\alpha)) = 0 \qquad on\ \partial M - S_1 = S_2$$

are the Euler-Lagrange equation and natural boundary conditions respectively. Noting that

$$\eta = *C(\beta), \qquad \beta = d\alpha$$

the Euler-Lagrange equation and the natural boundary conditions state that the functional is stationary when

$$d\eta = \lambda \qquad in\ M$$

$$t\eta = 0 \qquad on\ S_2.$$

Thus it is seen that the paradigm problem is amenable to a variational formulation.

Before moving to the questions of existence and uniqueness of extremal, it is useful to mention how the interface conditions associated with the two integral laws of the paradigm problem are handled in the variational formulation, since this aspect has been ignored in the above calculations. Interface conditions are considered when the function C is discontinuous along some $(n-1)$-dimensional manifold S. In the variational formulation it is assumed that the potential α is continuous everywhere in M and differentiable in $M - S$. One can define an orientation locally on S and hence a plus side and a minus side. In this case if superscripts refer to a limiting value of a differential form from a particular side of S then

$$t\beta^{+} = t\beta^{-} \qquad on\ S$$

is the interface condition associated with the integral law

$$\int_{\partial c_{p+1}} \beta = 0 \qquad \text{for all } c_{p+1} \in C_{p+1}(M).$$

That this interface condition results as a consequence of the continuity requirements imposed on α is seen from the following argument. Since α is continuous in M one has

$$t\alpha^{+} = t\alpha^{-} \qquad on\ S.$$

Both sides of the above equation are differentiable with respect to the directions tangent to S because α is assumed differentiable in $M - S$. The exterior derivative in $C^{*}(S)$ involves only these tangential directions and therefore one has

$$dt\alpha^{+} = dt\alpha^{-} \qquad on\ S.$$

but

$$td\alpha^{+} = td\alpha^{-} \qquad on\ S$$

or

$$t\beta^{+} = t\beta^{-} \qquad on\ S$$

since exterior differentiation commutes with pull backs. Similarly, when λ has bounded coefficients, the interface condition

$$t\eta^{+} = t\eta^{-} \qquad on\ S$$

is associated with the integral law

$$\int_{\partial c_{n-p+1}} \eta = \int_{c_{n-p+1}} \lambda.$$

In order to see how this interface condition comes out of the variational formulation, notice that

$$d * C(d\alpha)$$

need not exist on S. Hence, if there are interfaces, then taking the variation of the functional one must use the integration by parts formula in $M - S$. When this is done, one obtains the exact same answer as before plus the following term:

$$-\int_S t\tilde{\alpha} \wedge t\left(*C(d\alpha^+) - *C(d\alpha^-)\right).$$

The arbitrariness of $t\tilde{\alpha}$ on S implies that

$$t * C(d\alpha^+) = t * C(d\alpha^-).$$

Thus identifying

$$\eta = *C(\beta), \qquad \beta = d\alpha$$

one has the desired result. This completes the discussion of the constitutive relation and the variational principle.

3.3 Gauge Transformations and Conservation Laws

The objective of this section is to develop a feeling for how nonunique the solution of the paradigm problem can be and to show how this nonuniqueness is related to the compatibility conditions which must be satisfied in order for a solution to the paradigm problem to exist. The approach taken in this section is basically due to Tonti [1968], however, it is more general than Tonti's in that the role of homology groups is considered. Every effort is made to avoid using the words local and global because the mathematical usage of the words local and global does not coincide with the meanings attributed to these words by physicists working in field theory.

For the paradigm problem being considered let us define a gauge transformation as a transformation on the potential α which leaves the following quantities untouched:

$$\beta = d\alpha \qquad in\ M$$

$$t\alpha \qquad on\ S_1.$$

The gauge transformation is assumed to have the following form

$$\alpha \to \alpha + \alpha_G \qquad in\ M$$

where $\alpha_G \in Z_c^{p-1}(M - S_1)$. It is obvious that α_G cannot lie in any bigger space since, by definition

$$Z_c^{p-1}(M - S_1) = \left\{\omega \mid \omega \in C_c^{p-1}(M - S_1), d\omega = 0\ in\ M\right\}.$$

By the orthogonal decomposition of Chapter 2, it is known that

$$Z_c^{p-1}(M - S_1) = B_c^{p-1}(M - S_1) \oplus \mathcal{H}^{p-1}(M, S_1)$$

where

$$\mathcal{H}^{p-1}(M, S_1) = \left\{\omega \mid \omega \in Z_c^{p-1}(M - S_1), n\omega = 0\ on\ S_2, \delta\omega = 0\ in\ M\right\}$$

and

$$\beta_{p-1}(M, S_1) = \dim \mathcal{H}^{p-1}(M, S_1).$$

This orthogonal decomposition enables one to characterise the space of the gauge transformations. In scalar potential problems, that is Cases 2, 5, 6, 7 in Table 1, p is equal to one and $\alpha_G \in \mathcal{H}^0(M, S_1)$ since $B_c^0(M - S_1)$ is the space containing only the zero vector. This situation is trivial to interpret since α_G is equal to some constant in each connected component of M which does not contain a subset of S_1. In problems where p is equal to two, that is Cases 1, 3, 4 in Table 1, one has

$$\alpha_G \in B_c^1(M - S_1) \oplus \mathcal{H}^1(M, S_1).$$

Thus it is expected that the gauge transformation can be described by a scalar function which vanishes on S_1 and $\beta_1(M, S_1)$ other degrees of freedom. The case where n is equal to three is treated explicitly in Kotiuga [1982] Sect. 4.2.2.

Since the gauge transformation is supposed to leave the differential form β invariant, one would hope that the gauge transformation would also leave the stationary value of the functional invariant. In order to formalise this intuition, suppose α is an extremal and let

$$\gamma : [0,1] \to C_c^{p-1}(M - S_1)$$

where

$$\gamma(s) = \alpha + s\alpha_G$$

and

$$\frac{\partial \gamma(s)}{\partial s} = \alpha_G \in Z_c^{p-1}(M - S_1) \qquad \text{for all } s \in [0, 1].$$

In this case recalling the definition of the variational functional for the paradigm problem one has

$$\begin{aligned} F(\alpha + \alpha_G) - F(\alpha) &= F(\gamma(1)) - F(\gamma(0)) \\ &= \int_0^1 \left(\int_M d\left(\frac{\partial \gamma(s)}{\partial s}\right) \wedge *C\,(d\gamma(s)) + (-1)^{p-1} \int_M \left(\frac{\partial \gamma(s)}{\partial s}\right) \wedge \lambda \right) ds \end{aligned}$$

$$= (-1)^{p-1} \int_0^1 \left(\int_M \alpha_G \wedge \lambda \right) ds \qquad \text{since } d\left(\frac{\partial \gamma(s)}{\partial s} \right) = 0$$

$$= (-1)^{p-1} \int_M \alpha_G \wedge \lambda.$$

Thus the gauge transformation leaves the value of the functional invariant if and only if

$$\int_M \alpha_G \wedge \lambda = 0 \qquad \text{for all } \alpha_G \in Z_c^{p-1}(M - S_1).$$

This condition can be rewritten as

$$\langle \alpha_G, *\lambda \rangle_{p-1} = 0 \qquad \text{for all } \alpha_G \in Z_c^{p-1}(M - S_1).$$

However, from the orthogonal decomposition theorem developed in the last chapter, it is known that

$$\left(Z_c^{p-1}(M - S_1) \right)^{\perp} = \widetilde{B}_{p-1}(M, S_2) = *B_c^{n-p+1}(M - S_2)$$

hence $*\lambda \in *B_c^{n-p+1}(M - S_2)$ or $\lambda \in B_c^{n-p+1}(M - S_2)$. This condition is precisely the compatibility condition which ensures that the equations

$$d\eta = \lambda \qquad in\ M$$

$$t\eta = 0 \qquad on\ S_2$$

are solvable for η. Thus it is seen that the Euler-Lagrange equation and the natural boundary conditions can be satisfied only when the stationary value of the functional is invariant under any gauge transformation.

The compatibility condition on λ is not amenable to direct verification in its present form. However, since

$$Z_c^{n-p+1}(M - S_2) \simeq B_c^{n-p+1}(M - S_2) \oplus H_c^{n-p+1}(M - S_2)$$

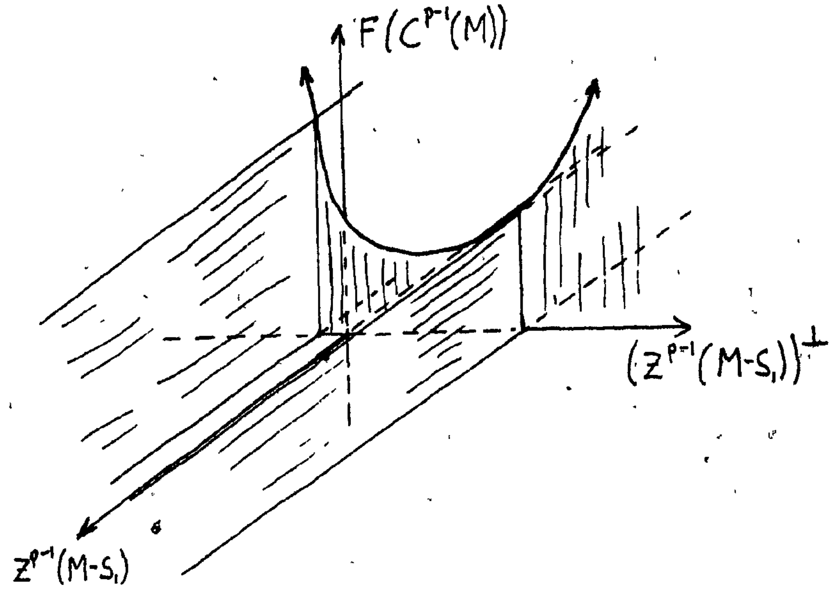


Fig.21

one sees that the compatibility condition can by verified by checking

$$\left.\begin{array}{ll} d\lambda = 0 & in\ M \\ t\lambda = 0 & on\ S_2 \end{array}\right\} \Rightarrow \lambda \in Z_c^{n-p+1}(M - S_2)$$

and then verifying that the periods of λ vanish on a set of generators of $H_{n-p+1}(M, S_2)$. This, in particular confirms the results given in Kotiuga [1982] which were considered in Example 22. This method of verifying the compatibility condition on λ also shows that the duality theorem

$$H_c^{p-1}(M - S_1) \simeq H_c^{n-p+1}(M - S_2)$$

plays a crucial role in interrelating degrees of freedom in the gauge transformation and degrees of freedom in λ constrained by the compatibility condition.

It is worth mentioning that

$$\lambda \notin B_c^{n-p+1}(M - S_2)$$

implies that the value of the functional is not invariant under every gauge transformation and that the Euler-Lagrange equation or the natural boundary conditions cannot be satisfied. In this case the functional has no extremum and it is useful to have a geometrical picture of the situation. Consider the diagram given in Fig. 21. The graph of the functional in the $F(C^{p-1}(M))$-$\left(Z_c^{p-1}(M - S_1)\right)^\perp$ "plane" is convex upward whenever the Riemannian structure on M is positive definite. This comes about as a result of the strict monotonicity assumption on the constitutive relation which is a valid assumption to make in all of the cases of the paradigm problem listed in Table 1 with the exception of electromagnetism in four dimensions. For simplicity, in the remainder of this section the discussion will focus on the case of convex functionals.

When the functional is invariant under gauge transformations, moving in the direction of $Z_c^{p-1}(M - S_1)$ does not change the value of the functional so that the graph looks like an infinitely long level trough which is convex upward in the "plane" $F(C^{p-1}(M))$-$\left(Z_c^{p-1}(M - S_1)\right)^\perp$. However, when the functional is not invariant under gauge transformations, that is, $\lambda \notin B_c^{n-p+1}(M - S_2)$ the trough is tilted and the functional has no stationary point. In this case the graph in the $F(C^{p-1}(M))$-$\left(Z_c^{p-1}(M - s_1)\right)^\perp$ "plane" remains the same but the slope in the $Z_c^{p-1}(M - S_1)$ direction has a nonzero value depending on the value of the projection

$$\frac{\int_M \alpha_G \wedge \lambda}{\sqrt{\langle \alpha_G, \alpha_G \rangle_{p-1}}}.$$

Thus, the interplay between gauge conditions and conservation laws arises from the above projection and gives a geometrical picture as to what happens when conservation laws are violated.

It has been seen that the compatibility condition $\lambda \in B_c^{n-p+1}(M - S_2)$ is necessary for the functional to have a minimum. In the case of a linear constitutive relation the Euler-Lagrange equation is a linear operator equation so that if the spaces in question are chosen so that the range of the operator is closed then the condition

$$\langle \alpha_G, *\lambda \rangle_{p-1} = 0 \quad \text{for all } \alpha_G \in Z_c^{p-1}(M - S_1)$$

is sufficient to ensure the solvability of the Euler-Lagrange equation (see Tonti [1968]) since the Fredholm alternative is applicable in this case. In the case of a nonlinear strictly monotone constitutive relation, the resulting convex functional may fail to have an extremum even if the above orthogonality condition holds. The extra condition which is required is

$$\frac{\langle C(\omega), \omega \rangle_p}{\sqrt{\langle \omega, \omega \rangle_p}} \to \infty \quad \text{as } \langle \omega, \omega \rangle_p \to \infty$$

for all $\omega \in C_c^p(M)$. The reason why this condition is necessary is best understood in terms of an example.

Example 36 (A convex function without a minimum.)

Let

$$f(\xi) = \sqrt{1 + \xi^2} - l\xi, \qquad l, \xi \in \mathbb{R}^1.$$

It is readily verified that

$$f'(\xi) = \xi(1 + \xi^2)^{-1/2} - l$$

$$f''(\xi) = (1 + \xi^2)^{-3/2}.$$

Since the second derivative of this function is always positive, it is seen that the function is convex for all values of λ. However if ξ is the minimum value of this function then ξ must satisfy

$$f'(\xi) = 0 = \xi(1 + \xi^2)^{-1/2} - l$$

or

$$\xi = l(1+\xi^2)^{1/2} \Rightarrow \xi = \frac{l}{\sqrt{1-l^2}}.$$

Thus, this convex function has no minimum if $|l| > 1$. In order see how this example relates to the above condition the identifications

$$f(\xi) = \int_0^{\xi} C(\tau)d\tau - l\xi,$$

$$(C(\tau),\tau) = \tau C(\tau)$$

are made so that

$$C(\xi) = f'(\xi) + l = \frac{\xi}{\sqrt{1+\xi^2}}$$

and in this case

$$\lim_{|\xi|\to\infty} \frac{C(\xi)\xi}{|\xi|} = \frac{\xi^2}{\sqrt{1+\xi^2}|\xi|} = \frac{|\xi|}{\sqrt{1+\xi^2}} = 1 < \infty$$

so that the extra condition imposed on the constitutive relation is violated.

End of Example 36

Example 36 shows that in the paradigm problem being considered, if

$$\lim_{\sqrt{\langle\omega,\omega\rangle_p}\to\infty} \frac{\langle C(\omega),\omega\rangle_p}{\sqrt{\langle\omega,\omega\rangle_p}} < \infty$$

then one expects that for some $\lambda \in B_c^{n-p+1}(M - S_2)$ with sufficiently large norm, the functional of the paradigm problem may fail to have a minimum. The interpretation of this extra condition in terms of the trough picture is as follows. Suppose $\lambda \in B_c^{n-p+1}(M-S_2)$ and consider the graph of the functional in the $F\left(\left(Z_c^{p-1}(M-S_1)\right)^{\perp}\right)$-$\left(Z_c^{p-1}(M-S_1)\right)^{\perp}$ plane as a function of the norm of λ as show in Fig. 22. This diagram illustrates how the minimum value of the functional may tend to minus infinity as the norm of λ increases and the condition

$$\lim_{\|\omega\|_p \to \infty} \frac{\langle C(\omega), \omega \rangle}{\|\omega\|_p} = \infty$$

is violated. Thus, when thinking of the graph of $F(\alpha)$ as a trough, one sees that the trough is tilted in the $Z_c^{p-1}(M - S_1)$ direction when λ violates some conservation law, and the trough "rolls over" when the above condition is not satisfied and λ is chosen in a suitable way.

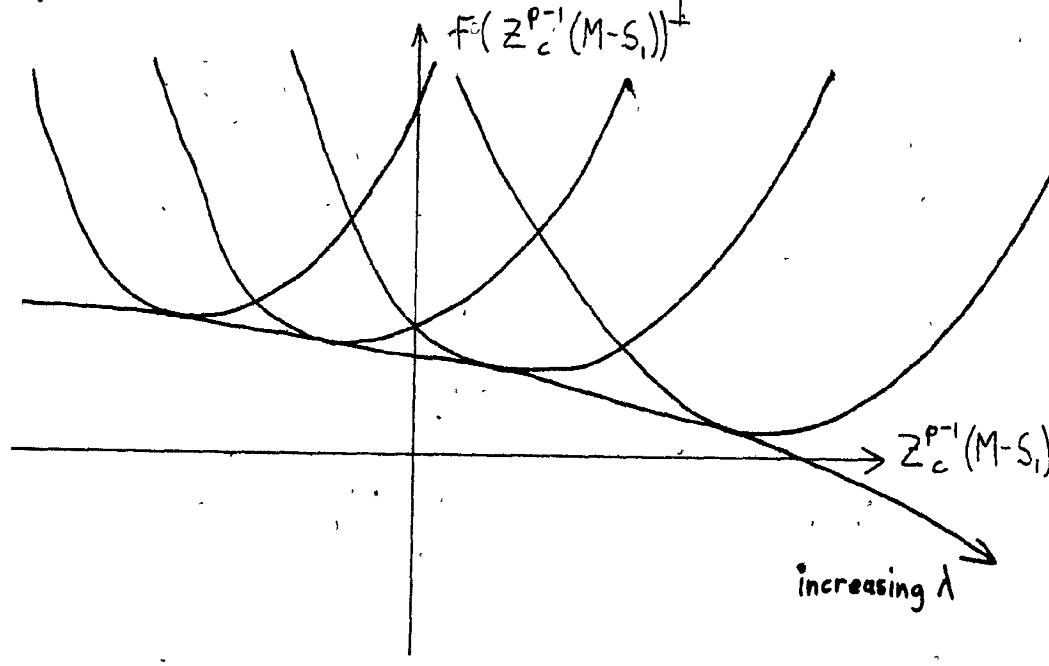


Fig. 22

For the purposes of numerical work, one would like a variational principle whose extremum always exists and is unique. The variational principle for the paradigm problem has a unique solution if and only if the space $Z_c^{p-1}(M - S_1)$ which is homologous to $B_c^{p-1}(M - S_1) \oplus H_c^{p-1}(M - S_1)$ contains only the null vector. By the above direct sum decomposition this happens in practical problems where $p = 1$ (so that $B_c^0(M - S_1) = 0$) and there is a Dirichlet condition imposed on some part of the boundary of each connected component of M, (so that $H_c^0(M - S_1) = 0$). When the extremal of the functional is nonunique, the usual algorithms for minimising convex functionals can be generalised to the case where the extremum of the functional is nonunique. For example, Newton's method as described by Luenberger [1969] Section 10.4 can be generalised as in Altman [1955]. However, in such cases it is usually easier to reformulate the variational principle for the paradigm problem in such a way that there always exist a unique solution. There are two basic approaches to this problem which will be considered next.

3.4 Modified Variational Principles

The purpose of this section is to formulate variational principles for the paradigm problem for which the potential α is unique. Such variational principles will have interesting consequences for conservation laws since a unique solution for the potential α implies that there is no gauge transformation which in turn implies that there is no conservation law which is naturally associated with the functional.

The first approach to the problem is to note that once the principal boundary conditions have been imposed on S_1 the space of admissible variations of the extremal is $C_c^{p-1}(M - S_1)$ and the space of gauge transformations is $Z_c^{p-1}(M - S_1)$. Hence if the space of admissible variations of the functional and the domain of the functional is restricted to

$$\left(Z_c^{p-1}(M - S_1)\right)^{\perp} \cap C_c^{p-1}(M - S_1)$$

then the functional's previous minimum can still be attained but the solution is now unique. By the orthogonal decomposition developed in the last chapter, one has

$$\left(Z_c^{p-1}(M - S_1)\right)^{\perp} = \widetilde{B}_{p-1}(M, S_2)$$

hence the space of admissible variations becomes

$$\widetilde{B}_{p-1}(M,S_2)\cap C_c^{p-1}(M-S_1)=\{\tilde{\alpha}\mid t\tilde{\alpha}=0 \text{ on } S_1, \tilde{\alpha}=\delta_p\omega \text{ in } M$$
$$\text{for some } \omega\in C_c^p(M) \text{ with } n\omega=0 \text{ on } S_2\}.$$

This procedure raises an interesting question. By the observations of Tonti [1968] one knows that the number of degrees of freedom in the gauge transformation is equal to the number of degrees of freedom by which the source, described by λ, is constrained by a conservation law. Hence in this case where the domain of the functional is constrained, so that the extremal is unique, one expects that the variational principle is completely insensitive to violations of the conservation law $\lambda\in B_c^{n-p+1}(M-S_2)$. In order to see why this is so, consider the unique decomposition:

$$\lambda=\lambda_{cons}+\lambda_{nonc}$$

where

$$\lambda\in C_c^{n-p+1}(M)$$
$$\lambda_{cons}\in B_c^{n-p+1}(M-S_2)$$
$$\lambda_{nonc}\in\left(B_c^{n-p+1}(M-S_2)\right)^{\perp}.$$

What is required is to show that the extremal of the functional is independent of the way in which λ_{nonc} is prescribed. Considering the functional of the paradigm problem as a function of λ, when α is restricted as above, one has

$$\begin{aligned}F_\lambda(\alpha)-F_{\lambda_{cons}}(\alpha)&=(-1)^{p-1}\int_M\alpha\wedge\lambda_{nonc}\\&=(-1)^{(p-1)-(n-p+1)(p-1)}\int_M\lambda_{nonc}\wedge\alpha\\&=(-1)^{(p-1)}\int_M\lambda_{nonc}\wedge**\alpha\\&=(-1)^{(p-1)}\langle\lambda_{nonc},*\alpha\rangle_{n-p+1}.\end{aligned}$$

However, one has

$$*\alpha \in *\left(\left(Z_c^{p-1}(M - S_1)\right)^{\perp}\right) = *\widetilde{B}_{p-1}(M, S_2)$$
$$= B_c^{n-p+1}(M - S_2)$$

but

$$\lambda_{nonc} \in \left(B_c^{n-p+1}(M - S_2)\right)^{\perp}.$$

and combining the above two results, one has

$$\langle \lambda_{nonc}, *\alpha \rangle_{n-p+1} = 0$$

so that

$$F_\lambda(\alpha) = F_{\lambda_{cons}}(\alpha).$$

Thus, by restricting the class of admissible variations of the functional's extremal, one obtains a variational formulation whose unique extremal is insensitive to violations of the compatibility condition $\lambda \in B^{n-p+1}(M - S_1)$. This approach to the problem is useful in the context of direct variational methods such as the Ritz method or the finite element method only when it is possible to find basis functions which ensure that

$$\alpha \in \left(Z_c^{p-1}(M - S_1)\right)^{\perp} = \widetilde{B}_{p-1}(M, S_2).$$

The second method for obtaining a variational formulation of the paradigm problem in which the extremal is unique is inspired by Kotiuga [1982] Chapter 5. In this method, which at first sight resembles the "penalty function method" (see Luenberger [1969] Sect 10.11), the domain of the functional before principal boundary conditions are imposed is $C_c^{p-1}(M)$. The method involves finding a functional $F^{\perp}(\alpha)$ whose graph looks like a trough perpendicular to the trough of $F(\alpha)$ as is illustrated in Fig. 23.

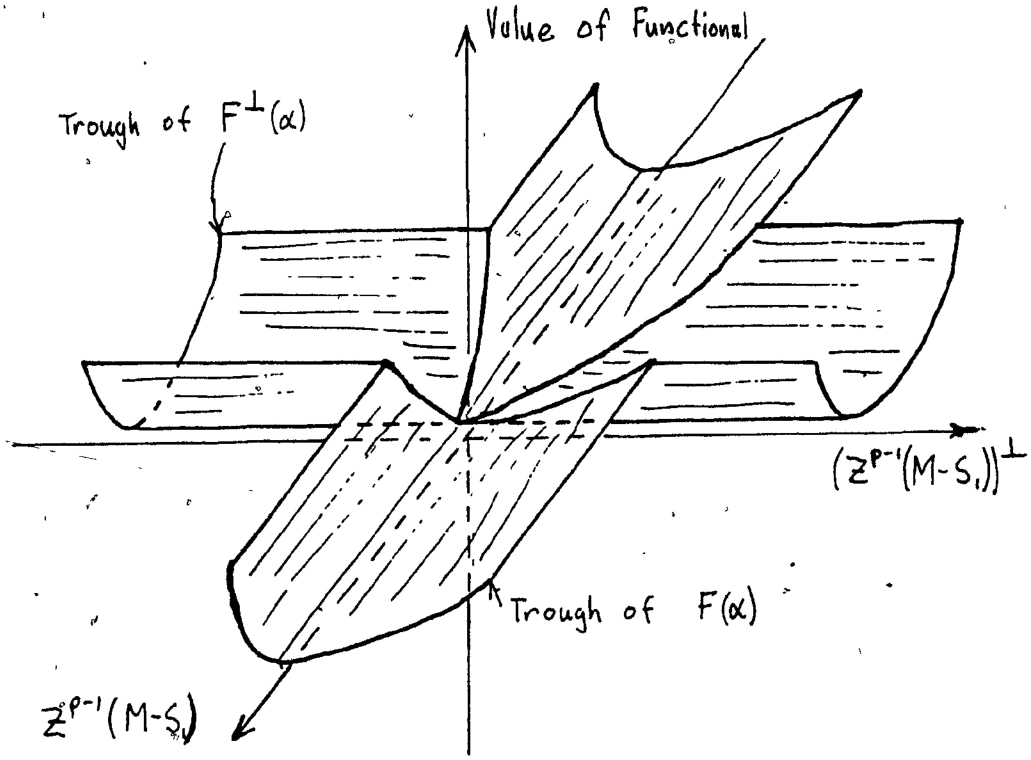


Fig. 23

In this scheme the functional

$$G(\alpha) = F(\alpha) + F^{\perp}(\alpha)$$

has a unique minimum which lies above the $\left(Z_c^{p-1}(M - S_1)\right)^{\perp}$ "axis" whenever the trough associated with $F(\alpha)$ is not tilted. That is if $F^{\perp}(\alpha)$ is designed so that its minimum is the $\left(Z_c^{p-1}(M - S_1)\right)^{\perp}$ "axis" then the minimum of $G(\alpha)$ should lie above the $\left(Z_c^{p-1}(M - S_1)\right)^{\perp}$ "axis" whenever $\lambda \in B_c^{n-p+1}(M - S_2)$. It is also desired that the contrapositives of these statements are also true in the following sense. If $\lambda \notin B_c^{n-p+1}(M - S_2)$ so that the trough associated with $F(\alpha)$ is "tilted" then the distance of the extremum of the functional $G(\alpha)$ to the $G(\alpha) - \left(Z_c^{p-1}(M - S_1)\right)^{\perp}$ plane measures, in some sense, the value of the projection

$$\frac{\max\limits_{\alpha_G \in Z_c^{p-1}(M-S_1)} \int_M \alpha_G \wedge \lambda}{\|\alpha_G\|_{p-1}}.$$

Having this picture in mind, the first thing to do is construct a functional with the properties desired of $F^{\perp}(\alpha)$. In order to find a functional which is definite on $Z_c^{p-1}(M -$

S_1) and level on $\left(Z_c^{p-1}(M-S_1)\right)^{\perp}$. one notes that by the orthogonal decomposition of last chapter

$$\left(Z_c^{p-1}(M-S_1)\right)^{\perp} = \widetilde{B}_{p-1}(M,S_2)$$

$$Z_c^{p-1}(M-S_1) = \left(\widetilde{B}_{p-1}(M,S_2)\right)^{\perp}.$$

Hence, one actually wants a functional $F^{\perp}(\alpha)$ which is level on $\widetilde{B}_{p-1}(M,S_2)$ and convex on $\left(\widetilde{B}_{p-1}(M,S_2)\right)^{\perp}$.

As a prelude to the construction of $F^{\perp}(\alpha)$, let K be a map

$$K : C^{p-2}(M) \to C^{p-2}(M)$$

satisfying the same conditions associated with the constitutive mapping. That is, for $\omega, \omega_1 \omega_2 \in C^{p-2}(M)$ the following three properties are assumed to hold.

1) Strict monotonicity:

$$\langle K(\omega_1) - K(\omega_2), \omega_1 - \omega_2 \rangle_{p-2} \geq 0$$

with equality if and only if $\omega_1 = \omega_2$.

2) Symmetry: defining the functional

$$f_\omega(\omega_1) = \langle K(\omega), \omega_1 \rangle_{p-2}$$

and denoting the Gateaux variation of this functional by

$$f'_\omega(\omega_1, \omega_2) = \langle K'_\omega(\omega_2), \omega_1 \rangle_{p-2},$$

it is required that this function is a symmetric bilinear function of ω_1 and ω_2. That is

$$\langle K'_\omega(\omega_1), \omega_2 \rangle_{p-2} = \langle K'_\omega(\omega_2), \omega_1 \rangle_{p-2}.$$

3) Asymptotic property:

$$\lim_{\|\omega\|_{p-2}\to\infty} \frac{\langle K(\omega),\omega\rangle_{p-2}}{\|\omega\|_{p-2}} = \infty.$$

In addition to these usual properties the mapping K will also be assumed to satisfy the following condition:

4) $K(0) = 0$ where 0 is the differential form whose coefficients vanish relative to any basis.

Given a mapping K which satisfies the above four conditions, consider the functional $F_0(\alpha)$ defined as follows

$$F_0 : \widetilde{C}_{p-1}(M, S_2) \to \mathbb{R}$$

where if

$$\gamma : [0,1] \to \widetilde{C}_{p-1}(M, S_2),$$

then one has

$$F_0(\gamma(1)) = F_0(\gamma(0)) + \int_0^1 \left\langle K(\delta\gamma(s)), \frac{d}{ds}(\delta\gamma(s)) \right\rangle_{p-2} ds.$$

By construction, this functional is convex in the subspace

$$\left(\widetilde{Z}_{p-1}(M, S_2)\right)^{\perp} \cap \widetilde{C}_{p-1}(M, S_2)$$

and "level" in the subspace $\widetilde{Z}_{p-1}(M, S_2)$. Furthermore $F_0(\alpha) \geq F_0(0)$ with equality if and only if $\alpha \in \widetilde{Z}_{p-1}(M, S_2)$.

At this stage, the construction of $F^{\perp}(\alpha)$ is actually simple. Considering the orthogonal decomposition of the previous chapter, the following diagrams are readily seen to be true

$$\begin{array}{ccccccc}
B_c^{p-1}(M-S_1) & \subset & Z_c^{p-1}(M-S_1) & = & B_c^{p-1}(M-S_1) & \oplus & \mathcal{H}^{p-1}(M,S_1) \\
\| & & \| & & \| & & \| \\
\left(\tilde{Z}_{p-1}(M,S_2)\right)^{\perp} & \subset & \left(\tilde{B}_{p-1}(M,S_2)\right)^{\perp} & = & \left(\tilde{Z}_{p-1}(M,S_2)\right)^{\perp} & \oplus & \mathcal{H}^{p-1}(M,S_1)
\end{array} \tag{1}$$

and

$$\begin{array}{ccccccc}
\tilde{B}_{p-1}(M,S_2) & \subset & \tilde{Z}_{p-1}(M,S_2) & = & \tilde{B}_{p-1}(M,S_2) & \oplus & \mathcal{H}^{p-1}(M,S_1) \\
\| & & \| & & \| & & \| \\
\left(Z_c^{p-1}(M-S_1)\right)^{\perp} & \subset & \left(B_c^{p-1}(M-S_1)\right)^{\perp} & = & \left(Z_c^{p-1}(M-S_1)\right)^{\perp} & \oplus & \mathcal{H}^{p-1}(M,S_1)
\end{array} \tag{2}$$

where

$$*\mathcal{H}^{n-p+1}(M,S_2) = \mathcal{H}^{p-1}(M,S_1) = Z_c^{p-1}(M-S_1) \cap \tilde{Z}_{p-1}(M,S_2).$$

Looking at these diagrams, it is seen that in (1) $F^{\perp}$ is supposed to be convex on the spaces listed in the second column while F_0 is convex in the spaces listed in the first column. Similarly in (2), $F^{\perp}$ is invariant with respect to variations in the spaces listed in the first column while F_0 is invariant with respect to variations in the spaces listed in the second column. Thus, observing the direct sum decompositions in the third column of (1) and (2) it is obvious that the functional F_0 meets all of the specifications of $F^{\perp}$ except on the space $\mathcal{H}^{p-1}(M,S_1)$. More specifically, the functional $F^{\perp}$ is required to constrain the periods of a $p-1$ form in $\mathcal{H}^{p-1}(M,S_1)$ while the functional F_0 does not. In order to fix this discrepancy, let

$$z_i, \qquad 1 \le i \le \beta_{n-p+1}(M,S_2)$$

be a set of generators of $H_{n-p+1}(M,S_2)$, and

$$k_i, \qquad 1 \le i \le \beta_{n-p+1}(M,S_2)$$

be a set of positive constants. If

$$\gamma : [0,1] \to C_c^{p-1}(M)$$

then the functionals

$$F_i : C_c^{p-1}(M) \to \mathbb{R}, \qquad 1 \le i \le \beta_1(M, S_1)$$

defined by

$$F_i\left(\gamma(1)\right) = F_i\left(\gamma(0)\right) + k_i \int_0^1 \left(\int_{z_i} *\gamma(s)\right)\left(\int_{z_i} *\frac{\partial \gamma(s)}{\partial s}\right) ds$$
$$= F_i\left(\gamma(0)\right) + \frac{k_i}{2}\left[\left(\int_{z_i} *\gamma(1)\right)^2 - \left(\int_{z_i} *\gamma(0)\right)^2\right]$$

have the property

$$F_i(\alpha) \ge F_i(0)$$

with equality if and only if the integral of $*\alpha$ over z_i vanishes. Next, consider the "candidate" functional

$$F_{cand}^{\perp}(\alpha) = \sum_{i=0}^{\beta_{n-p+1}(M,S_2)} F_i(\alpha).$$

Immediately, from the definitions of the F_i, one has

$$F_{cand}^{\perp}(\alpha) - F_{cand}^{\perp}(0) = \sum_{i=0}^{\beta_{n-p+1}(M,S_2)} \left(F_i(\alpha) - F_i(0)\right) \ge 0$$

with equality if and only if

$$F_i(\alpha) = F_i(0), \qquad 0 \le i \le \beta_{n-p+1}(M, S_2).$$

This last condition is equivalent to

$$\begin{cases} \alpha \in \widetilde{Z}_{p-1}(M, S_2) \\ \int_{z_i} *\alpha = 0, \quad 1 \le i \le \beta_{n-p+1}(M, S_2) \end{cases}$$

or

$$\begin{cases} *\alpha \in *\widetilde{Z}_{p-1}(M, S_2) = Z_c^{n-p+1}(M - S_2) \\ \displaystyle\int_{z_i} *\alpha = 0, \qquad 1 \le i \le \beta_{n-p+1}(M, S_2) \end{cases}$$

which, by the relative de Rham isomorphism, is equivalent to

$$*\alpha \in B_c^{n-p+1}(M - S_2)$$

and finally, this is equivalent to

$$\alpha \in *B_c^{n-p+1}(M - S_2) = \widetilde{B}_{p-1}(M, S_2) = \left(Z_c^{p-1}(M - S_1)\right)^{\perp}.$$

Hence, in summary

$$F_{cand}^{\perp}(\alpha) - F_{cand}^{\perp}(0) \ge 0$$

with equality if and only if

$$\alpha \in \left(Z_c^{p-1}(M - S_1)\right)^{-}$$

Furthermore, by construction, $F_{cand}^{\perp}$ is convex when its domain is restricted to the space $Z_c^{p-1}(M - S_1)$. Thus interpreting $F_{cand}^{\perp}$ as a trough, it is seen that it satisfies the requirements of $F^{\perp}$ and hence can be used to define $F^{\perp}$. Hence if one defines a curve

$$\gamma : [0, 1] \to \widetilde{C}_{p-1}(M, S_2)$$

then one can define the functional

$$F^{\perp} : \widetilde{C}_{p-1}(M, S_2) \to \mathbb{R}$$

as follows:

$$\begin{aligned} F^{\perp}(\gamma(1)) - F^{\perp}(\gamma(0)) &= F_{cand}^{\perp}(\gamma(1)) - F_{cand}^{\perp}(\gamma(0)) \\ &= \sum_{i=0}^{\beta_{n-p+1}(M,S_2)} \left(F_i(\gamma(1)) - F_i(\gamma(0))\right) \\ &= \int_0^1 \left\langle K(\delta\gamma(s)), \delta\left(\frac{\partial\gamma(s)}{\partial s}\right)\right\rangle_{p-2} ds \\ &\quad + \sum_{i=1}^{\beta_{n-p+1}(M,S_2)} k_i \int_0^1 \left(\int_{z_i} *\gamma(s)\right)\left(\int_{z_i} *\frac{\partial\gamma(s)}{\partial s}\right) ds. \end{aligned}$$

Finally one can complete the quest for a variational formulation of the paradigm problem in which the extremal of the functional is unique by letting

$$G(\alpha) - G(0) = (F(\alpha) - F(0)) + \left(F^{+}(\alpha) - F^{-}(0)\right)$$

while respecting the following principal boundary conditions:

$$t\alpha \qquad \text{prescribed on } S_1$$

$$n\alpha = 0 \qquad \text{on } S_2.$$

In order to define $G(\alpha)$ more explicitly, consider a continuous differentiable curve

$$\gamma : [0,1] \to \tilde{C}_{p-1}(M, S_2)$$

with

$$\gamma(0) = \alpha_0 \qquad \text{some initial state}$$

$$\gamma(1) = \alpha_1 \qquad \text{an extremal}$$

and in order to respect the principal boundary conditions, one has

$$t\alpha_0 = t\alpha = t\gamma(s) \qquad on\ S_1$$

$$n\alpha_0 = n\alpha = n\gamma(s) = 0 \qquad on\ S_2$$

for all $s \in [0,1]$. No other constraints are placed on γ so that

$$\frac{\partial \gamma}{\partial s} \in \tilde{C}_{p-1}(M, S_2) \cap C_c^{p-1}(M - S_1) \qquad \text{for all } s \in [0,1]$$

and the variation of the extremal

$$\left.\frac{\partial \gamma}{\partial s}\right|_{s=1} = \tilde{\alpha}$$

can be chosen to be any admissible variation where the space of admissible variations is $\tilde{C}_{p-1}(M, S_2) \cap C_c^{p-1}(M - S_1)$. Thus writing out the functional $G(\alpha)$ explicitly one has

$$G(\alpha) - G(\alpha_0) = G\left(\gamma(1)\right) - G\left(\gamma(0)\right)$$

$$= \int_0^1 \left[\left\langle C\left(d\gamma(s)\right), \frac{\partial}{\partial s}\left(d\gamma(s)\right) \right\rangle_p + (-1)^r \left\langle *\lambda, \frac{\partial\gamma(s)}{\partial s} \right\rangle_{p-1} + \left\langle K\left(\delta\gamma(s)\right), \frac{\partial}{\partial s}\left(\delta\gamma(s)\right) \right\rangle_{p-2} \right] ds$$

$$+ \sum_{i=1}^{\beta_{n-p+1}(M,S_2)} k_i \int_0^1 \left(\int_{z_i} *\gamma(s) \right) \left(\int_{z_i} * \frac{\partial\gamma(s)}{\partial s} \right) ds$$

where $r = (n-p+1)(p-1) + (p-1)$. To investigate the stationary point of the functional one recalls that

$$\alpha = \gamma(1) \in C_c^{p-1}(M - S_1), \qquad \tilde{\alpha} = \left. \frac{\partial\gamma}{\partial s} \right|_{s=1} \in C_c^{p-1}(M - S_1) \cap \tilde{C}_{p-1}(M, S_2)$$

and insists that

$$\frac{\partial}{\partial\varepsilon} G\left(\gamma(1-\varepsilon)\right) \Big|_{\varepsilon=0} = 0$$

for all admissible $\tilde{\alpha}$. Doing this shows that the following identity must be true for all α.

$$0 = \langle C(d\alpha), d\tilde{\alpha} \rangle_p + (-1)^r \langle *\lambda, \tilde{\alpha} \rangle_{p-1} + \langle K(\delta\alpha), \delta\tilde{\alpha} \rangle_{p-2} + \sum_{i=1}^{\beta_{n-p+1}(M,S_2)} \left(\int_{z_i} *\alpha \right) \left(\int_{z_i} *\tilde{\alpha} \right).$$

It is in general not possible to integrate by parts to obtain an Euler-Lagrange equation in the usual sense because of the integral terms which constrain the integrals of $*\alpha$ on a set of generators of $H_{n-p+1}(M, S_2)$. Furthermore, in the present case it is not necessary to derive an Euler-Lagrange equation since the functional is designed to be extremised by direct variational methods. What is necessary to verify is the geometric picture developed when thinking about the troughs associated with the graphs of the functionals $F(\alpha)$ and $F^{\perp}(\alpha)$. That is, it must be verified that when λ obeys the conservation law

$$\lambda \in B_c^{n-p+1}(M - S_2)$$

the extremal of $G(\alpha)$ provides a physically meaningful solution to the paradigm problem and the projection of the extremal into $Z_c^{p-1}(M - S_1)$ vanishes. Alternatively, when the conservation law is violated one hopes that the extremal of $G(\alpha)$ can be interpreted as providing a "least squares" solution to the nearest physically meaningfull problem where the conservation law is not violated and that the projection of the extremal into $Z_c^{p-1}(M - S_1)$ measures in some sense the extent by which the conservation law is violated. Hence let λ be prescribed in some way which does not necessarily respect a conservation law and consider the orthogonal decomposition

$$\lambda = \lambda_{cons} + \lambda_{nonc}$$

where

$$\lambda_{cons} \in B_c^{n-p+1}(M - S_2)$$

$$\lambda_{nonc} \in \left(B_c^{n-p+1}(M - S_2)\right)^{\perp} = \widetilde{Z}_{n-p+1}(M, S_1).$$

From this orthogonal decomposition, it follows immediately that

$$*\lambda = *\lambda_{cons} + *\lambda_{nonc}$$

where

$$*\lambda_{cons} = *B_c^{n-p+1}(M - S_2) = \widetilde{B}_{p-1}(M, S_2)$$

$$*\lambda_{nonc} = *\widetilde{Z}_{n-p+1}(M, S_1) = Z_c^{p-1}(M - S_1).$$

Similarly for $\alpha \in \widetilde{C}_{p-1}(M, S_2)$ one has the orthogonal decomposition

$$\widetilde{C}_{p-1}(M, S_2) = \widetilde{B}_{p-1}(M, S_2) \oplus \left(Z_c^{p-1}(M - S_1) \cap \widetilde{C}_{p-1}(M, S_2)\right)$$

and α can be expressed as

$$\alpha = \alpha_0 + \alpha_G$$

where

$$\alpha_0 \in \widetilde{B}_{p-1}(M, S_2)$$

$$\alpha_G \in Z_c^{p-1}(M - S_1) \cap \widetilde{C}_{p-1}(M, S_2).$$

Finally, for $\widetilde{\alpha} \in \widetilde{C}_{p-1}(M, S_2) \cap C_c^{p-1}(M - S_1)$ one has the orthogonal decomposition

$$\widetilde{C}_{p-1}(M, S_2) \cap C_c^{p-1}(M - S_1) = \left(\widetilde{B}_{p-1}(M, S_2) \cap C_c^{p-1}(M - S_1)\right) \oplus \left(Z_c^{p-1}(M - S_1) \cap \widetilde{C}_{p-1}(M, S_2)\right)$$

and $\widetilde{\alpha}$ can be expressed as

$$\widetilde{\alpha} = \widetilde{\alpha}_0 + \widetilde{\alpha}_G$$

where

$$\widetilde{\alpha}_0 \in \widetilde{B}_{p-1}(M, S_2) \cap C_c^{p-1}(M - S_1)$$

$$\widetilde{\alpha}_G \in Z_c^{p-1}(M - S_1) \cap \widetilde{C}_{p-1}(M, S_2).$$

Before returning to the condition that ensures that the functional G is stationary at α, note that expressing α_0 and $\widetilde{\alpha}_0$ as

$$\alpha_0 = \delta\theta, \qquad \widetilde{\alpha}_0 = \delta\widetilde{\theta}$$

it becomes apparent that

$$\int_{z_i} *\alpha_0 = \int_{z_i} *\delta\theta = (-1)^p \int_{z_i} d * \theta = 0, \qquad 1 \le i \le \beta_{n-p+1}(M - S_2)$$

$$\int_{z_i} *\widetilde{\alpha}_0 = \int_{z_i} *\delta\widetilde{\theta} = (-1)^p \int_{z_i} d * \widetilde{\theta} = 0, \qquad 1 \le i \le \beta_{n-p+1}(M - S_2)$$

since the integral of a coboundary on a cycle vanishes. Next, recall the identity which must be satisfied for all $\widetilde{\alpha} \in C_c^{p-1}(M - S_1) \cap \widetilde{C}_{p-1}(M, S_2)$ in order for the functional G to be stationary at α:

$$0 = \langle C(d\alpha), d\widetilde{\alpha}\rangle_p + \langle K(\delta\alpha), \delta\widetilde{\alpha}\rangle_{p-2} + (-1)^r \langle *\lambda, \widetilde{\alpha}\rangle_{p-1} + \sum_{i=1}^{\beta_{n-p+1}(M,S_2)} k_i \left(\int_{z_i} *\alpha\right)\left(\int_{z_i} *\widetilde{\alpha}\right).$$

Substituting the above orthogonal decompositions into this identity and recalling the definitions of the spaces involved gives

$$0 = \langle C(d\alpha_0, d\widetilde{\alpha}_0\rangle_p + \langle K(\delta\alpha_G, \delta\widetilde{\alpha}_G\rangle_{p-2} + (-1)^r \langle *\lambda_{cons} + *\lambda_{nonc}, \widetilde{\alpha}_0 + \widetilde{\alpha}_G\rangle_{p-1} + \sum_{i=1}^{\beta_{n-p+1}(M,S_2)} k_i \left(\int_{z_i} *\alpha_G\right)\left(\int_{z_i} *\widetilde{\alpha}_G\right).$$

Keeping in mind that the spaces $Z_c^{p-1}(M - S_1)$ and $\widetilde{B}_{p-1}(M, S_2)$ are mutually orthogonal, the inner product involving the source term and the variation of the extremal can be simplified to yield

$$\begin{aligned} 0 = \langle C(d\alpha_0), d\widetilde{\alpha}_0 \rangle_p + (-1)^r \langle *\lambda_{cons}, \widetilde{\alpha}_0 \rangle_{p-1} \\ + \langle K(\delta\alpha_G), \delta\widetilde{\alpha}_G \rangle_{p-2} + (-1)^r \langle *\lambda_{nonc}, \widetilde{\alpha}_G \rangle_{p-1} \\ + \sum_{i=1}^{\beta_{n-p+1}(M,S_2)} k_i \left(\int_{z_i} *\alpha_G \right) \left(\int_{z_i} *\widetilde{\alpha}_G \right). \end{aligned}$$

It is obvious by the independence of $\widetilde{\alpha}_0$ and and $\widetilde{\alpha}_G$ that the above condition is equivalent to the following two conditions:

1.

$$0 = \langle C(d\alpha_0), d\widetilde{\alpha}_0 \rangle_p + (-1)^r \langle *\lambda_{cons}, \widetilde{\alpha}_0 \rangle_{p-1}$$

for all $\widetilde{\alpha}_0 \in \widetilde{B}_{p-1}(M, S_2) \cap C_c^{p-1}(M - S_1)$;

2.

$$0 = \langle K(\delta\alpha_G), \delta\widetilde{\alpha}_G \rangle_{p-2} + (-1)^r \langle *\lambda_{nonc}, \widetilde{\alpha}_G \rangle_{p-1} + \sum_{i=1}^{\beta_{n-p+1}(M,S_2)} \left(\int_{z_i} \alpha_G \right) \left(\int_{z_i} \widetilde{\alpha}_G \right)$$

for all $\widetilde{\alpha} \in Z_c^{p-1}(M - S_1) \cap \widetilde{C}_{p-1}(M, S_2)$.

Thus in order to deduce the properties of the extremal $\alpha = \alpha_0 + \alpha_G$ of the functional G, one can look at the consequences of the above two identities. This can be done in two steps as follows.

Consequences of (1)

Condition (1) is precisely the criterion for the orginal functional F to be stationary at $\alpha_0 \in \left(Z_c^{p-1}(M-S_1)\right)^{\perp}$ and where the source is λ_{cons}. Previous calculations show that the above identity implies

$$d * C(d\alpha_0) = \lambda_{cons} \qquad in\ M$$

$$t * C(d\alpha_0) = 0 \qquad in\ S_2$$

so that the potential which makes $G(\alpha)$ stationary gives a solution to the paradigm problem where λ is replaced by λ_{cons}. By the definition of λ_{cons} it follows that

$$\min_{\xi \in B_c^{n-p+1}(M-S_2)} \| \lambda - \xi \|_{n-p+1} = \| \lambda - \lambda_{cons} \|_{n-p+1} .$$

Hence one can say that the extremal of G provides a solution to the nearest physically meaningful paradigm problem.

Consequences of (2)

Noticing that $\widetilde{\alpha}_G$ and α_G both belong to the space $Z_c^{p-1}(M-S_1) \cap \widetilde{C}_{p-1}(M, S_2)$ and $\widetilde{\alpha}_G$ is arbitrary, one can let $\widetilde{\alpha}_G$ be equal to α_G so that the identity (2) becomes

$$-(-1)^r \langle *\lambda_{nonc}, \alpha_G \rangle_{p-1} = \langle K(\delta\alpha_G), \delta\alpha_G \rangle_{p-2} + \sum_{i=1}^{\beta_{n-p+1}(M,S_2)} k_i \left(\int_{z_i} *\alpha_G \right)^2 \geq 0$$

with equality if and only if $\alpha_G \in \widetilde{B}_{p-1}(M, S_2)$. Since α_G is an element of $Z_c^{p-1}(M-S_1)$ it is seen that the expression becomes an equality if and only if $\alpha_G = 0$. Thus it is apparent that

$$*\lambda_{nonc} = 0 \qquad \text{implies} \qquad \alpha_G = 0$$

and

$$\alpha_G \neq 0 \qquad \text{implies} \qquad *\lambda_{nonc} \neq 0.$$

In order to prove the converses of these statements it is necessary to show that it is possible to find a $\widetilde{\alpha}_G$ such that if $\lambda_{nonc} \neq 0$ then

$$\langle * \lambda_{nonc}, \widetilde{\alpha}_G \rangle_{p-1} \neq 0$$

and that α_G is such an $\widetilde{\alpha}_G$. Unfortunately,

$$* \lambda_{nonc} \in Z_c^{p-1}(M - S_1)$$

and

$$\widetilde{\alpha}_G \in Z_c^{p-1}(M - S_1) \cap \widetilde{C}_{p-1}(M, S_2)$$

hence $\widetilde{\alpha}_G$ can be selected to reflect the projection of λ_{nonc} in $Z_c^{p-1}(M-S_1) \cap \widetilde{C}_{p-1}(M, S_2)$ and nothing more. Note however that if one imposes with complete certainty

$$t\lambda = 0 \qquad on\ S_2$$

then $\lambda \in C_c^{n-p+1}(M - S_2)$ and hence

$$* \lambda_{nonc} \in * \widetilde{Z}_{n-p+1}(M, S_1) \cap * C_c^{n-p+1}(M - S_2) = Z_c^{p-1}(M - S_1) \cap \widetilde{C}_{p-1}(M, S_2).$$

In this case α_G, $\widetilde{\alpha}_G$ and $* \lambda_{nonc}$ all belong to the space

$$Z_c^{p-1}(M - S_1) \cap \widetilde{C}_{p-1}(M, S_2)$$

and it is always possible to find an $\widetilde{\alpha}_G$ such that if $* \lambda_{nonc} \neq 0$ then

$$\langle * \lambda_{nonc}, \widetilde{\alpha}_G \rangle_{p-1} \neq 0.$$

However, by Equation (2) this implies that $\alpha_G \neq 0$ and since Equation (2) is valid for all possible $\widetilde{\alpha}_G$, one can set $\widetilde{\alpha}_G$ equal to α_G to obtain

$$-(-1)^r \langle * \lambda_{nonc}, \alpha_G \rangle_{p-1} = \langle K(\delta \alpha_G), \delta \alpha_G \rangle_{p-2} + \sum_{i=1}^{\beta_{n-p+1}(M, S_2)} k_i \left(\int_{z_i} * \alpha_G \right)^2 > 0.$$

Hence

$$\lambda_{nonc} \neq 0 \quad \text{implies} \quad \alpha_G \neq 0$$

or

$$\alpha_G = 0 \quad \text{implies} \quad \lambda_{nonc} = 0 \ .$$

and it is proved that

$$\alpha_G = 0 \quad \text{if and only if} \quad *\lambda_{nonc} = 0.$$

It is seen that the identities (1) and (2) adequately describe what happens in a neighbourhood of the extremal α in $\widetilde{C}_{p-1}(M, S_2)$ when one thinks in terms of tilted troughs.

Two final points are in order. The first point is that the value of the functional $F^{\perp}$ evaluated at the extremal of G provides an a posteriori estimate of how large λ_{nonc} is. This is apparent from the trough picture. The second point is that when there is a pseudo-Riemannian structure on the manifold M, the expression $\langle \ , \ \rangle_k$ is no longer positive definite, hence the functionals considered are no longer convex and the trough picture is no longer valid. Although the orthogonal decomposition of the last chapter is still a legitimate direct sum decomposition and when $\beta_{n-p+1}(M, S_2) = 0$ the functional G still provides an effective way of imposing the Lorentz gauge

$$\delta\alpha = 0$$

whenever the conservation of charge

$$d\lambda = 0$$

is respected, it is not clear what the exact properties of G are. From the point of view of computational electromagnetics, there is little motivation for pursuing this question and so the case of pseudo-Riemannian structures is ignored.

3.5 Tonti Diagrams

In this final section, Tonti diagrams and the associated framework for complementary variational principles will be considered. This work is well known to people in the field of computational electromagnetics and an overview of the literature in this context is given in the thesis of Fraser [1982] and, more recently, in the paper by Penmann and Fraser [1984]. In this connection the author also found the seminar paper by Cambrell [1983] most useful. The basis of the following discussion are the papers of Enzo Tonti [1972a, 1972b] where certain short exact sequences associated with differential operators appearing in field equations are recognized as being a basic ingredient in formulating a common structure for a large class of physical theories. This work of Tonti fits hand in glove with the work of J.J. Kohn [1972] on differential complexes. The point of view taken here is that for the practical problems described by the paradigm problem being considered in this chapter, the interrelationship between the work of Tonti and Kohn is easily seen by considering the complexes associated with the exterior derivative and its adjoint on a Riemannian manifold with boundary. The idea of introducing complexes and various concepts from algebraic topology into Tonti diagrams is not new and is developed in the companion papers of Branin [1977] and Tonti [1977].

The main conclusion to be drawn from the present approach is that the differential complexes associated with the exterior derivative give, when applicable, a deeper insight into Tonti diagrams than is usually possible since the de Rham isomorphism enables one to give concrete and intuitive answers to questions involving the (co)homology of the differential complexes. More precisely, the usual development of Tonti diagrams involves differential complexes where the symbol sequence of the differential operators involved is exact while what is actually desired is that the (co)homology of the complex be trivial. That is, if the (co)homology of the differential complex is nontrivial, then reasoning with the exactness of the symbol sequence alone may lead to false conclusions concerning the existence and uniqueness of solutions to equations. To the best of the author's knowledge, the only differential complexes of practical use for which something concrete can be said about (co)homology, are the differential complexes associated with the exterior derivative since in this case the de Rham isomorphism applies.

In order to formulate a Tonti diagram for the paradigm problem, consider first the paradigm problem and suppose that $H_c^{p-1}(M - S_1)$ is trivial. In this case,

$$Z_c^{p-1}(M - S_1) = B_c^{p-1}(M - S_1)$$

and

$$\alpha \rightarrow \alpha + d\chi, \qquad \chi \in C_c^{p-2}(M - S_1)$$

is a gauge transformation which describes the nonuniqueness in the potential α. Next, when dealing with complementary variational principles, it is necessary to find an

$$\eta_{part} \in C_c^{n-p}(M - S_2) = *\widetilde{C}_p(M, S_2)$$

such that

$$d\eta_{part} = \lambda$$

and

$$\eta - \eta_{part} \in B_c^{n-p}(M - S_2) = *\widetilde{B}_p(M, S_2).$$

In this case the forms β and η are determined by reducing the problem to a boundary value problem for

$$\nu \in C_c^{n-p-1}(M - S_2) = *\widetilde{C}_{p+1}(M, S_2)$$

where ν is defined by

$$d\nu \doteq \eta - \eta_{part}.$$

This boundary value problem for ν is deduced from the equations

$$d\beta = 0 \qquad in\ M$$

$$t\beta = 0 \qquad on\ S_1$$

$$d\nu + \eta_{part} = *C(\beta) \qquad in\ M.$$

From these equations the boundary value problem is seen to be

$$d\left(C^{-1}\left((-1)^{p(n-p)} * (\eta_{part} + d\nu)\right)\right) = 0 \qquad in\ M$$
$$t\left(C^{-1}\left((-1)^{p(n-p)} * (\eta_{part} - d\nu)\right)\right) = 0 \qquad on\ S_1$$
$$t\nu = 0 \qquad on S_2.$$

The variational formulation for this problem is obtained by considering a curve

$$\gamma : [0,1] \to C_c^{n-p-1}(M - S_2)$$

and defining the functional for the complementary problem as follows:

$$J(\gamma(1)) - J(\gamma(0))$$
$$= -(-1)^{p(n-p)} \int_0^1 \left\langle C^{-1}\left((-1)^{p(n-p)} * (\eta_{part} + d\gamma(s))\right), * \left(\eta_{part} + d\left(\frac{\partial\gamma(s)}{\partial s}\right)\right)\right\rangle_p ds$$
$$= -\int_0^1 \int_M \left(C^{-1}\left((-1)^{p(n-p)} * (\eta_{part} - d\gamma(s))\right)\right) \wedge \left(\eta_{part} + d\left(\frac{\partial\gamma(s)}{\partial s}\right)\right) ds.$$

In order to verify that this is indeed the correct functional let

$$\gamma(1) = \nu$$
$$\left.\frac{\partial\gamma(s)}{\partial s}\right|_{s=1} = \tilde{\nu}$$

be the extremal and any variation of the extremal of the functional where the space of admissible variations is $C_c^{n-p-1}(M - S_2) = *\widetilde{C}_{p+1}(M, S_2)$. The functional is stationary when

$$\left.\frac{\partial J}{\partial \varepsilon}(\gamma(1-\varepsilon))\right|_{\varepsilon=0} = 0$$

for all admissible variations of the extremal. This conditions amounts to

$$0 = \int_M C^{-1}\left((-1)^{p(n-p)} * (\eta_{part} + d\gamma(1))\right) \wedge d\left(\left.\frac{\partial\gamma(s)}{\partial s}\right|_{s=1}\right)$$

or

$$0 = \int_M C^{-1}\left((-1)^{p(n-p)} * (\eta_{part} + d\nu)\right) \wedge d\tilde{\nu}$$

for all admissible $\tilde{\nu}$. Integrating this expression by parts and using the fact that

$$t\nu = 0 \quad \text{on } S_2$$

one obtains

$$\begin{aligned} 0 = &\int_M d\left(C^{-1}\left((-1)^{p(n-p)} * (\eta_{part} + d\nu)\right)\right) \wedge \tilde{\nu} \\ &- \int_{S_1} t\left(C^{-1}\left((-1)^{p(n-p)} * (\eta_{part} + d\nu)\right)\right) \wedge t\tilde{\nu} \end{aligned}$$

from which it is apparent that the functional is the desired one since $\tilde{\nu}$ can be taken to be any admissible variation. In this formulation, the extremal of the functional J is unique up an to an element of $Z_c^{n-p-1}(M - S_2)$ and the nonuniqueness can be described by a gauge transformation

$$\nu \to \nu + \nu_G \qquad \text{where } \nu_G \in Z_c^{n-p-1}(M - S_2).$$

Hence whenever there is a Riemannian structure on M which induces the inner product $\langle\ ,\ \rangle_k$ on k-forms, the functional J is convex on

$$\begin{aligned} \left(Z_c^{n-p-1}(M - S_2)\right)^{\perp} &= \tilde{B}_{n-p-1}(M, S_1) \\ &= *B_c^{p+1}(M - S_1) \end{aligned}$$

and level on the space

$$Z_c^{n-p-1}(M - S_2) = *\tilde{Z}_{p+1}(M, S_2).$$

Just as the construction of a functional $F^{\perp}$ enabled one to modify the functional F in order to construct a variational formulation involving a functional

$$G(\alpha) = F(\alpha) + F^{\perp}(\alpha)$$

for which the resulting extremal is unique, one can construct a functional $J^{\perp}(\nu)$ such that

$$I(\nu) = J(\nu) + J^{\perp}(\nu)$$

is a functional whose unique extremal is also an extremal of the functional J. This, of course, happens when the functional $J^{\perp}$ is constructed so that it is convex on the space

$$Z_c^{n-p-1}(M - S_2) = *\widetilde{Z}_{p+1}(M, S_2)$$

and level on

$$\begin{aligned}\left(Z_c^{n-p-1}(M - S_2)\right)^{\perp} &= \widetilde{B}_{n-p-1}(M, S_1)\\ &= *B_c^{p+1}(M - S_1).\end{aligned}$$

Thus, again, one is led to a situation involving two troughs as shown in Fig. 24.

Having this picture in mind, the functional $J^{\perp}$ can be constructed in analogy with the construction of $F^{\perp}$. Consider first a mapping

$$K' : C^{n-p-2}(M) \rightarrow C^{n-p-2}(M)$$

which satisfies the same symmetry, monotonicity, and asymptotic properties required of the function K used in the construction of $F^{\perp}$. Define a functional $J^{\perp}$ as follows. Given

$$\gamma : [0, 1] \rightarrow \widetilde{C}_{n-p-1}(M, S_1)$$

let

$$\begin{aligned}J^{\perp}(\gamma(1)) - J^{\perp}(\gamma(0)) = &-\int_0^1 \left\langle K'^{-1}(\delta\gamma(s)), \delta\left(\frac{\partial\gamma(s)}{\partial s}\right)\right\rangle_{n-p-2}\\ &- \sum_{i=0}^{\beta_{p+1}(M,S_1)} l_i \int_0^1 \left(\int_{z_i} *\gamma(s)\right)\left(\int_{z_i} *\frac{\partial\gamma(s)}{\partial s}\right) ds\end{aligned}$$

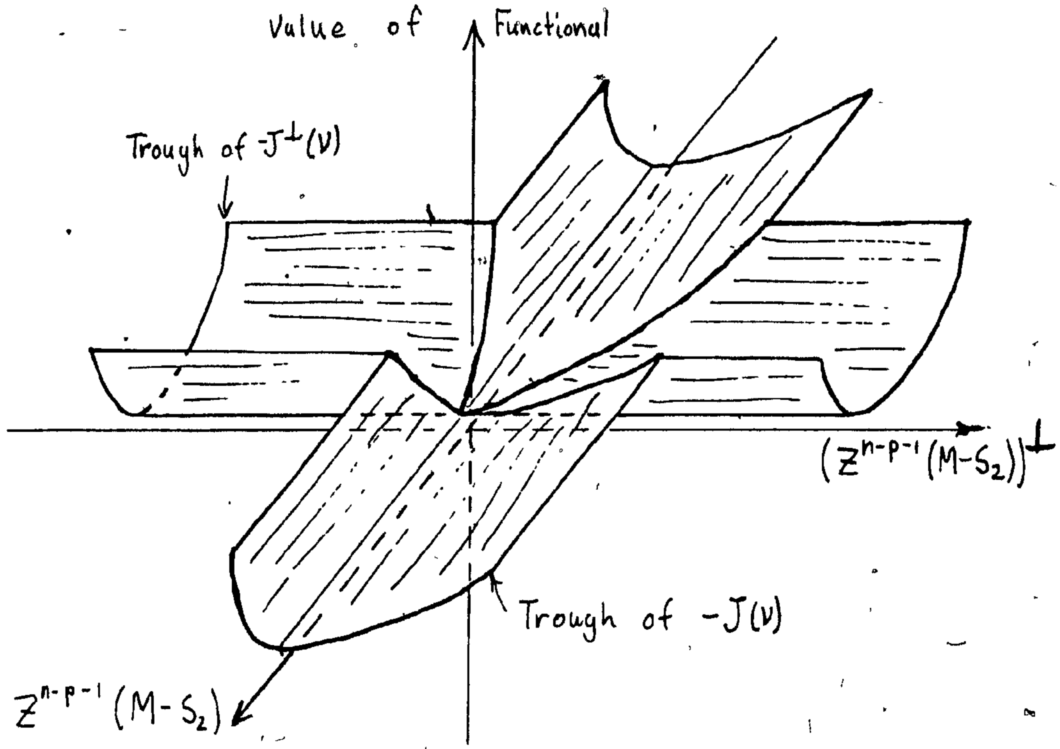


Fig. 24

where the z_i are associated with generators of the homology group $H_{p+1}(M, S_1)$ and the l_i are positive constants. The functional $J^\perp$ thus defined is convex on the space

$$Z_c^{n-p-1}(M - S_2) = \left(\widetilde{B}_{n-p-1}(M, S_1)\right)^\perp$$

and level on the space

$$\left(Z_c^{n-p-1}(M - S_2)\right)^\perp = \widetilde{B}_{n-p-1}(M, S_1).$$

This fact is easily seen to be true since the situations involving $F^\perp$ and $J^\perp$ become identical if one interchanges the following symbols:

$$J^\perp \leftrightarrow F^\perp$$

$$S_1 \leftrightarrow S_2$$

$$K' \leftrightarrow K$$

$$n - p \leftrightarrow p.$$

Hence the functional I defined as

$$I(\nu) = J(\nu) + J^{\perp}(\nu)$$

with domain $\widetilde{C}_{n-p-1}(M, S_1) \cap C_c^{n-p-1}(M - S_2)$, has a unique extremum.

Finally in order to finish this prelude to the Tonti diagram, note that if

$$\beta_{n-p-1}(M, S_2) = 0$$

then the nonuniqueness in the complementary potential ν, when the variational formulation involving the functional J is used, can be described by a gauge transformation

$$\nu \rightarrow \nu + d\theta$$

where

$$\theta \in C_c^{n-p-2}(M - S_2) = *\widetilde{C}_{p+2}(M, S_2).$$

Furthermore, when considering the Tonti diagram it is convenient to assume that β may be related to some type of source ρ through the equation

$$d\beta = \rho,$$

where in the present case $\rho = 0$. Hence, in terms of the notation introduced so far, the above formulation of the complementary variational principle for the paradigm problem is summarised by the following Tonti type diagram used extensively in Fraser [1982] and Penman and Fraser [1984]:

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\Big\downarrow d^{p-3} & & \Big\uparrow d^{n-p+2} \\
\xi & & d\lambda \\
\Big\downarrow d^{p-2} & & \Big\uparrow d^{n-p+1} \\
\alpha & \xrightarrow{\text{primal}} & \lambda \\
\Big\downarrow d^{p-1} & & \Big\uparrow d^{n-p} \\
\beta & \xrightarrow{*C} & \eta \\
\Big\downarrow d^{p} & & \Big\uparrow d^{n-p-1} \\
\rho & \xleftarrow{\text{dual}} & \nu \\
\Big\downarrow d^{p+1} & & \Big\uparrow d^{n-p-2} \\
d\rho & & \theta \\
\Big\downarrow d^{p+2} & & \Big\uparrow d^{n-p-3} \\
\vdots & & \vdots
\end{array}
$$

For the purposes of this thesis, the above diagram presents a simplistic view of the paradigm problem since boundary conditions and domains of definition of operators have been ignored. Thus it is impossible to get a clear understanding of how homology groups come into play. In order to remedy this situation, one must realize that when boundary conditions are imposed, the left hand side of the above diagram is associated with the complex $C_c^*(M - S_1)$ while the right hand side of the diagram is associated with the complex $C_c^*(M - S_2) = *\widetilde{C}_*(M, S_2)$. Thus to be more explicit, the above diagram should be rewritten as

$$
\begin{array}{ccccc}
\downarrow d^{p-3} & & \uparrow d^{n-p+2} & & \uparrow \delta_{p-2} \\
C_c^{p-2}(M-S_1) & & C_c^{n-p+2}(M-S_2) & \simeq & \tilde{C}_{p-2}(M,S_2) \\
\downarrow d^{p-2} & & \uparrow d^{n-p+1} & & \uparrow \delta_{p-1} \\
C_c^{p-1}(M-S_1) & & C_c^{n-p+1}(M-S_2) & \simeq & \tilde{C}_{p-1}(M,S_2) \\
\downarrow d^{p-1} & & \uparrow d^{n-p} & & \uparrow \delta_{p} \\
C_c^{p}(M-S_1) & \xrightarrow{\star C} & C_c^{n-p}(M-S_2) & \simeq & \tilde{C}_{p}(M,S_2) \\
\downarrow d^{p} & & \uparrow d^{n-p-1} & & \uparrow \delta_{p+1} \\
C_c^{p+1}(M-S_1) & & C_c^{n-p-1}(M-S_2) & \simeq & \tilde{C}_{p+1}(M,S_2) \\
\downarrow d^{p+1} & & \uparrow d^{n-p-2} & & \uparrow \delta_{p+2} \\
C_c^{p+2}(M-S_1) & & C_c^{n-p-2}(M-S_2) & \simeq & \tilde{C}_{p+2}(M,S_2) \\
\downarrow d^{p+2} & & \uparrow d^{n-p-3} & & \uparrow \delta_{p+3}
\end{array}
$$

Once this structure has been identified, it is apparent from the previous sections of this chapter that questions of existence and uniqueness of potentials and questions of existence and uniqueness of solutions to boundary value problems are easily handled by using the orthogonal decomposition developed in the last chapter. Though these questions have been considered in detail in the case of the potential α and the results for the complementary potential ν follow analogously, it is useful to outline the role played by various cohomology groups (vector spaces). Specifically, the role of the following pairs of groups and isomorphisms will be summarised:

i) $H_c^{p-1}(M-S_1) \simeq H_c^{n-p+1}(M-S_2)$,

ii) $H_c^{p}(M-S_1) \simeq H_c^{n-p}(M-S_2)$,

iii) $H_c^{p+1}(M-S_1) \simeq H_c^{n-p-1}(M-S_2)$,

Consequences of i)

Once $t\alpha$ is prescribed on S_1, the group $H_c^{p-1}(M - S_1)$ was seen to describe the nonuniqueness of α in the paradigm problem which cannot be described by a gauge transformation of the form

$$\alpha \to \alpha + d\chi, \qquad \chi \in C_c^{p-2}(M - S_1).$$

In other words, the nonuniqueness of α is described by $Z_c^{p-1}(M - S_1)$ while the above gauge transformation involves $B_c^{p-1}(M - S_1)$, hence the difference is described by $H_c^{p-1}(M - S_1)$ since by definition

$$H_c^{p-1}(M - S_1) = Z_c^{p-1}(M - S_1)/B_c^{p-1}(M - S_1).$$

Dually, $H_c^{n-p+1}(M - S_2)$ was seen to be associated with the global conditions which ensure that $\lambda \in B_c^{n-p+1}(M - S_2)$ once it is known that $\lambda \in Z_c^{n-p+1}(M - S_2)$. Finally, the isomorphism

$$H_c^{p-1}(M - S_1) \simeq H_c^{n-p+1}(M - S_2)$$

expresses the duality between the global degrees of freedom in the nonuniqueness (gauge transformation) of α and the solvability condition (conservation law) involving λ. This isomorphism is exploited in the construction of the functional $F^{\perp}$ and its interpretation is best appreciated by using the de Rham isomorphism to reduce the above isomorphism to

$$H_{p-1}(M, S_1) \simeq H_{n-p+1}(M, S_2)$$

and to interprete this isomorphism in terms of the intersection numbers of the generators of these two homology groups as in Chapter 1.

Consequences of ii)

The group $H_c^p(M - S_1)$ is associated with global conditions which ensure that $\beta \in B_c^p(M - S_1)$ once it is determined that $\beta \in Z_c^p(M + S_1)$. Furthermore it gives insight into the conditions which α must satisfy on S_1 if $\beta = d\alpha$. Dually the group $H_c^{n-p}(M - S_2)$ is associated with global conditions which η_{part} must satisfy in order for there to be a $\nu \in C_c^{n-p-1}(M - S_2)$ such that

$$\left.\begin{aligned} d\eta_{part} &= \lambda \\ d\nu &= \eta - \eta_{part} \end{aligned}\right\} in\ M.$$

Thus the cohomology group $H_c^p(M - S_1)$ is used in formulating a primal variational principle while the cohomology group $H_c^{n-p}(M - S_2)$ is used in formulating a dual variational principle and the isomorphism

$$H_c^p(M - S_1) \simeq H_c^{n-p}(M - S_2)$$

then expresses the fact that the number of global conditions is the same in both the original and complementary formulations. Note that for most problems, the periods of closed forms on the generators of

$$H_p(M, S_1), \quad H_{n-p}(M, S_2)$$

have the interpretation of a lumped parameter current, potential difference, or flux as was seen in Examples 12, 13, 14, and 21. Thus in these examples the isomorphism in homology has a direct interpretation.

Consequences of iii)

Had β not been a closed form but rather tied to an equation of the form

$$d\beta = \rho$$

then if $\rho \in Z_c^{p+1}(M - S_1)$ the group $H_c^{p+1}(M - S_1)$ is associated with the conditions which ensure that $\rho \in B_c^{p+1}(M - S_1)$. Thus the group $H_c^{p+1}(M - S_1)$ is associated with the global conditions which ensure the solvability of the equations for the extremal ν of the complementary variational principle. Dually, the group $H_c^{n-p-1}(M - S_2)$ describes the nonuniqueness in ν which cannot be described by a gauge transformation of the form

$$\nu \rightarrow \nu + d\theta, \qquad \theta \in C_c^{n-p-2}(M - S_2).$$

In other words, the nonuniqueness of ν is described by $Z_c^{n-p-1}(M - S_2)$ while the above gauge transformation involves $B_c^{n-p-1}(M - S_2)$ and the difference is characterised by $H_c^{n-p-1}(M - S_2)$. Finally the isomorphism

$$H_c^{p+1}(M - S_1) \simeq H_c^{n-p-1}(M - S_2)$$

expresses the duality between the global degrees of freedom in the nonuniqueness (gauge transformation) of ν and the solvability condition (conservation law) $\rho \in B_c^{p+1}(M - S_1)$. Thus the above isomorphism plays the same role in the complementary variational formulation as the isomorphism in i) played in the primal variational formulation. This shows how the above isomorphism played a role in the construction of the functional $J^{\perp}$.

Having considered the role of homology and cohomology groups in the context of the Tonti diagram for the paradigm problem, one of the principal aims of this thesis has been achieved. Furthermore the Tonti diagram for the paradigm problem includes as special cases electrostatics, magnetostatics and electromagnetics hence it unifies all of the cases considered in the paper by Penman and Fraser [1984] and makes explicit the role of homology groups in this context.

CHAPTER 4

Final Remarks

4.1 Future Work

The principal goal of this thesis is to demonstrate the usefulness of the formalism of differential forms and homology theory in the context of Maxwell's equations and, to an extent, this goal is achieved in the formulation and analysis of the paradigm problem of Chapter 3. It is however useful to mention which other routes could have been followed in order to reach this goal. That is, it is useful to mention other links between homology theory and electromagnetics which could have been developed. In particular the author believes that the following three areas look promising.

Singular Homology Theory and Finite Elements

Although the homology groups have played a central role in this thesis, no algorithm for their computation has been given. It is however a very fortunate coincidence that one of the most popular methods for computing numerically electromagnetic fields is the finite element method while one of the easiest ways of computing the homology groups of a manifold is done by computing the homology groups associated with a triangulation of the manifold by means of the techniques of singular homology theory. This coincidence is fortunate since the cell complexes (triangulations) of singular homology theory are essentially finite element meshes. In the context of the T-Ω method, this connection is made very clear in the paper by Brown [1984] while the papers of Eastman and Preiss [1984] and Mantyla [1983] are useful in the context of solid modeling. Background material for computing homology groups from triangulations can be found in the recent books by Munkres [1984], Chapter 1 and Massey [1980], Chapter 4 while the paper by Pinkerton [1966] implements a computer program for the computation of Betti numbers of manifolds. Along another route, it is useful to note that the theory of polynomial differential forms over triangulations and associated orthogonal decompositions have been studied by Baker [1982], Komorowski [1975], and Dodziuk [1976] which leads the author to believe that a fundamental connection between homology groups and finite element interpolation will be recognized in the next few years.

Eddy Current Problems

In the paper of Bossavit [1982] and Bossavit and Vérité [1983], the connection between several results in homology theory and boundary value problems for eddy current is made. In particular, it is seen that orthogonal decompositions for differential forms on manifolds with boundary and the Alexander duality theorem play an essential role in the global formulation of boundary value problems for eddy currents whenever the magnetic field is described by some hybrid set of potentials such as in the case of the T-Ω method. The computation of eddy currents in complicated three dimensional geometries is a very important engineering problem. For this reason it is important to further the study of the formulation of such problems.

Electrodynamics in Four Dimensions

Of the various instances of the paradigm problem listed in Table 1 of Section 3.1, the case of electromagnetics in four dimensions is often neglected. The author believes that this state of affairs is due primarily to two main obstacles, first because there is no systematic notation for vector calculus in four dimensions analogous to div, grad, and curl in three dimensions, and second because it is impossible to visualize the topological problems inherent in boundary value problems over four dimensional regions. The first obstacle is of course removed through the use of differential forms since the formalism of differential forms is suitable for any n-dimensional manifold. In the case of the second obstacle, it is instructive to note that many boundary value problems are set over a product manifold

$$M = M_s \times M_t$$

where M_s is a three dimensional space manifold and M_t is a one dimensional time manifold which, for example, may be S^1 (the circle) in problems involving stationary boundaries and periodic excitations. The Kunneth formula (see for example Munkres [1984] Section 58) enables one to rephrase many questions about the homology of the product manifold M in terms of the homology of M_s and M_t and hence the topological intricacies associated with M can be resolved by "staring at pictures" and the second obstacle mentioned above is removed. It is important to note that the London equations of superconductivity are stated in a four dimensionally covariant way and the topological aspects have been investigated by Post [1978 and 1984].

4.2 Outline of Original Contributions

The purpose of this thesis is to show how systematic use of homology groups and orthogonal decompositions of differential forms facilitate the formulation and solution of many theoretical problems associated with variational boundary value problems of electromagnetics. A major contribution of the thesis is then the formulation, in Chapter 3, of the paradigm problem in terms of differential forms. This formulation makes transparent the interpretation of the Tonti diagram in terms of two differential complexes associated with the exterior derivative. The (co)homology of these complexes is easily interpreted through the use of the de Rham isomorphism and the relevance of homology groups and duality theorems in variational boundary value problems of electromagnetics becomes undeniable once this connection is made. The construction of the modified variational principles in Chapter 3 should not be considered as a major contribution since the method used and the geometric pictures emphasized (troughs) are an adaption to the present context of material found in Kotiuga [1982]. Undoubtably, the main original contribution in Chapter 3 is the summary, in Section 3.5, of the role played by relative homology groups and duality theorems and their relation to the Tonti diagram.

Though the geometric intuition behind the construction of the modified variational principles in Chapter 3 is obtained by thinking in terms of "troughs", the justification of the arguments hinges on the orthogonal decomposition and duality theorems which were derived in Section 2.13. The validity of the orthogonal decomposition was apparent once the complex $\widetilde{C}_*(M, S)$ was introduced and the author believes that the introduction of this complex is something new. Furthermore, the use of the Hodge star operator for proving the duality theorem

$$\mathcal{H}^p(M, S_1) \simeq \mathcal{H}^{n-p}(M, S_2)$$

is also something which the author has never seen before.

Finally, the implicit use of homology theory in engineering electromagnetics is widespread but seldom is the connection between these two subjects made explicit. For this reason it is very hard to make confident claims of originality for the material in Chapter 1. In addition to the general synthesis of ideas, there are two uses of homology theory in Chapter 1 which the author believes are original and demonstrate an essential role for homology theory. They are:

1. The use of the long exact homology sequence, in Examples 12, 13, and 14, in order to resolve the global considerations in prescribing the tangential components of a

vector potential on part of a surface where it is known that the normal component of the curl of the potential must vanish.

2. The use of duality theorems and intersection numbers in making "cuts" which modify a region so as to make some relative homology group trivial. Though this technique is evident in Maxwell [1891], Section 22, the author knows of no place in the electromagnetics literature where mixed boundary conditions have been studied so extensively by means of relative homology groups. Furthermore, the limitations of the method in the context of nonorientable surfaces has not been exposed as in Example 20.

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