TOPICS IN MATRIX COSMOLOGY

Written By

Samuel Laliberte

Department of Physics

McGill University, Montreal

February, 2025

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of

Doctor of Philosophy, Physics

Aknowledgements

I would like to, first and foremost, thank my PhD supervisor Professor Robert Brandenberger for his boundless support during the last seven years of my Master's and PhD. Professor Brandenberger's optimism, insight, and availability greatly facilitated my journey through different projects, and his encouragement to attend various events and conferences played an important role in shaping my interests.

Second to Robert, I would like to thank Suddhasattwa (Suddho) Brahma for his notable contribution to many ideas discussed as part of this thesis. I have thoroughly enjoyed working with Suddho, and also spending time with him at various locations around the globe, including at the University of Edinburgh where he currently carries his research. I thank him for his hospitality during my visit.

In addition, I would like to thank my colleagues, who have helped define my experience at McGill during my PhD: Heliudson Bernardo, Bryce Cyr, Ziwei Wang, Aline Favero, Matteo Puel, Matteo Blamart, Paul Robert Chouha, Jiao Hao, Guilherme Franzmann, Jerome Quintin, Guillaume Payeur and Julia Pasiecznik. Among them, I feel that Heliudson Bernardo and Bryce Cyr deserve a special mention. Through his willingness to engage in many interesting debates, Heldiudson helped build the conceptual compass have in physics today. By means of constant witty banter, Bryce contributed to making my office life at McGill entertaining and memorable. Also, thanks to him, I will eternally know what "real physics" is supposed to be.

I would also like to acknowledge a grant from de Fonds de recherche du Québec - Nature et Technology (FRQNT), which has allowed me to bring the present project to fruition. Complementary to this grant, partial support from the Natural Sciences and Engineering Research Council of Canada (NSERC) and from the Canada Research Chair program through my supervisor's grant, and partial support from the McGill Space Institute are also acknowledged.

In margin of the academic world, I would like to thank my longtime friends Khalil Hassan and Félix Lévesques-Nicol for all the time we spent together and for making a big part of what the Montreal life is to me. I would also like to thank my jam friends Darwin Decena, Elvis Wu, and blues maestro extraordinaire Mathias Dallaire (aka Mr. Pinstripes), who brought a new musical dimension to my life recently. I hope to be able to continue this musical journey across the globe wherever the academia life may take me.

Last but not least, I would like to thank my former partner, Laurence St-Amour, for her incredible support throughout the years. Against all odds, Laurence remained an incredible source of mutual support in my life and helped me go through incredibly difficult times these past two years. I wish her the best in her future endeavors.

Abstract

String Gas Cosmology is a non-singular early universe scenario in which the universe begins described by the thermal properties of a gas of strings. While this scenario leads to many interesting predictions such as the emergence of three large dimensions and a scale-invariant spectrum of cosmological perturbations, it still lacks a complete description. In this thesis, which contains five published articles, we take steps toward obtaining a complete scenario for String Gas Cosmology by considering matrix theory as a possible description. In particular, we consider the BFSS and IKKT models, which are both non-perturbative descriptions of superstring theory. Our work is divided into four parts. In the first one, we consider features of an emergent universe as described by the BFSS and IKKT model at finite temperatures. We find that thermal fluctuations in both models can source a scale-invariant spectrum of cosmological perturbations, in line with expectations from String Gas Cosmology. In the second part, we elaborate on features of the background in the context of the IKKT model. More precisely, we formulate the notion of a space-time metric in this model based on features from numerical simulation. In the third part, we investigate a dynamical mechanism in which the IKKT model acquires a mass term, potentially realizing early universe solutions. In the fourth part, we probe symmetry breaking in the BFSS model as potential evidence for the emergence of three large dimensions. We find evidence that symmetry breaking can occur at high temperatures using the Gaussian expansion methods. In conclusion of this thesis, we find good evidence from matching experimental predictions and symmetry-breaking processes that matrix theory can provide a framework for String Gas Cosmology, with remaining aspects such as a complete background description left to be worked on.

Abrégé

La cosmologie à gaz de cordes est un scénario dans lequel l'univers est initialement décrit par les propriétés thermiques d'un gaz de cordes. Bien que ce scénario conduise à de nombreuses prédictions intéressantes, telles que l'émergence d'un universe à trois dimensions et un spectre de perturbations cosmologiques invariant d'échelle, une description complète de ce scenario n'a toujours pas été formulée. Dans cette thèse, qui contient cinq articles publiés, nous faisons des progrès vers une description complete du scénario à gas de cordes en considérant la théorie des matrices comme description potentielle. En particulier, nous examinons les modèles BFSS et IKKT, qui sont tous deux des descriptions non-perturbatives de la théorie des supercordes. Notre travail est divisé en quatre parties. Dans la première, nous étudions les caractéristiques d'un univers émergent tel que décrit par les modèles BFSS et IKKT à une temperature fixte. Nous constatons que les fluctuations thermiques dans ces deux modèles peuvent être à l'origine d'un spectre de perturbations cosmologiques invariant d'échelle, en accord avec la cosmologie à gaz de cordes. Dans la deuxième partie, nous étudions les charactéristiques évolutives de l'espace-temps dans le cadre du modèle IKKT. Plus précisément, nous formulons la notion d'une métrique de l'espace-temps dans ce modèle en nous basant sur des caractéristiques issues de simulations numériques. Dans la troisième partie, nous investiguons un méchanisme dans lequel le modèle IKKT acquière one masse, ce qui pourrais mener à des solutions d'univers relié au début de l'univers dans ce model. Dans la quatrième partie, nous étudions le possible bris de symétrie dans le modèle BFSS comme une preuve potentielle de l'émergence de trois dimensions. Nous trouvons des indices démontrant un possible bris de symétrie à haute températures en utilisant la méthode d'expansion Gaussienne. En conclusion de cette thèse, nous trouvons de solides indications, issues de la correspondance entre les prédictions expérimentales et un potentiel processus

de bris de symétrie, que la théorie des matrices peut fournir un cadre pour la cosmologie à gaz de cordes. Par contre, certains aspects tels qu'une description complète de l'évolution cosmologique, restent à approfondir.

Statement of contribution

This manuscript-based thesis contains five published articles that can be considered original scholarship and distinct contributions to knowledge. Each manuscript is presented in its original form, including the bibliography. In compliance with standards in the field of theoretical physics, the names of the authors are presented in alphabetical order. Only one exception to this rule was made for the manuscript "IKKT thermodynamics and early universe cosmology", in order to reflect the dominant contribution of the author of the present thesis. Contribution-wise, in [1] and [2], the author of the present thesis is the first author. In [3], [4] and [5], the author of the present thesis is co-first author. We state below the contribution of the author to each of the included works.

Contributions of the author

[3] S. Brahma, R. Brandenberger and S. Laliberte, "Emergent cosmology from matrix theory," JHEP **03**, 067 (2022) [arXiv:2107.11512 [hep-th]].

This article is presented in Chapter 3. As co-author of this paper, I contributed to the discussions that led to the idea of the paper, performed the computations with the assistance of Prof. Brahma, and helped with part of the writing process. Prof. Brandenberger helped in the discussions and writing process.

[1] S. Laliberte and S. Brahma, "IKKT thermodynamics and early universe cosmology," JHEP **11**, 161 (2023) [arXiv:2304.10509 [hep-th]].

This article is presented in Chapter 4. As the first author of this publication, I carried out all the computations with the supervision of Prof. Brahma, and wrote the paper from start to finish.

[4] S. Brahma, R. Brandenberger and S. Laliberte, "Emergent metric space-time from matrix theory," JHEP **09**, 031 (2022) [arXiv:2206.12468 [hep-th]].

This article is presented in Chapter 5. As co-author of this paper, I contributed to the discussions that led to the idea of the paper, performed the computations with the assistance of Prof. Brandenberger, and helped with part of the writing process. Prof. Brahma helped in the discussions and writing process.

[2] S. Laliberte, "Effective mass and symmetry breaking in the Ishibashi-Kawai-Kitazawa-Tsuchiya matrix model from compactification," Phys. Rev. D **110**, no.2, 026024 (2024) [arXiv:2401.16401 [hep-th]].

This article is presented in Chapter 6. As the first and only author of this publication, I carried out all the computations and wrote the paper from start to finish.

[5] S. Brahma, R. Brandenberger and S. Laliberte, "Spontaneous symmetry breaking in the BFSS model: analytical results using the Gaussian expansion method," Eur. Phys. J. C **83**, no.10, 904 (2023) [arXiv:2209.01255 [hep-th]].

This article is presented in Chapter 7. As co-author of this paper, I contributed to the discussions that led to the idea of the paper, performed the computations with the assistance of Prof. Brahma, and helped with part of the writing process. Prof. Brandenberger helped in the discussions and writing process.

Additional Content

The five articles we present in this thesis are written for an audience with knowledge of cosmology, string gas cosmology, and matrix theory. To introduce the reader to these topics, a review section covering the knowledge necessary to understand our contributions was included in this thesis. Here is a list, sorted by topic, of documents that helped write this review section.

Standard Big Bang Cosmology

- [6] W. H. Kinney, "TASI Lectures on Inflation," [arXiv:0902.1529 [astro-ph.CO]].
- [7] D. Baumann, "Inflation," [arXiv:0907.5424 [hep-th]].
- [8] D. Baumann, "Primordial Cosmology," PoS **TASI2017**, 009 (2018) [arXiv:1807.03098 [hep-th]].
- [9] R. H. Brandenberger, "Inflationary cosmology: Progress and problems," [arXiv:hep-ph/9910410 [hep-ph]].

String Gas Cosmology

- [10] R. H. Brandenberger and C. Vafa, "Superstrings in the Early Universe," Nucl. Phys. B **316** (1989), 391-410
- [11] R. H. Brandenberger, "String Gas Cosmology," [arXiv:0808.0746 [hep-th]].
- [12] R. H. Brandenberger, A. Nayeri, S. P. Patil and C. Vafa, "String gas cosmology and structure formation," Int. J. Mod. Phys. A **22**, 3621-3642 (2007) [arXiv:hep-th/0608121 [hep-th]].

Matrix theory

- [13] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, "A Large N reduced model as superstring," Nucl. Phys. B **498**, 467-491 (1997) [arXiv:hep-th/9612115 [hep-th]].

- [14] K. L. Zarembo and Y. M. Makeenko, “An introduction to matrix superstring models,” Phys. Usp. **41**, 1-23 (1998)

- [15] S. W. Kim, J. Nishimura and A. Tsuchiya, “Expanding (3+1)-dimensional universe from a Lorentzian matrix model for superstring theory in (9+1)-dimensions,” Phys. Rev. Lett. **108**, 011601 (2012) [arXiv:1108.1540 [hep-th]].

- [16] B. Ydri, “Review of M(atrix)-Theory, Type IIB Matrix Model and Matrix String Theory,” [arXiv:1708.00734 [hep-th]].

- [17] S. W. Kim, J. Nishimura and A. Tsuchiya, “Late time behaviors of the expanding universe in the IIB matrix model,” JHEP **10**, 147 (2012) [arXiv:1208.0711 [hep-th]].

- [18] D. N. Kabat and G. Lifschytz, “Approximations for strongly coupled supersymmetric quantum mechanics,” Nucl. Phys. B **571**, 419-456 (2000) [arXiv:hep-th/9910001 [hep-th]].

- [19] J. Nishimura and F. Sugino, “Dynamical generation of four-dimensional spacetime in the IIB matrix model,” JHEP **05**, 001 (2002) [arXiv:hep-th/0111102 [hep-th]].

Contents

List of Figures

Part I

Introduction and Theoretical Background

Chapter 1

Introduction

One of the greatest challenges in theoretical physics today is to understand the origin of the universe and the mechanisms at play during its formation. A major complication arises from the fact that our current framework for the origin and evolution of the universe, Standard Big Bang Cosmology (see [8] for a review), predicts that the universe begins with an initial singularity. Conceptually, this singularity signals the breakdown of Einstein's theory of gravitation, meaning that general relativity cannot fully describe the true beginning of the universe. To address this issue, new physics or a more fundamental theory is required.

Currently, string theory (see [21] for an introduction) is the leading candidate for this task, as it provides a consistent theory of quantum gravity and its coupling to matter. In string theory, there exists a fundamental length scale, the string scale, below which singularities are expected to be resolved by stringy effects. Within this framework, one may hope that the initial big bang singularity is properly resolved. Many scenarios inspired by string theory have been proposed to resolve the singularity problem. Among them, String Gas Cosmology (see [11] for a review) presents an elegant solution. In this model, the early universe evolves according to the thermal properties of a gas of strings, which possesses a maximum temperature, the Hagedorn Temperature, linked to the string scale, hence imposing a maximum density. As a result, the initial space-time curvature in String Gas Cosmology must be finite at the beginning of the universe, hence avoiding the singularity. After this initial phase, the universe transitions into the standard Big Bang scenario, leaving behind two key predictions. First, only three of the nine spatial dimensions in string theory become large, offering an explanation for why we live in a three dimensional world out of the nine string theory predicts. Second, thermal fluctuations of the string gas generate a scale-invariant spectrum of cosmological perturbations, providing a mechanism for the current distribution of matter in the universe.

Despite these promising features, the string gas scenario poses some challenges. Like many string-inspired models, one must go to the non-perturbative (high-energy) regime of string theory in order to fully describe the physics of the Big Bang. However, to this day, the dynamics of string theory in this regime are not fully understood. As a result, performing a full dynamical analysis of String Gas Cosmology remains a challenging task. There are, however, descriptions of string theory that might show promise in taming non-perturbative effects in the theory. This is the case of matrix models, for example. In matrix models, the dynamics of string theory can be studied without prior notion of strings or knowledge on how to carry out a perturbative expansion in string theory. Rather, strings, membranes, interactions, and space-time are emergent in the theory. Moreover, matrix descriptions of string theory are in general naturally non-commutative, which may give a hint on how to resolve singularities. In light of these properties, matrix theory is an interesting candidate to study the dynamics of the early universe.

In this thesis, which contains five published articles, we investigate matrix theory as a potential non-perturbative framework for string gas cosmology. Our approach builds on the BFSS [22] and IKKT [13] matrix models, which are respectively non-perturbative descriptions of M-theory in the Discrete Light Cone Quantisation limit, and Type IIB string theory. Our findings are presented in four parts, each summarized as follows. In the first part, we investigate thermal fluctuations of the BFSS and IKKT model at finite temperatures. We find that these fluctuations can produce a scale-invariant spectrum of cosmological perturbation, in agreement with the results of string gas cosmology. In the second part, we elaborate on features of emergent cosmological backgrounds in the context of the IKKT model, in an attempt to describe emergent cosmological solutions. More precisely, we formulate the notion of a space-time metric in this model based on features from numerical simulation. In the third part, we investigate a mechanism in which the IKKT model acquires an effective mass term following compactification on a six-torus where fermions have anti-symmetric boundary conditions. It is known that the IKKT model with a mass term has cosmological solutions. For this reason, the effective mass term acquired from compactification may lead to interesting early universe solutions. Finally, in the fourth part, we probe symmetry breaking in the BFSS model, as potential evidence for the emergence of three large dimensions using the Gaussian Expansion Method [18]. Using this method, we find evidence that symmetry breaking can occur at high temperatures, just like in String Gas Cosmology. In this case, the symmetry of the system after such a transition is still an open question.

Chapter 2

Review of key concepts

Before going into detail on how matrix theory may be related to String Gas Cosmology, let us first review some important notions related to Standard Big Bang Cosmology, String Gas Cosmology, and matrix cosmology. In what follows, we will review how Standard Big Bang Cosmology describes our universe. We will also review how Standard Big Bang Cosmology faces three challenges, namely the horizon, flatness, and singularity problem, and how a mechanism is needed to explain the distribution of matter in our universe on large scales. We will then review String Gas Cosmology, and how it can address some of the problems of Standard Big Bang Cosmology. Moreover, we will review how thermal string fluctuations can describe the observed distribution of matter on large scales. Finally, we will introduce matrix theory and matrix cosmology. As part of this last section, we will introduce the IKKT and BFSS matrix models, how strings and interactions are emergent in these models, how cosmological solutions emerge in matrix theory, how a mass term can lead to cosmological solutions, and how the Gaussian Expansion Method can be used to probe for symmetry breaking in the IKKT model. These concepts will then be used later in the thesis to draw comparisons between matrix theory and String Gas Cosmology.

2.1 Standard Big Bang Cosmology

Standard Big Bang Cosmology, also known as the ΛCDM (Lambda Cold Dark Matter) model, is currently the leading model to explain the evolution of the universe from its early times to today. As the name suggests, the model assumes that the expansion of the universe is driven by three forms of energy: dark energy (Lambda), cold matter (which includes regular matter and dark matter), and radiation, which is not included in the name but is also assumed to be present. In the next subsection, we will explain the framework of Standard Big Bang Cosmology. Moreover, we will go over observational aspects, namely the CMB spectrum, which we will try to explain in this thesis, along with some problems that we will also try to address.

2.1.1 Dynamics of Standard Big Bang Cosmology

In Standard Big Bang Cosmology, the evolution of the universe is driven by Einstein's equations of motion

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu} \,, \tag{2.1}$$

as derived from the Einstein action

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}\, R + S_{matter} \tag{2.2}$$

coupled to matter. In Einstein's field equations, the field strength tensor $T_{\mu\nu}$ induces curvature of space-time, whose information is encoded in the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R. The strength of the induced curvature is mediated by Newton's constant G_N.

In the context of cosmology, one usually imposes the cosmological principle to obtain cosmological solutions using the Einstein field equations. This principle asserts that on sufficiently large scales, the universe is homogeneous (the same at every point) and isotropic (the same in all directions). Under this assumption, the stress-energy tensor $T_{\mu\nu}$, must take the following form:

$$T^{\mu}_{\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p\,. \end{pmatrix} \tag{2.3}$$

Here, ρ is the energy density per unit volume of matter in the universe, and p is the pressure of matter in the universe.

The cosmological principle also constrains the possible forms the space-time metric $g_{\mu\nu}$ can take. For a homogenous and isotropic universe, the most general form of the metric is

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + sin^2\theta d\phi^2 \right) \right) . \tag{2.4}$$

Here, $a(t)$ is a scale factor that describes how the universe expands (or contracts, in certain cases), and k is a constant related to the spatial curvature of our universe. In the present case, k can only take three values, leading to cosmological solutions with different characteristics: $k = 0$ (flat universe), $k = -1$ (closed universe), and $k = 1$ (open universe).

Substituting our ansatz for the stress tensor and the metric (equations 2.4 and 2.3), we obtain the Friedmann-Robertson-Walker (FRW) equations for the evolution of the universe:

$$\frac{8\pi G_N}{3}\rho = \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\,, \tag{2.5}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3p)\,. \tag{2.6}$$

In addition to these equations of motion, one usually assumes energy-momentum conservation, which implies that

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{2.7}$$

This equation, named continuity equation, follows directly from imposing that the matter component of the Einstein action with matter (equation 2.2) is invariant under an infinitesimal change of coordinates (diffeomorphism invariance). For our choice of metric and stress-energy tensor, the continuity equation specifically takes the following form:

$$\dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)(\rho + p) = 0 \tag{2.8}$$

In Standard Big Bang cosmology, it is usually assumed that the expansion of the universe is driven by three matter entities: radiation, cold matter, and dark energy. Each of these entities obeys an equation of state of the form

$$p = \omega\rho\,, \tag{2.9}$$

where $\omega = 1/3$ for radiation, $\omega = 0$ for matter, and $\omega = -1$ for dark energy. Under this assumption, the FRW equations and the continuity equation can be simultaneously solved, leading to the solutions

$$\rho \propto a^{-3(1+\omega)} \qquad a(t) \propto \begin{cases} t^{2/[3(1+\omega)]} & \omega \neq -1 \\ e^{Ht} & \omega = -1 \end{cases}, \tag{2.10}$$

In the present case, which of the three entities contributes to the Einstein equation is determined by which one dominates the energy density at a given point in time. At early times, the energy is dominated by radiation ($\rho \sim a^{-4}$). The radiation era then leaves place to the matter-dominated era ($\rho \sim a^{-3}$), then the dark energy-dominated era ($\rho \sim$ const.). This transition between energies can be seen more accurately in Figure 2.1. This paints the picture

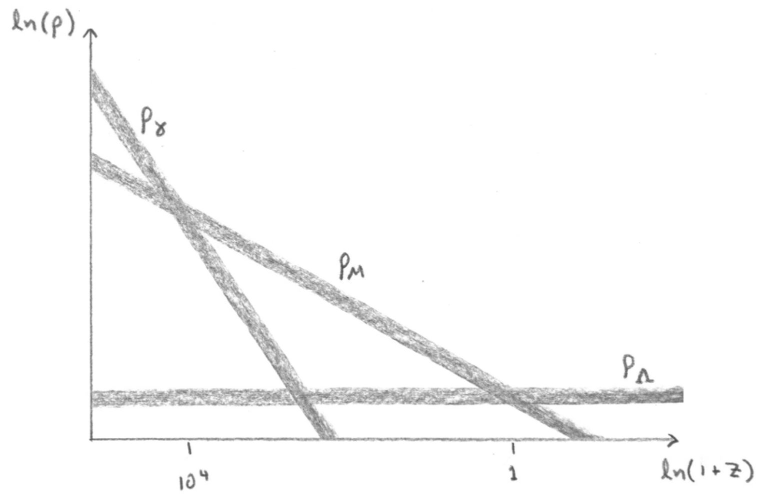

Figure 2.1: Visual representation of how each type of matter scale as a function of the redshift z. At early times, radiation dominates followed by matter and vacuum energy. Here, the redshift z is related to the scale factor $a(t)$ of the universe via $a(t) = a(t_0)/(1+z)$, where t_0 is the age of the universe at the present time. The original figure can be found in [6].

of a universe that begins in a radiation-dominated era, transitions to a matter-dominated era, and then ends eternally inflating in a dark energy-dominated era.

2.1.2 Observational status of Standard Big Bang Cosmoology

For the longest time, the energy content of the universe was largely unknown to cosmologists. However, this all changed with the discovery of the Cosmic Microwave Background in 1965, and subsequent studies of this background which sparked a new era of precision cosmology.

These observations revolve around a period of the universe named the time of recombination. This period describes a time of the universe where matter at high temperature, which was then in the form of a plasma of protons and electrons, cooled enough under the expansion of the universe to combine into a gas of hydrogen. Since plasma is opaque to light, this period marks the time at which light first became "free" in the universe, allowing for observation. Consequently, the time of recombination is the oldest epoch of the universe one can study using electromagnetic radiation. The residual radiation that can be observed from this era bears the name Cosmic Microwave Background (CMB), coming from the fact that this radiation can be observed in the microwave portion of the electromagnetic spectrum. A picture of this background can be found in Figure 2.2. In the CMB, one observes

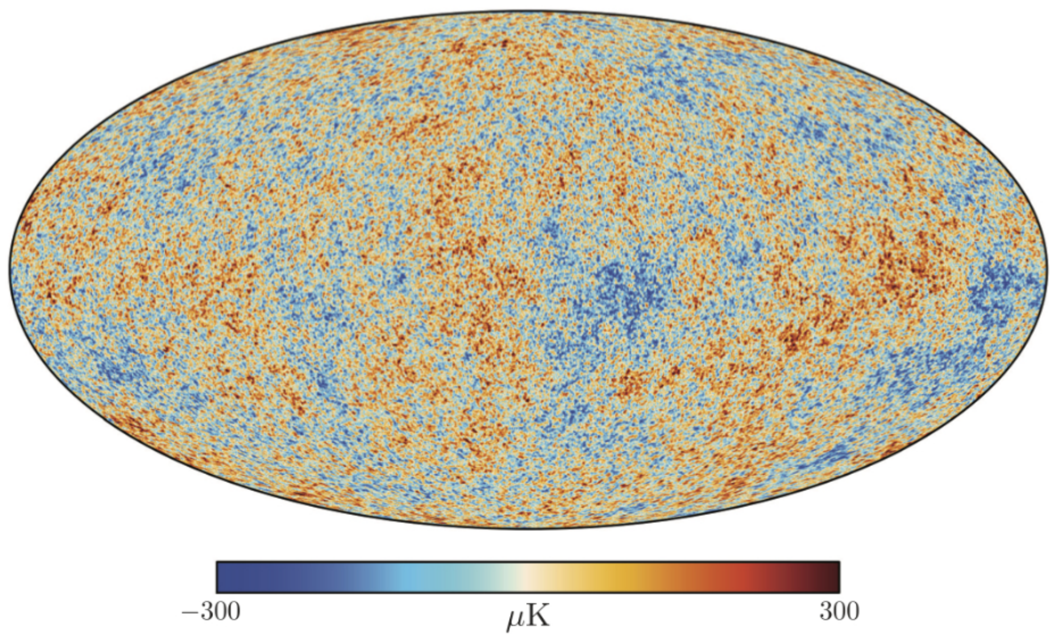

Figure 2.2: Map of temperature fluctuations in the Cosmic Microwave Backgorund, as shown in the Plank 2015 survey [20].

a background temperature with a value around 2.7 K with temperature fluctuations on the order of

$$\frac{\delta T}{T} \sim 10^{-4} \,. \tag{2.11}$$

This shows remarkable agreement with the cosmological principle, which demands that the universe must be homogeneous and isotropic.

From CMB spectrum data, along with data from galaxy surveys and measurements on the current Hubble expansion rate H_0, it is possible to infer the energy content of the universe. For an energy species i, the relative abundance is often expressed as the ratio

$$\Omega_i \equiv \frac{\rho_i}{\rho_c} \tag{2.12}$$

of the energy density ρ_i of the species with respect to the critical density

$$\rho_c = \frac{3H_0^2}{8\pi G_N}\,, \tag{2.13}$$

which is related to the sum of all energy species in a flat universe. According to the most recent observations from the Planck satellite (Planck 2018 observations taking into account distance measurements from baryon acoustic oscillations [23, 24]), our universe has the following energy content

$$\begin{aligned}
\Omega_\Lambda &= 0.6889 \pm 0.0056\,,\\
\Omega_m &= 0.3111 \pm 0.0056\,,\\
\Omega_r &\sim 10^{-4}\,.
\end{aligned}$$

Here, Ω_Λ is the energy fraction of dark energy, Ω_m is the energy fraction of matter, and Ω_r is the energy fraction of radiation. Using observations, it is also possible to evaluate how much energy in the universe is expressed in the form of curvature, as related to the curvature parameter k. This is usually done by measuring the energy fraction of curvature Ω_k, which is defined as

$$\Omega_k \equiv \Omega - 1 = \frac{\rho - \rho_{\text{crit}}}{\rho_{\text{crit}}} \tag{2.14}$$

Here, ρ is the total energy density of the universe, which includes dark energy, matter, radiation and curvature. In the present case, $\Omega = 1$ describes a flat universe ($\rho = \rho_c$). According to the Planck 2018 survey [24], on obtains

$$\Omega_k = 0.0007 \pm 0.0019\,, \tag{2.15}$$

which is consistent with a flat universe ($k = 0$).

Along with precision measurement of the energy content of the universe, valuable information about the beginning of the universe can be found by studying the fluctuations around the CMB background temperature. This analysis can be done by looking at the amplitude of different Fourrier modes of oscillation. To extract these Fourrier modes, one usually assumes the following spherical harmonics expansion

$$\Theta(\hat{n}) = \frac{\Delta T(\hat{n})}{T_0} = \sum_{lm} a_{lm} Y_{lm}(\hat{n})\,.$$

where

$$a_{lm} = \int d\Omega Y^*_{lm}(\hat{n})\Theta(\hat{n})\,, \tag{2.16}$$

is the amplitude of the various harmonics, and the $Y_{lm}(\hat{n})$'s are spherical harmonic fonctions in a direction $\hat{n}$ in the sky. Taking the norm of these amplitudes, one obtains the angular power spectrum

$$C_l = \frac{1}{2\pi}\sum_m \langle a^*_{lm} a_{lm}\rangle$$

which for the CMB can be plotted as in Figure 2.3. In Figure 2.3, there are three scales

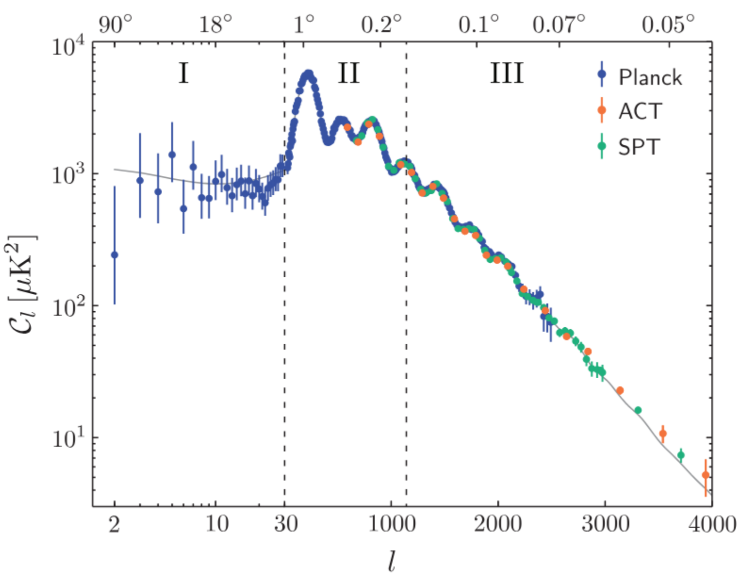

Figure 2.3: Angular power spectrum C_l of the CMB as a function of the angular frequency l. The original figure can be found in [8].

of interest, which are highlighted by Region I, II , and III. On large (super-Hubble) scales

(Region I), fluctuations are frozen. These fluctuations are not related to matter oscillations, and therefore must be explained by physics near the Big Bang. At intermediate scales (Region II), we observe peaks related to oscillations of the matter fluid at the moment of recombination. Finally, on small scales (Region III), fluctuations are damped because their wavelengths are smaller than the mean free path of the photons.

In principle, any scenario that describes the early universe should be able to explain the features of the CMB. Most importantly, it should explain the scale-invariance of perturbations on large scales, which cannot be explained by conventional matter perturbations. Fortunately, there exist scenarios, such as String Gas Cosmology, where these features can be explained. In the next section, we will see how these fluctuations can be realized in this context. In Part II, we will also see how this can be realized in matrix cosmology.

2.1.3 Problems of Standard Big Bang Cosmology

Despite the success of Standard Big Bang Cosmology in describing our universe, the model suffers from a few conceptual issues. These issues, named the horizon problem, the flatness problem, and the singularity problems, are summarised below.

The horizon problem

As we saw in the previous section, one observes in the CMB that the temperature of the universe is approximately the same in all directions, signaling a homogeneous and isotropic universe. A priori, this should mean that all parts of the universe came in causal contact at some point before the time of recombination, and thermalized. However, if we look back at the time evolution of the universe in Standard Big Bang cosmology, this simply cannot be the case. This fact can be seen from computing the past light cone $l_p(t_{rec})$ from the present time t_0 to the time t_{rec} of recombination, and the future light cone $l_f(t_{rec})$ from the time of the big bang to the time of recombination, which gives

$$l_p(t_{rec}) = \int_{t_{rec}}^{t_0} a(t)^{-1} dt \approx 3t_0^{1/3} \left(1 - \left(\frac{t_{rec}}{t_0} \right)^{\frac{1}{3}} \right), \tag{2.17}$$

$$l_f(t_{rec}) = \int_0^{t_{rec}} a(t)^{-1} dt \approx 3\, t_{rec}{}^{\frac{1}{3}} \,. \tag{2.18}$$

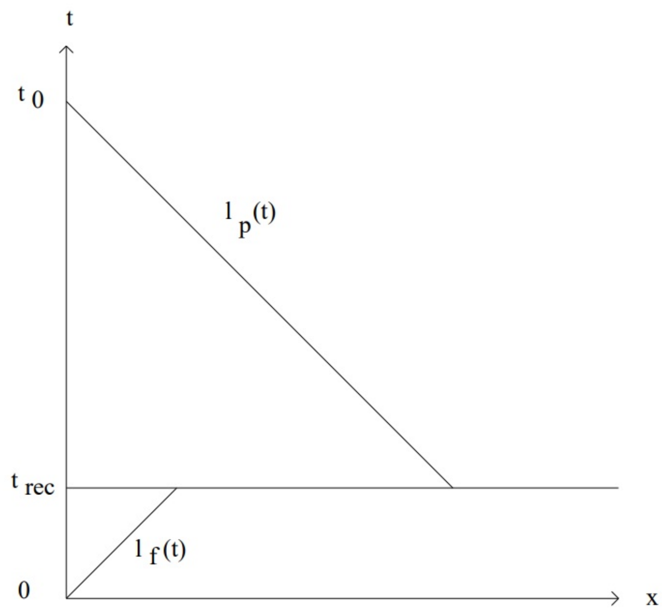

Figure 2.4: Schematic description of the past light cone l_p and the future light cone at the time of creation l_f. As we can see, the past light cone is much larger than l_f at t_{rec}, the time of recombination. The original figure can be found in [9].

In the above, we assumed that the universe evolves in a matter-dominated phase, neglecting the subdominant contribution from the radiation-dominated phase. Plotting the results, which can be seen in Figure 2.4, we see that the past light cone is much larger than the future light cone at the time of recombination. Consequently, distant patches in the early universe cannot come into causal contact before recombination, leaving the present homogeneity of the universe unexplained.

The flatness problem

The flatness problem emerges from the fact that in a radiation or matter universe, the universe deviates from flatness as a function of time. Consequently, we must impose incredibly flat initial conditions on the universe in order to obtain the level of flatness observed today.

To see why this is the case, we can look at the curvature parameter defined in Section 2.1.2. Using Friedmann's equations (Equation 2.6), along the solutions for the scale factor

$a(t)$ (Equation 2.10), we find that

$$\Omega_k = \frac{k}{(aH)^2} \sim k\, t^{(1+3\omega)/[3(1+\omega)]}. \tag{2.19}$$

As we can see, if $1 + 3\omega > 0$, which is the case in a radiation or matter-dominated universe, the energy fraction related to the curvature k increases as a power law as a function of time. For this reason, for the energy fraction Ω_k to be as small as it is measured today, it must have been extremely small at the moment of the big bang. Without a mechanism that can drive the universe to flatness, or imposing extremely flat initial conditions, Standard Big Bang Cosmology cannot explain these extremely fine-tuned initial conditions.

The singularity problem

Finally, the last problem one encounters in Standard Big Bang Cosmology, or at least the last we will discuss here, is the singularity problem. As one approaches the Big Bang in this scenario, the density of the universe diverges as

$$\rho \sim a^{-4} \quad , \quad a(t) \sim t^{1/2} \tag{2.20}$$

which is the case in a radiation-dominated universe. This leads to a curvature of the universe which diverges as

$$R \sim t^{-2}\,. \tag{2.21}$$

as $t \to 0$. This singularity signals the breakdown of general relativity. For this reason, to explain the beginning of the universe, one must turn to new physics.

2.2 String Gas Cosmology

Many scenarios involving new physics have been suggested to answer the problems of Standard Big Bang Cosmology, along with the scale-invariance of the angular power spectrum on large scales. In the present thesis, we will focus on only one of these scenarios, namely String Gas Cosmology. String Gas Cosmology is a scenario in which the beginning of the universe is explained by a gas of string at finite temperature. Below, we review the key aspects of String Gas Cosmology, how it solves some of the problems of Standard Big Bang Cosmology, and how it can explain the observed features of the CMB.

2.2.1 Dynamics of String Gas Cosmology

In the String Gas Cosmology scenario (see [10] for original work), the nine spatial dimensions of the universe begin compactified on a torus of radius R. This assumption, which is made in order to simplify the study of string thermodynamics, is not in contradiction with the observed properties of our universe if one assumes that, in three of the directions, the size of the compact space is at least equal to the size of the observed universe. This will be a natural consequence of the present scenario as we will see shortly. In such a space-time, closed strings acquire a spectrum that contains momentum modes, winding modes, and oscillatory modes:

$$M^2 = \underbrace{\left(\frac{n}{R}\right)^2}_{\text{Momentum modes}} + \underbrace{\left(\frac{mR}{\alpha'}\right)^2}_{\text{Winding modes}} + \underbrace{\frac{2}{\alpha'}\left(N + \tilde{N} - 2\right)}_{\text{Oscillatory modes}} .$$

In the spectrum above, the momentum modes are the stringy analogue of the Kaluza-Klein modes, which quantum fields usually acquire when compactified on a torus, the winding modes come from strings winding around the torus, and the oscillatory modes are the usual modes found in closed strings in a non-compact space-time. In the expression above, α' is a constant related to the string tension T via $\alpha' = 1/(2\pi T)$, and N and $\tilde{N}$ are integers related to the quantization of the string's oscillatory modes.

The spectrum above has an important property named T-duality. This duality implies that the momentum modes and the winding modes of closed strings can be interchanged under the following relations

$$R \longleftrightarrow \frac{\alpha'}{R} \quad , \quad n \longleftrightarrow m\, .$$

As a result of T-duality, string gases acquire the following dynamical properties, which are summarised in Figure 2.5. At large R, the winding modes are suppressed, and all the energy is in the momentum modes. As one shrinks the size of R, the momentum modes become heavy, the winding modes become light, the temperature increases, and the thermal energy starts flowing in the winding and oscillatory modes. At some point near $R \sim \sqrt{\alpha'}$, the system will reach a limiting temperature, namely the Hagedorn temperature $T_H \sim /\sqrt{\alpha'}$ [25], which arises from the fact that the density of states of the closed string oscillatory modes grows exponentially as one reaches this temperature causing the partition function of closed

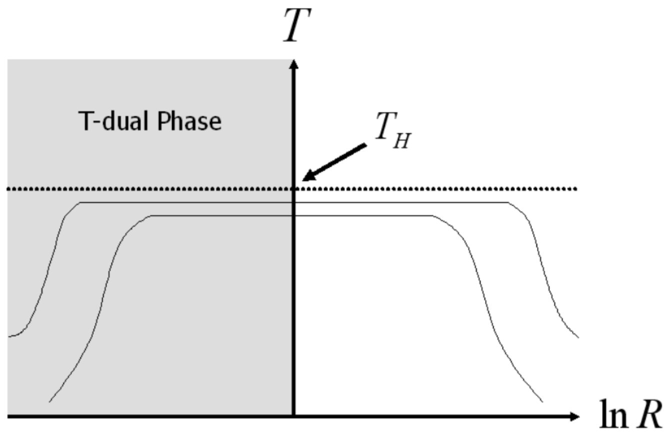

Figure 2.5: The temperature (vertical axis) as a function of radius (horizontal axis) of a gas of closed strings in thermal equilibrium. The original figure can be found in [11].

strings to diverge. Then, as one shrinks R further, the temperature will become low enough that the oscillatory modes are no longer excited, and all the energy will be found in the winding modes, which are now light.

Given the aforementioned thermal properties of string gases, String Gas Cosmology assumes that the universe starts in a meta-stable phase near the Hagedorn temperature. During this time, this metastable equilibrium is maintained by counterbalancing the positive pressure of the string momentum modes, and the negative pressure of strings winding around the torus.

During the meta-stable phase, the negative pressure of the winding modes will prevent space from expanding. There is, however, a nice dimension counting argument that can explain how winding modes can annihilate, allowing the universe to leave the Hagedorn phase and expand. In more than four dimensions, string world-sheets have measure zero interaction probability. This comes from the fact that strings swipe two-dimensional world-sheets, and that randomly placed strings will generally miss each other unless they overlap. For this reason, closed strings with corresponding winding number and anti-winding number will in general not find each other and annihilate. However, in a space-time with three spacial dimensions or less, the world sheet of strings will generally meet each other, allowing winding modes to annihilate (see Figure 2.6 for a visual representation of this process). For this reason, String Gas Cosmology predicts that three spatial dimensions must become large

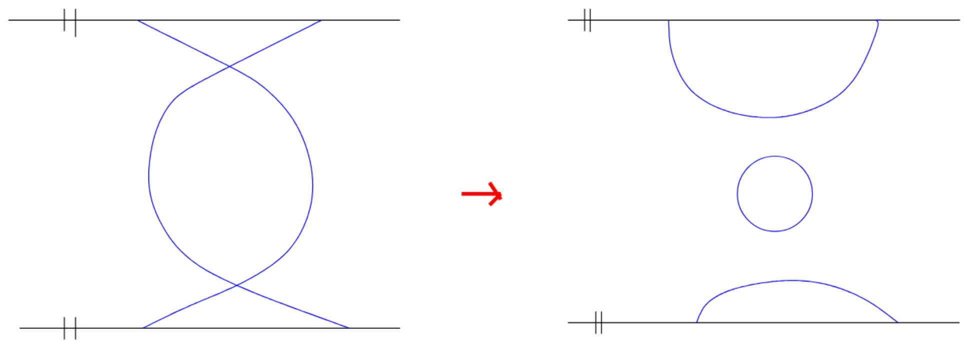

Figure 2.6: The process by which string loops are produced via the intersection of winding strings. The top and bottom lines are identified and the space between these lines represents space with one toroidal dimension un-wrapped. The original figure can be found in [11].

following the Hagedorn phase, leading to a four-dimensional emergent space-time.

It is important to note that supersymmetry plays a crucial role here in this dimension counting argument. Given that strings are supersymmetric objects, they do not attract nor repel each other since the attraction force from the string tension and the repulsion force from the charge cancel out. In abscence of attraction between the strings, the string-crossing probability determines the interaction probability.

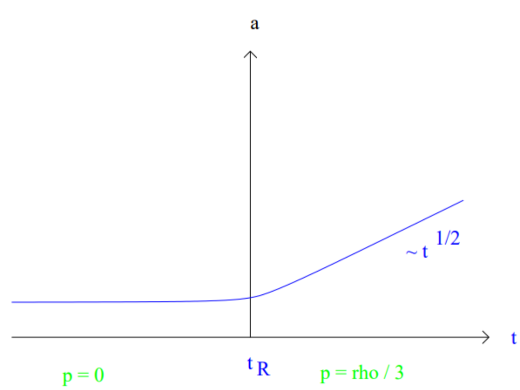

Figure 2.7: Evolution of the scale factor in string gas cosmology. The vertical axis represents the scale factor of the universe, the horizontal axis is time. The original figure can be found in [11].

Compiling all the processes we have overviewed so far, String Gas Cosmology paints the following picture for the evolution of the early universe (see Figure 2.7). At early times, the universe is in a meta-stable state near the Hagedorn temperature, and the scale factor is approximately constant. Following this metastable state, the winding modes of closed strings around a torus annihilate, and three dimensions become emergent. At this moment, the scale factor of the universe starts increasing, and one transitions to the radiation-dominated phase of Standard Big Bang Cosmology.

2.2.2 Resolution of the Standard Big Bang problems in String Gas Cosmology

So far, we saw that the early-time evolution of the universe in String Gas Cosmology has key differences from Standard Big Bang Cosmology. These differences allow String Gas Cosmology to address some of the problems encountered in Standard Big Bang Cosmology. We summarize answers to some of these problems below.

Singularity problem

In Standard Big Bang Cosmology, the singularity problem emerges because the density of matter is allowed to become infinite, leading to infinite curvature. In String Gas Cosmology, there is a limiting temperature called the Hagedorn temperature. For this reason, one expects there exists a limiting density of strings in the early universe, and therefore a limiting curvature. Consequently, the beginning of the universe in String Gas Cosmology is non-singular. This then follows with a transition to Standard Big Bang Cosmology.

Horizon problem

In Standard Big Bang Cosmology, the Horizon problem arises because different regions of the universe do not become in causal contact prior to the time of recombination. For this reason, it is a puzzle as to why the universe is allowed to thermalize prior to CMB observations. In String Gas Cosmology, this issue is resolved by the existence of a meta-stable state before the Standard Big Bang evolution. During this phase, the physical distance between any two points remains small for an extended duration. For this reason, assuming the meta-stable state lasts for a sufficiently long amount of time, the universe will be able to thermalize.

Once the universe starts expanding, the properties of this initial thermal state, including homogeneity, will be preserved along different expanding patches, explaining the observed homogeneity of our universe.

Flatness problem

The resolution of the flatness problem remains to this day one of the key limitations of string gas cosmology, as it does not provide a mechanism to explain the observed flatness. One should expect, however, that a non-perturbative description of String Gas Cosmology should provide answers in this regard. This problem remains a topic of investigation.

All-in-all, the main strength of String Gas Cosmology is that it can solve the singularity and horizon problems of Standard Big Bang Cosmology using quite general properties of string theory. The flatness problem, however, is still to this day unanswered, and the subject of ongoing research.

2.2.3 Experimental predictions of String Gas Cosmology

In addition to solving some of the problems of Standard Big Bang Cosmology, String Gas Cosmology can explain the observed features of the CMB (see [12] for original work). This is done in the following way. First, thermal fluctuations of strings, which are scale-invariant, exit the Hubble radius during the Hagedorn phase and become frozen. At a later time, during the Standard Big Bang phase, some of these modes then re-enter the Hubble horizon, start oscillating again, and source the oscillation peaks of the CMB. Perturbations that remain super-Hubble at the time of recombination, in their case, remain frozen and showcase the same scale-invariant spectrum that was sourced when the modes exited the Hubble horizon (see Figure 2.8).

The most important part of this process is that modes that exit the Hubble horizon must be scale-invariant. For thermal fluctuations of strings, this scale invariance can be shown in the following way. We first fluctuate around a cosmological background of the FRW form. Here, we will assume a metric of the form

$$ds^2 \;=\; (1+2\Phi)dt^2 - a(t)^2\left[(1-2\Phi)\delta_{ij} + h_{ij}\right]dx^i dx^j\,. \tag{2.22}$$

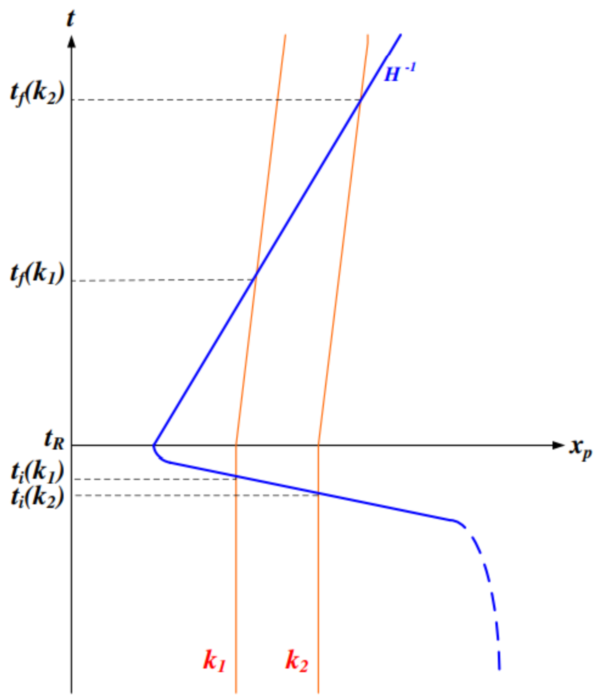

Figure 2.8: Sketch of how modes of oscillations of thermal strings exit and re-enter the Hubble radius in String Gas Cosmology. The original figure can be found in [11].

where Φ describes the scalar component of the fluctuations, and h_{ij} describes the tensor components of the fluctuations. We then relate Φ and h_{ij} to thermal properties of matter using the linearized Einstein's equations

$$\nabla^2\phi = 4\pi G_N \delta T^0_0 \quad , \quad \nabla^2 h_{ij} = -4\pi G_N \delta T^i_j \,. \tag{2.23}$$

In order to obtain the above, we assumed that transverse components δT^0_i are zero. We then move to Fourrier space in order to find expressions for $\langle|\Phi(k)|^2\rangle$ and $\langle|h(k)|^2\rangle$. Using equations 2.23, we find

$$\langle|\Phi(k)|^2\rangle \;=\; 16\pi^2 G^2 k^{-4} \langle\delta T^0_0(k)\delta T^0_0(k)\rangle \,, \tag{2.24}$$

and

$$\langle|h(k)|^2\rangle \;=\; 16\pi^2 G^2 k^{-4} \langle\delta T^i_j(k)\delta T^i_j(k)\rangle \,,\; i \neq j \,. \tag{2.25}$$

In the above, the scalar spectrum and tensor perturbations relate to the expectation value of density perturbations, and the expectation value of transverse excitations of the stress-energy tensor. Both quantities can be computed from the thermal partition function Z of matter in the following way. First, we use the fact that thermal fluctuations of the stress tensor can be encoded into the following tensor

$$\begin{aligned} C^{\mu\sigma}_{\nu\,\lambda} &\equiv \big\langle T^{\mu}_{\nu} T^{\sigma}_{\lambda}\big\rangle - \big\langle T^{\mu}_{\nu}\big\rangle\big\langle T^{\sigma}_{\lambda}\big\rangle \\ &= 2\frac{g^{\mu\alpha}}{\sqrt{-g}}\frac{\partial}{\partial g^{\alpha\nu}}\Big(\frac{g^{\sigma\delta}}{\sqrt{-g}}\frac{\partial \ln Z}{\partial g^{\delta\lambda}}\Big) + 2\frac{g^{\sigma\alpha}}{\sqrt{-g}}\frac{\partial}{\partial g^{\alpha\lambda}}\Big(\frac{g^{\mu\delta}}{\sqrt{-g}}\frac{\partial \ln Z}{\partial g^{\delta\nu}}\Big)\,, \end{aligned} \tag{2.26}$$

where $\big\langle T^{\mu}_{\nu}\big\rangle$ can be computed from

$$\langle T^{\mu}_{\nu}\rangle \;=\; 2\frac{g^{\mu\lambda}}{\sqrt{-g}}\frac{\partial \ln Z}{\partial g^{\nu\lambda}}\,. \tag{2.27}$$

We then consider fluctuations in a three-dimensional box of characteristic size $R = k^{-1}$, and move to the position basis using

$$C^{\mu\sigma}_{\nu\,\lambda}(R) = k^{-3}\, C^{\mu\sigma}_{\nu\,\lambda}\,. \tag{2.28}$$

This allows us to use the following expression for the density perturbations

$$\langle \delta T^0_0(k)\delta T^0_0(k)\rangle_R = C^{00}{}_{00} = \frac{T^2}{R^6} C_V\,, \tag{2.29}$$

along with the following expression for the transverse stress-energy perturbations

$$\langle \delta T^i_j(k)\delta T^i_j(k)\rangle_R \;=\; C^{ij}{}_{ij} \;=\; \alpha\frac{T}{R^2}\frac{\partial \tilde{p}}{\partial R}\,. \tag{2.30}$$

These expressions depend on well-known properties of thermal systems. First, we can see that the density perturbations depend directly on the heat capacity

$$C_V \;=\; \left(\frac{\partial E}{\partial T}\right)_V\,, \tag{2.31}$$

of the system. Additionally, the transverse stress-energy perturbation depend directly on transverse pressure perturbations $\tilde{p}$, which can be computed from the free energy as follows

$$\tilde{p} \;=\; -\frac{1}{V}\frac{\partial \mathcal{F}}{\partial \ln R}\,. \tag{2.32}$$

Consequently, the only two ingredients we need to compute the power spectra are the heat capacity of the system C_V, and the transverse pressure perturbations $\tilde{p}$.

For closed strings at a finite temperature near the Hagedorn temperature T_H, it has been shown [12] that the heat capacity of closed strings in a three dimensional box with characteristic size R obeys

$$C_V \approx \frac{R^2/l_s^3}{T(1-T/T_H)}\,, \tag{2.33}$$

with a pressure satisfying

$$p \approx n_H T_H - \frac{2}{3}\frac{(1-T/T_H)}{l_s^3 R}\ln\left[\frac{l_s^3 T}{R^2(1-T/T_H)}\right]\,. \tag{2.34}$$

Using these two quantities, we can find the power spectrum $P_\Phi(k)$ of scalar fluctuations and the power spectrum $P_h(k)$ of tensor fluctuations, which are related to the two-point functions of Φ and h_{ij} via

$$P_\Phi(k) = k^3\langle|\Phi(k)|^2\rangle \quad , \quad P_h(k) = k^3\langle|h(k)|^2\rangle\,. \tag{2.35}$$

Using the thermal properties of strings, we find

$$P_\Phi = 16\pi^2 G^2 k^2 T^2 C_V = 16\pi^2 G_N^2 \frac{T}{l_s^3}\frac{1}{1-T/T_H}\,,$$

for the power spectrum of scalar perturbations, and

$$P_h \sim 16\pi^2 G_N^2 \frac{T}{l_s^3}(1-T/T_H)\ln^2\left[\frac{1}{l_s^2 k^2}(1-T/T_H)\right]\,,$$

for the power spectrum of tensor perturbations. In both cases, the power spectrum is independent of k, and is therefore scale-invariant. In String Gas Cosmology, the scale-invariance of scalar fluctuations is explained by the fact that the heat capacity has "holographic" scaling, meaning that for box of size R^3, the heat capacity scales as $C_V \sim R^2$ instead of $C_V \sim R^3$. Given this scaling, one obtains a perfect cancelation of the powers of k in Equation 8.1, leading to a scale-invariant spectrum of scalar perturbations. A similar phenomenon also happens with the transverse pressure fluctuations of the system. In this case, a pressure that scales as $p \sim 1/R$, which is characteristic of a "holographic" scaling of the free energy ($\mathcal{F} \sim R^2$), leads to a scale-invariant spectrum of tensor perturbations. Given that fluctuation of thermal strings exit the Hubble radius with a scale-invariant power spectrum, one obtains the expected behavior for the power spectrum measured in the CMB.

There is, however, a subtlety here that leads to a slight deviation from scale invariance for both scalar and tensor perturbations. Let us first notice here that the Expressions 8.1

and 2.2.3 depend on a temperature, which is defined here as the temperature $T \equiv T(k)$ at horizon crossing for a given mode k. As this temperature decreases in the Hagedorn phase, modes exiting the Hubble horizon at different times will contribute to a slight tilt in the spectrum (modes with larger values of k exit the Hubble radius at a later time, as pictured in Figure 2.8). For the power spectrum of scalar perturbations, the factor of $(1-T/T_H)$ in the denominator is responsible for giving the spectrum a slight red tilt. For the power spectrum of tensor perturbations, this factor of $(1-T/T_H)$ is in the numerator, and is responsible for giving the spectrum a slight blue tilt.

2.3 Matrix cosmology

Altough String Gas Cosmology is a promising scenario to explain the early universe, it still lacks a complete description. In the present section, we will introduce matrix theory, and review key concepts and results that inspired our study of matrix theory as a possible non-perturbative basis for String Gas Cosmology. First, we will review the IKKT and BFSS models, and how strings and interactions are emergent in these models. Then, we will see how emergent universe solutions can arise in the IKKT model from numerical simulations, how cosmological solutions can be found from adding a mass term to the theory, and how symmetry breaking can be probed in the IKKT model using the Gaussian expansion method. These concepts will later be used in Chapter 3, 4, 5, 6 and 7 to examine aspects of String Gas Cosmology in matrix theory.

2.3.1 Introduction to the IKKT and BFSS matrix models

The first matrix model that will be of interest in this thesis is the BFSS model [22], which is described by the action

$$S_{BFSS} = \frac{1}{2g^2} \int dt \mathrm{Tr} \left((D_t A^i)^2 - \frac{1}{2}[A^i, A^j]^2 + \bar{\psi}\Gamma^0 D_t \Psi - i\bar{\Psi}\Gamma^i [A_i, \Psi] \right) . \tag{2.36}$$

In this action, the nine A^i matrices decribe space, Ψ is an associated fermionic matrix superpartner to A^i, and t is an explicit time parameter. This model is often viewed as a description of M-theory in the Discrete Light Cone Quantisation (DLCQ) limit, which amounts to a Type IIA a limit of M-theory. Hence, this model describes $1+9$ space-time dimensions, as opposed to $1+10$ space-time dimensions as a description of M-theory would suggest.

The second matrix model we will be interested in is the IKKT matrix model [13], which is described by the action

$$S_{IKKT} = -\frac{1}{4g^2}\mathrm{Tr}[A^\mu, A^\nu]^2 - \frac{1}{2g^2}\mathrm{Tr}\Psi\Gamma^\mu[A_\mu, \Psi]\,. \tag{2.37}$$

As opposed to the BFSS model, there are no free parameters in this theory. In this action, ten A^μ matrices describe space-time, and Ψ is an associated fermionic matrix superpartner, and there is nothing else. This makes the IKKT model a prime candidate to describe time and space in a single consistent non-perturbative framework. Similarly to the BFSS model, the IKKT model also describes superstring theory, but this time in the Type IIB limit.

In matrix theory, string theory emerges without prior notion of strings, and irrespectively of a string coupling expansion. Rather, the fundamental strings, string interactions, and space-time are an emergent phenomenon. To see how this is the case, let us take a closer view at the IKKT model. In the IKKT model, strings arise as solitonic excitations which are found by minimizing the action. This is done by solving the equations of motion of the system, which in the absence of fermions read

$$[A_\mu, [A^\mu, A^\nu]] = 0\,. \tag{2.38}$$

In the large N limit, where N is the size of the matrix, these equations of motion admit solutions of the form

$$A_0 = \frac{T}{\sqrt{2\pi n}}q \equiv p_0\,, \tag{2.39}$$

$$A_1 = \frac{L}{\sqrt{2\pi n}}p \equiv p_1\,, \tag{2.40}$$

$$\text{other } A_\mu\text{'s} = 0\,, \tag{2.41}$$

where $[q,p]=i$. Given that the eigenvalue distribution of the operators q and p extend in the A_0 and A_1 direction, this solution can be viewed as a static D_1-string stretching in the A_1 direction. To see how interactions between multiple strings arise, one can also consider a block-wise solution of the IKKT model of the form

$$A_0 = \begin{pmatrix} \frac{T}{\sqrt{2\pi n}} q & 0 \\ 0 & \frac{T}{\sqrt{2\pi n}} q \end{pmatrix} \equiv p_0\,, \tag{2.42}$$

$$A_1 = \begin{pmatrix} \frac{L}{\sqrt{2\pi n}} p & 0 \\ 0 & -\frac{L}{\sqrt{2\pi n}} p \end{pmatrix} \equiv p_1\,, \tag{2.43}$$

$$A_2 = \begin{pmatrix} \frac{d}{2} & 0 \\ 0 & -\frac{d}{2} \end{pmatrix} \equiv p_2, \tag{2.44}$$

$$\text{other } A_\mu\text{'s} = 0\,, \tag{2.45}$$

which describe two anti-parallel strings extending along the A_1 direction and a distance d from another. We then fluctuate around this background by considering the following ansatz

$$A_\mu = p_\mu + a_\mu, \tag{2.46}$$

$$\psi = \chi + \varphi, \tag{2.47}$$

where a_μ and φ contain off-diagonal contributions in the block diagonal structure showcased in Equation 2.45. In the string theory language, adding these off-diagonal elements can be viewed as adding open string degrees of free stretching between the two D_1-branes. Integrating out these open string degrees of freedom then allows us to get the effective potential for the interaction between the two membranes. To do this, we follow the procedure in [13] and expand the IKKT action to obtain

$$S_{IKKT} = S_0 + S_2 + ... \tag{2.48}$$

where S_0 is a zeroth order piece and S_2 is a piece quadratic in a_μ and φ. Since the present system is invariant under the gauge symmetry

$$\delta A_\mu = i[A_\mu, \alpha] \quad , \quad \delta\psi = i[\psi, \alpha]\,, \tag{2.49}$$

we will fix this symmetry in order to simplify computations. This will be done by adding ghost in order to impose the condition $[p_\mu, a^\mu] = 0$. Adding the contribution from ghosts, we obtain

$$\tilde{S}_2 = Tr(\frac{1}{2} a_\mu (P_\lambda^2 \delta_{\mu\nu} - 2iF_{\mu\nu}) a_\nu - \frac{1}{2}\bar{\varphi}\Gamma^\mu P_\mu \varphi + bP_\lambda^2 c), \tag{2.50}$$

where b and c are the ghost matrices, and P_μ and $F_{\mu\nu}$ are adjoint operators defined by

$$[p_\mu, X] = P_\mu X \,, \tag{2.51}$$

$$[f_{\mu\nu}, X] = F_{\mu\nu} X \,, \tag{2.52}$$

$$f_{\mu\nu} = i[p_\mu, p_\nu]. \tag{2.53}$$

To get the interaction potential, we then integrate out a_μ and φ to obtain the following effective action

$$\begin{aligned} W &= -\log \int da d\varphi dc db \, e^{-\tilde{S}_2} && (2.54)\\ &= \frac{1}{2} Tr \log(P_\lambda^2 \delta_{\mu\nu} - 2iF_{\mu\nu}) - \frac{1}{4} Tr \log((P_\lambda^2 + \frac{i}{2} F_{\mu\nu}\Gamma^{\mu\nu})(\frac{1+\Gamma_{11}}{2})) && (2.55)\\ &\quad - Tr \log(P_\lambda^2) + i\theta && (2.56)\\ &= -8n(\frac{TL}{2\pi n})^3 \frac{1}{d^6} + O(\frac{1}{d^8}) \,. && (2.57) \end{aligned}$$

Here, the projector $\frac{1+\Gamma_{11}}{2}$ imposes the Weyl condition on the fermions. This is a requirement for the system to have supersymmetry. As we can see, we then obtain the correct interaction potential between two long anti-parallel strings in ten dimensions (see [26] for a detailed comparison). Here, let us highlight that choosing that the strings are anti-parallel (minus sign in the second block of A_1 in Equation 2.45) was crucial to obtain a non-zero interacting potential. When choosing to work with parallel strings, the interaction potential vanishes as a result of supersymmetry.

Similar to the D_1-string, it can also be shown that other brane solutions can arise in the IKKT model. Moreover, their interaction potential can be computed in a similar way [26,27]. A similar computation can also be made in the BFSS matrix model, where one computes the interaction potential between D_0-brane degrees of freedom [28, 29].

2.3.2 Emergent cosmological solutions in matrix theory

In addition to string and membranes, the IKKT model admits emergent universe solutions. These solutions have first been observed in numerical simulations using Monte-Carlo methods [15], and remain a topic of study today in simulations using Complex Langevin methods (see [30] for a review of recent progress). We will here summarize features observed as a result of these simulations.

In numerical simulations that show emergent cosmological solutions, one usually takes the temporal matrix A^0 as being diagonal, where the temporal eigenvalues are chosen to be in increasing order. This choice is merely a choice of basis, which we are allowed to do in matrix theory. In the case where A^0, the spatial matrices A^i acquire a band-diagonal structure of the form shown in Figure 2.9.

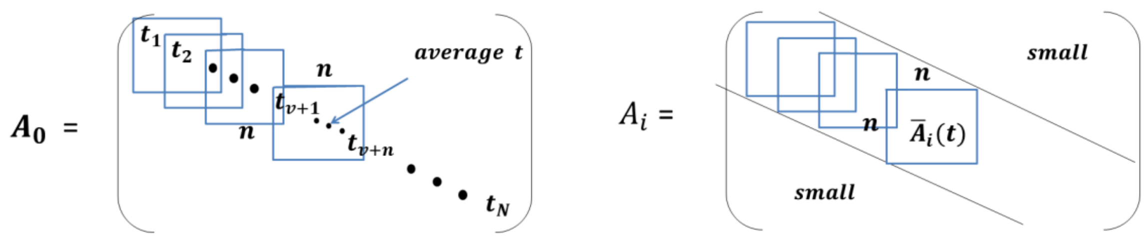

Figure 2.9: Band diagonal structure acquired by the matrices A^0 and A^i as a result of the numerical simulations. The original figure can be found in [16].

The reason why the matrices acquire such a structure is a current topic of study. A possibility is that the simulations describe group-like solutions similar to the one we will explore a bit later in Section 2.3.3. In these group-like solutions, some of the matrices usually acquire off-diagonal elements associated with the form of raising and lowering operators in certain bases, explaining the band-diagonal structure. Another possibility is that off-diagonal elements fall off as a result of the Riemann-Lebesque Lemma. This possibility will be explored later in Chapter 5.

Given the matrices described in Figure 2.9, one can define time evolution as follows. To begin, we consider a submatrix

$$\bar{A}_i^{(ab)}(m) = \langle t_{m+a}|A_i|t_{m+b}\rangle \,, \tag{2.58}$$

of size $n \times n$ located at a position m along the diagonal. Here, n is taken to be the characteristic size of the band-diagonal, beyond which the values of the off-diagonal elements become negligible. Then, we take time to be defined as the average of the temporal eigenvalues in this submatrix

$$t(m) = \frac{1}{n}\sum_{a=1}^{n} t_{m+a}\,. \tag{2.59}$$

Since the temporal eigenvalues are chosen to be in increasing order, it is then possible to probe the time evolution of the system by moving the location m of the submatrix along the diagonal. We can then use various trace operators to describe the properties of the system at a given time. For example, one can probe the "size" of space at a given time using the extent of space parameter

$$R(m)^2 = \frac{1}{n}\mathrm{Tr}\bar{A}_i(m)^2\,. \tag{2.60}$$

This operator is the root-mean-square of the location of the spatial eigenvalues in a given time interval n. For this reason, $R(t)$ acts as a scale factor for the extent of space at a time $t(m)$. In addition to the extent of space parameter $R(t)$, one can probe the extent of space in different directions by considering the moment of intertia tensor

$$T_{ij}(m) = \frac{1}{n}\mathrm{Tr}\left[\bar{A}_i(m)\bar{A}_j(m)\right]\,. \tag{2.61}$$

In this case, the eigenvalues of $T_{ij}(m)$ are related to the characteristic size of space in these different directions.

In numerical simulations, one obtains the following results in solutions related to an emergent three-dimensional space-time. At early times, $R(m)$ and the eigenvalues of $T_{ij}(m)$ are approximately constant, with the eigenvalues of $T_{ij}(m)$ being approximately the same size. Then, after a critical time t_c, the extent of space $R(m)$ starts increasing, and three of the nine eigenvalues of $T_{ij}(m)$ become large. This suggests that at the critical time t_c, the initial SO(9) symmetry of the system is broken, and three dimensions become large while six stay small.

So far, the conditions that lead to the emergent solutions in numerical simulations are still a topic of study. A possibility is that the fermions play a role in the symmetry breaking of the system, which leads to three emergent dimensions. This possibility was investigated in [19], which we will review in Section 2.3.4. In margin of this study, there are also some known conditions that lead to cosmological solutions in the IKKT model. This is the case, for example, when including a mass term in the theory (we will review this in Section 2.3.3).

Perhaps the contributions from the fermions can be related to the present solutions in a similar way.

Despite the fact that the present solutions remain to be fully understood, the emergent features of these solutions are, of course, very appealing from the point of view of String Gas Cosmology. Assuming the emergence of the three large dimensions can be understood in matrix theory, one may hope that matrix theory can offer a framework that can explain early universe dynamics in the String Gas Cosmology scenario.

2.3.3 Cosmological solutions from a mass term in matrix theory

In the present section, we will review some results found by imposing a cutoff on the extent of time in the theory, which can be done by adding a Lagrange multiplier that plays to role of a mass term (see [17] for the original work). We will see that, in this case, one obtains solutions related to a bouncing or emergent universe.

To see how a mass term leads to consmological solutions, let us consider the action

$$\tilde{S} = \mathrm{Tr}\left(-\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{\tilde{\lambda}}{2}(A_0^2 - \kappa L^2) - \frac{\lambda}{2}(A_i^2 - L^2)\right) \ , \tag{2.62}$$

which corresponds to the IKKT model where we have imposed the constraints

$$\frac{1}{N}\mathrm{tr}(A_0^2) = \kappa L^2 \ , \tag{2.63}$$

$$\frac{1}{N}\mathrm{tr}(A_i^2) = L^2 \ , \ , \tag{2.64}$$

using Lagrange multipliers λ and $\tilde{\lambda}$. In general, the Lagrange multipliers will break the $SO(1,9)$ symmetry of the system. However, we will later see cases where $\lambda = \tilde{\lambda}$ and the ten-dimensional Lorentz symmetry of the system is restored. This model has the equations of motion

$$-[A_0, [A_0, A_i]] + [A_j, [A_j, A_i]] - \lambda A_i = 0 \ , \tag{2.65}$$

$$[A_j, [A_j, A_0]] - \tilde{\lambda} A_0 = 0 \ , \tag{2.66}$$

which can be found by extremizing equation 2.62 with respect to A^μ. To solve these equations, we will restrain ourselves to the ansatz

$$\begin{aligned} A'_0 &= A_0 \otimes 1_k \ , \\ A'_i &= A_1 \otimes \mathrm{diag}(r_i^{(1)}, r_i^{(2)}, \cdots, r_i^{(k)}) \ , \end{aligned} \tag{2.67}$$

$$\text{where} \quad r_i^{(m)2} = 1 \ \ (m = 1, \cdots, k) \ , \tag{2.68}$$

which amounts to a direct sum of multiple solutions involving A_0 and A_1. Under this ansatz, the equations of motion reduce to the commutation relations

$$[A_0, A_1] = iE \ , \quad [A_0, E] = i\lambda A_1 \ , \quad [A_1, E] = i\tilde{\lambda} A_0 \ , \tag{2.69}$$

where E is an auxiliary matrix defined by $[A_0, A_1] = iE$. These commutation relations are related to different algebras depending on the sign of λ and $\tilde{\lambda}$. However, in the Lorentz invariant case $\lambda = \tilde{\lambda}$, the solutions are always related to the SU(1,1) algebra

$$[T_0, T_1] = iT_2 \ , \quad [T_2, T_0] = iT_1 \ , \quad [T_1, T_2] = -iT_0 \ . \tag{2.70}$$

The most interesting of these cases arise when $\lambda, \tilde{\lambda} < 0$, and the matrices A^0 , A^1, and E can be associated with the SU(1,1) algebra via

$$\begin{aligned} &A_0 = aT_0 \ , \quad A_1 = bT_1 \ , \quad E = cT_2 \ , \\ &\lambda = -a^2 \ , \quad \tilde{\lambda} = -b^2 \ , \quad ab = c \ . \end{aligned} \tag{2.71}$$

In this case, the temporal matrix A_0 coincides with the generator T_0 of the SU(1,1) group. Since for most representations of the SU(1,1) group T_0 can be taken to be diagonal, we can then find explicit descriptions of the matrices where the temporal matrix A_0 is diagonal, and the spatial matrices A^i have a band-diagonal structure.

To see how this is the case, let us consider the Primary Unitary Series Representation (PUSR) of the SU(1,1) group. In this case, the submatrices running describing the time evolution of the system, as defined in Section 2.3.2, take the form

$$\bar{A}_0(n) = a \begin{pmatrix} n-1+\epsilon & 0 & 0 \\ 0 & n+\epsilon & 0 \\ 0 & 0 & n+1+\epsilon \end{pmatrix} , \tag{2.72}$$

$$\bar{A}_1(n) = \frac{ib}{2} \begin{pmatrix} 0 & n+i\rho-\frac{1}{2}+\epsilon & 0 \\ -n+i\rho+\frac{1}{2}-\epsilon & 0 & n+i\rho+\frac{1}{2}+\epsilon \\ 0 & -n+i\rho-\frac{1}{2}-\epsilon & 0 \end{pmatrix} , \tag{2.73}$$

where $\epsilon = 0$ or $1/2$, ρ is a non-negative number, and n is an integer related to the position of the submatrices along the block-diagonal. In this case, we extent of space parameter, as defined in Equation 2.60, takes the form

$$R(n) = \sqrt{\frac{b^2}{3}\left(n^2+\rho^2+\frac{1}{4}\right)} . \tag{2.74}$$

To see how time evolution emerges in the system we can then define a time $t = an$, and take the continuum limit $a \to 0$ while keeping a parameter $\alpha = \frac{b}{a}$ fixed. In this case, we obtain

$$R(t) = \sqrt{\frac{\alpha^2}{3}(t^2+t_0^2)} , \tag{2.75}$$

where $t_0 = a\rho$ and ρ is tuned in a way that t_0 is fixed. As shown in Figure 2.10, this extent of space parameter describes a bouncing universe solution where $R(t_0) = \sqrt{\frac{\alpha^2 t_0^2}{3}}$ is the lowest value of the extent of space parameter at the bouncing point. Another similar solution can be found by considering the discrete series representation of the SU(1,1). In this case, the submatrices running describing the time evolution of the system take the form

$$\bar{A}_1(n) = \frac{ib}{2} \begin{pmatrix} 0 & \sqrt{(n+\tau)(n-\tau-1)} & 0 \\ -\sqrt{(n+\tau)(n-\tau-1)} & 0 & \sqrt{(n-\tau)(n+\tau+1)} \\ 0 & -\sqrt{(n-\tau)(n+\tau+1)} & 0 \end{pmatrix} , \tag{2.76}$$

where $\tau = -1, -2, -3, ...$ for $\epsilon = 0$, and $\tau = -1/2, -3/2, -5/2, ...$ for $\epsilon = 1/2$. In this case, the extent of space parameter reads

$$R(n) = \sqrt{\frac{b^2}{3}(n^2-\tau(\tau+1))} , \tag{2.77}$$

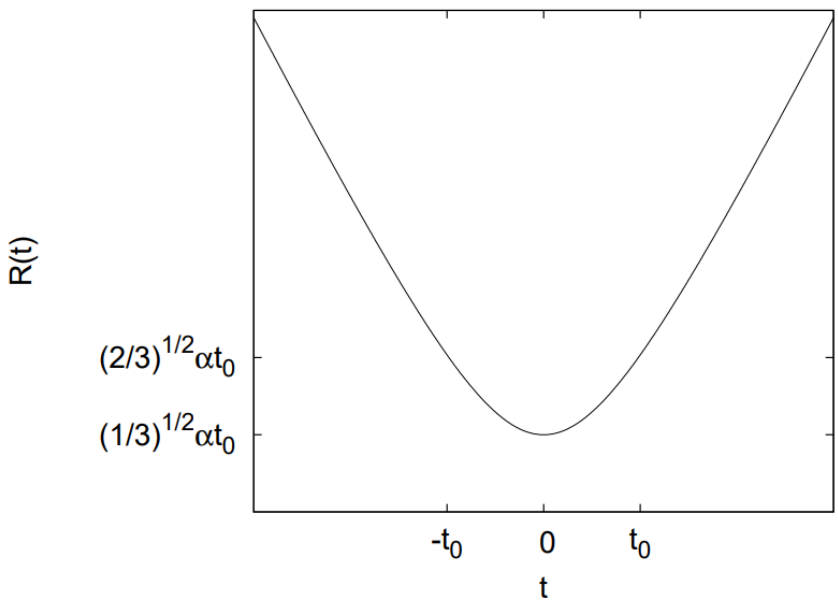

Figure 2.10: Evolution of the extent of space parameter $R(t)$ for solutions related to the Primary Unitary Series Representation of SU(1,1). The original figure can be found in [17].

and the continuum limit leads us to

$$R(t) = \sqrt{\frac{\alpha^2}{3}(t^2 - t_0^2)} \ , \tag{2.78}$$

where $t_0 = a\tau(\tau+1)$ is held fixed. In this case, the extent of space parameter $R(t)$ describes a universe that emerges at t_0, and expands indefinitely (see Figure 2.11).

2.3.4 Gaussian Expansion Method

In Section 2.3.2, we saw that three dimensions become emergent after a critical time t_c in the simulation. Following these results, the role of fermions in this symmetry breaking was investigated using the Gaussian Expansion Method [18]. Roughly speaking, the Gaussian Expansion Method is a mathematical trick that consists of adding and subtracting a Gaussian term in the theory in order to facilitate perturbative computations. This trick is particularly useful in theories that do not have a quadratic term, such as the IKKT model, and has proven useful in probing the strong coupling regime of theories. In the present section, we will introduce the Gaussian Expansion Method by using the simple example of a ϕ^4 theory

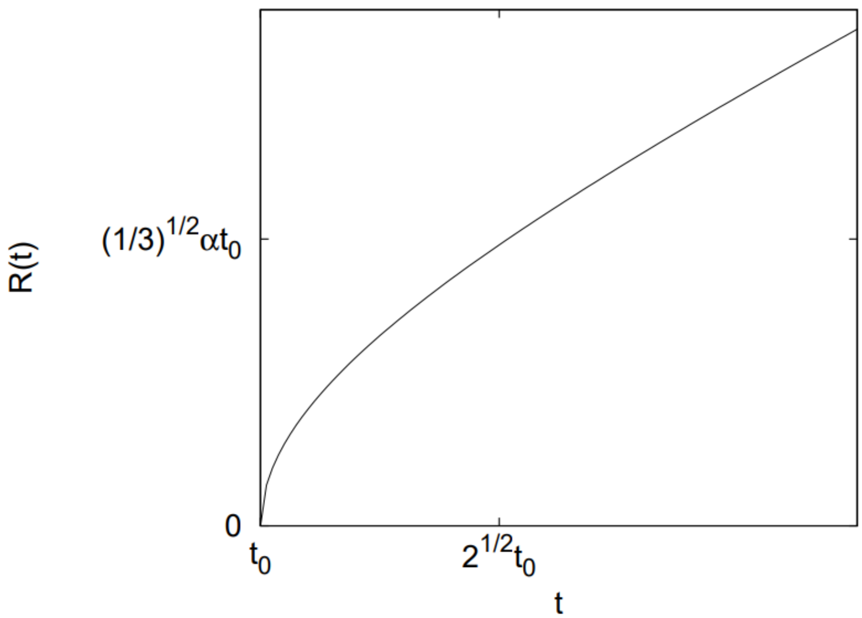

Figure 2.11: Evolution of the extent of space parameter $R(t)$ for solutions related to the Discrete Series Representation of SU(1,1). The original figure can be found in [17].

in $0 + 0$ dimensions. We will then review how this method can be used to probe symmetry breaking in the IKKT model, following the approach in [19].

Simple example : ϕ^4 theory in 0 + 0 dimensions

One may think that by "adding and subtracting" a Gaussian term in the theory, we are not doing much as we are simply adding zero. However, this mathematical trick has shown to be useful to probe the strong coupling limit of theories, while respecting the fact that the added terms should not change the nature of the theory.

To see how this is the case, let us follow the example presented by Kabat and Lifschiytz [18] and consider the simple case of a ϕ^4 theory, which is exactly solvable in $0 + 0$ dimensions. Here, we will consider a theory with an action of the form

$$S = \frac{1}{g^2}\left(\frac{1}{2}\phi^2 + \frac{1}{4}\phi^4\right) , \tag{2.79}$$

where ϕ are simple scalars. Given the simplicity of this theory, the partition function, and therefore the free energy, can be computed exactly. One obtains

$$Z = e^{-BF} = \int_{-\infty}^{\infty} d\phi\, e^{-S} = \frac{1}{\sqrt{2}} e^{1/8g^2} K_{1/4}\left(\frac{1}{8g^2}\right), \tag{2.80}$$

where $K_\nu(x)$ is the Kelvin function. Given that we have an exact expression, we can make a suitable expansion to see what the theory would look like in the weak coupling limit and the string coupling limit. In the $g^2 \ll 1$ limit, one obtains

$$\beta F = -\frac{1}{2}\log\left(2\pi g^2\right) + \frac{3}{4}g^2 - 3g^4 + \frac{99}{4}g^6 + \mathcal{O}(g^8), \tag{2.81}$$

which is the result one would expect using well-known perturbation theory techniques. In the opposite regime $g^2 \gg 1$, one can also expand the exact solution to find

$$\beta F = -\frac{1}{2}\log g - \log\frac{\pi}{\Gamma(3/4)} + \mathcal{O}(1/g). \tag{2.82}$$

However, in this case, perturbation theory offers very little hope to obtain this expression. This is where the perspective of new perturbative tools that can give us a window on the string coupling regime, such as the Gaussian Expansion Method, becomes appealing.

To showcase the power of the Gaussian Expansion Method, let us consider the contribution of a Gaussian piece

$$S_0 = \frac{1}{2\sigma^2}\phi^2, \tag{2.83}$$

with an associated partition function

$$Z_0 = e^{-\beta F_0} = \int_{-\infty}^{\infty} d\phi\, e^{-S_0}. \tag{2.84}$$

Here, the width of the Gaussian σ^2 is for the moment left arbitrary. In the ϕ^4 theory, this Gaussian piece can be added and subtracted to yield

$$Z = \int d\phi\, e^{-S_0} e^{-(S-S_0)}. \tag{2.85}$$

One can then expand the free energy of the system in terms of expectation values

$$\langle\, .\, \rangle_0 = \frac{1}{Z_0}\int_{-\infty}^{\infty} d\phi\, .\, e^{-S_0} \tag{2.86}$$

related to S_0 to obtain

$$\beta F = \beta F_0 \; - \langle e^{-(S-S_0)} - 1\rangle_{C,0} \tag{2.87}$$

$$= \beta F_0 + \langle S - S_0\rangle_0 - \frac{1}{2}\langle (S-S_0)^2\rangle_{C,0} + \cdots . \tag{2.88}$$

Here, the subscript $C, 0$ denotes expectation values related to connected diagrams in the expansion. Evaluating each piece, we obtain

$$\beta F_0 = -\frac{1}{2}\log 2\pi\sigma^2 \tag{2.89}$$

$$\langle S - S_0\rangle_0 = \frac{1}{2}\left(\frac{\sigma^2}{g^2} - 1\right) + \frac{3\sigma^4}{4g^2} \tag{2.90}$$

$$-\frac{1}{2}\langle (S - S_0)^2\rangle_{C,0} = -\frac{1}{4}\left(\frac{\sigma^2}{g^2} - 1\right)^2 - \frac{3\sigma^4}{2g^2}\left(\frac{\sigma^2}{g^2} - 1\right) - \frac{3\sigma^8}{g^4}\,. \tag{2.91}$$

As we can see, these contributions depend on the width σ^2 of the Gaussian. Since we have merely added zero, our answer should not depend on this parameter, and we must find a way to remove it. To achieve this task, one can use the Schwinger-Dyson equations of the theory, which in this case equate to

$$\langle \frac{1}{g^2}\left(\phi^2 + \phi^4\right)\rangle_0 = 1\,. \tag{2.92}$$

Evaluating this expression, we obtain a gap equation

$$\frac{1}{\sigma^2} = \frac{1}{g^2} + \frac{3\sigma^2}{g^2}\,, \tag{2.93}$$

which relates the Gaussian width σ^2 to the coupling g^2 of the theory. Solving this gap equation at weak and strong coupling then allows us to obtain explicit expressions for the free energy that do not depend on σ^2. At weak coupling the gap equation implies $\sigma^2 = g^2 + \mathcal{O}(g^4)$, and we obtain

$$\beta F_0 = -\frac{1}{2}\log 2\pi g^2 + \mathcal{O}(g^2)\,, \tag{2.94}$$

which matches Equation 2.81. At strong coupling, where $\sigma^2 = \frac{g}{\sqrt{3}} + \mathcal{O}(1)$, on also obtain the correct expression

$$\beta F_0 = -\frac{1}{2}\log g - \frac{1}{2}\log\frac{2\pi}{\sqrt{3}} + \mathcal{O}(1/g)\,, \tag{2.95}$$

which matches with Equation 2.82.

All-in-all, we find that the Gaussian Expansions Method gives the correct behavior at both strong and weak coupling. Note that this method should be, moreover, independent of the Gaussian term added. Different Gaussian terms will give us different gap equations. However, the solution of these gap equations will lead to the same result for the free energy.

Symmetry breaking in the IKKT model

Following the success of the Gaussian Expansion Method in probing the strong coupling regime of theories, this method was used in the Euclidean IKKT model to probe for symmetry breaking in matrix theory. This was done by using the gap equation, and studying how the gap parameters in this gap equation can be indicative of symmetry breaking. In the Gaussian Expansion Method, the gap equations can be found by the Schwinger-Dyson equations, or equivalently by minimizing the free energy of the system. Consequently, finding symmetry-breaking solutions to the gap equations is equivalent to finding a symmetry that is energetically favored in the system. In what follows, we summarise the results found in [19]. More precisely, we will showcase the possible role of fermions in symmetry breaking.

Let us first start with the bosonic IKKT model, which is described by the action

$$S = -\frac{1}{4} N \,\mathrm{Tr}[X_\mu, X_\nu]^2 \ . \tag{2.96}$$

To probe symmetry breaking, we will add and subtract the following symmetry-breaking Gaussian piece

$$S_0 = \sum_{\mu=1}^{D} \frac{N}{v_\mu} \mathrm{Tr}\,(X_\mu X_\mu) \quad , \quad v_\mu > 0 \ , \tag{2.97}$$

where the v_μ's are arbitrary parameters. Just like we did for the ϕ^4 theory case, we can then evaluate the free energy, which we will define as

$$Z = e^{-F} = \int dX e^{-(S-S_0)} e^{-S_0} \ , \tag{2.98}$$

from the partition function Z. To first order, the free energy then reads

$$F = F_0 + \langle S - S_0 \rangle_0 + \ldots \ , \tag{2.99}$$

and evaluating each piece gives us

$$F_0 = -\frac{1}{2}(N^2-1) \sum_{\mu=1}^{D} \ln v_\mu \ , \tag{2.100}$$

$$F_1 = \langle S - S_0 \rangle_0 \tag{2.101}$$

$$= \frac{1}{8}(N^2-1) \left(\sum_{\mu \neq \nu} v_\mu v_\nu + 4D \right) \tag{2.102}$$

Minimizing the free energy then gives us the gap equation

$$0 = \frac{1}{N^2-1}\frac{\partial}{\partial v_\mu}(F_0+F_1) = -\frac{1}{2v_\mu} + \frac{1}{4}\sum_{\nu\neq\mu} v_\nu \ . \tag{2.103}$$

which has the isotropic solution

$$v_1 = v_2 = \cdots = v_D = \sqrt{\frac{2}{D-1}} \ . \tag{2.104}$$

Given these isotropic solutions, we conclude that there is no symmetry breaking at first order in the Eucledian IKKT. This picture changes, however, if one adds fermions in the picture. To see this, let us consider the supersymmetric version of the IKKT model, which has the action

$$S = -\frac{1}{4}N\,\mathrm{Tr}[X_\mu, X_\nu]^2 - \frac{i}{2}N\,\mathrm{Tr}\Big(\Psi_\alpha(\widetilde{\Gamma}_\mu)_{\alpha\beta}[A_\mu,\Psi_\beta]\Big) \,. \tag{2.105}$$

In this case, we will consider the Gaussian term

$$S_0 = \sum_{\mu=1}^{D}\frac{N}{v_\mu}\mathrm{Tr}\,(X_\mu X_\mu) + \frac{N}{2}\sum_{a=1}^{N^2-1}\Phi^a_\alpha\,\mathcal{A}_{\alpha\beta}\,\Phi^a_\beta\,, \tag{2.106}$$

where $\mathcal{A}_{\alpha\beta}$ is an anti-symmatric matrix. In the same way as for the bosonic case, we can compute the gap equations by minimizing the free energy of the system at first order. In this case, we obtain

$$0 = -\frac{1}{2v_\mu} + \frac{1}{4}\sum_{\nu\neq\mu} v_\nu - \frac{1}{4}\rho_\mu \ , \tag{2.107}$$

$$0 = -\frac{1}{2}\,\mathrm{Tr}(\mathcal{A}^{-1}\mathcal{B}_{\mu\nu\lambda}) + \frac{1}{8}\sum_{\mu} v_\mu \mathrm{Tr}\Big\{(\mathcal{A}^{-1}\widetilde{\Gamma}_\mu)^2\mathcal{A}^{-1}\mathcal{B}_{\mu\nu\lambda}\Big\} \ . \tag{2.108}$$

In the above, ρ_μ is defined as

$$\rho_\mu = \frac{1}{4}\mathrm{Tr}\left(\mathcal{A}^{-1}\tilde{\Gamma}_\mu\right)^2 \ , \tag{2.109}$$

and we have chosen $\mathcal{A}$ to be parametrised as

$$\mathcal{A} = \frac{i}{3!}\sum_{\mu\nu\lambda} w_{\mu\nu\lambda}\,\mathcal{B}_{\mu\nu\lambda}\ ; \qquad \mathcal{B}_{\mu\nu\lambda} = \mathcal{C}\,\Gamma_\mu\,\Gamma^\dagger_\nu\,\Gamma_\lambda \ , \tag{2.110}$$

where $\omega_{\mu\nu\lambda}$ is a complex totally anti-symmetric tensor. Because of the contribution from ρ_μ in Equation 2.107, solutions of the form $v_1 = v_2 = ... = v_D$ are no longer possible. For this reason, one expects the preferred symmetry to be something other than SO(10), indicating symmetry breaking. In [19], possible preferred symmetries of the system have been investigated by choosing values of v_μ and $w_{\mu\nu\lambda}$ that preserve these symmetries, and computing the free energy related to these different symmetries. It was found, to third order in perturbation theory, that $SO(4)$ plus some discrete subgroup of $SO(10)$ is the configuration of the system that gives the minimum free energy. These results support the conjectured scenario that a four-dimensional space-time can be generated dynamically in the IKKT model. A similar symmetry breaking will be observed in Chapter 7 for the BFSS model. In this case, we will also find a similar symmetry breaking, with the preferred symmetry of the system being left to future work.

Part II

Experimental predictions of matrix cosmology

Chapter 3

Emergent Cosmology from Matrix Theory

Suddhasattwa Brahma, Robert Brandenberger, and Samuel Laliberte
Department of Physics, McGill University, Montréal, QC, H3A 2T8, Canada

Addendum for thesis

In Section 2.2, we saw how thermal fluctuations of strings can source a scale-invariant spectrum of cosmological perturbations in String Gas Cosmology. Assuming matrix cosmology is a possible description of String Gas Cosmology, one should hope to reproduce a similar scale-invariant spectrum in this context. In the present chapter, we compute the spectrum of cosmological perturbations sourced by thermal fluctuations of the BFSS matrix model at finite temperature. Leading to this computation, we first explain how a cosmological background similar to the one reviewed in Section 2.3.2 can arise in a BFSS context, namely by treating the zero mode as encoding information about the background evolution of the system. Treating the non-zero modes as encoding information about thermal fluctuations, we then proceed with the computation of the scalar and tensor power spectrum in the same way as for String Gas Cosmology. We find a spectrum of scalar and tensor perturbations that is scale-invariant, in agreement with results from string gas cosmology and CMB observations on large scales.

Erratum

There are a few typos in this article. First, there are missing parentheses in equations 3.12 and 3.14. These equations should read

$$x_i(t)^2 \equiv \left\langle \frac{1}{n} \mathrm{Tr}((\bar{A}_i)(t))^2 \right\rangle , \tag{3.1}$$

and

$$T_{ij} \equiv \left\langle \frac{1}{n} \mathrm{Tr}((\bar{A}_i)(t)(\bar{A}_j)(t)) \right\rangle \tag{3.2}$$

respectively. Second, there is a missing plus sign in equation 3.49. The free energy should read

$$C_V = \frac{3N^2}{4}\chi_2 + \frac{3N^4}{8}\left(\frac{d-1}{12} - \frac{p}{8}\right)\left(\chi_2 - \frac{1}{d}\chi_2 - \frac{4}{N^2}\right)\chi_1 T^{-3/2} \tag{3.3}$$

instead of

$$C_V = \frac{3N^2}{4}\chi_2 \frac{3N^4}{8}\left(\frac{d-1}{12} - \frac{p}{8}\right)\left(\chi_2 - \frac{1}{d}\chi_2 - \frac{4}{N^2}\right)\chi_1 T^{-3/2} . \tag{3.4}$$

This sign was properly taken into account in our computations. Concerning the quantity above, we will highlight that the second term is responsible for the scale invariant spectrum of cosmological perturbations. The first term, while being the leading term in the perturbative expansion, is subdominant on large scales. Hence, its contribution to the large-scale spectrum can be neglected.

Abstract

Matrix theory is a proposed non-perturbative definition of superstring theory in which space is emergent. We begin a study of cosmology in the context of matrix theory. Specifically, we show that matrix theory can lead to an emergent non-singular cosmology which, at late times, can be described by an expanding phase of Standard Big Bang cosmology. The horizon problem of Standard Big Bang cosmology is automatically solved. We show that thermal fluctuations in the emergent phase source an approximately scale-invariant spectrum of cosmological perturbations and a scale-invariant spectrum of gravitational waves. Hence,

it appears that matrix theory can lead to a successful scenario for the origin of perturbations responsible for the currently observed structure in the universe while providing a consistent UV-complete description.

3.1 Introduction

The inflationary scenario [1] is not the only early universe paradigm consistent with current observations. As already pointed out a decade before the development of the inflationary scenario [2], what is required in order to explain the origin of acoustic oscillations in the angular power spectrum of the cosmic microwave background (CMB) and the *Baryon Acoustic Oscillations* in the matter power spectrum is an early phase in the evolution of the universe which generates an approximately scale-invariant spectrum of nearly adiabatic and nearly Gaussian curvature fluctuations. Inflation is one way to obtain such a spectrum [3], but there are others (see e.g. [4] for a comparative review of several scenarios).

Specifically, in [5] an *emergent scenario* was proposed in which the universe originates in a quasi-static phase ("Hagedorn phase") of a gas of strings at a temperature close to the Hagedorn temperature [6], the limiting temperature of a gas of closed strings [1]. Via a phase transition (which in String Gas Cosmology is determined by the decay of string winding modes, or more generally speaking, by the spontaneous breaking of the T-dual symmetry of the state), this emergent phase connects to the radiation phase of Standard Big Bang (SBB) cosmology. As shown in [7, 8] (see [9] for a review), thermal fluctuations of the gas of strings lead to an almost scale-invariant spectrum of cosmological fluctuations with a slight red tilt [7], and of gravitational waves with a slight blue tilt [8]. Additionally, the String Gas scenario yields a non-singular cosmology. Note that, unlike in inflationary cosmology where the cosmological fluctuations (which are observed today) are generated in the early universe in their quantum vacuum state, in String Gas Cosmology the initial state is assumed to be a thermal one and the fluctuations are hence of thermal origin [2]. However, in [5] no dynamics for the Hagedorn phase were provided.

[1] In the following, we will call this scenario *String Gas Cosmology*.

[2] Note that there is a variant of standard inflation, namely *warm inflation* [10] in which the fluctuations also emerge thermally.

Recently, a very interesting proposal appeared [11] postulating that the early phase is a topological phase, and demonstrating that the predictions of String Gas Cosmology for the spectrum of curvature fluctuations can be recovered. Here, we propose an alternative view of the emergent phase, a model based on *matrix theory*. In analogy to what is assumed in String Gas Cosmology, the fluctuations are of thermal origin, and lead to power spectra of scalar and tensor modes which are consistent with current observations.

Matrix theory is the suggestion that certain large N matrix models can provide non-perturbative definitions of superstring theory. There are two main proposals for matrix theory, the BFSS model [12] and the IKKT proposal [13] (see [14] for a recent review of these and other matrix models). In the BFSS model, the matrices are functions of time, and space emerges from the properties of the matrices which will be discussed in the following section [3]. In the IKKT proposal, time is emergent as well. The scenarios are related in the sense that the high temperature limit of the BFSS model yields the IKKT action (compactification on a thermal circle).

Our starting point will be the BFSS model, conjectured to be the non-perturbative proposal for M-theory. Making use of the equivalence between the high temperature limit of the BFSS model and the IKKT action, and starting in a thermal state, we will use the results of detailed numerical studies of the IKKT model [17] to show that a background develops in which there is a separation between three spatial dimensions which become large, and six which remain compact, similar to what was argued to happen in String Gas Cosmology [5]. In this background, we then compute the thermal fluctuations of the energy-momentum tensor and determine the resulting spectra of curvature fluctuations and of gravitational waves.

We will consider a $D = d + 1$ dimensional space-time, where d is the number of spatial dimensions (which is $d = 9$ for superstring theory and $d = 3$ after the phase transition in the IKKT model). Roman indices will be used to refer to the spatial directions. When discussing late time cosmology, we denote the cosmological scale factor by $a(t)$, where t is physical time, and use comoving spatial coordinates x. The Hubble expansion rate is given

[3]See also the $c = 1$ matrix model of [15] which yields a non-critical string theory with one spatial dimension.

by $H \equiv \dot{a}/a$, an overdot denoting the derivative with respect to time. The inverse of H is the Hubble radius which plays a key role in the evolution of cosmological perturbations. As usual, the commutator of two matrices A and B is denoted by $[A, B]$.

3.2 Background

Our starting point will be the BFSS matrix model [12]. The basic objects in this model are $d = 9$ bosonic $N \times N$ Hermitian matrices $X^i_{a,b}$ and their sixteen fermionic superpartners $\theta_{a,b}$ which transform as spinors under the SO(9) group of spatial rotations. There is a $U(N)$ gauge symmetry, and A is the associated gauge field, another $N \times N$ matrix. The Lagrangian is given by

$$L = \frac{1}{2g^2}\left[\mathrm{Tr}\left\{\frac{1}{2}\left(D_t X_i\right)^2 - \frac{1}{4}\left[X_i, X_j\right]^2\right\} - \theta^T D_t \theta - \theta^T \gamma_i \left[\theta, X^i\right]\right]$$

where $D_t := \partial_t - i\left[A(t), \cdot\right]$ is the usual covariant derivative. The large-N limit corresponds to taking $N \to \infty$ while holding the 't Hooft coupling $\lambda := g^2 N$ fixed. The proposal of [12] is that in the large-N limit the action (5.23) yields a non-perturbation definition of M-theory (see [16] for a detailed review of this proposal using the discrete light-cone quantization in the 'infinite-momentum frame'). In particular, in this limit space is emergent. The spatial coordinates are related to the eigenvalue distribution of the matrices X^i in a similar way to how the one spatial dimension arises in the $c = 1$ matrix model of non-critical string theory [15].

As mentioned, there are $d+1$ bosonic matrices $A(t)$, $X_i(t)$ $(i = 1, 2, \ldots, d)$, each of which is an $N \times N$ Hermitian matrix. At finite temperature T, the bosonic part of the BFSS action is given by

$$S(\beta) = \frac{1}{g^2}\int_0^\beta \mathrm{d}t \ \mathrm{Tr}\left\{\frac{1}{2}\left(D_t X_i\right)^2 - \frac{1}{4}\left[X_i, X_j\right]^2\right\}, \tag{3.5}$$

where $\beta = 1/T$. One can set $\lambda = 1$ without any loss of generality, and thus we can trade $1/g^2$ for N in front of the action. We choose a unit convention in which the mass dimension of X is 1 and that of g^2 is 3 (we have set $l_s = 1$ for now).

At high temperatures, the BFSS model reduces to the IKKT model [29, 30]. Since we will use results from studies of the IKKT setup to establish our cosmological background, we recall the key points of the latter model.

The IKKT matrix model [13,17] (see [18] for a recent review) is defined by the following action

$$S = -\frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{4}\left[A^a, A^b\right]\left[A_a, A_b\right] + \frac{i}{2}\bar{\psi}_\alpha \left(\mathcal{C}\Gamma^a\right)_{\alpha\beta}\left[A_a, \psi_\beta\right]\right), \tag{3.6}$$

where ψ_α and A_a ($\alpha = 1, \ldots, 16$, $a = 1, \ldots, 10$) are $N \times N$ fermionic and bosonic Hermitian matrices, respectively, the Γ^α are the gamma-matrices for $D = 10$ dimensions, and $\mathcal{C}$ is the charge conjugation matrix. While a is a ten-dimensional vector index, α is a spinor index such that ψ_α plays the role of a ten-dimensional Majorana-Weyl spinor. This action can be seen as a matrix regularization of the worldsheet action of Type IIB superstring theory in the Schild gauge. Depending on the metric which is used to raise and lower the indices (either Euclidean or Minkowski), the action can be viewed as that of the Euclidean or Lorentzian type IIB matrix model. If we choose to have the same coupling g^2 for the IKKT model, as before, with mass dimension 3, the matrices A^a will have mass dimension 3/4.

The action of the Euclidean matrix model [13] is given by the following functional integral over the bosonic and fermionic fields

$$Z = \int dA d\psi\, e^{-S}, \tag{3.7}$$

while the action of the Lorentzian model is defined by [17]

$$Z = \int dA d\psi\, e^{iS}. \tag{3.8}$$

In the IKKT approach, both space and time are emergent, time being related to the matrix A_D, while space results from the other matrices. The $SU(N)$ symmetry can be used to diagonalize the matrix A_{10}

$$A_{10} = \mathrm{diag}(\alpha_1, ..., \alpha_N), \tag{3.9}$$

where without loss of generality the α_i can be ordered in ascending magnitude. As was shown in [17], the spatial matrices A_i then have a band-diagonal structure in the sense that there is an integer $n \ll N$ such that the matrix elements $(A_i)_{ab}$ for $|a - b| > n$ are much smaller than those for $|a - b| \leq n$. Note that n is a fixed fraction of N and hence goes to infinity in the limit $N \to \infty$.

A time variable t (which will be the time variable of our emergent phase) can then be defined by averaging the diagonal elements α_i over n elements

$$t(m) \equiv \frac{1}{n}\sum_{l=1}^{n} \alpha_{m+l} \, , \tag{3.10}$$

where the index m runs from 1 to $N - n$. Time-dependent spatial matrices $(\bar{A}_i)_{I,J}(t)$ of dimension $n \times n$ can then be defined via

$$(\bar{A}_i)_{I,J}(t(m)) \equiv (A_i)_{m+I,m+J} \, . \tag{3.11}$$

In the limit $N \to \infty$ we have $n \to \infty$ and $(\bar{A}_i)_{I,J}(t(m))$ becomes the emergent continuum space.

It is then natural to define the extent x_i of a given spatial dimension i at time t by

$$x_i(t)^2 \equiv \left\langle \frac{1}{n} \mathrm{Tr}(\bar{A}_i)(t))^2 \right\rangle \, , \tag{3.12}$$

where the pointed brackets stand for the quantum expectation value in the state defined by the partition function. Then, the total extent $R(t)$ of space at time t is

$$R^2(t) = \sum_{i=1}^{9} x_i(t)^2 \, . \tag{3.13}$$

Following [18], it is more convenient to define the *moment of inertia tensor*

$$T_{ij} \equiv \left\langle \frac{1}{n} \mathrm{Tr}(\bar{A}_i)(t)\bar{A}_j)(t)) \right\rangle \tag{3.14}$$

which is a symmetric 9×9 matrix whose eigenvalues can be denoted by λ_i.

A numerical analysis of this system shows [17] that as a function of time the $SO(9)$ spatial symmetry is spontaneously broken. Of the nine eigenvalues $\lambda_i(t)$, three of them become large, while six remain close to the original size. This is the same symmetry breaking pattern obtained in String Gas Cosmology [5] from considerations of the annihilation of string winding modes which can only liberate three spatial dimensions. What is observed in matrix theory can be viewed as the non-perturbative picture of the scenario of [5].

A similar symmetry breaking pattern was first studied in the Euclidean framework. There, the emergence of three large spatial dimensions can be seen both numerically and using a Gaussian expansion method in which the free energy is computed when approximating the functional integral via a Gaussian expansion about particular configurations, and the resulting free energy is compared for different chosen configurations, to find that the free energy is minimized for $d = 3$. See [19] for a selection of papers on this topic. On the other hand, the analyses for the Lorentzian IKKT model is more subtle. Firstly, numerical investigations are much more technically involved due to the 'sign problem'. Although this was averted using some approximations involving assuming a Gaussian action for the bosonic part of the IKKT action in [17], it was soon realized that the expanding spacetime has only two independent large eigenvalues (the so-called Pauli-matrix structure) [20]. In subsequent work, this was found to be a pathology of the approximation which was used to solve the sign problem and the numerical 'complex Langevin method' was introduced using two parameters to denote Wick rotations on both the worldsheet and the target space. This culminated in finding a true (smooth) $(3+1)$-d emergence of the background from the Lorentzian IKKT model [21], where the presence of fermionic matrices turn out to be essential.

In the high temperature limit, the BFSS model reduces to the (Euclidean) IKKT scenario. The BFSS gauge field matrix $A(t)$ corresponds to the IKKT matrix $A_D(t)$ and the BFFS spatial matrices $X_i(t)$ become the matrices $A_i(t)$ in the IKKT model. Hence, taking the results from the analysis of the IKKT model described above back to the BFSS side, we argue that in the high temperature limit (which is relevant for our discussion of the emergent phase) the background which minimizes the free energy will experience spontaneous symmetry breaking in which three of the spatial dimensions (given by the quantum expectation values of eigenvalue distribution of the matrices X_i) become large compared to the other six. Note that this symmetry breaking occurs during the emergent phase, and not only at the end of it.

Irrespective of whether one begins with the Euclidean or the Lorentzian version of the IKKT model, this emergence can be understood more generally. Since the eigenvalues of the matrices denote the target space coordinates, at very early times before the symmetry-breaking phase transition takes place, the nine eigenvalues are of equal size and of a microscopic scale, on which there does not exist a smooth geometric picture of spacetime. One way to see this is that the eigenvalues denote positions of D-branes and the matrices, corresponding to these, commute only in the limit after the symmetry breaking when the eigenvalues of three matrices become large. The target space coordinates are inherently non-commutative at very early times. This is a *non-geometric phase* from which our $(3+1)$-d universe emerges in the matrix model. As the emergent phase proceeds and the three large spatial dimensions grow in size, we will reach a point when the effective field theory description via Einstein gravity yields a good approximation for the infrared modes which we are interested in when considering cosmological perturbations measured at late times. Let us for now consider this transition to take place at a fixed time t_c, and we return to a discussion of how this transition happens later on.

Let us summarize the background cosmology which we are proposing. We start in a high temperature state of the BFSS matrix model which is equivalent to the IKKT matrix model (as $T \to \infty$). After Wick rotating the IKKT model, we can diagonalize the A_{10} matrix, and the diagonal elements determine our emergent time variable. The diagonal blocks of the $A_i, i = 1, .., 9$ matrices in this basis define an evolving space. All spatial dimensions (measured as described above in terms of the expectation values of the spatial matrices) are of the typical microscopic scale (the string scale). The spatial matrices evolve in time and the emergent space undergoes symmetry breaking in which three spatial dimensions become large and the other ones remain microscopic. The $SO(9)$ symmetry of space is broken to $SO(3) \times SO(6)$. As the three-dimensional space expands, General Relativity becomes a good description of the low energy dynamics of the three dimensional space [4] A transition to the expanding phase of Standard Big Bang cosmology occurs. Our subsequent analysis of fluctuations is independent of the specifics of this transition in the same way that the analysis of fluctuations in inflationary cosmology are in general insensitive to the details of reheating.

At this point, let us give some justification for using General Relativity (GR) as the low-energy limit of the matrix theory. How do we know that the low-energy gravity theory is going to be GR and not something else? Firstly, note that the BFSS model is a proposal for M-theory and therefore, we are guaranteed to have GR as the low-energy limit of (the

[4] Note that the Lorentz symmetry of the effective theory is a result of the $SO(10)$ symmetry of the original Euclidean IKKT matrix model.

gravitatiuonal part of) this theory. Furthermore, within the IKKT model, it has been shown that the underlying diffeomorphism symmetry of GR emerges naturally from this [22]. But the most direct way to note this was shown within the operator interpretation of matrices in the IKKT model, in which one could derive the vacuum Einstein equations starting from the classical equations of motion of the IKKT model [23]. In this approach, matter and gauge fields appear as fluctuations on top of a gravitational background and thus all fields of different spins, including the graviton, emerge from the same IKKT model. Going to higher-order corrections, one can find different quantum fields sourcing Einstein's equations. However, the exact dynamics of quantum fields are yet to be understood in this interpretation and, therefore, we approach the problem from a different perspective. We consider a natural state for our cosmological model, namely a thermal state, which yields the emergent background from the IKKT model discussed above. Since the state is a thermal state, it includes thermal fluctuations which yield source terms for late time cosmological perturbations. Based on the above arguments, and the fact that after the time t_c we are in the low-energy limit of superstring theory, we use Einstein's equations sourced by the thermal state to compute the cosmological perturbations whose properties we calculate below.

In contrast to Standard Cosmology and the Inflationary paradigm, our proposed cosmology does not suffer from an initial singularity problem because the early phase is a nonsingular quantum mechanical matrix model which cannot be described by Einstein gravity. In particular, there is no beginning of time in the sense of General Relativity. Furthermore, recall that the BFSS matrix model is a quantum mechanical model and does not suffer from field-theoretic divergences one has to contend within GR.

The origin of our proposed cosmology as a quantum mechanical matrix model also provides a solution to the Horizon Problem of Standard Big Bang cosmology. The initial thermal state of the quantum mechanical matrix model automatically generates correlations over the entire emergent spatial section. From the point of view of an emergent scenario it is very reasonable to assume that we start in a thermal state [5]. Thus, like in String Gas Cosmology, the cosmological fluctuations and primordial gravitational waves will be of thermal origin,

[5] Note that it is only in the high temperature limit that the correspondence between the BFFS and IKKT models has been established.

unlike in inflationary cosmology where the inhomogeneities emerge from quantum vacuum fluctuations.

Let us also point out a difference between our cosmological model emerging from matrix theory and String Gas Cosmology. Unlike the latter, our model cannot be thought of as a *free* collection of stringy objects, such as D-branes, whose thermal properties describe the thermal state sourcing cosmological perturbations. It is tempting to interpret our results as a collection of 'free' $D0$-branes in a box since our starting point is the BFSS model. Similarly, one might be led to presume that a box of 'free' $D(-1)$ branes would explain the background dynamics due to its origins in the IKKT model. If this were to be true, one could have studied the thermodynamics of free D-branes just like one does for a box of strings in String Gas Cosmology. However, the crucial point to realize is that the BFSS (or, similarly, the IKKT) model is not just any collection of $D0$-branes but a specific bound state configuration of them. This gives rise to a very specific theory which allows us to do our computations in a thermal state that, quite remarkably, gives rise to scale-invariant perturbations in the early-universe, as we shall show later on. In fact, the thermal properties of a collection of free $D0$-branes do not have the same properties as can be seen from [24].

The cosmological fluctuations and gravitational waves which we can measure today in cosmological experiments have a length scale which even at the end of the emergent phase is in the far infrared compared to the typical energy scale of the emergent phase. Hence, the evolution of fluctuations on these scales will be described by the usual linear cosmological perturbation theory based on Einstein gravity. Hence, as in the case of String Gas Cosmology [9], the metric fluctuations will be determined by the correlation functions of the energy-momentum tensor in the thermal state of the emergent phase. In the following section we turn to the computation of these fluctuations.

3.3 Fluctuations

3.3.1 Formalism

In the previous section we have described our model for the background of the emergent period which results from matrix theory. Via a phase transition, the emergent period will connect to the radiation phase of the SBB in $3 + 1$ space-time dimensions. In this section we will compute the spectra of cosmological fluctuations and gravitational waves which arise from our background. In the framework of an emergent cosmology, and in contrast to the situation in an inflationary model, the length scales on which we currently observe the fluctuations were always many orders of magnitude larger than the typical microscopic scales, e.g. the Planck length. Specifically, if the energy scale at which the transition from the emergent phase to the radiation phase of the SBB occurs is 10^{16}GeV, then the wavelengths at that time were of the order of 1mm or larger. Hence, the description of fluctuations using the usual theory of linear cosmological perturbations (see e.g. [25, 26] for reviews) will apply.

We will write the metric of our four dimensional space-time (which we assume to be spatially flat) in longitudinal gauge, i.e. in the form

$$ds^2 \;=\; (1+2\Phi)dt^2 - a(t)^2 \left[(1-2\Phi)\delta_{ij} + h_{ij}\right] dx^i dx^j \,, \tag{3.15}$$

where $\Phi(x, t)$ is the relativistic generalization of the Newtonian gravitational potential, and the transverse and traceless tensor h_{ij} represents the gravitational waves. Specifically, a gravitational wave with dimensionless polarization tensor ϵ_{ij} will have an amplitude $h(x, t)$. We are neglecting the contribution of vector modes since these modes decay in an expanding universe.

Note that in a thermal state, the fluctuations on the typical microscopic state may be large in amplitude, but on the infrared scales relevant to cosmological observations they will be Poisson suppressed and hence small in amplitude such that linear cosmological perturbation theory applies and all Fourier mode of the fluctuating fields evolve independently.

According to the theory of linear cosmological perturbations, the curvature fluctuation on a scale k (where k denotes comoving wave number) is given by the energy density perturbations on that scale via

$$\langle|\Phi(k)|^2\rangle \;=\; 16\pi^2 G^2 k^{-4}\langle\delta T^0_0(k)\delta T^0_0(k)\rangle\,, \tag{3.16}$$

where T^μ_ν is the energy-momentum tensor of matter (also evaluated in longitudinal gauge), and G is Newton's gravitational constant. Similarly, the amplitude $h(k)$ of a gravitational wave mode is determined by the off-diagonal pressure fluctuations via [6]

$$\langle|h(k)|^2\rangle \;=\; 16\pi^2 G^2 k^{-4}\langle\delta T^i_j(k)\delta T^i_j(k)\rangle\,,\ i\neq j \tag{3.17}$$

On sub-Hubble scales, matter fluctuations dominate over the induced curvature fluctuations. Hence, following the logic used in String Gas Cosmology in [7, 8], we will first use the partition function Z of our model to determine the correlation functions of the energy-momentum tensor. Using these results, we apply (3.16) and (3.17) to determine the initial amplitude of the curvature fluctuations and gravitational waves when the length mode k exits the Hubble radius at the end of the emergent phase. From then on until the present time the usual evolution of the cosmological fluctuations applies.

In a thermal state, the fluctuations in the energy-momentum tensor in a box of radius R are determined in terms of the finite temperature partition function of the system. Specifically, since

$$\langle T^\mu_\nu\rangle \;=\; 2\frac{g^{\mu\lambda}}{\sqrt{-g}}\frac{\partial \ln Z}{\partial g^{\nu\lambda}} \tag{3.18}$$

the fluctuations of the energy-momentum tensor in a box of radius R are given by (see [9] for details)

$$\begin{aligned} C^{\mu\sigma}_{\nu\,\lambda} &\equiv \big\langle T^\mu_\nu T^\sigma_\lambda\big\rangle - \big\langle T^\mu_\nu\big\rangle\big\langle T^\sigma_\lambda\big\rangle \\ &= 2\frac{g^{\mu\alpha}}{\sqrt{-g}}\frac{\partial}{\partial g^{\alpha\nu}}\Big(\frac{g^{\sigma\delta}}{\sqrt{-g}}\frac{\partial \ln Z}{\partial g^{\delta\lambda}}\Big)+2\frac{g^{\sigma\alpha}}{\sqrt{-g}}\frac{\partial}{\partial g^{\alpha\lambda}}\Big(\frac{g^{\mu\delta}}{\sqrt{-g}}\frac{\partial \ln Z}{\partial g^{\delta\nu}}\Big)\,, \end{aligned} \tag{3.19}$$

where Z is the partition function restricted to the box. Specifically, the energy density fluctuations are determined by

$$C^{00}{}_{00} = \delta\rho^2 \;=\frac{T^2}{R^6}C_V \tag{3.20}$$

[6] The notation is a bit loose here: the indices i and j correspond to the polarization state of the gravitational wave.

where C_V is the specific heat capacity in a box of radius R and is given by the partial derivative of the internal energy $E(\beta)$ with respect to temperature T at constant volume V:

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V . \tag{3.21}$$

The gravitational waves, in turn, are given by

$$C^{ij}{}_{ij} = \langle T^{i\,2}_j \rangle - \langle T^i_j \rangle^2 \quad i \neq j \tag{3.22}$$

where the indices i and j are related to the polarization tensor of the wave which is being considered.

In the case of String Gas Cosmology, the thermal correlation functions for a gas of closed strings in the high temperature Hagedorn phase have holographic scaling, i.e. $C_V \sim R^2$, and correspondingly for the other correlation functions [27]. This result can be understood heuristically from the fact that strings look like point particles in one lower spatial dimension. This then leads to the scale-invariance of the spectrum of cosmological perturbations and gravitational waves. The fact that the temperature is a slightly decreasing function of time towards the end of the Hagedorn phase (when scales exit the Hubble radius) leads to a slight red tilt of the spectrum of cosmological perturbations (modes with larger values of k exit the Hubble radius later). The fact that the pressure is an increasing function of time towards the end of the Hagedorn phase leads to a characteristic slight blue tilt in the spectrum of gravitational waves [8] [7].

3.3.2 Cosmological Perturbations in Matrix Cosmology

We will now perform the calculation of the spectra of cosmological perturbations and gravitational waves in our scenario. Specifically, we are interested whether a scale-invariant spectrum of cosmological perturbations emerges [8]. We cannot apply the abovementioned

[7] Recall that inflation models in the context of General Relativity (with matter obeying the usual energy conditions) always leads to a slight red tilt of the spectrum of gravitational waves. This characteristic blue tilt of the spectrum of gravitational waves is also obtained [28] in the recently proposed version of the Ekpyrotic scenario in which an S-brane motivated by superstring theory leads to a nonsingular transition between an Ekpyrotic contracting phase and the radiation phase of the SBB.

[8] In the case of the proposal of [11] such a spectrum emerges because of the conformal invariance of the emergent topological phase.

heuristic argument for such a spectrum since our calculation is not based on classical string degrees of freedom. On the other hand, since our scenario in the perturbative limit will reduce to perturbative string theory, it would not be surprising if the holographic scaling of the specific heat capacity emerges.

Our starting point is the finite temperature action $S(\beta)$ (3.5) of the BFSS matrix theory. The resulting finite temperature partition function is given by the functional integral

$$Z(\beta) \;=\; \int [\mathcal{D}A]_\beta \, [\mathcal{D}X]_\beta \;\, e^{-S(\beta)} \,, \tag{3.23}$$

where the subscript β implies that the fields to be integrated are over this range. Given this partition function, the internal energy E is given by

$$E \;=\; -\frac{\mathrm{d}}{\mathrm{d}\beta} \ln Z(\beta) \,. \tag{3.24}$$

One can now calculate this internal energy as follows:

$$E \;=\; -\frac{1}{Z(\beta)} \lim_{\Delta\beta\to 0} \frac{Z(\beta') - Z(\beta)}{\Delta\beta} \,, \tag{3.25}$$

where $\beta' = \beta + \Delta\beta$. $Z(\beta')$ is given by

$$Z(\beta') = \int [\mathcal{D}A']'_\beta \, [\mathcal{D}X']'_\beta \;\, e^{-S(\beta')} \,, \tag{3.26}$$

where S' can be obtained from the action by replacing $\beta,\ t,\ A(t),\ X_i(t)$ with $\beta',\ t',\ A'(t'),\ X'_i(t')$. While the measures remain invariant, $[\mathcal{D}A']'_\beta = [\mathcal{D}A]_\beta$ and $[\mathcal{D}X']'_\beta = [\mathcal{D}X]_\beta$, the fields and time-parameter are related by the transformations

$$t' = \frac{\beta'}{\beta} t \,, \;\; A'(t') = \frac{\beta}{\beta'} A(t) \,, \;\; X'_i(t') = \sqrt{\frac{\beta'}{\beta}} X_i(t) \,. \tag{3.27}$$

Under this transformation, the kinetic term remains invariant while the interaction term in S' is related to that in S as follows:

$$\int_0^{\beta'} \mathrm{d}t' \, \mathrm{Tr} \left\{ \left[X'_i(t'), X'_j(t') \right]^2 \right\} = \left(\frac{\beta'}{\beta} \right)^3 \int_0^{\beta} \mathrm{d}t \, \mathrm{Tr} \left\{ [X_i(t), X_j(t)]^2 \right\} \,. \tag{3.28}$$

One can now write down the relation

$$\begin{aligned} Z(\beta') \;&=\; \int [\mathcal{D}A']'_\beta \, [\mathcal{D}X']'_\beta \;\, e^{-S} \;\exp\left(-N \int_0^\beta \mathrm{d}t \, \frac{1}{4} \mathrm{Tr} \left\{ [X_i, X_j]^2 - \left(\frac{\beta'}{\beta} \right)^3 [X_i, X_j]^2 \right\} \right) \\ &=\; \int [\mathcal{D}A']'_\beta \, [\mathcal{D}X']'_\beta \;\, e^{-S} \left\{ 1 - \frac{3}{4} N \int_0^\beta \mathrm{d}t \; \mathrm{Tr} \left([X_i, X_j]^2 \right) \frac{\Delta\beta}{\beta} + \mathcal{O}((\Delta\beta)^2 \right\} \\ &=\; Z(\beta) \left(1 - N^2 \Delta\beta \langle \mathcal{E} \rangle + \mathcal{O}((\Delta\beta)^2) \right) \,, \end{aligned} \tag{3.29}$$

where

$$\mathcal{E} = -\frac{3}{4}\frac{1}{N\beta}\int_0^\beta \mathrm{d}t \ \mathrm{Tr}\left([X_i, X_j]^2\right) . \tag{3.30}$$

Therefore, we can write down the internal energy as

$$E = N^2\langle\mathcal{E}\rangle , \tag{3.31}$$

where $\langle\cdot\rangle$ is the expectation value calculated with respect to the partition function $Z(\beta)$.

It has been shown in [29] that the BFSS action reduces to the IKKT one at high temperatures. If one Fourier expands the fields as

$$X_i = \sum_n X_i^n \, e^{in\omega t} , \tag{3.32}$$

where $\omega = 2\pi/\beta$ are the Matsubara frequencies, then the BFSS action becomes

$$S_{\mathrm{BFSS}} = S_0 + S_{\mathrm{kin}} + S_{\mathrm{int}} . \tag{3.33}$$

Let us first consider the leading order behaviour of the action, in the high temperature limit, which is given by the zero modes of the Fourier expansion, and therefore

$$S_0 \equiv -N\beta \ \mathrm{Tr}\left\{\frac{1}{2}\left([A, X_i^0]\right)^2 + \frac{1}{4}\left([X_i^0, X_j^0]\right)^2\right\} . \tag{3.34}$$

We rescale the zero modes as [9]

$$A_i := T^{-1/4}X_i^0 \ (i = 1, 2, \ldots, d), \ \ A_D := T^{-1/4}A , \tag{3.35}$$

where $D = d + 1$. This tells us that

$$S_0 = \frac{1}{4} N \ \mathrm{Tr}\left(F_{\mu\nu}\right) =: S_{\mathrm{IKKT}} , \ \ F_{\mu\nu} := -i\left[A_\mu, A_\nu\right] . \tag{3.36}$$

Here we assume $(\mu, \nu = 1, 2, \ldots, D)$. On the other hand, the kinetic term of the action go as

$$S_{\mathrm{kin}} \equiv N\beta \ \mathrm{tr}\left\{\frac{1}{2}\sum_{n\neq 0}(n\omega)^2 X_{-n}^i X_n^i\right\} \tag{3.37}$$

and the interaction terms as

[9]Note the consistency of the mass dimensions using the conventions introduced at the beginning of Section II.

$$
\begin{aligned}
S_{\rm int} \;&= -N\beta\ {\rm tr}\Big\{ \textstyle\sum_{n\neq 0} n\omega X^i_{-n}[A, X^i_n] + \frac{1}{2}\sum_{n\neq 0}[A, X^i_{-n}][A, X^i_n] \\
&\qquad\qquad + \textstyle\frac{1}{4}\sum_{npq}[X^i_{-n-p-q}, X^j_n][X^i_p, X^j_q] \Big\}\,,
\end{aligned}
\tag{3.38}
$$

where the $n = p = q = 0$ term is excluded in the last sum. We only show the bosonic terms above and ignore the terms corresponding to the fermionic and the ghost fields to avoid clutter (see [29] for more details). While we use S_0 to calculate the leading order terms, $S_{\rm kin}$ and $S_{\rm int}$ become important when calculating the next-to-leading order terms.

Given this, we can calculate two quantities of interest to us – the extent of the eigenvalue distribution and the internal energy. The extent of the eigenvalue distribution is given by

$$
R^2 := \frac{1}{N\beta}\int_0^\beta {\rm d}t\ {\rm Tr}\,(X_i(t))^2\ , \tag{3.39}
$$

We begin by taking its expectation value with the BFSS partition function and we find that

$$
\langle R^2\rangle_{\rm BFSS} \simeq \chi_1 T^{1/2}\,, \tag{3.40}
$$

where

$$
\chi_1 := \left\langle \frac{1}{N}\ {\rm Tr}(A_i)^2 \right\rangle_{\rm IKKT}\ . \tag{3.41}
$$

Note that although expression (3.39) is an exact definition, (3.40) is the leading order (in temperature) relation between R and χ_1. The leading term is given by the zero modes. Note that the extent of space parameter in the IKKT model is given by the quantity χ_1 defined above (compare with (3.13) above). The dependence of the extent of eigenvalue distribution on the spatial volume is characterized by χ_1, whereas we explicitly separate its dependence on temperature as above. The numerical simulations of the IKKT model tell us how the extent of space parameter (3.13) evolves with time. The temperature (coming from the Euclideanized time direction) can be assumed to be constant in the model and χ_1 will give us the value of the background volume for this given time.

More relevantly for us, the internal energy $E = N^2\langle\mathcal{E}\rangle_{\rm BFSS}$ can be calculated to get

$$
E \simeq \frac{3N^2}{4}\,\chi_2\,T\,, \tag{3.42}
$$

where

$$\chi_2 := \left\langle \frac{1}{N}\,\mathrm{Tr}(F_{ij})^2 \right\rangle_{\mathrm{IKKT}} \,. \tag{3.43}$$

This shows that the internal energy, to the leading order in temperature, does not depend on the spatial volume (as this expression does not depend on χ_1).

Let us make a quick note regarding the dimensions involved in the above equations. Recall that since we have set the 't hooft coupling $\lambda = 1$, we need to replace factors of N by $1/g^2$, which has mass dimension of -3. The Yang-Mills coupling g is related to the string length l_s by the dimensionless string coupling constant g_s, via $g^2 = g_s/l_s^3$. Therefore, for restoring appropriate dimensions, one needs to insert proper powers of g in the expressions above. For instance, for the internal energy to have mass dimension 1, we therefore require that $\chi_2 \sim l_s^{-6}$ (due to the factor of $N^2 \sim 1/g^4$ in (3.42) above). Similarly, the extent of space parameter R^2 must have a factor of l_s^7 in its definition (3.40) for it to have mass dimension of -2. We continue to suppress these dimensional parameters for simplicity and would reinstate them in the final expression for the power spectra.

Going to the next to leading order in temperature, one can calculate the internal energy to be (we follow the conventions of [29]):

$$E \;\simeq\; \frac{3N^2}{4}\,\chi_2\,T - \frac{3N^2}{4}\left(\frac{d-1}{12} - \frac{p}{8}\right)\left(\chi_5 - \chi_6 - 4\chi_1\right)\,T^{-1/2}\,, \tag{3.44}$$

where

$$\begin{aligned}\chi_5 &:= \left\langle \mathrm{Tr}\,(F_{ij})^2\,.\,\mathrm{Tr}\,(A_k)^2 \right\rangle_{\mathrm{IKKT}}\\ \chi_6 &:= \left\langle \mathrm{Tr}\,(F_{ij})^2\,.\,\mathrm{Tr}\,(A_D)^2 \right\rangle_{\mathrm{IKKT}}\,.\end{aligned} \tag{3.45}$$

In the above expression, p denotes the number of fermionic superpartners of the d bosonic matrices.

Let us calculate χ_5 and χ_6 using the following approximation

$$\begin{aligned}\chi_5 &= \left\langle \mathrm{Tr}\,(F_{ij})^2\,.\,\mathrm{Tr}\,(A_k)^2 \right\rangle_{\mathrm{IKKT}}\\ &\simeq \left\langle \mathrm{Tr}\,(F_{ij})^2 \right\rangle_{\mathrm{IKKT}} \left\langle \mathrm{Tr}\,(A_k)^2 \right\rangle_{\mathrm{IKKT}}\\ &= N^2\,\chi_2\,\chi_1\end{aligned} \tag{3.46}$$

and

$$\begin{aligned}\chi_6 &= \left\langle \mathrm{Tr}\,(F_{ij})^2 \,.\, \mathrm{Tr}\,(A_D)^2 \right\rangle_{\mathrm{IKKT}} \\ &\simeq \left\langle \mathrm{Tr}\,(F_{ij})^2 \right\rangle_{\mathrm{IKKT}} \left\langle \mathrm{Tr}\,(A_D)^2 \right\rangle_{\mathrm{IKKT}} \\ &= \left(\frac{N^2}{d}\right) \chi_2\, \chi_1 \,. \end{aligned} \tag{3.47}$$

Let us make two observations regarding the above calculation. Firstly, the next to leading order values for these quantities can be evaluated explicitly by considering the propagators, from the kinetic term in the action, and from the interaction terms. However, although we only showed the bosonic terms in (3.37) and (3.38) for simplicity, one also needs to take into account the fermionic fields (and the ghost terms corresponding to our gauge-fixing) to carry out the explicit calculation. And finally, one needs to integrate out *only* over the non-zero modes in order to arrive at the above-mentioned results. The zero modes (in the Matsubara frequencies) are what gives rise to the IKKT action and therefore, we express our results in terms of quantities evaluated in the IKKT model and the temperature T. Note that since we have set the 'tHooft coupling $\lambda = g^2 N = 1$, our only dimensionful parameter for perturbation theory is $T^{-3/2}$ [31]. In other words, once one integrates out non-zero modes using perturbation theory, the leftover integration over the zero modes can be thought of as taking the expectation value of Green's functions using the bosonic part of the IKKT action.

We now have the ingredients needed to evaluate the power spectrum of energy density fluctuations in our scenario using (3.20) and (3.21). Let us consider a comoving momentum scale k. The associated volume is $\langle R^2 \rangle_{\mathrm{BFSS}}^{3/2}$ which we will in the following abbreviate by R^3. The dimensionless power spectrum $P(k)$ on the scale R related to the wavenumber k via $R = 2\pi k^{-1}$ is given by

$$\begin{aligned} P(k) &\sim k^3 \langle |\Phi(k)|^2 \rangle \\ &= 16\pi^2 G^2 k^{-1} \langle \delta T^0_0(k) \delta T^0_0(k) \rangle \\ &= 16\pi^2 G^2 k^{-4} (\delta\rho)^2 \\ &= 16\pi^2 G^2 k^{-4} T^2 C_V R^{-6} \,, \\ &= 16\pi^2 G^2 k^2 T^2 C_V (kR)^{-6} \end{aligned} \tag{3.48}$$

where the factor of k^{-3} in going from the second to the third line comes from converting momentum space to position space density.

Thus, the scalar power spectrum depends mostly on the specific heat C_V. Let us calculate it to the next-to-leading order in the high temperature limit. From (3.44), we find

$$C_V \;=\; \frac{3N^2}{4}\chi_2 \frac{3N^4}{8}\left(\frac{d-1}{12}-\frac{p}{8}\right)\left(\chi_2-\frac{1}{d}\chi_2-\frac{4}{N^2}\right)\chi_1 T^{-3/2}\,. \tag{3.49}$$

Note that the $C_V > 0$ for $d = 3$, $p = 4$ and the thermodynamics is well-defined in this case.

The first term hence yields a contribution to the power spectrum proportional to k^2. Since $\chi_1 \sim k^{-2}$, the second term yields a scale invariant contribution. On microscopic scales, the first term dominates. It corresponds to a Poisson spectrum and is what we expect for thermal fluctuations on scales close to the correlation length of the system. On infrared scales relevant for current cosmological fluctuations, however, it is the second term which dominates, and it corresponds to a scale-invariant spectrum, and its value is

$$P(k) \;=\; 16\pi^2 G^2 k^2 (kR)^{-6} T^{1/2} N^2 \chi_1 \frac{3}{8}\big(\frac{d-1}{12}-\frac{p}{8}\big)\big(N^2\chi_2-\frac{N^2}{d}\chi_2-4\big)\,. \tag{3.50}$$

Substituting for χ_1 making use of (3.40), and reinstating dimensional parameters, yields

$$P(k) \;=\; 16\pi^2\,(kR)^{-4}\left(\frac{1}{l_s m_{pl}}\right)^4\left(\frac{3}{8}\right)\left(\frac{d-1}{12}-\frac{p}{8}\right)\left(\frac{(d-1)^2}{d}\left(1-\frac{1}{N^2}\right)-4\right)\,, \tag{3.51}$$

from which it follows that the amplitude of the spectrum is given by

$$\mathcal{A} \;\sim\; (l_s m_{pl})^{-4}\,, \tag{3.52}$$

the same scaling as in String Gas Cosmology [7]. In (8.5), we have used the explicit expression for χ_2 [32]:

$$\chi_2 \;=\; (d-1)\left(1-\frac{1}{N^2}\right)\,, \tag{3.53}$$

where all dimensional factors have been accounted for and there is no further g dependence coming from the N^2 term. Note that this result does not depend on the exact dynamics of how the background volume expands with time (beyond the general evidence the numerical analysis provides for the emergence of 3 large spatial dimensions).

3.3.3 Gravitational Waves in Matrix Cosmology

Tensor perturbations are sourced by the off-diagonal pressure perturbations, as described in (3.22). Specifically, the dimensionless power spectrum of gravitational waves on a comoving momentum scale k is given by

$$P_h(k) \,=\, 16\pi^2 G^2 k^{-4} C^{ij}_{ij}(R(k))\,, \tag{3.54}$$

where we recall that $C^{ij}_{ij}(R(k))$ is the position space expectation value of the square of the off-diagonal pressure perturbation ($i \neq j$), and $R(k)$ is the length scale corresponding to k. In a thermal state we expect the off-diagonal pressure perturbations to be smaller but of similar magnitude as the diagonal pressure contribution. We will denote the suppression factor of the off-diagonal term compared to the diagonal term by a positive constant $\alpha < 1$. Hence,

$$C^{ij}{}_{ij} \,=\, \alpha \frac{T}{R^2} \frac{\partial \tilde{p}}{\partial R}\,, \tag{3.55}$$

where the pressure $\tilde{p}$ is given by

$$\tilde{p} \,=\, -\frac{1}{V} \frac{\partial \mathcal{F}}{\partial \ln R}\,. \tag{3.56}$$

To calculate the pressure, let us begin with the free energy of our system, calculated up to next-to-leading order

$$\mathcal{F} \,=\, \frac{3N^2}{4\beta} \left[\chi_2 \ln\beta \,-\, \frac{2}{3} \left(\frac{d-1}{12} - \frac{p}{8} \right) (\chi_5 - \chi_6 - 4\chi_1)\, \beta^{3/2} \right]\,.$$

We can use the same approximations as before to write χ_5 and χ_6 in terms of χ_1 and χ_2. We then obtain

$$C^{ij}{}_{ij} \,=\, \alpha \frac{T^{1/2}}{R^4} N^2 \big(\frac{d-1}{12} - \frac{p}{8} \big) (N^2 \chi_2 - \frac{N^2}{d} \chi_2 - 4)\,, \tag{3.57}$$

from which it follows that the dimensionless power spectrum of gravitational waves will also be scale-invariant with an amplitude $P_h(k)$ given by (on restoring the dimensional factors, and using (3.53), as before):

$$P_h(k) \,=\, \alpha\, 16\pi^2\, (kR)^{-4} \left(\frac{1}{l_s m_{pl}} \right)^4 \left(\frac{3}{8} \right) \left(\frac{d-1}{12} - \frac{p}{8} \right) \left(\frac{(d-1)^2}{d} \left(1 - \frac{1}{N^2} \right) - 4 \right)\,. \tag{3.58}$$

Comparing the results (8.7) and (3.50) for the tensor and scalar power spectra, we find that the tensor to scalar ratio r is given by

$$r \,=\, \frac{8}{3} \alpha\,. \tag{3.59}$$

In order to be consistent with the current observational bound on r, the value of α needs to be of the order $\mathcal{O}(10^{-2})$ or smaller. Note that although the off-diagonal elements are naturally suppressed compared to the diagonal ones, for thermal fluctuations, they are not expected to get fine-tuned to be extremely small. In other words, we expect the parameter α to be a smaller than 1 but not by many orders of magnitude [7]. Note that in String Gas Cosmology the value of r is suppressed by the ratio between the pressure and the energy density in the Hagedorn phase [8]. In the topological phase model of [11], no primordial gravitational waves are generated to leading order in the analysis. However, since α is not expected to be many orders of magnitude smaller than 1 for our model, we expect to find an observable signal for primordial gravity waves in our model. This is a significant difference between our model and those other approaches to early universe cosmology. It is hence important to estimate the value of α which results from our matrix theory model.

3.4 Conclusions and Discussion

In this paper we have suggested a concrete realization of a non-singular *emergent cosmology* based a matrix theory, a proposed non-perturbative definition of superstring theory in which space is emergent. The starting point is a gauge action for nine Hermitean $N \times N$ matrices X_i. The covariant derivative involves another $N \times N$ matrix A. We consider this matrix model in a finite temperature state. Space is emergent in the sense that in the large N limit, the expectation values of X_i^2 yield the size of the i'th spatial dimension. We have used results of numerical and analytical studies of matrix theory to show that a spontaneous breaking of the $SO(9)$ spatial symmetry takes place, and that exactly three dimensions of space become large. We have argued that at late times, a phase transition to the radiation phase of Standard Big Bang cosmology takes place, signalling the end of the emergent phase. Our scenario automatically solves some problems of Standard Big Bang cosmology such as the horizon problem, in the same way that they are solved in the proposal of [11]. A quick way to see this is to realize that the emergent spatial dimensions appear from the early non-geometric phase when the matrices are not commuting and their eigenvalues cannot be said to describe a smooth $(3+1)$-d spacetime. Thus, the entire emergent space is born out of the same matrix action and is interacting with each other in the non-geometric phase, naturally resolving the horizon problem [10].

However, background dynamics is not sufficient for understanding the properties of the emergent cosmology derived from matrix theory. One needs to calculate the spectrum of primordial perturbations in this model and this is where the novelty of our work lies. We have computed the thermal correlation functions of the energy-momentum tensor in the emergent phase. These determine the spectrum of cosmological density fluctuations and gravitational waves. In analogy to what is assumed in String Gas Cosmology, the fluctuations are of thermal origin. They do not originate as quantum vacuum perturbations as they do in canonical inflationary models. We find that the spectrum of cosmological fluctuations have two components, one of which has Poisson scaling and dominates on small scales, the other one being scale-invariant which dominates on scales relevant to cosmological observations. The spectrum of gravitational waves is also scale-invariant. We have computed the tensor to scalar ratio r on large scales. The resulting amplitude is given by the ratio of the off-diagonal to the diagonal pressure fluctuations, a ratio which we denote by α in the text. In order not to exceed the observational upper bound on r, the value of α needs to be sufficiently small. An open problem is to derive the value of α from our matrix theory model.

Note that the spectrum of both density perturbations and primordial tensor modes is not expected to be exactly scale-invariant on observable scales. Small deviations from scale-invariance, and a corresponding small tilt, naturally appear in our model when one goes to the next order in temperature. In addition, the processing of the fluctuations through the phase transition can induce a tilt, as it does in String Gas Cosmology. One can calculate the bispectrum and other higher order moments from higher order calculations in perturbation theory for the thermal state under consideration. We leave these topics for future work.

Note that our scenario does not involve a period of inflationary expansion. Since it is based on a non-perturbative approach to superstring theory, the scenario is free from any *swampland* constraints, consistency conditions which rule out many inflationary models (see [33] for reviews of the swampland program, and [34] for applications to inflation). The scenario is clearly consistent with the *trans-Planckian censorship conjecture* (TCC) [35] since

[10]Since space is emergent, the very concept of causality is also emergent in this theory.

the wavelengths of fluctuation modes which we observed today were never smaller than the Planck length (the fluctuations are generated towards the end of the emergent phase on scales which are macroscopic compared to the string length). This is another difference compared to the inflationary scenario, where the TCC sets a very restrictive upper bound on the energy scale of inflation [36], a bound which most models of inflation fail to satisfy.

Lastly, note that there are other approaches to obtaining space-time and cosmology from matrix theory. For example, Steinacker has a research program (see [37] for some original articles and [38] for a review) in which matrices satisfying the equations of motion derived from the matrix action are represented on a Poisson manifold. Specifically, one can choose the Poisson manifold to have space-time dimension four. This corresponds to choosing a background matrix set with $X^a = 0$ for $a \neq 0, 1, 2, 3$. Matrix fluctuations about this background then yield an action for gauge fields and scalar fields, and fermions if one starts from a supersymmetric matrix model. Gravity is induced on the background. Our work is different in that we obtain space-time directly from the matrix theory.

We would also like to mention recent work of Klinkhamer [39] which further develops some of the ideas of [19] for the Lorentzian matrix model, argues that matrix theory will yield a nonsingular emergent cosmology, and extracts space and the cosmological scale factor from a numerical analysis of the model. However, no attempt is made to compute cosmological perturbations and compare with observations.

The most important open issue for our scenario is the study of the transition from the emergent phase analyzed in this paper to the radiation phase of Standard Big Bang cosmology. In the case of String Gas Cosmology, the transition proceeds via the annihilation of string winding modes, resulting in the generation of string loops which lead to radiation. The transition is smooth and a high density radiation bath is automatically generated, obviating the need of a separate reheating phase, a phase which is needed in inflationary cosmology. In the same way, in our scenario the exit from the emergent phase will automatically lead to a high density radiation bath. The details of the transition, however, are not known, and these details will be important in order to be able to make precise predictions for the slopes of the spectra of scalar and tensor modes. Work on this issue is in progress.

Acknowledgments

The research at McGill is supported in part by funds from NSERC and from the Canada Research Chair program. SB is supported in part by the NSERC (funding reference CITA #490888-16) through a CITA National Fellowship and by a McGill Space Institute fellowship SL is supported in part by funds from a Templeton Foundation sub-contract. We are grateful to Sumit Das and Keshav Dasgupta for comments on a draft of this paper.

Bibliography

[1] A. H. Guth, "The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems," Phys. Rev. D **23**, 347 (1981) [Adv. Ser. Astrophys. Cosmol. **3**, 139 (1987)];

R. Brout, F. Englert and E. Gunzig, "The Creation Of The Universe As A Quantum Phenomenon," Annals Phys. **115**, 78 (1978);

A. A. Starobinsky, "A New Type Of Isotropic Cosmological Models Without Singularity," Phys. Lett. B **91**, 99 (1980);

K. Sato, "First Order Phase Transition Of A Vacuum And Expansion Of The Universe," Mon. Not. Roy. Astron. Soc. **195**, 467 (1981).

[2] R. A. Sunyaev and Y. B. Zeldovich, "Small scale fluctuations of relic radiation," Astrophys. Space Sci. **7**, 3 (1970);

P. J. E. Peebles and J. T. Yu, "Primeval adiabatic perturbation in an expanding universe," Astrophys. J. **162**, 815 (1970). doi:10.1086/150713

[3] V. Mukhanov and G. Chibisov, "Quantum Fluctuation And Nonsingular Universe. (In Russian)," JETP Lett. **33**, 532 (1981) [Pisma Zh. Eksp. Teor. Fiz. **33**, 549 (1981)].

[4] R. H. Brandenberger, "Cosmology of the Very Early Universe," AIP Conf. Proc. **1268**, 3 (2010) [arXiv:1003.1745 [hep-th]];

R. H. Brandenberger, "Alternatives to the inflationary paradigm of structure formation," Int. J. Mod. Phys. Conf. Ser. **01**, 67 (2011) [arXiv:0902.4731 [hep-th]].

[5] R. H. Brandenberger and C. Vafa, "Superstrings In The Early Universe," Nucl. Phys. B **316**, 391 (1989).

[6] R. Hagedorn, "Statistical thermodynamics of strong interactions at high-energies," Nuovo Cim. Suppl. **3**, 147 (1965).

[7] A. Nayeri, R. H. Brandenberger and C. Vafa, "Producing a scale-invariant spectrum of perturbations in a Hagedorn phase of string cosmology," Phys. Rev. Lett. **97**, 021302 (2006) [hep-th/0511140].

[8] R. H. Brandenberger, A. Nayeri, S. P. Patil and C. Vafa, "Tensor Modes from a Primordial Hagedorn Phase of String Cosmology," Phys. Rev. Lett. **98**, 231302 (2007) [hep-th/0604126];
R. H. Brandenberger, A. Nayeri and S. P. Patil, "Closed String Thermodynamics and a Blue Tensor Spectrum," Phys. Rev. D **90**, no. 6, 067301 (2014) [arXiv:1403.4927 [astro-ph.CO]].

[9] R. H. Brandenberger, "String Gas Cosmology," String Cosmology, J.Erdmenger (Editor). Wiley, 2009. p.193-230 [arXiv:0808.0746 [hep-th]].

[10] A. Berera, "Warm inflation," Phys. Rev. Lett. **75**, 3218 (1995) [astro-ph/9509049].

[11] P. Agrawal, S. Gukov, G. Obied and C. Vafa, "Topological Gravity as the Early Phase of Our Universe," arXiv:2009.10077 [hep-th].

[12] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, "M theory as a matrix model: A Conjecture," Phys. Rev. D **55**, 5112 (1997) [hep-th/9610043].

[13] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, "A Large N reduced model as superstring," Nucl. Phys. B **498**, 467 (1997) [hep-th/9612115].

[14] B. Ydri, "Review of M(atrix)-Theory, Type IIB Matrix Model and Matrix String Theory," [arXiv:1708.00734 [hep-th]].

[15] S. R. Das and A. Jevicki, "String Field Theory and Physical Interpretation of $D = 1$ Strings," Mod. Phys. Lett. A **5**, 1639 (1990).

[16] W. Taylor, "M(atrix) Theory: Matrix Quantum Mechanics as a Fundamental Theory," Rev. Mod. Phys. **73**, 419-462 (2001) [arXiv:hep-th/0101126 [hep-th]].

[17] S. W. Kim, J. Nishimura and A. Tsuchiya, "Expanding (3+1)-dimensional universe from a Lorentzian matrix model for superstring theory in (9+1)-dimensions," Phys. Rev. Lett. **108**, 011601 (2012) [arXiv:1108.1540 [hep-th]].

[18] J. Nishimura, "New perspectives on the emergence of (3+1)D expanding space-time in the Lorentzian type IIB matrix model," PoS CORFU **2019**, 178 (2020) [arXiv:2006.00768 [hep-lat]].

[19] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, "Space-time structures from IIB matrix model," Prog. Theor. Phys. **99**, 713 (1998) [hep-th/9802085];
J. Nishimura and G. Vernizzi, "Spontaneous breakdown of Lorentz invariance in IIB matrix model," JHEP **0004**, 015 (2000) [hep-th/0003223];
K. N. Anagnostopoulos and J. Nishimura, "New approach to the complex action problem and its application to a nonperturbative study of superstring theory," Phys. Rev. D **66**, 106008 (2002) [hep-th/0108041];
J. Nishimura and F. Sugino, "Dynamical generation of four-dimensional space-time in the IIB matrix model," JHEP **0205**, 001 (2002) [hep-th/0111102];
Y. Ito, J. Nishimura and A. Tsuchiya, "Power-law expansion of the Universe from the bosonic Lorentzian type IIB matrix model," JHEP **1511**, 070 (2015) doi:10.1007/JHEP11(2015)070 [arXiv:1506.04795 [hep-th]].

[20] T. Aoki, M. Hirasawa, Y. Ito, J. Nishimura and A. Tsuchiya, "On the structure of the emergent 3d expanding space in the Lorentzian type IIB matrix model," PTEP **2019**, no. 9, 093B03 (2019) [arXiv:1904.05914 [hep-th]].

[21] J. Nishimura and A. Tsuchiya, "Complex Langevin analysis of the space-time structure in the Lorentzian type IIB matrix model," JHEP **06**, 077 (2019) [arXiv:1904.05919 [hep-th]].

[22] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, "Space-time structures from IIB matrix model," Prog. Theor. Phys. **99**, 713-746 (1998) [arXiv:hep-th/9802085 [hep-th]];
S. Iso and H. Kawai, "Space-time and matter in IIB matrix model: Gauge symmetry

and diffeomorphism," Int. J. Mod. Phys. A **15**, 651-666 (2000) [arXiv:hep-th/9903217 [hep-th]].

[23] M. Hanada, H. Kawai and Y. Kimura, "Describing curved spaces by matrices," Prog. Theor. Phys. **114**, 1295-1316 (2006) [arXiv:hep-th/0508211 [hep-th]].

[24] J. Ambjorn, Y. M. Makeenko and G. W. Semenoff, "Thermodynamics of D0-branes in matrix theory," Phys. Lett. B **445**, 307-315 (1999) [arXiv:hep-th/9810170 [hep-th]].

[25] V.F. Mukhanov, H.A. Feldman and R.H. Brandenberger, "Theory of Cosmological Perturbations" Physics Reports **215**, 203 (1992).

[26] R. H. Brandenberger, "Lectures on the theory of cosmological perturbations," Lect. Notes Phys. **646**, 127 (2004) [hep-th/0306071].

[27] N. Deo, S. Jain, O. Narayan and C. I. Tan, "The Effect of topology on the thermodynamic limit for a string gas," Phys. Rev. D **45**, 3641 (1992).

[28] R. Brandenberger and Z. Wang, "Nonsingular Ekpyrotic Cosmology with a Nearly Scale-Invariant Spectrum of Cosmological Perturbations and Gravitational Waves," Phys. Rev. D **101**, no. 6, 063522 (2020) [arXiv:2001.00638 [hep-th]].

[29] N. Kawahara, J. Nishimura and S. Takeuchi, "High temperature expansion in supersymmetric matrix quantum mechanics," JHEP **12**, 103 (2007) [arXiv:0710.2188 [hep-th]].

[30] I. K. Kostov and P. Vanhove, "Matrix string partition functions," Phys. Lett. B **444** (1998), 196-203 [arXiv:hep-th/9809130 [hep-th]].

[31] N. Kawahara, J. Nishimura and S. Takeuchi, "Phase structure of matrix quantum mechanics at finite temperature," JHEP **10**, 097 (2007) [arXiv:0706.3517 [hep-th]].

[32] T. Hotta, J. Nishimura and A. Tsuchiya, "Dynamical aspects of large N reduced models," Nucl. Phys. B **545**, 543-575 (1999) [arXiv:hep-th/9811220 [hep-th]].

[33] C. Vafa, "The String landscape and the swampland," hep-th/0509212;
T. D. Brennan, F. Carta and C. Vafa, "The String Landscape, the Swampland, and the

Missing Corner," PoS TASI **2017**, 015 (2017) [arXiv:1711.00864 [hep-th]];
E. Palti, "The Swampland: Introduction and Review," Fortsch. Phys. **67**, no. 6, 1900037 (2019) [arXiv:1903.06239 [hep-th]];
M. van Beest, J. Calderon-Infante, D. Mirfendereski and I. Valenzuela, "Lectures on the Swampland Program in String Compactifications," arXiv:2102.01111 [hep-th].

[34] P. Agrawal, G. Obied, P. J. Steinhardt and C. Vafa, "On the Cosmological Implications of the String Swampland," Phys. Lett. B **784**, 271 (2018) [arXiv:1806.09718 [hep-th]].

[35] A. Bedroya and C. Vafa, "Trans-Planckian Censorship and the Swampland," JHEP **2009**, 123 (2020) [arXiv:1909.11063 [hep-th]].

[36] A. Bedroya, R. Brandenberger, M. Loverde and C. Vafa, "Trans-Planckian Censorship and Inflationary Cosmology," Phys. Rev. D **101**, no. 10, 103502 (2020) [arXiv:1909.11106 [hep-th]].

[37] H. Steinacker, "Emergent Gravity from Noncommutative Gauge Theory," JHEP **0712**, 049 (2007) [arXiv:0708.2426 [hep-th]];
H. Grosse, H. Steinacker and M. Wohlgenannt, "Emergent Gravity, Matrix Models and UV/IR Mixing," JHEP **0804**, 023 (2008) [arXiv:0802.0973 [hep-th]];
H. Steinacker, "Emergent Gravity and Noncommutative Branes from Yang-Mills Matrix Models," Nucl. Phys. B **810**, 1 (2009) [arXiv:0806.2032 [hep-th]].

[38] H. Steinacker, "Emergent Geometry and Gravity from Matrix Models: an Introduction," Class. Quant. Grav. **27**, 133001 (2010) [arXiv:1003.4134 [hep-th]].

[39] F. R. Klinkhamer, "On the emergence of an expanding universe from a Lorentzian matrix model," PTEP **2020**, 103 [arXiv:1912.12229 [gr-qc]];
F. R. Klinkhamer, "IIB matrix model: Emergent spacetime from the master field," PTEP **2021**, no. 1, 013B04 (2021) [arXiv:2007.08485 [hep-th]];
F. R. Klinkhamer, "On the emergence of an expanding universe from a Lorentzian matrix model," PTEP **2020**, no. 10, 103B03 (2020).

Chapter 4

IKKT Thermodynamics and early universe cosmology

Samuel Laliberte[1] and Suddhasattwa Brahma[2]

[1]*Department of Physics, McGill University, Montréal, QC, H3A 2T8, Canada*

[2]*Higgs Centre for Theoretical Physics, School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3FD, UK*

Addendum for thesis

Following the steps in Chapter 3, we explore here another way of obtaining a scale-invariant spectrum from a string-theoretic matrix model. In the present case, we explain how a scale-invariant spectrum of cosmological perturbations can be realized in the context of the IKKT model at finite temperature. We begin by formulating a description of the IKKT model at finite temperature, taking inspiration from string theory. In string theory, a description of strings at finite temperature can be found by compactifying the Eucledian target space of strings on a thermal circle where the radius of the thermal circle is identified with the inverse temperature. To define the IKKT model at finite temperature, we proceed in the same way as in string theory, and compactify the Euclidean time direction of the theory on a thermal circle with radius identified with the inverse temperature of the system. We obtain that the action of the IKKT model becomes exactly the action of the BFSS model up to T-duality. Motivated by this fact, we then compute the spectrum of cosmological perturbations of the IKKT model in the same way as in Chapter 3. Like for the BFSS model, we find a scale-invariant spectrum of scalar and tensor perturbations in agreement with string gas cosmology and CMB observations on large scales.

Abstract

Matrix theory is a proposed non-perturbative definition of superstring theory in which space is emergent. Recently, it was shown that space-time can emerge with a scale-invariant spectrum of cosmological perturbations which is sourced by thermal fluctuations of the BFSS model at finite temperature. Inspired by these results, we begin a study of the IKKT model at finite temperature. We find that in this model, which is closely related to the BFSS model at finite temperature, thermal fluctuations can also source a scale-invariant spectrum of scalar and tensor fluctuations.

4.1 Introduction

Superstring theory is a promising candidate for a self-consistent unified theory of quantum gravity and particle physics. From the perspective of a cosmologist, a promising aspect of string theory is that it can potentially describe the physics of the early universe where the Standard Model of Particles Physics and Einstein gravity are known to break down. Specifically, in [1], an emergent scenario based on string theory (String Gas Cosmology) was proposed in which the universe emerges from a quasi-static phase, the Hagedorn phase, as a gas of strings at a temperature close to its limiting temperature, also known as the Hagedorn temperature [2]. In this emergent phase, thermal fluctuations of the gas of strings lead to an almost scale-invariant spectrum of cosmological perturbations with a slight red tilt [4], and of gravitational waves with a slight blue tilt [3]. These thermal fluctuations provide a source for structure formation as the emergent phase transitions to the radiation-dominated phase of Standard Big Bang Cosmology. This model provides an interesting alternative to inflation, which is known to be hard to realize in string theory [5]. Moreover, the String Gas

Scenario yields a non-singular cosmology.

One problem of String Gas Cosmology, however, is that the background evolution of the universe near the Hagedorn temperature, which leads to the resolution of the singularity, is not well understood. This is the case because string theory is usually considered at the perturbative level of an an effective field theory on a classical background space-time. Such description is known to break down at very high densities and curvatures, and in particular in the very early universe. Therefore, to study the early universe in the high curvature regime, one should consider a non-perturbative formulation of string theory. There have been many proposals for non-perturbative formulations of string theory, most of which rely on matrix models. The idea behind these models is that certain system of $N \times N$ matrices can provide non-pertubative description of superstring theory in the large N limit. There are two main proposals for matrix theory, the BFSS model [6] and the IKKT model [7]. In the BFSS model, the eigenvalue distribution of the matrices describe space and depend on a continuous parameter t which plays the role of time. In the IKKT model, space-time is fully described by large N matrices: one holds information about time and the others hold information about space[1]. These models provide a non-perturbative description of M-theory and the Type IIB string respectively.

Recently, a novel emergent scenario was suggested which makes use of the BFSS model [8]. In this scenario, the universe emerges in a thermal state described by the BFSS model at finite temperature. While doing so, thermal fluctuations of the BFSS model lead to an almost scale-invariant spectrum of scalar and tensor perturbations, just like in the case of String Gas Cosmology. The universe then transitions to the radiation dominated phase of Standard Big Bang Cosmology, and thermal perturbations source structure formation. Steps have also been made to understand the time evolution of the metric at early times in the universe in matrix theory [9], which is something that was as hitherto out of reach of most perturbative approaches (See [10] for a review on these two topics).

Given the recent success of the BFSS model in explaining structure formation, we now suggest an alternative scenario, in which the IKKT model at finite temperature sources structure formation. In the new scenario that we are proposing in this paper, the universe emerges

[1] A review of this model will be provided in section 4.2

from a non-perturbative phase of the supersymmetric IKKT model[2] at finite temperature, and thermal fluctuations source the spectra of scalar and tensor perturbations which are both (approximately) scale-invariant. When space-time emerges, the universe transitions to the radiation dominated phase of Standard Big Bang Cosmology, and thermal perturbations source structure formation as was the case in the BFSS scenario.

Our new early universe scenario makes use of the IKKT model at finite temperature which, to the knowledge of the authors, is a system that has not yet been studied in the literature. Consequently, the first portion of the paper will be dedicated to defining a thermal state for the IKKT model, and studying its properties. Our approach to defining a thermal state for the IKKT model will build on known properties of string theory, where strings at finite temperature can be studied by compactifying the Euclidean time component of the string target space on a circle where space-time fermions acquire suitable anti-periodic boundary conditions (see [11] for relevant papers on this topic). Since the IKKT model provides a non-perturbative description of the Type IIB string target space, we postulate that the IKKT model at finite temperature can be studied by compactifying the Euclidean time component of the matrices on a circle where the fermionic matrices acquire anti-periodic boundary conditions. Following this prescription, we will evaluate the free energy and energy of the system, and study the thermal fluctuations which lead to structure formation.

The present paper is structured as follows. In section 4.2, we review some aspects of the IKKT matrix model and how space-time emerges from it. In section 4.3, we obtain the IKKT action at finite temperature by compactifying the Euclidean time component of the model on a thermal circle where fermions acquire boundary conditions. In section 4.4, we evaluate the free energy of the system at finite temperature in the low temperature regime. Finally, in section 4.5, we show how space-time can emerge with a scale-invariant spectrum of scalar and tensor perturbations.

[2]This model is described by bosonic matrices which describe space-time, and fermionic matrices which describe space-time fermions.

4.2 Review of the matrix models and emergent space

The IKKT model is a proposed non-perturbative formulation of Type IIB string theory. The idea behind this model is to describe the world sheet of the Type IIB string using large N matrices. To see how this can be done, let us start with the Green-Schwarz action of the Type IIB string:

$$S_{GS} = -T \int d^2\sigma \left(\sqrt{-h} + 2i\epsilon^{ab}\partial_a X^\mu \bar{\psi}\Gamma_\mu \partial_b \psi \right) , \qquad h_{ab} = \partial_a X^\mu \partial_b X_\mu \,. \tag{4.1}$$

Equivalently, the action above can be rewritten in the following "Schild" form

$$S_{Schild} = -T \int d\sigma^2 \sqrt{g} \left(-\frac{\alpha}{4}\{X^\mu, X^\nu\}^2 + \frac{i}{2}\bar{\psi}\Gamma^\mu \{X_\mu, \psi\} + \beta \right) , \tag{4.2}$$

where the Poisson brackets are defined by:

$$\{X^\mu, X^\nu\} = \frac{1}{\sqrt{g}} \epsilon^{ab}\partial_a X^\mu \partial_b X^\nu \,. \tag{4.3}$$

This new action depends on an auxiliary field $\sqrt{g}$, which satisfies the equation of motion

$$\sqrt{g} = \sqrt{\frac{\alpha}{2\beta}}\sqrt{-h} \quad , \quad h = \frac{1}{2}\{X^\mu, X^\nu\}^2 \tag{4.4}$$

Substituting the result above in the Schild action gives us

$$S_{Schild} = -T \int d^2\sigma \left(\sqrt{2\alpha\beta}\sqrt{-h} + \frac{i}{2}\bar{\psi}\Gamma^\mu \{X_\mu, \psi\} \right) , \tag{4.5}$$

which is equivalent to the Nambu-Goto action of the Type IIB string provided that $2\alpha\beta = 1$ and that we normalize the fermions. In the Schild formalism, the string dynamics is determined by the Poisson bracket $\{X^\mu, X^\nu\}$. By analogy with quantum mechanics, the Schild action can be discretised by replacing the Poisson brackets by a commutator and the integral by a trace.

$$\{\,,\,\} \implies -i[\,,\,] , \qquad \int d\sigma^2 \sqrt{g} \implies \mathrm{Tr}\,. \tag{4.6}$$

In this case, the target space coordinates X^μ and associated fermions ψ[3] are now described by large $N \times N$ matrices, and we obtain the IKKT model action

$$S_{IKKT} = -\frac{T}{4}\alpha\,\mathrm{Tr}\,[X^\mu, X^\nu]^2 - \frac{T}{2}\mathrm{Tr}\,\bar{\psi}\,\Gamma^\mu\,[X_\mu, \psi] - \beta\,\mathrm{Tr}1\,. \tag{4.7}$$

[3] We will suppress the spinor indices here and everywhere below to avoid clutter.

The last term in the action above is non-dynamical and can be neglected. Hence, by convention, we will use the following form for the IKKT model action

$$S_{IKKT} = -\frac{1}{4g^2}\mathrm{Tr}\,[A^\mu, A^\nu]^2 - \frac{1}{2g^2}\mathrm{Tr}\,\bar{\psi}\,\Gamma^\mu\,[A_\mu, \psi]\,, \tag{4.8}$$

where we have defined $X^\mu \equiv A^\mu$. The action above can also be obtained by dimensionally reducing the action of a 10-dimensional super Yang-Mills theory to a point. In all cases, the resulting action is invariant under SU(N) transformation. In the action above, the μ, ν is contracted with respect to the flat metric $\eta_{\mu\nu} = \mathrm{diag}(+, -, ..., -)$. Hence, this model is often referred to as the Lorentzian IKKT model. For the action to be supersymmetric in 10 dimensions, the fermions must be described by Majorana-Weyl spinors, which satisfy $\bar{\psi} = \psi\, C_{10}$. Here, C_{10} is the charge conjugation operator in ten dimensions, which we will take to satisfy $C_{10}\Gamma^\mu C_{10}^{-1} = -\Gamma^T$ and $C_{10}^T = -C_{10}$.

4.3 Compactification, SUSY breaking and thermodynamics

In superstring theory, one can obtain a thermal state of the string by compactifying the time direction of a Euclidean target space on a torus together with suitable anti-periodic boundary conditions for the target space fermions.[4] This antiperiodic boundary condition for the fermions is required to break supersymmetry, which is expected to occur in any thermal system. Given that the IKKT model describes the target space of the Type IIB string, one should expect to be able to construct a thermal state of the IKKT model by compactifying the matrices on such a torus. The precise sense in which this can be done has already been explored by Taylor [13] (symmetric boundary conditions for the fermions), Banks [14] (anti-symmetric boundary conditions for the fermions) and many other authors ([15], [16] and more). In the present section, we will build on the approach of these authors to compactify the IKKT model on a Eucledian time circle, and study the thermodynamic properties of this system.

[4] This choice of compactification is closely related to the Scherk-Schwartz orbifolding procedure [12].

4.3.1 Compactification and thermodynamics

Let us start by Euclideanising the IKKT action. For a Lorentzian metric in the mostly minus signature $(+,-,...,-)$, this can be done by the changes $A^0 \to iA^0$ and $\Gamma^i \to i\Gamma^i$ in the IKKT action (Equation 4.8). We obtain

$$S_{IKKT} = -\frac{1}{4g^2}\mathrm{Tr}[A^\mu, A^\nu]^2 - \frac{i}{2g^2}\mathrm{Tr}\left(\psi C_{10}\Gamma^\mu[A_\mu, \psi]\right) , \tag{4.9}$$

where the indices are now contracted with respect to the Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$. We will then compactify the time direction in the Euclidean IKKT model on a circle where the fermions acquire anti-periodic boundary conditions. To do this, let us presume the existence of an operator U which translates the matrices by an amount $2\pi\beta$ in the A^0 direction, where $\beta = 1/T$ is identified as the inverse temperature of the system. To impose the desired boundary conditions, we want this operator and the matrices A^μ and ψ to satisfy:

$$U^{-1}A^0U = A^0 + 2\pi\beta \tag{4.10}$$

$$U^{-1}A^iU = A^i \tag{4.11}$$

$$U^{-1}\psi U = -\psi\,. \tag{4.12}$$

In other words, A^i and ψ must have periodic and anti-periodic boundary conditions respectively, and A^0 must be periodic up to the circumference $2\pi\beta$ of the Euclidean time circle. These constraint equations can be solved by using operators q and p that satisfy the Heisenberg commutation relations $[q,p] = i$[5]. Let us consider the unitary operator

$$U = 1 \otimes e^{-i2\pi q}e^{-ip}\,, \tag{4.13}$$

which translates a state a eigenvector $|q\rangle$ of the operator q to $|q+1\rangle$ up to a phase $e^{-i2\pi q}$, and assume A^i and ψ take the form

$$A^i = \sum_n A^i_n \otimes e^{inp} \quad , \quad \psi = \sum_r \psi_r \otimes e^{irp}\,. \tag{4.14}$$

[5] Note that contrary to some papers in the literature, we will not assume that q is an integer. Here, q can take any value on the real axis just like in quantum mechanics. Let us remember that this is something we are allowed to do if A^μ and ψ are sufficiently large matrices, which we will be assuming here.

Here, we will assume the A_n^i's and ψ_r's are large $M \times M$ matrices that live in a Hilbert space different from the one of q and p, where we will take the latter to be described by large $N \times N$ matrices. Hence, A^i and ψ will be described by large $MN \times MN$ matrices. Using the ansatz of equation 6.21 and 4.14, the constraint equations 4.11 and 6.19 can easily be solved by invoking that n and r must be integers and half integers respectively. This result follows from the fact that the phase factor $e^{-i2\pi q}$ behaves like a fermion parity operator $(-1)^F$ in the sense that it commutes with the bosonic matrices A^μ and anti-commutes with the fermionic matrices ψ given the equation 4.14.[6] Hence, we obtain the desired boundary conditions for the bosonic and fermionic matrices. A^0 can be found in a similar way by adding a term of the form $2\pi\beta q$, which transforms like $U^{-1}(2\pi\beta q)U = 2\pi\beta$ under the action of U. We obtain

$$A^0 = \sum_{n \in \mathbb{Z}} A_n^0 \otimes e^{inp} + 1 \otimes 2\pi\beta q \,. \tag{4.15}$$

The resulting matrices A^μ and ψ, in the q-basis, can be written as

$$A^0_{q'q} = \langle q'|A^0|q\rangle = \sum_{n \in \mathbb{Z}} A_n^0 \otimes \delta_{q'-n,q} + 2\pi\beta q \delta_{q'q} \tag{4.16}$$

$$A^i_{q'q} = \langle q'|A^i|q\rangle = \sum_{n \in \mathbb{Z}} A_n^i \otimes \delta_{q'-n,q} \tag{4.17}$$

$$\psi_{q'q} = \langle q'|\psi|q\rangle = \sum_{r \in \mathbb{Z}+1/2} \psi_r \otimes \delta_{q'-r,q} \,. \tag{4.18}$$

The matrices above have (or at least partly in the case of A^0) some sort of Toeplitz structure in the sense that they satisfy $A^i_{q'q} = A^i_{q'-q}$ when q is an integer and $\psi_{q'q} = \psi_{q'-q}$ when q is a half-integer. This structure describes a system of mirror D-objects that live on the diagonal of $A^\mu_{q'q}$ and $\psi_{q'q}$. The system of mirror objects is composed of a fundamental region, described by A_0^μ and ϕ_0, which is translated by a distance $2\pi\beta$ for each adjacent diagonal block (See Figure 4.1). The off-diagonal elements, on the other hand, are related to interactions between each fundamental region.

These off-diagonal elements can also be associated to modes of a string winding around a circle. Independently of the q-basis chosen above, this can be seen by substituting the

[6] Notice that this is where our approach departs from Tom Banks approach in [14]. The matrices A^μ and ψ we obtain are slightly different in our case. However, they satisfy the desired properties given by equation 4.10, 4.11 and 6.19.

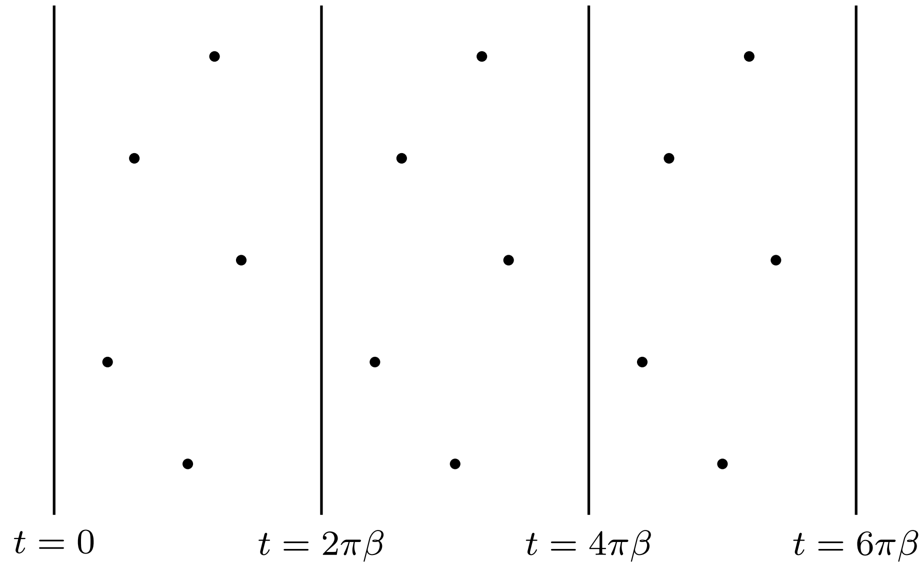

Figure 4.1: Sketch of mirror D-instantons in the duplicated fundamental regions along the Euclidean time direction. Each fundamental region, where the distribution of D-instantons is described by A_0^μ, has an infinite number of mirror regions located at a distance $t = 2\pi\beta n$ from each other along the A^0 direction, where n is an integer.

constrained values of A^μ and ψ in the IKKT action. We obtain

$$S_{IKKT} = \frac{N}{2g^2}\mathrm{Tr}\left(\sum_n (2\pi\beta n)^2 A^i_{-n}A^i_n + i\sum_r 2\pi\beta r\psi_{-r}C_{10}\Gamma^0\psi_r \right. \tag{4.19}$$

$$+\sum_{nm} 4\pi\beta n[A^0_{-n-m}, A^i_m]^2 A^i_n - \sum_{nml}[A^0_{-n-m-l}, A^i_l][A^0_m, A^i_n] \tag{4.20}$$

$$\left. -\frac{1}{2}\sum_{nml}[A^i_{-n-m-l}, A^j_l][A^i_m, A^j_n] - i\sum_{rn}\psi_{-r-n}C_{10}\Gamma^0[A^0_n, \psi_r] - i\sum_{rn}\psi_{-r-n}C_{10}\Gamma^i[A^i_n, \psi_r]\right) . \tag{4.21}$$

Here, we made use the identities

$$[q, e^{inp}] = -ne^{inp}\,, \qquad \mathrm{Tr}e^{i(n\pm m)p} = N\delta(n\pm m)\,, \tag{4.22}$$

and traced over the q, p degrees of freedom to simplify the sums over n and r. The compactified IKKT model action has the structure of a mode expansion, where the first two terms describe the contribution of bosonic and fermionic winding modes with frequencies

$\omega_n = 2\pi\beta n$ ($n \in \mathbb{Z}$ and $n \in \mathbb{Z} + 1/2$ for fermions), and the other term describe interactions between the different modes. In the decompactification limit $\beta \to \infty$, the non-zero modes become heavy and decouple from the system. The system is then well approximated by the bosonic IKKT model action

$$S_{IKKT} = -\frac{N}{4g^2}\mathrm{Tr}[A_0^\mu, A_0^\nu]^2 \,. \tag{4.23}$$

In the mirror objects picture, this means the mirror region are far from each other that the interactions between each region can be neglected. We recover N copies of the fundamental region, as reflected by the extra factor of N in the action. It's also interesting to note that fermions are projected away in the decompactification limit, which is a consequence of supersymmetry breaking.

4.3.2 Relation to the BFSS model at finite temperature

Given that the IKKT model is a description of the Type IIB string, it should be possible to link Equation 4.21 to the mode expansion of the Euclidean BFSS model, which is related to the Type IIA string, using T-duality. Let us start with the Euclidean BFSS action

$$S_{BFSS} = \frac{1}{2g^2}\int d\tau \mathrm{Tr}\left((D_\tau X^i)^2 - \frac{1}{2}[X^i, X^j]^2 + \bar{\psi}\Gamma^0 D_\tau\psi - i\bar{\psi}\Gamma^i[X^i, \psi]\right) , \tag{4.24}$$

where the covariant derivative is defined by

$$D_\tau X = \partial_t X - i[X^0, X] \,, \tag{4.25}$$

and the fermions are Majorana-Weyl spinors that satisfy $\bar{\psi} = C\psi$. Substituting the mode expansion

$$X^0 = \sum_n X_n^0 e^{i\omega nt} \quad , \quad X^i = \sum_n X_n^i e^{i\omega nt} \quad , \quad \psi = \sum_r \psi_r e^{i\omega rt} \tag{4.26}$$

in the BFSS action, we obtain

$$S_{BFSS} = \frac{\beta}{2g^2}\mathrm{Tr}\Bigg(\sum_n (2\pi Tn)^2 X_{-n}^i X_n^i + i\sum_r 2\pi Tr\psi_{-r}C_{10}\Gamma^0\psi_r \tag{4.27}$$

$$+\sum_{nm} 4\pi Tn[X_{-n-m}^0, X_m^i]^2 X_n^i - \sum_{nml}[X_{-n-m-l}^0, X_l^i][X_m^0, X_n^i] \tag{4.28}$$

$$-\frac{1}{2}\sum_{nml}[X_{-n-m-l}^i, X_l^j][X_m^i, X_n^j] - i\sum_{rn}\psi_{-r-n}C_{10}\Gamma^0[X_n^0, \psi_r] - i\sum_{rn}\psi_{-r-n}C_{10}\Gamma^i[X_n^i, \psi_r]\Bigg) , \tag{4.29}$$

where we have used $\omega = 2\pi/T$. Letting $T \to 1/T$, we recover the mode expansion of the IKKT model (equation 4.21) up to a normalization of the gauge coupling g^2. This implies that the thermodynamics of the IKKT and BFSS are related by T-duality. For example, the high-temperature limit ($T \to \infty$) will be related to the low-temperature limit ($T \to 0$) of the IKKT model. For the BFSS model, the high-temperature limit is a perturbative limit. Similarly, the low-temperature limit of the IKKT model will also be a perturbative limit, which we will explore in the following section.

Notice also that the diagonal elements of the compact IKKT action become the Matsubara frequencies of the BFSS model under T-duality. This property is a consequence of the Toeplitz structure of the compactified A^μ and ψ matrices, which was recently utilized to rewrite quantum field theories compactified on a Toroidal space-time in terms of Toeplitz matrices [17].

4.4 Free energy of the IKKT model at finite temperature

In the IKKT model, space-time time emerges from the bosonic A^μ matrices. As we saw in the previous section, this picture becomes different once the IKKT model is compactified. Instead of describing a single system, the IKKT describes an infinite number of copies of the same system that interact more and more with each other as we increase the temperature. In the zero temperature limit, one recovers N copies of the fundamental region which are far away from each other, and hence do not interact with each other. To understand the thermodynamics of this system, we will treat the low temperature limit of the IKKT model in the same way as was done in [18] for the high temperature limit of the BFSS model. We will integrate out the interactions (non-zero winding modes) in order to obtain the effective free energy felt in the fundamental regions (zero modes) as a function of temperature. This will later allow us to study thermodynamic properties of the fundamental regions (*e.g.,* thermal fluctuations), which are relevant for cosmology.

4.4.1 Gauge fixing and other considerations

Before computing the free energy, there are some considerations we have to make. First, let us choose an appropriate gauge fixing to evaluate the path integral. To do this, note that the compactification procedure is equivalent to studying fluctuations of the matrices A^μ and ψ around a background where $A^0 = 2\pi\beta q$, $A^i = 0$ and $\psi = 0$. In other words, we are imposing

$$A^\mu = X^\mu + \tilde{A}^\mu \tag{4.30}$$

$$\psi = \xi + \tilde{\psi}\,, \tag{4.31}$$

where we choose

$$X^0 = 2\pi\beta q \quad , \quad X^i = 0 \quad , \quad \tilde{A}^\mu = \sum_n A^\mu_n e^{inp} \tag{4.32}$$

$$\xi = 0 \quad , \quad \tilde{\psi} = \sum_r \psi_r e^{irp}\,. \tag{4.33}$$

Here, the only difference is that we are also imposing that the fluctuation matrices $\tilde{A}^\mu$ and $\tilde{\psi}$ are symmetric and anti-symmetric under the action of the unitary operator in equation 6.21. Such expansions have been studied extensively in [7] and [19]. In these cases, the appropriate gauge fixing condition is

$$P_\mu A^\mu = 0\,, \tag{4.34}$$

where P^μ is the adjoint operator

$$P_\mu Y = [X_\mu, Y] \tag{4.35}$$

associated to the background matrices X^μ. In the case at hand, imposing the gauge condition projects out the non-zero modes of X^0, giving us $X^0 = A^0_0$. This gauge can be fixed by adding a ghost part

$$S_{gh} = -\frac{1}{g^2}\mathrm{Tr}\left([X_\mu, \bar{c}][A^\mu, c]\right) \tag{4.36}$$

to the IKKT action. Here, the ghost matrices c will also be compact and hence take the form

$$c = \sum_{n\in\mathbb{Z}} c_n e^{inp} \tag{4.37}$$

Tracing out the q and p degrees of freedom in the ghost action, we obtain the following mode expansion

$$S_{gh} = \frac{N}{g^2}\sum_n (2\pi\beta n)^2 \mathrm{Tr}(\bar{c}_n c_n) - \frac{N}{g^2}\sum_{nm} 2\pi\beta(n+m)\mathrm{Tr}(\bar{c}_{n+m}[A^0_m, c_n])\,. \tag{4.38}$$

As a second consideration, we will assume the gamma matrices Γ^μ and the charge conjugation operator C_{10} are in the following representation:

$$\Gamma^0 = 1_{16} \otimes \sigma_1 \tag{4.39}$$

$$\Gamma^i = \gamma^i \otimes \sigma_2 \tag{4.40}$$

$$C_{10} = C_9 \otimes i\sigma_2 \tag{4.41}$$

Here, the γ^i's are a set of nine-dimensional (Euclidean) gamma matrices which satisfy $\{\gamma^i, \gamma^j\} = 2\delta_{ij}$, and C_9 is the associated charge conjugation matrix satisfying $C_9\gamma^i C_9^{-1} = \gamma^{iT}$. We will also choose γ^i to be in the Majorana representation (where the nine γ^i are taken to be real and symmetric), in which case the charge conjugation matrix takes the simple form $C_9 = 1_{16}$. Finally, we will impose the following choice of Majorana-Weyl spinor:

$$\psi = \phi \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}\,. \tag{4.42}$$

Here, ϕ is a sixteen-component Majorana fermion.

Given the considerations above, the compactified IKKT action, including the ghosts, can be rewritten as a sum of the zero mode action S_0, winding mode terms S_w and interaction terms S_{int} in the following way

$$S_{IKKT} = S_0 + S_w + S_{int} \tag{4.43}$$

$$S_0 = -\frac{N}{4g^2}\mathrm{Tr}[A_0^\mu, A_0^\nu]^2 \tag{4.44}$$

$$S_w = \frac{N}{g^2}\mathrm{Tr}\left(\frac{1}{2}\sum_{n\neq 0}(2\pi n\beta)^2 A^0_{-n}A^0_n + \frac{1}{2}\sum_{n\neq 0}(2\pi n\beta)^2 A^i_{-n}A^i_n + \frac{i}{2}\sum_r 2\pi r\beta\phi_{-r}\phi_r + \sum_n (2\pi n\beta)^2\bar{c}_n c_n\right) \tag{4.45}$$

$$S_{int} = -\frac{N}{2g^2}\mathrm{Tr}\left(\sum_{n=m\neq 0} 4\pi n\beta[A^0_{-n-m}, A^i_m]^2 A^i_n + \sum_{n=m=l\neq 0}[A^0_{-n-m-l}, A^i_l][A^0_m, A^i_n]\right. \tag{4.46}$$

$$+\sum_{n=m=l\neq 0}\frac{1}{2}[A^i_{-n-m-l}, A^j_n][A^i_m, A^j_l] + i\sum_{rn}\phi_{-r-n}[A^0_n, \phi_r] - \sum_{nr}\phi_{-r-n}\gamma^i[A^i_n, \phi_r] \tag{4.47}$$

$$\left.+\sum_{nm} 4\pi(n+m)\beta\bar{c}_{n+m}[A^0_m, c_n]\right)\,, \tag{4.48}$$

where $\sum_{n=m\neq 0}$ and $\sum_{n=m=l\neq 0}$ respectively imply that the $m = n \neq 0$ and $m = n = l = 0$ terms are excluded from the sum. Notice that we added a winding mode term for A^0 in the action. We can do this because the gauge condition $P_\mu A^\mu = 0$ imposes that all A^0_n's are zero when $n \neq 0$. Hence, adding the winding mode term for A^0 is equivalent to adding zero in the action. Since we recovered a winding mode term for A^0, it's useful to rewrite S_{IKKT} in the more condensed form

$$S_{IKKT} = S_0 + S_w + S_{int} \tag{4.49}$$

$$S_0 = -\frac{g^2}{4N}\mathrm{Tr}[A_0^\mu, A_0^\nu]^2 \tag{4.50}$$

$$S_w = \mathrm{Tr}\left(\frac{1}{2}\sum_{n\neq 0}(2\pi\beta n)^2 A^\mu_{-n}A^\mu_n + \frac{i}{2}\sum_r 2\pi\beta r\phi_{-r}\phi_r + \sum_n(2\pi\beta n)^2\bar{c}_n c_n\right) \tag{4.51}$$

$$S_{int} = -\frac{1}{2}\sqrt{\frac{g^2}{N}}\sum_{n=m\neq 0} 4\pi\beta n\mathrm{Tr}([A^0_{-n-m}, A^i_m]^2 A^i_n) \tag{4.52}$$

$$-\frac{1}{4}\frac{g^2}{N}\sum_{n=m=l\neq 0}\mathrm{Tr}([A^\mu_{-n-m-l}, A^\nu_l][A^\mu_m, A^\nu_n]) - \frac{i}{2}\sqrt{\frac{g^2}{N}}\sum_{nr}\mathrm{Tr}(\phi_{-r-n}[A^0_n, \phi_r]) \tag{4.53}$$

$$+\frac{1}{2}\sqrt{\frac{g^2}{N}}\sum_{nr}\mathrm{Tr}(\phi_{-r-n}\gamma^i[A^i_n, \phi_r]) - \frac{1}{2}\sqrt{\frac{g^2}{N}}\sum_{nm}4\pi(n+m)\mathrm{Tr}(\bar{c}_{n+m}[A^0_m, c_n])\,, \tag{4.54}$$

were we made the redefinitions $A^\mu_n \to \sqrt{g^2/N} A^\mu_n$, $\psi_r \to \sqrt{g^2/N}\psi_r$ and $\psi_r \to \sqrt{g^2/N}\psi_r$ to get rid of the N and g^2 dependance in S_w. The form above will be useful when evaluating the free energy. As we can see from the action above, the SO(10) symmetry of the system is explicitly broken when the temperature of the system is non-zero. The symmetry is restored when $T \to 0$ and the zero modes dominate the effective action.

4.4.2 Free energy at leading order

Let us now derive the free energy of the system at leading order. We start with the partition function

$$Z = \prod_{lrmn} \int DA^\mu_l D\phi_r D\bar{c}_m Dc_n \, e^{-S_0 - S_w - S_{int}} \tag{4.55}$$

$$= Z_0 Z_w \langle e^{-S_{int}} \rangle \,, \tag{4.56}$$

of the compact IKKT action. As shown above, this partition function can be split into a contribution from the zero-mode partition function Z_0, the winding modes partition function Z_w and the expectation value of the interaction terms $\langle e^{-S_{int}} \rangle$. Here, Z_0, Z_w and $\langle . \rangle$ are defined as.

$$Z_0 = \int DA^\mu_0 e^{-S_0} \quad , \quad Z_w = \prod_{l \neq 0} \prod_{rmn} \int DA^\mu_l D\phi_r D\bar{c}_m Dc_n \, e^{-S_w} \tag{4.57}$$

$$\langle \, . \, \rangle = \frac{1}{Z_0 Z_w} \prod_{lrmn} \int DA^\mu_l D\phi_r D\bar{c}_m Dc_n \;.\; e^{-S_0 - S_w} \tag{4.58}$$

Given the partition functions above, the free energy

$$F = -T \ln Z \tag{4.59}$$

$$= -T \ln Z_0 - T \ln Z_w - T \ln \langle e^{-S_{int}} \rangle \,, \tag{4.60}$$

can be evaluated perturbatively. At leading order in perturbation theory, only the first two terms contribute significantly to the free energy. The first term, which depends on $\ln Z_0$, is not very interesting since $\ln Z_0$ does not depend on β. Hence, we won't pay much attention to it. For the second term, the contribution to the energy can be found by carrying out a series of Gaussian integrals. We first split the winding modes partition function in a bosonic part Z_b, a fermionic part Z_f, and a ghost part Z_{gh} in the following way:

$$Z_w = Z_b Z_f Z_{gh}\,, \tag{4.61}$$

$$Z_b = \prod_{n\neq 0}\int DA_n^\mu \, e^{-\frac{1}{2}\mathrm{Tr}\left(\sum_{n\neq 0}(2\pi\beta n)^2 A^\mu_{-n}A^\mu_n\right)}\,, \tag{4.62}$$

$$Z_f = \prod_r \int D\phi_r \, e^{-i\frac{1}{2}\mathrm{Tr}\left(\sum_r 2\pi\beta r\phi_{-r}\phi_r\right)}\,, \tag{4.63}$$

$$Z_{gh} = \prod_{nm} D\bar{c}_n Dc_m \, e^{-\mathrm{Tr}\left(\sum_n (2\pi\beta n)^2\bar{c}_n c_n\right)}\,. \tag{4.64}$$

The Gaussian integrals above can be carried out to obtain the expression below:

$$Z_b = \left(\prod_{n\neq 0}(2\pi\beta n)^2\right)^{-DM^2/2}\,, \tag{4.65}$$

$$Z_f = \left(\prod_r 2\pi\beta i r\right)^{pM^2/2}\,, \tag{4.66}$$

$$Z_{gh} = \left(\prod_{n\neq 0}(2\pi\beta n)^2\right)^{M^2}\,. \tag{4.67}$$

Here, D is the number of space-time dimensions and p is the dimension of the ϕ spinors, which are respectively $D = 10$ and $p = 16$ in the present case. However, we will keep D and p arbitrary for the sake of generality. The products above are manifestly divergent. However, these divergences can be tamed using the identities

$$\prod_{n=1}^{\infty}\left(\frac{2\pi n}{\alpha}\right)^{-2} = \frac{1}{\alpha} \quad , \quad \prod_{n=1}^{\infty}\left(\frac{2\pi(n-1/2)}{\alpha}\right)^{2} = 2 \tag{4.68}$$

found by zeta function regularisation (See Appendix 4.7). Using the identities above, we obtain

$$Z_b = \beta^{DM^2/2}\,, \tag{4.69}$$

$$Z_f = 2^{pM^2/2}\,, \tag{4.70}$$

$$Z_{gh} = \beta^{-M^2}\,. \tag{4.71}$$

Including all the terms above, we obtain the following contribution to the free energy at leading order

$$F_{leading} = -T\ln Z_0 - T\ln Z_w \tag{4.72}$$

$$= -T\ln Z_0 - TM^2(D-2)\ln(\beta) - \frac{1}{2}TpM^2\ln 2\,, \tag{4.73}$$

and the following expression for the energy:

$$E_{leading} = -\partial_\beta \ln Z_0 - \partial_\beta \ln Z_w = -M^2(D-2)T\,. \tag{4.74}$$

Here, $\ln Z_0$ is a constant that does not depend on the temperature of the system, and hence does not contribute to the energy. Note that unlike for the BFSS model, the leading order contribution to the energy is negative. This is a consequence of the fact that the Matsubara frequencies of the system are winding modes, and not Kaluza-Klein modes of a field compactified on a thermal circle like in thermal field theory. The positive sign found for the BFSS model can be recovered by letting $T \to 1/T$ in the partition function, and computing the energy again using equation 4.74. Also note that, similar to the BFSS model (or all supersymmetric theories for that matter), the breaking of supersymmetry plays an important role in obtaining an non-vanishing contribution to the energy at leading order. If supersymmetry was restored by giving the fermions periodic boundary conditions, the contribution from the bosonic sector, the fermionic sector and the ghosts would cancel giving $\ln Z_w = 0$ and $E_{leading} = 0$, and leaving only a contribution from $\ln Z_0$ to the free energy.

This is to be expected since, as we mentioned before, our chosen compactification is equivalent to perturbing the action around a background X^μ where $X^0 = 2\pi\beta q$, $X^i = 0$ and $\xi = 0$, and imposing that the bosonic and fermionic fluctuations have periodic and anti-periodic boundary conditions respectively. Such backgrounds describe a distribution of D-instantons that satisfy the BPS condition $F_{\mu\nu} = i[P_\mu, P_\nu] = 0$. In this case, the one-loop partition function is known to vanish as a consequence of supersymmetry. However, supersymmetry is broken here because of our choice of boundary conditions, so we obtain a non-vanishing contribution to the energy.

4.4.3 Free energy at next to leading higher order

The next to leading order terms in perturbation theory can be evaluated by expanding $\langle e^{-S_{int}}\rangle$ and evaluating the expectation values. To obtain an effective description of the zero modes of the theory, we will evaluate the expectation values $\langle\langle\,.\,\rangle\rangle$ associated to the non-zero modes of theory in order to obtain an expression that depends on the expectation values $\langle . \rangle_0$ related to the zero modes in the theory. The resulting expression will give us corrections to the zero-mode effective action. Here, the expectation value $\langle\langle\,.\,\rangle\rangle$ with respect to the non-zero modes is defined by

$$\langle\langle\,.\,\rangle\rangle = \frac{1}{Z_w}\prod_{l\neq 0}\prod_{rmn}\int DA_l^{\mu} D\phi_r D\bar{c}_m Dc_n\ .\ e^{-S_w}\,. \tag{4.75}$$

and the expectation value $\langle . \rangle_0$ with respect to the zero modes A_0^{μ} is defined by

$$\langle\,.\,\rangle_0 = \frac{1}{Z_0}\int DA_0^{\mu} e^{-S_0}\,. \tag{4.76}$$

To evaluate the correction terms, it's useful to write down the two-point functions associated to the bosonic matrices, the fermionic matrices and the ghosts:

$$\langle\langle (A_m^{\mu})_{ab}(A_n^{\nu})_{cd}\rangle\rangle = \frac{\delta_{\mu\nu}\delta_{n+m,0}\delta_{ad}\delta_{bc}}{(2\pi\beta n)^2} \tag{4.77}$$

$$\langle\langle (\phi_r^{\alpha})_{ab}(\phi_s^{\beta})_{cd}\rangle\rangle = \frac{\delta_{\beta\alpha}\delta_{r+s,0}\delta_{ad}\delta_{bc}}{2\pi\beta i r} \tag{4.78}$$

$$\langle\langle (\bar{c}_m)_{ab}(c_n)_{cd}\rangle\rangle = \frac{\delta_{nm}\delta_{ad}\delta_{bc}}{(2\pi\beta n)^2}\,. \tag{4.79}$$

Here, α, β and a, b, c, d are respectively spinor and matrix indices. We will also separate the interaction part of the action into five different interaction terms in the following way:

$$S_{int} = \sum_{p=1}^{5} V_p\,. \tag{4.80}$$

Here, each interaction term can be written as:

$$V_1 = -\frac{g^2}{4N} \sum_{n=m=l\neq 0} \mathrm{Tr}\left([A^\mu_{-n-m-l}, A^\nu_l][A^\mu_m, A^\nu_n]\right) \,, \tag{4.81}$$

$$V_2 = -\frac{1}{2}\sqrt{\frac{g^2}{N}} \sum_{n=m\neq 0} 4\pi\beta n \mathrm{Tr}\left([A^0_{-n-m}, A^i_m]A^i_n\right) \,, \tag{4.82}$$

$$V_3 = -\frac{i}{2}\sqrt{\frac{g^2}{N}} \sum_{nr} \mathrm{Tr}\left(\phi_{-r-n}[A^0_n, \phi_r]\right) \tag{4.83}$$

$$V_4 = \frac{1}{2}\sqrt{\frac{g^2}{N}} \sum_{nr} \mathrm{Tr}\left(\phi_{-r-n}\gamma^i[A^i_n, \phi_r]\right) \tag{4.84}$$

$$V_5 = -\sqrt{\frac{g^2}{N}} \sum_{nm} 2\pi(n+m)\beta \mathrm{Tr}(\bar{c}_{n+m}[A^0_m, c_n]) \,. \tag{4.85}$$

Using the interaction terms above, the corrections to the effective action at two-loop order can be written as

$$\ln\langle e^{-S_{int}}\rangle = -\langle V_1\rangle + \frac{1}{2}\langle V_2^2\rangle + \frac{1}{2}\langle V_3^2\rangle + \frac{1}{2}\langle V_4^2\rangle + \frac{1}{2}\langle V_5^2\rangle + \ldots \,, \tag{4.86}$$

where each expectation value in the expansion above is given by

$$\langle V_1\rangle = \frac{(D-1)}{12}\frac{MT^2g^2}{N}\mathrm{Tr}\langle A^\mu_0 A^\mu_0\rangle_0 + \mathcal{O}(T^4) \tag{4.87}$$

$$\langle V_2^2\rangle = \frac{(D-1)}{3}\frac{MT^2g^2}{N}\mathrm{Tr}\langle A^0_0 A^0_0\rangle_0 + \frac{1}{6}\frac{MT^2g^2}{N}\mathrm{Tr}\langle A^i_0 A^i_0\rangle_0 + \mathcal{O}(T^4) \tag{4.88}$$

$$\langle V_3^2\rangle = -\frac{p}{4}\frac{MT^2g^2}{N}\mathrm{Tr}\langle A^0_0 A^0_0\rangle_0 + \mathcal{O}(T^4) \tag{4.89}$$

$$\langle V_4^2\rangle = \frac{p}{4}\frac{MT^2g^2}{N}\mathrm{Tr}\langle A^i_0 A^i_0\rangle_0 + \mathcal{O}(T^4) \tag{4.90}$$

$$\langle V_5^2\rangle = -\frac{1}{6}\frac{MT^2g^2}{N}\mathrm{Tr}\langle A^0_0 A^0_0\rangle_0 + \mathcal{O}(T^4) \,. \tag{4.91}$$

Here, we made use of the sum identities

$$\sum_{n\neq 0}\frac{1}{(2\pi n)^2} = \frac{1}{12} \quad , \quad \sum_r \frac{1}{(2\pi r)^2} = \frac{1}{4} \tag{4.92}$$

to evaluate the expectation values. Moreover, we only kept the terms to quadradic order in temperature, which provides the dominant next to leading order contribution. Adding all the terms together and restoring the initial dimensions of the zero modes by letting

$A_0^\mu \to \sqrt{N/g^2} A_0^\mu$, we obtain the following expression for the corrections to the effective action at two-loop order:

$$\ln\langle e^{-S_{int}}\rangle = \left(\frac{D-2}{12} - \frac{p}{8}\right) MT^2 \left(\mathrm{Tr}\langle A_0^0 A_0^0\rangle_0 - \mathrm{Tr}\langle A_0^i A_0^i\rangle_0\right) + \mathcal{O}(T^4)\,. \tag{4.93}$$

Here again, the contribution above is a consequence of broken supersymmetry. If we were to restore supersymmetry by giving the fermions periodic boundary conditions, the $p/8$ prefactor in the expression above would get replaced by $p/24$, leading to a cancelation of the term above.

Note that the expression above can be explicitly related to the expectation value of the extent of space $\langle R^2\rangle_0 = \frac{1}{M}\mathrm{Tr}\langle A_0^i A_0^i\rangle_0$ of the system and what we could define as the expectation value of the extent of time $\langle T^2\rangle_0 = \frac{1}{M}\mathrm{Tr}\langle A_0^0 A_0^0\rangle_0$. Here, $\langle R^2\rangle_0$ and $\langle T^2\rangle_0$ should be respectively viewed as the characteristic size of space and duration of time in the fundamental regions. Since $SO(10)$ symmetry is preserved at tree level, we expect the distribution of eigenvalues of A^0 to be similar to the distribution of eigenvalues of A^i. Consequently, we expect that $\langle T^2\rangle_0$ can be approximated by

$$\langle T^2\rangle_0 \approx \frac{1}{D-1}\langle R^2\rangle_0\,. \tag{4.94}$$

Substituting the expression above in the effective action, we obtain the following correction to the free energy

$$F_{next} = -T\ln\langle e^{-S_{int}}\rangle = \left(\frac{D-2}{12} - \frac{p}{8}\right)\frac{D-2}{D-1} M^2 T^3 \langle R^2\rangle_0 \tag{4.95}$$

and the following correction to the energy of the system

$$E_{next} = -\partial_\beta \ln\langle e^{-S_{int}}\rangle = -2\left(\frac{D-2}{12} - \frac{p}{8}\right)\frac{D-2}{D-1} M^2 T^3 \langle R^2\rangle_0\,. \tag{4.96}$$

4.5 Application to early universe cosmology

In the last section, we studied the IKKT model at finite temperature and derived its free energy up to the next to leading order. Let us now consider a scenario where the universe emerges in a thermal state described by the IKKT model at finite temperature. Here, we will assume a 4 dimensional universe emerges and that thermal fluctuations are sourced by a four-dimensional version of the IKKT model ($D = p = 4$).

If the theory of linear cosmological perturbations apply, we can show that the spectrum of scalar and tensor perturbations sourced by thermal fluctuations of the IKKT model is scale invariant following the prescription given in [4]. To do this, let us assume space-time is described by the following longitudinal gauge metric

$$ds^2 = (1 + 2\Phi)dt^2 - a(t)^2[(1 - 2\Phi)\delta_{ij} + h_{ij}]dx^i dx^j \tag{4.97}$$

where Φ is the relativistic generalisation of the Newtonian gravitational potential, h_{ij} is a transverse traceless tensor which describes excitations of the metric due to gravitational waves, and $a(t)$ is the scale-factor of an arbitrary cosmological background.

According to linear cosmological perturbation theory, the amplitude of curvature fluctuations on a scale k , where k denotes the comoving wave number, is related to energy fluctuations on that scale via

$$\langle|\Phi(k)|^2\rangle = 16\pi^2 G^2 k^{-4} \langle \delta T^0_0(k) \delta T^0_0(k)\rangle \,, \tag{4.98}$$

where T^μ_ν is the energy-momentum tensor of matter, and G is Newton's gravitational constant. Similarly, the amplitude $h(k)$ of tensor perturbations can be related to transverse pressure fluctuations via

$$\langle|h(k)|^2\rangle = 16\pi^2 G^2 k^{-4} \langle \delta T^i_j(k) \delta T^i_j(k)\rangle \quad , \quad i \neq j \,. \tag{4.99}$$

If matter is in a thermal state, then the amplitude of density and transverse pressure perturbations in a box of radius R can be found from the finite temperature partition function of the system. Specifically, since

$$\langle T^\mu_\nu \rangle = 2 \frac{g^{\mu\lambda}}{\sqrt{-g}} \frac{\partial \ln Z}{\partial g^{\nu\lambda}} \,, \tag{4.100}$$

the fluctuations of the stress-energy tensor can be expressed as

$$\langle \delta T^\mu_\nu(k) \delta T^\sigma_\lambda(k)\rangle \equiv \langle T^\mu_\nu T^\sigma_\lambda \rangle - \langle T^\mu_\nu \rangle \langle T^\sigma_\lambda \rangle \tag{4.101}$$

$$= 2 \frac{g^{\mu\alpha}}{\sqrt{-g}} \frac{\partial}{\partial g^{\alpha\nu}} \left(\frac{g^{\sigma\delta}}{\sqrt{-g}} \frac{\partial \ln Z}{\partial g^{\delta\lambda}} \right) + 2 \frac{g^{\sigma\alpha}}{\sqrt{-g}} \frac{\partial}{\partial g^{\alpha\lambda}} \left(\frac{g^{\mu\delta}}{\sqrt{-g}} \frac{\partial \ln Z}{\partial g^{\delta\nu}} \right) \tag{4.102}$$

The expression above may seem complicated. However, component-wise, it can be expressed in terms of rather simple thermodynamic observables. To obtain these expressions, we first move to the position space representation of the matter perturbation correlation functions using

$$\langle \delta T^0_0(k) \delta T^0_0(k) \rangle = k^{-3} \langle \delta T^0_0(k) \delta T^0_0(k) \rangle_R \,, \tag{4.103}$$

$$\langle \delta T^i_j(k) \delta T^i_j(k) \rangle = k^{-3} \langle \delta T^i_j(k) \delta T^i_j(k) \rangle_R \,, \tag{4.104}$$

where $\langle . \rangle_R$ denotes the expectation value in the position space representation. Then, the energy density correlation function can be expressed in terms of the heat capacity

$$C_V = \left(\frac{\partial E}{\partial T} \right)_V \tag{4.105}$$

of the system in the following way

$$\langle \delta T^0_0(k) \delta T^0_0(k) \rangle_R = \langle \delta \rho^2 \rangle_R = \langle \rho^2 \rangle_R - \langle \rho \rangle_R^2 = \frac{T^2}{R^6} C_V \,. \tag{4.106}$$

Similarly, the correlation function of the off-diagonal spatial components of the stress tensor

$$\langle \delta T^i_j(k) \delta T^i_j(k) \rangle_R = \langle (T^i_j)^2 \rangle_R - \langle T^i_j \rangle_R^2 \quad , \quad i \neq j \tag{4.107}$$

can be related to transverse pressure perturbations in the following way

$$\langle \delta T^i_j(k) \delta T^i_j(k) \rangle_R = \alpha \frac{T}{R^2} \frac{\partial \tilde{p}}{\partial R} \,, \tag{4.108}$$

where the pressure $\tilde{p}$ is related to the free energy F of the system via

$$\tilde{p} = -\frac{1}{V} \frac{\partial F}{\partial \ln R} \,. \tag{4.109}$$

Here, α is a suppression factor of the transverse pressure perturbations compared to the diagonal pressure perturbations which satisfies $|\alpha| < 1$.

If we assume that matter behaves in the way described by the thermodynamics of the IKKT model, then it's possible to find the spectrum of cosmological perturbations sourced by the IKKT model from the effective action derived in Section 4.4. Let us start with the power spectrum of scalar cosmological perturbations, which is defined by

$$P_\Phi(k) = k^3 \langle |\Phi(k)|^2 \rangle \,. \tag{4.110}$$

Making use of equations 4.98 and 4.104, the power spectrum can then be related to the position space density fluctuations via

$$P_\Phi(k) = 16\pi^2 G^2 k^{-4} \langle \delta\rho^2(k) \rangle_R \,. \tag{4.111}$$

As we saw before, the density perturbations can be related to the heat capacity of the system via equation 4.106. This gives us

$$P_\Phi(k) = 16\pi^2 G^2 k^2 T^2 C_V (kR)^{-6} \,. \tag{4.112}$$

As we can see above, the spectrum of fluctuations can be scale invariant as long as $C_V \sim k^{-2}$. This feature is known to arise in String Gas Cosmology, and in an emergent scenario involving the BFSS model as shown recently. Indeed, we will now show that this feature also arises when considering the thermodynamics of the IKKT model. Using the expression for the energy derived in section 4.4, find the heat capacity

$$C_V = -N^2(D-2) + 6\left(\frac{p}{8} - \frac{D-2}{12}\right)\frac{D-2}{D-1} M^2 T^2 \langle R^2 \rangle_0 \,. \tag{4.113}$$

The first term gives a contribution to the power spectrum proportional to k^2, which is subdominant on large scales. The second term, however, gives us a scale-invariant contribution since $\langle R^2 \rangle_0 \sim k^{-2}$. Putting everything together, the scale-invariant contribution to the power spectrum gives us

$$P_\Phi(k) = 96\pi^2 G^2 (kR)^{-4} \left(\frac{p}{8} - \frac{D-2}{12}\right)\frac{D-2}{D-1} M^2 T^4 \tag{4.114}$$

Similarly, one can evaluate the power spectrum of tensor fluctuations from the position space representation of transverse matter perturbations using

$$P_h(k) = k^3 \langle |h(k)|^2 \rangle \,, \tag{4.115}$$

which, using equations 4.99 and 4.104, can be related to the transverse matter fluctuations in the following way:

$$P_h(k) = 16\pi^2 G^2 k^{-4} \langle \delta T^i_j(k) \delta T^i_j(k) \rangle_R \,. \tag{4.116}$$

Making use of equations 4.108 and 4.109 and the free energy derived in section 4.4, we obtain

$$\langle \delta T^i_j(k) \delta T^i_j(k) \rangle_R = 2\alpha \left(\frac{D-2}{12} - \frac{p}{8}\right)\frac{D-2}{D-1} M^2 T^4 R^{-4} \,. \tag{4.117}$$

This gives us a contribution to the spectrum of cosmological perturbations of the form

$$P_h(k) = 32\pi^2 G^2 (kR)^{-4} \alpha \left(\frac{D-2}{12} - \frac{p}{8} \right) \frac{D-2}{D-1} M^2 T^4 \,, \tag{4.118}$$

which is also scale-invariant. Note that for equation 8.11 to be positive, α must be a negative quantity. Hence, we expect the diagonal and off-diagonal pressure perturbations to have opposite sign in this system.

Comparing the results (8.9) and (8.11), we find that the tensor-to-scalar ratio is given by

$$r = \frac{|\alpha|}{3} \,. \tag{4.119}$$

In order to be consistent with the current observational bound on r, the value of α needs to be of the order $\mathcal{O}(10^{-1})$. This is slightly better than for cosmological perturbations sourced from thermal fluctuations of the BFSS model, where we need α to be of the order $\mathcal{O}(10^{-2})$ to obtain a result consistent with observations. Recall that although the transverse pressure perturbations are naturally smaller compared to the diagonal ones for thermal perturbations, they are not expected to get fine-tuned to be extremely small. Hence, we expect α to be smaller than one, but not by many orders in magnitude. In this sense, the IKKT model at finite temperature is more likely to source perturbations with the correct value of r than the BFSS model at finite temperature.

4.6 Conclusion

In this paper, we began a study of matrix model thermodynamics and suggested an emergent scenario in which a non-singular cosmology emerges from a thermal system described by the IKKT model at finite temperature. Inspired by string thermodynamics, we defined the IKKT model at finite temperature by compactifying its Euclidean time matrix on a circle where fermions acquire anti-periodic boundary conditions. We found that if the early universe emerges in a thermal state of the IKKT model, then structure formation can be sourced by thermal fluctuations of the IKKT model at finite temperature, which yield scale invariant scalar and tensor perturbations.

So far, we have assumed that the universe transitions to the radiation-dominated phase of Standard Big Bang cosmology after the emergent phase. However, as discussed in section

4.3, the low-temperature regime of the theory is dominated by the bosonic IKKT action. Hence, it's possible that the late-time dynamics of the system can be described by known cosmological solutions of the bosonic IKKT model (e.g. [20]) with perhaps some thermal corrections. This scenario could share interesting similarities with new numerical results [21] involving the IKKT model which suggests space-time can emerge accompanied by a transition from a Euclidean space-time metric to a Lorentzian space-time metric. In our scenario, the Euclidean portion of our space-time could be described by a thermal state of the IKKT model which transitions into a Lorentzian space-time described by the bosonic IKKT model plus some thermal corrections. The details of this late time transition and the subsequent time evolution of the universe needs to be worked out in future work.

Our early universe model shares many properties of String Gas Cosmology, where the evolution of the universe is driven by a gas of strings at finite temperature. In comparison, our model relies on a matrix description of strings at finite temperature, which yields similar results. Namely, we obtain an emergent 4-dimensional space-time from superstring theory, which lives in 10 dimensions, and thermal fluctuations in the emergent universe yield a scale-invariant spectrum of perturbations. The details of the symmetry breaking have not been discussed here. However, numerical evidence suggests that such a process is realizable in matrix theory [22]. The details of this transition are currently under study (see [23] for progress on this topic, also see for an [24] alternate approach to trying to solve this problem). It could be interesting to figure out the connection between these results for the traditional world sheet description of the superstring, and its matrix description. In doing so, perhaps matrix theory could give us a better understanding of String Gas Cosmology at early times.

In addition to it's resemblance with String Gas Cosmology, our new scenario shares interesting similarities with another recent emergent scenario [25] where the early universe begins in a topological phase, which then transitions to Standard Big Bang Cosmology with a scale-invariant spectrum of cosmological perturbations. Large N matrix models are known to have interesting topological properties [26]. Hence, it would be interesting to understand how (and, if) the two scenarios are related to one another. This could be the another subject of further study.

Acknowledgements

S.L. and S.B. wish to thank Simon Caron-Huot for useful discussions that inspired them to write this paper, and Robert Brandenberger and Keshav Dasgupta for useful comments on the draft. S.L. is supported in part by the Fonds de recherche du Québec (FRQNT). S.B. is supported in part by the Higgs Fellowship and by the STFC Consolidated Grant "Particle Physics at the Higgs Centre". The research at McGill is supported in part by funds from NSERC and the Canada Research Chair program.

Appendix

4.7 Zeta function regularisation

While evaluating the contribution of the winding modes to the path integral, we used the identities

$$\prod_{n=1}^{\infty}\left(\frac{2\pi n}{\alpha}\right)^{-2}=\frac{1}{\alpha}\quad,\quad\prod_{n=1}^{\infty}\left(\frac{2\pi(n-1/2)}{\alpha}\right)^{2}=2 \tag{4.120}$$

to regulate various divergent products obtained by integrating over A_n^μ, ψ_r, and c_n at one loop order. Such products show up frequently in quantum mechanics and quantum field theory, and can be tamed using Zeta function regularisation. To obtain the first identity in equation 4.120, let us define the function

$$\zeta_b(s)=\sum_{n=1}^{\infty}\left(\frac{2\pi n}{\alpha}\right)^{-2s}=\left(\frac{\alpha}{2\pi}\right)^{2s}\zeta(2s) \tag{4.121}$$

where

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s} \tag{4.122}$$

is the Riemann Zeta function. Taking the derivative of $\zeta_b(s)$ with respect to s, we obtain

$$\zeta_b'(s)=\sum_{n=1}^{\infty}\left(\frac{2\pi n}{\alpha}\right)^{-2s}\ln\left(\frac{2\pi n}{\alpha}\right)^{-2} \tag{4.123}$$

$$=2\left(\frac{\alpha}{2\pi}\right)^{2s}\left(\ln\left(\frac{\alpha}{2\pi}\right)\zeta(2s)+\zeta'(2s)\right)\,. \tag{4.124}$$

The expression above can be used to express the product $\prod_{n=1}^{\infty}\left(\frac{2\pi n}{\alpha}\right)^{-2}$ as a function of the Zeta function and its derivative. In the limit where $s=0$, we obtain

$$e^{\zeta_b'(0)} = \prod_{n=1}^{\infty}\left(\frac{2\pi n}{\alpha}\right)^{-2} = \left(\frac{\alpha}{2\pi}\right)^{2\zeta(0)} e^{2\zeta'(0)}\,. \tag{4.125}$$

Using the known values $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$ of the Zeta function, we recover the desired identity:

$$\prod_{n=1}^{\infty}\left(\frac{2\pi n}{\alpha}\right)^{-2} = \frac{1}{\alpha}\,. \tag{4.126}$$

The section identity in equation 4.120 can be obtained in a similar way. We first define the function

$$\zeta_f(s) = \sum_{n=1}^{\infty}\left(\frac{2\pi(n-1/2)}{\alpha}\right)^{-s} = \left(\frac{\alpha}{2\pi}\right)^{s}\zeta(s,1/2)\,, \tag{4.127}$$

where

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s} \tag{4.128}$$

is the Hurwitz Zeta function. Then, we take the derivative of $\zeta_f(s)$ to obtain

$$\zeta_f'(s) = \sum_{n=1}^{\infty}\left(\frac{2\pi(n-1/2)}{\alpha}\right)^{-s}\ln\left(\frac{2\pi(n-1/2)}{\alpha}\right)^{-1} \tag{4.129}$$

$$= \left(\frac{\alpha}{2\pi}\right)^{s}\left(\ln\left(\frac{\alpha}{2\pi}\right)\zeta(s,1/2) + \zeta'(s,1/2)\right)\,. \tag{4.130}$$

Here again, the expression above can be related to the product $\prod_{n=1}^{\infty}\left(\frac{2\pi(n-1/2)}{\alpha}\right)^{2}$ via the expression

$$e^{-2\zeta_f'(0)} = \prod_{n=1}^{\infty}\left(\frac{2\pi(n-1/2)}{\alpha}\right)^{2} = \left(\frac{\alpha}{2\pi}\right)^{-2\zeta(0,1/2)} e^{-2\zeta'(0,1/2)}\,. \tag{4.131}$$

Using the value $\zeta(0,1/2) = 0$ and $\zeta'(0,1/2) = -\frac{1}{2}\ln(2)$ of the Hurwitz Zeta function, we obtain

$$\prod_{n=1}^{\infty}\left(\frac{2\pi(n-1/2)}{\alpha}\right)^{2} = 2\,, \tag{4.132}$$

as desired.

Bibliography

[1] R. H. Brandenberger and C. Vafa, "Superstrings in the Early Universe," Nucl. Phys. B **316**, 391-410 (1989) doi:10.1016/0550-3213(89)90037-0

[2] R. Hagedorn, "Statistical thermodynamics of strong interactions at high-energies," Nuovo Cim. Suppl. **3**, 147-186 (1965) CERN-TH-520.

[3] R. H. Brandenberger, A. Nayeri, S. P. Patil and C. Vafa, "Tensor Modes from a Primordial Hagedorn Phase of String Cosmology," Phys. Rev. Lett. **98**, 231302 (2007) doi:10.1103/PhysRevLett.98.231302 [arXiv:hep-th/0604126 [hep-th]];
R. H. Brandenberger, A. Nayeri and S. P. Patil, "Closed String Thermodynamics and a Blue Tensor Spectrum," Phys. Rev. D **90**, no.6, 067301 (2014) doi:10.1103/PhysRevD.90.067301 [arXiv:1403.4927 [astro-ph.CO]].

[4] A. Nayeri, R. H. Brandenberger and C. Vafa, "Producing a scale-invariant spectrum of perturbations in a Hagedorn phase of string cosmology," Phys. Rev. Lett. **97**, 021302 (2006) doi:10.1103/PhysRevLett.97.021302 [arXiv:hep-th/0511140 [hep-th]].

[5] G. Obied, H. Ooguri, L. Spodyneiko and C. Vafa, "De Sitter Space and the Swampland," [arXiv:1806.08362 [hep-th]];
A. Bedroya and C. Vafa, "Trans-Planckian Censorship and the Swampland," JHEP **09**, 123 (2020) doi:10.1007/JHEP09(2020)123 [arXiv:1909.11063 [hep-th]];
A. Bedroya, R. Brandenberger, M. Loverde and C. Vafa, "Trans-Planckian Censorship and Inflationary Cosmology," Phys. Rev. D **101**, no.10, 103502 (2020) doi:10.1103/PhysRevD.101.103502 [arXiv:1909.11106 [hep-th]].

[6] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: A Conjecture,” Phys. Rev. D **55**, 5112-5128 (1997) doi:10.1103/PhysRevD.55.5112 [arXiv:hep-th/9610043 [hep-th]].

[7] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, “A Large N reduced model as superstring,” Nucl. Phys. B **498**, 467-491 (1997) doi:10.1016/S0550-3213(97)00290-3 [arXiv:hep-th/9612115 [hep-th]].

[8] S. Brahma, R. Brandenberger and S. Laliberte, “Emergent cosmology from matrix theory,” JHEP **03**, 067 (2022) doi:10.1007/JHEP03(2022)067 [arXiv:2107.11512 [hep-th]].

[9] S. Brahma, R. Brandenberger and S. Laliberte, “Emergent metric space-time from matrix theory,” JHEP **09** (2022), 031 doi:10.1007/JHEP09(2022)031 [arXiv:2206.12468 [hep-th]].

[10] S. Brahma, R. Brandenberger and S. Laliberte, “BFSS Matrix Model Cosmology: Progress and Challenges,” [arXiv:2210.07288 [hep-th]].

[11] J. J. Atick and E. Witten, “The Hagedorn Transition and the Number of Degrees of Freedom of String Theory,” Nucl. Phys. B **310**, 291-334 (1988) doi:10.1016/0550-3213(88)90151-4;

R. Blumenhagen, C. Kneissl and A. Makridou, “De Sitter quantum breaking, swampland conjectures and thermal strings,” JHEP **10**, 157 (2021) doi:10.1007/JHEP10(2021)157 [arXiv:2011.13956 [hep-th]];

E. Alvarez and M. A. R. Osorio, “Superstrings at Finite Temperature,” Phys. Rev. D **36**, 1175 (1987) doi:10.1103/PhysRevD.36.1175;

K. R. Dienes, M. Lennek and M. Sharma, “Strings at Finite Temperature: Wilson Lines, Free Energies, and the Thermal Landscape,” Phys. Rev. D **86**, 066007 (2012) doi:10.1103/PhysRevD.86.066007 [arXiv:1205.5752 [hep-th]].

[12] J. Scherk and J. H. Schwarz, “Spontaneous Breaking of Supersymmetry Through Dimensional Reduction,” Phys. Lett. B **82**, 60-64 (1979) doi:10.1016/0370-2693(79)90425-8

[13] W. Taylor, "D-brane field theory on compact spaces," Phys. Lett. B **394**, 283-287 (1997) doi:10.1016/S0370-2693(97)00033-6 [arXiv:hep-th/9611042 [hep-th]].

[14] T. Banks and L. Motl, "A Nonsupersymmetric matrix orbifold," JHEP **03**, 027 (2000) doi:10.1088/1126-6708/2000/03/027 [arXiv:hep-th/9910164 [hep-th]].

[15] A. Connes, M. R. Douglas and A. S. Schwarz, "Noncommutative geometry and matrix theory: Compactification on tori," JHEP **02**, 003 (1998) doi:10.1088/1126-6708/1998/02/003 [arXiv:hep-th/9711162 [hep-th]].

[16] P. Schreivogl and H. Steinacker, "Generalized Fuzzy Torus and its Modular Properties," SIGMA **9**, 060 (2013) doi:10.3842/SIGMA.2013.060 [arXiv:1305.7479 [hep-th]].

[17] Y. Yargic, J. Lanier, L. Smolin and D. Wecker, "A Cubic Matrix Action for the Standard Model and Beyond," [arXiv:2201.04183 [hep-th]].

[18] N. Kawahara, J. Nishimura and S. Takeuchi, "High temperature expansion in supersymmetric matrix quantum mechanics," JHEP **12**, 103 (2007) doi:10.1088/1126-6708/2007/12/103 [arXiv:0710.2188 [hep-th]].

[19] A. Fayyazuddin, Y. Makeenko, P. Olesen, D. J. Smith and K. Zarembo, "Towards a nonperturbative formulation of IIB superstrings by matrix models," Nucl. Phys. B **499**, 159-182 (1997) doi:10.1016/S0550-3213(97)00321-0 [arXiv:hep-th/9703038 [hep-th]];
Y. Kitazawa and H. Takata, "D-brane scattering in IIB string theory and IIB matrix model," Nucl. Phys. B **551**, 617-649 (1999) doi:10.1016/S0550-3213(99)00073-5 [arXiv:hep-th/9810004 [hep-th]].

[20] S. W. Kim, J. Nishimura and A. Tsuchiya, "Late time behaviors of the expanding universe in the IIB matrix model," JHEP **10**, 147 (2012) doi:10.1007/JHEP10(2012)147 [arXiv:1208.0711 [hep-th]].

[21] J. Nishimura, "Signature change of the emergent space-time in the IKKT matrix model," PoS **CORFU2021**, 255 (2022) doi:10.22323/1.406.0255 [arXiv:2205.04726 [hep-th]];
K. N. Anagnostopoulos, T. Azuma, K. Hatakeyama, M. Hirasawa, Y. Ito, J. Nishimura,

S. K. Papadoudis and A. Tsuchiya, "Progress in the numerical studies of the type IIB matrix model," [arXiv:2210.17537 [hep-th]].

[22] S. W. Kim, J. Nishimura and A. Tsuchiya, "Expanding (3+1)-dimensional universe from a Lorentzian matrix model for superstring theory in (9+1)-dimensions," Phys. Rev. Lett. **108** (2012), 011601 doi:10.1103/PhysRevLett.108.011601 [arXiv:1108.1540 [hep-th]].

[23] S. Brahma, R. Brandenberger and S. Laliberte, "Spontaneous symmetry breaking in the BFSS model: Analytical results using the Gaussian expansion method," [arXiv:2209.01255 [hep-th]].

[24] H. C. Steinacker, "Gravity as a quantum effect on quantum space-time," Phys. Lett. B **827**, 136946 (2022) doi:10.1016/j.physletb.2022.136946 [arXiv:2110.03936 [hep-th]]; H. C. Steinacker, "One-loop effective action and emergent gravity on quantum spaces in the IKKT matrix model," [arXiv:2303.08012 [hep-th]].

[25] P. Agrawal, S. Gukov, G. Obied and C. Vafa, "Topological Gravity as the Early Phase of Our Universe," [arXiv:2009.10077 [hep-th]].

[26] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," Nucl. Phys. B **72**, 461 (1974) doi:10.1016/0550-3213(74)90154-0

Part III

Background in matrix cosmology

Chapter 5

Emergent metric space-time from matrix theory

Suddhasattwa Brahma[1], Robert Brandenberger[2], and Samuel Laliberte[2]

[1] *Higgs Centre for Theoretical Physics, School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3FD, UK*

[2] *Department of Physics, McGill University, Montréal, QC, H3A 2T8, Canada*

Addendum for the thesis

In Section 2.3.2, we saw how numerical simulations suggest the existence of emergent cosmological solutions where three dimensions become large, and six stay small in the IKKT model. In the present chapter, we investigate how to define a metric tensor in the IKKT model, in an attempt to describe these emergent cosmological solutions. To define such a metric, we consider smaller $n_i \times n_i$ submatrix of the submatrices showcased in Figure 2.9 as describing space in a comoving interval of size n_i. We then take $l_{i,\rm phys}(t, n_i) = \langle \mathrm{Tr}(\bar{A}_i^{n_i}(t))^2 \rangle$ to be the physical length of this comoving space interval. These assuptions then allows us to compute the g_{ii} component of the metric in the i'th direction via the expression $g_{ii}^{1/2}(t, n_i) = \frac{d}{dn_i} l_{i,\rm phys}(t, n_i)$. For the emergent cosmological solutions, we find the metric to be of the form $g_{ii}(t, n_i) = \mathcal{A}(t)\delta_{ii}$ for the three large dimensions, indicative of a homogeneous and isotropic metric $g_{ij}(t, n_i) = \mathcal{A}(t)\delta_{ij}$ for the emergent three large dimensions.

Abstract

The IKKT matrix model yields an emergent space-time. We further develop these ideas and give a proposal for an emergent metric. Based on previous numerical studies of this model, we provide evidence that the emergent space-time is continuous and infinite in extent, both in space and in time, and that the metric is spatially flat. The time evolution describes the transition from a string-theoretic emergent phase to a phase in which the $SO(9)$ symmetry of the model is spontaneously broken to $SO(6) \times SO(3)$, with three dimensions of space expanding, becoming classical and at later times evolving like in a radiation-dominated universe, and the remaining six dimensions of space stabilized at the string scale. We speculate on how this analysis can be extended to yield an early universe cosmology which, in addition to the above-mentioned properties, also leads to a roughly scale-invariant spectrum of cosmological fluctuations and gravitational waves.

5.1 Introduction

Evidence is mounting that in order to obtain a model of the very early universe which is consistent with the required quantum treatment of matter, we must go beyond a description based on naive effective field theory [1]. One of the reasons is based on unitarity problems of any effective field theory description of an expanding universe [1, 2]. A ultraviolet cutoff is required to make sense of such a model, and to maintain this ultraviolet cutoff at a fixed physical scale in an expanding universe, continuous creation of effective field theory modes is required. This time-dependence of the low-energy Hilbert space obviously implies the breakdown of unitarity. Demanding that, during the time evolution of the system, no such modes which are created ever exit the Hubble horizon (when these modes freeze out, become squeezed and can classicalize) leads to the *Trans-Planckian Censorship Conjecture* (TCC) [3] which results in serious constraints on inflationary cosmology [4] (see e.g. [5] for recent discussions).

[1]By *naive* effective field theory we mean the usual approach to (Fock)-quantizing fields in which the fields are expanded into comoving modes, and each such mode is canonically quantized like the standard quantization of a harmonic oscillator.

Another problem for an effective field theory analysis of cosmology is the following: Since each of the modes of the effective field theory has a ground state energy, such an effective field theory analysis leads to the famous cosmological constant problem, the fact that the predicted value of the vacuum energy is many orders of magnitude larger than what is consistent with observations if the vacuum energy gravitates.

Keeping these problems in mind, it is worth considering that although the *inflationary scenario* [6] has become the standard paradigm for early universe cosmology, there are alternative scenarios which are also consistent with current data. One class involves *bouncing cosmologies* (see e.g. [7] for reviews) in which the universe is initially contracting and then undergoes a bounce transition to an expanding phase, while another class is the *emergent scenario* (see e.g. [8] for a review), in which the universe begins in a quasi-static phase and then undergoes a phase transition to an expanding radiation phase. To realize bouncing and emergent cosmologies, however, physics beyond standard effective field theory of matter and Einstein gravity is required.

To obtain a consistent description of the early universe, we hence need to start with a model which is well defined and has no problems at high energy scales. For a long time, the hope has been that superstring theory will be able to provide such a description. The phenomenology of string theory is, however, usually explored in an effective field theory limit, and in this limit the aforementioned conceptual problems cannot be resolved. Thus, instead of trying to evaluate four-dimensional effective potentials using string-theoretic quantum effects, we believe a more fruitful way would be to start with the full theory itself and give a prescription for coarse-graining the cosmological degrees of freedom from it.

In this paper we provide some indications that a viable emergent cosmology will emerge from matrix model descriptions of string theory. Specifically, we consider the IKKT matrix model [9], a proposed non-perturbative definition of Type IIB superstring theory. This is a quantum mechanical model of Hermitean $N \times N$ matrices (with, a priori, no space and no time). We will indicate how in the $N \to \infty$ limit a continuous space-time with three large spatial dimensions and with infinite extent both in space and time emerges. We provide a prescription for an emergent metric for the $(3+1)$-dimensional space-time involving the three large spatial dimensions, and provide indications that the emergent metric is spatially flat, and described by a cosmological scale factor $a(t)$ which has a late time limit which corresponds to a radiation dominated universe. The effective cosmological constant vanishes in this model. [2].

In the following we first review the IKKT matrix model [9]. Over the past two decades there has been a lot of numerical work on this model (see e.g. [10, 11] for reviews, and [12] for more recent numerical studies), and we will summarize the results which are relevant for our analysis. In Section 3, we then show how continuous space-time with infinite extent of both space and time variables emerges, give a proposal for an emergent metric, and show that the resulting cosmological metric corresponds to a spatially flat manifold. In Section 4, we speculate that the same result will also emerge in the BFSS matrix model, a matrix theory (with an intrinsic time, but no space) which was proposed [13] as a non-perturbative definition of M-theory[3]. As shown in previous work [20], thermal fluctuations in a high temperature state of the BFSS model yield scale-invariant spectra of curvature fluctuations and gravitational waves, with a Poisson component of the curvature perturbations on short distance scales.

We will be working with units in terms of which the speed of light, Boltzmann's constant and Planck's constant are all set to 1.

5.2 Review of the IKKT Matrix Model and Emergence of Continuous Time

The IKKT matrix model [9] (see [10, 11] for recent reviews) has been proposed as a non-perturbative definition of Type IIB superstring theory. It is a pure matrix theory (no space and no time), given by the action

$$S = -\frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{4}\left[A^a, A^b\right]\left[A_a, A_b\right] + \frac{i}{2}\bar{\psi}_\alpha\left(\mathcal{C}\Gamma^a\right)_{\alpha\beta}\left[A_a, \psi_\beta\right]\right), \tag{5.1}$$

[2] The late time analysis does not take into account the presence of fermionic matrices, and matter will arise from that sector.

[3] For other related approaches to emergent space in the context of matrix models see e.g. [14], [15], [16], [17], [18]. For early work on matrix models as a means to quantize a theory of membranes see [19].

where A_a and ψ_α ($a = 0, \ldots, 9$, $\alpha = 1, \ldots, 16$,) are $N \times N$ are bosonic and fermionic Hermitian matrices, respectively, the Γ^α are the gamma-matrices for $D = 10$ dimensions, and $\mathcal{C}$ is the charge conjugation matrix. Note that a is a ten-dimensional vector index, while α is a spinor index. The vector indices are raised and lowered with the Minkowski symbol η_{ab}. g is a gauge-theory coupling constant. It is in the limit $N \to \infty$ with $\lambda \equiv g^2 N$ held fixed that this action leads to a non-perturbative definition of Type IIB superstring theory.

The action of the Lorentzian matrix model [9] is given by the following functional integral over the bosonic and fermionic matrices (with the standard measures)

$$Z = \int dA d\psi e^{iS} \, . \tag{5.2}$$

Since the matrices are Hermitean, it is possible to choose a basis in which A_0 is diagonal. We can also label the basis elements such that the eigenvalues α_a are ordered such that $\alpha_a < \alpha_b$ if $a < b$. Numerical studies of the theory show that for large values of N [21]

$$\frac{1}{N} \left\langle \mathrm{Tr} A_0^2 \right\rangle \sim \kappa N \, , \tag{5.3}$$

where κ is a constant ($\kappa < 1$), where the pointed brackets in $\langle \mathcal{O} \rangle$ indicate the expectation value of the operator $\mathcal{O}$

$$\langle \mathcal{O} \rangle \equiv \frac{1}{Z} \int dA \, d\psi \, \mathcal{O} \, e^{iS} \, . \tag{5.4}$$

This implies that in the $N \to \infty$ limit, the total extent of time becomes infinite, time running from $-\infty$ to $+\infty$. More precisely, time runs from $-t_m$ to $+t_m$ with t_m scaling as $\sqrt{N}$. To see this, assume for concreteness that the temporal eigenvalues are evenly spaced, with spacing Δt. In this case, the expectation value on the left hand side of (5.3) becomes the sum of squares of integers from 1 to $N/2$, multiplied by $(\Delta t)^2$. Making use of the formula for the sum of squares of integers, we find that the left hand side of (5.3) is proportional to $N^2 (\Delta t)^2$. Hence, time becomes continuous with

$$\Delta t \sim \frac{1}{\sqrt{N}} \, , \tag{5.5}$$

and the total extent of space becomes infinite, as $N \to \infty$

$$t_m \sim N \Delta t \sim \sqrt{N} \, . \tag{5.6}$$

Turning our attention now to the spatial matrices A_i, numerical work [21] has shown that these matrices have band-diagonal structure in the sense that if we consider the expectation values of the off-diagonal elements of the matrix, then they decay to zero if the distance n from the diagonal exceeds a critical value n_c, *i.e.*

$$\sum_i \left\langle |A_i|^2_{ab} \right\rangle \to 0 \;\; \text{for} \;\; \mathrm{n} \equiv |\mathrm{a}-\mathrm{b}| > \mathrm{n_c} \,. \tag{5.7}$$

The numerical studies [21] also show that

$$n_c \sim \sqrt{N} \,. \tag{5.8}$$

This result can also be understood by first performing the partial functional integral dA_{ab} in (5.4). Schematically,

$$\left\langle |A_i|^2_{ab} \right\rangle \sim \int d|A_i|_{ab} \, \frac{|A_i|^2_{ab}}{Z} \, e^{i/2g^2(\alpha_a - \alpha_b)^2 |A_i|^2_{ab}} \tag{5.9}$$

where

$$Z = \int \prod_a d\alpha_a \prod_{a>b} \left(\alpha_a - \alpha_b\right)^2 \int dA_i e^{i/2g^2(\alpha_a-\alpha_b)^2|A_i|^2_{ab}} \tag{5.10}$$

For values of $|A_i|^2_{ab} \left(\alpha_a - \alpha_b\right)^2 / g^2 > 1$, *i.e.* when $|a-b| > n_c$, the integrand becomes very rapidly oscillating, and by the Riemann-Lebesgue Lemma the integral hence tends to zero (in the sense of generalized functions). In the following we will assume that

$$\sum_i \left\langle |A_i|^2_{ab} \right\rangle \sim \text{const} \;\; \text{for} \;\; \mathrm{n} \equiv |\mathrm{a}-\mathrm{b}| < \mathrm{n_c} \,, \tag{5.11}$$

which is supported by the same consideration of the dA_{ab} integral. From the above (5.9), it is also easy to check that the scale n_c scales as (for $a - b = n_c$):

$$\frac{\left(\alpha_a - \alpha_b\right)^2}{g^2} \sim 1 \Rightarrow n_c \sim g\sqrt{N} \,, \tag{5.12}$$

where we have used the relation $\alpha_a - \alpha_b = n_c \Delta t \sim n_c/\sqrt{N}$ (from (5.5)).

Let us briefly return to the temporal matrix A_0. A time variable $t(m)$ corresponding to the m'th temporal eigenvalue can then be defined by averaging the diagonal elements α_i over n elements [22]

$$t(m) \equiv \frac{1}{n} \sum_{l=1}^{n} \alpha_{m+l} \,, \tag{5.13}$$

Time-dependent spatial matrices $(\bar{A}_i)_{I,J}(t)$ of dimension $n \times n$ can then be defined via [23]

$$(\bar{A}_i)_{I,J}(t(m)) \equiv (A_i)_{m+I,m+J} \, . \tag{5.14}$$

It is then natural to define the extent x_i of a given spatial dimension i at time t by [24]

$$R_i(t)^2 \equiv \left\langle \frac{1}{n} \mathrm{Tr}(\bar{A}_i)(t)^2 \right\rangle , \tag{5.15}$$

where the pointed brackets stand for the quantum expectation value in the state given by the partition function.

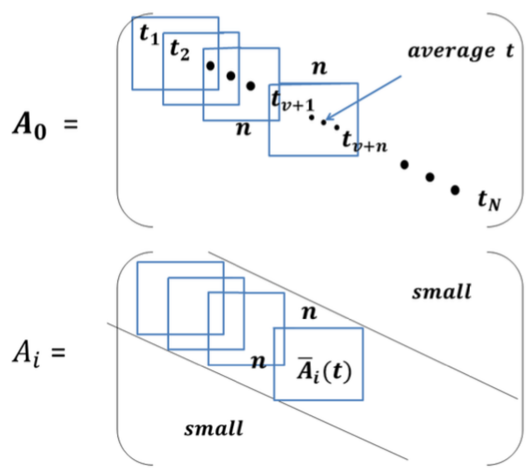

Figure 5.1: Temporal matrix $A \equiv A_0$ (top panel) and spatial matrices A_i (bottom panel) in the basis in which the temperal matrix is diagonal. The spatial matrices A_i (bottom panel) have "block-diagonal form" and can be used to define the sizes of the spatial dimensions at time t via sub-matrices $\bar{A}_i(t)$ of A_i centered a "distance" t along the diagonal of A_i. This figure is taken from [11] with permission.

Various numerical studies [25] of the IKKT mode indicate the as t increases, the initial $SO(9)$ symmetry of the matrix model is spontaneously broken to $SO(6) \times SO(3)$, with three of the $R_i(t)$ increasing in time in a nearly isotropic manner, while the other six remain small. If we identify $R_i(t)$ as the *extent of space parameter* in direction i as a function of tiime (which is done in the work on the IKKT matrix model), then we find the emergence of precisely three large spatial dimensions. Using Gaussian expansion analyses, one can verify that the state with $SO(6) \times SO(3)$ symmetry has a smaller free energy than the state with the full $SO(9)$ symmetry, and that this symmetry breaking pattern is in fact preferred over other ones.

This symmetry breaking which leads to three large spatial dimensions with the other six remaining small occurs in *String Gas Cosmology* [26]. In that model, a gas of closed strings in considered on a nine dimensional spatial torus. At high densities, the string gas will contain, in additional to the center of mass momentum modes and the string oscillatory modes, winding modes – strings winding the torus. It is then argued that for space to be able to expand, the winding modes need to annihilate into string loops, and this cannot happen in more than 3 large spatial dimensions since for more than three large spatial dimensions the strings would have vanishing probability to meet. We conjecture that there might be a relationship between the symmetry breaking dynamics in the matrix model and in String Gas Cosmology: there is evidence that the strings are coherent states of the matrix model in which one spatial matrix is excited. Such a coherent state will prevent space from expanding in the same way that a string does in the String Gas Cosmology picture.

5.3 Emergent Space and Metric

Let us now return to the spatial matrices A_i for the dimensions which become large. As already indicated in the previous section, we now consider a $n_i \times n_i$ submatrix $\bar{A}_i^{n_i}(t)$ centered a distance t down the diagonal, as indicated in Figure 5.1. We let n_i range from 0 to n_c, and we propose to view n_i as a comoving spatial coordinate in direction i [4]. Then, we adapt (5.15) and define the physical length of a curve along the i coordinate axis from $n = 0$ to n as

$$l_{i,\mathrm{phys}}^2(t, n_i) \equiv \left\langle \mathrm{Tr}(\bar{A}_i{}^{n_i})(t))^2 \right\rangle . \tag{5.16}$$

[4] The comoving coordinate thus defined runs from 0 to n_c. With a slight change of the definition we can extend the range to $-n_c < n_i < n_c$. We focus on the upper (lower) diagonal submatrix of the $|n_i| \times |n_i|$ matrix for positive (negative) values of n_i. The resulting length (and metric) will be symmetric under $n_i \to -n_i$.

Making use of (5.11), we see that the quantity (5.16) scales as n_i^2 (there are n_i eigenvalues to sum over, and each eigenvalue obtains contributions from n_i matrix elements). Thus, the total physical extent of space out to a comoving distance n_c scales as $n_c \sim N^{1/2}$, and the physical distance δx between neighboring comoving coordinate values scales as

$$\delta x \,\sim\, \frac{N^{1/2}}{N} \,\sim\, N^{-1/2}\,. \tag{5.17}$$

Thus, we obtain continuous space with infinite spatial extent in the $N \to \infty$ limit.

Since for a metric space the physical length of a line along the i axis between comoving coordinates 0 and x allows us to obtain the g_{ii} metric component via

$$l_{i,\mathrm{phys}}(t,x) \,=\, \int_{y=0}^{x} \sqrt{g_{ii}(t,y)}dy\,, \tag{5.18}$$

we propose to define the emergent metric as

$$g_{ii}^{1/2}(t,n_i) \,=\, \frac{d}{dn_i} l_{i,\mathrm{phys}}(t,n_i)\,. \tag{5.19}$$

Making use of (5.16) we get

$$g_{ii}^{1/2}(t,n_i) \,=\, \frac{1}{2}\frac{\left(\frac{d}{dn_i}\left\langle \mathrm{Tr}\left(\bar{A}_i^{\,n_i}\right)^2(t)\right\rangle\right)}{\left(\left\langle \mathrm{Tr}\left(\bar{A}_i^{\,n_i}\right)^2(t)\right\rangle\right)^{1/2}}\,. \tag{5.20}$$

Since (based on (5.11) we have seen that (5.16) scales as n_i^2, we find that the metric component is independent of n_i. This is a very important finding as it tells us that the emergent spacetime is spatially flat. Note that this crucially depends on the block diagonal structure of the spatial matrices with respect to our coarse-grained time. If, indeed, the spatial matrices were all diagonal or of some other form, then the scaling of the metric of n_i^2 would have been very different and would not have led to this result. The block-diagonal form, in turn, depends sensitively on the Lagrangian of the IKKT model and this type of an emergent structure would not appear in any matrix model but rather one that comes from string theory.

In the end, we obtain

$$g_{ii}(n_i,t) \,=\, \mathcal{A}(t)\delta_{ii}\,, \tag{5.21}$$

where $\mathcal{A}(t)$ is the time-dependent amplitude. This result corresponds to a homogeneous and commutative space for the three large dimensions. Making use of the $SO(3)$ symmetry of the system we then obtain a homogeneus and isotropic cosmological metric

$$g_{ij}(t) = \mathcal{A}(t)\delta_{ij} . \tag{5.22}$$

We identify the amplitude $\mathcal{A}(t)$ with the cosmological scale factor $a(t)$ of the three-dimensional space which becomes large. At early times (before the symmetry breaking phase transition) the scale factor is constant and corresponds to a state of string density. After the phase transition, the scale factor increases. Since quantum effects are expected to become negligible once the amplitude $\mathcal{A}(t)$ is large (by Ehrenfest's Theorem), the late time dependence of $\mathcal{A}(t)$ can be obtained by solving the classical equations of motion. This has been done in [27] with the result that $\mathcal{A}(t) \sim t^{1/2}$ which corresponds to the expansion of space in the radiation phase of Standard Big Bang cosmology.

We have thus obtained a first principles realization of the String Gas Cosmology (SGC) scenario put forwards in [26]. The SGC scenario is obtained by considering matter to be a thermal gas of strings on a nine-dimensional background space which admits long-lived winding string states. There is a maximal temperature of a gas of closed strings, the Hagedorn temperature [28] T_H. For a large range of energy densities, the temperature T remains close to T_H, and it is not unreasonable to assume that this phase (the *Hagedorn phase*) is quasi-static. At some point, however, a symmetry breaking phase transition occurs [26], allowing three dimensions of space to become large. This transition is triggered by the decay of winding modes. When the world sheet of two winding strings with opposite orientations meet, they can interact and produce string loops, thus eliminating the winding which prevents space from expanding, and leading to a radiation-dominated phase of expansion. Since string world sheets have zero probability for intersecting in more than four large space-time dimensions, it is precisely three spatial dimensions which can become large, while the others remain at the string scale. The weak point of the SGC scenario is the assumption of a quasi-static Hagedorn phase. Such a phase cannot be obtained using effective field theory techniques.

We have argued that matrix theory can provide a first principles realization of the dynamics of space-time assumed in SGC. The universe begins in a quasi-static phase in which all nine spatial extent parameters are the same, time independent and of string scale. This corresponds to the Hagedorn phase of SGC. There is a phase transition in which the $SO(9)$ symmetry of the Lagrangian is spontaneously broken to $SO(6) \times SO(3)$. The same transition also occurs in SGC. After the phase transition, the three dimensions which become large expand like in the radiation phase of Standard Big Bang cosmology, the same dynamics as once again occurs in SGC.

Note that there are other approaches to obtaining SGC dynamics from first principles. In the approach of [29], the analog of the Hagedorn phase of SGC is a topological phase. In the context of Double Field Theory [30], there have also been recent studies on how to obtain an initial cosmological phase which has properties in common with the Hagedorn phase of SGC [31].

Note that, as already pointed out in [32], the cosmological constant problem is absent in this model. The quantization of the model does not involve an effective field theory analysis in which fields are expanded in Fourier modes, and the ground state energy of the Fourier modes then adds up to yield a cosmological constant which is many orders of magnitude too large. Here, we are quantizing a matrix model Lagrangian, extracting an effective late time metric, and observing that the time evolution is inconsistent with the presence of a cosmological constant. We thus see that the cosmological constant problem may be an artefact of an effective field theory point of view.

5.4 Discussion

At about the same time that the IKKT matrix model was proposed, there was another proposal for a non-perturbative definition of string theory, namely the BFSS [13] matrix model. In contrast to the IKKT model, this model contains a time variable t, and is given by a Lagrangian involving 9 spatial Hermitean $N \times N$ matrices $X_i(t)$ and a temporal Hermitean $N \times N$ matrix $A_0(t)$:

$$S = \frac{1}{2g^2} \int dt \mathrm{Tr} \left[\frac{1}{2}(D_t X_i)^2 - \frac{1}{4}[X_i, X_j]^2 - \theta^T D_t \theta - \theta^T \gamma_i [\theta, X^i] \right] , \tag{5.23}$$

where $D_t \equiv \partial_t - i[A_0(t), ..]$ is a covariant derivative operator. The θ are sixteen fermionic superpartners which are spinors of $SO(9)$, and g^2 is a coupling constant. In the large N limit, (with the 't Hooft coupling $\lambda \equiv g^2 N = g_s N l_s^{-3}$ (where g_s and l_s are string coupling constant and string length, respectively) held fixed, this model was argued to yield a non-perturbative definition of M-theory.

In the high temperature limit, the leading term in the action of the bosonic sector of the BFSS matrix yields the bosonic part of the IKKT action. This is seen by expanding the X_i matrices in terms of Matsubara frequencies

$$X_i(t) \,=\, \sum_{m=0}^{\infty} X_i^m e^{im\omega t}\,, \tag{5.24}$$

with $\omega = 2\pi/\beta$, β being the inverse of the temperature T, and m running over the positive semi-definite integers. The BFSS action becomes

$$S \,=\, S_0 + S_1\,, \tag{5.25}$$

where S_0 contains the terms which only depend on the zero modes X_i^0. At high temperature, the terms in S_1 are small amplitude correction terms, and under the rescaling

$$A_i \,\equiv\, T^{1/4} X_i^0\,, \tag{5.26}$$

the action S_0 becomes the bosonic part of the IKKT action.

Starting with the BFSS model in a high temperature thermal state, we recently showed (using results from [33]) that the thermal fluctuations which are inevitably present in such a state lead to scale-invariant spectra of curvature fluctuations and gravitational waves, with a Poisson tail which dominates in the ultraviolet in the case of the curvature fluctuations [20]. The results parallel those obtained in the case of String Gas Cosmology [34,35], in which case the scale-invariance can be traced back to the fact that the fluctuations have holographic scaling as a function of the radius of the box in which the fluctuations are computed. Note that it is the correction terms S_1 in the high temperature expansion of the BFSS action which play an important role in determining the fluctuations.

We now suggest the following scenario: starting from the BFSS matrix model and taking a high temperature thermal state, we extract an emergent space-time and an emergent metric using the results from the IKKT model presented above. In order to be able to apply these arguments, we need to make sure that the $SO(9) \rightarrow SO(6) \times SO(3)$ symmetry breaking also occurs in the case of the BFSS model. Since the fermions appear to play an important role in the phase transition in the IKKT model [25], this is a non-trivial assumption. However, we have recently shown [36] that a phase transition which breaks the $SO(9)$ symmetry indeed occurs in the BFFS model [36]. Next, we also need to verify that the evolution of the expectation values $\langle \mathrm{Tr}|A_0|^2\rangle$ and $\langle |A_i|^2_{ab}\rangle$ is not changed. If successful, we would have a first principles realization of an emergent early universe cosmology in which infinite range continuous time, infinite range continuous space (with exactly three large spatial dimensions) and a homogeneous and spatially flat metric which leads to a radiation-dominated three dimensional expanding space all naturally emerge from the matrix model. Thermal fluctuations then lead to scale-invariant spectra of density fluctuations and gravitational waves with an ampitude which is set by $(\eta_s/m_{pl})^4$, where η_s is the string energy scale, and m_{pl} is the Planck mass [20].

In the appendix, we give a simple toy model realization of late-time dynamics for the BFSS action. Instead of going through the steps mentioned above, we use the classical equations of motion of the BFSS model to find cosmological solutions using a time gauge that had been overlooked earlier to find a universe with radiation-dominated expansion. If nothing else, this serves as an indication that a realization of a ultraviolet-complete model, as above, maybe possible in the BFSS model.

5.5 Conclusions

We have reviewed how continuous and infinte range time and space emerge from the IKKT matrix model in the $N \rightarrow \infty$ limit. As has been shown, this model undergoes symmetry breaking between an early stage of $SO(9)$ symmetry in which the *extent of space parameters* in all nine spatial directions are microscopic, and a stage when the $SO(9)$ symmetry breaks to $SO(6) \times SO(3)$, allowing exactly three spatial dimensions to become large. We have proposed a definition of comoving distance coordinates and corresponding physical distance, with which it becomes possible to extract an emergent metric for the four large space-time dimensions. The resulting metric is homogeneous, isotropic and spatially flat. At late times, the three large dimensions expand like in the radiation-dominated Friedmann model. Hence, the matrix model leads to vanishing cosmological constant, as already pointed out in [32] [5].

We have suggested that the method of extracting time, space and a metric from a matrix model also holds if we start from the BFSS matrix theory. Based on this starting point, we also have a mechanism by which thermal fluctuations lead to scale-invariant spectra of curvature fluctuations and gravitational waves.

Acknowledgments

The authors thank Simon Caron-Huot, Keshav Dasgupta, Hikaru Kawai, Jun Nishimura, Katsuta Sakai, Lee Smolin and Cumrun Vafa for comments on an earlier version of this draft. We also thank B. Ydri for permission to use the figure (5.1) which is taken from [11].

SB is supported in part by the Higgs Fellowship. SL is supported in part by FRQNT. The research at McGill is supported in part by funds from NSERC and from the Canada Research Chair program.

Appendix: Late-time evolution in the BFSS model

Although we have described how one can extract a metric for the IKKT model in this paper, let us show some evidence that, at late times, one does find a cosmological evolution for the BFSS model on the lines of what we have conjectured above. Our main assumptions in deriving such dynamics are as follows:

1. We can examine the classical equations of motion of the BFSS model in order to gain some insight into late-time evolution. We expect that this assumption certainly breaks down before the critical time when we expect the spontaneous symmetry-breaking to take place.

2. We assume a *homothetic anstaz*, as have been previously chosen in [39].

[5]For other approaches towards extracting cosmology from the IKKT matrix model see [37] and [38].

The classical equations for the bosonic matrices from the action (5.23) is given by

$$\ddot{X}^i + \left[X^j, \left[X^j, X^i\right]\right] = 0\,, \tag{5.27}$$

where repeated indices are summed over irrespective of their position. Since we are assuming these equations to hold only after the symmetry-breaking takes place, it is natural to suppose that the i index now runs from $(1, 2, 3)$. The homothetic ansatz implies

$$X^i = a(\tau)\,\Theta^i\,. \tag{5.28}$$

Before solving (5.27) with our ansatz above (5.28), let us make a few remarks. Firstly, note that the BFSS model implies classical equations for a Galilei invariant system. In doing so, we depart from previous studies where it was customary to add a mass term for the matrices which led to the symmetry group being that of a Newton-Hooke system (the non-relativistic contraction of de Sitter). Not only would adding such a mass term be unnatural from the point of view of the BFSS model (as derived from M-theory), requiring a positive cosmological constant would, in fact, imply having a negative (or tachyonic) mass term.

Secondly, note that the time parameter is intrinsic in the BFSS model unlike its IKKT counterpart. Therefore, it is not clear how to relate the time parameter appearing in these equations with the different choices of time gauges one can choose in General Relativity. However, this is where we make our most important conjecture – what if the equation of motion written above *describes dynamics with respect to conformal time?* Although, at this point, this is simply a hypothesis on our part, note that this choice is just as good as choosing the time parameter to be equivalent to cosmic time, as has been typically assumed in the past. In future work, we will explore more into this question of which time gauge does this time parameter correspond to for our cosmological scenario. For now, we simply identify with proper time and go on to explore the consequences.

Finally, as in well known for the BFSS model, the dynamics described by (5.27) is topped off by the following (Gauss) constraint on initial conditions:

$$\left[\dot{X}^i, X^i\right] = 0\,. \tag{5.29}$$

Since all the time-dependence of our ansatz (5.28) is contained in the overall pulsating function, which we have suggestively called $a(\tau)$ to coincide with the scale factor, (5.27) can be written as

$$\frac{\ddot{a}(\tau)}{a^3(\tau)} + \left[\Theta^j, \left[\Theta^j, \Theta^i\right]\right] \left(\Theta^i\right)^{-1} = 0\,, \tag{5.30}$$

where both the terms on the left hand side must be independent of time now. (Keep in mind that an overdot denotes a derivative with respect to conformal time in our convention.) Let us choose the constant for the separation of variables to be λ, *i.e.*

$$\begin{aligned} \ddot{a}(\tau) &= \lambda\, a^3(\tau)\,, &(5.31)\\ \left[\Theta^j, \left[\Theta^j, \Theta^i\right]\right] &= -\lambda \Theta^i\,. &(5.32) \end{aligned}$$

Let us begin analyzing the equation (5.31) as the Raychaudhuri equation for the scale factor, whose first integral gives the more familiar Friedmann equation:

$$\frac{d}{d\tau}\left(\dot{a}(\tau)\right) = \lambda a^3(\tau) \Rightarrow \frac{\dot{a}^2(\tau)}{2} = \frac{\lambda a^4(\tau)}{4} + K\,, \tag{5.33}$$

where K is a constant of integration. Next, we can switch over to cosmic time, recalling $d/d\tau = a(t)d/\mathrm{d}t$, and doing a little bit of algebra to find

$$\frac{1}{a^2}\left(\frac{da}{dt}\right)^2 = \frac{K}{a^4} + \frac{\lambda}{2}\,. \tag{5.34}$$

This is the standard Friedmann equation with a radiation component as well as a cosmological constant term (the sign of λ is yet to be determined).

However, recall how we expect these classical equations to only be valid for late times in which regime we expect the bosonic matrices to commute with each other since they are far separated and give rise to ordinary smooth spacetime geometry. In this case, for near-commuting matrix degrees of freedom, it is natural to have $\lambda = 0$ as can be seen by inspecting (5.32). If we plug it into the above Friedmann equation (5.34), we get

$$3M_{\mathrm{Pl}}^2 H^2 = \frac{C}{a^4}\,, \tag{5.35}$$

where $C = 3M_{\mathrm{Pl}}^2 K$ is the constant for the radiation energy density. In fact, this is where the fact that we consider that this equation is satisfied after the symmetry-breaking phase comes into play. For a 9-d universe, the equation of state corresponding to the Friedmann equation (5.35) would be given by $p = -(5/9)\,\rho$. However, for $d = 3$, this is given by $p = \rho/3$, as expected. We will come back to this point later on.

We could have also set $\lambda = 0$ directly into the Raychaudhuri equation (5.31) to find $\ddot{a} = 0$, whose solutions are given by

$$a(\tau) = a_0 + a_1 \tau \,. \tag{5.36}$$

As expected, this is indeed the solution for the scale factor for a radiation dominated universe along with a possible quasi-static phase. It is interesting that this is indeed the type of cosmological history predicted by the String Gas scenario (a pressureless fluid corresponding to the quasistatic phase followed by radiation)! Of course, we should only trust this solution after the symmetry-breaking phase and only find the description of the emergent (or non-geometric) phase as quasistatic to be intriguing.

Let us compare this result with previous investigations of BFSS cosmology [39]. Since it was always assumed that the time parameter corresponds to cosmic time in existing literature, this inevitably led to unphysical matter components along with a possible curvature term. Sometimes, a mass term was added by hand to the BFSS Lagrangian which led to a cosmological constant term. Apart from the radiation phase, what is remarkable is that using our interpretation of the time parameter as conformal time, one immediately finds that there is *no spatial curvature term* in the Friedmann equation above! This is consistent with our finding of a flat metric from the IKKT model as described in the previous section. Moreover, this conclusion is completely independent of our natural supposition that the matrices become commuting at late times. Even if we choose to keep a non-zero λ, this would act as an effective cosmological constant and not behave as a curvature term in the Friedmann equation. This is one of our main findings from the BFSS model – unless there are additional terms put in by hand, the Friedmann equations describe a flat, radiation dominated universe at late time with our choice of the time variable.

Finally, let us comment about choosing to work with $d = 3$ in interpreting the above equations. At first sight, this might seem like an extremely restrictive condition and the reader might view this as our most drastic assumption. However, this is not the case. To begin with, as explained, it only makes sense to use the classical equations to describe the dynamics at late times and this would naturally be after the symmetry-breaking phase. So, the real question is whether there is any such symmetry-breaking phase in the BFSS model, and this has recently been shown to be the case [36]. But, more importantly, note that if we allow for a parametric separation between the magnitude of the scale-factor $a(t)$ of the external cosmological background and those of the 6-dimensional internal spacetime, say $\tilde{a}(t)$, then it is break-up the original equations of motion (5.27) into two parts – one for the external and one for the internal spacetimes. Given this, it is perfectly possible that the internal spacetimes *do not* become commuting at late time and indeed has a $\tilde{\lambda}$ that is nonzero. That would simply imply that the Friedmann equation for the internal scale factor $\tilde{a}$, has an additional cosmological constant term in $\tilde{\lambda}$. In fact, as has been shown in [39], this constant is typically negative and takes the value $\tilde{\lambda} = -2$ if we assume the internal dimensions to have an $SO(3) \times SO(3)$ form. For an $SO(6)$ symmetry, λ would also be negative but a little more complicated to evaluate. What is important is that in deriving our Friedmann equation (5.35) above, we do not need to assume a BFSS toy model for $d = 3$. All we need is the knowledge that the solution of the classical BFSS equations of motion would be satisfied by an anisotropic ansatz since the scale factors of the internal and external dimensions must be parametrically separated due to the symmetry-breaking phase preceding it. However, we are still working with the full BFSS model, as derived from M-theory, for arriving at our solution for the $(3 + 1)$-d cosmological scale factor.

Bibliography

[1] N. Weiss, "Constraints on Hamiltonian Lattice Formulations of Field Theories in an Expanding Universe," Phys. Rev. D **32**, 3228 (1985).

[2] J. Cotler and A. Strominger, "The Universe as a Quantum Encoder," [arXiv:2201.11658 [hep-th]].

[3] A. Bedroya and C. Vafa, "Trans-Planckian Censorship and the Swampland," JHEP **2009**, 123 (2020) [arXiv:1909.11063 [hep-th]].

[4] A. Bedroya, R. Brandenberger, M. Loverde and C. Vafa, "Trans-Planckian Censorship and Inflationary Cosmology," Phys. Rev. D **101**, no. 10, 103502 (2020) [arXiv:1909.11106 [hep-th]].

[5] R. Brandenberger, "String Cosmology and the Breakdown of Local Effective Field Theory," [arXiv:2112.04082 [hep-th]];
R. Brandenberger and V. Kamali, "Unitarity Problems for an Effective Field Theory Description of Early Universe Cosmology," [arXiv:2203.11548 [hep-th]].

[6] A. H. Guth, "The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems," Phys. Rev. D **23**, 347 (1981) [Adv. Ser. Astrophys. Cosmol. **3**, 139 (1987)];
R. Brout, F. Englert and E. Gunzig, "The Creation Of The Universe As A Quantum Phenomenon," Annals Phys. **115**, 78 (1978);
A. A. Starobinsky, "A New Type Of Isotropic Cosmological Models Without Singularity," Phys. Lett. B **91**, 99 (1980);

K. Sato, "First Order Phase Transition Of A Vacuum And Expansion Of The Universe," Mon. Not. Roy. Astron. Soc. **195**, 467 (1981).

[7] R. H. Brandenberger, "Alternatives to the inflationary paradigm of structure formation," Int. J. Mod. Phys. Conf. Ser. **01**, 67-79 (2011) [arXiv:0902.4731 [hep-th]];
R. Brandenberger and P. Peter, "Bouncing Cosmologies: Progress and Problems," Found. Phys. **47**, no.6, 797-850 (2017) [arXiv:1603.05834 [hep-th]].

[8] R. H. Brandenberger, "String Gas Cosmology," String Cosmology, J.Erdmenger (Editor). Wiley, 2009. p.193-230 [arXiv:0808.0746 [hep-th]].

[9] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, "A Large N reduced model as superstring," Nucl. Phys. B **498**, 467 (1997) [hep-th/9612115].

[10] J. Nishimura, "New perspectives on the emergence of (3+1)D expanding space-time in the Lorentzian type IIB matrix model," PoS CORFU **2019**, 178 (2020) [arXiv:2006.00768 [hep-lat]];
J. Nishimura, "Signature change of the emergent space-time in the IKKT matrix model," [arXiv:2205.04726 [hep-th]].

[11] B. Ydri, "Review of M(atrix)-Theory, Type IIB Matrix Model and Matrix String Theory," [arXiv:1708.00734 [hep-th]].

[12] M. Hirasawa, K. Anagnostopoulos, T. Azuma, K. Hatakeyama, Y. Ito, J. Nishimura, S. Papadoudis and A. Tsuchiya, "A new phase in the Lorentzian type IIB matrix model and the emergence of continuous space-time," PoS **LATTICE2021**, 428 (2022) [arXiv:2112.15390 [hep-lat]];
K. Hatakeyama, K. Anagnostopoulos, T. Azuma, M. Hirasawa, Y. Ito, J. Nishimura, S. Papadoudis and A. Tsuchiya, "Relationship between the Euclidean and Lorentzian versions of the type IIB matrix model," PoS **LATTICE2021**, 341 (2022) [arXiv:2112.15368 [hep-lat]];
K. Hatakeyama, K. Anagnostopoulos, T. Azuma, M. Hirasawa, Y. Ito, J. Nishimura, S. Papadoudis and A. Tsuchiya, "Complex Langevin studies of the emergent space-time in the type IIB matrix model," [arXiv:2201.13200 [hep-th]];

[13] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: A Conjecture,” Phys. Rev. D **55**, 5112 (1997) [hep-th/9610043].

[14] S. R. Das and A. Jevicki, “String Field Theory and Physical Interpretation of $D = 1$ Strings,” Mod. Phys. Lett. A **5**, 1639-1650 (1990).

[15] R. Dijkgraaf, E. P. Verlinde and H. L. Verlinde, “Matrix string theory,” Nucl. Phys. B **500**, 43-61 (1997) [arXiv:hep-th/9703030 [hep-th]].

[16] B. Craps, S. Sethi and E. P. Verlinde, “A Matrix big bang,” JHEP **10**, 005 (2005) [arXiv:hep-th/0506180 [hep-th]];
M. Blau and M. O’Loughlin, “DLCQ and Plane Wave Matrix Big Bang Models,” JHEP **09**, 097 (2008) [arXiv:0806.3255 [hep-th]].

[17] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, “Strings in flat space and pp waves from N=4 superYang-Mills,” JHEP **04**, 013 (2002) [arXiv:hep-th/0202021 [hep-th]];
J. M. Maldacena and C. Nunez, “Towards the large N limit of pure N=1 superYang-Mills,” Phys. Rev. Lett. **86**, 588-591 (2001) [arXiv:hep-th/0008001 [hep-th]].

[18] L. Smolin, “M theory as a matrix extension of Chern-Simons theory,” Nucl. Phys. B **591**, 227-242 (2000) [arXiv:hep-th/0002009 [hep-th]];
Y. Yargic, J. Lanier, L. Smolin and D. Wecker, “A Cubic Matrix Action for the Standard Model and Beyond,” [arXiv:2201.04183 [hep-th]].

[19] J.Hoppe, Quantum Theory of a Massless Relativistic Surface, MIT Ph.D. Thesis 1982, http://dspace.mit.edu/handle/1721.1/15717;
B. de Wit, J. Hoppe and H. Nicolai, “On the Quantum Mechanics of Supermembranes,” Nucl. Phys. B **305**, 545 (1988)

[20] S. Brahma, R. Brandenberger and S. Laliberte, “Emergent cosmology from matrix theory,” JHEP **03**, 067 (2022) [arXiv:2107.11512 [hep-th]].

[21] Y. Ito, J. Nishimura and A. Tsuchiya, “Power-law expansion of the Universe from the bosonic Lorentzian type IIB matrix model,” JHEP **1511**, 070 (2015) [arXiv:1506.04795

[hep-th]];

K. Hatakeyama, A. Matsumoto, J. Nishimura, A. Tsuchiya and A. Yosprakob, "The emergence of expanding space–time and intersecting D-branes from classical solutions in the Lorentzian type IIB matrix model," PTEP **2020**, no.4, 043B10 (2020) [arXiv:1911.08132 [hep-th]].

[22] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, "Space-time structures from IIB matrix model," Prog. Theor. Phys. **99**, 713 (1998) [hep-th/9802085].

[23] S. W. Kim, J. Nishimura and A. Tsuchiya, "Expanding (3+1)-dimensional universe from a Lorentzian matrix model for superstring theory in (9+1)-dimensions," Phys. Rev. Lett. **108**, 011601 (2012) [arXiv:1108.1540 [hep-th]].

[24] T. Hotta, J. Nishimura and A. Tsuchiya, "Dynamical aspects of large N reduced models," Nucl. Phys. B **545**, 543-575 (1999) [arXiv:hep-th/9811220 [hep-th]].

[25] J. Nishimura and G. Vernizzi, "Spontaneous breakdown of Lorentz invariance in IIB matrix model," JHEP **0004**, 015 (2000) [hep-th/0003223];

J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, "Monte Carlo studies of the IIB matrix model at large N," JHEP **07**, 011 (2000) [arXiv:hep-th/0005147 [hep-th]];

J. Nishimura and F. Sugino, "Dynamical generation of four-dimensional space-time in the IIB matrix model," JHEP **0205**, 001 (2002) [hep-th/0111102];

H. Kawai, S. Kawamoto, T. Kuroki, T. Matsuo and S. Shinohara, "Mean field approximation of IIB matrix model and emergence of four-dimensional space-time," Nucl. Phys. B **647**, 153-189 (2002) [arXiv:hep-th/0204240 [hep-th]];

T. Aoyama, J. Nishimura and T. Okubo, "Spontaneous breaking of the rotational symmetry in dimensionally reduced super Yang-Mills models," Prog. Theor. Phys. **125**, 537-563 (2011) [arXiv:1007.0883 [hep-th]].;

J. Nishimura, T. Okubo and F. Sugino, "Systematic study of the SO(10) symmetry breaking vacua in the matrix model for type IIB superstrings," JHEP **10**, 135 (2011) [arXiv:1108.1293 [hep-th]].

[26] R. H. Brandenberger and C. Vafa, "Superstrings In The Early Universe," Nucl. Phys. B **316**, 391 (1989).

[27] T. Aoki, M. Hirasawa, Y. Ito, J. Nishimura and A. Tsuchiya, "On the structure of the emergent 3d expanding space in the Lorentzian type IIB matrix model," PTEP **2019**, no. 9, 093B03 (2019) [arXiv:1904.05914 [hep-th]].

[28] R. Hagedorn, "Statistical thermodynamics of strong interactions at high-energies," Nuovo Cim. Suppl. **3**, 147 (1965).

[29] P. Agrawal, S. Gukov, G. Obied and C. Vafa, "Topological Gravity as the Early Phase of Our Universe," arXiv:2009.10077 [hep-th].

[30] W. Siegel, "Two vierbein formalism for string inspired axionic gravity," Phys. Rev. D **47**, 5453 (1993) [hep-th/9302036];
W. Siegel, "Superspace duality in low-energy superstrings," Phys. Rev. D **48**, 2826 (1993) [hep-th/9305073];
W. Siegel, "Manifest duality in low-energy superstrings," hep-th/9308133;
C. Hull and B. Zwiebach, "Double Field Theory," JHEP **0909**, 099 (2009) [arXiv:0904.4664 [hep-th]].

[31] H. Bernardo, R. Brandenberger and G. Franzmann, "String cosmology backgrounds from classical string geometry," Phys. Rev. D **103**, no.4, 043540 (2021) [arXiv:2005.08324 [hep-th]];
H. Bernardo, R. Brandenberger and G. Franzmann, "Solution of the Size and Horizon Problems from Classical String Geometry," JHEP **10**, 155 (2020) [arXiv:2007.14096 [hep-th]].

[32] S. W. Kim, J. Nishimura and A. Tsuchiya, "Late time behaviors of the expanding universe in the IIB matrix model," JHEP **10**, 147 (2012) [arXiv:1208.0711 [hep-th]].

[33] N. Kawahara, J. Nishimura and S. Takeuchi, "High temperature expansion in supersymmetric matrix quantum mechanics," [arXiv:0710.2188].

[34] A. Nayeri, R. H. Brandenberger and C. Vafa, “Producing a scale-invariant spectrum of perturbations in a Hagedorn phase of string cosmology,” Phys. Rev. Lett. **97**, 021302 (2006) [hep-th/0511140].

[35] R. H. Brandenberger, A. Nayeri, S. P. Patil and C. Vafa, “Tensor Modes from a Primordial Hagedorn Phase of String Cosmology,” Phys. Rev. Lett. **98**, 231302 (2007) [hep-th/0604126];

R. H. Brandenberger, A. Nayeri and S. P. Patil, “Closed String Thermodynamics and a Blue Tensor Spectrum,” Phys. Rev. D **90**, no. 6, 067301 (2014) [arXiv:1403.4927 [astro-ph.CO]].

[36] S. Brahma, S. Laliberte and R. Brandenberger, to be published.

[37] F. R. Klinkhamer, “On the emergence of an expanding universe from a Lorentzian matrix model,” PTEP **2020**, 103 [arXiv:1912.12229 [gr-qc]];

F. R. Klinkhamer, “IIB matrix model: Emergent spacetime from the master field,” PTEP **2021**, no. 1, 013B04 (2021) [arXiv:2007.08485 [hep-th]];

F. R. Klinkhamer, “On the emergence of an expanding universe from a Lorentzian matrix model,” PTEP **2020**, no. 10, 103B03 (2020);

F. R. Klinkhamer, “Towards a Numerical Solution of the Bosonic Master-Field Equation of the IIB Matrix Model,” Acta Phys. Polon. B **53**, no.1, 1 (2022) [arXiv:2110.15309 [hep-th]].

[38] H. Steinacker, “Emergent Gravity from Noncommutative Gauge Theory,” JHEP **0712**, 049 (2007) [arXiv:0708.2426 [hep-th]];

H. Grosse, H. Steinacker and M. Wohlgenannt, “Emergent Gravity, Matrix Models and UV/IR Mixing,” JHEP **0804**, 023 (2008) [arXiv:0802.0973 [hep-th]];

H. Steinacker, “Emergent Gravity and Noncommutative Branes from Yang-Mills Matrix Models,” Nucl. Phys. B **810**, 1 (2009) [arXiv:0806.2032 [hep-th]];

H. Steinacker, “Emergent Geometry and Gravity from Matrix Models: an Introduction,” Class. Quant. Grav. **27**, 133001 (2010) [arXiv:1003.4134 [hep-th]];

H. C. Steinacker, “Gravity as a quantum effect on quantum space-time,” Phys. Lett. B **827**, 136946 (2022) [arXiv:2110.03936 [hep-th]].

[39] D. Z. Freedman, G. W. Gibbons and M. Schnabl, “Matrix cosmology,” AIP Conf. Proc. **743**, no.1, 286-297 (2004) [arXiv:hep-th/0411119 [hep-th]];
J. Hoppe, “Some classical solutions of membrane matrix model equations,” NATO Sci. Ser. C **520**, 423-427 (1999) [arXiv:hep-th/9702169 [hep-th]];
J. Hoppe and H. Nicolai, “RELATIVISTIC MINIMAL SURFACES,” Phys. Lett. B **196**, 451-455 (1987).

Part IV

Dynamical mechanism for emergent solutions in matrix cosmology

Chapter 6

Effective mass and symmetry breaking in the Ishibashi-Kawai-Kitazawa-Tsuchiya matrix model from compactification

Samuel Laliberte

[1]*Department of Physics, McGill University, Montréal, QC, H3A 2T8, Canada*

Addendum for the thesis

In Section 2.3.3, we saw how adding a mass term in the IKKT model can lead to cosmological solutions. Despite the fact that these solutions are appealing, the mass term is not a feature known to naturally arise in the IKKT model. In the present chapter, we investigate a mechanism that can realize such a mass term in the IKKT model. Our approach consists in compactifying six dimensions of the IKKT model where the fermions acquire anti-periodic boundary conditions. In this case, the IKKT model acquires an effective mass term that breaks the $SO(1,9)$ symmetry of the IKKT model to SO(1,3) $\times$ SO(6). Given this symmetry-breaking mass term, a cosmological solution where three dimensions become large and six stay small may exist in this setup. This remains to be investigated in future work.

Abstract

The IKKT model is a promising candidate for a non-perturbative description of Type IIB superstring theory. It is known from analytic approaches and numerical simulations that the IKKT matrix model with a mass term admits interesting cosmological solutions. However, this mass term is often introduced by hand, and serves as a regulator in the theory. In the present paper, we show that an effective mass matrix can arise naturally in the IKKT model by imposing a toroidal compactification where the space-time fermions acquire anti-periodic boundary conditions. When six spatial dimensions are chosen to be compact, the effective mass matrix breaks the SO(1,9) space-time symmetry of the IKKT model to SO(1,3) × SO(6). This paves the way for space-time solutions of the IKKT model where SO(1,9) symmetry is naturally broken to SO(1,3) × SO(6).

Erratum

There is a typo in this article. Below equation 6.20, there is a mistake in the text. The statement "X is a $M \times M$ matrix" should read "Y is a $M \times M$ matrix" instead. Moreover, below 6.8 and above equation 6.29, one should read $V = L^{-6}$ since the parameter σ lies in the interval $[0, L^{-1}[$.

6.1 Introduction

Superstring theory is a promising candidate for a self-consistent unified theory of quantum gravity. An interesting feature of the theory is that the dimensionality of space-time is not arbitrary, but comes from the consistency of the theory. Specifically, the theory is only consistently defined in ten space-time dimensions. For this theory to describe our world, one must impose that six out of the nine spatial dimensions are compactified. This can be done in many ways, resulting in a vast landscape of effective descriptions of string theory in four dimensions. In addition to the four dimensional vacua, there exist other ways to consistently compactify string theory to an arbitrary number of dimensions, which results in vacua that are not four dimensional. Clarifying why four dimensional vacua are prefered in the theory remains an open question, which to this day does not have an answer in perturbative string theory.

Another area where perturbative string theory lacks predictive power is in the context of cosmology. While string theory can be used to make predictions about the universe at late time, the cosmic stringularity is not resolved generally in perturbative string theory [1–4]. Therefore, in order to study the very early universe, and explain the dimensionality of our world, we definitely a non-perturbative description.

There have been many proposals for a non-perbative descriptions of string theory, most of them relying on matrix models [5–7]. Among these theories, the IKKT model [7], a non-perturbative description of Type IIB superstring theory, stands out as a natural choice to explain the birth of the universe. This model is built around the action

$$S_{IKKT} = -\frac{1}{4g^2}\mathrm{Tr}[A^M, A^N]^2 - \frac{1}{2g^2}\mathrm{Tr}\bar{\psi}\Gamma^M[A_M, \psi]\,, \tag{6.1}$$

where large bosonic matrices A^M's encode information about space-time, and large fermionic matrices ψ's are added to preserve supersymmetry. In the action above, g is a gauge coupling which is related to the string scale l_s via $g \sim l_s^2$, and the indices are contracted using the minkowski metric in the mostly minus sign convention $\eta_{MN} = \mathrm{diag}(1, -1, -1, ..., -1)$. Given the causal structure of the space, A^0 encodes information about time and A^i encodes information about space, where $i \in \{1, ..., 9\}$ labels the nine space dimensions.

Over the years, there have been many attempts to find solutions of the IKKT model that correspond to an emergent four-dimensional universe. The first steps towards finding these solutions were done by analysing the model using Gaussian expansion method [8, 9] to investigate symmetry breaking in the theory. Using this method, it was shown that the SO(10) symmetry of the Euclidean version of this model can be spontaneously broken to SO(4) [10–13]. Monte Carlo simulations have also shown consistant results [14, 15], and a recent analysis in the context of the BFSS model has also shown progress in this direction [16]. Then, Monte-Carlo simulations of the Lorentzian model showed that an expanding (1+3)-dimensional space-time can emerge from an SO(1,9) symmetric state of the model after a critical time [17–20]. To achieve this result, approximations were made to avoid the sign problem of the Lorentzian theory. However, further studies have shown that these approximations which are no longer valid when the four space-time dimensions emerge [21]. Since then, the Lorenzian model has been studied without this approximation using the Complex Langevin Method, where the emergence of four space-time dimensions remain a topic of study [22–27].

Despite the challenges encountered in studying the IKKT model from a numerical point of view, it was found in [17] that an important feature seems to be required to obtain the emergence of (1+3)-dimensions. When (1+3)-dimensions become large, certain bounds used to regulate theory,

$$\frac{1}{N}\mathrm{Tr}(A_0)^2 \leq \kappa\frac{1}{N}\mathrm{Tr}(A_i)^2 \quad , \tag{6.2}$$

$$\frac{1}{N}\mathrm{Tr}(A_i)^2 \leq \kappa L^2 \, , \tag{6.3}$$

become saturated. Saturating the constraints above is equivalent to adding the following piece to the IKKT action

$$S_{const.} = \frac{\tilde{\lambda}}{2}\mathrm{Tr}\left(A_0^2 - \kappa L^2\right) - \frac{\lambda}{2}\mathrm{Tr}\left(A_i^2 - L^2\right) \, , \tag{6.4}$$

where λ and $\tilde{\lambda}$ are Langrange multipliers. Minimising the IKKT action in the presence of the constraint above, it was shown analytically [28] and numerically [29] that various cosmological solutions of the equation of motions can be found. An important point to notice is that adding the constraint piece in equation 6.4 to the IKKT action is equivalent to adding a mass term to the theory, which may or may not be Lorentz invariant depending on the choice of λ and $\tilde{\lambda}$. Hence, adding a mass term to the theory can lead to interesting cosmological solutions. In fact, various analyses of the Lorentzian IKKT model with a mass term have been done before, in which case it was shown that cosmological solutions can also be found [30–33].[1]

Since an effective mass term arises as a possible explanation for the emergence of cosmological solutions, it seems natural to ask what conditions are necessary for a mass term to naturally appear in the theory, and what causes the symmetry of space-time to break from SO(1,9) to SO(1,3) $\times$ SO(6). In the present paper, we explore this question by studying

[1] See [34–38] for other deformations of the IKKT model which admit cosmological solutions, and [39–41] for recent progress in the study of cosmological solutions in the IKKT model.

compactifications of the IKKT model. We find that if six spatial dimensions are compactified in a way that supersymmetry is broken, space-time fermions are quenched and the IKKT model action develops an effective mass matrix that breaks the SO(1,9) symmetry of the model to SO(1,3) $\times$ SO(6). This leads the way for solutions of the IKKT model where the SO(1,9) symmetry of space-time is naturally broken to SO(1,3) $\times$ SO(6).

6.1.1 Outline

To obtain the mass matrix, we will proceed as follows. In section 6.2, we will Wick rotate the Lorentzian IKKT model by imposing the change of variables $A^0 \to iA^0$ and $\Gamma^i \to i\Gamma^i$ to obtain the Euclidean IKKT model action

$$S_{IKKT} = -\frac{1}{4g^2}\mathrm{Tr}[A^M, A^N]^2 - \frac{i}{2g^2}\mathrm{Tr}\bar{\psi}\Gamma^M[A_M, \psi]\,. \tag{6.5}$$

This transformation to Euclidean space will be done to simplify computations. Then, we will compactify this action on a six-dimensional torus where the space-time fermions ψ acquire anti-periodic boundary conditions, hence breaking supersymmetry. As a result, the IKKT model action under compactification will become equivalent to a six-dimensional Yang-Mills theory with the following action

$$S_C = \frac{1}{2g_{eff}^2}\int \frac{d\sigma^6}{V}\mathrm{Tr}\left(\frac{1}{2}F_{ab}F^{ab} + D_aA_\mu D^aA^\mu - \frac{1}{2}[A^\mu, A^\nu]^2 + \bar{\psi}\Gamma^a D_a\psi \right. \tag{6.6}$$

$$\left. -i\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right)\,, \tag{6.7}$$

where we have substituted the mode expansion

$$A^M = \sum_{n^a\in\mathbb{Z}^6} A^M(n^a)e^{in^a\sigma^a} \quad , \quad \psi = \sum_{r^a\in\mathbb{Z}^6+1/2}\psi(r^a)e^{ir^a\sigma^a}\,. \tag{6.8}$$

Here, $\mu \in \{0,...,3\}$ labels the non-compact directions, $a \in \{4,5,..,9\}$ labels the compact directions, $V = (2\pi L)^6$ is the volume of the internal space, $g_{eff}^2 = g^2/N$ is an effective gauge coupling, and N is a large integer that we will introduce later. In this six-dimensional Yang-Mills theory, the zero modes describe non-compact degrees of freedom, and the non-zero models describe interactions between these non-compact degrees of freedom. Hence, integrating out the non-zero modes in the theory, one can obtain a Wilsonian effective action for the non-compact degrees of freedom in the theory. In section 6.3, we will compute this Wilsonian effective action from the expression

$$S^0_{eff} = -\ln\left(\prod_{n^a\in\mathbb{Z}^6}{}' \prod_{r^b\in\mathbb{Z}^6+1/2} \int \mathcal{D}A^M(n^a)\mathcal{D}\psi(r^b)\, e^{-S_E}\right) . \tag{6.9}$$

Here, $\prod'$ means that we are not integrating over the zero modes $n^a = 0$ of the theory. This computation will be done in the decompactification limit $L \gg g_{eff}^{1/2}$, where perturbation theory is valid and we expect to obtain a result close to the IKKT action without the compactification constraint (equation 6.1). Carrying out the computation to leading order in perturbation theory and Wick rotating back to Lorentzian space, we will find that the effective action takes the form

$$S^0_{eff} = -\frac{1}{4g^2_{eff}}\mathrm{Tr}[A^M(0), A^N(0)]^2 + \frac{1}{2}M^2_{MN}\mathrm{Tr}(A^M(0)A^N(0))^2 + \ldots , \tag{6.10}$$

where the mass matrix

$$M^2_{MN} = \begin{bmatrix} \eta_{\mu\nu}M_4^2 & 0 \\ 0 & \eta_{ab}M_6^2 \end{bmatrix} \tag{6.11}$$

arises as a first order correction which breaks SO(1,9) symmetry to SO(1,3) $\times$ SO(6). In the expression above, the masses M_4^2 and M_6^2 take the values

$$M_4^2 = 16\,(S_{F_1} - S_{B_1})\,\frac{NM}{L^2} , \tag{6.12}$$

$$M_6^2 = \frac{32}{3}\,(S_{F_1} - S_{B_1})\,\frac{NM}{L^2} , \tag{6.13}$$

where the constants S_{B_1} and S_{F_1} are determined by the following sums

$$S_{B_1} = \sum_{n^a\in\mathbb{Z}^6}{}' \frac{1}{(2\pi n^a)^2} \quad , \quad S_{F_1} = \sum_{r^a\in\mathbb{Z}^6+1/2} \frac{1}{(2\pi r^a)^2} . \tag{6.14}$$

The sums S_{B_1} are S_{F_1} are divergent in the large n^a and r^a limit. However, the difference between these two sums is finite and takes the value $S_{F_1} - S_{B_1} \approx 0.0397887$ when evaluated numerically.

The reason why we obtain equation 8.17 and not equation 6.1 in the decompactification limit is because of broken supersymmetry. Since the fermions have anti-periodic boundary conditions, the fermionic zero modes are projected away in the mode expansion. Hence, the fermionic sector does not enter the zero-mode effective action. We are left with the bosonic part of the IKKT action, and a mass matrix coming from integrating out interactions between the zero modes degrees of freedom in the theory. If supersymmetry is restored by imposing that fermions have periodic boundary conditions, r^a becomes summed over $\mathbb{Z}^6$ instead of $\mathbb{Z}^6 + 1/2$ in the sum S_{F_1}. In this case, the masses M_4^2 and M_6^2 vanish since $S_{B_1} = S_{B_2}$, the fermions acquire a zero-mode term, and we obtain the IKKT model action (equation 6.1) with an effective gauge coupling g_{eff}.

6.2 Compactification of the IKKT model

Compactifying a matrix model presents a different challenge than compactifying a field theory. For one, there are no free parameters in the matrix model that we can choose to be compact. Hence, we must impose conditions on the matrices themselves. To overcome this challenge, we will make use of the method of mirror images, which was first brought forward by Washington Taylor in the context of D-brane mechanics [42]. This method proved successful to explain graviton scattering under toroidal compactification of the BFSS model [43], and has recently been used to explain three gravition amplitudes [44] and soft theorems [45] in this same model.

This method builds on the fact that toroidal compactification is equivalent to duplicating a fundamental region of the target space an infinite number of time along said direction. For example, let us suppose we wish to compactify the real line $x \in \mathbb{R}$ on a circle S^1 of radius R. One option would be to confine the real line to an interval $x \in [0, 2\pi R[$ where we impose periodic boundary condition. Another would be to invoke the fact that periodic boundary conditions are equivalent to duplicating the interval $[0, 2\pi R[$ an infinite number of times along the real line. In other words, each point on the real line can be associated to a point a distance $x \to x + 2\pi R$ away from this point. The mathematical term for this operation is called going to the universal cover of the circle.

The same procedure can be applied to matrix models to impose a compactification. Since the matrix model describes a target space, we can impose that the target space contains duplicated objects in the direction we want to compactify in an attempt to replicate the effects of a compact space. To see how this is done in the context of the IKKT model, let us first Wick rotate the Lorentzian IKKT model to Euclidan space by imposing the change of variables $A^0 \to iA^0$ and $\Gamma^i \to i\Gamma^i$ in the Laurentzian IKKT model action. We obtain

$$S_{IKKT} = -\frac{1}{4g^2}\mathrm{Tr}[A^M, A^N]^2 - \frac{i}{2g^2}\mathrm{Tr}\bar{\psi}\Gamma^M[A_M, \psi]\,. \tag{6.15}$$

As previously mentioned, we will be interested in configurations of the IKKT model where six spatial directions A^a are compact, and where fermions acquire anti-periodic boundary conditions. Such compactifications were first studied in the BFSS model [46], and have more recently been used to obtain a thermal state of the IKKT model [48]. In the present case, we will generalise the approach taken in [48] to the case where six dimensions are compactified. To do this, we will invoke the existence of unitary operators U^a, which generate a translation in the A^a direction of the target space. In addition, we impose that these operators commute with each other,

$$U^a U^b = U^b U^a\,, \tag{6.16}$$

so that translations in different compact directions can be made independently of each other. Following our previous discussion, capactifying the target space on a six-dimensional torus where fermions acquire anti-periodic boundary conditions should be equivalent to imposing the conditions

$$(U^b)^{-1}A^\mu U^b = A^\mu \tag{6.17}$$

$$(U^b)^{-1}A^a U^b = A^a + 2\pi L\delta_{ab} \tag{6.18}$$

$$(U^b)^{-1}\psi U^b = -\psi\,, \tag{6.19}$$

where L is the torus radius. Here, μ labels the non-compact space-time directions and a labels the compact space directions. To solve the constraint equation above, we will use a approach similar to the one in [49] and assume that the Hilbert space that the A's and ψ's act on has the tensor product form

$$X = Y \otimes Z\,, \tag{6.20}$$

where X is a $M \times M$ matrix that will remain invariant under the translation, and Z is a $N \times N$ matrix associated to the Hilbert space the translations act on. We will then invoke that U^a takes the following form

$$U^a = \mathbb{I}_M \otimes e^{-i2\pi q^a} e^{-ip^a} \,, \tag{6.21}$$

where $\mathbb{I}_M$ is the M-dimensional identity operator and q^a and p^b are operators that satisfy the Heisenberg algebra $[q^a, p^b] = i\delta_{ab}$. With the form above, the unitary operator U^a satisfies $(U^a)^{-1} q^a U^a = q^a + 1$, and generates a shift from q^a to $q^a + 1$. The extra factor of $e^{-i2\pi q^a}$ does not affect this shift. However, it will play a role in achieving the anti-periodic boundary conditions for the fermions. Next, we will note that a matrix of the form

$$B = \sum_{n^a} B(n^a) \otimes e^{in^a p^a} \,, \tag{6.22}$$

satisfies $(U^a)^{-1} B U^a = B$ if n^a is an integer, and $(U^a)^{-1} B U^a = -B$ if n^a if a half-integer. Consequently, it's possible to solve the constraint equations by imposing that the matrices A^μ, A^a, and ψ take the following form

$$A^\mu = \sum_{n^b \in \mathbb{Z}^6} A^\mu(n^b) \otimes e^{in^b p^b} \tag{6.23}$$

$$A^a = \sum_{n^b \in \mathbb{Z}^6} A^a(n^b) \otimes e^{in^b p^b} + 2\pi L \, \mathbb{I}_M \otimes q^a \tag{6.24}$$

$$\psi = \sum_{r^b \in \mathbb{Z}^6 + 1/2} \psi(r^b) \otimes e^{ir^b p^b} \,. \tag{6.25}$$

In the expressions above, n^b and r^b are summed over N integers and half-integers respectively, where N is taken to be large but finite. It's possible to show that, when written in the $|q^a\rangle$ basis, that the matrices above take the block Toeplitz form depicted in Figure 6.1. In this block Toeplitz form, the diagonal blocks describe the distribution of objects within an interval $[0, 2\pi L[$, and their interactions. The off-diagonal blocks, on their side, describe interactions between the duplicated fundamental regions. Substituting the matrices above in the IKKT model action and using the identities

$$[q^a, e^{inp^b}] = -n e^{inp^b} \delta_{ab} \quad , \quad \mathrm{Tr} e^{i(n\pm m)p^b} = N\delta(n \pm m) \,, \tag{6.26}$$

we obtain the momentum space representation of the Yang-Mills action

$$S_C = \frac{1}{2g_{eff}^2} \int \frac{d\sigma^6}{V} \mathrm{Tr} \left(\frac{1}{2} F_{ab} F^{ab} + D_a A_\mu D^a A^\mu - \frac{1}{2}[A^\mu, A^\nu]^2 + \bar{\psi}\Gamma^a D_a \psi \right. \tag{6.27}$$

$$\left. -i\bar{\psi}\Gamma^\mu [A_\mu, \psi] \right) \tag{6.28}$$

$$A^\mu = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & A^\mu(0) & A^\mu(1) & A^\mu(2) & \dots \\ \dots & A^\mu(-1) & A^\mu(0) & A^\mu(1) & \dots \\ \dots & A^\mu(-2) & A^\mu(-1) & A^\mu(0) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$A^a = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & A^a(0) - 2\pi L & A^a(1) & A^a(2) & \dots \\ \dots & A^a(-1) & A^a(0) & A^a(1) & \dots \\ \dots & A^a(-2) & A^a(-1) & A^a(0) + 2\pi L & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

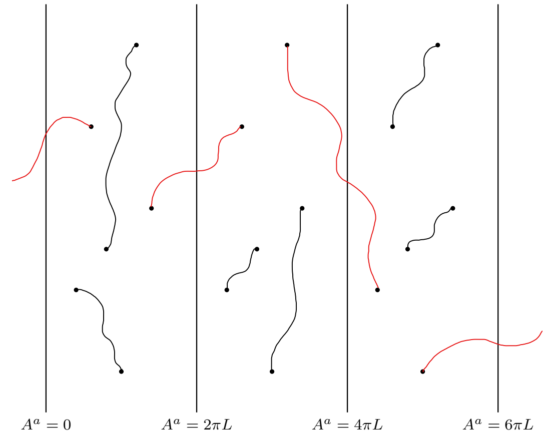

Figure 6.1: Left: The diagonal blocks (black) describe the distribution of objects and their interactions in the duplicated regions, and the off-diagonal blocks (red) describe interactions between the duplicated regions. Right: Sketch of the duplicated regions along a compact direction A^a. The line between the black dots depict interactions inside (black) and across duplicated regions (red).

where $g_{eff}^2 = g^2/N$ is an effective gauge coupling, $V = (2\pi L)^6$ is the volume of the internal space and A^M and ψ are expanded using the mode decomposition

$$A^M = \sum_{n^a \in \mathbb{Z}^6} A^M(n^a)\, e^{in^a\sigma^a} \quad , \quad \psi = \sum_{r^a \in \mathbb{Z}^6 + 1/2} \psi(r^a)\, e^{ir^a\sigma^a} \, . \tag{6.29}$$

Here, σ takes values inside the interval $[0, L^{-1}[$. Moreover, A^M and ψ respectively satisfy periodic and anti-periodic boundary conditions. In the mode expansion above, the zero modes are related to the distribution of objects and their interactions in the fundamental regions, and the non-zero modes are associated to interaction between fundamental regions [50]. In the decompactification limit $L \gg g_{eff}^{1/2}$, one should expect the fundamental regions to be far away from each other. In the case, interactions will be suppressed, and we should obtain a theory which is approximately described by the dynamics of the zero-modes of the theory. This can be seen by looking at the mode expansion

$$S_C = -\frac{1}{4g_{eff}^2}\mathrm{Tr}[A^M(0), A^N(0)]^2 + \frac{1}{2g_{eff}^2}\sum_{n^a \in \mathbb{Z}^6} (2\pi L n^a)^2 \mathrm{Tr}\left(A^M(-n^a)A^M(n^a)\right) \tag{6.30}$$

$$+ \frac{1}{2g_{eff}^2}\sum_{r^a \in \mathbb{Z}^6 + 1/2} (2\pi L r^a i)\mathrm{Tr}\left(\bar{\psi}(r^a)\Gamma^a\psi(r^a)\right) + \dots , \tag{6.31}$$

of the compact IKKT action. In the $L \gg g_{eff}^{1/2}$ limit, the non-zero winding modes $\omega_{n^a} = 2\pi L n^a$ and $\omega_{r^a} = 2\pi L r^a$ associated to the second and third term become heavy, and interactions become suppressed in the path integral. As a result, we expect the compact IKKT action to be effectively described by the zero modes of the system. This means we should recover the bosonic IKKT model action

$$S_C = -\frac{1}{4g_{eff}^2}\mathrm{Tr}[A^M(0), A^N(0)]^2 + \ldots, \tag{6.32}$$

and possible corrections coming from interactions between the fundamental regions. The fermions, in this case, do not contribute since their zero mode are projected away by the anti-periodic boundary conditions. As the radius of compactification L decreases, one should expect that interactions become important, leading to more corrections to equation 6.32. In the following sections, we will derive the leading corrections to equation 6.32 by evaluating a Wilsonian effective action for the zero modes $A^M(0)$ of the theory. In the limit where $L \gg g_{eff}^{1/2}$, we will see that the effective action of the zero modes aquires a mass matrix as first order correction, leading to symmetry breaking in the theory.

6.3 Wilsonian effective action

To compute an effective action for the zero modes of the theory, which describes the non-compact degrees of freedom, we will adopt a Wilsonian approach. This appraoch will consist in integrating out the non-zero modes in the path integral in order to obtain an action that depends exclusively on the zero modes of the theory.

To see how this can be done, let us remind ourselves that a Wilsonian effective action can be used to find an effective description of the low energy modes of a theory by integrating out high energy modes above a cutoff Λ. For example, let us consider the action $S[\Phi]$ associated to a scalar field Φ. To obtain the low energy effective action for some long wavelength modes Φ_L, we can split the scalar field $\Phi = \Phi_L + \Phi_S$ into the contributions from Φ_L and the short wavelength component Φ_S. Then, the contribution of the short wavelength modes Φ_S can be integrated out in the partition function in the following way

$$Z = \int \mathcal{D}\Phi e^{-S[\Phi]} \tag{6.33}$$

$$= \int \mathcal{D}\Phi_L \left(\int \mathcal{D}\Phi_S e^{S[\Phi_L + \Phi_S]} \right) \tag{6.34}$$

$$= \int \mathcal{D}\Phi_L e^{-S_{eff}[\Phi_L]}\,, \tag{6.35}$$

to obtain a Wilsonian effective action $S_{eff}[\Phi_L]$ of the short wavelength component Φ_L. This Wilsonian effective action can be then computed from the expression

$$S_{eff}[\Phi_L] = -\ln \left(\int \mathcal{D}\Phi_S e^{S[\Phi_L + \Phi_S]} \right)\,. \tag{6.36}$$

In the present case, we want to obtain an effective action of the zero modes $A^M(0)$ of the theory. This means that, in the Wilsonian sense, we must integrate out all the non-zero modes $A^M(n^a)$ for $n^a \neq 0$ and $\psi(r^a)$ in the path integral. To do this, we can split $A^M = A^M(0) + \sum'_{n^a \in \mathbb{Z}^6} A^M(n^a) e^{in^a\sigma^a}$ into the zero-mode component $A^M(0)$ and and the non-zero-mode component $\sum'_{n^a \in \mathbb{Z}^6} A^M(n^a) e^{in^a\sigma^a}$. Here, $\sum'$ means that we don't sum over the zero modes $n^a = 0$. We will then integrate out the non-zero-modes in the partition function in same way as for our scalar field example. For the compact IKKT action (equation 6.28), this gives us

$$Z = \prod_{n^a r^b \in \mathbb{Z}^6} \int \mathcal{D}A^M(n^a) \mathcal{D}\psi(r^b)\, e^{-S_C} \tag{6.37}$$

$$= \int \mathcal{D}A^M(0) \left(\prod'_{n^a \in \mathbb{Z}^6} \prod_{r^b \in \mathbb{Z}^6 + 1/2} \int \mathcal{D}A^M(n^a) \mathcal{D}\psi(r^b)\, e^{-S_C} \right) \tag{6.38}$$

$$= \int \mathcal{D}A^M(0) e^{-S^0_{eff}}\,, \tag{6.39}$$

where

$$S^0_{eff} = -\ln \left(\prod'_{n^a \in \mathbb{Z}^6} \prod_{r^b \in \mathbb{Z}^6 + 1/2} \int \mathcal{D}A^M(n^a) \mathcal{D}\psi(r^b)\, e^{-S_C} \right)\,, \tag{6.40}$$

can be identified as the zero-mode effective action. Here again, we remind the reader that $\prod'$ means we integrate over all the modes $n^a \in \mathbb{Z}^6$ except the zero modes $n^a = 0$ of the theory. This means that S^0_{eff} will depend exclusively on the zero modes $A^M(0)$ that haven't been integrated over. The goal of the next sections will be to compute the quantity above. This will be done using standard perturbative methods.

6.3.1 Choice of gamma matrix representation and gauge fixing

As a first step towards computing equation 6.40, we will choose a convenient representation for the gamma matrices that reflects the fact that $SO(10)$ symmetry is broken to $SO(4) \times SO(6)$ by our choice of compactification. We will do this in a way to preserves the Majorana and Weyl conditions

$$\Gamma_{11}\psi = \psi \quad , \quad \bar{\psi} = \psi^T C_{10} \tag{6.41}$$

which the fermions must satisfy for the theory to be supersymmetric. Here, Γ_{11} and C_{10} are respectively the chirality operator and the charge conjugation operator in 10 dimensions. In the present case, we will use the representation introduced in [51] and consider Gamma matrices of the form

$$\Gamma^a = \tilde{\Gamma}^a \otimes 1 \quad , \quad \Gamma^\mu = \tilde{\Gamma}_7 \otimes \gamma^\mu \,, \tag{6.42}$$

where $\tilde{\Gamma}^a$ are SO(6) gamma matrices, $\tilde{\Gamma}_7$ is the chirality operator these matrices and γ^μ are SO(4) gamma matrices (see [52] for other convenient representations). We will further require that the SO(4) gamma matrices are in the Weyl representation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{6.43}$$

where σ^μ and $\bar{\sigma}^\mu$ are Pauli 4-vectors which satisfy

$$\bar{\sigma}_0 = \sigma_0 = 1 \quad , \quad \bar{\sigma}_i = -\sigma_i \quad , \quad \{\sigma_i, \sigma_j\} = -2\delta_{ij} \,. \tag{6.44}$$

In this representation, the chirality and charge conjugation operator for the 10 dimensional Gamma matrices take the form

$$\Gamma_{11} = \tilde{\Gamma}_7 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad C_{10} = C_6 \otimes \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \,. \tag{6.45}$$

Therefore, the Majorana and Weyl conditions reduce to

$$\psi = \begin{pmatrix} \psi_+^A \\ \psi_-^A \end{pmatrix} \quad , \quad \tilde{\Gamma}_7 \psi_\pm^A = \pm \psi_\pm^A \quad , \quad \psi_\pm^A = \pm \epsilon^{AB} C_6 (\bar{\psi}_\pm^A)^T \,, \tag{6.46}$$

where $A = 1, 2$. Given our choice of gamma matrices, the compact IKKT action takes the form

$$S_C = \frac{1}{2g_{eff}^2} \int \frac{d\sigma^6}{V} \mathrm{Tr} \left(\frac{1}{2} F_{ab} F^{ab} + D_a A_\mu D^a A^\mu - \frac{1}{2} [A^\mu, A^\nu]^2 + \frac{1}{2} \bar{\psi}_+^A \tilde{\Gamma}^a \partial_a \psi_+^A \right. \tag{6.47}$$

$$+ \frac{1}{2} \bar{\psi}_-^A \tilde{\Gamma}^a \partial_a \psi_-^A - \frac{i}{2} \bar{\psi}_+^A \tilde{\Gamma}^a [A_a, \psi_+^A] - \frac{i}{2} \bar{\psi}_-^A \tilde{\Gamma}^a [A_a, \psi_-^A] + \frac{i}{2} \bar{\psi}_+^A (\sigma^\mu)^{AB} [A_\mu, \psi_-^B] \tag{6.48}$$

$$\left. - \frac{i}{2} \bar{\psi}_-^A (\bar{\sigma}^\mu)^{AB} [A_\mu, \psi_+^B] \right) . \tag{6.49}$$

In addition to our choice of gamma matrices, we will choose to work in the Lorenz gauge $\partial_a A^a = 0$. This choice can be imposed by adding the ghost term

$$S_{gh} = \frac{1}{g_{eff}^2} \int \frac{dx^6}{V} \mathrm{Tr} \left(\partial^a \bar{c} D_a c \right) . \tag{6.50}$$

to the compact IKKT action.

6.3.2 Mode expansion

Next, we will decompose the compact IKKT action into its different Fourier modes and separate the zero mode and the non-zero mode of the action. To do this, we will first separate the compact IKKT action $S_c = S_{kin} + S_{int}$ in a kinetic part

$$S_{kin} = \int \frac{dx^6}{V} \mathrm{Tr} \left(\frac{1}{2} \partial_a A_N \partial^a A^N + \frac{1}{2} \bar{\psi}_+^A \tilde{\Gamma}^a \partial_a \psi_+^A + \frac{1}{2} \bar{\psi}_-^A \tilde{\Gamma}^a \partial_a \psi_-^A + \partial_a \bar{c} \partial^a c \right) , \tag{6.51}$$

and an interaction part

$$S_{int} = \int \frac{dx^6}{V} \mathrm{Tr} \left(-i \partial_a A_N [A^a, A^N] - \frac{1}{4} [A^M, A^N]^2 - \frac{i}{2} \bar{\psi}_+^A \tilde{\Gamma}^a [A_a, \psi_+^A] \right. \tag{6.52}$$

$$\left. - \frac{i}{2} \bar{\psi}_-^A \tilde{\Gamma}^a [A_a, \psi_-^A] + \frac{i}{2} \bar{\psi}_+^A (\sigma^\mu)^{AB} [A_\mu, \psi_-^B] - \frac{i}{2} \bar{\psi}_-^A (\bar{\sigma}^\mu)^{AB} [A_\mu, \psi_+^B] - i \partial_a \bar{c} [A^a, c] \right) . \tag{6.53}$$

Then, we will rescale the gauge fields, the fermions and the ghosts to make them dimensionless using the change of variable

$$A^M \to \lambda L A^M \quad , \quad \psi_+ \to \lambda L^{3/2} \psi_+ \quad , \quad \psi_- \to \lambda L^{3/2} \psi_- \quad , \quad c \to \lambda L c \, . \tag{6.54}$$

Here, λ is a dimensionless parameter defined by $\lambda^2 \equiv g_{eff}^2/L^4$, which will play a role later in the perturbative expansion of equation 6.40. Finally, we will substitute the mode expansion

$$A^M = \sum_{n^a \in \mathbb{Z}^6} A^M(n^a) e^{i2\pi L n^a \sigma^a}\,, \tag{6.55}$$

$$\psi = \sum_{r^a \in \mathbb{Z}^6 + 1/2} \psi(r^a) e^{i2\pi L r^a \sigma^a}\,, \tag{6.56}$$

$$c = \sum_{n^a \in \mathbb{Z}^6} c(n^a) e^{i2\pi L n^a \sigma^a}\,, \tag{6.57}$$

in $S_C = S_{kin} + S_{int}$. After this sustitution, the compact IKKT action can be written in the form $S_C = S_0 + S'_{kin} + S'_{int}$, where

$$S_0 = -\frac{\lambda^2}{4} \mathrm{Tr}[A^M(0), A^N(0)]^2 \tag{6.58}$$

is the zero mode part of the action, and

$$S'_{kin} = \frac{1}{2} \sum_{n^a \in \mathbb{Z}^6}{}' (2\pi n^a)^2 \mathrm{Tr}\left(A_M(n^a) A^M(-n^a)\right) + \frac{1}{2} \sum_{r^a \in \mathbb{Z}^6 + 1/2} (2\pi r^a i) \bar{\psi}_+^A(r^a) \tilde{\Gamma}^a \psi_+^A(r^a) \tag{6.59}$$

$$+ \frac{1}{2} \sum_{r^a \in \mathbb{Z}^6 + 1/2} (2\pi r^a i) \bar{\psi}_-^A(r^a) \tilde{\Gamma}^a \psi_-^A(r^a) + \sum_{n^a \in \mathbb{Z}^6}{}' (2\pi n^a)^2 \mathrm{Tr}\left(\bar{c}(n^a) c(n^a)\right) \tag{6.60}$$

is the kinetic part where the zero modes, which do not contribute, are not summed over. The final term, corresponding to the interaction part where the zero modes have been removed, takes the form

$$S'_{int} = \sum_{i=1}^{5} V_i\,, \tag{6.61}$$

where the V_i's are given by

$$V_1 = -\frac{\lambda^2}{4} \sum_{n^a m^a l^a \in \mathbb{Z}^6}{}' \operatorname{Tr}\left([A^M(-n^a - m^a - l^a), A^N(n^a)][A_M(m^a), A^N(l^a)]\right) \tag{6.62}$$

$$V_2 = -\lambda \sum_{n^a m^a \in \mathbb{Z}^6}{}' 2\pi(n^a + m^a)\operatorname{Tr}\left(A_M(-n^a - m^a)[A^a(n^a), A^M(m^a)]\right) \tag{6.63}$$

$$V_3 = -\frac{i}{2}\lambda \sum_{r^a \in \mathbb{Z}^6 + 1/2\,,\, n^a \in \mathbb{Z}^6}{}' \operatorname{Tr}\left(\bar{\psi}_+^A(r^a + n^a)\tilde{\Gamma}^b[A_b(n^a), \psi_+^A(r^a)]\right) \tag{6.64}$$

$$V_4 = -\frac{i}{2}\lambda \sum_{r^a \in \mathbb{Z}^6 + 1/2\,,\, n^a \in \mathbb{Z}^6}{}' \operatorname{Tr}\left(\bar{\psi}_-^A(r^a + n^a)\tilde{\Gamma}^b[A_b(n^a), \psi_-^A(r^a)]\right) \tag{6.65}$$

$$V_5 = \frac{i}{2}\lambda \sum_{r^a \in \mathbb{Z}^6 + 1/2\,,\, n^a \in \mathbb{Z}^6}{}' \operatorname{Tr}\left(\bar{\psi}_+^A(r^a + n^a)(\sigma^\mu)^{AB}[A_\mu(n^a), \psi_-^B(r^a)]\right) \tag{6.66}$$

$$V_6 = -\frac{i}{2}\lambda \sum_{r^a \in \mathbb{Z}^6 + 1/2\,,\, n^a \in \mathbb{Z}^6}{}' \operatorname{Tr}\left(\bar{\psi}_-^A(r^a + n^a)(\bar{\sigma}^\mu)^{AB}[A_\mu(n^a), \psi_+^B(r^a)]\right) \tag{6.67}$$

$$V_7 = -\lambda \sum_{n^a \in \mathbb{Z}^6 + 1/2\,,\, m^a \in \mathbb{Z}^6}{}' 2\pi(n^a + m^b)\operatorname{Tr}\left(\bar{c}(n^a + m^a)[A_a(n^a), c(m^a)]\right) \,. \tag{6.68}$$

6.3.3 Zero mode effective action

We are now in a position to evaluate the Wilsonian effective action for the zero modes of the theory. Before taking on the task of evaluating equation 6.40, let us pause and notice that the only free parameter in equation 6.58 to 6.68 is the dimensionless quantity λ. In the computation that follows, λ will play the role of expansion parameter. Since S_0 is an $\mathcal{O}(\lambda^2)$ quantity, we will only be concerned with corrections to S_0 that contribute at $\mathcal{O}(\lambda^2)$ order, neglecting the higher order corrections. This approximation is valid when $\lambda \ll 1$, or in other words when $L \gg g_{eff}^{1/2}$. In the IKKT model, g^2 is related to the string scale l_s via $g^2 \sim l_s^4$. Hence, our approximation will be valid when the compactification radius L is much larger than the string lenght l_s.

To evaluate 6.40, we will first substitute $S_E = S_0 + S'_{kin} + S'_{int}$ in our definition for the zero mode effective action of the theory (equation 6.40). We obtain

$$S^0_{eff} = -\ln\left(\prod_{n^a\in\mathbb{Z}^6}{}' \prod_{r^b\in\mathbb{Z}^6+1/2}\int \mathcal{D}A^M(n^a)\mathcal{D}\psi(r^b)\, e^{-S_C}\right) \tag{6.69}$$

$$= -\ln\left(e^{-S_0}\prod_{n^a\in\mathbb{Z}^6}{}' \prod_{r^b\in\mathbb{Z}^6+1/2}\int \mathcal{D}A^M(n^a)\mathcal{D}\psi(r^b)\, e^{-S'_{kin}-S'_{int}}\right) \tag{6.70}$$

$$= S_0 - \ln Z_{kin} - \ln\langle e^{-S'_{int}}\rangle\,, \tag{6.71}$$

where we have defined

$$Z_{kin} = \prod_{n^a\in\mathbb{Z}^6}{}' \prod_{r^b\in\mathbb{Z}^6+1/2} \mathcal{D}A^M(n^a)\mathcal{D}\psi(r^b)\, e^{-S'_{kin}}\,, \tag{6.72}$$

$$\langle\,,\,\rangle = \frac{1}{Z_{kin}}\prod_{n^a\in\mathbb{Z}^6}{}' \prod_{r^b\in\mathbb{Z}^6+1/2}\int \mathcal{D}A^M(n^a)\mathcal{D}\psi(r^b)\,.\, e^{-S'_{kin}}\,. \tag{6.73}$$

As expected, the first term lets us recover the bosonic part of the IKKT action. The second term, on its side, does not depend on $A^M(0)$ and is non-dynamical. For this reason, we will simply ignore it. Finally, we have the term $-\ln\langle e^{-S_{int}}\rangle$ which is dynamical and will bring correction to the bosonic IKKT action. This term can be evaluated perturbatively by expanding it in the form

$$-\ln\langle e^{-S_{int}}\rangle = \langle S_{int} - \frac{1}{2}S^2_{int} + ...\rangle_c \tag{6.74}$$

$$= \langle V_1\rangle_c - \frac{1}{2}\langle V_2^2\rangle_c - \frac{1}{2}\langle V_3^2\rangle_c - \frac{1}{2}\langle V_4^2\rangle_c - \langle V_5 V_6\rangle_c - \frac{1}{2}\langle V_7^2\rangle_c + ... \tag{6.75}$$

where $\langle . \rangle_c$ denotes the fact that only connected diagrams contribute to the expectation value. In the expression above, we have only kept the terms that contribute to leading order ($\mathcal{O}(\lambda^2)$). All other contributions from the vertex terms (equation 6.63 to 6.68) either vanish, contribute at next to leading ($\mathcal{O}(\lambda^4)$) order, or at a higher order in the expansion parameter

λ. To evaluate the quantities above, it's useful to write down the two point functions

$$\langle A^M(n^a)A^N(m^a)\rangle = \frac{\delta_{MN}\delta_{n^a+m^a,0}}{(2\pi n^a)^2} \tag{6.76}$$

$$\langle \bar{\psi}^A_{+\alpha}(r^a)\psi^B_{+\beta}(s^a)\rangle = -i\frac{2\pi r^a\tilde{\Gamma}^a_{\alpha\beta}\delta_{AB}\delta_{r^a,s^a}}{(2\pi r^a)^2} \tag{6.77}$$

$$\langle \bar{\psi}^A_{-\alpha}(r^a)\psi^B_{-\beta}(s^a)\rangle = -i\frac{2\pi r^a\tilde{\Gamma}^a_{\alpha\beta}\delta_{AB}\delta_{r^a,s^a}}{(2\pi r^a)^2} \tag{6.78}$$

$$\langle \bar{c}(n^a)c(m^a)\rangle = \frac{\delta_{n^a,m^b}}{(2\pi n^a)^2}\,, \tag{6.79}$$

for the gauge fields, the fermions and the ghosts.[2] Using the two point functions above, we find

$$\langle V_1\rangle_c = 9\lambda^2 M S_{B_1}\mathrm{Tr}(A^N(0))^2 \tag{6.80}$$

$$\langle V_2^2\rangle_c = 2\lambda^2 M\left((17S_{B_2}+S_{B_1})\,\mathrm{Tr}(A_0^a)^2 + S_{B_1}\mathrm{Tr}(A^\mu(0))^2\right) \tag{6.81}$$

$$\langle V_3^2\rangle_c = -8\lambda^2 M(2S_{F_2}-S_{F_1})\mathrm{Tr}(A^a(0))^2 \tag{6.82}$$

$$\langle V_4^2\rangle_c = -8\lambda^2 M(2S_{F_2}-S_{F_1})\mathrm{Tr}(A^a(0))^2 \tag{6.83}$$

$$\langle V_5V_6\rangle_c = 8\lambda^2 M S_{F_1}\mathrm{Tr}(A^\mu(0))^2 \tag{6.84}$$

$$\langle V_7^2\rangle_c = -2\lambda^2 M S_{B_2}\mathrm{Tr}(A^a(0))^2 \tag{6.85}$$

where S_{B_1} , S_{B_2} , S_{F_1} and S_{F_2} are defined as follows

$$S_{B_1} = {\sum_{n^a\in\mathbb{Z}^6}}' \frac{1}{(2\pi n^a)^2} \quad , \quad S_{F_1} = \sum_{r^a\in\mathbb{Z}^6+1/2}\frac{1}{(2\pi r^a)^2} \tag{6.86}$$

$$S_{B_2} = {\sum_{n^a\in\mathbb{Z}^6}}' \frac{(2\pi n^1)^2}{(2\pi n^a)^4} \quad , \quad S_{F_2} = \sum_{r^a\in\mathbb{Z}^6+1/2}\frac{(2\pi r^1)^2}{(2\pi r^a)^4}\,. \tag{6.87}$$

Adding each term in the expansion, we find

$$-\ln\langle e^{iS_{int}}\rangle = -8\left(S_{F_1}-S_{B_1}\right)\lambda^2 M\mathrm{Tr}(A^\mu(0))^2 \tag{6.88}$$

$$-8\left(S_{F_1}-S_{B_1}-2\left(S_{F_2}-S_{B_2}\right)\right)\lambda^2 M\mathrm{Tr}(A^a(0))^2 + \mathcal{O}(\lambda^4)\,. \tag{6.89}$$

[2] In equation 6.76 to 6.79, we did not write down the matrix indices to avoid cluttering the notation. Here, the two point functions of any two matrices A_{ab} and B_{cd} should take the form $\langle A_{ab}B_{cd}\rangle \sim \delta_{ad}\delta_{bc}$, where a, b, c and d are matrix indices.

The expression above can be simplified by noting that $S_{B_1} = 6S_{B_2}$ and $S_{F_1} = 6S_{F_2}$. Adding the corrections terms to S_0, the zero mode effective action at $\mathcal{O}(\lambda^2)$ takes the form

$$S^0_{eff} = \Bigg(-\frac{1}{4}\mathrm{Tr}[A^M(0), A^N(0)]^2 - 8\,(S_{F_1} - S_{B_1})\, M\mathrm{Tr}(A^\mu(0))^2 \tag{6.90}$$

$$-\frac{16}{3}\,(S_{F_1} - S_{B_1})\,\mathrm{Tr}(A^a(0))^2 \Bigg)\,\lambda^2 + \mathcal{O}(\lambda^4)\,. \tag{6.91}$$

Hence, we find that at leading order, the corrections to S_0 take the form of two mass terms: one associated to the non-compact directions A^μ, and one associated to the compact directions A^a. As expected, these corrections break the $SO(10)$ symmetry of the target space to $SO(3) \times SO(6)$. This is to be expected since by making the choice to compactify six spatial dimensions, we are picking six special directions in space. The zero mode effective action at leading order in perturbation theory reflects this fact.

The expression above can be more neatly written after undoing our previous change of variable via $A^M \to \lambda^{-1}L^{-1}A^M$. In passing, we will also go back to Lorentzian signature by imposing $A^0 \to -iA^0$. In this case, the effective action takes the form

$$S^0_{eff} = -\frac{1}{4g^2_{eff}}\mathrm{Tr}[A^M(0), A^N(0)]^2 + \frac{1}{2}M^2_{MN}\mathrm{Tr}(A^M(0)A^N(0))^2 + \dots\,, \tag{6.92}$$

where we have defined a mass matrix

$$M^2_{MN} = \begin{bmatrix} \eta_{\mu\nu}M_4^2 & 0 \\ 0 & \eta_{ab}M_6^2\,, \end{bmatrix} \tag{6.93}$$

which includes two mass terms

$$M_4^2 = 16\,(S_{F_1} - S_{B_1})\,\frac{NM}{L^2}\,, \tag{6.94}$$

$$M_6^2 = \frac{32}{3}\,(S_{F_1} - S_{B_1})\,\frac{NM}{L^2}\,. \tag{6.95}$$

In the expression above, the sums S_{B1} and S_{F1} individually diverge in the limit where N is large. However, it's possible to isolate the divergence in these sums by rewriting them as an integral and using Poisson resummation. What we find is rather interesting. It turns out that S_{B1} and S_{F_1} have the same divergent piece which is canceled by the difference $S_{F_1} - S_{B_1}$. It is then possible to evaluate difference numerically, which gives $S_{F_1} - S_{B_1} \approx 0.0397887$. A detailed derivation of this result can be found in Appendix 6.5.

Notice that the breaking of supersymmetry plays a crucial role in obtaining non-vanishing masses M_4^2 and M_6^2. If supersymmetry is restored by imposing that fermions have periodic boundary conditions, then r^a becomes summed over $\mathbb{Z}^6$ instead of $\mathbb{Z}^6 + 1/2$, and the masses vanish since $S_{B_1} = S_{B_2}$. Moreover, the fermions are indeed projected away by the anti-periodic boundary conditions, as expected. When suppersymmetry is restored, the fermions have zero modes terms that will appear at leading order in perturbation theory, and we recover the non-compact IKKT model action (equation 6.1) with an effective gauge coupling g_{eff}.

Moreover, notice that the mass term correction arise at leading order when integrating out the non-zero modes of the theory. This means that, in the decompactification limit, one cannot ignore residual interactions between duplicated regions. This potentially implies that interactions between regions are long ranged, and cannot be ignored even at large distances.

A consequence of this phenomenon seems to be the breaking of gauge invariance in the fundamental regions. Since interactions between regions cannot be ignored, the theory develops an effective potential that takes the form of a mass term. This mass term, which impacts the distribution of objects and their interactions in the fundamental regions, also breaks the gauge invariance of the theory.[3] One may view this as being problematic since, naively, it should be expected that gauge invariance is preserved in the decompactification limit. This intuition comes from the fact that in the decompactification limit, we should recover the same theory we started with, along with the same symmetries. However, we should remind ourself that this is not the case when compactifying matrix theories. Instead of recovering the initial system, we recover a large N number of copies of the initial system, as reflected by the overall factor of N in the equation which is absorbed in the effective coupling $g_{eff}^2 = g^2/N$. These copies come from the fact that we have duplicated a fundamental region N times along the compact directions. Since we don't recover the same system we stated with, it's possible that some symmetries of the original system are not preserved. In the present case, we find that gauge symmetry in the fundamental regions is dependant on the structure of the interaction between them. If supersymmetry is preserved, interactions

[3]The IKKT model action is invariant under the gauge variations $\delta A^M = i[A^M, \alpha]$ and $\delta\psi = i[\psi, \alpha]$, where α is an arbitrary matrix. Including a mass term in the theory breaks this symmetry.

vanish and gauge symmetry is preserved. If supersymmetry is broken, the gauge symmetry is broken.

It is worth noting that compactifying a matrix theory on a higher dimensional torus can lead to some issues. For example, in the BFSS matrix model, decoupling breaks down when the theory is compactified on T^k where $k > 5$ (see [47] for more detail). However, this problem only arises when the compactification radius L is taken to be small, and the system starts behaving like a dual quantum field theory. In the present case, the compactification radius is taken to be large, and the obtained system is closer to the IKKT model than a dual quantum field theory. Hence, we do not expect this issue to arise here. [4]

6.4 Conclusion and discussion

In this paper, we compactified the IKKT matrix model on a six-dimensional torus where the space-time fermions acquire anti-periodic boundary conditions, and we found that the Wilsonian effective action for the non-compact degrees of freedom in the theory acquires an effective mass term which breaks the SO(1,9) symmetry of the IKKT model to SO(1,3) $\times$ SO(6). This mass matrix arises as a result of broken supersymmetry. If supersymmetry is restored, the conventional IKKT action (equation 6.1) is recovered.

It would be interesting to see if the equations of motion of the effective action we have found have interesting cosmological solutions. Given that the SO(1,9) space-time symmetry of the IKKT model is broken to SO(1,3) $\times$ SO(6), one may expect there exist solutions where three space dimension expand, and the six other stay small. In this case, it might be possible that a SUSY breaking compactification is responsible for the emergence of three large space dimensions in recent numerical simulations of the IKKT model.

Assuming interesting cosmological solutions exist, it might be possible to use them to test recent predictions in matrix cosmology, one of them being the scale invariance of cosmological perturbations [48, 53] (see for [54] a summary of progress and challenges in these scenarios). Another avenue of research would be to test a recent space-time metric proposal in the IKKT matrix model [55] using these solutions, or repeat our anaylsis in the BFSS matrix

[4] We thank Savdeep Sethi for bringing this point to our attention.

model. In this case, one may find a possible connection with cosmological scenarios found in non-supersymmetric string theories [56].

Another exciting perspective is that higher order correction to the Wilsonian effective allow for Fuzzy de Sitter space solutions [57–60]. For example, Fuzzy dS_4 is described by four "Pauli-Lubanski" vectors that act as Casimir operators of the SO(1,4) group. Since these operators are built out of Lorentz generators of the SO(1,4) group, they satisfy well-known commutation relations. It would be interesting to see if these commutation relations are solutions of the IKKT model under compactification when higher order corrections are considered.

Finally, it is worth to mention that effective mass terms have been found in matrix models before, notably in the following work [61, 62]. However, in this case, the analysis was done for bosonic (1+D) and (2+D)-dimensional Yang-Mills theories where all all but one or two of the space-time matrices are integrated out. (1+D) and (2+D)-dimensional Yang-Mills theories can be viewed as a (1+D) and (2+D)-dimensional IKKT model where one or two dimensions are compactified on a torus. Hence, our analysis can be viewed as a special case of this work in which we consider a (6+4)-dimensional Yang-Mills theory where fermions are included, supersymmetry is broken, and all bosonic matrices are remain in the effective action. Contrary to [61, 62], we have restrained ourselves to the limit where the compactification radius L is large but finite. It would be interesting to see if phase transitions appear as we decrease the compactification radius.

Acknowledgements

The author wishes to thank Robert Brandenberger, Keshav Dasgupta, Suddhasattwa Brahma and Savdeep Sethi for useful comments on the draft of this paper. Discussions with Herman Verlinde and Dionysios Anninos are also aknowledged. Additionally, the author wishes to thank the Erwin Schrodinger International Institute for Mathematics and Physics (ESI) for the hospitality during the workshop "Large-N Matrix Models and Emergent Geometry" in Vienna, where many good discussions with other participants helped shape the direction of this project. The author is supported in part by the Fonds de recherche du Québec

(FRQNT). The research at McGill is supported in part by funds from NSERC and the Canada Research Chair program.

Appendix

6.5 Epstein Series Regularisation

When deriving the zero mode effective action of the IKKT model, we encountered the sums S_{B1} and S_{F1}. These sums involve Epstein and Epstein-Hurwitz series that are divergent in the limit where $N \to \infty$. In the present section, we show that the divergent part of these sums can be isolated by introducing a regulator in the sums. When this is done, we find that S_{B1} and S_{F1} have the same divergent part, which is canceled in the difference $S_{F1} - S_{B1}$. Evaluating the difference numerically, we find $S_{F1} - S_{B1} \approx 0.0397887$.

6.5.1 Bosonic Sum

We will first start by treating the divergence of the bosonic sum

$$S_{B_1} = \sum_{\vec{n}\in\mathbb{Z}^6}{}' \frac{1}{|2\pi\vec{n}|^2} \,. \tag{6.96}$$

Here, we will use the vector notation $n^a = \vec{n}$ for simplicity. This sum involves the Epstein series

$$E_B = \sum_{\vec{n}\in\mathbb{Z}^d}{}' \frac{1}{|\vec{n}|^2} \,, \tag{6.97}$$

which diverges when $d/2 > 1$. To treat the divergence, we will modify the sum to include a UV regulator. Let us consider the expression

$$\sum_{\vec{n}\in\mathbb{Z}^d}{}' \frac{e^{-\alpha^2|\vec{n}|^2}}{|\vec{n}|^2} \,. \tag{6.98}$$

Here, α^2 plays the role of cutoff which truncates the modes above $N \sim \alpha^{-1}$ out of the sum, hence taming the divergence. In the limit where $\alpha^2 \to 0$, all modes contridute to the sum and the expression above reduces to E_B. Equation 6.98 can rewritten in integral form using the property

$$\frac{1}{|\vec{n}|^2} = \int_0^\infty dt e^{-t|\vec{n}|^2} \,. \tag{6.99}$$

We obtain

$$\sum_{\vec{n}\in\mathbb{Z}^d}{}' \frac{e^{-\alpha^2|\vec{n}|^2}}{|\vec{n}|^2} = \int_0^\infty dt \sum_{\vec{n}\in\mathbb{Z}^d}{}' e^{-(t+\alpha^2)|\vec{n}|^2} \tag{6.100}$$

$$= \pi \int_{\alpha^2/\pi}^\infty dt \left(\theta^d(t) - 1\right) \,, \tag{6.101}$$

where we made use of the function

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi t n^2} \,. \tag{6.102}$$

Since $\theta(t) \sim t^{-1/2}$ when $t \to 0$, the integrand in equation 6.101 diverges in the limit when the regulator α^2 goes to zero. To deal with the divergent part of the integral, we will rewrite the part of the integral in the interval $t \in [\alpha^2/\pi, 1]$ by making use of the property

$$\theta(t) = \frac{1}{t^{1/2}}\theta(1/t) \,, \tag{6.103}$$

which can be derived using Poisson's resummation formula. Substituting the expression above in 6.101, we obtain

$$\int_{\alpha^2/\pi}^1 dt \left(\theta^d(t) - 1\right) = \int_1^{\pi/\alpha^2} dt\, t^{d/2-1} \left(\theta^d(t) - 1\right) - \frac{d}{d-2} + \alpha^2 + \frac{2}{d-2}\left(\frac{\pi}{\alpha^2}\right)^{d/2-1} \,. \tag{6.104}$$

In the expression above, the integral is finite for all values of d. Hence, for $d/2 > 1$, the only divergent piece when $\alpha^2 \to 0$ comes from the last term which is inversely propotional to α^2. Piecing everything together and letting α^2 go to zero, we obtain

$$E_B = \pi \left(\int_1^\infty dt \left(1 + t^{d/2-1}\right) \left(\theta^d(t) - 1\right) - \frac{d}{d-2} + \frac{2}{d-2}\left(\frac{\pi}{\alpha^2}\right)^{d/2-1} \right) \,. \tag{6.105}$$

When $d = 6$, which is the case we are interested in, substituting the value of the E_B in S_{B_1} gives us

$$S_{B_1} = \frac{1}{4\pi} \left(\int_1^\infty dt \left(1 + t^2\right) \left(\theta^6(t) - 1\right) - \frac{3}{2} + \frac{1}{2}\left(\frac{\pi}{\alpha^2}\right)^2 \right) \,. \tag{6.106}$$

6.5.2 Fermionic Sum

Finally, we will evalutate the fermionic sum

$$S_{F_1} = \sum_{\vec{n} \in \mathbb{Z}^6 + 1/2} \frac{1}{|2\pi\vec{n}|^2}\,, \tag{6.107}$$

where the vector notation $n^a = \vec{n}$ is used for simplicity. In this case, we will be interested in the Epstein-Hurwitz series

$$E_F = \sum_{\vec{n} \in \mathbb{Z}^d} \frac{1}{|\vec{n} + a|^2}\,, \tag{6.108}$$

when $a \neq 0$. Here again, we will modify the sum to include a regulator α^2, which trunctates the modes above $N \sim \alpha^{-1}$ out of the sum. In the present case, the expression of interest will be

$$\sum_{\vec{n} \in \mathbb{Z}^d} \frac{e^{-\alpha^2 |\vec{n}+a|^2}}{|\vec{n} + a|^2}\,, \tag{6.109}$$

which reduces to E_F when α^2 goes to zero. Making use of equation 6.99, the sum above can be rewritten as an integral. We obtain

$$\sum_{\vec{n} \in \mathbb{Z}^d} \frac{e^{-\alpha^2 |\vec{n}+a|^2}}{|\vec{n} + a|^2} = \int_0^\infty dt \sum_{\vec{n} \in \mathbb{Z}^d} e^{-(t+\alpha^2)|\vec{n}+a|^2} \tag{6.110}$$

$$= \pi \int_{\alpha^2/\pi}^\infty dt\, \theta^d(t|a)\,, \tag{6.111}$$

where we defined the function

$$\theta(t|a) = \sum_{n=-\infty}^{\infty} e^{-\pi t (n+a)^2}\,. \tag{6.112}$$

Just like $\theta(t)$, the function above can be approximated as $\theta(t|a) \sim t^{-1/2}$ when $t \to 0$, so the integrand in equation 6.111 diverges in the limit when the regulator α^2 goes to zero. To treat this divergence, we will rewrite the divergent part of the integral by making use of the property

$$\theta(t|a) = \frac{e^{-\pi a^2 t}}{t^{1/2}} \theta(1/t|iat) \tag{6.113}$$

which can be derived using Poisson's resummation formula. In this case, the divergent part of the integral can be written as

$$\int_{\alpha^2/\pi}^1 dt\, \theta^d(t|a) = \int_1^{\pi/\alpha^2} dt\, t^{d/2-1} \left(e^{-\pi d a^2/t} \theta^d(t|ia/t) - 1 \right) - \frac{2}{d-2} \tag{6.114}$$

$$+ \frac{2}{d-2} \left(\frac{\pi}{\alpha^2} \right)^{d/2-1}\,. \tag{6.115}$$

The integral on the right-hand side of the expression above is convergent for all values of d. Hence, when $d/2 > 1$, the only divergent piece comes from the last term which is inversely proportional to α^2. Piecing everything together and letting α^2 go to zero, we obtain

$$E_F = \pi \left(\int_1^\infty dt\, \theta^d(t|a) + \int_1^\infty dt\, t^{d/2-1} \left(e^{-\pi d a^2/t} \theta^d(t|ia/t) - 1 \right) - \frac{2}{d-2} \right. \tag{6.116}$$

$$\left. + \frac{2}{d-2} \left(\frac{\pi}{\alpha^2} \right)^{d/2-1} \right) . \tag{6.117}$$

Letting $d = 6$ and $a = 1/2$, we can finally evaluate S_{F_1} by substituting E_F in equation 6.107. We obtain

$$S_{F_1} = \frac{1}{4\pi} \left(\int_1^\infty dt\, \theta^6(t|1/2) + \int_1^\infty dt\, t^2 \left(e^{-\frac{3\pi}{2t}} \theta^6(t|i(2t)^{-1}) - 1 \right) - \frac{1}{2} + \frac{1}{2} \left(\frac{\pi}{\alpha^2} \right)^2 \right) . \tag{6.118}$$

As we can see, the divergent piece in S_{F1} is the same one that we obtained for S_{B1}. This is decause in the $t \to 0$ limit, the theta function in the integrand (6.111) behaves as $\theta(t|a) \sim t^{-1/2}$ independantly of a. Consequently, integrating $\theta(t|a)^d$ in the viscinity of $t \to 0$ yields the same divergent piece regardless of if a takes the value zero (in the bosonic case) or $1/2$ (in the fermionic case). This means that substracting S_{B_1} from S_{F_1} should give a finite value, which can be otained by carrying out each integrals in S_{B_1} and S_{F_1}. Carrying out the integrals numerically, we obtain

$$S_{F1} - S_{B_1} = 0.0397887 . \tag{6.119}$$

Bibliography

[1] A. Lawrence, "On the Instability of 3-D null singularities," JHEP **11**, 019 (2002) doi:10.1088/1126-6708/2002/11/019 [arXiv:hep-th/0205288 [hep-th]].

[2] H. Liu, G. W. Moore and N. Seiberg, "Strings in time dependent orbifolds," JHEP **10**, 031 (2002) doi:10.1088/1126-6708/2002/10/031 [arXiv:hep-th/0206182 [hep-th]].

[3] G. T. Horowitz and J. Polchinski, "Instability of space - like and null orbifold singularities," Phys. Rev. D **66**, 103512 (2002) doi:10.1103/PhysRevD.66.103512 [arXiv:hep-th/0206228 [hep-th]].

[4] M. Berkooz, B. Craps, D. Kutasov and G. Rajesh, "Comments on cosmological singularities in string theory," JHEP **03**, 031 (2003) doi:10.1088/1126-6708/2003/03/031 [arXiv:hep-th/0212215 [hep-th]].

[5] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, "M theory as a matrix model: A Conjecture," Phys. Rev. D **55**, 5112-5128 (1997) doi:10.1103/PhysRevD.55.5112 [arXiv:hep-th/9610043 [hep-th]].

[6] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, "Strings in flat space and pp waves from N=4 superYang-Mills," JHEP **04**, 013 (2002) doi:10.1088/1126-6708/2002/04/013 [arXiv:hep-th/0202021 [hep-th]].

[7] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, "A Large N reduced model as superstring," Nucl. Phys. B **498**, 467-491 (1997) doi:10.1016/S0550-3213(97)00290-3 [arXiv:hep-th/9612115 [hep-th]].

[8] S. Oda and F. Sugino, "Gaussian and mean field approximations for reduced Yang-Mills integrals," JHEP **03**, 026 (2001) doi:10.1088/1126-6708/2001/03/026 [arXiv:hep-th/0011175 [hep-th]].

[9] F. Sugino, "Gaussian and mean field approximations for reduced 4-D supersymmetric Yang-Mills integral," JHEP **07**, 014 (2001) doi:10.1088/1126-6708/2001/07/014 [arXiv:hep-th/0105284 [hep-th]].

[10] J. Nishimura and F. Sugino, "Dynamical generation of four-dimensional space-time in the IIB matrix model," JHEP **05**, 001 (2002) doi:10.1088/1126-6708/2002/05/001 [arXiv:hep-th/0111102 [hep-th]].

[11] H. Kawai, S. Kawamoto, T. Kuroki, T. Matsuo and S. Shinohara, "Mean field approximation of IIB matrix model and emergence of four-dimensional space-time," Nucl. Phys. B **647**, 153-189 (2002) doi:10.1016/S0550-3213(02)00908-2 [arXiv:hep-th/0204240 [hep-th]].

[12] T. Aoyama and H. Kawai, "Higher order terms of improved mean field approximation for IIB matrix model and emergence of four-dimensional space-time," Prog. Theor. Phys. **116**, 405-415 (2006) doi:10.1143/PTP.116.405 [arXiv:hep-th/0603146 [hep-th]].

[13] J. Nishimura, T. Okubo and F. Sugino, "Systematic study of the SO(10) symmetry breaking vacua in the matrix model for type IIB superstrings," JHEP **10**, 135 (2011) doi:10.1007/JHEP10(2011)135 [arXiv:1108.1293 [hep-th]].

[14] K. N. Anagnostopoulos, T. Azuma, Y. Ito, J. Nishimura and S. K. Papadoudis, "Complex Langevin analysis of the spontaneous symmetry breaking in dimensionally reduced super Yang-Mills models," JHEP **02**, 151 (2018) doi:10.1007/JHEP02(2018)151 [arXiv:1712.07562 [hep-lat]].

[15] K. N. Anagnostopoulos, T. Azuma, Y. Ito, J. Nishimura, T. Okubo and S. Kovalkov Papadoudis, "Complex Langevin analysis of the spontaneous breaking of 10D rotational symmetry in the Euclidean IKKT matrix model," JHEP **06**, 069 (2020) doi:10.1007/JHEP06(2020)069 [arXiv:2002.07410 [hep-th]].

[16] S. Brahma, R. Brandenberger and S. Laliberte, "Spontaneous symmetry breaking in the BFSS model: analytical results using the Gaussian expansion method," Eur. Phys. J. C **83**, no.10, 904 (2023) doi:10.1140/epjc/s10052-023-12082-w [arXiv:2209.01255 [hep-th]].

[17] J. Nishimura, "(3+1)-dimensional expanding universe from a Lorentzian matrix model for superstring theory in (9+1)-dimensions," Int. J. Mod. Phys. A **28**, 1340002 (2013) doi:10.1142/S0217751X13400022 [arXiv:1209.6386 [hep-th]].

[18] Y. Ito, S. W. Kim, J. Nishimura and A. Tsuchiya, "Monte Carlo studies on the expanding behavior of the early universe in the Lorentzian type IIB matrix model," PoS **LATTICE2013**, 341 (2014) doi:10.22323/1.187.0341 [arXiv:1311.5579 [hep-lat]].

[19] Y. Ito, J. Nishimura and A. Tsuchiya, "Power-law expansion of the Universe from the bosonic Lorentzian type IIB matrix model," JHEP **11**, 070 (2015) doi:10.1007/JHEP11(2015)070 [arXiv:1506.04795 [hep-th]].

[20] Y. Ito, J. Nishimura and A. Tsuchiya, "Large-scale computation of the exponentially expanding universe in a simplified Lorentzian type IIB matrix model," PoS **LATTICE2015**, 243 (2016) doi:10.22323/1.251.0243 [arXiv:1512.01923 [hep-lat]].

[21] T. Aoki, M. Hirasawa, Y. Ito, J. Nishimura and A. Tsuchiya, "On the structure of the emergent 3d expanding space in the Lorentzian type IIB matrix model," PTEP **2019**, no.9, 093B03 (2019) doi:10.1093/ptep/ptz092 [arXiv:1904.05914 [hep-th]].

[22] M. Hirasawa, K. Anagnostopoulos, T. Azuma, K. Hatakeyama, Y. Ito, J. Nishimura, S. Papadoudis and A. Tsuchiya, "A new phase in the Lorentzian type IIB matrix model and the emergence of continuous space-time," PoS **LATTICE2021**, 428 (2022) doi:10.22323/1.396.0428 [arXiv:2112.15390 [hep-lat]].

[23] K. Hatakeyama, K. Anagnostopoulos, T. Azuma, M. Hirasawa, Y. Ito, J. Nishimura, S. Papadoudis and A. Tsuchiya, "Relationship between the Euclidean and Lorentzian versions of the type IIB matrix model," PoS **LATTICE2021**, 341 (2022) doi:10.22323/1.396.0341 [arXiv:2112.15368 [hep-lat]].

[24] K. Hatakeyama, K. Anagnostopoulos, T. Azuma, M. Hirasawa, Y. Ito, J. Nishimura, S. Papadoudis and A. Tsuchiya, "Complex Langevin studies of the emergent space-time in the type IIB matrix model," doi:10.1142/9789811261633_0002 [arXiv:2201.13200 [hep-th]].

[25] K. N. Anagnostopoulos, T. Azuma, K. Hatakeyama, M. Hirasawa, Y. Ito, J. Nishimura, S. K. Papadoudis and A. Tsuchiya, "Progress in the numerical studies of the type IIB matrix model," doi:10.1140/epjs/s11734-023-00849-x [arXiv:2210.17537 [hep-th]].

[26] M. Hirasawa, K. N. Anagnostopoulos, T. Azuma, K. Hatakeyama, J. Nishimura, S. K. Papadoudis and A. Tsuchiya, "The emergence of expanding space-time in a novel large-N limit of the Lorentzian type IIB matrix model," PoS **LATTICE2022**, 371 (2023) doi:10.22323/1.430.0371 [arXiv:2212.10127 [hep-lat]].

[27] M. Hirasawa, K. N. Anagnostopoulos, T. Azuma, K. Hatakeyama, J. Nishimura, S. Papadoudis and A. Tsuchiya, "The emergence of expanding space-time in the Lorentzian type IIB matrix model with a novel regularization," PoS **CORFU2022**, 309 (2023) doi:10.22323/1.436.0309 [arXiv:2307.01681 [hep-th]].

[28] S. W. Kim, J. Nishimura and A. Tsuchiya, "Late time behaviors of the expanding universe in the IIB matrix model," JHEP **10**, 147 (2012) doi:10.1007/JHEP10(2012)147 [arXiv:1208.0711 [hep-th]].

[29] K. Hatakeyama, A. Matsumoto, J. Nishimura, A. Tsuchiya and A. Yosprakob, "The emergence of expanding space–time and intersecting D-branes from classical solutions in the Lorentzian type IIB matrix model," PTEP **2020**, no.4, 043B10 (2020) doi:10.1093/ptep/ptaa042 [arXiv:1911.08132 [hep-th]].

[30] H. C. Steinacker, "Quantized open FRW cosmology from Yang–Mills matrix models," Phys. Lett. B **782**, 176-180 (2018) doi:10.1016/j.physletb.2018.05.011 [arXiv:1710.11495 [hep-th]].

[31] M. Sperling and H. C. Steinacker, "The fuzzy 4-hyperboloid H_n^4 and higher-spin in Yang–Mills matrix models," Nucl. Phys. B **941**, 680-743 (2019) doi:10.1016/j.nuclphysb.2019.02.027 [arXiv:1806.05907 [hep-th]].

[32] M. Sperling and H. C. Steinacker, "Covariant cosmological quantum space-time, higher-spin and gravity in the IKKT matrix model," JHEP **07**, 010 (2019) doi:10.1007/JHEP07(2019)010 [arXiv:1901.03522 [hep-th]].

[33] J. L. Karczmarek and H. C. Steinacker, "Cosmic time evolution and propagator from a Yang–Mills matrix model," J. Phys. A **56**, no.17, 175401 (2023) doi:10.1088/1751-8121/acc61e [arXiv:2207.00399 [hep-th]].

[34] A. Stern, "Matrix Model Cosmology in Two Space-time Dimensions," Phys. Rev. D **90**, no.12, 124056 (2014) doi:10.1103/PhysRevD.90.124056 [arXiv:1409.7833 [hep-th]].

[35] A. Chaney, L. Lu and A. Stern, "Lorentzian Fuzzy Spheres," Phys. Rev. D **92**, no.6, 064021 (2015) doi:10.1103/PhysRevD.92.064021 [arXiv:1506.03505 [hep-th]].

[36] A. Chaney, L. Lu and A. Stern, "Matrix Model Approach to Cosmology," Phys. Rev. D **93**, no.6, 064074 (2016) doi:10.1103/PhysRevD.93.064074 [arXiv:1511.06816 [hep-th]].

[37] A. Chaney and A. Stern, "Fuzzy CP^2 spacetimes," Phys. Rev. D **95**, no.4, 046001 (2017) doi:10.1103/PhysRevD.95.046001 [arXiv:1612.01964 [hep-th]].

[38] A. Stern and C. Xu, "Signature change in matrix model solutions," Phys. Rev. D **98**, no.8, 086015 (2018) doi:10.1103/PhysRevD.98.086015 [arXiv:1808.07963 [hep-th]].

[39] E. Battista and H. C. Steinacker, "On the propagation across the big bounce in an open quantum FLRW cosmology," Eur. Phys. J. C **82**, no.10, 909 (2022) doi:10.1140/epjc/s10052-022-10874-0 [arXiv:2207.01295 [gr-qc]].

[40] E. Battista and H. C. Steinacker, "Fermions on curved backgrounds of matrix models," Phys. Rev. D **107**, no.4, 046021 (2023) doi:10.1103/PhysRevD.107.046021 [arXiv:2212.08611 [hep-th]].

[41] E. Battista and H. C. Steinacker, "One-loop effective action of the IKKT model for cosmological backgrounds," JHEP **01**, 125 (2024) doi:10.1007/JHEP01(2024)125 [arXiv:2310.11126 [hep-th]].

[42] W. Taylor, "D-brane field theory on compact spaces," Phys. Lett. B **394**, 283-287 (1997) doi:10.1016/S0370-2693(97)00033-6 [arXiv:hep-th/9611042 [hep-th]].

[43] D. Berenstein and R. Corrado, "M(atrix) theory in various dimensions," Phys. Lett. B **406**, 37-43 (1997) doi:10.1016/S0370-2693(97)00671-0 [arXiv:hep-th/9702108 [hep-th]].

[44] A. Herderschee and J. Maldacena, "Three Point Amplitudes in Matrix Theory," [arXiv:2312.12592 [hep-th]].

[45] A. Herderschee and J. Maldacena, "Soft Theorems in Matrix Theory," [arXiv:2312.15111 [hep-th]].

[46] T. Banks and L. Motl, "A Nonsupersymmetric matrix orbifold," JHEP **03**, 027 (2000) doi:10.1088/1126-6708/2000/03/027 [arXiv:hep-th/9910164 [hep-th]].

[47] T. Banks, "TASI lectures on matrix theory," [arXiv:hep-th/9911068 [hep-th]].

[48] S. Laliberte and S. Brahma, "IKKT thermodynamics and early universe cosmology," JHEP **11**, 161 (2023) doi:10.1007/JHEP11(2023)161 [arXiv:2304.10509 [hep-th]].

[49] O. J. Ganor, S. Ramgoolam and W. Taylor, "Branes, fluxes and duality in M(atrix) theory," Nucl. Phys. B **492**, 191-204 (1997) doi:10.1016/S0550-3213(97)00100-4 [arXiv:hep-th/9611202 [hep-th]].

[50] R. Helling, "Scattering in supersymmetric M(atrix) models," Fortsch. Phys. **48**, 1229-1310 (2000)

[51] S. Yokoyama, "Supersymmetry Algebra in Super Yang-Mills Theories," JHEP **09**, 211 (2015) doi:10.1007/JHEP09(2015)211 [arXiv:1506.03522 [hep-th]].

[52] L. Brink, J. H. Schwarz and J. Scherk, "Supersymmetric Yang-Mills Theories," Nucl. Phys. B **121**, 77-92 (1977) doi:10.1016/0550-3213(77)90328-5

[53] S. Brahma, R. Brandenberger and S. Laliberte, "Emergent cosmology from matrix theory," JHEP **03**, 067 (2022) doi:10.1007/JHEP03(2022)067 [arXiv:2107.11512 [hep-th]].

[54] S. Brahma, R. Brandenberger and S. Laliberte, "BFSS Matrix Model Cosmology: Progress and Challenges," [arXiv:2210.07288 [hep-th]].

[55] S. Brahma, R. Brandenberger and S. Laliberte, "Emergent metric space-time from matrix theory," JHEP **09**, 031 (2022) doi:10.1007/JHEP09(2022)031 [arXiv:2206.12468 [hep-th]].

[56] E. J. Martinec, D. Robbins and S. Sethi, "Non-Supersymmetric String Theory," JHEP **10**, 078 (2011) doi:10.1007/JHEP10(2011)078 [arXiv:0904.3498 [hep-th]].

[57] M. Buric and J. Madore, "Noncommutative de Sitter and FRW spaces," Eur. Phys. J. C **75**, no.10, 502 (2015) doi:10.1140/epjc/s10052-015-3729-6 [arXiv:1508.06058 [hep-th]].

[58] M. Buric, D. Latas and L. Nenadovic, "Fuzzy de Sitter Space," Eur. Phys. J. C **78**, no.11, 953 (2018) doi:10.1140/epjc/s10052-018-6432-6 [arXiv:1709.05158 [hep-th]].

[59] M. Buric and D. Latas, "Discrete fuzzy de Sitter cosmology," Phys. Rev. D **100**, no.2, 024053 (2019) doi:10.1103/PhysRevD.100.024053 [arXiv:1903.08378 [hep-th]].

[60] B. Brkic, M. Buric and D. Latas, "Fuzzy de Sitter and anti-de Sitter spaces," PoS **CORFU2021**, 274 (2022) doi:10.22323/1.406.0274

[61] G. Mandal, M. Mahato and T. Morita, "Phases of one dimensional large N gauge theory in a 1/D expansion," JHEP **02**, 034 (2010) doi:10.1007/JHEP02(2010)034 [arXiv:0910.4526 [hep-th]].

[62] G. Mandal and T. Morita, "Phases of a two dimensional large N gauge theory on a torus," Phys. Rev. D **84**, 085007 (2011) doi:10.1103/PhysRevD.84.085007 [arXiv:1103.1558 [hep-th]].

Part V

Symmetry breaking in matrix cosmology

Chapter 7

Spontaneous symmetry breaking in the BFSS model: Analytical results using the Gaussian expansion method

Suddhasattwa Brahma[1], Robert Brandenberger[2], and Samuel Laliberte[2]

[1]*Higgs Centre for Theoretical Physics, School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3FD, UK*

[2]*Department of Physics, McGill University, Montréal, QC, H3A 2T8, Canada*

Addendum for the thesis

In Section 2.3.4, we saw how the Gaussian Expansion method can be used to probe symmetry breaking in the Euclidean IKKT matrix model. We found that for the bosonic IKKT model, there is no symmetry breaking. We then found that for the supersymmetric IKKT model, the contribution of fermions breaks the SO(10) symmetry of the system. In the present section, we explore a potentially similar symmetry breaking in the Euclidean BFSS matrix model. Using the Gaussian Expansion Method, we find that the bosonic BFSS matrix model retains its SO(9) symmetry, while the contribution from fermions breaks the SO(9) symmetry of the system in the supersymmetric case. This symmetry-breaking may relate to a similar symmetry-breaking process in String Gas Cosmology, where three dimensions become large following the Hagedorn phase. However, further studies remain to be made to confirm this hypothesis.

Abstract

We apply the Gaussian expansion method to the BFSS matrix model in the high temperature limit. When the (Euclidean) BFSS action is expanded about a Gaussian ansatz, it is shown that the SO(9) symmetry is spontaneously broken, analogous to what happens in the IKKT model. The analysis of the free energy, using the set of gap equations which determines the width of the Gaussian terms, is sufficient to show that this symmetry breaking happens *only* when the fermionic terms are included and is absent in the bosonic case.

7.1 Introduction

Understanding the quantum origin of space and time in the very early universe requires starting from a consistent non-perturbative theory which is complete in the ultraviolet (UV), *i.e.* in the high energy limit. String theory is the most promising candidate for such a theory. To understand string theory at the high densities of the very early universe, it is crucial to start from a non-perturbative approach[1]. In the late 1990s it was realized that large N matrix models could provide non-perturbative definitions of string theory. Two key examples are the BFSS [5] and the IKKT [6] matrix models.

Recently, there has been a new proposal to understand the origin of space and time in the context of these matrix models [7]. It has been shown how, starting from abstract matrix degrees of freedom, one not only obtains emergent space and time, but one can also extract variables which meaningfully depict the dynamical history of our universe. The resulting scenario has aspects in common with *String Gas Cosmology* (SGC) [8], a model of early time cosmology in which the universe was taken to begin in a high temperature equilibrium

[1]Effective field theory approximations run into serious conceptual problems (see e.g. [1] for an overview), and effective field theories consistent with string theory are constrained by several criteria such as the "swampland" conditions [2] (see [3] for recent reviews) and the "Trans-Planckian Censorship Conjecture" [4].

state of a gas of strings, and the thermal fluctuations were shown [9] to lead to scale-invariant spectra of both cosmological perturbations and gravitational waves, again like what results in SGC [10]. What was missing in SGC, however, was an embedding of the model into a consistent dynamical framework coming from string theory. Our present model provides such an embedding [2]

Let us summarize our new findings in a bit more detail. Firstly, it was shown in [9] that, assuming a thermal state of the BFSS matrix model, one can extract an early universe cosmology in which the thermal fluctuations lead to roughly scale-invariant spectra for both the primordial gravitational waves and the (observable, infrared) curvature fluctuations. Moreover, the amplitude of these perturbations is given by the ratio of the string scale to the Planck scale, exactly as in SGC, and agrees with current observations for a string scale of the order of that of Grand Unification (a string scale which in agreement with what is expected from string particle phenomenology in heterotic superstring theory [12]). The remarkable bit about this result is that it arises solely due to the form of the BFSS Lagrangian itself and the result is independent of any fine-tuning (as compared with the freedom offered by the choice of the scalar field potential for inflation). The choice of a thermal state for primordial perturbations is also natural, as we work in the Euclidean BFSS model at high temperatures. The main leap of faith in our model was that, at high temperatures, the BFSS model was assumed to exhibit a symmetry-breaking pattern in which there would emerge exactly three large spatial dimensions. The intuition behind this assumption comes from the IKKT model, where it has been long demonstrated [13] that a large $(3+1)$-d universe emerges from the full 10-d theory due to a spontaneous symmetry breaking (SSB) of the isometry group (see [14] for more recent results). In fact, this was one of the motivating factors for choosing a high temperature state in the BFSS model, since it is well known [15] that the BFSS model approaches the IKKT model as $T \to \infty$[3].

Although it was well-motivated, there were a couple of strong assumptions in the afore-

[2]See also [11] for another approach to an emergent early universe scenario based on fundamental principles of string theory which connects with SGC.

[3]Coincidentally, it is indeed in this limit that we have a well-defined perturbative expansion for the thermodynamic quantities of interest for the BFSS model since the perturbation parameter g^2N/T^3 is small in this limit.

mentioned calculation. Firstly, at high temperatures, the fermions decouple in the BFSS model and one only recovers the bosonic IKKT action in this limit. On the other hand, it has been argued from different directions that the presence of fermions is crucial for the $SO(10)$ symmetry-breaking in the IKKT model. Secondly, there has been a combination of analytical and numerical results which have demonstrated that although all the eigenvalues of the bosonic matrices start out small (on the string scale) for the IKKT model, the SSB leads to 3 of them becoming large after the symmetry-breaking time. In the absence of any such evidence for the BFSS model, it is a rather strong assumption to say that there is a similar dynamical explanation happening for the BFSS action.

It is precisely this lacuna which we seek to (at least, partially) fill in this paper. We will show using well-established methods that there is a SSB of the $SO(9)$ group for the Euclidean BFSS model, in the presence of fermions. This will be done by employing the Gaussian expansion method, [16] in which we will approximate the BFSS Lagrangian with a Gaussian action and then systematically calculate the corrections to this ansatz, by expanding about the Gaussian term. The Gaussian action contains a host of free parameters (or "masses") which are determined by solving the so-called 'gap' or 'self-consistency' equations. When truncated to any finite order, this constitutes an approximation, as mentioned above. To put it differently, in this method one adds and subtracts a Gaussian piece S_0 to the action and then regards $S + S_0$ as the classical action and the $-S_0$ as the one-loop counter-term [17]. Our goal is to then calculate the free energy for such an expanded action, and see if it gets minimized for a specific choice of the Gaussian variables. Naturally, for any finite truncation, the result of the Gaussian expansion will depend on the "mass" parameters in the ansatz. Nevertheless, in a lot of models which are exactly solvable, it has been shown that there exist regions of parameter space where the action is independent of the Gaussian parameters. These regions signal local minima of the effective action and identifying the region with the smallest free energy enables one to predict the true vacuum of the theory.

In order to examine the symmetry-breaking pattern of the model, we will start with an ansatz that preserves the $U(N)$ gauge group of the BFSS model but not the isometry group. The goal is to show that there exists a region of parameter space where the free energy is minimized, and that this occurs for a configuration which corresponds to the breaking of

$SO(9)$ to $SO(6)$. In this work, we will be content with studying the gap equations which deal with the bosonic Gaussian parameters[4], and show that such a gap equation cannot be solved by a $SO(9)$-symmetric solution in the presence of fermionic terms. In other words, we will demonstrate that as long as there are some fermionic contributions to the Gaussian ansatz, there must be a SSB of the BFSS model. Our result mimics previous computations of the IKKT model although the presence of an intrinsic time parameter complicates the application of the Gaussian method to the BFSS action. Thus, we find first evidence of SSB in the BFSS model which is at the same footing as in the IKKT model.

In the next section, we review the results of the applying the Gaussian expansion method to the IKKT model in order to introduce this technique and highlight the minimum ingredients required for showing that there must be a SSB in such a matrix model. In Sec-3, we show that, analogous to what happens in the IKKT model, there is no symmetry breaking for the bosonic BFSS action. Finally, we present our main result in Sec-4 demonstrating why there *must* be SSB for the full BFSS model in the presence of fermions. We conclude in Sec-5, highlighting future directions and why our result has deep consequences for matrix cosmology.

7.2 Review of the IKKT results

Let us briefly review the results of applying the Gaussian expansion technique to the Euclidean IKKT model [18, 19]. The main goal of this section is to show what the precise requirements are to obtain spontaneous breaking of the $SO(10)$ symmetry. For the IKKT model, the analytical arguments have been supplemented with numerical studies [13] which show that the $SO(10)$ symmetry in fact breaks to some $SO(4)$ group (cross some other symmetry group for the internal dimensions). We are at the moment unable to carry out the full analyses for the BFSS model, since it would require a considerable amount of numerics to do so. As a result, we are not yet in the position to determine the end state of the symmetry breaking. Nevertheless, by identifying the necessary ingredients required to identify

[4] By this term, we will throughout refer to the "mass" terms assigned to the bosonic matrices of the model since the spatial directions emerge from the eigenvalue distribution of these matrices in the BFSS model.

the symmetry-breaking, we will be able to show definitively that the $SO(10)$ symmetry must necessarily be broken in the BFSS model.

The partition function of the Euclidean IKKT model is given by (we will closely follow the notation of [18] for consistency)

$$Z = \int \mathrm{d}X \mathrm{d}\Phi \, e^{-S_{\mathrm{IKKT}}} \,, \tag{7.1}$$

where the IKKT action is defined as:

$$S_{\mathrm{IKKT}} = -\frac{1}{4} N \,\mathrm{Tr}\, [X_\mu, X_\nu]^2 - \frac{i}{2} N \,\mathrm{Tr} \left(\Phi_\alpha \left(\tilde{\Gamma}_\mu \right)_{\alpha\beta} [X_\mu, \Phi_\beta] \right) =: S^{(b)}_{\mathrm{IKKT}} + S^{(f)}_{\mathrm{IKKT}} \,. \tag{7.2}$$

In the above expressions we have rescaled the 10 bosonic matrices $A_\mu \to \lambda^{1/4} X_\mu$ and their 16 fermionic superpartners $\Psi_\alpha \to \lambda^{3/8} \Phi_\alpha$ by some powers of the 't Hooft coupling $\lambda = g^2 N$, so as to make the action independent of the Yang-Mills coupling g. The $10-d$ gamma matrices in the Weyl basis, Γ_μ, are multiplied by the charge conjugation matrix $\mathcal{C}$, to get $\tilde{\Gamma}_\mu = \mathcal{C}\Gamma_\mu$. From now on, we will assume the number of bosnic matrices to be d (*i.e.*, $\mu = 1, 2, \ldots, d$), while the fermionic ones to be p (*i.e.*, $\alpha = 1, 2, \ldots, p$), to be completely general. The reason for separating out the bosonic and fermionic parts of the IKKT action will be clear later on.

The general idea of the Gaussian expansion method [16, 17] is to approximate the above action by a Gaussian ansatz which is the most general $SU(N)$-invariant action and yet does not assume an $SO(d)$ symmetry. One then expands the action around this Gaussian ansatz and calculates quantities such as the free energy to whichever order in perturbation theory is desired, and minimizes the free energy by solving a set of gap equations which fixes the parameters of the Gaussian terms. In general, for large-N theories, it is known that planar diagrams are the only ones which contribute to the gap equations when calculating corrections to the Gaussian approximation.

More concretely, we introduce a Gaussian action of the form

$$S_0 = \sum_{\mu=1}^{d} \left(\frac{N}{v_\mu} \right) \mathrm{Tr}\, (X_\mu X_\mu) + \left(\frac{N}{2} \right) \sum_{a=1}^{N^2-1} \Phi^a_\alpha \, \mathcal{A}_{\alpha\beta} \, \Phi^b_\beta =: S^{(b)}_0 + S^{(f)}_0 \,, \tag{7.3}$$

where the Gaussian parameters are $v_\mu > 0$ can, a priori, take d distinct values and $\mathcal{A}$ is a $p \times p$ matrix, and we have separated out the bosonic and fermionic Gaussian terms. [5]

[5] We have expanded the fermionic fields Φ_α in terms of the $N^2 - 1$ $SU(N)$ generators, the Φ^a_α being the expansion coefficients.

7.2.1 No symmetry breaking for bosonic IKKT

In order to understand the full power of the Gaussian expansion method in exploring the symmetry breaking pattern of the model, let us first apply it to the bosonic part of the IKKT model, *i.e.* to the $S^{(b)}_{\text{IKKT}} = -\frac{1}{4} N \operatorname{Tr}[X_\mu, X_\nu]^2$ part of the action. Naturally, we will now only consider the part of the Gaussian ansatz $S_0^{(b)}$ that depends on the "bosonic mass" parameters v_μ.

First, let us express the partition function in terms of an expansion around $S_0^{(b)}$, namely

$$Z^{(b)}_{\text{IKKT}} = Z_0^{(b)} \left\langle e^{-\left(S^{(b)}_{\text{IKKT}} - S_0^{(b)}\right)} \right\rangle_0 , \tag{7.4}$$

where $Z_0^{(b)} \sim \int \mathrm{d}X e^{S_0^{(b)}}$ and the expectation values (denoted by the angled brackets) are taken with respect to $Z_0^{(b)}$ and we have ignored factors of λ in the prefactor as they are of no consequence to us. Although one can systematically calculate the free energy of the bosonic part of the IKKT action from the partition function above $F^{(b)}_{\text{IKKT}} = -\ln Z^{(b)}_{\text{IKKT}}$ by expanding it in a power series [18, 19], for our purposes, it will be sufficient to consider only the first order correction.

More concretely, once the bosonic matrices are expanded with respect to $SU(N)$ generators and the measure of the integrals written appropriately in terms of them, it is easy to carry out a series of arduous but straight-forward Gaussian integrals to express the free energy as[6]:

$$F_0^{(b)} := -\ln Z_0^{(b)} = \frac{1}{2}\left(N^2-1\right)\left[C_1 - \sum_{\mu=1}^{d} \ln v_\mu\right] , \tag{7.5}$$

$$F_1^{(b)} := \left\langle S^{(b)}_{\text{IKKT}} \right\rangle_0 - \left\langle S_0^{(b)} \right\rangle_0 = \frac{1}{8}\left(N^2-1\right)\left[\sum_{\mu\neq\nu} v_\mu v_\nu - C_2\right] , \tag{7.6}$$

where C_1 and C_2 are constants which depend on the λ and d and contain some numerical factors.

[6]See [18] for details of these calculations. More explicitly, we need to do a bunch of Gaussian integrals after writing out the measure as $\mathrm{d}X = \prod_{a=1}^{N^2-1}\prod_{\mu=1}^{d} \mathrm{d}x^a_\mu$. Specifically, the prefactor of (N^2-1) arises since the gauge group is $SU(N)$ and has N^2-1 generators. Note that we have ignored some factors of π throughout.

Our simple goal is to minimize the (bosonic) free energy, to this order, by varying the Gaussian parameters v_μ, *i.e.*

$$\frac{\partial}{\partial v_\mu}\left(F_0^{(b)} + F_1^{(b)}\right) = 0\,, \tag{7.7}$$

which gives us the bosonic gap equations:

$$-\frac{1}{2v_\mu} + \frac{1}{4}\sum_{\mu\neq\nu} v_\mu = 0\,. \tag{7.8}$$

Note that these are actually d equations written in a compact form. What is important is that, given that $v_\mu \geq 0$ in our ansatz (which is still the most general Gaussian action one can choose), we find that the solution of the above equation is given by a $SO(d)$ symmetric solution:

$$v_1 = v_2 = \cdots = v_d = \sqrt{\frac{2}{d-1}}\,. \tag{7.9}$$

Therefore, although we did not start with an $SO(d)$ symmetric ansatz for the Gaussian term, the bosonic part of the IKKT action is such that the solution which minimizes the free energy is $SO(d)$ symmetric thereby indicating that the symmetry is unbroken. However, looking ahead, note that we will always look at the bosonic gap equations to tell us if there is a symmetry breaking in the theory as it is indeed the eigenvalue distribution of the bosonic matrices which is conjectured to give us the emergent spacetime in this theory.

7.2.2 Symmetry breaking and the role of the fermionic terms

Let us now go back to the full IKKT action, including the fermionic terms, and therefore include the fermionic part of the Gaussian ansatz $S_0^{(f)}$. The Gaussian parameters are encapsulated in the $p \times p$ anti-symmetric matrix $\mathcal{A}$:

$$\mathcal{A}_{\alpha\beta} := \frac{i}{3!}\sum_{\mu\nu\lambda} \omega_{\mu\nu\lambda}\left(\mathcal{B}_{\mu\nu\lambda}\right)_{\alpha\beta}\,, \tag{7.10}$$

where $\mathcal{B}_{\mu\nu\lambda} \sim \mathcal{C}\Gamma_\mu\Gamma_\nu^\dagger\Gamma_\lambda$ and we have suppressed the spinor indices in the last expression to avoid clutter. Note that $\mathcal{A}$ could have had a term of the form $\omega_\mu\Gamma_\mu$ which is absent due to the Majorana nature of the fermions in 10-d. However, this is not crucial for our arguments below. In fact, the explicit form of $\mathcal{A}$ will not enter our expressions and all we willneed is its $SO(d)$ index structure.

Before going through the calculation, note that we will have to collect the expanded terms in the free energy in a slightly different way due to the fermionic terms. For instance, we have to count terms of the form $S_0^{(f)}$ and $\left(S_{\rm IKKT}^{(f)}\right)^2$ to be of the same order in the reorganized expansion [20]. Keeping this in mind, and once again expanding only to first order, one finds that

$$\begin{aligned} F_0 &:= -\ln Z_0 = \frac{1}{2}\left(N^2-1\right)\left[C_3 - \sum_{\mu=1}^{d} \ln v_\mu - \ln\left(\mathrm{Pf}\mathcal{A}\right)\right] , \qquad (7.11) \\ F_1^{(b)} &:= \left\langle S_{\rm IKKT}^{(b)} \right\rangle_0 - \left\langle S_0^{(b)} \right\rangle_0 - \frac{1}{2}\left\langle \left(S_{\rm IKKT}^{(f)}\right)^2 \right\rangle_0 \qquad (7.12) \\ &= \frac{1}{8}\left(N^2-1\right)\left[\sum_{\mu\neq\nu} v_\mu v_\nu + C_4 - 4\sum_\mu \rho_\mu v_\mu\right] , \end{aligned}$$

where the constants C_3 and C_4 have new contributions (compared to the corresponding constants C_1 and C_2 in (7.5)) coming from the fermionic terms which, like before, depend on N, d and now on p. All the expectation values are now taken with respect to the full Gaussian action (but we still keep the same subscript 0 to denote this). We have also introduced the Pfaffian of the Gaussian matrix $\mathcal{A}$, defined as $\mathrm{Pf}\mathcal{A} := \det\mathcal{A}^{1/2}$. More importantly, we have defined

$$\rho_\mu := \frac{1}{4}\mathrm{Tr}\left[\left(\mathcal{A}^{-1}\Gamma_\mu\right)^2\right] , \qquad (7.13)$$

where we have taken the trace over p-dimensional spinor indices α, β. The details of the above terms will not be important for our argument. We will only need to focus on the bosonic gap equation (the self-consistency equation according to the definition of [18,19]) at first order:

$$\frac{\partial}{\partial v_\mu}\left(F_0 + F_1\right) = 0 \qquad (7.14)$$

which yields

$$-\frac{1}{2v_\mu} + \frac{1}{4}\sum_{\mu\neq\nu} v_\mu - \frac{1}{4}\rho_\mu = 0 \, . \qquad (7.15)$$

Just by inspecting this equation (7.15), we can conclude that there must be a breaking of the $SO(d)$ symmetry. The argument simply depends on realizing that the ρ_μ, as defined in (7.13), are different for different choices of μ due to the appearance of the Γ function. Therefore, this bosonic gap equation cannot have a solution of the form $v_1 = v_2 = \cdots = v_d$, as before. As promised, the explicit form of the matrix $\mathcal{A}$ did not play a role in this derivation. This point was already pointed out in [18].

The main reason for the symmetry breaking is the form of the interaction between the fermionic and bosonic matrices, as encoded in $S^{(f)}_{\mathrm{IKKT}}$ in (7.2). The appearance of the Γ function in that term is the reason why the bosonic gap equation gets a "source" term for the v_μ which is not μ-independent What is important for us is that inspecting the bosonic gap equation is sufficient to detect the presence (or absence) of symmetry-breaking. Of course, the contributions of the fermionic terms to this gap equation is what turned out to be crucial. In fact, symmetry breaking is guaranteed as long as $\mathcal{A}$ has some non-zero entries, *i.e.* the Gaussian parameters $\omega_{\mu\nu\lambda}$ are not all zero. And that is why we never even needed to write down the fermionic gap equation to draw this conclusion.

Of course, without inspecting the full system of gap equations, it will not be possible to conclude what the $SO(d)$ symmetry break into. In the case of the IKKT model, numerical tools were required to demonstrate what the symmetry of the solution which minimizes the free energy is. But at the moment we do not have the required tools to study this question for the BFSS model, at least not in this work. It would be nice to gain a good physical understanding for why the resulting symmetry of the Lorentzian matrix model is $SO(3)$, and why the solution corresponds to three expanding dimensions while six remain microscopical. This is a question we are currently working on. But in the following, we will address the restricted question of showing, in analogy with what was described above in the case of the IKKT model, that there is symmetry breaking in the BFSS model as long as the contributions of the fermionic terms are taken into account.

7.3 Gaussian expansion method for the bosonic BFSS model

Just like for the Euclidean IKKT model, we will start with the bosonic BFSS action first and apply the Gaussian expansion technique to it. As we described in the case of the symmetry breaking analysis in the IKKT model, we will not assume a $SO(D)$ symmetric ansatz to begin with and will consider the most general $U(N)$-invariant Gaussian ansatz[7] which is, however, not $SO(D)$ symmetric.

The bosonic part of the Euclidean BFSS action is given by (now we follow the notation of [16] for easy comparison):

$$S^{(b)}_{\text{BFSS}} = \frac{1}{g^2}\int \mathrm{d}\tau\, \mathrm{Tr}\left\{\frac{1}{2}D_\tau X^i D_\tau X^i - \frac{1}{4}\left[X^i, X^i\right]^2\right\}\,. \tag{7.16}$$

There are $D = 9$ $SU(N)$ bosonic matrices for the BFSS model, denoted by X^i above. We have changed the indices from Latin μ to Roman i to indicate that $i = 1, 2, \ldots, d-1$, in the notation of the previous section. Let us keep the dimension arbitrary as before and denote $D = d - 1$ to make connection with the previous section. τ here denotes the Euclidean time direction and $\beta := 1/\tau$. We would typically be interested in the high-temperature (confined) behaviour of the model since this is the regime which is interesting for early-universe cosmology [9].

The first new complication of the BFSS model is to have a gauge-covariant derivative $D_\tau := \partial_\tau + i\,[A_0, \cdot]$. We fix the gauge as $\partial_\tau A_0 = 0 \Rightarrow A_0 := A_{00}/\sqrt{\beta} = \text{const.}$. On introducing the necessary ghost fields $\alpha, \bar{\alpha}$, we can write the action as:

$$\begin{aligned} S^{(b)}_{\text{BFSS}} = \frac{1}{g^2}\int \mathrm{d}\tau\, \mathrm{Tr}\Big\{&\frac{1}{2}\partial_\tau X^i \partial_\tau X^i - \frac{1}{2}\left[A_0, X^i\right]^2 + i\left[A_0, X^i\right]\left(\partial_\tau X^i\right) - \frac{1}{4}\left[X^i, X^i\right]^2 \\ &+\partial_\tau \bar{\alpha} D_\tau \alpha\Big\}\,. \end{aligned} \tag{7.17}$$

Then we can Fourier expand all the fields (in their Matsubara frequencies in units of $\omega =$

[7]There is going to be a minor difference between the calculations done here compared to what was done in the previous section in the choice of the gauge group. While we chose it to be $SU(N)$ in the case of the IKKT analysis summarized above, we will stick to $U(N)$ for the BFSS model as was done in [16].

$2\pi/\beta$) as

$$X_i(\tau) = \sum_l X_l^i \, e^{il\omega\tau} \,, \qquad \alpha(\tau) = \sum_{l\neq 0} \alpha_l e^{il\omega\tau} \quad \text{and} \quad \bar{\alpha}(\tau) = \sum_{l\neq 0} \bar{\alpha}_l e^{-il\omega\tau} \,, \tag{7.18}$$

using which, we can expand the BFSS action to get

$$\begin{aligned} S^{(b)}_{\mathrm{BFSS}} \;=\; & \frac{1}{2g^2}\sum_l \left(\frac{2\pi l}{\beta}\right)^2 \mathrm{Tr}\left(X_l^i X_{-l}^i\right) + \frac{1}{g^2}\sum_{l\neq 0}\left(\frac{2\pi l}{\beta}\right)\mathrm{Tr}\left(\bar{\alpha}_l \alpha_l\right) \\ & - \frac{1}{g^2\sqrt{\beta}}\sum_l \left(\frac{2\pi l}{\beta}\right)^2 \mathrm{Tr}\left(X_l^i\left[A_{00}, X_{-l}^i\right]\right) + \frac{1}{g^2\sqrt{\beta}}\sum_{l\neq 0}\left(\frac{2\pi l}{\beta}\right)\mathrm{Tr}\left(\bar{\alpha}_l\left[A_{00}, \alpha_{-l}\right]\right) \\ & + \frac{1}{2g^2\beta}\sum_l \mathrm{Tr}\left(\left[A_{00}, X_l^i\right]\left[A_{00}, X_{-l}^i\right]\right) - \frac{1}{4g^2\beta}\sum_{l+m+n+p=0}\mathrm{Tr}\left(\left[X_l^i, X_m^j\right]\left[X_n^i, X_p^j\right]\right) . \end{aligned} \tag{7.19}$$

It is easy to see that the bosonic part of the Euclidean BFSS action, written as above, is the same as the one used in [9, 15] when we rescale some the fields by some factors of β. Moreover, the sums over the Fourier modes are arranged in a slightly different way than in [15] , and the Mastsubara frequencies have been written out explicitly. Finally, the 't Hooft coupling has not been set to one, as is typically done.

Let us now introduce our Gaussian ansatz for the above bosonic action as:

$$S_0^{(b)} = -\frac{N}{\Lambda}\mathrm{Tr}\left(U + U^\dagger\right) + \sum_l \sum_{i=1}^{D} \frac{1}{2v_{l,i}}\mathrm{Tr}\left(X_l^i X_{-l}^i\right) - \sum_{l\neq 0}\frac{1}{s_l}\mathrm{Tr}\left(\bar{\alpha}_l \alpha_l\right) \,, \tag{7.20}$$

where $U := \mathcal{P}e^{i\oint \mathrm{d}\tau A_0} = e^{i\sqrt{\beta}A_{00}}$ is the holonomy of the gauge connection A_{00} and takes its value in the $SU(N)$ group. Clearly, the first term corresponding to U is not a quadratic term, and so the action is not strictly-speaking a Gaussian one. However, this is the appropriate term to include for angular variables as argued in [16, 17]. The Gaussian parameter corresponding to this is given by Λ whereas we keep calling the 'bosonic mass' parameters $v_{l,i}$ in accordance with the previous section (and differing from what has been done in [16]).. However, recall that $v_{l,i} > 0$ for us. More importantly, we have added an i index to this Gaussian parameter to allow for a breaking of the $SO(D)$ symmetry, as was done for the IKKT model. This is a necessary new generalization for us (say, as compared to [17]) since we are interested in studying symmetry-breaking. However, as we will show later, we can derive the $SO(D)$-symmetric solution of [16], which was found by studying the gap equations in the bosonic case.

From a practical point of view, what complicates the case for the BFSS model is the appearance of the l index due to the presence of an intrinsic time parameter which necessitates expansion in terms of Fourier modes. Finally, the ghost fields have the usual form, with s_l denoting the Gaussian parameters. Already before going into the calculations, we can see that the gap equations even for the purely bosonic case will be of three types – one for the Wilson loops, one for the bosonic matrices and one for the ghost terms. Contrast this with the single type of gap equations we had for the bosonic IKKT model (7.8). However, as we will show explicitly below, some of the gap equations will decouple and allow us to study the bosonic gap equations, relevant for discovering symmetry-breaking patterns, in isolation.

The necessary ingredients for doing the calculations are to calculate propagators for the bosonic and ghost field, as well as the expectation values of the Wilson-loop operators. Let us begin by diagonalizing the holonomy $U = \text{diag.}\,(e^{i\alpha_1}, e^{i\alpha_2}, \ldots, e^{i\alpha_N})$, and then quoting the one-plaquette partition function for the holonomy [17, 21]:

$$Z_\square = e^{-\beta F_\square} = \begin{cases} \exp\left(N^2\left(-\frac{2}{\Lambda} - \frac{1}{2}\ln\frac{\Lambda}{2} + \frac{3}{4}\right)\right) & \Lambda \le 2 \\ \exp\left(-N^2/\Lambda^2\right) & \Lambda \ge 2\,, \end{cases} \tag{7.21}$$

with a phase-transition at $\Lambda = 2$. Similarly, the expectation value of the Wilsonian loop is

$$\langle \mathrm{Tr}\, U\rangle_\square = \begin{cases} N(1-\Lambda/4) & \Lambda \le 2 \\ N/\Lambda & \Lambda \ge 2\,. \end{cases} \tag{7.22}$$

These will be useful for calculating the free energy to the first order. Next, it is easy ot calculate the propagators of the gauge field, the bosonic and the ghost fields (using the Gaussian action (8.21)):

$$\begin{aligned} \langle (A_{00})_{AB}\,(A_{00})_{CD}\rangle_0 &= \rho_0^2\,\delta_{AD}\,\delta_{BC}\,, \\ \left\langle \left(X_l^i\right)_{AB}\left(X_m^j\right)_{CD}\right\rangle_0 &= v_{l,i}\,\delta^{ij}\,\delta_{l+m}\,\delta_{AD}\,\delta_{BC}\,, \\ \langle (\bar\alpha_l)_{AB}\,(\alpha_m)_{CD}\rangle_0 &= s_l\,\delta_{lm}\,\delta_{AD}\,\delta_{BC}\,, \end{aligned} \tag{7.23}$$

where the expectation values are taken with respect to the Gaussian action in (8.21), and we have defined ρ_0 in terms of the eigenvalues of the holonomy α as:

$$\rho_0^2 = \frac{1}{\beta N}\int \mathrm{d}\alpha\,\alpha^2\rho_\square(\alpha)\,. \tag{7.24}$$

The explicit expression for the above can be found in [16] but this is not important for our purposes.

Before going ahead with the calculation, let us make one observation. In the IKKT model (7.2), there are no quadratic (or kinetic) terms. However, this is not the case in the BFSS model as can be seen from the presence of terms of the form:

$$\frac{1}{g^2}\left[\frac{1}{2}\sum_l\left(\frac{2\pi l}{\beta}\right)^2 X^i_{-l}X^i_l+\sum_{l\neq 0}\left(\frac{2\pi l}{\beta}\right)^2\bar{\alpha}_l\alpha_l\right]. \tag{7.25}$$

Naturally, these terms will also give contributions to the propagators for the bosonic and the ghost field, of the form:

$$\begin{aligned}\left\langle \left(X^i_l\right)_{AB}\left(X^j_m\right)_{CD}\right\rangle &= \left(\frac{g\beta}{2\pi l}\right)^2\delta^{ij}\,\delta_{l+m}\,\delta_{AD}\,\delta_{BC}\,,\\ \left\langle \left(\bar{\alpha}_l\right)_{AB}\left(\alpha_m\right)_{CD}\right\rangle &= \left(\frac{g\beta}{2\pi l}\right)^2\delta_{lm}\,\delta_{AD}\,\delta_{BC}\,,\end{aligned} \tag{7.26}$$

where the form of the delta functions $\delta_{AD}\delta_{BC}$ can be understood by writing out the matrices explicitly in some basis of generators of the $U(N)$ group. The angled brackets above refer to calculating the expectation values with respect to the BFSS partition function, and not with respect to the Gaussian one. Crucially for us, to the next-to-leading order that we will consider for calculating the free energy, we will not need the above propagators to do the computation. More explicitly, for evaluating (F_0+F_1), we only need $\langle S\rangle_0$ and $\langle S_0\rangle_0$ – none of which will require going beyond the propagators given in (7.23).

Given this background, we can write the free energy to zero'th order as:

$$\beta F_0=\beta F_{\Box}(\Lambda)-\frac{N^2}{2}\sum_{i=1}^{D}\sum_l \ln v_{l,i}+N^2\sum_{l\neq 0}\ln s_l\,. \tag{7.27}$$

The details of this calculation has been delegated to Appendix (7.6.1). To calculate the first order correction to the free energy, we recall that

$$F_1=\langle (S-S_0)\rangle_0\,. \tag{7.28}$$

Once again, we have left the details to Appendix (7.6.2), and only quote the final result here:

$$\begin{aligned} F_1 &= F_{\Box}^{(1)}(\Lambda) + \frac{N^2}{2}\sum_{i=1}^{D}\sum_{l}\left[\frac{1}{g^2}\left(\frac{2\pi l}{\beta}\right)^2 v_{l,i}\right] - \frac{N^2 D}{2}\sum_{l} 1 \\ &\quad + N^2\sum_{l\neq 0}\left[\frac{1}{g^2}\left(\frac{2\pi l}{\beta}\right)^2 s_l + 1\right] + \frac{N^3}{g^2\beta}\rho_0^2\sum_{i=1}^{D}\sum_{l} v_{l,i} \\ &\quad + \frac{N^3}{2g^2\beta}\underbrace{\sum_{i=1}^{D}\sum_{j=1}^{D}}_{j\neq i}\sum_{l,k} v_{l,i}\, v_{k,j}\,, \end{aligned} \tag{7.29}$$

where

$$F_{\Box}^{(1)}(\Lambda) = \frac{N}{\Lambda}\left\langle \mathrm{Tr}\left(U+U^{\dagger}\right)\right\rangle_{\Box} = \begin{cases} \dfrac{2N^2}{\Lambda}\left(1-\frac{\Lambda}{4}\right)\,, & \Lambda \leq 2 \\ \dfrac{N^2}{\Lambda^2}\,, & \Lambda \geq 2 \end{cases} \tag{7.30}$$

Once we have the free energy calculated to the next-to-leading order, we can evaluate the gap equations as:

$$\frac{\partial}{\partial v_{l,i}}\left(F_0+F_1\right) = 0 \tag{7.31}$$

$$\frac{\partial}{\partial s_l}\left(F_0+F_1\right) = 0 \tag{7.32}$$

$$\frac{\partial}{\partial \Lambda}\left(F_0+F_1\right) = 0\,. \tag{7.33}$$

While the first equation is the one of interest for us, as it gives the gap equation for the bosonic Gaussian parameter, the second one refers to the gap equation for the ghosts and the third one for the Wilson loop parameters. Since the full free energy takes the form

$$\begin{aligned} F_0+F_1 &= \beta F_{\Box}(\Lambda) - \frac{N^2}{2}\sum_{i=1}^{D}\sum_{l}\ln v_{l,i} + N^2\sum_{l\neq 0}\ln s_l + \frac{N}{\Lambda}F_{\Box}^{(1)}(\Lambda) \\ &\quad + \frac{N^2}{2}\sum_{i=1}^{D}\sum_{l}\left[\frac{1}{g^2}\left(\frac{2\pi l}{\beta}\right)^2 v_{l,i}\right] - \frac{N^2 D}{2}\sum_{l} 1 + N^2\sum_{l\neq 0}\left[\frac{1}{g^2}\left(\frac{2\pi l}{\beta}\right)^2 s_l + 1\right] \\ &\quad + \frac{N^3}{g^2\beta}\rho_0^2\sum_{i=1}^{D}\sum_{l} v_{l,i} + \frac{N^3}{2g^2\beta}\underbrace{\sum_{i=1}^{D}\sum_{j=1}^{D}}_{j\neq i}\sum_{l,k} v_{l,i}\, v_{k,j}\,, \end{aligned} \tag{7.34}$$

the gap equations for the ghosts can be written as:

$$\frac{N^2}{s_l} + N^2\left(\frac{2\pi l}{\beta}\right)^2\frac{1}{g^2} = 0 \qquad \Rightarrow \qquad \frac{g^2}{s_l} = -\left(\frac{2\pi l}{\beta}\right)^2 \qquad l\neq 0\,. \tag{7.35}$$

This shows that, to this order, the ghost fields are completely decoupled from the rest as their solutions are free from the other parameters. This, of course, greatly simplifies our analysis. The gap equations obtained by varying the holonomy parameter Λ can be written as[8]:

$$-\left(1-\frac{2}{\Lambda}\right)\ln\left(\frac{1-\Lambda}{2}\right)+1 = \frac{Ng^2\beta}{4\sum_{i=1}^{D}\langle (R^i)^2\rangle}\,, \qquad \Lambda \le 2 \tag{7.36}$$

$$\Lambda = \frac{Ng^2\beta}{2\sum_{i=1}^{D}\langle (R^i)^2\rangle}\,, \qquad \Lambda \ge 2\,, \tag{7.37}$$

where we have introduced the 'extent of space parameters' for the eigenvalue distribution of the bosonic matrices, and therefore gives an estimate of the "size" of the system, as:

$$\left\langle (R^i)^2\right\rangle := \frac{1}{N}\mathrm{Tr}\left\langle \left(X^i(\tau)\right)^2\right\rangle = \frac{N}{\beta}\sum_l v_{l,i}\,. \tag{7.38}$$

Importantly, note that there is no sum over the i index in the above equation. These are the same extent of parameters which one uses in extracting cosmological solutions from the BFSS model and when spatial isotropy of the large dimensions is assumed (which is why we will simply get factors of D in the denominator in front of $\langle (R^i)^2\rangle$ in (7.36)). We will have more to say about this later on. However, for now note that the bosonic parameters $v_{l,i}$ are indeed coupled to Λ through the term $N^3/(g^2\beta)\,\rho_0^2\sum_{i=1}^{D}\sum_l v_{l,i}$ in the free energy.

Finally, let us derive the gap equations for $v_{l,i}$, which are given as

$$-\frac{1}{v_{l,i}}+\frac{1}{g^2}\left(\frac{2\pi l}{\beta}\right)^2+\frac{2N}{g^2\beta}\,\rho_0^2(\Lambda)+\frac{2N}{g^2\beta}\sum_{\mathrm{J}\neq i}\sum_l v_{l,j}=0\,. \tag{7.39}$$

Although we did not choose a Gaussian ansatz which is $SO(D)$ symmetric to begin with, we can now search for solutions of the above equations to examine if $SO(D)$ symmetric solutions are allowed or not. Notice the remarkable similarity between this equation and the gap equation for the bosonic IKKT action given in (7.8). This encourages us to look for solutions of the form $v_{l,i}=v_{l,2}=\cdots=v_{l,D}=:v_l\,,\ \forall\ l\in\mathbb{Z}$, and plugging this ansatz into (8.22), we find

$$\frac{g^2}{v_l}=\left(\frac{2\pi l}{\beta}\right)^2+\frac{2N}{\beta}\,\rho_0^2(\Lambda)+\frac{2N(D-1)}{\beta}\sum_l v_l\,. \tag{7.40}$$

[8]The reason why the extent of space parameter appears in the gap equation for Λ is that there is a term in (7.34) which involves both $\rho_o(\Lambda)$ and $v_{l,i}$.

This is the same gap equation which had been derived in [16] assuming an $SO(D)$-symmetric Gaussian action. Therefore, we have proved that the $SO(D)$ group remains unbroken for the bosonic BFSS action[9]. Note that our proof neither requires the explicit form of $\rho_0(\Lambda)$ nor the analyses of the other gap equations (7.35) and (7.36), as was hinted earlier on.

However, for the sake of completeness, we will go on to show the solutions of (7.40), which can be written as (following the notations of [16]):

$$v_l = \frac{g^2}{\left(\frac{2\pi l}{\beta}\right)^2 + m_{\text{eff}}^2}\,, \tag{7.41}$$

where the effective thermal mass is defined as

$$m_{\text{eff}}^2 = \frac{2N}{\beta}\rho_0^2(\Lambda) + 2(D-1)\left\langle R^2 \right\rangle\,. \tag{7.42}$$

For the case when there is no symmetry breaking, it is easy to identify the usual extent of space parameter as

$$\left\langle (R^i)^2 \right\rangle := \frac{1}{N}\text{Tr}\left\langle \left(X^i(\tau)\right)^2 \right\rangle = \frac{N}{\beta}\sum_l v_l\,, \tag{7.43}$$

and is the same in all the i-directions[10].

Before ending this section, let us note that the gap equation for the Gaussian parameters corresponding to the bosonic fields in the bosonic BFSS model is very similar to the one we had derived in the bosonic IKKT model. The appearance of the (infinite number of) Fourier modes does not actually complicate the story for the $SO(D)$ symmetry-breaking, *i.e.* the v_l's for all the different i's are the same in the bosonic model. Thus, the main finding of this section is that the bosonic BFSS model has an unbroken $SO(D)$ symmetry, just like in the IKKT case. In hindsight, we could have guessed this conclusion from the results of [16]; however, our calculation by not assuming an $SO(D)$-symmetric Gaussian action will be extremely helpful in the next section when including the fermionic terms.

---

[9] Strictly speaking, we have proved that our more general gap equation allows for $SO(D)$-symmetric solutions and we have not proven the uniqueness of the ensuing solution. However, given that the gap equation is an algebraic one, it easily follows that the $SO(D)$-symmetric solution of [16] is the only one.

[10] The case for cosmology is a bit more subtle. In that case, we find a symmetry breaking to give us three large spatial dimensions which are expanding; however, one does assume spatial isotropy amongst these large external directions. In essence there is an unbroken $SO(3)$ symmetry in that case when defining the extent of space parameters.

7.4 Symmetry breaking in the BFSS model

The full BFSS model has the following additional terms due to the presence of fermionic matrices:

$$\begin{aligned} S^{(f)}_{\text{BFSS}} &= \frac{i}{2g^2}\sum_r \left(\frac{2\pi r}{\beta}\right)\psi_{-r}\psi_r - \frac{i}{2g^2\sqrt{\beta}}\sum_r \text{Tr}\left(\psi_{-r}\left[A_{00},\psi_r\right]\right) \\ &\quad - \frac{1}{2g^2\beta}\sum_{r,s}\text{Tr}\left(\psi_r\Gamma_i\left[X^i_{-r-s},\psi_s\right]\right) , \end{aligned} \tag{7.44}$$

where we have now Fourier expanded the fermionic fields as

$$\psi_\alpha(\tau) = \sum_r \psi^\alpha_r e^{ir\omega\tau} . \tag{7.45}$$

We have suppressed the spinor index α throughout in (7.44) above. The Gamma matrices are $p\times p$ symmetric matrices satisfying the anti-commutation relations $\{\Gamma_i,\Gamma_j\} = 2\delta_{ij}$. The above Fourier expanded terms comes from the two following fermionic terms in the Euclidean BFSS action:

$$S^{(f)}_{\text{BFSS}} \propto \frac{1}{2g^2}\int \mathrm{d}\tau \text{Tr}\left(\psi_\alpha D_\tau\psi_\alpha - \psi_\alpha\left(\Gamma_i\right)_{\alpha\beta}\left[X_i,\psi_\beta\right]\right) . \tag{7.46}$$

Including the fermionic terms in the BFSS action (7.44) requires adding to the Gaussian ansatz the following term:

$$S^{(f)}_0 = \sum^p_{\alpha,\beta}\sum_r \text{Tr}\left(\psi^\alpha_r\mathcal{A}_{\alpha\beta}\psi^\beta_{-r}\right) , \tag{7.47}$$

where we use the same symbol for the Gaussian parameter matrix $\mathcal{A}$ as we had done for the IKKT model. Recalling the wisdom gained from the IKKT case, we do not try to expand this matrix in terms of Gaussian parameters $\omega_{\mu\nu\lambda}$ and ω_μ since this was not important to explore the question of existence of symmetry-breaking in the model, and unnecessarily complicates the gap equations.

Before going through with the explicit calculations for the fermionic terms, let us make the following initial observations:

- As we see from (7.44), the fermions do not interact with the ghost fields and therefore, the gap equation for the ghost propagator will obviously remain decoupled.

- On the other hand, we find that the fermions couple to the bosonic fields X^i_l, A_{00} through cubic interactions. However, and this the most subtle part of the calculation, although cubic terms of the form $\mathrm{Tr}\left(X^i_l\left[A_{00}, X^i_{-l}\right]\right)$ and $\mathrm{Tr}\left(\bar{\alpha}_l\left[A_{00}, \alpha_l\right]\right)$ do not contribute to the free energy (to the order we are interested in), as shown in the previous section, things are a bit different with fermionic terms. In fact, $\left\langle\left(S^{(f)}_{\mathrm{BFSS}}\right)^2\right\rangle$ does contribute to F_1 [20]. Thus, the cubic interactions, involving the fermion bilinears, will end up affecting the gap equation for the bosonic Gaussian parameters ($v_{l,i}$).

- The purely fermionic terms, on the other hand, can *only* contribute to the gap equation for the fermionic parameters alone. So, we will ignore the contribution and focus only in the interaction terms which will have an effect on the $v_{l,i}$-gap equations.

Keeping the above general comments in mind, we will not try to calculate the full expression for the free energy, up to the next-to-leading order, including the fermionic terms. Instead, we will only try to identify the contribution which effect the $v_{l,i}$-gap equation (8.22). The main ingredients required for doing this calculation are the fermionic propagators, calculated with respect to the Gaussian action (8.24):

$$\left\langle\left(\psi^\alpha_r\right)_{AB}\left(\psi^\beta_s\right)_{CD}\right\rangle_0 = \left(\mathcal{A}^{-1}\right)^{\alpha\beta}\,\delta_{r,-s}\,\delta_{AD}\,\delta_{BC}\,. \tag{7.48}$$

As before, we will never need the expression for the propagator for the fermions calculated using the BFSS action itself, to the order we are considering, but we present that result here for completeness:

$$\left\langle\left(\psi^\alpha_r\right)_{AB}\left(\psi^\beta_s\right)_{CD}\right\rangle = \frac{g^2}{2\pi i r}\,\delta^{\alpha\beta}\,\delta_{r,-s}\,\delta_{AD}\,\delta_{BC}\,. \tag{7.49}$$

We begin with the following term: $-\frac{i}{2g^2\sqrt{\beta}}\sum_r \mathrm{Tr}\left(\psi_{-r}\left[A_{00}, \psi_r\right]\right)$, whose contribution to the free energy will scale as $\left\langle\left(S^{(f)}_{\mathrm{BFSS}}\right)^2\right\rangle$. Using the propagators from above, we can evaluate the contribution of this term as

$$\begin{aligned}
&\sim\ \sum_s\sum_r\left(A_{00}\right)_{AB}\left(\psi^\alpha_r\right)_{BC}\left(\psi^\beta_{-r}\right)_{CA}\delta_{\alpha\beta}\left(A_{00}\right)_{DE}\left(\psi^{\alpha'}_s\right)_{EF}\left(\psi^{\beta'}_{-s}\right)_{FD}\delta_{\alpha'\beta'}\\
&\sim\ \rho^2\,\delta_{AE}\,\delta_{BD}\,\delta_{DB}\,\delta_{CF}\,\delta_{FC}\,\delta_{EA}\left[(\mathcal{A})^{-1}_{\alpha\beta}\,\delta^{\alpha\beta}\right]^2\\
&\sim\ \rho^2\,N^3\left[\mathrm{Tr}\left(\mathcal{A}^{-1}\right)\right]^2\,.
\end{aligned} \tag{7.50}$$

Although we have dropped a lot of numerical prefactors in the above calculation, the important conclusions are the following:

1. There are no free $SO(D)$ or (i) indices in the above expression, as is expected from the structure of the term itself. This clearly implies that even if the above term is to somehow influence the gap equation for $v_{l,i}$, it will surely allow for a $SO(D)$-symmetric solution of the form $v_{l,1} = v_{l,2} = \cdots = v_{l,D}\,,\ \forall\, l \in \mathbb{Z}$.

2. However, more specifically in this case, the above term does not contain any factors of $v_{l,i}$, and can therefore not appear in the gap equation for $v_{l,i}$. The only way it can influence this gap equation is through ρ_0. Since the above equation does depend on ρ_0, and the latter appears in the $v_{l,i}$-gap equation, it can indirectly affect the solutions of $v_{l,i}$ through ρ_0. However, due to the argument mentioned above, it will not have any effect on the symmetry-breaking pattern.

3. This small calculation demonstrates that *any* type of a fermionic term will not lead to a $SO(D)$ symmetry breaking. In other words, having a matrix model action involving fermionic terms does not guarantee any symmetry-breaking and the structure of the Lagrangian itself is very important in exploring the pattern of symmetry-breaking.

We are now finally in the position to tackle the main term that will give us evidence of symmetry-breaking in the BFSS model, namely the term

$$-\frac{1}{2g^2\beta}\sum_{r,s}\mathrm{Tr}\left(\psi_r\Gamma_i\left[X^i_{-r-s},\psi_s\right]\right)\,. \tag{7.51}$$

Notice the striking similarity of this term with the (only) fermionic term in the IKKT model, which was ultimately responsible for symmetry-breaking in that case. Remembering that its contribution to the free energy is going to be at the quadratic order, we find

$$\begin{aligned}
&\sum_{i=1}^{D}\sum_{j=1}^{D}\sum_{p,q,r,s}\left\langle\left(\psi^\alpha_r\right)_{AB}(\Gamma_i)_{\alpha\beta}\left(X^i_{-r-s}\right)_{BC}\left(\psi^\beta_s\right)_{CA}\left(\psi^{\alpha'}_p\right)_{DE}(\Gamma_j)_{\alpha\beta}\left(X^j_{-p-q}\right)_{EF}\left(\psi^\alpha_r\right)_{FD}\right\rangle\\
\sim\ &\sum_{i=1}^{D}\sum_{j=1}^{D}\sum_{p,q,r,s}\delta_{r,-q}\,\delta_{s,-p}\,\delta_{-r-s+p+q}\,\delta^{ij}\,N^3\,v_{(-r-s),i}\,\left(\mathcal{A}^{-1}\right)^{\alpha\beta'}(\Gamma_i)_{\alpha\beta}\left(\mathcal{A}^{-1}\right)^{\beta\alpha'}(\Gamma_i)_{\alpha'\beta'}\\
\sim\ &\sum_{i=1}^{D}\sum_{p,q}N^3\,v_{(p+q),i}\,\mathrm{Tr}\left[\left(\mathcal{A}^{-1}\,\Gamma_i\right)^2\right]\,.
\end{aligned}\tag{7.52}$$

Since $(p,q) \in \mathbb{Z}/2$, and we have an infinite sum over both, we can replace this index by some $l \in \mathbb{Z}$, such that we now have

$$\sum_{i=1}^{D} \sum_{l} N^3 \, v_{l,i} \, \mathrm{Tr} \left[\left(\mathcal{A}^{-1} \, \Gamma_i \right)^2 \right] . \tag{7.53}$$

This, of course, depends explicitly on $v_{l,i}$ and would therefore contribute to the gap equation of interest. In fact, its contribution to the gap equation will be a term of the form

$$\frac{N}{g^2 \beta} \, \mathrm{Tr} \left[\left(\mathcal{A}^{-1} \, \Gamma_i \right)^2 \right] . \tag{7.54}$$

When we include this in (8.22), we get an equation of the form:

$$-\frac{1}{v_{l,i}} + \frac{1}{g^2} \left(\frac{2\pi l}{\beta} \right)^2 + \frac{2N}{g^2 \beta} \, \rho_0^2(\Lambda) + \frac{2N}{g^2 \beta} \sum_{\jmath \neq i} \sum_{l} v_{l,j} - \frac{N}{g^2 \beta} \, \mathrm{Tr} \left[\left(\mathcal{A}^{-1} \, \Gamma_i \right)^2 \right] = 0 \, . \tag{7.55}$$

The last term shows that this equation cannot have an $SO(D)$ symmetric solution as long as the matrix $\mathcal{A}$ has at least one non-zero entry. This is analogous to saying that the fermionic Gaussian parameters are not trivially zero, and for the reasons emphasized earlier for the IKKT model, we are assured that the solution to the above equation must break $SO(D)$ symmetry.

7.5 Conclusion

The recent developments in matrix cosmology have provided a promising new direction in understanding our early universe. For the IKKT model, it has been shown [7] how analytical methods can be used to extract a spacetime metric, with an infinite extent for both space and time, from the (block-diagonal) structure and dynamics of the matrices. This was shown to naturally solve the flatness problem of standard big bang cosmology, and hint towards a solution for the cosmological constant issue. On the other hand, a thermal state in the BFSS model has been shown to provide a natural solution to the horizon problem as well as predict an almost scale-invariant spectrum of primordial perturbations [9]. A crucial input input the scenario of [9] was the assumption that the spatial rotational symmetry of the BFSS Lagrangian is spontaneously broken in the state which minimizes the free energy. to a configuration in which the extent of space only becomes large in three spatial directions.

The existence of such a phase transition has been established in the IKKT model, but not yet in the BFSS scenario.

In this paper, we have provided first evidence for a $SO(9)$ symmetry-breaking in the BFSS model, similar to what happens in the IKKT case, by employing the Gaussian expansion method (the same methods which were used in the case of the IKKT model to show the existence of the symmetry breaking phase transition). The inclusion of the contribution of fermions is crucial to reach this conclusion. In the absence of fermions, the state which minimizes the free energy maintains the $SO(9)$ symmetry. In the case of the IKKT model, numerical studies (both full matrix model simulations and also numerical evaluations of the free energies) have shown that the energetically favored state has $SO(3)$ symmetry with three dimensions of space becoming large. A next step in our research program is to perform similar analyses in the case of the BSFF model in order to determine the symmetry and features of the configuration after the breaking of the $SO(9)$ symmetry. This would require going well-beyond the next-to-leading order calculations done here and cannot be achieved by analytical tools alone. However, since the corresponding calculations for the IKKT model have been manageable, it is only natural to push for examining if such a similar result can be obtained for the BFSS model. If possible, such a result would be prove that a large $(3+1)$-d universe can spontaneously emerge in the BFSS matrix model.

There has already been a wealth of similarities between results coming out of matrix cosmology and string gas cosmology. For instance, the amplitude of (scale-invariant) perturbations for the thermal state in the BFSS model is exactly the same as in string gas cosmology (see e.g. [22] for a review). It is well known that three large dimensions do emerge in the string gas model since space cannot expand unless the winding modes of the strings annihilate into string loops, the probability for which is zero only if there are more than three large spatial dimensions. We believe that this striking similarity in explaining the emergence of a large $(3+1)$-d universe from full string theory, in both string gas cosmology and matrix models, is not a mere coincidence and that the physical reason underlying both must be the same. In fact, there must be a well-defined sense in which one can recover the string gas model from the full dynamics of matrix theory. Another goal for the future will be to further explore the physical reason behind the SSB in the BFSS and IKKT models since this might point to the aforementioned connection.

Acknowledgements:

SB is supported in part by the Higgs Fellowship. SL is supported in part by FRQNT. The research at McGill is supported in part by funds from NSERC and from the Canada Research Chair program.

Appendix

7.6 Computation of the free energy for the bosonic BFSS action

In the appendix, we give details of the computation of the free energy, up to the first order, for the bosonic BFSS model which has been used in the main draft to derive the gap equation for the bosonic Gaussian parameters.

7.6.1 Derivation of F_0

We want to find the expressions for all the terms appearing in (7.27), which we reproduce here for convenience:

$$\beta F_0 = \beta F_{\Box}(\Lambda) - \frac{N^2}{2} \sum_{i=1}^{D} \sum_{l} \ln v_{l,i} + N^2 \sum_{l \neq 0} \ln s_l \,. \tag{7.56}$$

The first term is what it is by definition, and its explicit form can be read from (7.21). Let us begin by deriving the second term on the r.h.s. of the equation above which involves the second term in the Gaussian ansatz (8.21):

$$
\begin{aligned}
S_0^{\text{bosonic}} &= \sum_l \sum_{i=1}^{D} \frac{1}{2v_{l,i}} \operatorname{Tr}\left(X_l^i X_{-l}^i\right) \\
&= \sum_l \sum_{i=1}^{D} \sum_{a=1}^{N^2} \frac{1}{2v_{l,i}} \operatorname{Tr}\left(X_l^{i\,a} X_{-l\,a}^i\right) , \qquad (7.57)
\end{aligned}
$$

where we have explicitly written the trace in terms of the $U(N)$ generators, and a refers to the $U(N)$ index. Given this action, the corresponding partition function (for the above term in the Gaussian action) is given by:

$$
\begin{aligned}
Z_0^{\text{bosonic}} &= \int \prod_{a=1}^{N^2} \prod_{i=1}^{D} \mathrm{d}x_i^a \, e^{-S_0^{\text{bosonic}}} \qquad (7.58) \\
&= \int \prod_{a=1}^{N^2} e^{-\sum_l \frac{1}{2v_{l,1}}\left(x_1^a x_{1\,a}\right)} \int \prod_{a=1}^{N^2} e^{-\sum_l \frac{1}{2v_{l,2}}\left(x_2^a x_{2\,a}\right)} \ldots \int \prod_{a=1}^{N^2} e^{-\sum_l \frac{1}{2v_{l,D}}\left(x_D^a x_{D\,a}\right)} \\
&= \underbrace{\left(2\sum_l v_{l,1}\right)^{N^2/2} \left(2\sum_l v_{l,2}\right)^{N^2/2} \ldots \left(2\sum_l v_{l,D}\right)^{N^2/2}}_{D \text{ number of terms}} ,
\end{aligned}
$$

where we have omitted several factors of numerical constants (involving π in the multiple Gaussian integrals) as this will only give a constant contribution to the free energy which is irrelevant for the gap equation. We have also suppressed a factor of β which we will restore later by noting that $Z \sim e^{-\beta S}$. Thus, a factor of β will appear in the denominator after carrying out the Gaussian integrals and will finally cancel with the β from βF_0.

Moving forward, the free energy corresponding to this is given by

$$
\begin{aligned}
F_0^{\text{bosonic}} &= -\ln Z_0^{\text{bosonic}} \qquad (7.59) \\
&= -\frac{N^2}{2}\left[\sum_l \ln v_{l,1} + \sum_l \ln v_{l,2} + \ldots + \sum_l \ln v_{l,d}\right] \qquad (7.60) \\
&= -\frac{N^2}{2} \sum_{i=1}^{D} \sum_l \ln v_{l,i} \, . \qquad (7.61)
\end{aligned}
$$

A similar analysis for the ghost fields yields:

$$
F_0^{\text{ghost}} = N^2 \sum_{l \neq 0} \ln s_l \, . \qquad (7.62)
$$

7.6.2 Derivation of F_1

Let us calculate this term by term for the (bosonic part of the) BFSS action given in (7.19), as well as the Gaussian action, which we rewrite below and label the different terms, as follows:

$$\begin{aligned}
S^{(b)}_{\text{BFSS}} \;=\;& \underbrace{\frac{1}{2g^2}\sum_l \left(\frac{2\pi l}{\beta}\right)^2 \text{Tr}\left(X^i_l X^i_{-l}\right)}_{(1)} + \underbrace{\frac{1}{g^2}\sum_{l\neq 0}\left(\frac{2\pi l}{\beta}\right)\text{Tr}\left(\bar{\alpha}_l \alpha_l\right)}_{(2)} \\
& \underbrace{-\frac{1}{g^2\sqrt{\beta}}\sum_l \left(\frac{2\pi l}{\beta}\right)^2 \text{Tr}\left(X^i_l\left[A_{00}, X^i_{-l}\right]\right) + \frac{1}{g^2\sqrt{\beta}}\sum_{l\neq 0}\left(\frac{2\pi l}{\beta}\right)\text{Tr}\left(\bar{\alpha}_l\left[A_{00}, \alpha_{-l}\right]\right)}_{(6)} \\
& \underbrace{+\frac{1}{2g^2\beta}\sum_l \text{Tr}\left(\left[A_{00}, X^i_l\right]\left[A_{00}, X^i_{-l}\right]\right)}_{(3)} \underbrace{-\frac{1}{4g^2\beta}\sum_{l+m+n+p=0}\text{Tr}\left(\left[X^i_l, X^j_m\right]\left[X^i_n, X^j_p\right]\right)}_{(4)} .
\end{aligned} \tag{7.63}$$

$$S^{(b)}_0 = \underbrace{-\frac{N}{\Lambda}\text{Tr}\left(U + U^\dagger\right)}_{(5)} + \underbrace{\sum_l\sum_{i=1}^D \frac{1}{2v_{l,i}}\text{Tr}\left(X^i_l X^i_{-l}\right)}_{(1)} - \underbrace{\sum_{l\neq 0}\frac{1}{s_l}\text{Tr}\left(\bar{\alpha}_l\alpha_l\right)}_{(2)} , \tag{7.64}$$

We will repeatedly use the propagators written down in (7.23) in order to do the explicit calculation. As mentioned earlier, and shown below, we will never need to consider the propagators given in (7.26) for evaluating F_1.

The easiest to calculate are the (5) terms involving the Wilson loop operators which are given by

$$\langle (S - S_0)\rangle_0 = \frac{N}{\Lambda}\left\langle \text{Tr}\,\left(U + U^\dagger\right)\right\rangle_\square , \tag{7.65}$$

the expression for which has been given in (7.30).

Next we focus on the terms labelled by (1):

$$\langle S\rangle_0 \sim \sum_{i=1}^D\sum_l \frac{1}{2g^2}\left(\frac{2\pi l}{\beta}\right)^2 v_{l,i}\,\delta_{AA}\delta_{BB} = \sum_{i=1}^D\sum_l \frac{N^2}{2g^2}\left(\frac{2\pi l}{\beta}\right)^2 v_{l,i} , \tag{7.66}$$

and

$$\langle S_0\rangle_0 \sim \sum_{i=1}^D\sum_l \frac{1}{2v_{l,i}}\left\langle \text{Tr}\left(X^i_l X^i_{-l}\right)\right\rangle_0 = \frac{N^2}{2}\sum_{i=1}^D\sum_l 1 . \tag{7.67}$$

Although we could have carried out the sum over i in the second term above, we keep it in this form since in this way it is easier to organize the terms later on.

The terms labelled by (2) can be evaluated as:

$$\langle S \rangle_0 \sim \frac{N^2}{g^2} \sum_{l \neq 0} \left(\frac{2\pi l}{\beta} \right)^2 s_l \,, \tag{7.68}$$

and

$$\langle S_0 \rangle_0 \sim N^2 \sum_{l \neq 0} 1 \,. \tag{7.69}$$

Note that naively the sum over the Fourier modes give an infinite contribution to each of these terms, but this is not a problem for us since the terms are independent of the bosonic gap parameters.

Although the terms marked (3) look more complicated, they can easily be evaluated by keeping in mind the following considerations. Firstly, there are no such quartic terms in the Gaussian ansatz (7.64). And secondly, we need to only calculate the dominant (connected) term in the large-N limit (which corresponds to choosing the right contractions of the $U(N)$ indices A, B, C, D etc.):

$$\langle S \rangle_0 \sim \frac{1}{g^2 \beta} \sum_{i=1}^{D} \sum_{l} \left\langle (A_{00})_{AB} (A_{00})_{BC} (X_l^i)_{CD} (X_{-l}^i)_{DA} \right\rangle_0 = \frac{N^3}{g^2 \beta} \, \rho_0^2 \, (\Lambda) \sum_{i=1}^{D} \sum_{l} v_{l,i} \,. \tag{7.70}$$

Similarly, the term marked (4) can be evaluated to be:

$$\frac{N^3}{2 g^2 \beta} \underbrace{\sum_{i=1}^{D} \sum_{j=1}^{D}}_{j \neq i} \sum_{l} v_{l,i} \, v_{l,j} \,. \tag{7.71}$$

Finally, the terms marked (6) actually do not give any contribution to the free energy to the order we are considering at all. This comes from the simple observation that these terms are cubic and thus have zero contribution for the (quadratic) Gaussian propagators. However, the important observation in this regard is that this conclusion is only true for the X_l^i, A_{00} and α_l fields since they are all bosonic in nature, and would not be applicable for the fermionic fields as we will see later on.

Bibliography

[1] R. Brandenberger, "String Cosmology and the Breakdown of Local Effective Field Theory," [arXiv:2112.04082 [hep-th]];
R. Brandenberger and V. Kamali, "Unitarity Problems for an Effective Field Theory Description of Early Universe Cosmology," [arXiv:2203.11548 [hep-th]].

[2] H. Ooguri and C. Vafa, "On the Geometry of the String Landscape and the Swampland," Nucl. Phys. B **766**, 21-33 (2007) doi:10.1016/j.nuclphysb.2006.10.033 [arXiv:hep-th/0605264 [hep-th]];
G. Obied, H. Ooguri, L. Spodyneiko and C. Vafa, "De Sitter Space and the Swampland," [arXiv:1806.08362 [hep-th]].

[3] C. Vafa, "The String landscape and the swampland," hep-th/0509212;
T. D. Brennan, F. Carta and C. Vafa, "The String Landscape, the Swampland, and the Missing Corner," PoS TASI **2017**, 015 (2017) [arXiv:1711.00864 [hep-th]];
E. Palti, "The Swampland: Introduction and Review," Fortsch. Phys. **67**, no. 6, 1900037 (2019) [arXiv:1903.06239 [hep-th]];
M. van Beest, J. Calderon-Infante, D. Mirfendereski and I. Valenzuela, "Lectures on the Swampland Program in String Compactifications," arXiv:2102.01111 [hep-th].

[4] A. Bedroya and C. Vafa, "Trans-Planckian Censorship and the Swampland," JHEP **2009**, 123 (2020) [arXiv:1909.11063 [hep-th]];
A. Bedroya, R. Brandenberger, M. Loverde and C. Vafa, "Trans-Planckian Censorship and Inflationary Cosmology," Phys. Rev. D **101**, no. 10, 103502 (2020) [arXiv:1909.11106 [hep-th]].

[5] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, "M theory as a matrix model: A Conjecture," Phys. Rev. D **55**, 5112 (1997) [hep-th/9610043].

[6] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, "A Large N reduced model as superstring," Nucl. Phys. B **498**, 467 (1997) [hep-th/9612115].

[7] S. Brahma, R. Brandenberger and S. Laliberte, "Emergent Metric Space-Time from Matrix Theory," [arXiv:2206.12468 [hep-th]].

[8] R. H. Brandenberger and C. Vafa, "Superstrings In The Early Universe," Nucl. Phys. B **316**, 391 (1989).

[9] S. Brahma, R. Brandenberger and S. Laliberte, "Emergent cosmology from matrix theory," JHEP **03**, 067 (2022) [arXiv:2107.11512 [hep-th]].

[10] A. Nayeri, R. H. Brandenberger and C. Vafa, "Producing a scale-invariant spectrum of perturbations in a Hagedorn phase of string cosmology," Phys. Rev. Lett. **97**, 021302 (2006) [hep-th/0511140];
R. H. Brandenberger, A. Nayeri, S. P. Patil and C. Vafa, "Tensor Modes from a Primordial Hagedorn Phase of String Cosmology," Phys. Rev. Lett. **98**, 231302 (2007) [hep-th/0604126];
R. H. Brandenberger, A. Nayeri and S. P. Patil, "Closed String Thermodynamics and a Blue Tensor Spectrum," Phys. Rev. D **90**, no. 6, 067301 (2014) [arXiv:1403.4927 [astro-ph.CO]].

[11] P. Agrawal, S. Gukov, G. Obied and C. Vafa, "Topological Gravity as the Early Phase of Our Universe," arXiv:2009.10077 [hep-th].

[12] M. B. Green, J. H. Schwarz and E. Witten, "SUPERSTRING THEORY. VOL. 1: INTRODUCTION," (Cambridge Univ. Press, Cambridge, 1987);
M. B. Green, J. H. Schwarz and E. Witten, "SUPERSTRING THEORY. VOL. 2: LOOP AMPLITUDES, ANOMALIES AND PHENOMENOLOGY," (Cambridge Univ. Press, Cambrige, 1987).

[13] J. Nishimura and G. Vernizzi, "Spontaneous breakdown of Lorentz invariance in IIB matrix model," JHEP **0004**, 015 (2000) [hep-th/0003223];

J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, "Monte Carlo studies of the IIB matrix model at large N," JHEP **07**, 011 (2000) [arXiv:hep-th/0005147 [hep-th]];

S. W. Kim, J. Nishimura and A. Tsuchiya, "Expanding (3+1)-dimensional universe from a Lorentzian matrix model for superstring theory in (9+1)-dimensions," Phys. Rev. Lett. **108**, 011601 (2012) [arXiv:1108.1540 [hep-th]].

[14] M. Hirasawa, K. Anagnostopoulos, T. Azuma, K. Hatakeyama, Y. Ito, J. Nishimura, S. Papadoudis and A. Tsuchiya, "A new phase in the Lorentzian type IIB matrix model and the emergence of continuous space-time," PoS **LATTICE2021**, 428 (2022) [arXiv:2112.15390 [hep-lat]];

K. Hatakeyama, K. Anagnostopoulos, T. Azuma, M. Hirasawa, Y. Ito, J. Nishimura, S. Papadoudis and A. Tsuchiya, "Relationship between the Euclidean and Lorentzian versions of the type IIB matrix model," PoS **LATTICE2021**, 341 (2022) [arXiv:2112.15368 [hep-lat]];

K. Hatakeyama, K. Anagnostopoulos, T. Azuma, M. Hirasawa, Y. Ito, J. Nishimura, S. Papadoudis and A. Tsuchiya, "Complex Langevin studies of the emergent space-time in the type IIB matrix model," [arXiv:2201.13200 [hep-th]];

J. Nishimura, "Signature change of the emergent space-time in the IKKT matrix model," [arXiv:2205.04726 [hep-th]];

T. Aoki, M. Hirasawa, Y. Ito, J. Nishimura and A. Tsuchiya, "On the structure of the emergent 3d expanding space in the Lorentzian type IIB matrix model," PTEP **2019**, no. 9, 093B03 (2019) [arXiv:1904.05914 [hep-th]].

[15] S. Bal and B. Sathiapalan, "High temperature limit of the N=2 matrix model," Mod. Phys. Lett. A **14**, 2753-2766 (1999) [arXiv:hep-th/9902087 [hep-th]];

N. Kawahara, J. Nishimura and S. Takeuchi, "High temperature expansion in supersymmetric matrix quantum mechanics," JHEP **12**, 103 (2007) [arXiv:0710.2188 [hep-th]].

[16] D. N. Kabat and G. Lifschytz, "Approximations for strongly coupled supersymmetric quantum mechanics," Nucl. Phys. B **571**, 419-456 (2000) [arXiv:hep-th/9910001 [hep-th]].

[17] D. N. Kabat, G. Lifschytz and D. A. Lowe, "Black hole thermodynamics from calculations in strongly coupled gauge theory," Int. J. Mod. Phys. A **16**, 856-865 (2001) [arXiv:hep-th/0007051 [hep-th]];
D. N. Kabat, G. Lifschytz and D. A. Lowe, "Black hole entropy from nonperturbative gauge theory," Phys. Rev. D **64**, 124015 (2001) [arXiv:hep-th/0105171 [hep-th]].

[18] J. Nishimura and F. Sugino, "Dynamical generation of four-dimensional space-time in the IIB matrix model," JHEP **0205**, 001 (2002) [hep-th/0111102].

[19] H. Kawai, S. Kawamoto, T. Kuroki, T. Matsuo and S. Shinohara, "Mean field approximation of IIB matrix model and emergence of four-dimensional space-time," Nucl. Phys. B **647**, 153-189 (2002) [arXiv:hep-th/0204240 [hep-th]];
J. Nishimura, T. Okubo and F. Sugino, "Gaussian expansion analysis of a matrix model with the spontaneous breakdown of rotational symmetry," Prog. Theor. Phys. **114**, 487-508 (2005) [arXiv:hep-th/0412194 [hep-th]];
J. Nishimura, T. Okubo and F. Sugino, "Convergent Gaussian expansion method: Demonstration in reduced Yang-Mills integrals," JHEP **10**, 043 (2002) [arXiv:hep-th/0205253 [hep-th]];
T. Aoyama, J. Nishimura and T. Okubo, "Spontaneous breaking of the rotational symmetry in dimensionally reduced super Yang-Mills models," Prog. Theor. Phys. **125**, 537-563 (2011) [arXiv:1007.0883 [hep-th]].;
J. Nishimura, T. Okubo and F. Sugino, "Systematic study of the SO(10) symmetry breaking vacua in the matrix model for type IIB superstrings," JHEP **10**, 135 (2011) [arXiv:1108.1293 [hep-th]].

[20] F. Sugino, "Gaussian and mean field approximations for reduced 4-D supersymmetric Yang-Mills integral," JHEP **07**, 014 (2001) [arXiv:hep-th/0105284 [hep-th]].

[21] D. J. Gross and E. Witten, "Possible Third Order Phase Transition in the Large N Lattice Gauge Theory," Phys. Rev. D **21**, 446-453 (1980).

[22] R. H. Brandenberger, "String Gas Cosmology," String Cosmology, J.Erdmenger (Editor). Wiley, 2009. p.193-230 [arXiv:0808.0746 [hep-th]].

Part VI

Discussion and Conclusion

Chapter 8

Discussion

In the present chapter, we review the status of the matrix theory as a potential description of String Gas Cosmology in the light of the work Part II, Part III, Part IV, and Part V. The chapter will be divided in four different sections, each highlighting the results of the different parts, and how they align with predictions from String Gas Cosmology. In Section 8.1, we will compare the scale-invariant spectrum found from the BFSS and IKKT models at finite temperature to the one found from thermal fluctuations of closed strings in String Gas Cosmology. In Section 8.2, we will review the definition of the metric given in Chapter 5, compare with expectations from String Gas Cosmology, and point to possible improvements. In Section 8.3, we will investigate the similitudes and differences between the dynamical mechanism used to realize the mass term in Chapter 6, and String Gas Cosmology. Finally, in Section 8.4, we will review the symmetry breaking of the Euclidean BFSS model, as suggested by the Gaussian Expansion Method, and discuss how it aligns with expectations from String Gas Cosmology.

8.1 Comparison of experimental predictions in matrix cosmology and String Gas Cosmology

In the present section, we will aim to make a comparison between the experimental predictions of String Gas Cosmology and matrix cosmology. Before making this comparison, let us review the key results of String Gas Cosmology.

In String Gas Cosmology, the spectrum of cosmological perturbations is computed by considering a gas of closed string at finite temperature, and how its thermal fluctuations behave in an expanding background. The key aspect of this computation is that scalar perturbations are sourced by thermal density perturbations, which are directly related to the heat capacity

$$C_V \approx \frac{R^2/l_s^3}{T(1-T/T_H)}\,, \tag{8.1}$$

of close strings at finite temperature. Similarly, tensor perturbations are sourced by transverse energy density perturbations, which are related to the pressure perturbations

$$p \approx n_H T_H - \frac{2}{3}\frac{(1-T/T_H)}{l_s^3 R}\ln\left[\frac{l_s^3 T}{R^2(1-T/T_H)}\right]\,, \tag{8.2}$$

of the system. Using the heat capacity and the pressure of the system, we then obtained

$$P_\Phi = 16\pi^2 G^2 k^2 T^2 C_V = 16\pi^2 G_N^2 \frac{T}{l_s^3}\frac{1}{1-T/T_H}\,,$$

for the power spectrum of scalar perturbations, and

$$P_h \sim 16\pi^2 G_N^2 \frac{T}{l_s^3}(1-T/T_H)\ln^2\left[\frac{1}{l_s^2 k^2}(1-T/T_H)\right]\,,$$

for the power spectrum of tensor perturbations. In String Gas Cosmology, the key feature that leads to the scale invariance of the scalar fluctuation power spectrum is the "holographic" scaling of the heat capacity, which scales as $C_V \sim R^2$ instead of $C_V \sim R^3$ for thermal fluctuations of closed strings in a three-dimensional box. Given this scaling, one obtains a perfect cancelation of the powers of k, leading to a scale-invariant spectrum of scalar perturbations. Similarly, a pressure that scales as $p \sim 1/R$, which is characteristic of a "holographic" scaling of the free energy ($\mathcal{F} \sim R^2$), leads to a scale-invariant spectrum of tensor perturbations.

Let us now look at the results from the BFSS and IKKT models. In the BFSS model, we found that the heat capacity scales as

$$C_V = \frac{3N^2}{4}\chi_2 + \frac{3N^4}{8}\left(\frac{d-1}{12} - \frac{p}{8}\right)\left(\chi_2 - \frac{1}{d}\chi_2 - \frac{4}{N^2}\right)\chi_1\, T^{-3/2}\,, \tag{8.3}$$

where χ_2 is a constant and

$$\chi_1 := \left\langle \frac{1}{N}\,\mathrm{Tr}(A_i)^2 \right\rangle_{\mathrm{IKKT}} . \tag{8.4}$$

is the expectation value of the extent of space parameter related to the zero modes of the BFSS model, which are related to the IKKT model. Given that χ_1 scales as $\chi \sim R^2$, we obtained the scalar power spectrum

$$P(k) \;=\; 16\pi^2\,(kR)^{-4}\left(\frac{1}{l_s m_{pl}}\right)^4\left(\frac{3}{8}\right)\left(\frac{d-1}{12}-\frac{p}{8}\right)\left(\frac{(d-1)^2}{d}\left(1-\frac{1}{N^2}\right)-4\right), \tag{8.5}$$

which is scale-invariant. We also computed the pressure

$$\tilde{p} \;=\; -\frac{1}{V}\frac{\partial\mathcal{F}}{\partial \ln R}, \tag{8.6}$$

and found the free energy

$$\mathcal{F} \;=\; \frac{3N^2}{4\beta}\left[\chi_2\ln\beta \;-\; \frac{2}{3}\left(\frac{d-1}{12}-\frac{p}{8}\right)(\chi_5-\chi_6-4\chi_1)\,\beta^{3/2}\right].$$

In this case, we found a scale-invariant spectrum of tensor perturbations of the form

$$P_h(k) \;=\; \alpha\,16\pi^2\,(kR)^{-4}\left(\frac{1}{l_s m_{pl}}\right)^4\left(\frac{3}{8}\right)\left(\frac{d-1}{12}-\frac{p}{8}\right)\left(\frac{(d-1)^2}{d}\left(1-\frac{1}{N^2}\right)-4\right). \tag{8.7}$$

Similarly, in the IKKT model, we found a heat capacity of the form

$$C_V = -N^2(D-2) + 6\left(\frac{p}{8}-\frac{D-2}{12}\right)\frac{D-2}{D-1}M^2T^2\langle R^2\rangle_0\,. \tag{8.8}$$

In this case, the expectation value $\langle R^2\rangle_0$ of the extent of space parameter also scales as $\langle R^2\rangle_0 \sim R^2$. For this reason, we obtain a scale-invariant spectrum of scalar fluctuations of the form

$$P_\Phi(k) = 96\pi^2G^2(kR)^{-4}\left(\frac{p}{8}-\frac{D-2}{12}\right)\frac{D-2}{D-1}M^2T^4\,. \tag{8.9}$$

Using the next-to-leading order contribution

$$F_{next} = -T\ln\langle e^{-S_{int}}\rangle = \left(\frac{D-2}{12}-\frac{p}{8}\right)\frac{D-2}{D-1}M^2T^3\langle R^2\rangle_0\,, \tag{8.10}$$

for the free energy of the system, we then computed the spectrum of tensor perturbations and obtained

$$P_h(k) = 32\pi^2G^2(kR)^{-4}\alpha\left(\frac{D-2}{12}-\frac{p}{8}\right)\frac{D-2}{D-1}M^2T^4\,, \tag{8.11}$$

which is also scale invariant. Here, we dropped the leading contribution of the free energy because it does not depend on R, and hence does not contribute to the pressure and hence the power spectrum.

For both the BFSS and IKKT model a finite temperature, the features of the power spectrum of scalar and tensor perturbations align well with expectations from String Gas Cosmology. First, one obtains a scale-invariant spectrum of scalar and tensor perturbations in both the IKKT and BFSS models, which agrees with expectations from String Gas Cosmology. Second, the reason why the power spectrum is scale-invariant in the BFSS and the IKKT model seems to be the same as in String Gas Cosmology. In String Gas Cosmology, one obtains a "holographic" scaling for the heat capacity and the free energy of the system, meaning that we have $C_V, \mathcal{F} \sim R^2$ for closed strings in a three-dimensional box. This "holographic" scaling of the heat capacity and the free energy is also found in the BFSS and IKKT model, and leads to a scale-invariant spectrum for both scalar and tensor perturbations.

However, the IKKT and BFSS models may lead to slight differences in predictions for the tilt of the power spectrum, as reflected by the temperature dependence of the power spectrum which differs from the one computed in String Gas Cosmology. In String Gas Cosmology, one expects a slight red tilt for the power spectrum of scalar perturbations, as the power spectrum decreases as the temperature decreases. For the spectrum of tensor fluctuations, one expects a blue tilt since the power spectrum increases as the temperature decreases. This behavior has so far not been observed in the BFSS model since both the power spectrum of scalar perturbations and tensor perturbations do not depend on the temperature. However, it might be that corrections to the power spectrum at higher order in perturbation theory lead to the correct behavior. This topic remains to be investigated. In the IKKT model case, the power spectrum of scalar and tensor perturbations both depend on temperature, leading to a tilt. In both cases, the power spectrum decreases as a function of temperature, which leads to a red tilt. For the power spectrum of scalar perturbations, this red tilt is consistent with what one would expect in String Gas Cosmology. However, the observed red tilt in the power spectrum of tensor perturbations contrasts with the blue tilt expected from String Gas Cosmology. Perhaps higher-order corrections could change this result. However, this remains to be investigated.

8.2 Robustness of matrix theory as a description of the background in String Gas Cosmology

As mentioned in Chapter 2, an issue of String Gas Cosmology is that it currently lacks a complete non-perturbative description. In Chapter 5, we made progress towards such a non-perturbative description by defining a metric tensor in the IKKT model, in an attempt to describe the emergent cosmological solutions observed in simulations. To define such a metric, we considered smaller $n_i \times n_i$ submatrix of the submatrices showcased in Figure 2.9 as describing space in a comoving interval of size n_i. We then took

$$l_{i,\mathrm{phys}}(t, n_i) = \langle \mathrm{Tr}(\bar{A}_i^{n_i}(t))^2 \rangle \tag{8.12}$$

to be the physical length of this comoving space interval. These assumptions then allowed us to compute the g_{ii} component of the metric in the i'th direction via the expression

$$g_{ii}^{1/2}(t, n_i) = \frac{d}{dn_i} l_{i,\mathrm{phys}}(t, n_i)\,, \tag{8.13}$$

for which we obtained

$$g_{ii}^{1/2}(t, n_i) \;=\; \frac{1}{2} \frac{\left(\frac{d}{dn_i}\left\langle \mathrm{Tr}\left(\bar{A}_i{}^{n_i}\right)^2(t)\right\rangle\right)}{\left(\left\langle \mathrm{Tr}\left(\bar{A}_i{}^{n_i}\right)^2(t)\right\rangle\right)^{1/2}}\,. \tag{8.14}$$

For emergent cosmological solutions, we found

$$g_{ii}(t, n_i) = \mathcal{A}(t)\delta_{ii}\,, \tag{8.15}$$

for three large dimensions, indicative of a homogeneous and isotropic metric

$$g_{ij}(t, n_i) = \mathcal{A}(t)\delta_{ij} \tag{8.16}$$

for the emergent three large dimensions. In light of the present results, the present metric definition seems to properly describe the features of a universe emerging from a Hagedorn phase, as one would expect from String Gas Cosmology.

However, some progress remains to be made in order to put the present framework on a firmer basis. For one, the present metric proposal has only been formulated and tested using input from one background, namely the one described as a result of the present simulations. In order to put the present proposal on a firmer basis, one should hope that the present definition reproduces the expected features of other well-known backgrounds. This, however, remains to be tested.

Related to this issue, it would be helpful to use a background with a known Einstein gravity description in order to test the present metric description. Recently, there has been new work on a "polarised" Euclidean IKKT model [32–34], where the present IKKT model is deformed by a mass term that breaks the SO(10) symmetry of the system but preserves supersymmetry. This system has been proposed as a dual for a background describing a Euclidean D_1-brane in a finite, Euclidean, ellipsoidal cavity in Type IIB superstring theory. Perhaps this background could be used in some way to test the present metric definition.

To end the present section, it would also be good to formulate a framework in which Einstein's equations, or another description of gravity, naturally emerge with the present metric definition. So far, such a framework has yet to be formulated. However, there has been work on the IKKT model as a non-commutative description of gravity that may shed light on this question. In non-commutative gravity (see [35] for a review), Einstein's equations naturally emerge as a result of a mechanism called induced gravity [36]. Perhaps this mechanism may be used to put the present metric definition on a firmer basis.

8.3 Comments on the dynamical mechanism leading to a mass term in the IKKT model

In Chapter 6, we explored a dynamical mechanism by which the IKKT model acquires a mass term, potentially leading to emergent cosmological solutions. To obtain this mass term, we compactified the IKKT model on a six-dimensional torus where the fermions acquire anti-periodic boundary conditions. We then computed the effective action related to the zero modes of the compact IKKT model, which are related to the degrees of freedom of the non-compact theory. After proceeding with the computations of the zero modes effective action, by integrating out the non-zero modes of the theory, we found

$$S^0_{eff} = -\frac{1}{4g^2_{eff}}\mathrm{Tr}[A^M(0), A^N(0)]^2 + \frac{1}{2}M^2_{MN}\mathrm{Tr}(A^M(0)A^N(0))^2 + \ldots\,, \tag{8.17}$$

where

$$M^2_{MN} = \begin{bmatrix} \eta_{\mu\nu}M_4^2 & 0 \\ 0 & \eta_{ab}M_6^2 \end{bmatrix}\,, \tag{8.18}$$

is a mass matrix that results from integrating out the "compact" degree of freedom in the theory. Here, the elements of this mass matrix are given by

$$M_4^2 = 16\,(S_{F_1} - S_{B_1})\,\frac{NM}{L^2}\,, \tag{8.19}$$

$$M_6^2 = \frac{32}{3}\,(S_{F_1} - S_{B_1})\,\frac{NM}{L^2}\,. \tag{8.20}$$

This symmetry-breaking term and the potential cosmological solutions that could emerge by considering this term may have much in common with String Gas Cosmology. First, notice that the mass term breaks the symmetry of the theory from $SO(1,9)$ to $SO(1,3)\times SO(6)$, which is a result of the fact that we have chosen six spatial dimensions to be compactified. Given this symmetry of the system, one may expect that there exist solutions where three dimensions become large (the i directions) and six other directions stay small (the a directions). This could be the case, for example, if we showed to study solutions where the μ directions and a directions do not commute with each other $[A^\mu, A^a] = 0$. In this case, the equations of motions for A^i and A^a decouple, and one may recover solutions in the form of Equations 2.68 for A^μ.

However, there are key differences may pose a challenge in reconciliating the present mechanism with String Gas Cosmology. The most important one is that, in order to neglect subleading parts in Equation 8.17, we must work in the regime where six compact dimensions are large, and the others remain not compactified. These conditions are quite different from the initial conditions in String Gas Cosmology, where all dimensions are taken to be compactified and small. It is possible that the present conditions are related to the transition phase where three dimensions become large. However, a more involved analysis would be needed to establish this link. Related to this issue, in String Gas Cosmology, symmetry breaking occurs spontaneously without imposing any condition. However, here, we have imposed a condition, namely the compactification of six dimensions, in order to obtain the symmetry breaking. A more involved analysis would be required to find if the present conditions can be realized spontaneously, and hence realize a spontaneous symmetry breaking in the IKKT model.

Despite the challenges in linking the present mechanism to String Gas Cosmology, the present mechanism may have ties to current progress in non-supersymmetric string theory. For example, it was shown in [31] that compactifying one dimension on a torus where fermions acquire anti-periodic boundary conditions (Rohm circle [37]) can lead to interesting cosmological solutions in Type IIA or Type IIB string theory. It would be interesting to investigate the connection between these backgrounds and the cosmological solutions related to the present effective action.

8.4 Comparison of symmetry breaking in matrix cosmology and String Gas Cosmology

In Chapter 7, we investigated a possible symmetry breaking of the Euclidean BFSS model using the Gaussian Expansion Method. Before drawing a comparison with String Gas Cosmology, let us first review the results.

To begin, in the bosonic version of the Euclidean BFSS model, we considered adding and subtracting the following term

$$S_0^{(b)} = -\frac{N}{\Lambda}\mathrm{Tr}\left(U + U^\dagger\right) + \sum_l \sum_{i=1}^{D} \frac{1}{2v_{l,i}}\mathrm{Tr}\left(X_l^i X_{-l}^i\right) - \sum_{l\neq 0}\frac{1}{s_l}\mathrm{Tr}\left(\bar{\alpha}_l \alpha_l\right) , \tag{8.21}$$

in order to perform the Gaussian Expansion Method. By minimizing the energy of the system, we then found that the gap equation must be satisfied

$$-\frac{1}{v_{l,i}} + \frac{1}{g^2}\left(\frac{2\pi l}{\beta}\right)^2 + \frac{2N}{g^2\beta}\rho_0^2(\Lambda) + \frac{2N}{g^2\beta}\sum_{\jmath\neq i}\sum_l v_{l,j} = 0\,. \tag{8.22}$$

Moreover, we saw that the gap equation has the solution $v_{l,1} = v_{2,1} = ...v_{l,D} = v_l$, where v_l is given by

$$v_l = \frac{g^2}{\left(\frac{2\pi l}{\beta}\right)^2 + m_{\text{eff}}^2}\,. \tag{8.23}$$

Given that this solution preserves the $SO(D)$ symmetry of the system, we concluded that there is non-symmetry breaking in the bosonic BFSS model at first order.

We then added fermions in order to consider the supersymmetric BFSS model, and performed the Gaussian Expansion Method by also adding the following Gaussian term

$$S_0^{(f)} = \sum_{\alpha,\beta}^{p} \sum_{r} \mathrm{Tr}\left(\psi_r^\alpha \mathcal{A}_{\alpha\beta} \psi_{-r}^\beta\right) , \tag{8.24}$$

for the fermions. Minimizing the free energy, we then found the gap equation

$$-\frac{1}{v_{l,i}} + \frac{1}{g^2}\left(\frac{2\pi l}{\beta}\right)^2 + \frac{2N}{g^2\beta}\,\rho_0^2(\Lambda) + \frac{2N}{g^2\beta}\sum_{j\neq i}\sum_{l} v_{l,j} - \frac{N}{g^2\beta}\,\mathrm{Tr}\left[\left(\mathcal{A}^{-1}\,\Gamma_i\right)^2\right] = 0\,. \tag{8.25}$$

Just like for the IKKT model, we found that fermions contribute to a term that breaks $SO(D)$ invariance in the gap equations. For this reason, we expect the $SO(D)$ symmetry of the BFSS model to be broken by fermionic contributions.

Part of the work that remains to be done in the BFSS model is to determine which symmetry-breaking pattern is preferred in the theory. In order to perform this task, one would need to choose values of $v_{l,i}$ and $\mathcal{A}$ that preserve certain symmetries, and compute the free energy of the system for these symmetries. Admitting $SO(3) \times SO(6)$ is the symmetry with the lowest free energy, one would then find evidence that $SO(3)$ is the preferred symmetry of the system. This would support the hypothesis that a symmetry-breaking process where $SO(9)$ breaks to $SO(3) \times SO(6)$ can occur in the theory.

Despite the fact that the preferred symmetry of the system remains to be found in the BFSS model, the evidence for symmetry breaking in the model (including the fermions), seems to align well with expectations from String Gas Cosmology. In String Gas Cosmology, the universe begins described with $SO(9)$ symmetry. This symmetry is then spontaneously broken to $SO(3) \times SO(6)$ when winding modes decay, and three space dimensions become large. In the present case, we have a model, namely the BFSS model, which has intrinsic $SO(9)$ symmetry. We then found evidence that symmetry breaking can occur in the theory using the Gaussian Expansion Method. Assuming we can show that $SO(3) \times SO(6)$ is the preferred symmetry of the system, this would show a strong tie with String Gas Cosmology, and a potential realization of its dynamical symmetry-breaking mechanism.

The role of the fermions in the symmetry-breaking process is, moreover, interesting. In String Gas Cosmology, we saw that supersymmetry, and therefore the presence of fermions, is required in order for the symmetry-breaking process that leads to three large dimensions takes place. In the BFSS case, fermions also need to be required in order for the symmetry-breaking process to occur. This is furthermore evidence that the symmetry-breaking process in String Gas Cosmology and in the BFSS model may have something in common.

To find further similarities between symmetry breaking in the BFSS model and symmetry breaking in String Gas Cosmology, it would be interesting to investigate if a mechanism such as the decay of winding modes can be linked to symmetry breaking in the BFSS model. Recently, it has been shown that interactions between long winding strings in the IKKT models vanish in a background that which breaks the SO(9) symmetry of space in the IKKT model to SO(3) $\times$ SO(6) [38]. This shows evidence that winding modes of strings can decay in the IKKT matrix model, leading to a potentially emerging three-dimensional space just like in String Gas Cosmology. Assuming this effect can be shown in the BFSS model, it would be further evidence for symmetry breaking in this model, and the potential emergence of three dimensions.

Finally, to help with the present efforts in linking the features of symmetry breaking in the BFSS model to String Gas Cosmology, it would be interesting to use new tools, in a similar way as the Gaussian Expansion Method, to probe symmetry breaking. Lately, there has been interest in using bootstrap methods to find the allowed energy states in the BFSS model [39, 40], and other matrix theories [41]. The idea behind this method is to use the statement of positivity, along with several other constraints, in order to constrain possible values for the correlators in the system. These constraints can then be used to rule out values of the energy as a function of various correlators in the system. Given this fact, it would be interesting to probe the allowed values of quantities such as the expectation value of the moment of inertia tensor

$$\langle T_{ij} \rangle = \frac{1}{N} \langle \mathrm{Tr} X_i X_j \rangle \tag{8.26}$$

in the context of the BFSS model. As we saw in Section 2.3.2, the eigenvalues of the moment of inertia tensor are related to the characteristic size of space in different directions. For this reason, probing allowed values of $\langle T_{ij} \rangle$ could help constrain symmetries at different energies, and help shed light on a possible dynamical symmetry-breaking process in the BFSS model.

Chapter 9

Conclusion

In this thesis, we studied matrix theory as a possible description of String Gas Cosmology. First, we investigated if the BFSS and IKKT model can give rise to a scale-invariant spectrum of cosmological perturbations on large scales. In both cases, we find good evidence that a thermal state of the BFSS and IKKT model can source a scale-invariant spectrum on these distance scales, which agrees with expectations from String Gas Cosmology. Second, we provided a possible definition for a space-time metric in the IKKT model, and found that it reproduces some expected features such as isotropicity assuming some features from numerical simulations. Given its ties to simulations describing an emergent universe, this metric could be used in a non-perturbative description of the background in String Gas Cosmology. Third, we investigated a mechanism in which the IKKT model acquires a mass term as a result of a compactification of six spacial dimensions. Since this mechanism requires the compact dimensions to be large, it is unclear if it can be tied to cosmological solutions in String Gas Cosmology where all dimensions start small and then three expand. However, perhaps a connection to solutions in non-supersymmetric string cosmology exists. Fourth, we investigated symmetry breaking in the BFSS matrix model using the Gaussian expansion method. In agreement with String Gas Cosmology, we found evidence for symmetry breaking at high temperatures. However, work still needs to be done to understand if a $(3 + 6)$-dimensional space is the preferred symmetry of this system.

All-in-all, we have found good evidence that matrix theory can provide a description of String Gas Cosmology. The most conclusive piece of evidence comes from the emergence of a scale-invariant spectrum from thermal excitations of matrices. This result, which is realized in the same way as for string gas cosmology from a "holographic" scaling of the heat capacity and free energy, is strong evidence in support of a realization of String Gas Cosmology in matrix models. The second most promising piece of evidence is the realization of symmetry breaking in the BFSS matrix model. Although the preferred symmetry of the system still needs to be found, the evidence for symmetry breaking may point to an emergent lower dimensional universe in the theory, in agreement with String Gas Cosmology.

Despite this evidence, there are some important aspects that need to be improved in order to establish matrix theory as a non-perturbative framework for String Gas Cosmology. One of our main motivations for studying matrix theory was the promise of a fully non-perturbative description of the dynamics of the early universe. Despite some evidence from numerical simulations that this framework may exist, a full analytic understanding of how a three-dimensional universe emerges in matrix models remains lacking. A step we have made in this direction is the definition of a space-time metric in the IKKT model. However, this metric definition still needs to be thoroughly tested, and a complete framework for the gravitational equations of motion skill needs to be established. Ongoing work on matrix theory and their gravity duals in string theory may bring an answer to these issues. Associated to this background, work still remains to be done to understand why three large dimensions are emergent in matrix models. New tools, such as the matrix bootstrap, may be helpful to probe this symmetry breaking.

We will end by highlighting that the quest to understand how three dimensions become large in matrix theory is the subject of ongoing work at McGill. In addition to the present work, it has been shown that interactions between long winding strings in the IKKT models vanish in a background which breaks the SO(9) symmetry of space in the IKKT model to SO(3) $\times$ SO(6) [38]. This shows evidence that winding modes of strings can decay in the IKKT matrix model, leading to a potentially emerging three-dimensional space just like in String Gas Cosmology. In light of these results and ongoing work on the topic, one should expect the status of matrix theory with respect to String Gas Cosmology to become clearer in the upcoming years.

Bibliography

[1] S. Laliberte and S. Brahma, "IKKT thermodynamics and early universe cosmology," *JHEP*, vol. 11, p. 161, 2023.

[2] S. Laliberte, "Effective mass and symmetry breaking in the Ishibashi-Kawai-Kitazawa-Tsuchiya matrix model from compactification," *Phys. Rev. D*, vol. 110, no. 2, p. 026024, 2024.

[3] S. Brahma, R. Brandenberger, and S. Laliberte, "Emergent cosmology from matrix theory," *JHEP*, vol. 03, p. 067, 2022.

[4] S. Brahma, R. Brandenberger, and S. Laliberte, "Emergent metric space-time from matrix theory," *JHEP*, vol. 09, p. 031, 2022.

[5] S. Brahma, R. Brandenberger, and S. Laliberte, "Spontaneous symmetry breaking in the BFSS model: analytical results using the Gaussian expansion method," *Eur. Phys. J. C*, vol. 83, no. 10, p. 904, 2023.

[6] W. H. Kinney, "TASI Lectures on Inflation," 2 2009.

[7] D. Baumann, "Inflation," in *Theoretical Advanced Study Institute in Elementary Particle Physics: Physics of the Large and the Small*, pp. 523–686, 2011.

[8] D. Baumann, "Primordial Cosmology," *PoS*, vol. TASI2017, p. 009, 2018.

[9] R. H. Brandenberger, "Inflationary cosmology: Progress and problems," in *IPM School on Cosmology 1999: Large Scale Structure Formation Tehran, Iran, January 23-February 4, 1999*, 1999.

[10] R. H. Brandenberger and C. Vafa, "Superstrings in the Early Universe," *Nucl. Phys. B*, vol. 316, pp. 391–410, 1989.

[11] R. H. Brandenberger, "String Gas Cosmology," 8 2008.

[12] R. H. Brandenberger, A. Nayeri, S. P. Patil, and C. Vafa, "String gas cosmology and structure formation," *Int. J. Mod. Phys. A*, vol. 22, pp. 3621–3642, 2007.

[13] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, "A Large N reduced model as superstring," *Nucl. Phys. B*, vol. 498, pp. 467–491, 1997.

[14] K. L. Zarembo and Y. M. Makeenko, "An introduction to matrix superstring models," *Phys. Usp.*, vol. 41, pp. 1–23, 1998.

[15] S.-W. Kim, J. Nishimura, and A. Tsuchiya, "Expanding (3+1)-dimensional universe from a Lorentzian matrix model for superstring theory in (9+1)-dimensions," *Phys. Rev. Lett.*, vol. 108, p. 011601, 2012.

[16] B. Ydri, "Review of M(atrix)-Theory, Type IIB Matrix Model and Matrix String Theory," 8 2017.

[17] S.-W. Kim, J. Nishimura, and A. Tsuchiya, "Late time behaviors of the expanding universe in the IIB matrix model," *JHEP*, vol. 10, p. 147, 2012.

[18] D. N. Kabat and G. Lifschytz, "Approximations for strongly coupled supersymmetric quantum mechanics," *Nucl. Phys. B*, vol. 571, pp. 419–456, 2000.

[19] J. Nishimura and F. Sugino, "Dynamical generation of four-dimensional space-time in the IIB matrix model," *JHEP*, vol. 05, p. 001, 2002.

[20] R. Adam *et al.*, "Planck 2015 results. I. Overview of products and scientific results," *Astron. Astrophys.*, vol. 594, p. A1, 2016.

[21] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge Monographs on Mathematical Physics, Cambridge University Press, 12 2007.

[22] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, "M theory as a matrix model: A conjecture," *Phys. Rev. D*, vol. 55, pp. 5112–5128, 1997.

[23] N. Aghanim *et al.*, "Planck 2018 results. I. Overview and the cosmological legacy of Planck," *Astron. Astrophys.*, vol. 641, p. A1, 2020.

[24] N. Aghanim *et al.*, "Planck 2018 results. VI. Cosmological parameters," *Astron. Astrophys.*, vol. 641, p. A6, 2020. [Erratum: Astron.Astrophys. 652, C4 (2021)].

[25] R. Hagedorn, "Statistical thermodynamics of strong interactions at high-energies," *Nuovo Cim. Suppl.*, vol. 3, pp. 147–186, 1965.

[26] Y. Kitazawa and H. Takata, "D-brane scattering in IIB string theory and IIB matrix model," *Nucl. Phys. B*, vol. 551, pp. 617–649, 1999.

[27] A. Fayyazuddin, Y. Makeenko, P. Olesen, D. J. Smith, and K. Zarembo, "Towards a nonperturbative formulation of IIB superstrings by matrix models," *Nucl. Phys. B*, vol. 499, pp. 159–182, 1997.

[28] K. Becker and M. Becker, "A Two loop test of M(atrix) theory," *Nucl. Phys. B*, vol. 506, pp. 48–60, 1997.

[29] Y. Okawa and T. Yoneya, "Multibody interactions of D particles in supergravity and matrix theory," *Nucl. Phys. B*, vol. 538, pp. 67–99, 1999.

[30] M. Hirasawa, K. N. Anagnostopoulos, T. Azuma, K. Hatakeyama, J. Nishimura, S. Papadoudis, and A. Tsuchiya, "The emergence of expanding space-time in the Lorentzian type IIB matrix model with a novel regularization," *PoS*, vol. CORFU2022, p. 309, 2023.

[31] E. J. Martinec, D. Robbins, and S. Sethi, "Non-Supersymmetric String Theory," *JHEP*, vol. 10, p. 078, 2011.

[32] S. A. Hartnoll and J. Liu, "The Polarised IKKT Matrix Model," 9 2024.

[33] S. Komatsu, A. Martina, J. a. Penedones, A. Vuignier, and X. Zhao, "Einstein gravity from a matrix integral – Part I," 10 2024.

[34] S. Komatsu, A. Martina, J. Penedones, A. Vuignier, and X. Zhao, "Einstein gravity from a matrix integral – Part II," 11 2024.

[35] H. Steinacker, "Emergent Geometry and Gravity from Matrix Models: an Introduction," *Class. Quant. Grav.*, vol. 27, p. 133001, 2010.

[36] A. D. Sakharov, "Vacuum quantum fluctuations in curved space and the theory of gravitation," *Dokl. Akad. Nauk Ser. Fiz.*, vol. 177, pp. 70–71, 1967.

[37] R. Rohm, "Spontaneous Supersymmetry Breaking in Supersymmetric String Theories," *Nucl. Phys. B*, vol. 237, pp. 553–572, 1984.

[38] R. Brandenberger and J. Pasiecznik, "On the Origin of the $SO(9) \to SO(3) \times SO(6)$ Symmetry Breaking in the IKKT Matrix Model," 8 2024.

[39] H. W. Lin, "Bootstrap bounds on D0-brane quantum mechanics," *JHEP*, vol. 06, p. 038, 2023.

[40] H. W. Lin and Z. Zheng, "Bootstrapping ground state correlators in matrix theory. Part I," *JHEP*, vol. 01, p. 190, 2025.

[41] X. Han, S. A. Hartnoll, and J. Kruthoff, "Bootstrapping Matrix Quantum Mechanics," *Phys. Rev. Lett.*, vol. 125, no. 4, p. 041601, 2020.