Using multiple representation systems to deepen understanding of functional relationships in mathematics.

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Table of Contents

List of Tables

List of Figures

Abstract

This thesis describes an experiment using technology to develop conceptual understanding of functions through graphical representations. It examines the effects of including dynamic representations in a conceptual approach to the teaching of functions. The study was implemented over a 5-day period in a Grade 9 class in a small, generally working class, rural school in Eastern Massachusetts. Participating students were observed during class discussions and video analysis, and their written responses and created functions were analyzed. The procedure used in the experiment was based on the Theory of Didactic Situations and used the Didactic Engineering methodology. The structure and sequencing of the thesis is also based on these concepts. Conclusions are drawn regarding the effects of using multiple representations systems to deepen understanding of functional relationships and suggested improvements to the introduction of the function concept in high school instructional programs are given.

Résume

Cette thèse décrit une expérimentation qui utilise la technologie pour l'apprentissage du concept de fonction par l'intermédiaire des représentations graphiques. Elle examine les effets de l'intégration des représentations dynamiques dans une approche conceptuelle de l'enseignement des fonctions. Cette étude a été mise en œuvre sur une période de cinq jours dans une classe de Grade 9 dans une petite classe ordinaire d'une école de campagne à Massachusetts Est. Les étudiants participants ont été observés pendant des discussions de classe et des analyses de vidéo, et leurs réponses écrites et les fonctions créées ont été analysées. Le procédé utilisé dans l'expérimentation a été basé

sur la Théorie des Situations Didactiques et la méthodologie employée a été celle de l'ingénierie didactique. La structure et l'organisation de la thèse sont également basées sur ces concepts. Des conclusions sont tirées concernant les effets de l'utilisation des systèmes de représentations multiples pour approfondir la notion de relation fonctionnelle et des améliorations suggérées pour l'introduction du concept de fonction dans les programmes de l'enseignement secondaire sont données.

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Introduction

At the core of my research is the teaching and learning of functional relationships in mathematics with an emphasis on understanding. Functions are fundamental tools for modeling scientific and social phenomena. Debates over such international issues as those of global warming, population control, radioactive waste, inflation rates, and the national debt, often revolve around understanding the mathematical behavior of functions, especially how one quantity changes in relation to another, or how one quantity changes over time. The importance of this for an educated citizenry is indisputable. Furthermore, it will be critical for students of the 21st century to solve real-world problems by creating and testing models of situations involving physical and simulation data and analyzing the data by utilizing different forms of representation. There is a need to democratize access to the mathematical concepts that are required to understand, create, and test such models.

In my past research (Balyta, 1999), I showed that a conceptual understanding of some aspects of functions can indeed be developed in students at an earlier stage (grade 6) than is common in mathematics curricula. Using technologies, in particular, the motion detector technology proved to be beneficial for the development of the concept of piecewise-defined functions by creating dynamic graphing situations that allowed the students to see and to control physically relationships between dependent and independent variables. The goal of this research was to suggest that middle school instructional programs should include an emphasis on functions so that all students understand the mathematical behavior of functional relationships. Among my findings was that the use of motion detector technology created dynamic graphing situations allowing students to see relationships between dependent and independent variables as they occur in real time. I also reported that such powerful dynamic teaching and learning tools allowing students to physically control these relationships deepened their understanding of functions and their graphical representations as a foundation for deeper understanding in middle school.

This thesis will investigate whether didactic situations can be engineered in these new learning environments that improve student learning of traditional topics, render new topics more accessible, and increase active participation. Thus, the goal of my research is to determine how best to organize the didactic milieu to facilitate optimal interactions between the students and the milieu in order for them to develop a conceptual understanding of functional relationships. **Specifically, I will focus on motion and its associated functional relationships between position and time and the multiple forms of representation within two carefully selected representation systems, namely, graphical representations and simulated motion.**

Representation can also be viewed as the process of transforming the contents of consciences into a public form so that they can be stabilized, inspected, edited, and shared with others (Eisner, 1993). Also, since the forms of representation differ, the way understanding is acquired may also be different. As such, different forms of representation allow us to construct meanings that might otherwise elude us (ibid.) and there are relative strengths and weakness associated with each form of representation (NCTM, 2000). This is especially true when the representations which can be viewed at the same time are dynamically linked. For example, a change in one representation simultaneously results in a change in another and vice versa. The effective and appropriate use of multiple representations fosters the students' ability to acquire knowledge. Graphical representations are often effective in providing a clear picture of a function and are beneficial in helping students visualize certain phenomena. For example, graphical representations are often helpful in visualizing a complex relationship between variables. Also, by using graphs, students can explore aspects of a context that would otherwise not be apparent (Monk, 1994). Students often develop representations during problem solving inferring mental or internal representations from graphs that they have created and modified as they represented and interpreted problem situations (Goldin & Steingold, 2001).

The experiment presented in this thesis explores the effects of dynamic representation by incorporating additional technologies in the didactic milieu, allowing for richer representation systems and social interaction among learners, using carefully designed didactic situations that allow synergy to occur among networked handheld devices, motion detector technology, and the classroom computer. The intention is to open a rich opportunity space for learning about functions to allow exploring the active physical, linguistic, and social participation of students employing simulations and the effect of multiple representation systems on student understanding of functional relationships. Classroom connectivity enables students to share mathematical functions across diverse hardware platforms, and teachers to collect and aggregate these functions into a common classroom display of their aggregated functions. Specifically, I explore how to take advantage of the students' personal connection with their individual constructions in the aggregated and publicly displayed set of student constructions. I propose that this develops important coordination skills that would deepen students' understanding of the functional relationships involved in motion.

The thesis investigates the connection between formal mathematics and functional relationships that are made possible through the effective and appropriate use of selected technologies and their multiple representations.

Research Hypotheses

Motion is an important "live" context for functional relationships between position and time. Historically, it has been the basis for a considerable amount of mathematics. It is directly experienced, has immediate kinesthetic, cognitive and linguistic aspects that can be tapped into, and fits into classrooms. I am interested in learning about how motion in carefully constructed didactic situations in a milieu leveraging multiple representation systems enhances students' conceptual understanding of functions. Therefore, for my experiment, the milieu will include devices allowing the students to directly experience motion. They will interact with such a milieu and get immediate feedback from the milieu that will contribute to the building of understanding as explained in Section 2.1. It is my broad hypothesis that the use of multiple representation systems in a didactic milieu that allows for individual and aggregated mathematical constructions challenges students to coordinate multiple representations. Furthermore the representational strategies involved in coordinating multiple representations of the same functions (physical or simulation) will enhance the depth of learning about functional relationships. My experiment will allow me to investigate the following research hypotheses:

Hypothesis 1 Individual mathematical constructions that are directly experienced in a "live" context have immediate kinesthetic, cognitive and linguistic aspects that will help students develop an understanding of the relationship between distance and time in problems of motion.

Hypothesis 2 Individual mathematical constructions in a "live" context facilitate the development of understanding of independent and dependent variables.

Hypothesis 3 Multiple linked representations of the same function in a simulated environment allowing for manipulation by the students improve their learning about rate of change.

Hypothesis 4 Aggregated mathematical constructions challenge students to coordinate multiple representations and deepen their understanding of functional relationships.

As a result of my work, I hope to suggest new approaches to help democratize the learning of fundamental mathematical concepts required for students of the 21st century and, in doing so, hopefully spark the mathematical imagination of all students.

My research is directed by didactic engineering (Artigue, 1992), a research methodology which has been used exclusively for research in mathematics education, mostly in France, and unfamiliar outside France. Didactic engineering is a method of designing and evaluating the effect of instruction by a carefully structured series of steps: a preliminary analysis of the epistemological, cognitive, and didactic dimensions of the concept at stake, design and *a priori* analysis, experimentation, and *a posteriori* analysis and validation. This methodology is based on the Theory of Didactic Situations (Brousseau, 1997), and the whole dissertation is structured in the same way that an experiment based on didactic engineering is structured. Thus, after Chapter 1 has described and justified the Theory of Didactic Situations and didactic engineering, the next three chapters are structured in the same way as a didactic engineering methodology. Chapter 2 contains the preliminary analysis of the concept of function. The design of the didactic situations employed in this study required a clear definition of the meaning of the specific notions of function (the epistemological dimension), described in Chapter 2, and helped by previous research studies (Sfard, 1991; Sierpinska, 1992). Chapter 2 also describes the cognitive and didactic dimensions of functions. Chapter 3 presents the design of the didactic situations employed and their *a priori* analyses required by didactic engineering, and Chapter 4 will describe the results of experimentation with the *a posteriori* analysis and validation of the research hypotheses. This will be elaborated at the end of Chapter 1. Finally, Chapter 5 will present the conclusion and summary, including an assessment of the limitations of this research, and suggestions for further research.

Chapter 1 Theoretical framework and methodology

Since the Theory of Didactic Situations (Brousseau, 1997) constitutes the theoretical framework for this study, forms the basis of its methodology and also the structure of the thesis, this chapter will describe this theory in detail, then explore some research methodologies used in mathematics education, before justifying didactic engineering as the chosen methodology.

1.1 Theory of Didactic Situations (TDS)

The research makes use of tools for analyzing and designing situations aimed at realizing the goals of schools, where the actions of learning and teaching cannot be analyzed independently from each other. The students will be actively involved in constructing their knowledge through a carefully designed sequence of didactic situations that will be built on students' earlier knowledge and experiences. The Theory of Didactic Situations (Brousseau, 1986, 1997), which is based on the constructivist approach, will constitute the main theoretical framework of the research. I have also considered various issues related to the appropriate use of technology in mathematics classrooms, since the technology will play an important part of the didactic milieu.

At the heart of the TDS are interactions between the teacher, the student(s), and targeted knowledge in a didactic milieu, as outlined in Figure 1 below.

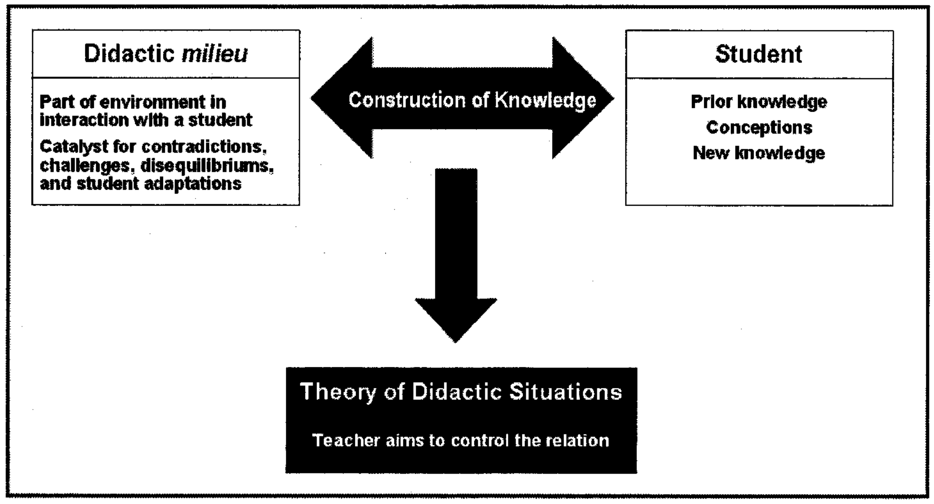

Figure 1. Interaction between a student and a milieu in the Theory of Didactic Situations

According to the TDS, a didactic milieu is a part of the environment with which the student interacts (Brousseau, 1986). It constrains the student's activity and, thus the evolution of his knowledge. In order to play its role properly, a milieu is subject to two constraints (Salin, 2002):

- The milieu acts as a catalyst for contradictions, challenges, and disequilibriums, and thus students' adaptations.
- The milieu has to allow the student to function autonomously.

Thus, a didactic milieu is the natural milieu of students in much the same way as a cliff is the natural milieu for a rock climber. In both cases, one must understand certain ground rules and develop strategies that will eventually result in a successful strategy, whether to solve a problem or climb a cliff. In this sense, knowledge is the means of understanding the ground rules and strategies associated with a milieu and the means of elaborating winning solutions. The teacher's role is critical in organizing the didactic milieu in such a way that the targeted knowledge becomes necessary for the student to 'survive' in it (to win or to solve the problem). The TDS assumes that learning in a school situation is largely an adaptation to the milieu (Sierpinska, 1999). As Sierpinska

(1999) explains, if the situations in a mathematics class are such that a certain type of social behavior is sufficient for survival in the class, without any use of mathematical knowledge, then it is the social behavior, not the mathematical knowledge that the students will learn. According to the TDS, knowledge is understood as the outcome of the interactions between the student and a specific milieu organized by the teacher in the framework of a didactic situation.

The teacher organizes a milieu by setting up values of a certain parameters, called *didactic variables*. A didactic variable is a parameter of a didactic situation that can be assigned several values. The modification of these values gives rise to changes of the students' strategies. The student needs to change his/her strategy as it became too long, too costly, too complex or erroneous.

Thus, the didactic variables on which a didactic situation appears to depend can be controlled by the teachers as elements of the didactic system. For example, the teacher will need to make choices about arranging the milieu such as choosing the type of task, the resources, and the tools put at the disposal of the students. The teacher will also need to make choices regarding his role in the milieu. Such variables help define the actual didactic situation.

In the TDS, teaching is the devolution to the student of an *adidactical* situation and learning is the student's adaptation to this situation (Brousseau, 1997). An *adidactical* situation is the setting up of a problem (that is, the organizing of the milieu) so that the student does not need the intervention by a teacher to construct knowledge (Artigue, 2005). The rules and strategies associated with the interactions between the teacher, student and the milieu which are needed to solve a problem and specific to the knowledge taught are referred to as a didactic contract (Brousseau, 1997). The TDS presumes the existence of a didactic contract that is inherently social in nature (Schoenfeld, 2002). The didactic contract can be explained as the idea that teacher and students enter a classroom with implicit understandings regarding the norms for their interactions and that these understandings shape the ways they act. The TDS is used both to identify what mathematical knowledge is being constructed by the teacher and the students in an actual lesson and also to engineer situations aimed at the construction of certain piece of knowledge by the students (Sierpinska, 1999).

Sierpinska (1999) has described different types of didactic situations, defined by Brousseau in his theory of didactic situations, in terms of the role of the teacher. In *situations of action*, the teacher organizes a milieu for students to engage with but then completely withdraws from the scene. Knowledge in this situation appears as a means for solving a problem or a class of problems and the knowledge is personalized and contextualized. In *situations of formulation*, the students exchange and compare observations between themselves while the teacher focuses on managing communication among the students. Knowledge, in this situation, appears as a result of a personal experience which needs to be communicated, and thus slightly de-personalized and de-contextualized, in order to be understood by others. In *situations of validation*, the teacher acts as a chair of a debate, only intervening to put some order in the debate among students. He helps draw attention to possible inconsistencies in student explanations and encourages more precision in the use of concepts. Knowledge has the dynamic features of a theory in the making, not of a finished, institutionalized theory. In *situations of institutionalization*, the teacher is the representative of the curriculum and the students receive the instruction with explicit instructions and rules. Knowledge in this situation is considered to be the understanding of the instructions and rules given by the teacher. It is by means of the institutionalization that the knowledge becomes completely de-personalized and de-contextualized.

1.2 Methodology

The research methodology that guided my research was that of didactic engineering which is based on the TDS. The didactic engineering is qualitative in nature and, as explained by Laborde (1989), has the following as a goal:

> to apprehend teaching situations globally in order to develop a model which encompasses their epistemological, social, and cognitive dimensions and which attempts to take into account the complexity of the interactions between knowledge, pupils and teacher within the context of a particular class, or more generally of an educational group.
>
> (p. 32)

Researchers are challenged with the important decision regarding which kinds of methods are appropriate in which circumstances, a challenge only exacerbated by the variety of methods currently available. Moreover, mathematics education research is a young discipline, having developed extensively in the last third of this past century (Lester & Lambdin, 2003). This serves, in large measure, to explain the diversity of perspectives and methods seen today (Schoenfeld, 2002). The Handbook of Qualitative Research in Education (LeCompte & al., 1992) and the Handbook of Research Design in

Mathematics and Science Education (Kelly & Lesh, 2000) are 881 and 993 pages long, respectively. Unfortunately the phenomenal growth of research methodologies over the past few decades has been largely chaotic, making it critical to analyze corresponding foundational assumptions and methods of investigation (English, 2002). The development of widely recognized standards for research has not kept pace with the development of new problems, new perspectives, and new research problems (Lesh, 2002).

Assuming one wants to conceive of and try a new way of teaching a piece of mathematical knowledge, as Sierpinska (1999) suggests, there are essentially two ways of going about that. One can use quantitative or qualitative research methodologies. I will start with a brief discussion of quantitative methodology and my reasons for rejecting it for this research. Next, I will situate didactic engineering in its qualitative research paradigm. The role of underlying assumptions in the conduct of this research and the implications of (implicit or explicit) choices of theoretical frameworks will also be outlined.

1.2.1 Quantitative methodologies

I have chosen to include this brief discussion on quantitative methodologies as part of the rationale to support my choice of a qualitative research methodology for my research. These are the comparative studies of experimental and control groups that dominated educational research until a few decades ago and which remain important to understand (Shoenfeld, 2002).

The objective of comparative studies in classic experimental research is to determine if desired outcomes are caused by specific actions (Campbell & Stanley, 1990). These comparisons are usually made between two groups: an experimental group treated by a set of actions and a control group receiving no such treatment (Romberg, 1992). Comparative studies usually begin with an outline of a lesson containing classroom activities, with a precise description of the role and actions for the teacher and the expected responses of the students. The lesson contains advice for the teacher in case the students make errors and mistakes of various types. The planning decisions pertaining to the choice of the mathematical activities and problems could be justified by curriculum prescriptions, some theory of learning, some principles of teaching, knowledge of mathematics and personal experience. However, the evaluation of the lesson will not be done on the basis of this justification. In fact, as Sierpinska (1999) reports, this justification may not even be written down or otherwise made explicit in the final report of the project. The project will be evaluated by testing the lesson on a group of students, with a control group for comparison. The control group will be taught the same mathematical content with traditional methods, and both groups will be administered identical pre-tests and post-tests. In the case of similar results on the pre-test, if the experimental group performs better on the post-test, then the teaching project will be evaluated as "effective" (Sierpinska, 1999).

Although the US Department of Education is now actively encouraging educational research communities and education technology companies to exclusively use scientifically based research, many educators believe that, for ethical reasons, it may be that such broad use of a research methodology with roots in the pharmaceutical industry is not such a good idea in the classroom. In fact, there is still a lot of debate on this issue (Zaritsky & al., 2003). Because of my disquiet with these ethical issues, and my wish for a richer data set than quantitative research can provide, I have rejected quantitative methodologies for this study.

1.2.2 Qualitative methodologies

Qualitative research begins with questions and qualitative researchers seek answers to their questions in the real world. They gather what they see, hear, and read from people and places and from events and activities. Essentially they do their research in natural settings rather that in laboratories or through written surveys (Rossman & Rallis, 1998). According to these authors, qualitative research has two unique features: (1) the research is the means through which the experiment is conducted, and (2) the purpose of qualitative research is learning about some piece of knowledge. Both these characteristics are integral to a view of learning that sees the learner as a constructor of knowledge rather than a receiver of it.

As Sierpinska (1999) explains, if one rejects the "quantitative study" methodology because one does not believe that it is possible to teach the same mathematical content with two different sets of mathematical activities and different pedagogical approaches, and if one does not believe that one can assess what the students have learned by counting their scores on a standardized test, then one should seriously consider using a methodology that supports instructional development. Qualitative methodologies require the researchers to make explicit the rationale behind all of their decisions, the specified theoretical perspective, an instructional theory and their theory of what it means to know the particular mathematical content that they plan to teach the students, because that will be the basis by which their research projects will be evaluated. Researchers using the qualitative paradigm also need to make predictions concerning the knowledge that the students should construct as a result of participating in the planned activities. After which, the researchers often attempt the scenario in a class with someone else as a teacher while sitting in the classroom as an observer. The researchers generally collect all possible documentation concerning the students' mathematical work by recording the classes, and collecting all the students' written work. The researchers analyze this material with the question: is the anticipated knowledge apparent in the students' productions? If not, then the researchers must try to determine what knowledge has developed. They need to determine whether or not the new knowledge can be explained in terms of the theoretical frameworks assumed *a priori*. The researchers need to determine if an alternative theory is needed, or if the theory needs to be modified. Finally, they need to analyze if the scenario can be improved to decrease the difference between the anticipated and the actual knowledge produced by the scenario. On the basis of this analysis, the researchers re-design their lesson and try it again.

If a researcher plans to use a qualitative research methodology as described above and the theoretical framework is based on the theory of didactic situations (Brousseau,1986), then the researcher would be using the methodology of didactic engineering in developing a teaching project in mathematics (Sierpinska, 1999). The concept of didactic engineering as a specific research methodology entered French mathematics education in the early 1980s and is gradually becoming known in North America (Kieran, 1998), although it has not been researched to any extent.

1.3 Didactic engineering

Didactic engineering can be viewed as both a product, resulting from an *a priori* analysis, and as a process, resulting from an adaptation to the implementation of the product in the dynamic conditions of a classroom (Douady, 1997). Didactic engineering seeks, among other things, to situate the possible actions of a teacher under the constraints of his or her classroom and to determine the course of action required to obtain a desired behavior. In order to do this, a researcher using didactic engineering must formulate his or her questions and transform them into hypotheses in a developed theoretical framework in order to construct an experiment. The results of the experiment are then "confronted" with the predetermined expected behaviors before decisions are made regarding the success of the experiment (Douady & al., 1987). This process of confronting results with predetermined expected behaviors is referred to as “internal validation”. According to didactic engineering, the validation of the research hypotheses is essentially internal in the sense that it is based on the confrontation of the *a priori* and *a posteriori* analyses, rather than external, based on the statistical comparison of the achievements of experimental and control groups (Laborde, 1989; Douady & al., 1987; Artigue, 1992; Artigue & Perrin-Glorian, 1991). It is predominately this type of internal validation that differentiates didactic engineering from other research methodologies in the field of mathematics education. As figure 2 shows, didactic engineering is essentially composed of four parts: preliminary analysis, design and *a priori* analysis, experimentation, and *a posteriori* analysis and validation.

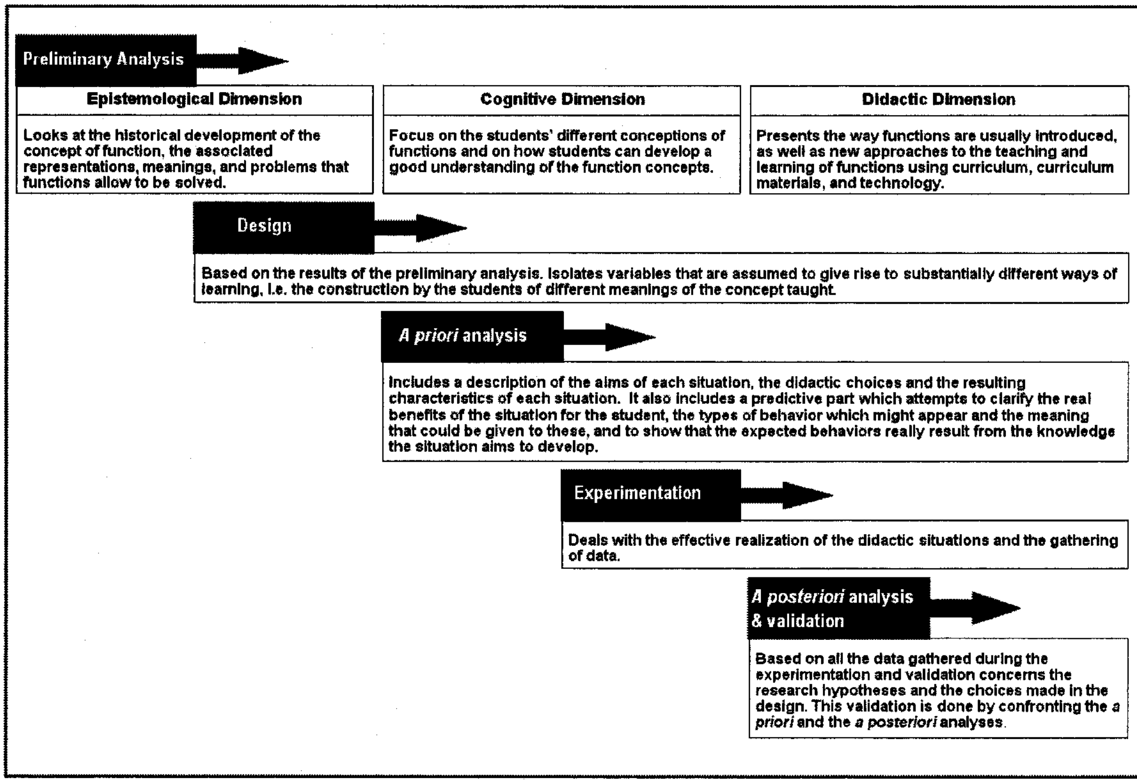

Figure 2. Didactic engineering research methodology

<u>1.3.1 Preliminary analysis</u>

The preliminary analysis in didactic engineering usually involves the consideration of three dimensions:

(1) an epistemological dimension which looks at the historical development of a concept, various aspects of a concept, problems based on the concept that may be solved;

(2) a didactic dimension which is mainly concerned with the usual introduction of the concept in question, its effects on students' achievements, the didactic constraints, and recent developments in the teaching and learning of the concept through curriculum and textbook studies;

(3) a cognitive dimension which deals with student conceptions about the concept.

1.3.2 Design

The design phase is based on the results of the preliminary analysis. This key phase of the methodology isolates a certain number of variables that are assumed to give rise to substantially different ways of learning, i.e. the construction by the students of different meanings of the concept taught.

1.3.3 *A priori* analysis

The *a priori* analysis of the designed didactic situations includes a description of the aims of each situation, the didactic choices and the resulting characteristics of each situation. It also includes a predictive part which attempts to clarify the real benefits of the situation for the student, the types of behavior which might appear and the meaning that could be given to these, and to show that the expected behaviors really result from the knowledge the situation aims to develop.

1.3.4 Experimentation

The experimentation phase deals with the effective realization of the didactic situations and the gathering of data. In this phase, the students are assumed to be learning, and data are gathered to assess whether this is true.

1.3.5 *A posteriori* analysis and validation

The *a posteriori* analysis is based on all the data gathered during the experimentation. This phase involves the validation of the research hypotheses and the choices made in the design of didactic situations. It is done by confronting the *a* *posteriori* analysis with the one done *a priori.*

1.4 Summary

Chapter 1 described the theoretical framework and methodology for this study. The Theory of Didactic Situations and didactic engineering was described and an explanation for the choice of this methodology was presented. After situating didactic engineering in the qualitative research paradigm, a detailed description of the phases involved in didactic engineering was presented.

As well as being the methodology for this study, didactic engineering is also the framework for the dissertation itself. Thus this dissertation will not follow the usual format employed in most dissertations, but will reflect the structure of a didactic engineering experiment, specifically as it relates to teaching functions.

In summary:

Chapter 2, Preliminary analysis, will analyze the epistemological, didactic, and cognitive dimensions of functions.

Chapter 3, Teaching sequence: design and *a priori* analysis, will describe the three teaching sequences employed, including the *a priori* analysis of the aims, characteristics and expected outcomes of each.

Chapter 4, Experimentation, will complete the sequence, with a description of the realization of the didactic situations and the *a posteriori* analysis of the data.

The next chapter will describe the preliminary analysis required by the process of didactic engineering and end with a discussion of the technology used in this research.

Chapter 2: Preliminary analysis

This chapter represents the preliminary analysis phase of the didactic engineering methodology which is essentially composed of epistemological, cognitive, and didactic dimensions.

2.1 Historical and epistemological dimension

In this section, I will provide an epistemological and historical analysis of the concept of function with special attention to the dynamic aspects of functions which are not easily observed in traditional paper/pencil based curriculum and outline the epistemological roots of functional relationships and their dependence on time.

The concept of function is considered one of the most important concepts in mathematics today (Luzin, 1998; Ponte, 1992; Youshkevitch, 1976). However, the concept was not discovered or conceived by a single individual or at a particular time. Instead, it evolved over a period of several centuries and continues to evolve today in response to important problems in a number of different fields both within and outside of mathematics. This is why, even today, no single formal definition can include a complete description of the function concept.

2.1.1 Historical development of the concept of function

Youschkevitch (1976) outlined three main stages in the development of the idea of function up to the middle of the 19th century. He outlined the stages as follows:

1. Antiquity - The stage in which the study of particular cases of dependencies between two quantities had not yet isolated general notions of variable quantities and functions. Examples of particular instances of functions from antiquity include counting,

which implies a correspondence between a set of given objects and a sequence of counting numbers; the four elementary arithmetical operations, which are functions of two variables; and the Babylonian tables of reciprocals, squares, square roots, cubes, and cubic roots which are also functions (Ponte, 1992). Because the literature of that time did not suggest an abstract and more general idea which unifies separate concrete dependences between quantities or numbers in whichever form these dependences happen to be considered, the concept of function was not attributed to antiquity (Youschkevitch, 1976).

2. The Middle Ages - The stage in which, in the European science of the 14th century, these general notions were first definitely expressed both in geometrical and mechanical forms, but in which, as in antiquity, each concrete case of dependence between two quantities was defined by a verbal description, or by a graph rather than a formula because the algebraic symbolism necessary to express functional relationship in the form of a formula had not been available until the 16^{th} century.

3. The Modern Period - The stage in which, beginning at the end of the 16th century, and especially during the 17th century with the development of algebra and its symbolism, analytical expressions of functions began to prevail. The class of analytic functions generally expressed by sums of infinite power series soon became the main class used.

Although the concept of function was not introduced until the 18th century, graphs of functions were used to analyze their properties as early as the 14th century. Oresme (1323-1382) was among the mathematicians who could be regarded as having come close to a modern definition of function concept in studying the latitude of forms.

Oresme developed this theory of latitudes of forms representing the distance covered by an object moving with variable velocity. The graphical representation given by Oresme to the latitude of forms, which was essentially a representation of the functional variation in velocity with time to study the motions of bodies under uniform acceleration, is one of the earliest instances in the history of mathematics using what we now call "the graph of a function" (NCTM, 1969). Although he did not state the law of falling bodies, which was later attributed to Galileo, Oresme essentially yielded that conclusion. The actual emergence of a notion of function as an individualized mathematical entity can be traced to the work of Descartes (1596-1650). Descartes clearly stated that an equation in two variables, geometrically represented by a curve, indicates a dependence between variable quantities (Ponte, 1992).

Newton (1642-1727) was one of the first mathematicians to show how functions could be developed in infinite power series, thus allowing for the intervention of infinite processes. He used the term "fluent" to designate independent variables, "relata quantitas" to indicate dependent variables, and "genita" to refer to quantities obtained from others using the four fundamental arithmetical operations (Ponte, 1992). Newton presented a clear kinematic-geometric interpretation of the basic conceptions which described conceptions of time and motion and of their geometrical presentation originating with Galileo and Oresme (Youschkevith, 1976). It was Leibniz (1646-1716) however, who first used and therefore introduced the term "function" (in unpublished documents) in 1673 and as such, the concept of function was generally attributed to him. Perhaps Newton illustrated the distinction between dependent and independent variables more clearly than Leibniz, but Leibniz was the inspiration of the eighteenth century because of the pedagogical quality of his work (NCTM, 1969).

The term "function" was adopted in the correspondence interchanged by Leibniz and Jean Bernoulli (1667-1748) between 1694 and 1698 when discussing the study of curves (Ponte, 1992). The actual term "function" first appeared in a scientific article written by Jean Bernoulli in 1698, while the first explicit definition of a function appeared in another written by Bernoulli in 1718 and was widely disseminated (Youschkevith, 1976). It contained his definition of a function of a variable as a quantity that is composed in some way from that variable and constants. In 1748, Euler (1707-1793), who was a former student of Bernoulli, later added his touch to this definition speaking of analytical expression instead of quantity thereby creating an association between the notion of function and the notion of analytical expression (Ponte, 1992).

The 19th century brought about lively interactions enlarging and clarifying the notion of function. The most significant argument revolved around the study of motion of a vibrating string outlined by Johann Bernoulli (1727) and questioned by d'Alembert (1717-1783). Both Euler and Daniel Bernoulli (1700-1782), Johann Bernoulli's son, attempted to find more general solutions to the vibrating string problem. For example, Euler began with a concept of function similar to that of Leibniz, but broadened it in his work on the vibrating string problem to include piecewise defined functions (Kaput, 1994). However, it was d'Alembert who gave an almost exhaustive solution of this problem in a famous paper published in 1747. The debate around the vibrating string continued for years with both Euler and Bernoulli providing their own ideas and alternative solutions. In 1759, Lagrange (1736-1813) entered the debate by taking sides with Euler and opposing both Bernoulli and d'Alembert and the debate lasted over 20 years without a final solution (Kaput, 1994). In 1807, Fourier (1768-1830) gave the rule for the coefficients of the trigonometric series representing an "arbitrarily given" function **f** known as the Fourier formulas (Luzin, 1998). He also observed that his functions included the piecewise-defined functions of earlier mathematicians. However, Fourier never gave a mathematical proof for his solution. The challenge of outlining this mathematical proof was later taken up by Lejeune Dirichlet (1805-1859) who succeeded in defining a function that could be represented by a Fourier series. In 1837, Dirichlet gave the following definition of a function: "if a variable **y** is so related to a variable **x** that when a numerical value is assigned to **x**, there is a rule according to which a unique value of **y** is determined, then **y** is said to be a function of the independent variable **x**."

Dirichlet also gave a well-known example of a function which is everywhere discontinuous to emphasize the generality of his definition. He introduced the following function $f: R \to R$:

$$f(x) = 1 \qquad \text{if x is a rational number;}$$

$$f(x) = 0 \qquad \text{if x is an irrational number.}$$

(Usiskin & al., 2003)

However, it was Dirichlet's 1829 definition of function that was most widely accepted at the turn of this century (Kleiner, 1989 and Malik, 1980). Function was defined by Dirichlet as follows:

> **y** is a function of a variable x defined on the interval $a<x<b$, if to every value of the variable **x** in this interval there corresponds a definite value of the variable **y**. Also, it is irrelevant in what way this correspondence is established.

(Kleiner, 1989, p. 291)

The historical development of the concept of function gives perspective to the current debate around the function concept in the 20th and 21st centuries. There has been a gradual evolution in the understanding of the function concept. It has evolved from Oresme's graph, to an algebraic formula, to a correspondence between numerical variables, to a mapping between ordered pairs during the 20th century.

It was not until Bourbaki, a well known proponent of abstract algebra, that the definition of function evolved further. In 1939, Bourbaki offered the following definition of function:

> Let E and F be two sets, which may or may not be distinct. A relation between a variable element **x** of E and a variable element **y** of F is called a *functional relation* in **y** if, for all **x** in E, there exists a unique **y** in F which is in the given relation with **x**.
>
> We give the name of *function* to the operation which in this way associates with every element **x** in E the element **y** in F which is in the given relation **x**; **y** is said to be the *value* of the function at the element **x**, and the function is said to be *determined* by the given functional relation. Two equivalent functional relations determine the *same* function.
>
> (cited in Kleiner, 1989, p.299)

The historical emergence of the function concept is intimately related to the study of motion (Biehler, 2005). It is ironic that this idea of function may be regarded as the longstanding attempt to downplay the idea of motion. For example, motion was one of the ideas that Lagrange intended to cancel from the theory of analytical functions (Laborde & Mariotti, 2002). The solution which was adopted consisted of substituting for the metaphor of motion, a more suitable one, which does not involve time. As such, the modern definition of function definitely abandoned the metaphor of motion. This definition de-contextualizes functions, and removes any dependencies on the use of motion as a metaphor for functions, although the connection to this metaphor was preserved in the idea of graph.

2.1.2 Summary of historical and epistemological dimension

The historical development of the concept of function gives a good overview of how different representations emerged and how they contributed to a definition of function that was commonly accepted.

The idea of motion played an important role in the emergence of the concept through debates that were focused on solving real problems. Important contributions to the concept of function and the notion of dependent and independent variables from leading mathematicians and scientists appear to have been contextualized in problems of motion. It is for these reasons that I have chosen the idea of motion as the context for my research on the understanding of functional relationships.

It is interesting to observe that in attempts to de-contextualize a definition for function, the important contribution of the idea of motion was suppressed. Even Euler, who devoted a large part of his life working with a concept of function that was contextualized in problems of motion, eventually focused on a more analytical view of functions. In fact, it was Euler's colleague, Lagrange who strove to remove the entire metaphor of motion from the definition of function. As a result, the modern definition of function no longer has grounding in the idea of motion. Luckily this important idea is still preserved in graphing.

There is no one representation of function that allows anyone to completely grasp the notion of function. Thus, the complementarity of the various aspects and representations of functions is very important. Therefore in my research, I will pay special attention to representing the concept of function from various points of view.

2.2 Cognitive dimension of function

2.2.1 Understanding

An underlying assumption regarding learning with understanding during my research is that such learning is generative. When students have an understanding of some newly acquired knowledge, they can apply that knowledge to learn new concepts and to solve new problems (Carpenter & Lehrer, 1999). It is also important to realize that understanding is not an all-or-none phenomenon. As such, understanding can be thought of as emerging or developing rather than presuming that someone either does or does not understand a given concept or process (Carpenter & Lehrer, 1999). As a consequence, understanding can be characterized in terms of mental activity that contributes to the development of understanding rather than as a static attribute of an individual's knowledge. Carpenter & Lehrer (1999) propose five forms of mental activity from which mathematical understanding emerges: a) constructing relationships, b) extending and applying mathematical knowledge, c) reflecting about experiences, d) articulating what one knows, and e) making mathematical knowledge one's own.

Formal mathematical concepts, such as the concept of function which forms the basis of the high school mathematics curriculum, should be given meaning by relating them to earlier intuition or ideas that students may have. Unless instruction constructs relationships between children's informal knowledge and targeted concepts they learn in school, they may develop two separate systems of mathematical knowledge: one they use in school and one they use outside school. Of course, developing understanding of functional relationships involves more than simply connecting new knowledge with prior knowledge: it involves developing relationships that reflect important mathematical principles. For example, an understanding of a functional relationship may be extended to more general forms of relationships between variables. Students should be able to extend their understanding of functional relationships by making transitions or connections between various representations of functions. Reflection about experiences involves the conscious examination of one's own activities and thoughts. Little reflection is needed during the routine application of skills. However, problem solving often involves consciously examining the relationship between one's existing knowledge and the condition of the problem situation. Students stand a better chance of acquiring this ability if reflection is part of the learning process.

The notion of the emerging nature of understanding is seen in students' developing ability to reflect on their thinking (Carpenter & Lehrer, 1999). Finally, the ability to communicate or articulate one's ideas is a benchmark of understanding. Understanding involves the construction of knowledge by individuals through their own activities so that they develop a personal investment in building knowledge.

2.2.1.1 Concept image and concept definition

The distinction between a person's concept definition and her/his concept image was introduced and analyzed by Vinner (1983) and further discussed by Vinner and Dreyfus. Their work (1989) serves as a foundation for much of the current research in the learning of functions.

Concept definition is the way teachers define function and how we expect students to define function. Developing a solid concept definition is often the primary focus of the high school mathematics curriculum in North America. Historically, mathematicians regarded function as an active process; typically, definitions described some kind of relationship between two variables or even a requirement that one variable is dependent on the other. This was especially true when the early mathematicians focused on solving real problems involving motion where time was the independent variable. This conception created an understanding of function that allowed continuous and smooth functions. As the need grew for a more sophisticated definition for function, mathematicians altered the definition of function to allow for wider examples of functions, such as split domain functions. The current definition of function, known as the Dirichlet-Bourbaki definition and explained earlier in this Chapter, allows for even non-constructible functions and is defined as follows:

> A *function* is a correspondence between two sets, known as the domain and the codomain, where each element in the domain corresponds to one element in the codomain.
>
> (Stafslien & al., 2001, p.32)

Clearly, all direct connection to variables and continuity disappear in the above definition. In the course of the last century, teachers have progressed from using the classical definition of a function to more abstract but powerful modern definitions. Therefore, a student's introduction to functions often involves a definition completely separate from any intuitions he or she might develop concerning functions.

The concept image of function, on the other hand, embodies how students are able to visualize and perceive functions in a variety of forms. The concept image comprises the visual representations, mental pictures, experiences and impressions evoked by the concept name (Thompson, 1994). As Thompson explains, this difference between the concept image and concept definition is similar to the difference in understanding we might have with the concept of "blue." Although we can potentially offer a definition for the concept of blue, we rarely use this definition when interacting with blue. For example, if we were to ask the reader, whether the type in this paper is blue, we would expect the reader to answer based on whether the type looks blue instead of analyzing the text based on some formal definition of blue. Similarly, when we ask a student if an expression is a function or could be a function, students typically answer based on their previous experience with functions and not with an analysis using the Dirichlet-Bourbaki definition of function. Ideally students use their concept image to inform their concept definition. In particular, the formal definition should be the final decision-making factor for solving a problem. Vinner (1983) and others argue that instead, a closer approximation to a typical student approach to problems is similar to observations of the color blue: students rely solely on concept image to formulate their thoughts on functions.

The process definition of a function is akin to the historical perspective on function. It acknowledges that function has a domain and a range, and views the function as a process of moving from the domain to the range. This understanding has increased flexibility and allows the student to include many more functions in their conception. As explained earlier, the process conception of function is an adequate view of function for most situations, excluding the life of a mathematician.

The most structural classification is the correspondence understanding. This category on the level of concept definition implies that the student defines function in the Dirichlet-Bourbaki fashion. Similarly, a correspondence understanding of concept image moves beyond understanding function as a complex process. The Dirichlet-Bourbaki definition, on the other hand, allows for even the existence of non-constructible functions. That is, there are an infinite number of functions that we will never be able to represent or construct. This last level of sophistication is rare in middle and high school students. Even Sfard (1992), who extensively worked with a class of students to develop their conception of function, could not convince most students of the existence of a non-constructible function. Therefore, though students often reach the point where their concept definition is of this type, their image conception rarely reaches this level.

By analyzing students' understanding based on the above spectrum, one can directly measure the depth of a student's understanding of the concept of function. In the literature, however, there seems to be an agreement that the concept definition aspect of a student's education should be significantly de-emphasized, so that their concept image has a better chance of developing. Sfard (1992), for example, argues that introducing a correspondence definition too early actually damages or hinders the student's development of the concept of function. She argues that we learn concepts only at the level at which we absolutely need them. That is, the curriculum in a general K-12 education does not cover mathematics advanced enough to necessitate a correspondence conception of function.

An alternative approach, then, is to let students develop as much power as they need to fit situations that they encounter in the curriculum. In fact, it may be beneficial to allow the concept image to develop at a higher level than the concept definition, reversing the standard practice. Yerushalmy, for example, has examined the effects of such an approach on seventh grade students. He gave seven students a problem involving a function of multiple variables. They were told to model the following situation (Yerushalmy, 1997):

> A rental car company charges 100 shekels for a day and an additional 5 shekels per kilometer. The company would like to have a clear description of the price that any client may have to pay when returning the car. Suggest such a description.

This was extended by the following exercise.

> You have won a 1000 shekels coupon from the rental car company. Provide a detailed description of all your options to spend the exact amount for renting and driving a car using the maximum of your winnings.
>
> (p.435)

The students had not received any formal definition for function, and only minimal notation. The resulting models that students invented were remarkable. While several students tried to graph the situation, their various methods of generalizing from the one-variable situation (which they had some experience in modeling) demonstrate how differently students can construct their knowledge. For instance, while one student graphed the solution on two two-dimensional graphs, another student constructed a three-dimensional coordinate system (ibid.). The students analyzed the situation, and seemed to decide against the three-dimensional graph. This suggested that for their situation, the three-dimensional graph was not yet useful. Having thought of it, however, it will be readily available when the student encounters situations where the two two-dimensional graphs give less information. The lack of a formal definition gave rise to a rich dialogue on concept image.

Concept definition and concept image are both aspects of a student's understanding of function, but the traditional method of beginning from the concept definition promotes a large disparity between both concept definition and concept image that can cause confusion and hinder development. On the other hand, if one initially focuses on concept image, it seems as though the concept definition naturally follows and students gain an appreciation of how and why these definitions develop. Mathematical experts come to use concept images and concept definitions dialectically. Over time, their images become tuned so that they are aligned with a conventionally accepted concept definition, which in turn allows intuition to guide and support reason. Not every student of mathematics attains equilibrium between definitions and images, however we can increase their chances of success by giving explicit attention to imagery as an important aspect of pedagogy and curriculum (Thompson, 1994).

2.2.1.2 The importance of prototypes

Confrey and Smith (1991) use "prototype" functions to introduce families of functions. They are in agreement with Schwartz and Yerushalmy (1992) that it is from the study of the characteristics of these base functions that students come to know the attributes of each family. The idea of prototype, applied to the concept of function by Schwarz and Hershkowitz (1999), ties together many of the ideas in the previous two sections. The idea of prototypes also provides a powerful mechanism for understanding how students develop an image of the function concept, and the use of various representations aids in the creation of beneficial prototypes.

It is important for students to develop a healthy concept image of function that not only permits them to recognize a variety of functions, but also permits them to move comfortably and wisely between appropriate function representations. The idea of prototypes suggests that we think of objects and concepts in terms of examples (Brawner, 2001). When faced with a new extension or generalization, we either reject it on the basis that it does not match our set of prototypes, or we adjust our prototypes to include the given extension. In regard to functions, it is important to develop an increasingly sophisticated palette of examples that are readily adaptable to new situations, yet middle school and high school curriculum rarely go beyond the family of quadratic functions.

We all have a favorite set of functions; we differ, however, in how we apply our examples of functions to different situations. For example, if we ask how many functions pass through the three points on the graph in figure 3,

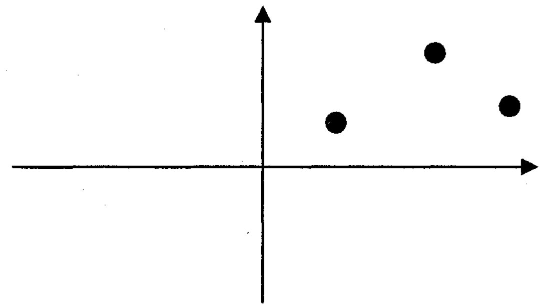

Figure 3. Three points (Schwarz and Herschkowitz, 1999)

students have various options at their disposal. They could use their concept definition of function, which is unlikely given the curricula they have probably followed, or they could attempt to imagine functions that might pass through these three points. Here, their previous experience with functions becomes critical. The errors that students frequently commit are to either claim that there are no functions that pass through all three points, or that there is only one function that passes through these points (Schwarz & Hershkowitz, 1999). In the first case, the students will typically justify their statement by claiming that since no linear function can be drawn through the three points, no function can be drawn through these three points. On the other hand, a common justification for there being exactly one such graph is that while one cannot draw a linear function, one can draw a quadratic function. The students apply their prototypes to the graph, and if the prototypes do not work, the students conclude that no functions work.

Without requiring that our students automatically have a vast warehouse of functions, it is nevertheless desirable to avoid this kind of static prototype. There is a difference between the student who has had rigorous instruction solely in regard to linear functions, but acknowledges this as a sample in the wide range of different functions, and the student who insists his prototype is the sole representative of functions. Some students use linearity as a popular prototype, whereas the other forces his prototype on the situation. Similar results were found by other research studies such as those of

Schwarz & Hershkowitz (1999).

Applying the concept of prototypes to functions is relatively new so there is much research yet to be done in uncovering how to best help students develop a good set of prototypes with an appropriate conception of function to expand their set. The idea of prototype, however, underlies concept understanding and so can hopefully aid teachers in understanding how their students develop a concept image of function.

2.2.1.3 Physical models and multiple representations

Monk (1994) investigated students' conceptualizations of classical situations having to do with related rates providing physical models for students' experimentation and asking questions about the situations that encouraged students to reason with the physical devices. Monk observed that students have difficulty in developing a coherent conceptualization of a physical model as a system of dependencies among quantities whose values vary - even while holding the devices in their hands and playing with them. Monk proposed that imagining situations as being functionally constituted was also part of seeing generality in geometric diagrams, and that we can actively promote this ability in students with carefully crafted curriculum and instruction. Attending to students' conceptualizations of situations is especially important when applied to models involving physical phenomena and physical quantities. The most effective situations will be those that require meaningful and contextual interpretation of representations in a problem-based approach (Coulombe & Berenson, 2001).

Physical phenomena such as motion taking place in a straight line can be represented through simulation. Doing so allows students to capture a physical phenomenon so that it may be analyzed, edited, and shared with others for meaningful interpretation in a problem-based approach. Algebraic representations on the other hand, almost always offer concise, true, and effective representations of patterns and mathematical models. Of course the inappropriate use of this representation may blur the mathematical meaning or the nature of the represented object and cause difficulties in students' interpretation of results (Friedlander & Tabach, 2001). Numeric representations can also be calculated or viewed on tables to provide detailed information regarding discrete data points or other aspects of graphs. Fluency with multiple representations in creating and testing models of mathematical relationships is critical in maximizing students' ability to acquire knowledge when solving real-world problems. Interpretation and translation of representations are skills that can, for example, extend students' algebraic thinking by helping them construct their mental images of patterns and functions (Moschkovich & al., 1993). The simultaneous display of multiple linked representations clearly illustrates interrelationships between the representations.

The core concept of function is not represented by any of what are commonly called the multiple representations of functions, but instead by the connections made among representational activities which produce a subjective sense of invariance, important in understanding the function concept. Relating different representations to each other is regarded as a basic element of meaningful teaching and learning of functions (Biehler, 2005). The subtlety of the function concept with its various representations and process-object duality proves to be highly complex, leading not only to a concept with wide ranging powers, but also with widespread misunderstandings among students. One of the main points found in the rich literature on this theme concerns the relationship between function and its graphical representation. In particular, it seems that, for students, there is a lack of explicit relationship between function and graph (Vinner & Dreyfus, 1989, Dreyfus & Eisenberg, 1983). Difficulties of interpreting graphic information in terms of function are widely reported; generally speaking students do not consider the graph of a function to be the representation of the relationship that exists between the variables. Therefore, it is important to focus on graphs, expressions, or tables as representations of something that, from the students' perspective, is *representable*, such as aspects of specific situations. The key issue outlined in the preliminary analysis then becomes twofold: (1) To find situations that are sufficiently rich that they can be represented in many different ways and (2) To orient students toward drawing connections among their representational activities in regard to the situation that initiated them. The situation being represented must be contextualized in such a way that it highlights the connections among the representations. It is helpful when the representations can be linked together such that a change to the function in one representation is immediately reflected in the other representation of the same function. Otherwise, students may only learn each topic in isolation from the others.

2.2.2 A constructivist approach

There is general agreement in the mathematics education community that a constructivist approach is based on the following principles (Dugast, 1991):

- A student constructs his knowledge rather than receiving it passively from a teacher.

- A new piece of knowledge is constructed based on a prior knowledge (therefore, a teaching sequence must take into account this prior knowledge; in order to take it into account, one must know students' conceptions of the targeted knowledge.
- A student constructs a new piece of knowledge while engaged in a problem solving activity where he experiences the limits of his current conceptions and realizes the need to develop new ones.
- Formulation is an important phase in the process of knowledge construction.
- The role of the group contributes to the learning process. The construction of knowledge in a classroom setting does not happen in isolation. Knowledge is constructed during membership in a group. Classroom debates, exchanges of ideas, procedures, and rationales cause students to modify their approach to a problem or even their thinking.

2.2.2.1 Students' conceptions of functions and obstacles

As stated earlier, no single formal definition can include the full description of the function concept and today, different conceptions associated with the function concept continue to evolve. Students also have difficulty distinguishing the concept of function from its graphical representation. They may in fact believe the graphical representation of the function is the actual function. For example, a curve may only be seen within its continuity, and the students do not realize that a continuous function should be considered a particular case of general function.

When students come to think of an expression as producing a result of a calculation, they have what several researchers have called an *action* conception of function (Dubinsky & Harel, 1992; Thompson, 1994). This conception views a function as a rule which applies to numbers. Students holding an action conception of function imagine that the rule remains the same across numbers, but that they must actually apply it to some number before the rule will produce anything. They do not necessarily view the rule as representing a result of its application. Sfard (1992) identifies two conceptions associated with the function concept following this action conception: the process conception and the object conception. The process conception of function views a function as a formula or rule for computation. When students build an image of "self-evaluating" expressions they have a process conception of function. From the perspective of students with a process conception of function, an expression stands for what you would get by evaluating it. They do not feel compelled to imagine actually evaluating an expression in order to think of the results of its evaluation. Therefore, it is not surprising that achieving a process conception of function is a non-trivial achievement for students, and that for many students it is not achieved without receiving instruction that focuses explicitly on its development (Dubinsky & Harel, 1992; Goldenberg & al., 1992). A process conception of function opens the door to a wealth of imagery. Goldenberg and Lewis (quoted in Dubinsky & Harel, 1992) have developed visual supports for students to envision functions as processes applied over a continuum. Once students are adept at imagining expressions being evaluated continually as they "run rapidly" over a continuum, the groundwork has been laid for them to reflect on a *set* of possible inputs in relation to a *set* of corresponding outputs.

A function viewed as a static entity on which operations can be performed demonstrates the object conception of function. At the point where students have solidified a process conception of function so that a representation of the process is sufficient to support their reasoning about it, they can begin to reason formally about functions – they can reason about functions as if they were objects (Thompson, 1994). To reason formally about functions seems to entail a scheme of conceptual operations which grows from a great deal of reflection on functional processes. Of most importance is the image of functional process as defining a correspondence between two sets: a set of possible inputs to the process and a set of possible outputs from the process. The many paths by which students achieve an object conception of function are long and complex (Ayers & al., 1988), and explanations of it draw on a long tradition in philosophy and epistemology regarding the notion of reflective abstraction (Dubinsky, 1991; von Glaserfeld, 1991). One hallmark of a student's object conception of functions is his[1] ability to reason about operations on sets of functions. It is easy to think that students are reasoning about functions as objects when it is actually the function's literal representation (i.e., marks on paper) that is the object of their reasoning (Sfard, 1992; Sierpinska, 1992). Sfard also notes that the object conceptions usually develop out of process conception.

Sfard (1992) identifies three components of the progression from the process to the object conceptions of function: interiorization, condensation, and reification. In the first stage, there is a process acting on an established object (interiorization). Then the process becomes more compact-whole (condensation). Finally Sfard explains that an ontological shift occurs when the student converts the condensed knowledge into an object in its own right (reification). Sfard advocates a process-focused method of teaching based on two principles: (a) Students must first develop a process conception of

[1] For the purpose of this thesis, the use of "his" includes "her" and "he" includes "she".

function. Specifically, the concept should not be introduced in structural terms. (b) An object conception of function should be delayed as long as students can do without it. Function as an entity should not be required until it provides an indispensable advantage over its computational view. Sfard claims that these two principles are necessary for reification to occur (1991). Sfard and Linchevski (1994) also explain that students must grow gradually through these perspectives.

Confrey and Costa (1996) took issue with reification theory, particularly with its hierarchical view of mathematics learning and they described the theory as an excessive and narrow orientation towards abstraction. In addition, Confrey and Costa (1996) wrote that reification theory tends to separate mathematical thinking from its origins in social contexts. They explained that context does not have to be stripped away as students move from concrete activities to the abstract and that connections should be made with everyday applications.

Schwartz and Yerushalmy (1992) believe that function is the primary object of algebra and that algebra courses should be restructured and re-sequenced in light of its importance. Thus, functions should be introduced from the onset and the constructs of algebra should build on the function concept. The multiple representations of functions should be emphasized. Students traditionally learn about functions by manipulating the symbols that represent them. However, representations such as graphs can provide a richer, deeper understanding through the use of graphical operations such as translations, reflections, and dilations. Through such operations with a small base of functions, one can see the consequences of the actions, symbolically and graphically. In addition, the direct manipulation offered by dynamic software such as *Geometer's SketchPad* (Jackiw, 1990), *Cabri Geometry* (Laborde, 2003) and *MathWorlds* (Roschelle & Kaput, 1996) allow for the direct manipulation of the graphical representations themselves providing opportunities for even deeper understanding. The perspective of Schwartz and Yerushalmy is guided by the following assumptions: (a) function as a process can most readily be seen through symbolic representation, (b) function as an object can most readily be seen through graphical representation, (c) some operations such as composition are best understood through symbolic representations, and (d) some operations such as translations are best seen through graphical representations.

Like Sfard (1992), Confrey and Smith (1991) discussed two traditions in the development of functions: as a co-variation between quantities (process conception) and as a correspondence between values of two quantities (object conception). But unlike Sfard, Confrey & Costa (1996) do not believe that reification should be the sole focus of teaching and learning because it minimizes the importance of alternative approaches involving the teaching of functions as a co-variation between quantities..

Herscovics (1989) explicated the notion of cognitive obstacle as it relates to learning mathematics. An obstacle is a way of knowing something that gets in the way of understanding something else. An obstacle can have various origins: epistemological when it is related to the notion itself; cognitive when it is related to the students capabilities; didactic when it is a consequence of the instruction. The origin of cognitive obstacles is developmental. It is interesting to note that Balacheff and Gaudin (2003) and their team no longer make reference to cognitive obstacles. Instead, they stress that obstacles are in fact, student conceptions and that such student conceptions would be replaced with new student conceptions of the concept being taught. In this context, learning takes place as students replace one conception by another one that is more general. Without disagreeing with this argument, I will continue to use the label cognitive obstacle in order to differentiate more easily student conceptions that get in the way of understanding something else from those that do not interfere with learning. This will be useful in completing the specific *a priori* analysis of my research project.

A typical obstacle that students demonstrate when moving from graphs to algebraic expressions is to interpret the graph as a picture instead of a graph. For example, Kerslake (quoted in Leinhardt & al., 1990) asked students to decide which of the graphs in figure 4 represent journeys and to describe what happens in each case.

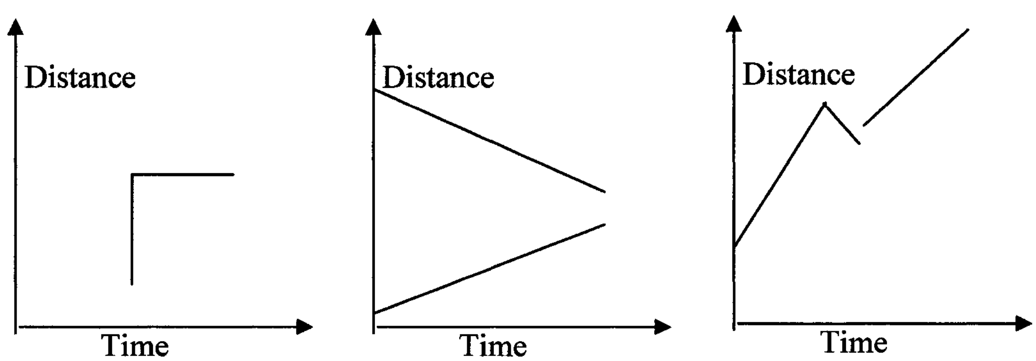

Figure 4. Journeys (Leinhardt & al., 1990)

Many students answered that all were descriptions of journeys. For example the first one apparently represents someone climbing a vertical wall, and the last climbing a mountain. This type of cognitive obstacle creates significant difficulties in conveying the connections between graphs and functions to students.

2.2.2.2 The importance of communication

When students acquire knowledge using the different forms of representation outlined above (as well as others) they rely on communication to share personal interpretations with others. Recall that this should be considered another form of representation – verbal representation. Students can build a shared understanding through joint reference to the representation of the phenomena within a context (Monk, 1994). Other research has demonstrated that representations, when used as rhetorical devices in collaborative environments, improve shared understanding (Kozma & al., 2000). By combining the use of familiar types of representations and analogies to familiar events, and communicating their understandings with others, students can acquire knowledge of even complex mathematical relationships.

2.2.3 Summary of the cognitive dimension

Understanding in the general sense as outlined above can be characterized in terms of mental activity that contributes to the development of understanding rather than as a static attribute of an individual's knowledge. The cognitive dimension focused on the student's different conceptions of functions and how students can develop a good understanding of the function concept. A good understanding of functions means that students should be able to make transitions or connections between various representations of functions and be able to choose the representation which is best adapted to solve a given problem. Different conceptions associated with the function concept continue to evolve in the mind of students as they think about functional relationships. Students develop a concept of function by first going through process conception. The object conception is developed from the process conception as a consequence. A correspondence definition of function is not suitable to introduce the notion of function. One of the defining characteristics of learning with understanding is that knowledge is learned in ways that clarify how it can be used. The construction of knowledge happens best in contextualized situations.

The concept of function is difficult for many students because it is often presented to them as a decontextualized mathematical definition. Historically, we can observe that the modern definition of function emerged only as a result of trying to solve real problems that were grounded in the idea of motion. Not providing students with situations that allow for the repeating of this pattern will surely make the concept difficult to grasp. The concept of function turns out to be difficult because of the obstacles presented above.

2.3 Didactic dimension

2.3.1 Concept of function in textbooks

In this section, I provide a didactical analysis on the teaching of functions involving data about motion. I will outline some of the traditional methods being used in schools today as well as new methods being supported by the National Council for the Teaching of Mathematics utilizing multiple forms of representations. Specifically, I will outline cases in which the appropriate and effective use of technology has allowed students to learn concepts that were not accessible to them using traditional tools.

In combination with the Dirichlet definition, the Bourbaki definition would eventually affect school mathematics curriculum for many years. Vinner and Dreyfus

(1989) introduced the Dirichlet-Bourbaki definition of function as a correspondence between two nonempty sets that assigns to every element in the first set (the domain) exactly one element in the second set (the codomain). Although the formal definition of function has been static since the 1950s, this Dirichlet-Bourbaki definition is still the most accepted definition, taught in the majority of mathematics curricula, and the majority of mathematics curricula continue to utilize this definition of function (Lloyd and Wilson, 1998). The notion of function and how it can be learned and taught is still evolving.

The modern definition has been expanded to include many relationships not previously considered such as functions defined on split domains, discontinuous functions, and piecewise defined functions. Since no single formal definition of function can include the full description of the function concept, researchers like James Kaput described these special types of functions in the larger context of the Dirichlet-Bourbaki definition in order to help students understand the concept of function in contexts that make sense to them. For example, Kaput described piecewise defined functions as follows:

> A piecewise defined function defined on an interval [a, b] subdivided into subintervals differs from a function globally defined over [a, b] only in that its values are defined independently on each subinterval. There may or may not be additional constraints imposed on those subinterval definitions, continuity being one example - which is the case for piecewise defined Position vs. Time functions in most (but not all) *MathWorlds* documents. Another is piecewise defined Linear function, which makes the function graphs into polygons (non-closed, obviously).

It is possible to define the values of the piecewise defined functions in various ways on the boundary points of the subintervals - that is, for a subinterval defined by **c** and **d** with $a<c<d<b$, they could be defined on (c, d], [c, d), [c, d] or (c, d). Of course, on complementary subintervals, the function needs to be defined appropriately.

Two other points:

(1) "Piecewise defined" is different from "Stepwise varying" - which is a description of BEHAVIOR rather than of definition.

(2) It is possible to have a globally defined (and hence not piecewise defined) function that is Stepwise varying: The greatest integer function, for example, which is a step-function defined for all integers (f(x) = greatest integer < x, or < or =).

J. Kaput (personal communication, August 3, 2004)

The purpose of this explanation is perhaps to avoid the conception "a function cannot have more than one rule; a piece-wise function corresponds to more than one function" (Mesa 2004, p. 278).

Using the above description of function to guide instruction will help the teacher situate learning about functions in contexts that are familiar and interesting to the students. Since function is defined by what is needed for application or development of new fields of study, then in determining the appropriate definition and context with which high school students should explore the function concept, teachers should constantly be examining the purpose of studying functions in their own classroom (Mesa 2004). It is important to focus on descriptions of functions that are appropriate for students according to where they are in their level of understanding and achievement. The research described in this thesis is guided by descriptions of functions similar to the one provided by Kaput above. Such descriptions of functions can be used by teachers to help students conceptualize the concept in a way that engages the student to learn more about the concept and be able to make the necessary transitions with other concepts.

Today, the teaching of functions essentially falls in one of two approaches used by educational publishers of school algebra texts in North America. Table 1 provides a basic overview of these common approaches to teaching about functions.

Table 1. Common publisher approaches to teaching about functions

Approach	Algorithmic	Conceptual
Definition of Function	A relationship between input and output where the output depends on the input. There is exactly one output for each input.	A relationship between dependent and independent variables.
Focus of learning	Basics of numerical and symbolic manipulation so that students have a foundation composed of the algorithms needed for further mathematics study.	A core set of concepts so that students have contexts and motivation for learning the algorithms.
Assumptions	Students will develop understanding of concepts later, as they learn to apply algorithms.	Development of the concepts of variable, rate of change, and functions will lead naturally to the development of core symbolic algorithms that students will eventually need.
Types of problems	Problems that require manipulation of algorithms. Many of the problems used in this approach are void of meaningful context for students.	Realistic problems grounded in contexts that are meaningful to students. They are often interesting problems that include hands-on activities.
Most frequent modes of representation	Symbolic representations, set notation, correspondence rule.	Words, graph, table, pattern, symbolic.

The first approach is rather algorithmic in nature considering algebra to be a set of basic symbolic algorithms which the student must master as a basis for further mathematical study. In this view of algebra, the learning of functions is about the basics of symbolic manipulation, much as middle school is about the basics of numerical manipulation. The students can develop understanding of the concepts later, as they learn to apply the algorithms. The second approach is more conceptual. In this approach, algebra is a set of core concepts, among which are number pattern, sequences, variables, functions, etc. Development of these concepts will lead naturally to development of the core symbolic algorithms that the student eventually needs to acquire. In other words, the goals are much the same as in the algorithmic approach, but the conceptual approach focuses on having students spend more time up front on the concepts so they have context and motivation for learning the algorithms. This approach tends to depend heavily on interesting problems and hands-on activities. Laying out an organized set of transitions in the student's thinking about key mathematical concepts, including the function concept, is essential for the learning of algebra. One of these important transitions in student conceptions involves moving from recognizing patterns of numbers, to systematic recursive definitions for the patterns, to explicit functional definitions of mappings between two sets of patterns, to the relationships between variables, to a complete concept of function. The successful implementation of the conceptual approach is essentially focused on helping students make such transitions in their thinking (moving from existing conceptions to more complete or accurate conceptions).

Both of these approaches are commonly used to cover many of the same topics. One could argue that they may even form a progression in the teacher's conception of important algebra concepts with the algorithmic approach preceding the conceptual. Although research and even the curricula envisioned by most states and provinces are basically conceptual in nature, the reality is that the majority of North American students still learn about important algebra concepts like functions through the algorithmic approach used by leading publishers like Pearson Education, McGraw-Hill Education, Houghton Mifflin, and Reed Elsevier, who have respectively approximately 40%, 35%, 20% and 5% of the market share in the North American market (Resnick & al., 2004).

2.3.2 Calls for reform in the teaching of mathematics and the influence of textbooks

As the concepts and transitions become more apparent in the teaching and learning of algebra, visualization and rich discourse become increasingly more important. One can train an individual to perform an algorithm without much discussion, however, developing understanding of concepts requires real discourse between people. The necessity of teaching the formal set definition of function at the school level is not obvious and many teachers feel that pedagogical considerations were ignored while designing the basic curriculum (Malik, 1980). More specifically, researchers question if students actually understand this formal definition of function (Markovits & al. 1986). Recall from the discussion of the cognitive dimension in Section 2.2, many researchers today find that the formal definition is not appropriate to introduce the concept of function.

Concerns from mathematics teachers and researchers about understanding and pedagogy were key issues in the NCTM, *Curriculum and Evaluation Standards for*

School Mathematics (1989) and in the NCTM, *Principles and Standards for School Mathematics* (2000). Since the publication of these two sets of Standards, there has been a renewed focus on the teaching of mathematics so that students gain an appreciation for its applications in the world around them. As a result of the emphasis being removed from the formal definition of function, the focus of the study of function as described by the NCTM (1989, 2000) became more conceptual and contextual in nature. Also, Froelich et al. (1991) explained that the basic idea of function is that two quantities are related in some way. Recall that this is how the concept of function was first developed by Galileo when he studied physical problems associated with motion. Current recommendations by the mathematics education research community with regards to the study of function in school mathematics, such as the effective use of modeling, data analysis, contextual and interdisciplinary applications, are not new ideas. These ideas were expressed from the very beginning of the development of the function concept. Neither are the new pedagogical goals outlined by researchers, new ideas. These ideas include developing connections within mathematics through the use of function, using function as a unifying theme, and engaging students to learn more about mathematics in new ways. The goals today are so similar to the calls of the mathematicians and educators of the early 1900s that, just as it did for Klein, the question comes to mind "have we come full circle in the study of functions?" It appears that in terms of recommendations for the intended curriculum for school mathematics, we have indeed come full circle (Brawner, 2001).

The study of function in school mathematics in the context of physical problems associated with motion is not a new idea, nor is the use of modeling and data analysis in real world contexts. However, if one assumes that today's recommendations for intended curriculum supporting the teaching and learning of functions have come full circle from the recommendations made in the 1900s, it can be concluded that research communities were simply not successful in influencing change necessary to make a difference in how the concept is taught. In today's environment, this would require successfully influencing the majority of publishers to change the way they present functions.

The recommendations and intentions for mathematics curriculum have been varied and at times unaligned with each other. As a result, many of the recommendations have had limited impact on the actual mathematics taught in schools. There has been a clear pattern throughout the history of curriculum reform efforts of misinterpretation or partial implementation of curriculum recommendations, leading Stanic and Kilpatrick (1992) to conclude that the achievement of intended outcomes of reform movements have been limited. Textbooks tend to influence classroom teaching dramatically. In fact, 90% of mathematics teachers in North America who use textbooks reportedly teach to the book (Mickey, 2003). It is therefore important to examine the definition of the function concept and the related definition of variable in the most popular textbooks today. The following table outlines how function and variable are defined in today's most popular textbooks.

Table 2. Common publisher definitions of variable and function

Textbook, Publisher, Year	**Definition of Function**	**Definition of Variable**
Algebra 1, Pearson-Prentice Hall, a division of Pearson Education, 2004	A function is a relation that assigns exactly one value in the range to each value in the domain. Note that a relation is defined on the preceding page as a set of ordered	A variable is a symbol, usually a letter, that represents one or more numbers.

	pairs.	
Algebra 1, Glencoe McGraw-Hill, 2005	A function is a relationship between input and output. In a function, the output depends on the input. There is exactly one output for each input.	Variables are symbols used to represent unspecified numbers of values. Any letter may be used as a variable.
Algebra 1, McDougal Littell, a division of Houghton Mifflin Company, 2004	A function is a rule that establishes a relationship between two quantities, called the input and the output. For each input, there is exactly one output. More than one input can have the same output.	A variable is a letter that is used to represent one or more numbers. The numbers are the values of the variable.

In these popular books, the function definition is presented without any context and is of little use for helping students solve real problems. The definitions are variations of the formal Bourbaki definition. The problems faced by the students in these books are mostly intended to help the students learn the definition rather than to deepen their understanding of the function concept. As explained in the cognitive dimension in Section 2.2, such a formal set definition is very abstract to students and therefore not appropriate to use when introducing the concept.

Publishing is a $8.4 Billion industry in North America with three clear market leaders that compete aggressively with each other for precious market share (Shea, 2004). Clearly, the top publishers do not see it as their role to change how functions should be learned by students. Rather than implement the recommendations of researchers, publishers rely extensively on critical feedback from teachers - their core customer base. In this competitive environment, textbook publishers cannot afford to promote function as a unifying theme in mathematics education unless requested by the great majority of their customers. Although researchers suggest that teachers should take responsibility for what happens in their classrooms and that teachers must pay attention to the recommendations being made by researchers, this is not a simple nor realistic task given the current reliance on the textbook and the reality of teachers' work.

Function is a powerful and unifying topic in secondary mathematics, as highlighted above, however currently no textbook can help teachers come to this understanding. Teachers do have access to a wealth of knowledge in helping them guide the study of functions in their classrooms. However, given what is known about the teacher's reliance on textbooks in mathematics, it may be more effective in reaching the masses for mathematics education researchers to mobilize, align themselves and approach the NCTM with explicit recommendations regarding the teaching and learning of the function concept in schools today. The NCTM could then conceivably influence its worldwide membership – many of whom are leaders in mathematics education - to consider new approaches to teaching such important concepts. Only after the majority of the teachers begin to adopt new approaches to teaching functions, will the large scale publishers change the way the concept is presented in their textbooks.

2.3.3 Function as change

Since the middle ages, function was explained in the context of real problems involving motion. Most of the discussions involved dependencies between quantities and associated rates. For example, the early work of Oresme resulted in a representation of the functional variation in velocity with time to study the motions of bodies under uniform acceleration.

One purpose of the function is to represent how things change (Tall, 1996). Based on this meaning it is natural to consider the important concept of rate of change. Grasping the idea of function requires grasping the idea of variation and that the idea of variation is easily understood when some continuity (in a naive sense) is involved in the variation (Laborde & Mariotti, 2002). Perception of change may be related to different modalities (sense of touch, sense of sight…) of perception, but certainly sight plays an essential role in learning about functions. It leads to the claim that space change over time (motion) can be considered as one of the basic primitive perceptions of dynamic and continuous variation.

> In our everyday conceptual system, change is understood metaphorically in terms of motion.
>
> (Lakoff & Núñez, 2000, p.406)

Variation is appropriately demonstrated by dynamic features of dynamic geometry environments, as is the dependency between two variables. Essentially, dynamic geometry environments are very effective in representing a functional dependency since constructions in such environments are expressed as functional dependencies between geometric objects. The representation in such environments can then be put in relation with other kinds of representations, such as algebraic or graphic representations, depending on the context. For example, it has been claimed that ideas related to variation (increase, decrease, constancy, maximum, minimum), and variation within variation (fast and slow variation, rate of change, smoothness, continuity, and discontinuity), are better grasped from graphical representations (Ponte, 1992). Clagett (1968) attempts to capture the variational nature of a quality's "intensity" (e.g., temperature) over position and time. Kaput (1994) extended Clagett's analysis to trace the evolution of today's ideas of variable and variability in the calculus, concluding that today's static picture of function hides many of the intellectual achievements that gave rise to our current conceptions.

Unfortunately there is very little emphasis on variation in today's K-12 mathematics curriculum in North America. In examining the most recent editions of the three most popular K-9 textbooks series in the U.S., we observe that the closest they come to examining variation is to have students construct tables of data, and even then there is a profound confusion between the ideas of random variable and variable magnitudes. This is in stark contrast to the Japanese elementary curriculum (Kodaira, 1992) which repeatedly provokes students to conceptualize literal notations as representing a continuum of states in dynamic situations (Thompson, 1994). It is also surprising that so little has been investigated in regard to students' concepts of variable magnitude – the focus instead being on variable as literal representation of number (Arcavi & Schoenfeld, 1987).

Students have difficulties grasping the idea of function as a relationship between variables (one depending on the other). They have a discrete view of a function, in which a function relates separate pairs of numbers with each number considered as an input giving another number as result; students consider that there is a relationship between numbers, but the relation is conceived separately for each pair. In any case, the relationship of dependency between the two variables is not visible in the graph, that remains a static representation of the couple (x,y) and does not afford the meaning of dependency between the two variables that rather play a symmetrical role.

2.3.4 Summary of didactic dimension

The didactic dimension outlined the way functions are usually introduced in North American middle and high schools today. It also supported new approaches to the teaching and learning of mathematics. The Dirichlet-Bourbaki definition of function as a correspondence between sets continues to be the basis for the presentation of function in the most utilized mathematics textbooks in the North American middle and high school textbooks. Although different researchers have made important contributions towards more modern definitions such as the notion of function as change, the Dirichlet-Bourbaki definition, which has been static since the 1950s, is still the most accepted definition today. The recommendations by the research community including the investigation of functions through problems involving motion, modeling and data analysis are not new ideas. It appears that we have come full circle in the study of function back to its very roots. However, given the teacher's reliance on popular textbooks in mathematics, it is increasingly important for researchers to approach the NCTM and departments and ministries of education with a consistent recommendation regarding the teaching and learning of the function concept in middle school and in high school classrooms in North America.

2.4 The role of technology in the teaching and learning of functions

Mathematics and, more specifically the study of functions, are rapidly changing due to new technologies available to students (Hegedus & Kaput, 2002). Technology provides students with the ability to learn about functions by providing easy access to multiple forms of representation. The use of technology provides room for more explorations, in a faster manner, of those different forms of representation (Brumbaugh & al., 2006). As explained in the preliminary analysis, it is important to understand that the way understanding happens may be different depending on the representation used. Studying multiple linked representations of a function is even more powerful because it makes the link between representations more dynamic. Direct manipulation revolutionized the teaching and learning of geometry. These same ideas now allow students to directly manipulate the representations of functions themselves.

Today, we can exploit the benefits of technology such as the graphing calculator and computer software in exploring the concept of function. Research shows that the use of technology in mathematics education causes students to become better problem solvers and achieve a better overall understanding of functions when compared to students that do not use technology in a traditional algebra curriculum (Brumbaugh & Rock, 2006). Therefore, we should assume that very shortly, the teaching of functions will increasingly involve the effective and appropriate use of various types of technologies. At the very least, a graphing calculator or computer environment can free students from tedious point-by-point plotting and move the instructional focus to understanding. Several technological developments have had a very significant role in the study of functions. For example, with the aid of such technology, students can readily examine a variety of functions and altered functions. The effective and appropriate use of graphing calculators and computers with appropriate software such as spreadsheets, graph plotters, and symbol manipulation programs help students to develop a deeper mathematical understanding of the function concept (Peressini & Knuth, 2005). Direct manipulation-enabled software allows for new dynamic representations of functions. In earlier multiple representation software, graphs were essentially static display representations. The results of any actions were presented as new representations of mathematical objects. Today, interactive software making effective and appropriate use of dynamic manipulations allows students to act within a representation by transforming objects dynamically (Kaput, 1992). Translations in functions, for example, are now permitted with Cartesian representations in software programs such as *Function Probe* (Confrey, 1992), *CabriGeometry IIPlus* (Laborde, 2003), *Geometer's SketchPad* (Jackiw, 1990), and *MathWorlds* (Roschelle & Kaput, 1996). Even more recent developments bring the power of direct manipulation to data. Using *Fathom* (Finzer, 2001) or *TinkerPlots* (Konold, 2004), students can now directly manipulate data while simultaneously seeing the effects on the graphical representations. Because of their affordability and capabilities, handheld graphing and data collection devices provide greatly increased access to the type of functionality provided by the above powerful software programs, especially in resource-challenged schools that may not have funds for extensive purchases of computer hardware (Berson & Balyta, 2004).

The interpretation of significant features of functions from their Cartesian graphs deserves a prominent place in mathematics curricula. To be mathematically literate means to be able to use mathematics concepts to make predictions, interpolate, and extrapolate. In the context of functions, it means to be able to establish relationships among different functions by superimposing graphs, to be able to construct regression curves that approximate relationships for empirically obtained data, and to estimate the degree of association between two variables (Gomes Ferreira, 1997).

U.S. Secretary of Education, Rod Paige, commented that schools are still struggling when it comes to truly integrating the appropriate use of technology.

> Many schools have simply applied technology on top of traditional teaching practices rather than reinventing themselves around the possibilities technology allows. The result is marginal - if any - improvement....Technology can not only improve instruction but transform what we think of as education.
>
> (U.S. Department of Commerce, 2002, p. 4)

Technology clearly still has the potential to transform education. If it is to do so, teachers must be able to take advantage of that potential. Therefore, to advance students' conceptual understanding and achievement, technology implementation requires integration into teaching practice and standards-based curricular materials. Many of the technologies being purchased by schools and school districts are business tools that have been repurposed for education. In contrast, handheld graphing and data collection devices are built specifically to support the effective teaching and learning of mathematical concepts that would otherwise not be accessible to most students, and in addition, their affordability allows many students access to the technology.

Many studies examined the constructive contribution made by the use of computers and by the dynamic visualization of functions used to overcome or at least lessen the classic problems of coordination between the various forms of representation (Duval, 2000; Mavarech & Kramarsky, 1997; Vinner, 1992). Others have concentrated on the possibilities offered by the emerging technologies in order to work on function as a mathematical object (Borba & Confrey, 1996; Dagher & Artigue 1993; Kieran 1994, 1998). However, many of these studies are pitched at higher grade levels. There are fewer studies on problems encountered by students approaching the function concept at lower grade levels.

Different proposals have been suggested in order to help students learn about functions by a conceptual approach. In many such cases, technology provides new environments to explore the function concept in very different ways compared to traditional algorithmic approaches using traditional tools like paper and pencil. For example, new technology is affording creation of new qualitatively different representations of functions. Kaput (1992) pointed out that historically, mathematical notation systems have been instantiated in static, inert media, but new technologies now afford a whole new class of dynamic, interactive notations of virtually any kind. When software is used to represent function concepts, it is usually done graphically, often with the option to represent them in table form. The way in which the graph is often drawn as a curve may cause students to see it as a whole object. Kaput claimed that dynamic technologies are also the natural "home" for variables, rather than static technology, which requires the user to apply much of the variation cognitively. Unlike the paper and pencil environment, the interactive environment can afford the representation of change through motion; the idea of variation is grounded in motion, so that it is possible to experience variation in the form of motion.

The key notions supported by the interactive environments such as *MathWorlds* are motion and the fact that motion preserves the links constructed between elements. As a consequence, such interactive environments incorporate and represent the idea of variation and that of functional dependency. This type of functional dependency constitutes a very particular instance of function, differing from numerical function, because it relates spatial elements instead of numbers. Thus interactive environments such as *MathWorlds* offer a powerful environment incorporating the semantic domain of space and time, where the notion of function can be grounded.

Some programs, such as *RandomGrapher* (Goldenberg & al., 1992) plot random function values to build the graph as a collection of points. Although this gives a set of points, further activities may be necessary to see the function process assigning to each value of x the value of y=f(x). The authors also created *DynaGraph*, a dynamic visual representation where the users can vary the variable having as feedback the value of the function.

Other programs allow for the linking of alternative forms of representations, for instance *Function Probe* (Confrey 1992) allows graphs to be directly manipulated, using the mouse to transform graphs by translating, stretching, and reflecting. Such an approach treats the graph as a single object to be transformed. *Function Probe,* developed by the Mathematics Education Research Group at Cornell University, can be considered a multiple representational (equation, graphs and tables) software tool that enables students to explore the idea of real functions. Its goal was to encourage students to enter mathematical thinking by using a tool built for them to investigate and model phenomena using mathematical functions (Confrey & Maloney, 1996). Students can explore functions with actions either within one representation or with links made between different representations. Thus, it preserves the integrity of each representation. Such dynamic representations made possible by software such as *Function Probe* give a new status to the Cartesian representations that become qualitatively different Cartesian representations instantiated in paper and pencil.

The primacy of numerical representation (Goldenberg & al., 1992) and the lack of experience with functional dependency in a qualitative way may be considered a source of students' difficulties. This is why it has been proposed that it is important to start in an environment providing a qualitative experience of functional dependency independently of a numerical setting (Laborde & Mariotti, 2002). Dynamic geometry incorporates functional dependency and working in a dynamic geometry environment fosters the thinking about geometrical links in terms of functional dependency. Usually functional dependency remains implicit, i.e. "in action" (Vergnaud, 1991), but once made explicit it provides a rich semantic context for the idea of function. *MathWorlds* also incorporates this qualitative type of functional dependency focusing on the variation between variables.

By assuming that the basic idea of function as outlined by NCTM (2000) is that two quantities are related in some way, Kaput (1992; 1993) and Nemirovsky (1993) approach the study of function in much the same way that Galileo did. In fact, Kaput and Nemirovsky focus their research on the way students have intuitive sense of concepts such as distance, velocity, acceleration, which can be utilized in conjunction with computer simulations such as the *MathWorlds* microworld (Roschelle & Kaput, 1996) to study different aspects of motion at an earlier age. They utilize technology-enabled simulations in familiar contexts linked to multiple representations of position versus time so that students make effective use of multiple representations in meaningful introductory contexts. Thus many of the important aspects of the function concept may be explored by students at an early age and the resulting initial image should be appropriate for a wide spectrum of students.

The technology framework underlying this research integrates hardware, specific software, and device-connectivity designed specifically for the teaching and learning of functions. The first representation system is the handheld graphing calculator in combination with a data collection device and the second system is a computer-based microworld designed specifically for the teaching and learning of the mathematics of motion in combination with a data projector. Both these representation systems provide multiple forms of representation. Interpretation and translation of representations are skills that can, for example, extend students' algebraic thinking by helping them construct their mental images of patterns and functions (Moschkovich et al., 1993). The most effective situations will be those that require meaningful and contextual interpretation of representations in a problem-based approach (Coulombe & Berenson, 2001).

I hypothesize that fluency with multiple representations in creating and testing models of mathematical relationships is critical in maximizing students' ability to acquire knowledge when solving real-world problems. One major similarity related to how students acquire knowledge using the different forms of representation outlined above (as well as others) is the reliance on communicating personal interpretations to others. Students can build a shared understanding through joint reference to the representation of the phenomena within a context (Monk, 1994). Other research has demonstrated that representations, when used as rhetorical devices in collaborative environments, advance shared understanding (Kozma & al., 2000). By combining the use of familiar types of representations and analogies to familiar events, students can acquire knowledge of even complex mathematical relationships.

2.4.1 Choice of technologies for this study

Graphing calculators are a special type of multiple representation tools and have many similarities with the technologies outlined above. However, graphing calculators are different from the programs discussed above which require a powerful and expensive computer. In the past, graphing calculators offered representations in numeric, graphic, algebraic, and table form, but today, they can also offer geometric representations. For example, the *TI-84Plus Silver Edition* graphing calculator from Texas Instruments and the *Classpad300* from Casio come preloaded with powerful geometry tools. Such environments can provide new opportunities for the teaching and learning of the notion of function, by making effective use of a dynamic geometry environment.

Traditionally, one of the largest challenges in the teaching of the connections between representations has been the time it takes for students to construct the appropriate tables and graphs associated with their functions. For example, by the time the students have constructed a table for a sine curve and plotted sufficient points to get a sense of how the curve acts, they lose much of the connection between the various representations. The introduction of easily accessible technologies like the graphing calculator has vastly changed this inhibiting factor in intuition development. A graphing calculator or computer can quickly convert an algebraic expression into tables and graphs, allowing the student to explore how minute changes to the function affect the graph of the function. Overall, graphing calculators or computers have increased the opportunity for students to work with and to interpret a variety of different representations efficiently.

With the use of graphing calculators or computers in the classroom, we must reevaluate our goals for instruction. Today, if the student is able to find an algebraic expression for the function, the initial graphing process is simple. Instead of evaluating the expressions for several numbers, students can now focus on understanding how certain graphs act under different transformations, but, while students using graphing calculators or computers no longer have difficulties graphing functions, there are far more critical elements in mathematics that they can and must consider. Without critical thought and interpretation and the ability to make connections among representations, the act of graphing functions on graphing calculators or computers becomes as meaningless as, or even more meaningless than, the original computation. By carefully designing activities, one also avoids the natural belief by students that the computer is omnipotent. Like any other tool, the graphing calculator, the computer and their outputs have limitations, and part of the student's task is how to combat the limitations of various representations to the best of his ability. Currently the graphing calculators do not provide for powerful linked multiple representations of the same phenomena on the same screen. Thus the incorporation of technology changes pedagogy by making new demands on students' thinking, while removing some of the mechanical factors that may impede learning.

In the didactic milieu employed in this experiment, the students and the teacher each had a *TI-84 Plus Silver Edition* handheld device from Texas Instruments loaded with the *MathWorlds* application from SimCalc Technologies. These tools were chosen because of the multiple forms of representation that they make available to the students and the fact that they constitute a technology that supports effective communication in the classroom. Also, in combination with *MathWorlds* and the *Calculator Based Ranger*™, *(CBR)*, the graphing calculator provides the "live" context and immediate feedback needed to test my research hypotheses H1, H2 and H3 (cf. p. 5). The teacher's version of the *TI-84Plus Silver Edition* calculator had a projection screen attached.

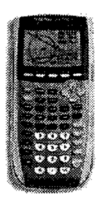

Figure 5. TI-84 Plus Silver Edition graphing calculator

Each graphing calculator is loaded with the *MathWorlds* application. *MathWorlds* is an environment rich in interactive motion simulations, visualization tools such as qualitative graphs, and a motion animator (that replays imported motions) to enliven and deepen understanding of important mathematical concepts. *MathWorlds* also extends the graphing calculator by providing for direct experiences on these ubiquitous devices. The software allows for the representation of motion in two different ways. The bottom portion of the calculator screen presents a graphical representation of a motion. The top portion of the calculator screen presents a simulation of the motion as a representation of motion going from left to right.

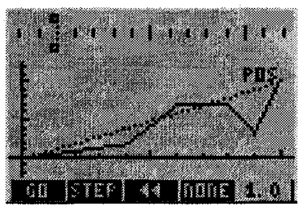

Figure 6. MathWorlds for the *TI-84 Plus Silver Edition*

In combination with the graphing calculator and the *CBR*, *MathWorlds* provided the students with a multiple representation system that allows for the direct experience, "live" context, and multiple linked representations needed to test the hypotheses.

The teacher's computer, loaded with *Connected MathWorlds* software and connected to a projection device, serves as a visualization, simulation coordination, and classroom discussion tool. The teacher can also construct a function (or choose from among student functions) and broadcast it to each student in the class. The software's ability to display any subset of student functions supporting comparison, contrast, reflection, and group analysis is particularly suitable for the situations of formulation and validation in reference to the theory of didactic situations employed in this experiment. *Connected MathWorlds* provides the second representation system in this experiment. Also, in combination with the *TI-Navigator* classroom network, it provides for the collection, aggregation, and viewing of mathematical constructions needed to test the fourth hypothesis which claims that aggregated mathematical constructions challenge students to coordinate multiple representations and deepens their understanding of functional relationships.

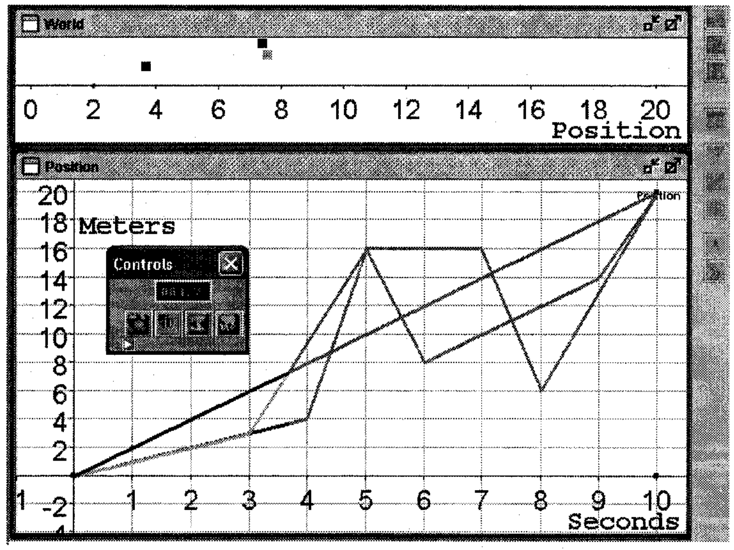

Figure 7. MathWorlds for the computer

The teacher makes use of a *CBR* (Calculator Based Ranger™) from Texas Instruments connected to his *TI-84 Plus Silver Edition* to collect and analyze real motion data. The *CBR* works by sending out ultrasonic pulses and then measuring how long it takes for those pulses to return after the have bounced off the closest object. The *CBR*, like any sonic motion detector, measures the time interval between transmitting an ultrasonic pulse and the first returned echo, but *CBR* has a built-in microprocessor that also calculates the distance of the object from the *CBR* using a speed-of-sound calculation. Then it computes the first and second derivatives of the distance data with respect to time to obtain velocity and acceleration data. It stores these measurements in lists within the graphing calculator for further analysis by teachers and students. The *CBR* therefore allows the students to explore mathematical and scientific relationships between distance, velocity, acceleration, and time using motion data collected from the activities they perform. In combination with *MathWorlds*, it also allows for the collection and animation of real student motion. In the past, students did an experiment, collected data, analyzed the data, and then learned something about phenomena or concepts. With this kind of technology, students can start analyzing real-world data while they are physically involved in collecting it.

Figure 8. The Calculator Based Ranger™

Finally, the *TI-Navigator* classroom network from Texas Instruments provides for connectivity in the didactic milieu. The physical layout of this technology framework is illustrated in Figure 9.

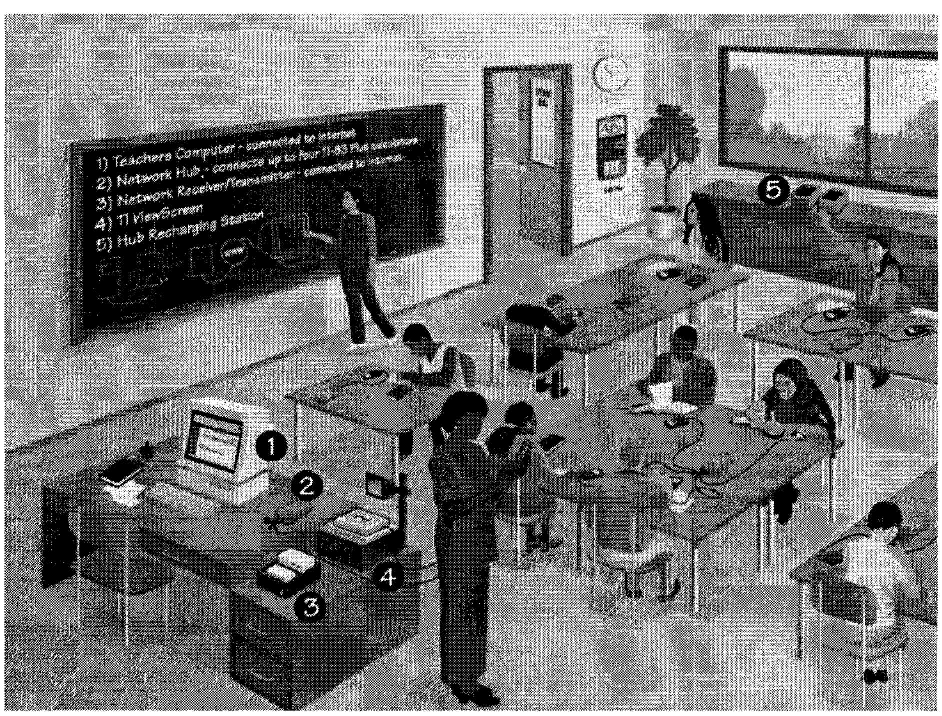

Figure 9. The *TI-Navigator* Classroom Network

The *TI-Navigator* network acts as an aggregation and broadcast server on a centralized computer that allows the teacher to harvest and combine students' individually constructed mathematical functions with those of their classmates.

The *TI-Navigator* Classroom Network depicted above has the following important components:

1. The teacher desktop computer loaded with *TI-Navigator* software.
2. Wireless hubs that network the *TI-84Plus Silver Edition* graphing calculator.
3. The access point.
4. The teacher overheard projector and overhead panel for the *TI-84Plus Silver Edition* graphing calculator.
5. Charging units for the wireless hubs.

The *TI-Navigator* Classroom Network will allow testing of the fourth research hypothesis H4.

By combining physically constructed motions using *CBR* motion detector technology, graphically constructible position vs. time functions and their animations in *MathWorlds* on the handheld graphing calculators with the power of *TI-Navigator* to collect and display them using *MathWorlds* on the teacher's computer, students and groups of students will be engaged in exciting new forms of mathematical activity, including a "mathematical performance", that bring new levels of engagement and excitement to learning critically important ideas. Thus the power of these different pieces of technology is integrated to establish a classroom environment which may enhance teaching and learning, and where the research hypotheses may be tested.

2.5 Summary

Chapter 2 provided an epistemological and historical analysis of the concept of function outlining the epistemological roots of functional relationships and their dependence on time. Understanding in the general sense was characterized in terms of mental activity that contributes to the development of understanding rather than as a static attribute of an individual's knowledge. Chapter 2 also outlined students' different conceptions of functions as well as what it means to have a good understanding of functions. The way functions are usually introduced using the Dirichlet-Bourbaki definition of function and support for new approaches to the teaching and learning of mathematics was also discussed. Chapter 2 concluded with an overview of the role of technology in the teaching and learning of functions today as well as a presentation of the technology framework underlying my research, complete with a description of the technologies used in this study.

The next chapter will describe the design of the teaching situations employed in this study to test the hypotheses.

Chapter 3 Teaching sequence: design and *a priori* analysis

Recall that the *a priori* analysis contains a predictive part which attempts to clarify the real benefits of the situation for the student, the types of behavior which might appear and the meaning that could be given to these, and to show that the expected behaviors really result from the knowledge the situation aims to develop. Since the following section describes how the *a priori* analysis was conducted, the future tense is used to report the process authentically. Also, it is important to remember that the design phase relies on the preliminary analysis of the concept, in this case, function, previously elaborated in Chapter 2.

3.1 Overview of design of teaching sequences

The teaching sequence is to be implemented in its entirety over a 5 day period. The goal of the first day is essentially to familiarize the students with the new additions to their regular learning environments, namely the research team and the five video cameras which will be present in their classroom. The actual teaching sequence will be implemented in the experimentation phase during the second, third, and fourth days, and composed of two didactic situations intended to have the students progress through situations of action, formulation, and validation in order for them to deepen their understanding of the concepts. Both didactic situations will essentially be composed of the following structure, according to the theory of Didactic Situations outlined in Chapter 1.

- A preliminary activity where the students become familiarized with the technology tools which are going to be used in the activities and the activities themselves.
- Situations of action where student are involved physically in the creation of graphical representations. The student searches for answers first individually, then in groups.
- Situations of formulation where students need to formulate their strategies in order to solve given problems involving functions. This is followed by the presentation of the answers to the whole class and discussion.
- Situation of validation where the students try to explain some phenomena, or to verify conjectures. This is essentially both a discussion and validation of the answers by the students.
- Conclusion or synthesis of the activity is led by the teacher who is helped by the students. This is the institutionalization phase.

The first series of didactic situations involves the *CBR* motion detector (Teaching sequence 1 described below in Section 3.2) and is used to provide physical grounding for the simulation-based activities that will engage the students to learn more about functions and be able to make the necessary transition with other concepts such as slope as rate of change. The goal of this teaching sequence is for students to develop an understanding of the relationship that exists between the motions and their representations and to develop a deeper understanding of independent and dependent variables.

The second series of didactic situations involving the creation and modeling of piecewise defined functions (Teaching sequence 2 described in Section 3.3) focuses on challenging students to coordinate multiple representations to deepen their understanding of functional relationship and the concept of rate and slope as rate of change, with emphasis on modeling situations that involve interesting variability. This teaching sequence will incorporate multiple representation systems in the didactic milieu that allow for individual and aggregated mathematical constructions. As hypothesized in the cognitive dimension of the preliminary analysis outlined in Chapter 2, the representational strategies involved in such a didactic milieu will enhance the depth of learning about functional relationships.

All of the activities included in the teaching sequences are designed to test the first three hypotheses:

Hypothesis 1 Individual mathematical constructions that are directly experienced in a "live" context have immediate kinesthetic, cognitive and linguistic aspects that will help students develop an understanding of the relationship between distance and time in problems of motion.

Hypothesis 2 Individual mathematical constructions in a "live" context facilitate the development of understanding of independent and dependent variables.

Hypothesis 3 Multiple linked representations of the same function in a simulated environment allowing for manipulation by the students improve their learning about rate of change.

Only the last two activities, which include the classroom network, will actually test the fourth and final hypothesis, namely, that aggregated mathematical constructions challenge students to coordinate multiple representations and deepen their understanding of functional relationships.

The last day (Day 5) will provide the research team with the opportunity to ask explicit questions to the students to probe their understanding of functions and their ability to transfer the knowledge acquired with technology to the traditional paper and pencil environment.

For all activities we precise initial required knowledge, milieu, variable, and expected outcomes (students' strategies, responses, behavior, difficulties, errors). As a result, a detailed *a priori* analysis will be presented for each activity comprising:

- goal of the activity
- the milieu (the role of the technology tool as a part of the milieu)
- didactic variables
- the type of situation and means of validation (when appropriate)
- class organization
- activity of the students (the instructions)
- role of the teacher
- prerequisites
- expected outcomes (behaviors, strategies, difficulties, and/or errors)

3.2 Teaching sequence 1: Exploring physical motion

The goal of this teaching sequence is for students to become familiarized with the motion detector and enhance their understanding of:

- the relationship that exists between the x- and y-axes;

- the concept of variable, and the notion of dependence between two variables;
- the concept of function as a relationship between a dependent variable and an independent variable.
- the relationship that exists between the motions and their representations.

The physical motion is explored under four aspects: the actual physical motion in front of the *CBR* motion detector, a verbal description of the motion, a graphical representation and a horizontal simulation of the motion. The teaching sequence is aimed at making relationships between verbal description, graphical representation and simulated horizontal representation of the motion. The physical motion is used as a means of validation of students' answers.

The students' understanding of these concepts and relationships will be developed if they are successful in constructing relationships between the new knowledge and knowledge that they already have. They will also have to be successful in articulating verbally or in writing, what they know about the concepts and relationships.

The role of the *CBR* motion detector in this lesson is to ground the notion of functions in the students' own physical motion by having them import their own physical motion data in a situation of action. In this situation of action, students use the *CBR* motion detector to track their motion and then use the *CBR* Animator in *MathWorlds* to plot and then animate their actual physical motion. This allows the students to compare how differences between a synthetically built graph and a physically built graph are reflected in differences between the motions, and vice-versa. It will be important for the teacher to chair the exchanges and highlight some of the students' formulations throughout the different phases in this lesson. After organizing the didactic milieu, it will be important for the teacher to present a problem that focuses on the relation between the graphs representing physical motion and the one representing synthetic motion. The students will have the means to construct a solution by themselves by physically walking in front of the *CBR* and then have the opportunity to try and explain their understanding of the relationship being investigated in a seminar type setting.

This teaching sequence is initially a teacher-led situation followed by a situation of action using the *CBR* motion detector and the *CBR* Animator, however, it quickly gets transformed into a situation of formulation. The class demonstration will involve two students, one to hold the *CBR* and the other to "walk" the motion. The rest of the class will be engaged in the discussion and will be allowed to make suggestions to the student doing the motion.

In order to appropriately participate in the discussion, the students will need to know the important components of a graph (i.e. identification of units and graduations on the axes). They will have been introduced to the concept of variable one month earlier by the teacher in the context of algebraic manipulations, using the definitions found in traditional textbooks (see table 2). The students will have received no formal introduction to functions or to functional relationships up to this point.

The following table summarizes the four activities that comprise the first teaching sequence. The goal, type of activity, and classroom organization are outlined for each activity.

Table 3. Four activities comprising the first lesson

	Goal	**Activity Type**	**Class Organization**
Activity 1	The students will familiarize	Action	Students working

(TS1,A1)	themselves with the tool that will be used throughout the teaching sequence. Find out the relationship between the physical motion in front of the *CBR* motion detector and the graph being displayed		individually and in groups. Whole class discussion.
Activity 2 (TS1,A2)	The students will identify the two variables involved in the representation of the motion: time and position.	Formulation	Students working in small groups (4 or 5 students). Whole class discussion.
Activity 3 (TS1,A3)	The students will identify the two variables involved in the representation of the motion: time and position, and the relationship between them.	Formulation	Students working in small groups (4 or 5 students). Whole class discussion.
Activity 4 (TS1,A4)	The students will reinforce the acquired knowledge. The students will also be able to coordinate multiple representations of the same motion – the graphical representation and the simulated horizontal representation The students will develop a good understanding of the notion of dependence and independence. For the teacher: assess what the students have learned	Formulation	Students working in small groups (4 or 5 students).

3.2.1 Activity 1 (TS1,A1)

Table 4. A priori analysis for Activity 1 of Teaching Sequence 1

Title	Getting Started
Description of the task	The students are asked to observe and make conjectures about the relationship between the motion walked physically in front of the *CBR* motion detector and the representations of the motion displayed.
Goal	The students will: - familiarize themselves with the technology tool that will be used throughout the teaching sequence. - find out the relationship between the physical motion in front of

	the *CBR* motion detector and the graph being displayed
Milieu	- *CBR* motion detector and *CBR* animator: graphical representation of the physical motion and horizontal simulation of the motion. - Real-time feedback from the graphical representation as the student walks in front of the *CBR*. - Other students' observations and conjectures.
Variables	- Two ways of representing a motion - graphical representation and simulation of the motion helps the understanding of the relationship because each representation of the motion accentuates different information regarding the motion. For example, the graphical representation explicitly represents time as one second for every graduation along the x-axis and distance in meters along the y-axis. The horizontal representation does not accommodate for the explicit visualization of time. Rather, it accentuates distance away from the *CBR* and the direction of the motion. Therefore, the two forms of representation have the students reflect on different aspects of the motion. Conversely, having the two representations of the same motion presented at once may confuse the students with too much information. It may also make it difficult for students to focus on the relationship between the physical motion in front of the *CBR* motion detector and the graph being displayed, which is the goal of this activity. - Number of physical motions in front of *CBR*
Type of situation	- Situation of action for the volunteering students
Means of validation	- Validation by the milieu of the student's conjectures: walking in front of the motion detector and making adjustments to understand the relationship between his movement and the graphical representation.
Classroom organization	- Students working individually and in groups. - Whole class discussion.
Student activity	(See Appendix 1 for detailed instructions to students) - Observe what happens with the graph when somebody moves in front of the *CBR*. - Write down the observations.
Role of teacher	(See Appendix 2 for detailed instructions to the teacher) - Manipulate the *CBR* motion detector and the *CBR* Animator. - Animate the discussion without giving any clues of the expected answer
Prerequisite(s)	None

Expected outcomes

Strategies and answers: The students will observe that the graph depends on the position of the student moving in front of the *CBR* and on the speed of the motion. They will first observe what happens and make conjectures. They will be able to verify their conjectures by moving themselves in front of the motion detector. As in the historical development of the concept of function, it is expected that the student will rely on verbal descriptions of the motion or the graph to describe the motion. Also, as outlined in the cognitive dimension of the preliminary analysis, the ability to articulate what one knows about the functional relationship involved in motion is an important indicator of understanding. An early indicator that the students are beginning to understand the relationship between distance and time in problems of motion will be their ability to contextualize the x and y-axes respectively as elapsed time and distance away from the *CBR*.

Four main difficulties are anticipated:

- Initial struggling with understanding the relationship that exists between the distance away from the *CBR* and the graphical representation of the motion. This may be observable when students make wrong conjectures regarding the slope of the graphical representation when walking towards the *CBR* and walking away from the *CBR*. It is anticipated that exploration with physical motion in front of the *CBR* and the direct feedback given to the students will resolve this issue.
- Students may have difficulty conceptualizing two different representations of the same motion (graphical and horizontal) because it may be too much information to process at once causing them to focus on only one representation.

- Students who will be focusing on the graphical representation of the motion will have difficulty understanding scale and intervals in the horizontal representation of the motion because they may choose to ignore this information.
- Because this is really an exploration activity for familiarizing students with the technology, it is anticipated that students may confuse how distance in front of the *CBR* is measured and represented. For example, students may believe that the starting point for this activity is some location far from the *CBR* and that the graph will measure distance away from the location. Therefore, it is also anticipated that students may have difficulty understanding the concept of position in relation to the *CBR*.

3.2.2 Activity 2 (TS1,A2)

Table 5. A priori analysis for Activity 2 of Teaching Sequence 1

Title	Activity 2
Description of the task	Given a graphical representation of a motion and a horizontal simulation of the physical motion, the students are asked to describe a motion that matches the target motion. The requirement to describe forces the students to reflect on precise attributes of the motion (where to start, how long to walk, how quickly, …).
Goal	- To identify the two variables involved in the representation of the motion: time and position.
Milieu	- *CBR* motion detector and *CBR* animator: graphical representations of the physical motion and horizontal simulations of the motion. Feedback provided by the milieu in the form of real-time representation of the physical motion in two forms (graphical and horizontal simulation) allows the students to see if their motion matches the targeted graph. - Other students in the group.
Variables	- The shape of B's motion (easy or not to reproduce by physically moving). - B's motion represented by a graph and a simulated horizontal representation of the motion: non-verbal representation, the students are given no indication of the variables involved in the situation.

	- As explained earlier, the two ways of representing a motion: graphical representation and simulation of the motion help or hinder the understanding of the relationship. - The fact that a student from another group is walking a physical motion: the description of the motion must be precise enough so that another student who had not participated in the group work could understand the instructions.
Type of situation	- Situation of formulation
Means of validation	- Validation by the milieu: walking the described physical motion by a student
Classroom organization	- Students working in small groups (4 or 5 students). - Whole class discussion.
Student activity	(See Appendix 1 for detailed instructions to students) - Find out how to walk a physical motion to match a given motion as closely as possible. - Compare two motions.
Role of teacher	(See Appendix 2 for detailed instructions to the teacher) - Manipulate the tool. - Make sure that all groups understand the problem and get involved in its solution. - The teacher will ask the students to be precise when describing the motion.
Prerequisite(s)	- Being familiar with the tool. - An understanding of the concept of variable.

Expected Outcomes

Strategies and Answers: The students should be able to describe the meaning of the *x*- and *y*-axes and in doing this, identify the two variables involved in the representations of the motion: time and position. It is expected that most of the students will not have difficulty identifying time as a variable in this activity because they have already been taught the concept of variable by their teacher. However, it is expected that some students will confuse speed with position as the second variable. The expectation is that this misunderstanding will be corrected as the students discuss the motion and the meaning of the axes and reflect on their thinking. The students should be able to describe the motion as a relationship between the *x*- and *y*-axes. Perfect matches are not expected.

Instead, it is expected that students will explain where the graph is the same and where it is different and why. An example of a partially complete description would look like: "From 2 to 4 seconds, B's graph was on top of A's and B was ahead of A. Then, A caught B at 4 seconds, where they were both at 2 meters." It is also expected that the students will start using descriptions involving slope defined as rate of change to describe the motions (eg. faster, slower, steeper, etc.).

Three main difficulties are anticipated:

- Students may have difficulty describing the motion in the context of the functional relationship between dependent and independent variables involved.
- It is expected that some students may still have difficulty with the concept of position in relationship to the *CBR*. However, it is also anticipated that the group will be able to help those understand this concept through group discussion.
- Another anticipated difficulty with this question would be the cognitive obstacle seeing the graph as a literal picture. The group discussion should help students overcome this obstacle.

3.2.3 Activity 3 (TS1,A3)

Table 6. A priori analysis for Activity 3 of Teaching Sequence 1

Title	Activity 3
Description of the task	Given a graphical representation and a simulated horizontal representation of a motion, the students are asked to describe a motion with constraints (e.g., slower than the given motion, catches the given motion at the end). In order to satisfy the constraints, the students need to make relationships between the graphical representation and attributes of the motion (position, time, speed).
Goal	- To identify the two variables involved in the representation of the motion: time and position, and to describe the relationship between them.
Milieu	- Other students in the group during the group work.

	- Other students in the class during the class exchanges and discussion.
Variables	- B's motion represented by a graph and a simulated horizontal representation, but additional constraints on A's motion given verbally: the students need to interpret these constraints in terms of the graphical representation and they are thus forced to make relationships between the graphical representation and attributes of the motion (position, time, speed) - Constraints allowing for multiplicity of motions: validation is not straightforward - As explained earlier, there are two ways of representing a motion (graphical representation and horizontal representation of simulated motion), which either help or hinder the understanding of the relationship.
Type of situation	- Situation of formulation
Means of validation	- Validation by the milieu: a student walking the described physical motion
Classroom organization	- Students working in small groups (4 or 5 students). - Whole class discussion.
Student activity	(See Appendix 1 for detailed instructions to students) - Find out how to walk a physical motion satisfying several constraints. - Compare two motions.
Role of teacher	(See Appendix 2 for detailed instructions to the teacher) - Manipulate the tool. - Make sure that all groups understand the problem and get involved in its solution.
Prerequisite(s)	- Being familiar with the tool. - Have an idea of the relationship between physical motion in front of the *CBR* motion detector and its representation in the tool.

Expected outcome

Strategies and Answers: At the end of this situation, the students should understand what variables are represented in the x- and y-axes and what is the relationship between them. They will show this understanding by correctly articulating their ideas regarding position and time. For example, position away from the CBR is represented by y values and time elapsed is represented by x values (along the x-axis). It is expected that some students will begin coordinating between the horizontal representation of motion and graphical representation of motion in order to describe and validate their motion among the group. For example, it is easier to be precise about distance and direction when referring to the simulated horizontal representation in discussions. This is because position away from the CBR is clearly identified in close proximity to the representation of the motion on the screen.

Three main difficulties are anticipated:

- It is expected that some students will still have difficulty seeing the connection between the horizontal representation of motion and graphical representation of motion. Students may be more comfortable with one form of representation than another. For example, it is expected that most students will naturally lean towards graphical representations because they have already used such representations in their books. Although they may initially have difficulty relating two forms of representation, their familiarity with one should help make connections with the other.
- Students will be able to describe the motion in the context of the functional relationship between distance away from the *CBR* and time elapsed. They will use terms such as steeper and faster to describe the differences in the rate of change. Some students will describe the motion in general terms while others will try and be more specific referring to units and rates of change.
- The students may have difficulty to recognize that there are many different motions that satisfy the requirements of the problem. The main requirement is that A starts off slower than B and that they both end in a tie. A lot of different variations in the motion could happen in between these two events.

3.2.4 Activity 4 (TS1,A4)

Table 7. A priori analysis for Activity 4 of Teaching Sequence 1

Title	Activity 4
Description of the task	The students are asked to draw a graph representing a motion and give a description of the physical motion.
Goal	For the students: reinforce the acquired knowledge - The students will also be able to coordinate multiple representations of the same motion – the graphical representation and the simulated horizontal representation - The students will also develop a good understanding of the notion of dependence and independence. For the teacher: see what the students have learned
Milieu	- Other students in the group during the group work. - Other students in the class during the class exchanges and discussion.
Variables	- The shape of B's motion is to be chosen freely by the students: initial sketches of graphs that cannot represent a motion may occur. However, the constraint that the graph must represent a possible motion forces the students to check for the possibility which requires making connections between the graph and attributes of the motion.
Type of situation	- Situation of formulation
Means of validation	- Validation by the milieu: walking the described physical motion by a student
Classroom organization	- Students working in small groups (4 or 5 students).
Student activity	(See Appendix 1 for detailed instructions to students) - Find out how to walk a physical motion to match a given motion as closely as possible. - Compare two motions.
Role of teacher	(See Appendix 2 for detailed instructions to the teacher) - Manipulate the tool. - Make sure that all groups understand the problem and get involved in its solution.
Prerequisite(s)	- Being familiar with the tool. - Have an idea of the relationship between physical motion in front of the *CBR* motion detector and its representation in the tool.

Expected Outcomes

Strategies and Answers: Two main strategies can be anticipated: either the students will start by drawing a graph and will then invent the story fitting the graph and perhaps adjusting it, or they will start by inventing a story and then draw a graph representing the motion involved in the story. Both strategies require making relations between graph and physical motion. It is expected that the groups will begin by discussing motions that could be created by simple graphs ("graph → description of motion" strategy). However, it is also expected that students will try to invent creative stories ("description of motion → graph" strategy) and will challenge the other teams. It is expected that students will be able to reproduce the shape of letters while walking in front of the *CBR*. An interesting discussion should arise about why some letters cannot be reproduced while walking in front of the *CBR*. It is planned that the notion of dependence and independence will surface during this activity and that students will construct meaning regarding the relationship between the two variables involved in motion much like it did for Newton in the 1600s. Specifically, the students will directly experience the concept of independent variable. For example, a student physically trying to reproduce the final portion of the letter *P* (slanted) will understand and even perhaps "feel" that he is unable to make time go backwards.

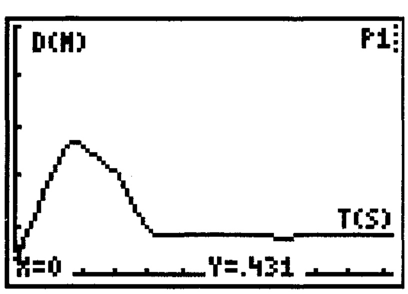

Figure 10. The letter *P* created using the *CBR*

This will be recognizable when the student moving towards the *CBR* slowly comes to a stop and starts leaning towards the *CBR* while watching the graphing representation continue to plot points further and further away from the *y*-axis. Another example would be attempts to reproduce the letter R. It would pose the same challenges as those found in the attempt to reproduce P. However, it would accentuate the impossibility of being in two places at once. This physical experience enables the mental activity for the student to construct the relationship between position and time, and to apply this knowledge directly in a milieu that allows him to get immediate feedback, reflect, adjust, and apply knowledge. By the end of this experience, it is expected that the students will articulate their understanding without being prompted by the teacher. Some students may even be creative about how to reproduce a letter which one would initially think is not possible to produce while walking in front of a *CBR* (i.e. not based on a functional relationship). For example, it is expected that students may show how to create a slanted ***J*** so that the motion respects the independence of time.

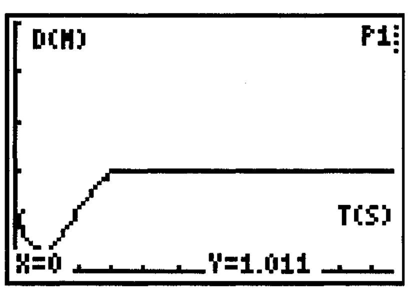

Figure 11. The letter J created using the *CBR*

Two main difficulties are anticipated:

- An anticipated difficulty associated with this question is that motivated students may want to try at all costs. Also, some students may still not completely understand the relationship between the distance away from the *CBR* and the time elapsed.
- As explained in the cognitive dimension of the preliminary analysis, it is also expected that students will have difficulty in developing a coherent conceptualization

of the physical models as a system of dependencies among quantities whose values vary. Therefore some students might initially struggle with the functional relationship between the dependent variable – distance away from the *CBR*, and the independent variable – time elapsed. Most of these anticipated difficulties should be overcome in the group discussions or in a class discussion.

3.3 Teaching sequence 2: Modeling and piecewise defined functions

This teaching sequence uses multiple representations systems in the didactic milieu to enable individual and aggregated mathematical constructions. The activities in this teaching sequence are designed to help students deepen their understanding about concepts and relationships by helping students connect new knowledge with prior knowledge, extend and apply the new mathematical knowledge, reflect on experiences, articulate what they know, and make the target knowledge their own through their interactions with the milieu. Specifically, students are challenged to coordinate the multiple representations to deepen their understanding of functional relationships and slope as rate of change in problems of motion. The representational strategies involved in such a didactic milieu as well as the effective use of technology will enhance the depth of learning about functional relationships and slope as rate of change.

The students will create and use piecewise-defined position vs. time graphs as a way of describing and controlling motion through individual mathematical performances and then aggregated functions. It is expected that in this teaching sequence, the students will:

- deepen their understanding of functional relationships as they coordinate multiple representation systems of the same functions (physical or simulation)
- deepen their understanding of the concept of variable and the notion of dependence between two variables.
- generate mathematics-based excitement as they deepen their understanding of slope as rate of change and functional relationships

This teaching sequence combines situations of institutionalization, action, formulation, and validation. In order to appropriately participate in the discussion, the students must have a strong understanding of the important components of a graph in the context of motion activities. Their prior experiences and teaching sequence 1 should provide appropriate prior knowledge. The students will need to be able to act out a motion based on a set of instructions.

The following table lists the four activities that comprise the second teaching sequence. The goal, type of activity, and classroom organization are outlined for each activity.

Table 8. Four activities comprising the second teaching sequence

	Goal	**Activity Type**	**Class Organization**
Activity 1 (TS2,A1)	- The students will familiarize themselves with the *MathWorlds* software on the TI-84 Plus. - The students will understand how motion could be represented by more than one form of representation and the differences between these representations. - A common framework and language for discussion of functional relationships will be institutionalized.	N/A	Class participation with demonstrations and discussions

Activity 2 (TS2,A2)	- The students will familiarize themselves with the graphical editing of piecewise defined functions through direct manipulation. - The students will enhance their understanding of the critical ideas of functions and slope as rate of change.	Formulation Validation	Group work with exchanges between groups and group discussion
Activity 3 (TS2,A3)	- The students will familiarize themselves with a new representational system that will be used through the rest of this teaching sequence. (This activity is essentially a continuation of the previous activity.)	Validation	Group/class discussion
Activity 4 (TS2,A4)	- The students will display an understanding of the concept of variable, and the notion of dependence between two variables in their descriptions. In this case, distance and time. The student will do this by creating and formulating a description of a motion given constraints in small groups. - The students will be challenged to coordinate multiple representation systems to deepen their understanding of functions as a relationship between dependent and independent variables and slope as rate of change. They will be able to explain how the created motion is similar and why some parts are different. - The students will be engaged personally with their mathematical work and students who might otherwise feel alienated from mathematics will be offered a chance to "perform" mathematically. - The students will be able to act out a motion based on the formulation of the motion given by a different group of students. - The students will also sharpen their focus modeling using Position vs. Time graphs.	Formulation Validation	Group work with exchanges between groups and group discussion

The activities that constitute this lesson do not have the same mathematical analyses, expected outcomes, or anticipated difficulties as teaching sequence 1. As a result, a detailed *a priori* analysis will be presented for each activity.

3.3.1 Activity 1 (TS2,A1)

Table 9. *A priori analysis for Activity 1 of Teaching Sequence 2*

Title	Getting Started
Description of the task	The students are asked to observe the simulation run by the teacher and answer a few questions.
Goal	The students will: - familiarize themselves with a new feature of the technology tool that will be used throughout the teaching sequence. - understand how motion could be represented by more than one form of representation and the differences between these representations. The new feature will focus the students' attention on the simulated horizontal representation. - be introduced to a common language for discussion of functional relationships.
Milieu	- *CBR* motion detector, *CBR* animator, stepping functionality and the marking functionality: graphical representation of the physical motion, simulation of the motion, ability to slow down the simulated horizontal representation and the graphical representation of the motion, and the ability to drop marks at regular time intervals during the simulated motion. - Other students' observations and conjectures.
Variables	- Two ways of representing a motion (graphical representation and simulated horizontal representations of the motion) help or hinder the understanding of the relationship - Ability to slow down the re-creation of the two representations of the motion helps the understanding of the relationship because it provides time for students to think about what is happening and to reflect on the representations or their created motion. - Ability to "drop marks" at regular time intervals during the simulated motion helps the understanding of the relationship. As the motion is simulated marks are "dropped" along the path at regular time intervals. Personalizing the tool in this way focuses attention on the simulated horizontal representation. It also lays the ground for interesting discussion on rate of change. For example, if marks are set to drop every second but the distance between drops grows, students will hopefully see that that means more distance was covered during those seconds. Personalizing

	the tool is such a way may also create difficulties for students if they choose not to experiment with this new feature by adjusting the rate at which marks are dropped. This may lead some students to believe that the rate of change is always the same. - Number of physical motions in front of *CBR*
Means of validation	- Validation by the milieu: walking in front of the motion detector and making adjustments to understand the relationship between student's movement and the graphical - Validation also occurs in the class discussion: the conjectures are validated or invalidated by peers, under the teacher's control.
Classroom organization	- Students working individually - Whole class discussion
Student activity	(See Appendix 1 for detailed instructions to students) - Observe how things slow down when stepping through the graph. - Observe what happens when marks are dropped at regular time interval. - Students will need to increase the level of analysis in order to answer the detailed questions from the teacher (i.e. **When is B going the *fastest*? When is B going the *slowest*? When does B seem to *change* speed? How far apart are they in the 3rd part of the trip? Exactly how fast is B moving during each part of the trip? Which part of the graph is the steepest and which part is the least steep?)**
Role of teacher	(See Appendix 2 for detailed instructions to the teacher) - Manipulate the tool by turning on the stepping feature and the dropping of marks feature. - Animate the discussion without giving any clues to the expected answer.
Prerequisite(s)	- Being familiar with the tool. - Have a good idea of the relationship between physical motion in front of the *CBR* motion detector and its representation by the tool. In order to successfully complete this activity, the students must have a strong understanding of the important components of a graph in the context of the motion activities. They must also understand how motion is represented in the context of both representations (horizontal and graphical). The students must also understand rate of change.

Expected Outcomes

Strategies and Answers: It is expected that the students will start using descriptions involving slope as rate of change in discussing these questions and in articulating their understanding. The new features being introduced in this activity (i.e. "stepping" and "dropping marks") will focus the students' attention on the simulated horizontal representation. Therefore, it is also expected that students will start utilizing information from both forms of representations of the motion to answer this question. By doing so, they will extract knowledge from newly formed relationships between the two forms of representation. For example, slowing down the motion by "stepping" through it is best represented on the graphical representation, while the "dropping of marks" is best contextualized on the horizontal representation of the motion where the marks are actually dropped every 1 second. It is also expected that students will reflect on their experiences with both tools and articulate their understanding to each other. At the end of this activity, the teacher will take a moment to summarize some of the highlights of the discussion, thereby establishing a common language for discussing functional relationships. For example, he will try to get the students to use the following terminology in their future conversations: dependent and independent variables, rate of change, etc. An example of expected correct behavior is the following: "B is going fastest in the middle section of the race. This is the section with the steepest slope with a rate of change of X meters/second. B is going slowest in the first section of the race where the rate of change is only Y meters/seconds compared to X meters/second in the middle section and Z meters/seconds in the last section. The first section is also the section that has the slope with the smallest incline."

Two main difficulties are anticipated:

- It is expected that students will need to adjust to the new level of precision made possible by the "stepping" and "dropping marks" as the motion is animated. This anticipated difficulty is desirable because it will encourage the students to refine the analyses of their observations. An anticipated difficulty is that some students may have trouble linking between the two representations of the motion. For example, they may have trouble understanding the significance of the marks (on the horizontal representation of the motion) in the context of the graphical representation of the motion.

- Another anticipated difficulty outlined in the cognitive dimension of the preliminary analysis is that students may have difficulty considering the actual graph to be the representation of the relationship that exists between the variables.

3.3.2 Activity 2 (TS2A2)

Table 10. A priori analysis for Activity 2 of Teaching Sequence 2

Title	Creating exciting sack races
Description of the task	The students are first asked to answer a few questions about attributes of a motion that is simulated and represented by a graph and a horizontal representation of the simulated motion. Next, they have to draw a graph of a motion described verbally. Finally, they have to sketch a graph and provide a description of a motion that can be represented by this graph.
Goal	The students will: - familiarize themselves with the graphical editing of piecewise defined functions through direct manipulation. - enhance their understanding of the critical ideas of functions and slope as rate of change. The core mathematical ideas being addressed and contextualized in this activity are the concepts of function as a relationship between distance and time and the qualitative idea of slope as rate of change where the rate in this case is velocity: positive vs. negative, steeper means faster (greater rate), zero slope means zero rate (zero velocity).
Milieu	- *CBR* motion detector, *CBR* animator, stepping functionality and the marking functionality, the ability to edit and directly manipulate piecewise defined functions: graphical representation of the physical motion and simulation of the motion. - Other students' observations and conjectures. - The role of the technology tool as a part of the milieu here is still to give the students an opportunity to examine the motions and graphs more closely and to help them deepen their understanding

	of functional relationships and slope as rate of change.
Variables	- Two ways of representing a motion (graphical representation and simulation of the motion) can help or hinder the understanding of the relationship. - As explained earlier, the ability to slow down the recreation of the two representations of the motion - help or hinders the understanding of the relationship. - The ability to drop marks at regular time intervals during the simulated motion helps or hinders the understanding of the relationship. - Ability to add piecewise defined functions to the graphical representation helps the understanding of the relationship because it is a means for students to articulate their understanding of the functional relationship through direct manipulation. - Ability to edit the existing graphical representation through direct manipulation (i.e. stretching a section increasing/decreasing of the graph and/or changing the slope) should help because it allows the students to make immediate changes to their original constructions based on their new understanding of the functional relationship involved.
Type of situation	- Situation of formulation: Requires the students to describe a motion so that another team could reproduce the same graph. - Situation of validation: The groups that receive the descriptions for motions described by another group will follow the instructions to validate the description.
Means of validation	- Validation by the milieu: walking in front of the motion detector following the verbal instructions and matching the graph.
Classroom organization	- Students working individually - Whole class discussion
Student activity	(See Appendix 1 for detailed instructions to students) - The students will be asked to reflect, discuss, and document a group response to a series of questions in the context of a race before collaborating to solve a problem. The series of questions are intended for students to exchange and compare observations regarding how the motions of A and B are different. - The students in their groups will then be challenged to reflect upon, discuss, sketch, and create a graph representing motion that would satisfy the new criteria. - The students in their groups are also instructed to be ready to explain their motion and have their animation assessed by another group.
Role of teacher	(See Appendix 2 for detailed instructions to the teacher) - Manipulate the tool by turning on the stepping feature and the dropping of marks feature. - Animate the discussion without giving any clues of the expected answer.

Prerequisite(s)	- Being familiar with the tool. - Have a good idea of the relationship between physical motion in front of the *CBR* motion detector and its representation in the tool. - In order to successfully complete this activity, the students understand that the horizontal axis represents time measured in seconds (Time), and the vertical axis and the motion represents distance measured in meters (Distance). They should understand that the objects both start at zero, which is indicated by the dashed tick mark where their respective right edges are in the animation.

Expected Outcomes

Strategies and Answers: It is expected that some of the groups will "step through" the motion when describing their motion to each other. This is because the motion will most likely go too quickly for most scripts. Such actions will help the students to better reflect on what is happening and to articulate their understanding more easily to each other. It is also expected that students will drop marks during the simulations to analyze the motion more carefully in terms of the script communicated to them from the other teams. It is expected that the groups will make their stories seem exciting. The mark of a good race is spontaneous applause when it ends! An example of a correct behavior is illustrated in Figure 12 below.

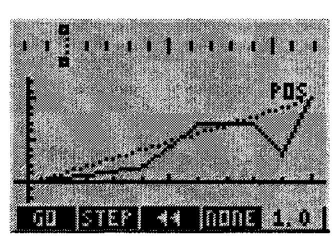

Figure 12. A student generated sack race

It is expected that the students in their groups will explore different ways of creating the graphical representations by directly manipulating the different segments of the graph. For example, students might add a segment and stretch it to the right by one second (so it is 2 seconds in Duration) and then upward until it is above A's graph by a decent amount. Students will have access to paper and pencil, but it is expected that most students will explore the creation of graphical representations by directly manipulating the different segments of the graph on the graphing calculator rather than on the graph paper provided to them. It is expected that some students will add segments and then adjust them so that they have zero slope. They may be slow to suggest this because it is a subtle idea. However, the students should be able to show this understanding of zero slope by applying the knowledge they learned earlier in satisfying this constraint. It is expected that students will discuss and experiment in their groups how to represent "going backwards" graphically. The correct behavior would be for students to explain that they need to add a new segment that must slope downward, or have negative slope. It is also expected that students will discuss how to represent "finishing in tie". The correct behavior would be for them to explain to each other that one more segment must be added and extended it to the right and upward as needed so that its right endpoint coincides with that of A's graph. The actual endpoint may be partially obscured by the label "**POS**" so it might take more than one try. By this time, it is also expected that the students can be autonomous while working in groups with the technology such that they can explore, make conjectures, verify them and then adjust their actions.

Main difficulty anticipated:

- Many of the criteria that the students must satisfy in creating their race will challenge their conceptual understanding of motion as a functional relationship between distance and time. As such, it is expected that some students may initially struggle with the misconception associated with interpreting or constructing a literal picture of a situation. This misconception was identified as an obstacle in the cognitive dimension of the preliminary analysis. Although extra time will be taken in the earlier activities to reduce the amount of misconceptions associated with graphical representations that students presently encounter, it is expected that the criteria given to the students in this activity may bring to the surface some of these misconceptions. For example, an anticipated difficulty associated with this activity is for some students to initially create a negative sloping segment going down to zero to represent the criterion – "Due to a wild burst of speed, B falls down for 2 seconds". The corresponding correct expected behavior would be for the other students in the group to discuss this criterion in the context of the race and in the context of the graphical representation of the motion. Also, some students will focus on the horizontal representation of the motion and quickly overcome the misconception. Through such group discussions and experimentation with both representations of the motion (graphical and horizontal), the misconceptions should quickly be resolved.

3.3.3 Activity 3 (TS2,A3)

Table 11. A priori analysis for Activity 3 of Teaching Sequence 2

Title	Find your exciting sack race
Description of the task	All the students' functions produced in activity 2 are collected by the teacher and displayed all together on the screen. The students are asked to predict the position of all slow B' and of all fast B', then to identify their own function.
Goal	- The students will familiarize themselves with a new representational system that will be used through the rest of this teaching sequence. (This activity is essentially a continuation of the previous activity.)
Milieu	- The *TI-84 Plus Silver Edition*, *CBR* motion detector, *CBR* animator.

- Graphical representation of the physical motion and simulated horizontal representation of the simulated motion.
- Stepping and marking functionality for analysis of graphical and horizontal representations.
- The ability to edit and directly manipulate piecewise defined functions.
- The TI Navigator classroom network for collecting and aggregating student races.
- Other students' observations and conjectures.
- The role of technology as a part of the milieu here is to challenge the students to coordinate multiple representation systems to "find" their individual constructions among the many representations in the shared space. They will need to use a deep understanding of the functional relationship involved in their graphical representation. The milieu here introduces students to the process of relating their personal constructions to the larger collection of objects that appears on the "big screen" when their work is aggregated with that of their peers. This process requires them to reflect upon and think through the kinds of issues that are at the heart of the mathematics we want them to learn.
- The milieu in this activity provides a safe environment by providing anonymity. For example, before asking the next question, the teacher will "hide" the identity of the functions and their "owners." Provision has been made to preserve student anonymity in the shared representation space – the teacher will click the box in the lower left corner of the screen where identifiers appear. Then no names will appear either here or when we hover over a dot or graph. The teacher will then ask: **Where are you? Can you find yourself?** If there is a position with a single dot, then a single student should be able to identify himself/herself. It can be confirmed by selecting it (by clicking on it) and then checking the box in the lower-left corner of the screen, where that student's identifier will appear.

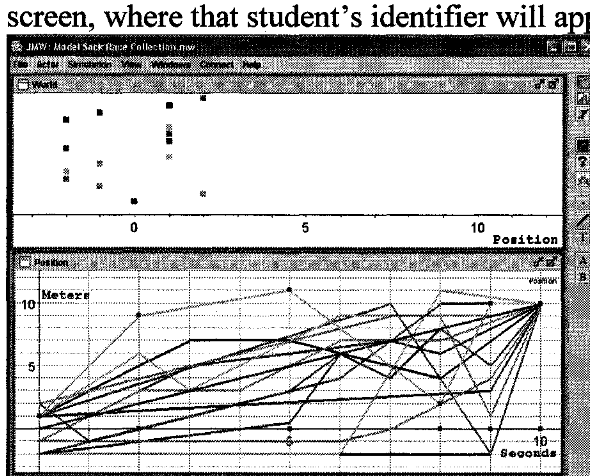

Figure 13. An aggregated view of multiple representations in *Connected*

	MathWorlds - It is usually fun to run the animation with all the dots showing (as illustrated in Figure 13 above). But the detailed analysis is best done with a smaller set of dots, which follows. - The milieu also provides for students identifiers and colors to facilitate the investigation of the connection between the motion and its graphical representation.
Variables	- Two ways of representing a motion on two different representation systems: graphical representation and simulation of the motion on the *TI-84 Plus Silver Edition* graphing calculator screen and in higher resolution and color using a data projector and *Connected MathWorlds* helps the understanding of the functional relationship. - The ability to slow down the recreation of the two representations of the motion helps the understanding of the relationship. - The ability to drop marks at regular time intervals during the simulated motion helps the understanding of the relationship. This should help because it provides the students with a tool to analyse rate of change during any specific time interval. - The ability to add piecewise defined functions to the graphical representation helps the understanding of the relationship. This should help because it is a means for students to articulate their understanding of the functional relationship through direct manipulation. - The ability to edit the existing graphical representation through direct manipulation (i.e. stretching a section increasing/decreasing of the graph and/or changing the slope) should help because it allows the students to make immediate changes to their original constructions based on their new understanding of the functional relationship involved. - The ability to display smaller set of motions to help the students identify the one they are looking for.
Type of situation	Situation of validation: The students are expected to make conjectures about the links between the motions (i.e. moving dots) and graphs and provide arguments to support these.
Means of validation	- Validation by the milieu: Students will try to find themselves on the new representation systems using the properties of their graphical representation. They will also walk in front of the motion detector following the verbal instructions and matching the graph - It is important for the teacher to let the students validate by the physically moving in front of the motion detector the description and the correspondence between the obtained graph and B's motion graph.
Classroom organization	- Students working individually. - Whole class discussion.

Student activity	(See Appendix 1 for detailed instructions to students) - The students are asked the following question by the teacher: **Where will all the Slow B's appear, and where will all the Fast B's appear?** - It is important that the teacher develop the habit of asking the students to predict before any display of action. Here the fast B's all appear to the left of 0 and the slow ones to the right, with those sharing the same initial position "stacked vertically." (Note that it is expected that the teacher will refer to the "students" and their "dots" interchangeably). - Next, the students are informed that the teacher will run the animation and that their job is to figure out which graph goes with which dot. Depending on how different the motions and graphs are, the teacher may need to Step through the motions. This is an important learning opportunity to examine subtle differences in the graphs and how they are reflected in differences in the motions, so the teacher should repeat the Stepping and encourage discussion till a consensus has developed.
Role of teacher	(See Appendix 2 for detailed instructions to the teacher) - Manipulate the *TI Navigator* classroom network and *Connected MathWorlds*. - Animate the discussion without giving any clues of the expected answer - The teacher will show "All" in the View Matrix World (Graph) column to show all of the students' dots. The milieu here introduces students to the process of relating their personal constructions to the larger collection of objects that appears on the "big screen" when their work is aggregated with that of their peers. This process requires them to think through the kinds of issues that are at the heart of the mathematics we want them to learn.
Prerequisite(s)	- Have a good idea of the relationship between physical motion in front of the *CBR* motion detector and its representations. - The students must have a strong understanding of the important components of a graph in the context of the motion activities. They must also understand how motion is represented in the context of both representations (graphical and simulated) and be familiar with the *MathWorlds* software on the device. The students must also understand rate of change.

Expected Outcomes

It is expected that the students will start identifying themselves with their objects by referring to those objects as "my dot". This is a desired behavior because it links the students psychologically to the mathematical object that they have built. This could be seen as evidence that students have developed a personal investment in building knowledge – another indicator of understanding. Because of differences such as size of screen, representation of the axes and intervals, resolution, and color, and the fact that several of the graphs looked very similar, it is expected that students will need to identify certain aspects of their mathematical objects on their graphing calculators and look for them in the public display. Putting the students in a situation where they will need to coordinate both systems in applying their mathematical knowledge regarding the same functional relationship in two representation systems should encourage the students to reflect more about the relationship, thereby increasing their understanding. Putting the students in this situation will also require them to be able to clearly articulate what they know about the critical ideas of functions and slope as rate of change when identifying themselves in the public display. The students whose graphs are <u>NOT</u> now displayed will also be able to determine which graph goes with which dot given their understanding of how the horizontal simulated representation and the graphical representation are linked. Relating students to their functions, and especially their motions to their graphs, is a powerful way of getting students engaged mathematically, it is also a place where the teachers' experience and knowledge of their students will directly come into play. The teacher will know who is likely to err, who is likely to be embarrassed, who enjoys attention, and so on. The teacher can also quickly review the student function graphs before making them public and not choose to display those that he feels would either be unproductive to examine or embarrassing to their creators. The technology amplifies the impacts of the teachers' pedagogical decisions.

3.3.4 Activity 4 (TS2,A4)

Table 12. A priori analysis for Activity 4 of Teaching Sequence 2

Title	Exciting races
Description of the task	The students are asked first to write a story describing their own race with **A** which ends in a tie, and to make a Position graph for **B** that makes their race happen. The stories are exchanged between groups and each group has to create a graph representing the story of another group.
Goal	The students will - display an understanding of the concept of variable, and the notion of dependence between two variables in their descriptions. In this case, distance and time. The student will do this by creating and formulating a description of a motion given constraints in small groups. - be challenged to coordinate multiple representation systems to deepen their understanding of functions as a relationship between dependent and independent variables and slope as rate of change. - be able to explain how the created motion is similar and validate why some parts are different. - be able to act out a motion based on the formulation of the motion given by a different group of students. - sharpen their focus modeling using Position vs. Time graphs.
Milieu	- The *TI-84 Plus Silver Edition*, *CBR* motion detector, *CBR* animator, and stepping functionality and the marking functionality, the ability to edit and directly manipulate piecewise defined functions: graphical representation of the physical motion and simulation of the motion. - The *TI Navigator* classroom network for collecting and aggregating student races. - Other students' observations and conjectures. - Stories written by other students. - The role of the technology tool as a part of the milieu here is to give the students an opportunity to examine the motions and graphs in two representations and to challenge the students to coordinate multiple representation systems (*MathWorlds* on the *TI-84Plus Silver Edition & Connected MathWorlds* on the *data projector*) to deepen their understanding of functional relationships and slope as rate of change.
Variables	- Two ways of representing a motion on two different representation systems: graphical representation and simulation of the motion on the *TI-84Plus Silver Edition* graphical calculator screen and in higher resolution and color using a data projector

	and *Connected MathWorlds* – helps the understanding of the relationship. - Scale: It is important to note that the scale has changed in this activity. - The ability to slow down the recreation of the two representations of the motion - helps or hinders the understanding of the relationship. - The ability to drop marks at regular time intervals during the simulated motion - helps or hinders the understanding of the relationship. - The ability to add piecewise defined functions to the graphical representation – helps or hinders the understanding of the relationship. - The ability to edit the existing graphical representation through direct manipulation (i.e. stretching a section (increasing/decreasing of the graph and/or changing the slope). - Instructions given to students in the form of questions created by other groups of students should help the understanding of the relationship because it gives students the opportunity to organize their understanding and articulate their ideas so that others can recreate the functional relationship.
Type of situation	- The second part of this activity is a situation of formulation. It then progresses to a situation of validation where the teacher will act as a chair of a scientific debate aiming at validating the students' answers only intervening to put some order in the debate among students. The teacher will also help draw attention to possible inconsistencies in student explanations and encourage more precision in the use of the vocabulary describing the motion.
Means of validation	- Validation by the milieu: Students will try to find themselves on the new representation systems using the properties of their graphical representation. - To validate the answer of B, the two graphs, both drawn on a paper, are compared by both groups A and B. If they are considerably different, the students have to find the reasons of the differences: the message of A is wrong? The interpretation of B is wrong? If there is no agreement between A and B, a physical moving in front of the motion detector will be used to validate.
Classroom organization	- Students working in small groups - Whole class discussion
Student activity	(See Appendix 1 for detailed instructions to students) - The students are asked **to write an exciting race story-script for their own race with A which ends in a tie, <u>and</u> to make a Position graph for B that makes their race happen.** - They are also instructed that the teacher will then collect their graph and run it for the whole class to see while they read their exciting story! They are asked to describe each segment of their

	graph. - A group (X) draws on a paper a graph of a motion another group (Y) will have to reproduce. Group X writes a description of the motion and gives this message to the group Y. The group Y tries to figure out the graph corresponding to the given motion. - Next, the students are informed that the teacher will run the animation and that their job is to figure out which graph goes with which dot. Depending on how different the motions and graphs are, the teacher may need to Step through the motions – use a Step-Time of 1 second (set by opening the bottom part of the Controls Window). This is an important learning opportunity to examine subtle differences in the graphs and how they are reflected in differences in the motions, so the teacher should repeat the Stepping and encourage discussion till a consensus has developed.
Role of teacher	(See Appendix 2 for detailed instructions to the teacher) - Manipulate the *TI Navigator* classroom network and *Connected MathWorlds*. - Animate the discussion without giving any clues of the expected answer.
Prerequisite(s)	- Same as the previous activity.

Expected Outcomes

- It is expected that the students will be autonomous while working in groups with the technology such that they can explore, make conjectures, verify them and then adjust their actions.
- It is expected that the groups will make their stories seem very exciting and difficult for members of other groups to recognize.
- It is expected that students will be able to formulate their motions correctly so that another team will be able to re-create the motion which meets the constraints of the story. This is important because in order to articulate their ideas, they must first reflect on them in order to identify and described critical elements. The ability to communicate or articulate one's ideas is a benchmark of understanding (Carpenter & Lehrer, 1999).

- It is expected that students will be able to recognize the graphical representations of their motion on the large projected screen. Most students will make the connection right away, while others might still need to see the simulation before grounding their motion in the new representation system offered by *MathWorlds* for *TI-Navigator*.
- It is expected that the students who successfully coordinate between their personal creation on the device and the more refined version of the graphical representation offered by the computer version of *MathWorlds* will have a deeper understanding of functions as a relationship between dependent and independent variables and slope as rate of change. Relating back to the cognitive dimension of the preliminary analysis, where understanding was characterized in terms of mental activity that contributes to the development of understanding, a deep understanding of functional relationships and slope means the following for this activity. First, this activity forces the students to construct relationships between the two representational systems in order to "find themselves". It also requires students to extend and apply their mathematical knowledge when comprising the two representational systems. It is expected that students initially might have difficulty articulating their ideas for selecting their graphical representation, however by struggling to articulate their rational, students develop the ability to reflect on and articulate their thinking. Therefore, another important characteristic of students' developing understanding reinforced by this activity is that they become increasingly able to reflect on their thinking. The ability for students to communicate their motions so that others may actually act out the targeted motion will also be seen as an important indicator of understanding.

- It is expected that students will enjoy learning about the functional relationship involved in motion in this way. This is important because understanding involves the construction of knowledge by individuals through their own activities so that they develop a personal investment in building knowledge.

3.4 Teaching sequence 3: Summarizing individual and group understanding

Recall that the goal of this sequence is to provide the research team with the opportunity to ask explicit questions to the students highlighting the conceptions of functions students have developed through the first two teaching sequences. As most of the activities are going to be worked in groups, we wish to assess each individual student's understanding of functions. The goal of this sequence is also to see to what extent the students are able to transfer the knowledge acquired in technology-based environment to the traditional paper and pencil environment. The outcomes of this sequence should give a good picture of the students' individual and group understanding of the concepts explored in the first two lessons.

The students will individually complete each activity and then take part in an open teacher-mediated discussion with the rest of the class to validate, defend, and/or refine their solutions. It is important for the teacher to focus on organizing a discussion around the correctness of students' descriptions rather than trying to simply get the correct desired response. It is expected that in this sequence, the students will demonstrate the understanding of the following concepts:

- Functional relationship between distance and time in problems of motion.

- The concept of variable and variability, and the notion of dependence between two variables.
- Rate of change and slope as rate of change.

It is expected that students will demonstrate this understanding by applying newly acquired knowledge to solve new problems of motion and by constructing relationships with past experiences. They will also demonstrate understanding by being able to clearly communicate their ideas on paper and in a public form.

The following table lists the three activities that comprise the third teaching sequence. The goal and classroom organization is outlined for each activity.

Table 13. Three activities comprising the third lesson

	Goal	**Class Organization**
Activity 1 (TS3,A1)	To determine if students have a good understanding of the functional relationship between distance and time in problems of motion and solid understanding of independent and dependent variables.	Individual work, followed by class participation with demonstrations and discussions
Activity 2 (TS3,A2)	To determine if students are able to transfer what they have learned about the functional relationship in problems of motion involving the motion detector to more general problems. Another goal is to determine if the students deepened their understanding of slope as rate of change.	Group work with exchanges between groups and group discussion
Activity 3 (TS3,A4)	To determine if students have a good understanding of the functional relationship between distance and time in problems of motion; a deep understanding of slope; and the ability to obtain important information by analyzing graphical representations of functional relationships between distance and time in a motion problem of a different context.	Group/class discussion

This sequence is different from the other two teaching sequences. The activities all contain a phase of individual work followed up with a phase of group discussion. The *milieu* is void of any use of technology. Instead they make use of the traditional tools like paper, pencils, chalk and blackboard.

3.4.1 Activity 1 (TS3,A1)

In this activity the students are presented with a story and are asked to select the appropriate graphical representation from four different graphs. The goal of this activity is to determine if the students have a good understanding of the functional relationship between distance and time in problems of motion and solid understanding of independent and dependent variables.

See Appendix 1 and 2 for detailed instructions to the teacher and the student respectively.

Expected Outcomes

- Strategies and Answers: It is expected that the students will select the correct graphical representation for this problem (d). The other three incorrect graphical representations were chosen as distracters because they each represent different possible literal interpretations of the problem – misconceptions that this experiment was designed to help students overcome.
- It is also expected that interesting discussions will arise when students are asked why the other representations were not selected. It is expected that the students will recognize that these graphical representations do not represent functional relationships. For example, students may comment that the graphs show time going backwards (b and c) or a person being at more than one location at one point in time

(a). Being able to communicate why the distracter graphs are not appropriate to represent the given situation is also an important indicator of understanding.

Two main difficulties are anticipated:

- It is anticipated that some students will have difficulty explaining their rationale for not choosing the other graphical representations for their choices. Here it will be important to engage the class in being specific about why the other representations were not appropriate. This is another important aspect of understanding. By struggling to communicate their ideas, students develop the ability to reflect on and articulate their own thinking. Articulation in this sense can be considered to be a public form of reflection.
- Some students may still have difficulties overcoming a literal representation of the graph. Classroom discussion should help those troubled students overcome this obstacle. However, as it was noted in the cognitive dimension of the preliminary analysis, this type of obstacle creates significant difficulties in conveying the connections between graphs and functions to students. As a result, should this obstacle persist in some students, more one-to-one remediation with these students will be required with those students following this research project.

<u>3.4.2 Activity 2 (TS3,A2)</u>

In this activity the students are asked to create a graphical representation of a motion given descriptions for specific segments. One goal for this activity is to determine if the students are able to transfer what they have learned about the functional relationship in problems of motion involving the motion detector to more general problems. Another goal is to determine if the students deepened their understanding of slope as rate of change.

Reflection plays an important role in the solving of unfamiliar problems and problem solving often involves consciously examining the relation between one's existing knowledge and the conditions of a problem situation. As such, the empty graph that the students are given for this activity has the following conditions. The x-axis is divided into fifteen intervals to represent time (no mention of units) and the y-axis is divided into seven intervals to represent distance (no mention of units).

There are three important didactic variables that are outlined below: (1) a very rough description of motion leaving a place for multiple correct graphs, which can destabilize students. The students are actually used to have a unique solution to a given problem, and exact based on the givens of the problem. It is not the case here. (2) the students need to analyze the situation in order to construct a correct representation of the motion: e.g., same pace means that the line segments representing these portions of the motion are parallel (this is something that was not explicitly addressed in the teaching sequences), (3) the grid is pre-constructed: the students must decide what a unit represents (e.g., if a unit represents 1 second, the whole journey would take only 15 seconds which is quite unlikely in reality, if a unit represents 1 minute, then it would be difficult to represent 1 second break).

See Appendix 1 and 2 for detailed instructions to the student and the teacher respectively.

Expected Outcomes

- Strategies and Answers: There are many possible graphical representations students could use to represent this problem of motion. However, the expected correct behavior is for students to use their understanding of functional relationships in problems of motion to correctly represent the given segments graphically. This would include the ability to effectively represent slope (positive, negative and zero) with different rates of change. An example of a correct behavior is illustrated in Figure 14 below.

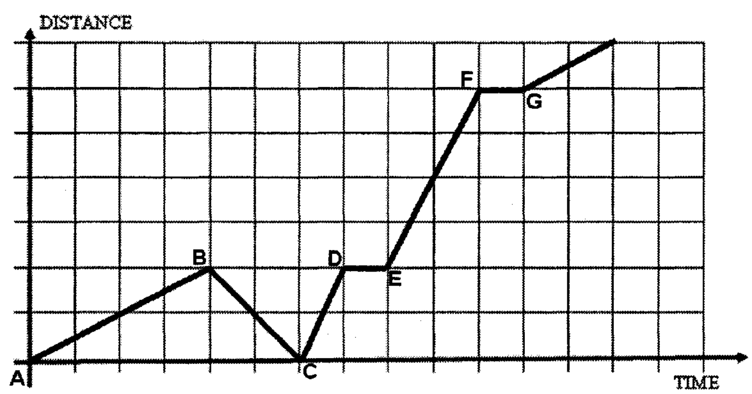

Figure 14. Example of a correct behavior for activity one of lesson three

Two main difficulties are anticipated:

- It is anticipated that some students may have difficulties with the relative nature of this activity (e.g. walking slowly, at a fast pace, etc.). It is also expected that many students will need to start their graph over several times in order to make effective use of the graph paper provided. For example, some students will get through several criteria and then observe that they will not have enough room on the provided graph paper to complete the problem. The constraints provided by the graph paper will push the students to focus on the important properties of the graphical representation.

It is also anticipated that some students will have difficulty with the units as they are not specified.

- An anticipated strategy is for some students to resort back to literal representations of segments of graphs when unsure of how to represent the trip correctly.

3.4.3 Activity 3 (TS3,A3)

This activity is similar to the first two in that its goals are: to determine if students have a good understanding of the functional relationship between distance and time in problems of motion; a deep understanding of slope; and the ability to obtain important information by analyzing graphical representations of functional relationships between distance and time in a motion problem of a different context. This activity requires the students to focus on reading graphs and providing accurate descriptions of the attributes of the motions represented.

See Appendix 1 and 2 for detailed instructions to the teacher and the student respectively.

Expected Outcomes

- Strategies and Answers: It is expected that almost all students will correctly respond to the first three questions. **A** starts at 0 km, **A** finishes his trip after 14 minutes, and **B** finishes his trip at 5 km. The expected correct behavior for question 4 is for students to answer **A** and by providing the following rationale. The students should explain that the slope is steeper and/or that **A** has a greater rate of change. The students might even provide the actual rates of change for each graph. They can also say that **A** rides less time and goes further than **B**. The expected correct behavior for question 5 is for students to identify **B** as the biker that has traveled farther after 5 minutes. It is expected that the students will focus on the dependent variable **distance** in answering this question. They will determine which graph has the greatest difference in distance over the given time frame. It is expected that almost all students will correctly answer questions 6 and 7. However, in question 7, the students will have to look at the intersection point and read the coordinates. The expected correct behavior for question 8 is for students to identify **A** as the biker that traveled the greater distance over its entire trip.

Three main difficulties are anticipated:

- One anticipated difficulty is for students to make the false assumption that the bikers started the trip from the same location. This would result in wrong answers to many of the questions. However, it is also expected that after spending more time reflecting on the question, even the students experiencing difficulty with the question, will display the correct behavior. It is also expected that some students may simply confuse **A** for **B** in answering some or all of the questions in this activity. This is not so much of a concern and is easy to recognize. Also students who make this mistake will still provide data that will help us validate the goals of this activity.
- Some students may still confuse the x and y-axes.
- It is also expected that some students may have difficulty reading the graph and interpreting the point of intersection.

3.5 Summary

Chapter 3 has described the design and *a priori* analysis of the didactic situations that was conducted as part of didactic engineering before conducting the experiment. It will be seen that the design and analysis involve a considerable amount of detailed thought and planning before entering the classroom. This detail allows difficulties to be anticipated and focuses attention on learning goals to be evaluated.

Chapter 4 Experimentation

This Chapter will describe the realization and the analysis of the teaching sequences and the last day assessment. The experiment took place in a grade 9 classroom in a small rural school in Eastern Massachusetts in December 2004. The quiet New England communities that feed into the local high school are generally working class with many of the families having ties to the fishing industry. Over 50% of the students in the high school are from low-income families as defined by the No Child Left Behind Act (NCLB; U.S. Public Law No. 107-110, 2002). Over 20% of the students had a first language that was not English and over 16% of the students were on Individualized Education Plans. According to United States federal law, adequate yearly progress (AYP) is defined as a measure of the extent to which students in a school demonstrate proficiency in English language arts and mathematics (Fusarelli, 2004). Based on an analysis of the performance and improvement this school demonstrated toward achieving AYP, it has been identified as being in need of improvement in each of the last 6 years.

4.1 Classroom setting

Twenty-three students participated in the experiment and most were already familiar with the graphing calculator. The classroom teacher was interviewed before the experiment and identified a wide range of understandings and capabilities among these students in this classroom. While many of the students were well engaged in the learning process and had a good track record of achievement, others had more difficulties. For example, nine of the 23 students were either failing or had unsatisfactory standing in the course. Two of these students had individual education plans and two others had been recommended by other teachers for evaluation. The school was aware that four students in the class were on medication for some form of attention deficit disorder. One of these students was also one of the two students with an individual education plan. Two of the students who were failing in the course were also documented as having severe behavior problems. Two days before the start of this experiment, a student in this class had been expelled from the school due to several behavior problems and the inability to follow his individual education plan. Thus the interview with the teacher before the experiment gave visibility to a group of students in the class that had a range of abilities and aptitudes and were quite representative of high school students. That is, they were not specially selected for the study on any criteria.

The mathematics teacher who delivered the teaching sequence was very familiar with the technology being introduced and had a good understanding of the concept of function. The teacher was also part of several *SimCalc* research projects using similar technologies conducted by the University of Massachusetts in Dartmouth, Massachusetts. As a result, the teacher was comfortable having a research team in his classroom.

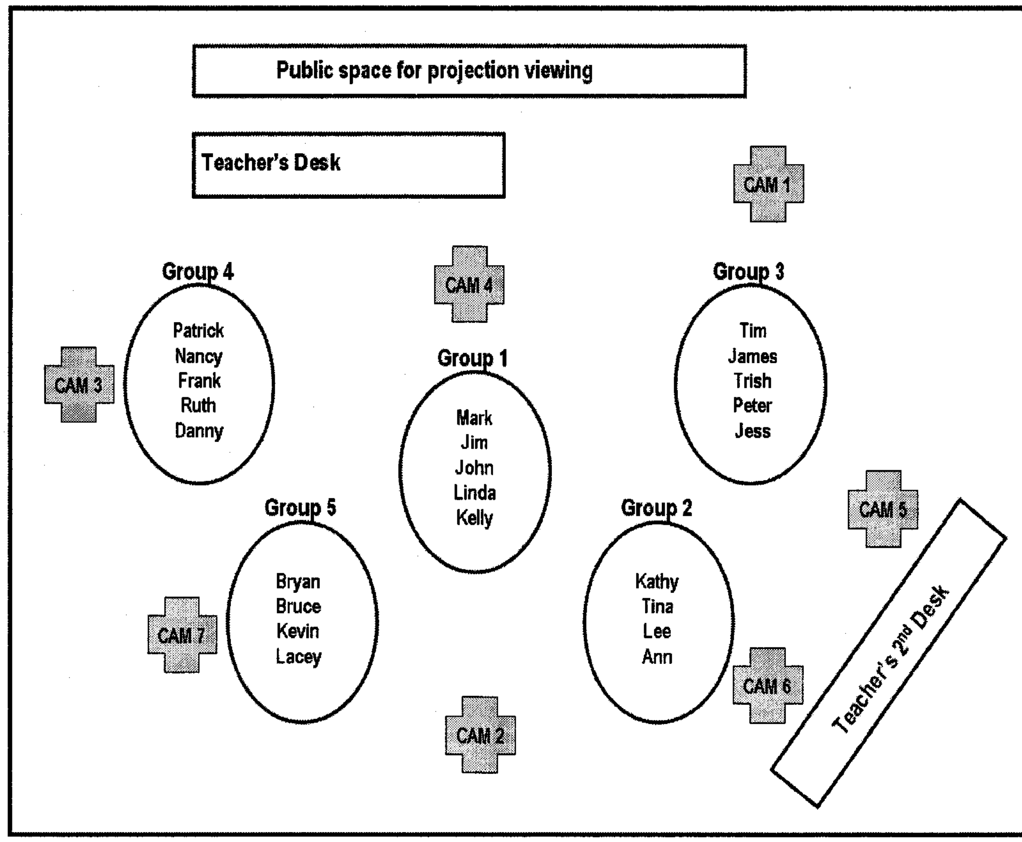

Figure 15. Organization of the class

The teaching sequence was implemented over a 5-day period as planned. The goal of the first day was essentially to familiarize the students with the new additions to their regular learning environments, namely the research team and the seven video cameras to be present in their classroom. The actual teaching sequence was implemented during the second, third, and fourth days. The research team had the opportunity to ask explicit questions of the students during the last day for additional validation of their responses against predictions made in the *a priori* analysis of the design.

For this teaching sequence, the class was divided into five groups of 4-5 students each, as shown in figure 15. The students are identified by pseudonyms. Each class was 83 minutes long, the norm in this particular school. The data presented below were obtained by videotapes recorded by five stationary video cameras (CAM 3, 4, 5, 6, 7), two mobile video cameras (CAM 1, 2), my field notes and those provided by four research assistants who assisted in the filming of the experiment. The principal data source was video of discussions and interactions between students and of discussions between the teacher and the students. Following each class, a short debriefing session was conducted to verbally exchange observations between myself, the teacher, and the research assistants. These debriefing sessions were also videotaped. In addition, data were obtained from student-written responses to worksheets (see Appendix 1) and from student-created functions in the form of *Connect MathWorlds* files.

The *TI-84Plus Silver Edition* graphing calculator with *MathWorlds* and *Connected MathWorlds* software loaded on the teacher's desktop computer were used throughout the experiment. The *CBR*™ motion detector in combination with the *TI-84Plus Silver Edition* graphing calculator overhead set-up was also used. The *TI-Navigator* classroom network provided the classroom connectivity between the students' graphing calculators and the teacher's desktop computer, and a data projector was used to project a public display. The research assistants also provided technical assistance when the need arose. For example, some of the graphing calculators needed new batteries and some of the *TI-Navigator* hubs needed to be re-set. The research assistants quietly reminded students to document their stories and graphs on their worksheets so that this physical data would also be available for analysis.

4.2 *A posteriori* analysis and validation

The *a posteriori* analysis has similar structure for each teaching sequence. The most significant findings from the teaching sequence, descriptions of important discussions and interactions that took place during the activities in small groups and the interventions of the teacher will all be presented. Extracts of student discussions, student created artifacts and teacher interventions will also be provided to highlight significant findings. For validation purposes, each teaching sequence will also include the highlighting of relevant information with links back to the *a priori* analysis presented in Chapter 3.

Recall that the following hypotheses were made in the theoretical framework outlined in Chapter 2:

Hypothesis 1 Individual mathematical constructions that are directly experienced in a "live" context, have immediate kinesthetic, cognitive and linguistic aspects that will help students develop an understanding of the relationship between distance and time in problems of motion.

Hypothesis 2 Individual mathematical constructions in a "live" context facilitate the development of understanding of independent and dependent variables.

Hypothesis 3 Multiple linked representations of the same function in a simulated environment allowing for manipulation by the students improves their learning about rate of change.

Hypothesis 4 Aggregated mathematical constructions challenge students to coordinate multiple representations and deepen their understanding of functional relationships.

4.3 Teaching sequence 1: Exploring physical motion

The first teaching sequence involved the *CBR* motion detector and was used to provide physical grounding for the simulation-based activities intended to engage the students in learning more about functions.

4.3.1 Activity 1 (ST1,A1)

The goal of this activity was for students to familiarize themselves with the technology tool that would be used over the teaching sequence and allow them to find out the relationship between the physical motion in front of the *CBR* motion detector and the graph being displayed.

There was a lot of discussion where students articulated conjectures and referred to the milieu for validation. As was anticipated in the *a priori* analysis and consistent with the historical development of the concept of function, the students relied heavily on verbal descriptions of the motion or its graphical representation to describe the functional relationship. The students engaged in rich discussions to help each other overcome some of the anticipated difficulties. For example, after some free exploration with one of the students (Kelly) walking in front of the motion detector, the students were able to contextualize the x and y-axes and agree on where the best starting point would be (in terms of distance away for the *CBR* motion detector). Coming to the realization that the x-axis represents time and the y-axis represents distance away from the *CBR* as described in the *a priori* analysis, is an early indicator that students are beginning to understand the relationship between distance and time in problems of motion. Also, the students demonstrated some understanding of the notion of dependence when they realized that the height of the graph directly depended on the students' location/position in front of the *CBR*. The milieu which gives the students real-time feedback helps the students validate or invalidate their conjectures. For example, in the extract below, Kelly's realization that she went too far and Jim's conjecture regarding the starting point, was validated by the milieu.

All of the extracts use the following code in order to situate them in the experiment. The first two characters identify the teaching sequence, the second two characters identify the activity, and the last two characters number the extract. For example, S1A1E1 identifies the first extract (E1) of the first activity (A1) in Teaching Sequence 1 (S1).

<u>S1A1E1</u> [extract of a discussion regarding the initial location of the volunteer]

Kathy: Look, Kelly is controlling the **A**. You (referring to Kelly) need to start a little further away. You're too close.
Kelly: Where do you want me to start? Here?
Kathy: A little further… Yes, that should be good.
Kelly: See, now I am too far (observing after the *CBR* started)
Kathy: Yes, but move forward now and get on the graph (matching Kelly's **A** motion with the animated **B** motion).
Kelly: Check it out! I'm right on, oops! There you go – back on.
Teacher: OK, now where should Kelly stand so that he is exactly where he needs to be at the beginning of the motion?
Jim: About 3 feet.
Teacher: Why do you say that?
Jim: There are 6 little tics marks on the bottom of the screen.
Kathy: They're not feet, they are meters.
Jim: Ok, then 3 meters.
Kelly: Let me try it again.
Jim: (After the start of the *CBR*) I was right. He was right on.

The effects on the representations of the forward and backward motions in front of the *CBR* motion detector were not obvious to everyone at first. Initially, there were opposing views on what would happen if one were to move towards the *CBR* motion detector. The majority of the students were correct in conjecturing that the graph would "go downwards" as a person moved towards the *CBR* motion detector and it would "go upwards" as a person walked away from it. However, one student, Mark, said that he believed that the graph "would go upwards" if a person walked towards the *CBR*. This temporary confusion was anticipated and described in the *a priori* analysis and was quickly resolved. After a volunteer actually walked the motion, Mark quickly saw that this initial conjecture was wrong. This was a good example of a milieu invalidating a conjecture. Mark needed evidence to be certain that the opposite was true. He did this by asking the volunteering student to move away from the *CBR* motion detector and commented:

S1A1E2

Teacher: So, what do you think will happen when Kelly moves towards the *CBR*?
Mark: The graph would go up.
Teacher: Kelly, give it a try. Move towards the *CBR*.
Mark: Oh, it's going down.
Kelly: It gets lower and lower as I get closer to the *CBR*.
Mark: Move back now (meaning away from the *CBR*). OK, I see it.

From this brief exchange and the visible immediate feedback from the *CBR*, Mark was able to examine his earlier thoughts and the conditions of the situation and was able to start understanding the relationship between position in front of the *CBR* and its representation on the *y*-axis. The students were also able to verbalize the effect of speed on the representations of the motion.

S1A1E3

Tim: You're not moving fast enough at the end.
Jess: I could do it.
Teacher: OK, Jess, explain what you will be doing.
Jess: I would start over here (approximately 0.5 meters away from the *CBR*), move

towards the *CBR* for a little bit and then sprint out.
Teacher: Does anyone have other suggestions or concerns.
Kelly: Ya, when Jess moves towards the *CBR*, he needs to do it slowly.
Jason: Mr. Y, you better open the classroom door so that he doesn't crash into it. He's going to be going very fast at the end.

Only one student near the end of this activity made reference to the horizontal representation of the simulated motion in describing the actual motion. This may be explained by the fact that the students were not familiar with this type of representation for motion. However, after a student's short explanation of the representation, other students seemed to pay more attention to it and use it in subsequent activities.

This activity allowed students to familiarize themselves with the technology tool. It was successful in having the students discover the relationship between physical motion in front of the *CBR* and the graph being analyzed. The technology was valuable because it gave the students immediate and direct feedback allowing them to reflect on their thinking and examine their thoughts. It also provided them with a means of validation for their conjectures.

4.3.2 Activity 2 (TS1,A2)

The goal of this activity was for students to identify the two variables involved in the representations of the motion: time and position.

This activity showed some of the expected difficulties outlined in the *a priori* analysis. These are addressed below. The teacher also made a judgment to add additional instructions which were somewhat counter to the design of the activity. In order for students to get to the level of detail targeted by this activity, the teacher asked students to be more precise. The teacher took it upon himself to make this specific intervention.

S1A2E1 [extract of unplanned teacher intervention]

Teacher: While you think about how you would walk a motion so that A matches B's motion, think about where you would need to be standing when you start, the direction you would need to walk, where you would need to change directions, etc.

It was discovered after the class that the teacher's reason for this intervention was to increase the likelihood that the research team would collect valuable data. This intervention was not necessary because the students would have realized they needed to be more specific. This situation was designed so that students would naturally increase the level of detail in their exchanges because the feedback they were getting from the milieu (other students and the technology in this case) required them to do so.

In their groups, the students also started discussing the meaning of the *x* and *y*-axes. All groups came to understand that the x-axis represented time and that the graduations along the *x*-axis represented seconds. It took longer for students to be able to articulate what was actually represented by the *y*-axis. After 3 minutes into this activity, two of the five groups were accurately describing the meaning of the *y*-axis as a representation of their position in front of the *CBR* motion detector. After 7 minutes into the activity, all but one of the groups was actually describing the meaning of the *y*-axis. The one group that was not able to do so focused on describing the speed and direction of motion over time without providing details regarding the position in terms of specific distances away from the *CBR* motion detector. It turned out that in their group discussions, this group focused more on the horizontal simulated representation of the motion which simply does not make use of the *y*-axis. Following this situation of formulation, the students in this group understood the need to be more specific with regards to position as it relates to the distance away from the *CBR* motion detector. For example, the group quickly realized that they needed to provide the volunteer with more details in order for him to match the motion accurately and they were able to do this using information from the horizontal representation. Therefore, even with the unplanned teacher intervention, the students still saw the need to increase the level of detail in their exchanges because of the feedback they were getting from the milieu. The majority of the students were able to identify time as a variable in the representation of the motion, however as anticipated in the *a priori* analysis, some students said that speed was the second variable in the motion. This misunderstanding of the representation of the motion was corrected as the students discussed the motion in the context of the *x*- and *y*-axes.

<u>S1A2E2</u> [extract of a discussion in Group 3 regarding the two variables involved in the representation]

Trish: Well time is a variable.
Tim: Right, and speed is the other variable.
Jess: No way. Distance is other variable.
Tim: Speed affects the steepness of the lines, not distance.
Trish: Look, distance is the other variable. The steep sections just say that you move more during a certain time.

By the end of the activity, the students were successful in identifying the two variables involved in the representation of the motion. Although the unplanned intervention by the teacher did take away for the planned interactions between the students and milieu, the students still learned from the milieu that they needed to articulate more detailed instructions to the person walking the motion.

4.3.3 Activity 3 (TS1,A3)

The goal of this activity was for students to identify the two variables involved in the representations of the motion: time and position, and the relationship between them.

The students still relied heavily on verbal descriptions of the motions and their representations. The teacher and the research assistant often had to remind the groups to write their descriptions on their worksheets. As anticipated, the groups quickly recognized that there are many different motions that satisfy the requirements of the problem.

In this activity, the students were beginning to show a better understanding of the meaning of the x- and y-axes and the variables that they represent. For example, in the conversation among the students in Group 3 in the extract below, the students are making reference to position and time and the relationship between them while describing the motions. Whereas Tim initially struggled with the definition of distance as the dependent variable above, Jess and Tim now show an understanding of how speed as rate of change is represented graphically. As explained earlier, the ability to communicate one's ideas is a benchmark of understanding.

S1A3E1 [a continuation of extract S1A2E2]

Tim:	Yes, but the steep parts show speed.
Jess:	You are saying the same thing as she is. The more you move during a certain time, the steeper the line. That's speed.
Tim:	That's what I said.
Trish:	No, you didn't
Tim:	It's what I meant to say.

Many more of the students started using the horizontal representation of the simulated motions in their discussions while describing the motion. As anticipated in the *a priori* analysis, many of the students used the horizontal representation to obtain detailed information regarding the distance and direction. This was because the distance intervals in the horizontal representation were clearly identified in close proximity to the graphical representation of the motion. It also made it easier for students to identify where both **A** and **B** were at the same position. This can be seen in the following extracts:

<u>S1A3E2</u> [extract of a discussion using the horizontal representation in Group 5 regarding the two variables involved in the representation]

Bruce: We could not move for 4 seconds and run until the dots are on top of each other.
Kevin: How do you fix it so that the dots are on top of each other at exactly 6 seconds?
Bryan: I could do it.
Lacey: So could I.

<u>S1A3E3</u> [extract of a discussion using the graphical representation in Group 2 regarding the two variables involved in the representation]

Lin: Look, all we need to do is start off slowly and then run fast to get on top of the line.
Ann: Where do we start? Oh, we start at the same place and as long as we walk really slowly, we will stay under the line. But we need to be at the same place after 6 seconds.
Lin: 6 seconds is just 6 tick marks away.
Kathy: So as long as we get there in time, we are good.

Also anticipated in the *a priori* analysis, many of the students were able to describe the motion in the context of the functional relationship between position (distance away from the *CBR* motion detector) and time elapsed. Most of the students used terms such as steeper and faster to describe the differences in the rate of change. For example, this can be seen in the extract below:

<u>S1A3E4</u> [extract of a discussion describing motion as a functional relationship between

position and time]

Frank: Let's start off by going really slow. Then we could sprint to the finish line and beat them.
Ruth: Right, but we don't want to beat them. We need to finish in a tie.
Frank: OK, why don't we start slow and go really fast and almost catch up. Wait a couple seconds and then floor it?
Patrick: So what would that look like?
Ruth: Almost a flat line at the beginning, then a really steep line going up, then a flat line, then a steep line going up and hitting the other line at the end.

Fewer students were able to provide a more detailed description of the motion by providing more specific information referring to units and rates of change. 2 of the 5 groups provided this level of information in their descriptions. By the end of this activity, the students were easily able to identify the two variables involved in the representation of motion. Most of the students were also able to show an understanding of the relationship between these two variables through their ability to correctly articulate the relationship between distance and time.

4.3.4 Activity 4 (S1A4)

The goal of this activity was for students to reinforce the acquired knowledge and to be able to coordinate multiple representations of the same motion – the graphical representation and the simulated horizontal motion. Another goal was for students to develop a good understanding of the notion of dependence and independence.

As expected, the students certainly came up with very interesting stories. In fact, many of the initial brainstorms from the groups were more centered on finding the craziest story. It took several minutes for students to ground themselves in reality. Once they did this, the activity really took off as expected. It was expected that the students would try to invent creative stories and would challenge the other teams. They certainly succeeded. Below is an example of discussion that took place in the initial brainstorm.

S1A4E1 [extract of a discussion in Group 4 in brainstorm mode]

Danny: You're on a skateboard going really fast before you hit the ramp and do a 360 [a 360 degree rotation in the air] before hitting the ground and wiping out.
Nancy: What about a car chase where one of the cars spins out of control?
Danny: What if I landed my 360 and then spun out of control?

However, after 2-3 minutes, the group brought themselves to reality when they started to discuss the implementation of some of these crazy stories in the context of the graphical and horizontal representations available to them. For example,

S1A4E2 [extract of a discussion in Group 4 in implementation mode]

Frank: Does anyone actually have a car or a skateboard and ramp that could be used to show this? Remember, Mr. Smith said that it had to be doable.
Ruth: Let's use Danny's skateboard idea without the skateboard.
Danny: What?
Ruth: Like Frank said, you don't have a skateboard here and it would not work without a ramp. You could pretend you are doing it on the skateboard.
Patrick: OK, so going really fast away from the *CBR* would look like this [holding a graph with a very steep and positive slope]. What happens when you hit your jump?
Danny: You get into your 360.
Patrick: Do you stop in the air to do it?
Danny: No, you get into the rotation as you go up and finish it just before your wheels hit the ramp.
Patrick: OK, so the next part looks like this [a very short section with zero slope followed by a section with very steep negative slope].
Ruth: Don't forget the crash at the end.
Danny: Right, that graph comes to a dead stop.
Patrick: What?
Ruth: No, when you hit the ground, your line goes flat.
Danny: Right, you don't move but the time keeps ticking.
Frank: He doesn't just stop. Even if he crashes, there's always a little slide before he stops.
Patrick: OK, so we make the last section look like this [very small negative slope followed by a long section with zero slope].

Also, as anticipated in the *a priori* analysis, some students offered graphs that did not represent motions but were functional relationships. In no instances did these graphs remain candidates for group submissions very long because they were void of any physical reality. In attempting to describe the physical motion, students quickly realized that no motion could represent the graph. However, only in situations where some students held firm to their beliefs did very rich discussions around the relationship between position and time and the notion of dependence and independence occur. For example, below is an excerpt of an explanation a student (Lee) was giving while attempting to physically create a motion that would be represented by graph in the shape of the letter P. Another student (Ann) used an anticipated argument in challenging Lee.

<u>S1A4E3</u> [extract of a discussion from Group 2]

Lee: Look, I am telling you, I could do the P. You just need to move very fast.
Ann: How are you going to create the bottom portion of the P?
Lee: That's where you need to be very fast and run back towards the thing.
Ann: Can you run so fast that you could make time go backwards? You can't make it go back.
Lee: What do you mean?
Ann: No matter what you do, the thing keeps going. You can't make it stop.
Lee: I am telling you, I could do it.
Ann: Whatever. Why don't you do it in front of the class?
Lee: I will.
Tina: Look guys, we got to get this thing done. Why don't we do something that you guys won't argue about?

After the groups had an opportunity to create other groups' scripts, the teacher asked if anyone wanted to volunteer to create the letter P or other difficult letters. Lee from group 2 raised his hand to volunteer.

<u>S1A4E4</u> [extract of exchange between the teacher and Lee]

Teacher: So Lee, let me know when you are ready. Make sure you are standing where you want to be when I hit start.

Lee: No prob. Mr. Smith. Ready.
Lee: (as he is walking away from the *CBR*) See, here we go with the first part…doing the curve at the top…coming around quickly to close… to close… come on now! (moving his upper body towards the *CBR* without moving his feet). I'm stuck.
Ann: Told you.
Teacher: Lee, why can't you finish the bottom portion of the P?
Lee: I can't make it stop.
Teacher: What can't you stop?
Lee: The time

The notion of dependence and independence surfaced naturally during this activity and the students appeared to better understand the relationship between the two variables involved in motion. The notion of dependence and independence in a problem of motion surfaced naturally in much the same way it did for Newton in the 1600s as outlined in the preliminary analysis. Specifically, the notion of dependence and independence really surfaced when the students were directly experiencing the concept of independent variable as they were physically creating the motion that was being represented. As reported above, the student physically trying to reproduce the final portion of the letter *P* (slanted) understood and even perhaps "felt" that he was unable to make time go backwards. This was directly observed when the student moving towards the *CBR* slowly came to a stop and started leaning towards the *CBR* while watching the graphing representation continue to plot points further and further away from the y-axis. The milieu in this activity provided the students with real-time feedback allowing them to quickly adjust their thinking and their actions.

One student was creative about how to reproduce a letter which one would initially think is not possible to produce while walking in front of a *CBR* (i.e. not based on a functional relationship). He showed how to create a slanted *J* so that the motion respected the independence of time by walking towards the *CBR*, slowing down, changing directions, then walking away from the *CBR* at an increasing pace.

This activity was successful in having the students develop a good understanding of the notion of dependence and independence. They were able to show this new understanding by being able to articulate their understanding of the relationship between the variables. Specifically, many of the students were able to clearly communicate (after their experience) that time was an independent variable. Their understanding of dependence was a little more subtle in that they were able to communicate that they were able to control the dependent variable. It should also be noted that few of the students actually made any reference to the horizontal motion in this activity.

4.4 Teaching sequence 2: Modeling and piecewise defined functions

The second teaching sequence leverages multiple representations in order to enable individual and aggregated mathematical constructions. The goal is to challenge students to coordinate multiple representations of functional relationships and slope as rate of change in problems of motion.

4.4.1 Activity 1 (TS2,A1)

The goal of this activity was for students to familiarize themselves with a new feature of the technology tool that could be used throughout the teaching sequence and for students to understand how motion could be represented by more than one form of representation and the differences between these representations. A common framework and language for discussing functional relationships was institutionalized throughout the activity.

This activity was essentially a teacher led discussion. The students participated by being part of the class discussion and/or through the action of the volunteering student. As was anticipated in the *a priori* analysis, this activity focused the students' attention on the simulated horizontal representation of the motion. Similar to the first "Getting Started" activity, the students relied heavily on verbal descriptions of the motion and their observations.

In order for students to get to the level of detail targeted by this activity, the teacher asked students to be more precise. For example, after a brief discussion regarding the students' initial reaction to what was going on in the demonstration, the teacher asked the students the following questions: When is B going the fastest?; When is B going the slowest?; When does B seem to change its speed?. These specific questions forced the students to be more specific with their responses:

<u>S2A1E1</u> [extract of a discussion regarding rates of change at different locations – a disagreement]

Bruce: The first section is going fast, the middle section is going fastest, and the last section is slowing down.
Ann: Mr. Smith I would say it differently. B is going fastest in the middle section and slowest in the first section. The middle section has the steepest slope and the first section has the smallest slope.
Teacher: And when does B change speeds?
Ann: There and there.
Teacher: Can you be more specific about when?
Ann: After about two and half seconds, it goes faster and slows down after about five seconds.
Frank: I think the last section is the slowest section.
Teacher: Let's see if we could investigate this more to determine the actual slowest section.

The introduction of the STEP feature allowed the students to place the time cursor on top of the vertical axis where each subsequent press caused the vertical line to slide along the animation one Step-Time value. This proved to be helpful in obtaining more detail during their investigations. This feature was a tool used to explore the functional relationship in the graphical representation.

The introduction of the MARKS feature allowed students to drop marks at regular time intervals during the animation in the horizontal representation. This led to rich discussion with students trying to justify their answers to the first three exploratory questions posed by the teacher.

<u>S2A1E3</u> [extract of a discussion regarding the actual slowest section of the graph]

Teacher: So how could we tell which section is the slowest?
Lee: The one with the most space between marks.
Teacher: What do the rest of you think about this?
John: Most distance between each section should be fastest. The second section is twice as fast as the first section. [See figure 16 below.]
Frank: Right, and the last section is a little faster than the first. I was wrong.
Teacher: No big deal. They were very close. Without dropping marks, it would have been tricky.

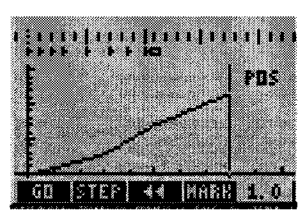

Figure 16. The MARKS feature

The students appeared to adjust to the new level of precision made possible by the "stepping" and "dropping marks" as the motion was animated. These new features encourage the students to refine the analyses of their observations. The students appeared to agree with the rationale provided by several students in answering the questions at the end of this activity.

An interesting discussion surfaced when the students were asked "exactly how fast is B moving during each part of the trip". Through the following exchange among three students, the notion of rate of change emerged quite naturally. In fact, it seems to have been a real eye opener for one particular student ("oh so that's what rate of change means"...).

<u>S2A1E4</u> [extract of a discussion regarding actual speeds during the sections]

Teacher: Now, I would like us to describe exactly how fast B is moving during each part of the trip.
Lee: The first section is moving one tick per second.
Teacher: One tick?
Mark: One meter per second.
Lee: Oh yes, the tics are for meters.
Teacher: How fast is B moving during the second section?
Peter: Two meters per second.
Teacher: How did you get that?
Peter: Well, there are two tics between each drop in that section.
Teacher: What's the difference between the rate in the first section and the rate in the second section?
Peter: That one [referring to the first section] has one tick per mark and the middle section has two tics per mark.
Teacher: What does this tell us?
Patrick: The second section is twice as fast as the first section.
Teacher: Right, the rate of change in the second section is twice as fast as that of the first section.
Linda: So that's what rate of change means...
Teacher: What did you think it meant?
Linda: Well, I never really thought about it.
Teacher: Well, I am glad that you understand it now. Can anyone tell us what the actual speed for the last section is?
Patrick: I would say one and half meters per second.
Teacher: What do the rest of you think about this?
Lee: That's about right.
Teacher: How can we be sure?
Lee: We could take out a ruler and measure.
Teacher: That would be a great idea. However, I think most of the class accepts 1.5 meters per section as the rate of change for the last section.

This activity achieved its goal of familiarizing the students with the new features of the technology. The students' understanding of how motion could be represented in more than one way and the differences between their representations surfaced when they used the new features to refine their analysis in the horizontal representations. They were able to clearly communicate what was happening in both representations.

4.4.2 Activity 2 (TS2,A2)

The goal of this activity was for students to enhance their understanding of the critical ideas of functions and slope as rate-of-change. The core mathematical ideas being addressed and contextualized in this activity are the concepts of function as a relationship between distance and time and the qualitative idea of slope as rate of change where the rate in this case is velocity: positive vs. negative, steeper means faster (greater rate), zero slope means zero rate (zero velocity). The students were to familiarize themselves with the graphical editing of piecewise defined functions through direct manipulation enabled by a new feature of the technology tool that was to be used throughout the teaching sequence.

As expected in the *a priori* analysis, the students were able to use the graphical editing features to help them analyze the first series of questions. Only two of the five groups were making use of the new features that they had been introduced to in the previous activity. This was largely because the motion was designed to go very quickly. All of the groups were able to quickly answer the first two questions. Just by looking at the representations, the students were able to observe that B was represented by the dot below the tick marks and by the shorter graph below the longer graph. The more interesting discussions happened when the students were asked to reflect, discuss, sketch, and create a graph.

Two groups used the Marks feature to "drop marks" during their motions to analyze the motion more carefully using the horizontal representation. The other four groups chose to focus only on the representations of the simulations without using the new tools. Both groups who started by using the Marks feature also switched to the more visual approach to answer the questions. All groups were able to correctly answer the questions. By being able to accurately articulate their ideas regarding slope as rate of change, they were able to demonstrate their understanding. The tool also allowed the two groups to develop their understanding by providing them with additional information that had them reflect on their own thinking.

S2A2E1 [Exploring distances and durations]
Lee: B goes slower than A. Look at the marks.
Kathy: Yes but we just need to say how long they each traveled for.
Jim: They want to know time and distance for each one.
Lee: Well that's easy.
Ann: A goes for ten seconds and covers ten meters.
Tina: B only goes for 4 seconds.
Lee: Right, and it goes on for only about 2 meters.

The students came up with very interesting graphs when it came time for students to reflect upon, discuss, sketch, and create a graph for a motion that would satisfy the given criteria. All the students in all the groups explored different ways of creating the graphical representations by directly manipulating the different segments of the graph on the graphing calculator rather than on the graph paper provided to them. Also, each of the groups made use of a section with a slope of zero. This was anticipated in the *a priori* analysis because it was anticipated that students would be able to apply their knowledge of zero slope. It was expected that students would discuss and experiment in their groups how to represent "going backwards" graphically. The correct behavior was for students to explain that they needed to add a new segment that must slope downward, or have negative slope. This is an example of how it was discussed:

<u>S2A2E2</u>

Lee:	So first, the graph has to go up very fast. Then it goes flat for 2 seconds....
Ann:	Going in the wrong direction would be like coming back. The graph would be coming down.
Kathy:	It's like moving towards the *CBR*.
Ann:	Right. And then we must end in a tie.
Lee:	That means we need to create a section that connects with A at the very end.

It was also expected that students would discuss how to represent "finishing in a tie". The correct behavior was for them to explain to each other that you must add one more segment and extend it to the right and upward as needed so that its right endpoint coincided with that of A's graph. Although the majority of the groups had little difficulty finishing in a tie, one particular group helped one student overcome a difficulty with the concept.

<u>S2A2E3</u> [A group helping a student who thought that a tie meant making sure the last segment went all the way to the end and that it did not need to coincide with **A**'s graph.]

Jess:	... OK, now we need to end in a tie.
Trish:	Let's create a section which goes to the end.
James:	Like this [showing a section which coincides with **A** at the end].
Trish:	That's good but so is this [showing a section which has **B** finish below **A**].
Peter:	No, that one does not work.
Tim:	It must be exactly on top of **A** at the end to finish in a tie.
Trish:	As long as it goes to the end, it's a tie. Oh no, you're right. I forgot about the distance. You are right. It does need to be on top of **A** at the very end.

Another student made a literal picture of **B** falling down as a segment sloping down to the x-axis. This incorrect conception has been identified as an obstacle in the cognitive dimension of the preliminary analysis and the students in his group were able to help this student overcome this obstacle by discussing the meaning of the x and y-axes in their contexts.

S2A2E4

Linda:	...and then the graph goes straight down for two seconds.
John:	Falls down is the same as stopping. It's just a flat line.
Linda:	Right. But the graph goes down before it goes flat.
Jim:	Going down means going back towards the start. Remember the *CBR* thing. When we walked away from it the graph went up and when we walk towards it, it went down.
Linda:	When we stood still, it was a flat line. OK, OK, I get it. I wasn't thinking.

This activity put students in a situation where they were required to reflect upon the functional relationship involved in motion and communicate their understanding. Some students like Trish and Linda, who at first struggled with their ideas, developed the ability to reflect on and articulate their knowledge. This activity was successful in familiarizing the students with the editing feature for the editing and creation of the piecewise defined functions. This was evident because all the students were using this new feature as they were creating their graphical representations while attempting to respect the constraints.

4.4.3 Activity 3 (TS2,A3)

The goal of this activity was for students to familiarize themselves with the new representational system that was to be used throughout the rest of this teaching sequence. The students were provided with the first experience to "see" all of the representations that had been collected, especially overlapping graphs and they will be challenged to coordinate multiple representation systems to "find" their constructions. This activity was essentially a continuation of the previous activity.

The role of technology as a part of the milieu in this activity was to challenge the students to coordinate multiple representation systems to "find" their individual constructions among the many representations in the shared space. The milieu here introduced students to the process of relating their personal constructions to the larger collection of objects that appeared on the "big screen" when their work was aggregated with that of their peers.

As was anticipated in the *a priori* analysis, the students immediately started creating psychological links with the mathematical objects that they created, referring to them as "my dot". This personal investment in the building of knowledge as outlined in the cognitive dimension of the preliminary analysis is an important indicator of understanding. Most students were able to successfully coordinate their mathematical objects in the two different representation systems – *Connected MathWorlds* in the shared space being projected to the class using the data projector and *MathWorlds* in the personal space on their graphing calculators. Because of differences such as size of screen, representation of the axes and intervals, resolution, and color, and the fact that several of the graphs looked very similar, students needed to identify certain aspects of their mathematical objects on their graphing calculators and look for them in the public display. The students were successful in overcoming this challenge by reflecting on and applying their knowledge of the functional relationship involved in problems of motion. Figure 17 is an aggregated view of all of the groups' constructions. The justification from the students below showed that they relied on a good understanding of the functional relationship represented in the graphical representation on their graphing calculator in order to apply this mathematical knowledge and locate the same mathematical object in the public display even though it looked a little different.

S2A3E1 [Group 2 identification of their construction.]

Lee: Look, there's our graph.
Teacher: How do you know for sure that that's your graph.
Lee: Because, we took off as fast as we could – that's why.
Ann: That's right. And I know that we fell down after only 1 second. No other graphs up there do that.
Teacher: You guys really know your graph.

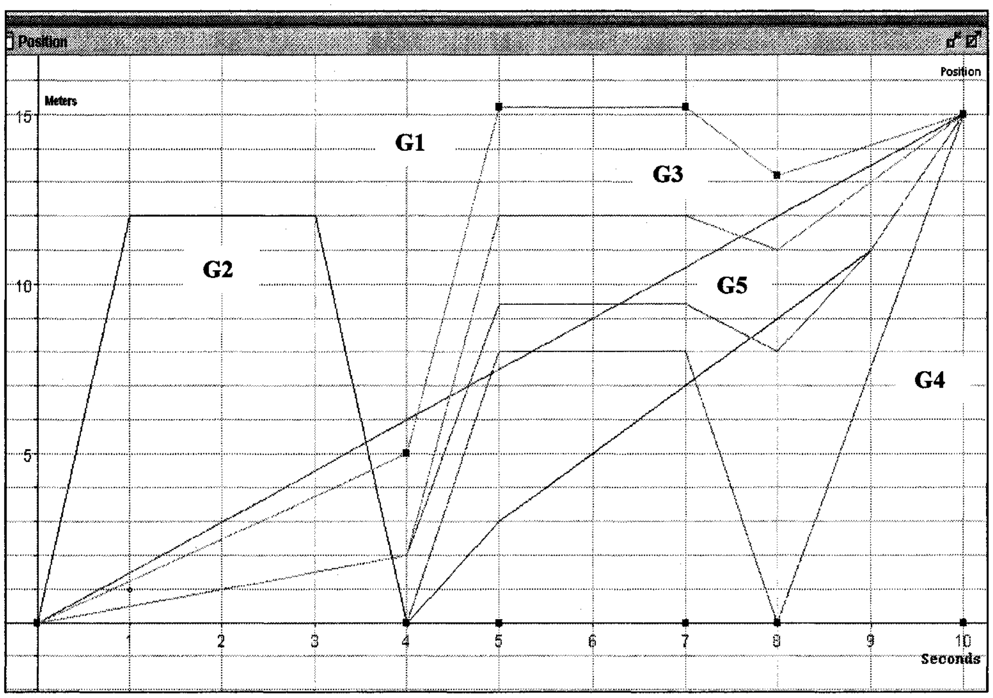

Figure 17. Aggregated view of group constructions

S2A3E2 [Group 1 & 3 identification of their constructions.]
Kelly: Our graph must be the orange one [G1 graph above was orange on the projected screen] because it goes the highest.
Mark: Yes, ours went off our graph on the graphing calculator.
Tim: No, the orange one is our graph. We went high also.
John: Could you see the graph on the graphing calculator.
Peter: Yes, we could. Our graph only went up to about 11 meters.
Tim: OK, right, so our graph is the green one [G3]… or the brown one (G5).
Peter: Ours is the green one [G3].
Teacher: Why do you say that?
Peter: Well we fell down farther than 10 meters away… it was more like..
James: 12 meters away.
Peter: Right it was about 12 meters away. Look the green (G3) one is exactly 12

meters away.
James: The brown (G5) one fell less than 10 meters away.

<u>S2A3E3</u> [Group 4 identification of their construction.]
Teacher: What other group would like to point out their construction for us?
Danny: We would.
Teacher: OK, which is it and why?
Danny: It's the pink one [G4] because we did nothing for the first few seconds.
Tina [from group 2]: You couldn't do that! You had to start with a wild burst of speed.
Ruth: It did not say that it had to start with that. We did our wild burst of speed after 4 seconds. Mr. Smith?
Teacher: Keep explaining why the pink one is your graph.
Danny: That's basically it. We were the only ones that started that way so it's easy to spot our graph. The rest of the pieces look very similar.
Ruth: Right. And we were the only ones to stop at 9 meters.

As a continuation of the previous activity, this activity was successful in having students enhance their understanding of function as a relationship between distance and time and the qualitative idea of slope as rate of change. The students displayed this by successfully applying their mathematical knowledge of functional relationships and of slope as rate of change. The students also demonstrated a personal investment in the building of knowledge as they constructed relationships between the mathematical representations they created on their graphing calculators and the representations of the same objects on the shared public display. Also, the students were successful in communicating their justification for the correct identification of their representations in the shared public display.

<u>4.4.4 Activity 4 (TS2,A4)</u>

The goal of this activity was for students to display an understanding of the concept of variable, and the notion of dependence between two variables in their descriptions - distance and time. The students were challenged to coordinate multiple representation systems, the personal graphing calculator/*CBR*/*MathWorlds* representation system and the public *TI-Navigator*/*Connected MathWorlds* representation system, to deepen their understanding of functions as a relationship between a dependent (distance) and an independent (time) variables. The goal was also for students to deepen their understanding of slope as rate of change. Implicitly there was still the goal to engage the students personally with their mathematical work and to reach students who might otherwise feel alienated from mathematics by offering them a chance to "perform" mathematically.

As was anticipated in the *a priori* analysis, the students were very autonomous while working in groups. They appeared excited about the activity and their stories were very exciting. Initially, most groups focused on verbalizing stories that they felt would be difficult to re-create. During this brainstorm, other members of the team tried to bring the brainstorm something that could be contextualized on their devices.

<u>S2A4E1</u> [Wild story brainstorm in group 5]
Bruce: …and then the rocket ship goes…
Lacey: Hey! Is that something that could be done on this screen?...
Bruce: Ya, but we need to think of something cool. What if we start by making a really ugly graph and then find a story for it?...OK?

The students used the technology to explore and to verify the stories and then adjusted their stories so that they fit the representations that they created. These students focused on the representations of the mathematical object first. They needed to have a good understanding of the relationships between distance and time over the different sections that they created in order to come up with story lines that reinforced the relationships during the appropriate times. The notion of dependence and independence was discussed in the exchange between these students below. Although this type of discussion was not anticipated in the *a priori* analysis, the arguments that were used in the discussion were predicted.

S2A4E2 [Discussion time and position in group 5]
Bruce: Look, the guy wants an exciting race, let's give him one.
Lacey: Yes, but you are talking about going back in time. That's not possible.
Bruce: Let's make it possible… How do we show it on the calculator?
Bryan: Well, uh, we could uh…
Lacey: Give an example of where it would work.
Bruce: OK, let's say you run super fast to get near the end and then you forgot your phone, you could stop the time, floor it to get back and zoom back to where you were, the time would start up again and we cross the finish line… How would you graph that?
Kevin: The graph would have a bunch of lines behind where you stopped.
Bruce: OK, how do we do it on this [calculator]?
Kevin: Oh, you can't because you can't stop time on this calculator.
Bruce: OK, you are right, we can't change that but we can change where we are – let's go all over the place as fast as we can.

Although this excerpt showed that one of the students got carried away trying to change the environment, the students did display a strong understanding of the independent variable in this activity – time. The above discussion did force the students to reflect on their initial ideas and their thinking about this story. In the end, they were able to apply their understanding of time as an independent variable and adjust their story.

Other students started from a story, explored and validated the story, and then adjusted the graphs so that they fit their story.

S2A4E3 [excerpt of a discussion in Group 2]
Lee: Let's start here…you need to go up quickly…
Ann: Yes but not as quickly as you'll need to go later – we start out kind of slow, we stop, and then take off to the finish
Kathy: So the third section needs to be steeper than the first section.
Ann & Lee: Right!

This excerpt demonstrates that the students had a good understanding of slope as rate of change. This was observable in their ability to communicate clearly about the concept.

The actual exercise of creating graphical representations for the exciting races that were delivered from the other groups also resulted in some great discussion displaying a deeper understanding of the functional relationships. The ability to formulate a clear description of the motion to other teams to follow in order to reproduce the graphical representation is an important indicator of understanding.

S2A4E5 [Group 3 - Creating a Position vs Time Graph for another group's story]
Jess: So they want us to start far away. Then they want us to move towards the *CBR*. They want us to be jogging.
Trish: Then they want us to sprint back to exactly where we started.
Peter: John probably thinks none of us could run as fast as he can.
Tim: I can. I want to do it.

All the groups were able to recognize their graphical representations after the teacher collected them all, aggregated them and projected them in the public display using a data projector. As anticipated in the *a priori* analysis, the students who successfully coordinated between their personal creation on the device and the more refined version of the graphical representation offered by the computer version of *MathWorlds* in the public display had a deeper understanding of functions as a relationship between dependent and independent variables and slope as rate of change. As anticipated in the *a priori* analysis, the development of understanding by the students in this activity emerged as the students constructed relationships, extended and applied mathematical knowledge, reflected about experiences, articulated what they knew, and made mathematical knowledge their own. This was also observed by the descriptions that they gave of the functional relationships being represented by the graphical representations.

Some students were more challenged than others in coordinating the representations on the two devices – the graphing calculator and the public display. For example, groups 4 and 5 had difficulty recognizing their graphical representations because some of the sections of both graphs overlapped making it difficult to see just one graph. The teacher helped by hiding some of the overlapping graphs and the students were able to find their own graphs.

This activity was successful in having the students display their understanding of the notion of dependence and independence. The challenge of coordinating the two representational systems also helped the students deepen their understanding of function as a relationship between distance and time in problems of motion.

4.5 Teaching sequence 3: Summarizing understanding

Recall that the goal of this sequence was to provide the research team with the opportunity to ask explicit questions to the students to probe their understanding of the concept of function and their ability to transfer knowledge acquired with technology to a traditional paper and pencil environment.

4.5.1 Activity 1 (TS3,A1)

The goal of this activity was to determine if the students had a good understanding of the functional relationship between distance and time in problems of motion and solid understanding of independent and dependent variables.

As was anticipated in the *a priori* analysis, the majority of the students (22 out of the 23) selected the correct graphical representation for this problem (d). It is difficult to know exactly why the one student chose the incorrect representation or what meaning he had constructed to do so. In the class discussion, this student might have been the first presenting his/her answer and the teacher could have asked the reasons for this choice. It would have allowed having an insight to the understanding of this student. However, assumptions for this have been made in the *a priori* analysis. Recall that the obstacle related to student difficulties overcoming literal representations of graphs was outlined in the cognitive dimension of the preliminary analysis and that this type of obstacle creates significant difficulties in conveying the connections between graphs and functions to students. As a result, more one-to-one interview by the teacher was required with this student following the research project.

An interesting class discussion took place when the teacher asked the class why the other representations were not selected. The students were able to recognize that the other graphical representations did not represent functional relationships. The expectation was that the students would comment that the incorrect graphs showed time going backwards or a person being at more than one location at one point in time. As described below, this did in fact happen. Also, recall that Frank was the student in the first lesson who strongly believed that he could create the letter P because he was a fast runner. His comment below shows that he now has a good understanding of the function relationship between distance and time in this problem of motion.

<u>S3A1E1</u> [Excerpt of discussion around the distracters]
Peter: You can't be at a bunch of places at exactly the same time
Frank: You can't go backwards in time.
Ann: How did you create these graphs anyway? It should not be possible for you to do these. What's up Mr. Smith?
Teacher: Well, these are actually fake graphs…

4.5.2 Activity 2 (TS3,A2)

One goal for this activity was to determine if the students were able to apply what they had learned about the functional relationship in problems of motion involving the motion detector in a different context without technology. Another goal was to determine if the students deepened their understanding of slope as rate of change.

As was anticipated in the *a priori* analysis, the majority of the students were able to use their understanding of functional relationships in problems of motion to correctly represent the given segments graphically. Specifically, 19 out of the 23 students were able to represent slope (positive, negative and zero) appropriately with different rates of change and displayed a good understanding of position in this activity.

As expected, many of the students needed to start their graph over several times in order to make effective use of the graph paper provided. Most of these students got through several criteria before realizing that they ran out graph paper to complete the problem. This limitation pushed the students to focus on representing important properties of the functional relationship graphically.

It was promising that no students appeared to resort back to literal representations of segments of graphs when unsure of how to represent parts of Santa's trip correctly. Most of the mistakes that were made on this problem dealt with accuracy. For example, Nancy had a short portion of a negative slope representing the front tire blow out instead of the zero slope that was expected. However, Nancy did have zero slope representing the one-second stop. Unfortunately, I did not have the opportunity to discuss this with Nancy, however, it may be that she thought the wagon had moved slightly backwards after getting the blow out. Linda, Bryan, and Kevin all had final sections (G) which had slopes that were slightly different from the slope in section A. Perhaps this is because the instructions for the last section were not clear enough. Specifically, perhaps the students had difficulty interpreting same rates as parallel line segments. This is not obvious and it was not addressed explicitly in the previous activities. Although all three of these students did not have the exact same slope as they did in the first section, they all had slopes that were relatively close.

The students were able to successfully transfer what they had learned about problems of motion involving the motion detector to a different context and to a paper and pencil environment. Also, the students' graphs communicated a good understanding of slope as rate of change demonstrating a greater depth of understanding than at the beginning.

4.5.3 Activity 3 (S3A3)

The goal of this activity was similar to the first two in that it attempted to determine if the students had a good understanding of the functional relationship between distance and time in problems of motion and a deep understanding of slope. This activity also attempted to determine if the students had the ability to obtain important information by analyzing graphical representations of functional relationships between distance and time in a motion problem set in a different context.

As was anticipated in the *a priori* analysis, the majority of the students were able to use their understanding of functional relationships and displayed the ability to obtain important information by analyzing graphical representations of functional relationships between distance and time in a motion problem set in a different context. Specifically, 13 out of the 23 students were able to answer all questions successfully. 5 out of 23 students had difficulty with number 5. As anticipated in the *a priori* analysis, this is most likely because these students made the false assumption that the bikers started the trip from the same location. Two of the students answered all of the questions matching A and B with the wrong graphical representations. Although, these students' answers were wrong, their answers were correct within the context of the matching they used.

This lesson was successful in providing a good picture of the students' individual and group understanding of the concepts explored in the first two lessons.

4.6 Summary

This chapter described the realization and the analysis of the teaching sequences and the last day assessment. The next chapter will present the conclusion and summary for this thesis.

Chapter 5 Conclusion and Summary

The goal of this experiment was to explore the effects of using multiple representation systems on student understanding of functional relationships involved in problems of motion. Historically, motion has been the basis for a considerable amount of mathematics, especially in the development of the concept of function, and I was interested in learning about how motion in a didactic milieu making effective use of multiple representation systems enhances students' understanding of functions. I will now consider each hypothesis and relate the results of the experiment to the hypotheses.

Hypothesis 1: Individual mathematical constructions that are directly experienced in a "live" context, have immediate kinesthetic, cognitive and linguistic aspects that will help students develop an understanding of the relationship between distance and time in problems of motion.

The selected technology included in the milieu did succeed in allowing the students to experience motion directly. As a result of their interactions with the milieu, the students were able to get immediate feedback from their actions allowing them to reflect on their thinking and examine their thoughts and consequently adjust their actions, thus contributing to the building of understanding of the relationship between distance and time in problems of motion. Consistent with the historical development of the concept of function, the students in this experiment relied heavily on verbal descriptions of the motion and its representations to describe the functional relationship. The students engaged in rich discussions helping each other overcome some of the anticipated difficulties. As outlined in the first chapter, the ability to articulate what one knows about the functional relationship involved in motion is an important indicator of understanding. The free exploration where each student walked in front of the *CBR* at least twice allowed the volunteering students to directly experience the relationship that exists between their motions and graphical representations that simultaneously were being graphed as they walked. By being directly involved in the creation of the motion and its graphical representation, the students were able to get immediate feedback from their physical actions. They were clearly able to analyze the results of their past physical motion with the targeted graphical representation and refine their motion to match the desired graph more closely on the second try. These activities encouraged the conscious examination of their own actions and thoughts, and as outlined in Chapter 2, the notion of the emerging nature of understanding is seen in the students' ability to reflect on their own thinking (Carpenter & Lehrer, 1999). Either in the context of entire class discussions or in smaller group discussions, students had the opportunity to observe the direct effect of motion on its graphical representation. Several of the activities involved having students formulate descriptions of motion for others to act out physically in front of the *CBR*. This required the students doing the formulation to think about the different aspects of the motion and refine their descriptions until the desired motion could be acted out.

The simulated horizontal representation of the motions that could be played back on demand by the students allowed them to see a simulated model of their physical motion representing someone walking in front of a *CBR* as often as they liked. The ability to play back the motion in the form of a simulation while simultaneously watching the graph of the motion being created allowed the students to explore the phenomena at their own pace. This really seemed to help students refine their thinking about functional relationships in problems of motion.

By the end of the second activity in Teaching Sequence 1, most of the students were able to show their understanding of this relationship between distance and time by correctly articulating their ideas regarding position and time. By the end of the third activity in Teaching Sequence 1, they were able to correctly articulate relationship between distance and time in problems of motion.

Hypothesis 2: Individual mathematical constructions in a "live" context facilitate the development of understanding of independent and dependent variables.

Students started demonstrating some understanding of the notion of dependence as early as the first activity in Teaching Sequence 1 when they realized that the height of the graph directly depended on the students' position in front of the *CBR*. The real-time kinesthetic feedback provided by the milieu helped the students validate or invalidate their conjectures and develop their understanding of independent and dependent variables. Also, the challenge provided to students in the final activity of Teaching Sequence 1 resulted in very creative stories and the students appeared to be very engaged in the mathematical experience. The major goal of the last activity of Teaching Sequence 1 was to have the students construct meaning regarding the relationship between the two variables in motion and the notion of dependence in much the same way early mathematicians did. The notion of dependence and independence surfaced when the students were directly experiencing the concept of independent variable by physically creating the motion and simultaneously seeing it represented on the screen. For example, the student physically trying to reproduce the final portion of the letter *P* (slanted) understood and even perhaps "felt" that he was unable to make time go backwards. This was observed when the student moving towards the *CBR* slowly came to a stop and started leaning towards the *CBR* while watching the graphing representation continue to plot points further and further away from the y-axis. One student was creative about how to reproduce a letter which one would initially think was impossible to produce while walking in front of a *CBR* (i.e. not based on a functional relationship). This student showed how to create a slanted *J* so that the motion respects the independence of time. The milieu in this activity provided the students with real-time feedback allowing them to quickly adjust their thinking and their actions.

Teaching sequence 2 was successful in having the students develop a good understanding of the notion of dependence and independence. They were able to show this new understanding by being able to articulate their understanding of the relationship between the variables. Specifically, after their experiences with the *CBR*, many of the students were able to clearly communicate that time was an independent variable. Their understanding of dependence was a little more subtle in that they were able to communicate that they were able to control the dependent variable – distance away from the *CBR*.

Hypothesis 3: Multiple linked representations of the same function in a simulated environment allowing for manipulation by the students improves their learning about rate of change.

As expected, some students had difficulty coordinating between the simulated horizontal representation of the motion and the graphical representation of motion. Most students were simply more familiar with the graphical representation than the simulated horizontal representation, but their familiarity with graphical representations helped them make connections. Seeing the same motion represented in different ways caused them to focus on the relationship involved in the motion, and the properties of the simulated horizontal representations allowed the students to refine their analyses of the motion. Specifically, it was observed in this experiment that the students were able to refine their thinking about distance away from the *CBR* and the direction of the motion.

The graphical representation had different benefits: the majority of the students chose the graphical representation when discussing and analyzing slope as rate of change. The third activity of Teaching Sequence 2 showed that these multiple linked representations of the same motion (i.e. graphical and horizontal simulated representations) allowed the students to improve their learning about the functional relationship involved in motion and about rate of change. Therefore, while analyzing the motion using two linked representations, the students could focus on different aspects of the functional relationship depending on the representation they were focusing on. This allowed the students to quickly refine their thinking, especially when they were learning about rate of change. By the end of the fourth activity in Teaching Sequence 2, the majority of the students were able to display a good understanding of slope as rate of change by being able to communicate clearly about the concept when describing their motions to other teams, so that they could reproduce the graphical representation. Teaching Sequence 3 also confirmed that students had a good understanding of slope as rate of change when 19 out of 23 students were able to represent slope (positive, negative and zero) appropriately with different rates of change and in the contexts of different situations of motion.

The direct experience resulting from deeper engagement with the mathematical objects – functions, allowed the students to link them to the representations that they had built. This became very evident in the third activity of Teaching Sequence 2 when the students began referring to their representations as "my dot", "my graph", or "that's me". As described in the cognitive dimension of the preliminary analysis, this personal investment in the building of knowledge is an important indicator of understanding.

Hypothesis 4: Aggregated mathematical constructions challenge students to coordinate multiple representations and deepen their understanding of functional relationships.

The final activity of Teaching Sequence 2 confirmed that students who can successfully relate their personal creation on the handheld device to the more refined version of the graphical representation offered by the computer version of *MathWorlds* in the public display, had a deeper understanding of functions as a relationship between dependent and independent variables and slope as rate of change. The development of understanding by the students in this activity appeared to emerge as the students constructed relationships, extended and applied mathematical knowledge, reflected about experiences, articulated what they knew, and made mathematical knowledge their own. This was also observed by the descriptions that they gave of the functional relationships being represented by the graphical representations. The challenge of coordinating the two representational systems also helped the students to deepen their understanding of function as a relationship between distance and time in problems of motion. Most students were able to successfully coordinate their mathematical objects in *Connected MathWorlds* in the shared space that was projected to the class using the data projector, and *MathWorlds* in the personal space on their graphing calculators. Because of differences such as size of screen, representation of the axes and intervals, resolution, and color, and the fact that several of the graphs looked very similar, students needed a way to relate their personal constructions to the larger collection of objects that appears on the "big screen" when their work was aggregated with that of their peers. Figure 17 from the third activity of Teaching Sequence 2 showed an aggregated view of all of the groups' constructions. The students identified certain aspects of their mathematical objects on their graphing calculators and looked for them in the public display. This internal process required the students to reflect on and apply their knowledge of the functional relationship involved in problems of motion.

Teaching Sequence 3 showed that the students were able to successfully transfer what they had learned about problems of motion involving the motion detector to problems involving motion in a different context. Also, the students' responses communicated a good understanding of slope as rate of change, a marked improvement over their understanding at the beginning of the experiment.

Finally, it was clear from the observations of the student interactions and participation during this activity that they really enjoyed learning about the functional relationship involved in motion in this way compared with the day of filming that preceded the implementation of this research design.

5.1 Limitations

In order to investigate the validity of my research hypotheses in a diverse rural high school setting, it was necessary to deal with the dynamics of a regular classroom. Therefore, in addition to implementing the experiment, the teacher needed to spend a lot of time managing the students' behaviors. The size of the classroom was another limitation. If the experiment could have been conducted in a much larger forum, the design could have included a station for each group so that each group would have had their CBR set ups. Doing so would have involved more of the students in the physical creation of the motion and associated explorations. Also, there were at least two students who required individual intervention and were in fact receiving individual interventions by the school district on a regular basis. The design of this experiment did not accommodate these needs.

It may be thought that a limitation of the experiment was that understanding was not assessed by formal testing, either before or after the instruction. However, the richness of the students' language and their ability to apply their knowledge to the new problems presented in Teaching Sequence 3 was clear evidence of their learning and of their understanding of the concept of function, so this evidence validated the *a priori* analysis of the teaching sequence. Since a goal of the study was to apply didactic engineering to a teaching sequence, it was designed according to the sequence *a priori* analysis → experimentation → *a posteriori* analysis. This parallels the classic quasi-experimental sequence of pre-test → intervention → post-test, so it meets the conditions of Messick's "Consequences as Validity Evidence." (Messick, 1995).

Because the effectiveness of this experiment was controlled by didactic engineering and therefore mediated by research-based teaching practices and the control of the variables, the scaling up of innovative-based mathematics to a wide variety of teachers and students and classroom settings is a concern. There are also concerns that only teachers who are comfortable with technology, interested in mathematics education research, and with a high level of support and guidance would be able to implement such innovations. It has already been observed that the teacher did not always follow instructions, which may have affected the outcome, but this factor is inevitable in the naturalistic setting of the real classroom and did not seem to influence the students unduly, and indeed, the teacher is a necessary component of the didactic milieu.

Additional limitations included the fact that the only variables used were time and distance functions, the activities addressed mostly piecewise linear functions, and that specific technology tools were used.

5.2 Practical contributions

Through didactic engineering of teaching sequences, students may construct, manipulate, and analyze graphical representations to important effect even in the absence of a shift to a learner-centered constructivist pedagogy by the teacher. Carefully designed sequences that take advantage of the affordances of specific representational technologies may increase the students' opportunities to learn.

Although historically, early notions of function were expressed in graphical forms representing a dependence between two quantities involved in motions, work done in the early 1900s to explore the concept attempted to downplay the idea of motion. It should not be surprising therefore, that the modern definition of function has abandoned the metaphor of motion. Helping students come to understand important mathematical concepts such as functions might be more effective if the concept were presented to them in contexts similar to the ones that housed real problems that early mathematicians debated while developing the concept. If important contributions to the concept of function and the notion of dependent and independent variables from leading mathematicians and scientists appear to have been contextualized in problems of motion, then students should be put in similar situation early on to help them develop a solid foundation for understanding functional relationships. This is a pattern of proven success that should be repeated.

Learning about functions by studying their multiple linked representations is very powerful because it makes the links between multiple representations more dynamic and therefore, more visible to the students. Whereas direct manipulation of conjecturing software has revolutionized the teaching and learning of geometry over the last ten years, these same ideas now have the opportunity to revolutionize the teaching and learning of functions. Teachers can use a variety of educational software products to help students learn about functions, as suggested in this experiment. Ubiquitous devices such as the graphing calculator greatly increase access to the functionality and capabilities that was once only provided by powerful mathematics software for computers (Berson & Balyta, 2004). The most appropriate technology will be the technology that can provide all students with meaningful and contextual interpretations of representations in a problem-based approach, and handheld graphing calculators are inexpensive enough and powerful enough to satisfy these criteria.

Taking advantage of the students' personal connection with their individual constructions in an aggregated and publicly displayed set of student constructions appeared to have helped the students develop important coordination skills that deepened their understanding of functional relationships involved in motion. As classrooms become more technologically advanced, the potential will exist to aggregate student work in this anonymous and powerful way.

5.3 Theoretical contributions

It appears that this experiment was the first doctoral dissertation in North America based on the Theory of Didactic Situations and didactic engineering. The Theory of Didactic Situations formed the theory base for the experiment, and the experiment showed that applying the process of didactic engineering in these new learning environments could result in improved student learning of functions, and increase active participation and interest. It is hoped that the success of the experiment might stimulate other teachers and other researchers to employ the methodology.

5.4 Recommendations for future research

Much of the preliminary analysis and the *a priori* analysis driving this research over the last several years has contributed to the new product development plans of a major educational technology company which will continue to research the effectiveness of its products on the teaching and learning of functions and other important mathematical concepts. It is recommended that future researchers and teachers take advantage of the multiple linked representations within such new integrated learning environments to determine if they are successful in deepening student understanding of functions. For example, the ability to view one function through several linked representations should deepen student understand of functions, both as a process and as an object. Therefore one important question to explore in the future is whether or not the appropriate and effective use of multiple linked representations could help students through the progression from the process to the object conception of functions as described in Chapter 2. Specifically, it would be valuable to learn if it helps students reach the reification stage where an ontological shift occurs when the student converts the condensed knowledge into an object in its own right (Sfard, 1992). It would also be interesting to learn how students could go through this progression.

Other questions related more directly to this experiment that can be addressed in the future such as long term effects of the sequences and the use of this particular technology on avoiding some of the well known obstacles to understanding functions.

Another recommendation was derived from the preliminary analysis. Given the pattern throughout the history of curriculum reform efforts resulting in misrepresentation or partial implementation of curriculum recommendations, textbooks continue to influence classroom teaching and learning dramatically. Because publishers rely on extensive feedback from teachers, it would be important for mathematics researchers to mobilize, align themselves, and become more effective in reaching the masses. This also implies the necessity of work done with teachers to influence their conceptions of teaching and learning. In addition to recommendations already made in the preliminary analysis, researchers in mathematics education might spend less time debating points on which they disagree, and more time making explicit recommendations on the issues on which they agree, like the introduction of the function concept. A large-scale quantitative collaborative research project led by several leading researchers in the field resulting in major adoptions of curriculum and pedagogical recommendations around the teaching and learning of functions should influence publishers to change the way the concept of function is introduced in their textbooks, thereby changing how the concept is taught in the majority of North American classrooms.

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Appendix 1 – Student documents

Group Name: ________________

Lesson 1: Exploring Physical Motion

Activity 1: Getting Started

Instructions for the students:
In the space provided below, write down your observations regarding what you think is happening. Following the demonstration, the teacher will ask you to share your observations and explain the relationship between the *CBR* motion detector and the graph.

Write your observations below.

Group Name: ____________________

Activity 2: Matching Motions

Walk a physical motion so that A matches B's motion as closely as possible.

Instructions for the students:

Describe how someone would walk a physical motion so that A matches B's motion as closely as possible.

In the space provided below, write down a description of a physical motion for A. (One description per group is sufficient.)

Write your description below.

Lesson 1: Exploring Physical Motion

Group Name: ____________________

Activity 3: Catch Up Motions

Walk a physical motion so that A starts off slower than B, but catches up to B at the end of the motion, at 6 seconds.

Instructions for the students:

Describe how someone would walk a physical motion so that A starts off slower than B, but catches up to B at the end of the motion, at 6 seconds.

In the space provided below, write down a description of a physical motion for A. (One description per group is sufficient.)

Write your description below.

Group Name: ____________________

Activity 4: The Challenge!

Groups will challenge each other to create graphs of interesting motions.

Instructions for the students:
Create a graph for a physical motion that you feel would be difficult for other teams to reproduce. However, make sure that it is possible to reproduce it.

In the space provided below, draw a detailed sketch of the graph and provide a story for the physical motion needed to create it. (One graph and description per group is sufficient.)

Sketch of the graph:

Story for the physical motion:

Lesson 2: Modeling and Piecewise Defined Functions

Group Name: ____________________

Activity 1: Getting Started

Instructions for the students:
This activity is designed to familiarize you with a new piece of software. It is very important to understand how this software works to be able to complete the other activities.

Lesson 2: Modeling and Piecewise Defined Functions

Group Name: ____________________

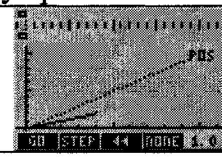
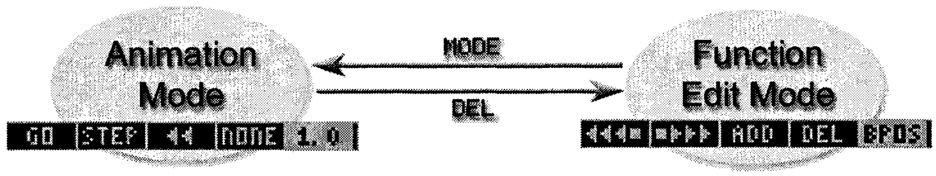
Activity 2: Creating Exciting Sack Races

Instructions for the students:

- Open menu item 1, "Piecewise Animations" from the Lesson 2 "L2: WarmUp" in the MathWorlds application.
- Run the animation by pressing the GO SoftKey and **reflect and discuss what is going on in your group**.
    - Reflect, discuss, and answer the following questions:

Which graph goes with which object?
How are the motions of A and B different?
How long does A travel – in time and in distance?
How long does B travel – in time and in distance?

- Reflect upon, discuss, sketch, and create a graph for a motion that would satisfy the following criteria:
    - Due to the wild burst of speed, B falls down for 2 seconds!
    - In the confusion of falling down, B gets up and goes in the wrong direction.
    - The race must end in an exciting tie!
    - Every team member needs to have a graph for a motion that satisfies the above criteria.
    - Be ready to explain your motion and have your animation assessed by another team.

This activity opened in Animation Mode.


You switch to Function Edit Mode to make changes to your function. Notice that the SoftKey labels change when you switch modes.
You'll use the blue Cursor Keys and HotSpots, shown above, to edit your graph.

In the space provided below, draw a detailed sketch of the graph and provide a story for the physical motion needed to create it. (One graph and story per group is sufficient.)

Sketch of the graph:

Story for the physical motion:

Group Name: ____________________

Activity 3: Find Your Exciting Sack Race

Instructions for the students:

- The teacher will collect your Sack-Races. It is important to listen to his instructions.

Group Name: ________________

Activity 4: Mathematical Performances – Exciting Races

Instructions for the students:

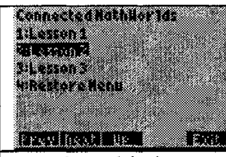
- In your groups, create an exciting sack race story-script for your own race with A which ends in tie, and create a Position vs Time Graph for B that makes your race happen.

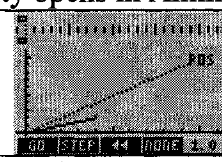
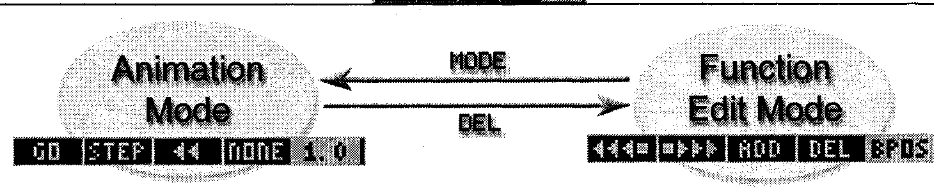
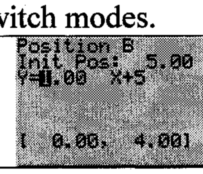
In the space provided below, write down a story for an exciting sack rack story-script for B. (One description per group is sufficient.)

Write your story below.

Creating a Position vs Time Graph to model your exciting sack race.

- You'll open MthWrlds from the APPS menu and choose the 1st of 2 activities – "Sack Races."

				
Choose MthWrlds	Press Enter Key	Select this item	Select this item	Select this item

This activity opens in Animation Mode.


You switch to Function Edit Mode to make changes to your function. Notice that the SoftKey labels change when you switch modes.

You'll use the blue Cursor Keys and HotSpots, shown above, to edit your graph.

1. Press the MODE or DEL Key to switch to Function Edit Mode.
   Make an exciting Position graph for B by adding and adjusting segments on B's Position graph.
   Scaling: Note that the scale of the "world" is now 1 m, so A travels 20 m in 10 sec. The vertical scale of the Position graph is no longer 1, but 2 m.
   Because the vertical scaling is now 2 m, there are times when you **ADD** a new segment that is 1 sec wide your new segment will look horizontal. However, when you stretch it to the right to 2 sec in width, you see that it slopes upward by 1 tic mark, which stands for 2m. Hence to make it flat, you need to drag it down by one tic mark.

2. Be careful not to extend your segments too far or add so many segments that your graph extends far off the screen. If this happens, then you will be adding, deleting or adjusting segments that are out of sight and it will seem like nothing is happening! You might then need to delete segments (using the **DEL** SoftKey, NOT the DEL hardkey) till you get back to something you can see! At least for now, try to keep most of your graph and motion on the screen.

3. You can adjust segments to the left of the last one by moving the Hotspot from segment to segment—use the two left-most SoftKeys that look like arrows, which move the HotSpot left or right.

4. When you think you have the Position-graph you want, return to Animation Mode to try out your race. To re-adjust it, go back to Function Edit Mode.

5. As you are testing and finalizing your race, write your script in the form of a list of descriptions, one for each segment of your graph. Draw your graph on paper to accompany your written story. It may be helpful to label your list using letters A, B, C, etc.

Sketch of the graph:

6. Bring your exciting sack race story-script to one of the other groups so that they could create a Position vs Time Graph to match it.
7. Create a Position vs Time Graph for the exciting sack race story-script that was delivered to you from another group. (Everyone in the group should create this Position vs Time Graph.)

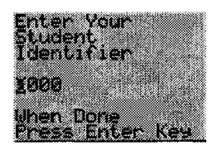
Sketch of the graph:

8. When you have the motion and function you want Pause MathWorlds so your teacher can collect this function.

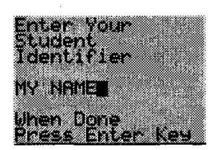
PAUSING MATHWORLDS TO HAVE YOUR FUNCTION COLLECTED

Make sure that your calculator is connected to a Hub WITH the black wire firmly plugged in at each end.

Pressing the 2nd Key followed by the STO Key takes you this screen where you enter your identifier:

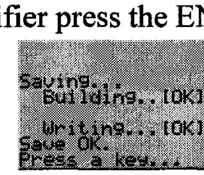


Now use the ALPHA Keys to enter your name, or your **Group Number-Count-Off Number** (like **0304** for **Group Number 3, Count-Off Number 4**) followed by your name, as directed by your teacher.



*Using the ALPHA Keys to enter letters is easier if you press 2nd Key followed by the **ALPHA** Key. This keeps the ALPHA Keys active till you press the **ALPHA** Key again.*

When you have entered your identifier press the ENTER Key. You'll see this screen:



Press the ENTER Key to Pause MathWorlds. The calculator will return to its Home Screen, ready for your teacher to collect your function.

KEEP THE FOLLOWING IN MIND

1. **DO NOT RUN MATHWORLDS WHILE YOUR FUNCTION IS BEING COLLECTED.** MathWorlds prevents network communication when it is running!
2. **DO NOT EXIT MATHWORLDS.** Exiting, rather than Pausing, will cause MathWorlds to delete the data it needs to Resume processing your work. If you Exit MathWorlds you will start at the Main Menu and you'll have to open the activity again.
3. **Leave the calculator in Pause, at least until you see the word SUCCESS appear on the display.**

Lesson 3: Summarizing Individual and Group Understanding

NAME: ____________________

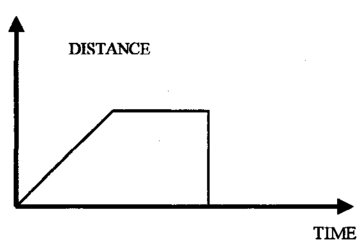
Activity 1: Distance vs. Time Graphs

Instructions for the students:

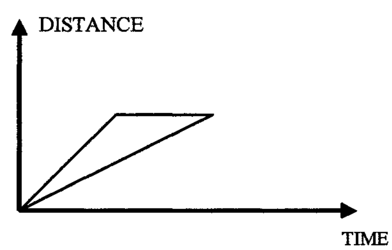
Circle the Distance vs. Time Graph That Goes with This Motion (If there's more than one, circle them all.)

Henri walks away from the zero-mark, stops for awhile, and then returns to his starting point.

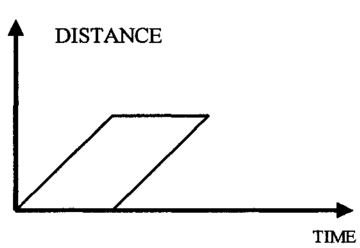
(a)



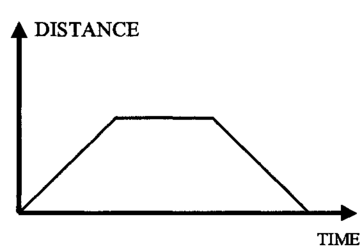
(b)



(c)



(d)



Lesson 3: Summarizing Individual and Group Understanding

Activity 2: Santa's Having a Bad Day.

NAME: ________________

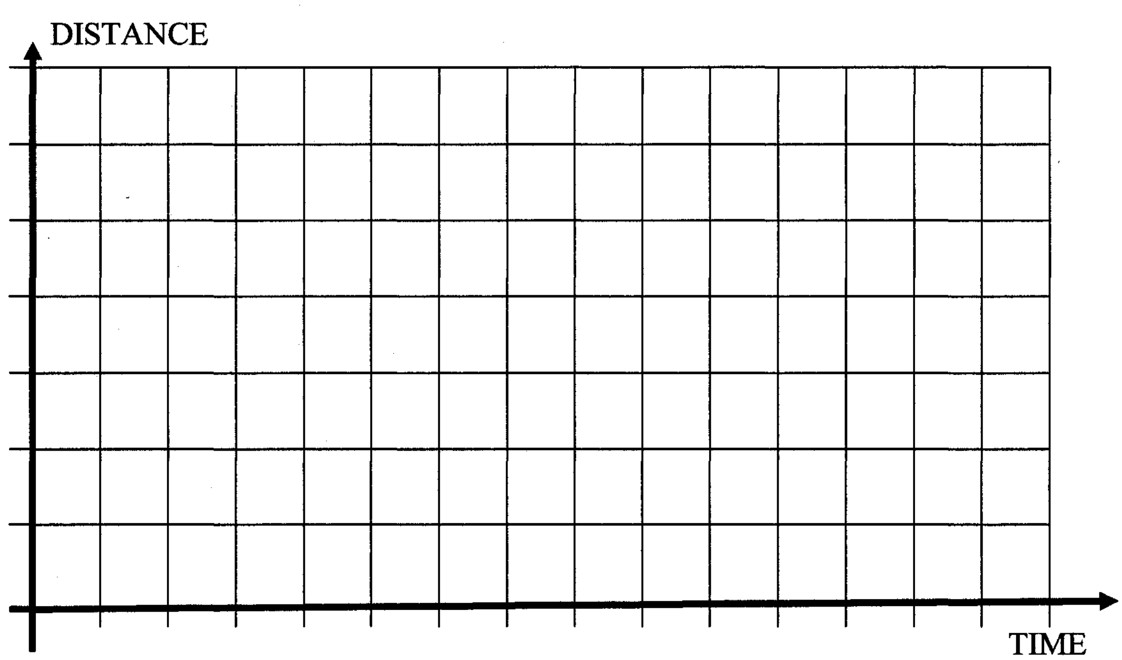
Instructions for the students:

Things are tough up at the North Pole. No snow, the reindeer broke out of their pen. Elves quit.

Santa made a 7-part trip as described below in A to G.

On the Distance vs. Time axis system below, graph his trip using segments. Label the segments with the letters A to G so we can see which part of your graph goes with which part of his trip.

A. Santa heads off on foot walking slowly with his heavy bag.
B. After awhile, he decides to drop the bag and rush back to his starting point to get his wagon-bike.
C. He jumps on the bike & heads out at a fast pace to pick up his bag.
D. He stops for 1 second to toss the bag in the wagon.
E. He then continues at his same fast pace till his front tire blows out.
F. He stops for one second to jump off the bike and grab the bag.
G. Finally, he heads off at his same slow walking pace again for the rest of his journey.



Lesson 3: Summarizing Individual and Group Understanding

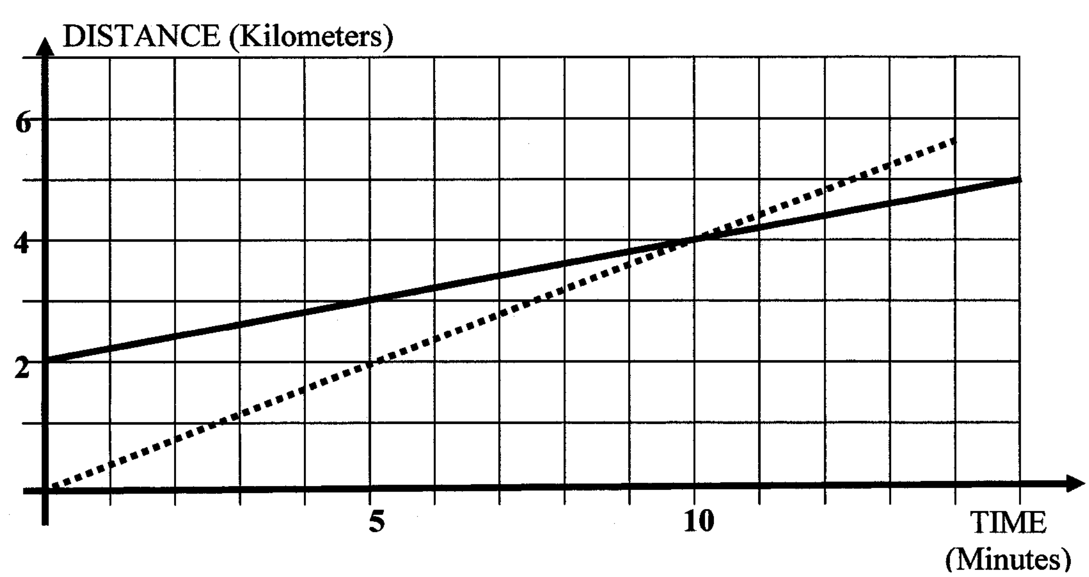
Activity 3: Comparing Bikers A and B

NAME: ________________

Instructions for the students:

Below are graphs of two bikers who start a trip at the same time. **A** has the dotted graph and **B** has the solid graph. Use these graphs to answer the questions below. Time is in minutes and distance is in km from the starting line, which is at zero km.

1. Where does **A** start? ____
2. When does **A** finish his trip? _________________________________
3. Where does **B** finish his trip? _________________________________
4. Which biker travels faster? ____Explain your answer:____________________
5. Which biker has traveled farther at 5 minutes? ___ Explain your answer:_____ ____________________
6. Who is ahead at 5 minutes?_____________
7. Is there a time when the 2 bikers are the same dist. from the starting line? Y N
   a. If your answer to #6 is Yes, what is that time? ________
   b. If your answer to #6 is Yes, what is the distance? ________
8. Which biker traveled the greater distance over its entire trip? ________



Appendix 2 – Teachers' guide

Lesson 1: Exploring Physical Motion

Activity 1: Getting Started

Instructions for the teacher:

- The teacher will open MthWrlds from the APPS menu and then choose Lessons 3, the *CBR* Lesson, which has one section. Then choose the 1st activity – "*CBR* Motions".
- Ask for one student volunteer and have that student move freely in front of the *CBR*. Ask all other students to observe what is happening. Repeat this with 2 or 3 (or even more, it depends on how this part goes on) other volunteer students.
- Make sure that all of the students are ready to make observations and conjectures about what the graph corresponds to.
- The teacher will ask questions in order to see what are the students' observations and conjectures about what is happening. The teacher must be as neutral as possible regarding the students suggestions. He may reformulate students observations, but must not give his opinion about their correctness.
- The teacher should make a short summary about what the students had observed (it may be a review of the various students' conjectures for example).

Instructions for the students:

In the space provided below, write down your observations regarding what you think is happening. Following the demonstration, the teacher will ask you to share your observations and explain the relationship between the *CBR* motion detector and the graph.

Write your observations below.

Activity 2: Matching Motions

Walk a physical motion so that A matches B's motion as closely as possible.

Instructions for the teacher:

- Have the students get into groups (4 or 5).
- Ask the students to reflect and discuss in their groups regarding how someone would have to **move in front of the CBR in order to match B's motion.**
- Have them document the motion in their workbooks.
- Ask for one volunteer student to create a motion.
- Have the spokesperson from a different group read their directions to the volunteer student who will walk their motion. (Please make sure that the volunteering students understand that they must try and follow the instructions as closely as possible.)
- Repeat so that all groups have an opportunity to give their directions.
- Let the students validate the description and the correspondence between the obtained graph and B's motion graph (situation of validation – the teacher should organize a discussion about the correctness of students' descriptions; the students are asked to say whether the reproduced graph is close to B's motion graph, and if not, say what was wrong: the instructions coming from the description or the student who was moving in front of *CBR* hadn't follow the instructions correctly).
- Ask the students in the class to **describe how the differences between A's Position graph and B's Position graph match specific differences in their motions.**

Instructions for the students:

Describe how someone would walk a physical motion so that A matches B's motion as closely as possible. One student from another group will move in front of *CBR* following your description and he should reproduce a graph matching B's motion as closely as possible (you can of course modify the formulation of the task, but it is important to add this to motivate the students to do a good work).

In the space provided below, write down a description of a physical motion for A. (One description per group is required.

Write your description below.

Lesson 1: Exploring Physical Motion

Activity 3: Catch Up Motions

Walk a physical motion so that A starts off slower than B, but catches up to B at the end of the motion, at 6 seconds.

Instructions for the teacher:

- Keep the students in their groups.
- Ask the students to reflect and discuss in their groups regarding how someone would have to move in front of the *CBR* in order to **walk a physical motion so that A starts off slower than B, but catches up to B at the end of the motion, at 6 seconds.**
- Have them document the motion in their workbooks.
- Ask for one volunteer student to create a motion.
- Have the spokesperson from a different group read their directions to the volunteer student who will walk their motion. (Please make sure that the volunteering students understand that they must try and follow the instructions as closely as possible.)
- Repeat so that all groups have an opportunity to give their directions.
- Ask the students in the class to **describe how the differences between A's Position graph and B's Position graph match specific differences in their motions.** (It is important that the teacher only intervenes to draw the students attention to possible inconsistencies, and to encourage more precision in the discussion around the concepts.)

Instructions for the students:

Describe how someone would walk a physical motion so that A starts off slower than B, but catches up to B at the end of the motion, at 6 seconds.

In the space provided below, write down a description of a physical motion for A. (One description per group is required.)

Write your description below.

Lesson 1: Exploring Physical Motion

Activity 4: The Challenge!

Groups will challenge each other to create graphs of interesting motions.

Instructions for the teacher:

- Keep the students in their groups (an even number of groups would be better here, so that you can have pairs of groups challenging each other).
- Ask the students to reflect and discuss in their groups on potential graphs that may be difficult to recreate while walking in front of the *CBR* (eg. the first letter of some names).
- Have each group decide on one graph to be used in the challenge. Have them sketch this graph in their workbooks. Remind them that it must be possible to reproduce.
- Ask one group to challenge another group.
- The challenged group must have one member get ready to walk a motion.
- Have one spokesperson from challenging group read their directions to the volunteer student who will walk their motion. (Please make sure that the volunteering students understand that they must try and follow the instructions as closely as possible. Also, remind the students that no one else should be talking. (easier said then done☺)
- Repeat so that all groups have an opportunity to give their directions.
- Ask the students in the class to **describe how the differences between A's Position graph and B's Position graph match specific differences in their motions.** (It is important that the teacher only intervenes to draw the students attention to possible inconsistencies, and to encourage more precision in the discussion around the concepts.)
- Ask students to volunteer to re-create the first letter of their names.

Instructions for the students:

Create a graph for a physical motion that you feel would be difficult for other teams to reproduce. However, make sure that it is possible to reproduce it.

In the space provided below, draw a detailed sketch of the graph and a description of the physical motion needed to create it. (One graph and description per group is sufficient.)

Sketch of the graph:	Description of the physical motion:

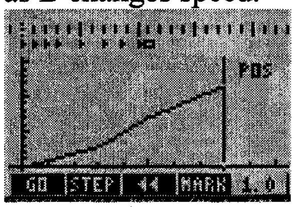
Lesson 2: Modeling and Piecewise Defined Functions

Activity 1: Getting Started

Instructions for the teacher:

- The teacher will open menu item 1, "Piecewise Animations" from the Lesson 2 "L2: WarmUp" in the MathWorlds application.
- The teacher will simply run the animation by pressing the GO SoftKey and **ask the students to explain what is going on.**
    - The goal is to get them to recognize that B's motion relates closely to the graph, and that, as the discussion proceeds, they should come to see it as a Position vs. Time graph for B's motion. The teacher will want to have the student bring up, via questions, discussion and by highlighting some formulations, that the vertical axis measures the position of B while the horizontal axis gives its time.
  - Following a discussion regarding B's motion, the teacher will establish a common framework and language, informing the students to treat the object on the lower part of the screen as B, which can be thought of as a person moving, and where the tic marks measure meters.
  - Ask:
    - **When is B going the *fastest*?**
    - **When is B going the *slowest*?**
    - **When does B seem to *change* its speed?**
  - The teacher can also step through the animation to slow things down a bit, giving the students an opportunity to examine the motions and graphs more closely and to help them create a common language and agree on some common meanings.
  - <u>Using the STEP SoftKey</u>. Reset the animation and then press the **Step** SoftKey. Notice that this first press of the **Step** SoftKey places the time cursor on top of the vertical axis (our motions begin at time equal to 0 seconds) making the vertical axis appear dashed. Each subsequent press advances the animation one Step-Time value, which is controlled by the right-most SoftKey. When the teacher reaches the end of the animation, they could press the reset, ◄◄, SoftKey to put the animation back to the beginning.
    The Step-Time is set to **1.0** now but can be changed by pressing the right-most Softkey (it cycles through **1.0**, **0.5**, **0.25**, and **2.0.**) It is recommended that the teacher leave the Step-Time at **1.0**.
    With each press of the **Step** SoftKey, the teacher should ask students: **How far does B move?**
  - <u>Dropping MARKS</u>. After a couple of runs and discussion of the questions, it will be helpful to have B "leave Marks." Marks are dropped by objects at regular time intervals, again controlled by the right-most Step-Time SoftKey, as they move.
  - The teacher will press the 4th SoftKey, labeled NONE, which changes it to MARK indicating that Marking is ON. This indicates that marks are dropped at the Step-Time shown in the rightmost *SoftKey* —when the animation is running or

during Stepping. Now, when B moves, it will leave Marks at each 1.0-second time interval during the trip so students can see more clearly how far B travels in each second. Students should note how the distance between the Marks changes as B changes speed.



The teacher will ask students the following questions:

- **How far apart are the Marks in the 1st part of the trip? How far apart are they in the 2nd part of the trip? How far apart are they in the 3rd part of the trip?**
- **Exactly how fast is B moving during each part of the trip?**
- **Which part of the graph is the steepest and which part is the least steep?**

• The teacher will then have the students debate and validate all of the answers.

Instructions for the students:
This activity is designed to familiarize you with a new piece of software. It is very important to understand how this software works to be able to complete the other activities.

Lesson 2: Modeling and Piecewise Defined Functions

Activity 2: Creating Exciting Sack Races

Instructions for the teacher:

- Keep the students in their groups.
- The students and teacher will open menu item 1, "Piecewise Animations" from the Lesson 2 "L2: WarmUp" in the MathWorlds application.
- The teacher will simply run the animation by pressing the GO SoftKey and **ask the students to reflect and discuss what is going on in their groups**. The students should also be encouraged to explore the animation by pressing the GO SoftKey on their own handheld.
  - Ask the students to reflect, discuss, and document a group response to the following questions:
    - Which graph goes with which object?
    - How are the motions of A and B different?
    - How long does A travel – in time and in distance?
    - How long does B travel – in time and in distance?
  - The teacher will then ask the students to exchange and compare observations regarding **how the motions of A and B different?** (For example, we want them to note how B's graph is shorter and B's duration is shorter.)
  - The same remark as the previous one regarding the debate and validation of the students' answers.
  - The teacher will then explain that **they now want to make a motion for B by extending B's Position vs. Time graph so that B enacts an exciting Sack Race with A which ends in a tie.**
    - In the context of the situation (from a race point of view), the teacher will ask the students: From the Race point of view, what is happening early in the race? (B is starting slowly, falling behind A.)
  - Ask the students to reflect, discuss, sketch, and create a graph for a motion that would satisfy the following criteria:
    - Due to the wild burst of speed, B falls down for 2 seconds!
    - In the confusion of falling down, B gets up and goes in the wrong direction.
    - The race must end in an exciting tie!
  - All students in a group should have a graph for the motion that satisfies the above criteria.
  - A representative from each group must take their animations to another group to validate that it meets the race criteria.
  - The teacher will then ask the students to exchange and compare observations regarding **how the motions were similar and how they were different.**

Instructions for the students:

- Open menu item 1, “Piecewise Animations” from the Lesson 2 “L2: WarmUp” in the MathWorlds application.
- Run the animation by pressing the GO SoftKey and **reflect and discuss what is going on in your group**.

    - Reflect, discuss, and answer the following questions:

Which graph goes with which object?

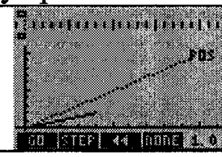
How are the motions of A and B different?

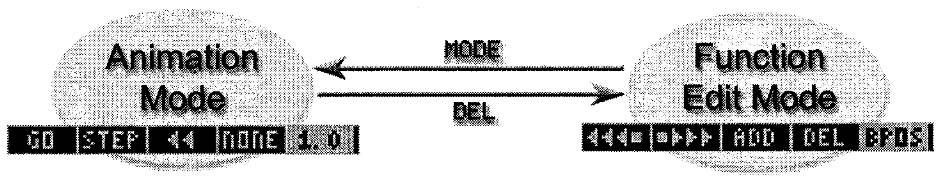
How long does A travel – in time and in distance?

How long does B travel – in time and in distance?

- Reflect upon, discuss, sketch, and create a graph for a motion that would satisfy the following criteria:
    - Due to the wild burst of speed, B falls down for 2 seconds!
    - In the confusion of falling down, B gets up and goes in the wrong direction.
    - The race must end in an exciting tie!
    - Every team member needs to have a graph for a motion that satisfies the above criteria.
    - Be ready to explain your motion and have your animation assessed by another team.

This activity opened in Animation Mode.





You switch to Function Edit Mode to make changes to your function. Notice that the SoftKey labels change when you switch modes.

You'll use the blue Cursor Keys and HotSpots, shown above, to edit your graph.

In the space provided below, draw a sketch a detailed sketch of the graph and provide a story for the physical motion needed to create it. (One graph and story per group is sufficient.)

Sketch of the graph:

Story for the physical motion:

Lesson 2: Modeling and Piecewise Defined Functions

Activity 3: Find Your Exciting Sack Race

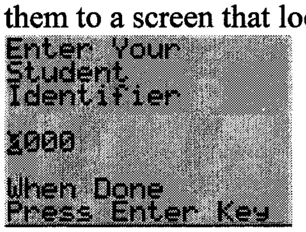
Instructions for the teacher:

- Students remain in their groups
- Collect the students' Sack-Race functions –their B functions, which they have just created in Activity 2.
- Display and animate them on the screen.

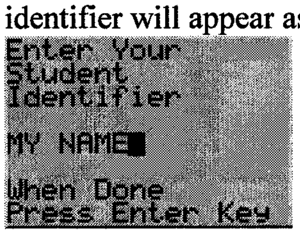
- **In order for the collection to be successful, the following instructions must be followed.**

1. Make sure that each calculator is connected to a Hub with the black wire firmly plugged in at each end.

2. First tell the students to **press the 2nd Key followed by the STO Key**. This takes them to a screen that looks like this:

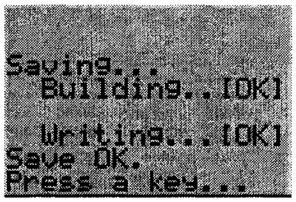


3. Student should use the DEL Key to erase what appears in the input field and then use the ALPHA Keys to input a short version of their name, or their initials. This identifier will appear as the name of the student's function in Java MathWorlds.



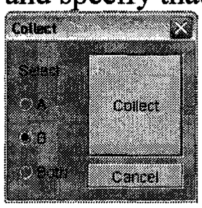
This student's B function will appear as "MY NAME_B", where the "B" is used to differentiate from the "A" function which may be collected in other activities.

4. When the Students have entered their names they must press the ENTER Key. They will see this screen:

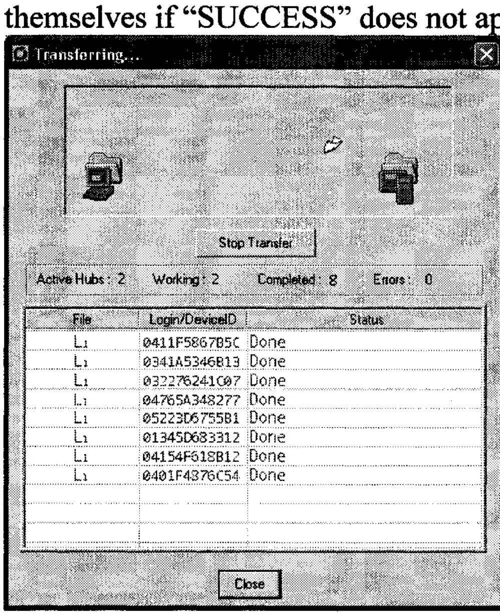


Press the ENTER Key a second time to Pause MathWorlds. Now the teacher can Collect your B-function!

5. Now, from the Connect Menu in Java MathWorlds, the teacher will select Collect, and specify that they are collecting B only.



The teacher will be able to identify how many functions have been collected by looking at the Transfer window during the transfer. Also, you can ask students to identify themselves if "SUCCESS" does not appear on their screen.

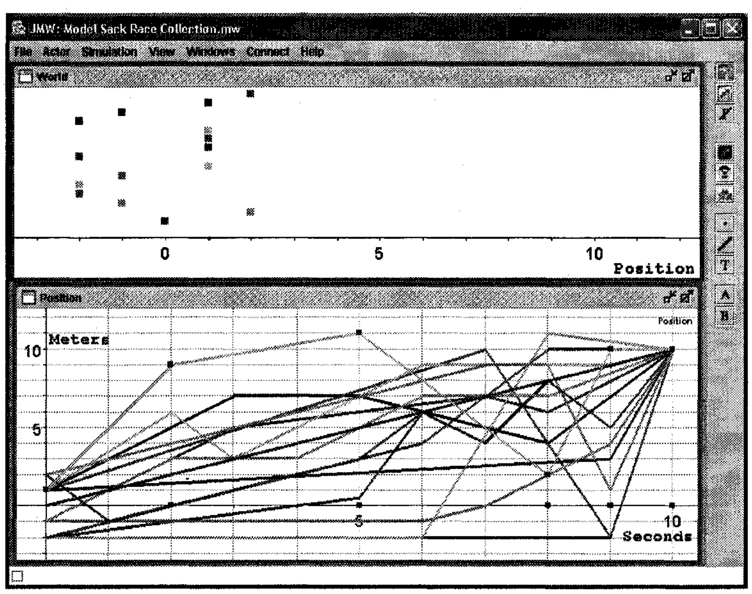


None of the collected functions or actors (in the Dots World) will be displayed until you decide to display them.

6. Showing All the Students' Dots – Viewing the Slow B's and Fast B's. The teacher will show "All" in the View Matrix World (Graph) column to show all of the students' dots. The goal here is to introduce students to the process of relating their personal constructions to the larger collection of objects that appears on the "big screen" when their work is aggregated with that of their peers. This process requires them to think through exactly the kinds of issues that are at the heart of the mathematics we want

them to learn. *Hence, before clicking **Apply**, the teacher will ask the students:*
Where will all the Slow B's appear, and where will all the Fast B's appear? It is important that the teacher instill the habit of getting a prediction from the students before any display action. Here the fast B's all appear to the left of 0 and the slow ones to the right, with those sharing the same initial position "stacked vertically." (Note that it is expected that the teacher will refer to the "students" and their "dots" interchangeably. It is expected that the students will be the ones to start using this terminology when identifying themselves with "their dot." This is a desired behavior because it links the student psychologically to the mathematical object that they have built.)

7. Anonymity and "Finding Yourself": before asking the next question, the teacher will "hide" the identity of the functions and their "owners." Provision has been made to preserve student anonymity – the teacher will click the box in the lower left corner of the screen where identifiers appear. Then no names will appear either here or when we hover over a dot or graph. The teacher will then ask:
Where are you? Can you find yourself?
If there is a position with a single dot, then a single student should be able to identify himself/herself. It can be confirmed by selecting it (by clicking on it) and then checking the box in the lower-left corner of the screen, where that student's identifier will appear.



It is usually fun to run the animation with all the dots showing. But the detailed analysis is best done with a smaller set of dots, which follows.

8. Preparing for the Graph-Motion Connection Investigation – Student Identifiers and Colors.
The teacher will need to look at their graphs to see if they can connect their respective graphs to their dot. This is important because it is likely to be the case that for most positions from –2 to +3 there will be more than one dot at that position.

9. Using the View Matrix to Narrow the Focus to a Few Students and Their Graphs.
The teacher will now open the View Matrix again so that they can hide all but the

relevant dots, plus A, and then show only the graphs of the chosen set of dots.

10. Relating Graphs to Motions: Here the idea is to get the students whose graphs are NOT now displayed to determine which graph goes with which dot. (Their owners presumably already know.) The teacher will tell the students:
I will run the animation and your job is to figure out which graph goes with which dot.
Now run the animation. Depending on how different the motions and graphs are, the teacher may need to Step through the motions – use a Step-Time of 1 second (which you can set by opening the bottom part of the Controls Window). This is an important learning opportunity to examine subtle differences in the graphs and how they are reflected in differences in the motions, so the teacher should repeat the Stepping and encourage discussion till a consensus has developed.

11. Relating Graphs to Motions For More Sets. The teacher will repeat the above process for another set of dots with the same initial position:
(1) Show all the dots, (2) pick a set, (3) make their colors the same, (4) display the dots and position graphs for that set, (5) run or step the animations as needed until the non-owners of the set have formed a consensus.

12. Dealing with Student Errors. While relating students to their functions, and especially their motions to their graphs, is a powerful way of getting students engaged mathematically, it is also a place where your experience as a teacher and your knowledge of your students directly come into play. You know who is likely to err, who is likely to be embarrassed, who enjoys attention, and so on. You can also quickly review the student function graphs before making them public and not choose to display those that you feel would either be unproductive to examine or embarrassing to their creators. The technology amplifies the impacts of your pedagogical decisions.
If it is a group production, it will perhaps be easier to deal with errors. There are usually not such psychological effects in groups. The students must be involved in the validation of the answers.

13. Resuming and Exiting the Application: When this activity is complete, the teacher should have the students resume the application if they haven't already, and then EXIT the application as described above (**2nd-QUIT** followed by **EXIT**). Otherwise the next time MathWorlds is selected in the APPS menu, MathWorlds will open to the prior state rather than to the menus, requiring a **2nd-QUIT** to get to the menus.

Students should now be ready for the next set of activities.

Instructions for the students:

- The teacher will collect your Sack-Races. It is important to listen to his instructions.

Activity 4: Mathematical Performances – Exciting Races

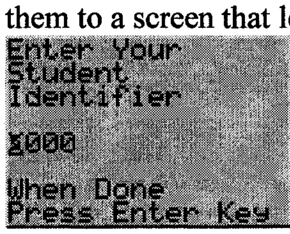
Instructions for the teacher:

- Collect the students' Sack-Race functions –their B functions, which they have created using another group's story-script.
- Display and animate them on the screen.

- **In order for the collection to be successful, the following instructions must be followed.**

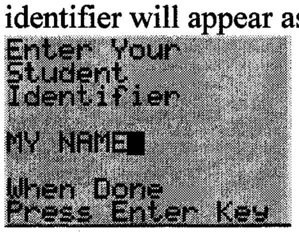
1. Make sure that each calculator is connected to a Hub with the black wire firmly plugged in at each end.

2. First tell the students to **press the 2nd Key followed by the STO Key**. This takes them to a screen that looks like this:



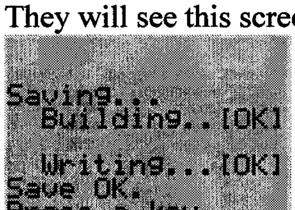
(This is where they enter their names so you can identify their function later. Names must be unique!)

3. Student should use the DEL Key to erase what appears in the input field and then use the ALPHA Keys to input a short version of their name, or their initials. This identifier will appear as the name of the student's function in Java MathWorlds.



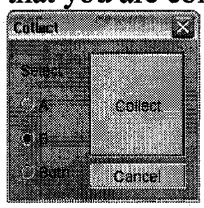
This student's B function will appear as "MY NAME_B", where the "B" is used to differentiate from the "A" function which may be collected in other activities.
(As usual, using the ALPHA Keys is easier if you press **2nd** Key followed by the **ALPHA** Key, which keeps the ALPHA Keys active till you press the **ALPHA** Key again.)

4. When the Students have entered their names they must press the ENTER Key. They will see this screen:



Press the ENTER Key a second time to Pause MathWorlds. Now the teacher can Collect your B-function!
(MathWorlds will put their function data into List 1 where it can be collected using Java MathWorlds. Their function data is stored in List 1 in active RAM, whereas the state of the calculator is archived as a list, SS0, that is reactivated when the students return to MathWorlds by choosing it from the APPS Menu.)

5. Now, from the Connect Menu in Java MathWorlds, select Collect, and specify that you are collecting B only.



When the teacher presses the Collect button, MathWorlds will collect the B function from each student whose TI-83 Plus is connected with a black wire to a hub and is not running MathWorlds. The teacher will be able to identify how many functions have been collected by looking at the Transfer window during the transfer.

There are likely to be widely varying functions of many shapes and here the teacher will need to use judgment regarding whose to show first, whose to ignore, etc. The teacher's choice will be informed both by the class dynamics, questioning of the students regarding what they did and wrote, looking at their calculator screens, etc. (Depending on the teacher's and the students' style, there could be some wild stories!) However, the teacher will also want to take a quick look using the View Matrix of the students' Position functions. Hence:

6. (Privately) Viewing All the Students' Position Graphs. The teacher will first view all of the students' Position graphs (privately) before making a choice regarding whose to display and run.

7. The teacher will then ask the student to read the story as they run the animation. The student will read it in advance, run the animation, and finally Step through the animation as the story is repeated because the animation is usually too quick to parallel the story.

Again, the students must be involved in the discussion of the correctness of the productions.

8. Looking at the Motion, Story and Graph More Closely, and Doing More Performances. Depending on the story and the graph, the teacher may be able to glean a lot of learning by analyzing things more closely once the performance aspect has occurred.

Using the Collected Functions to Support More Learning
Beyond the performance opportunity, typically there are many opportunities to pursue important mathematical ideas across the collected functions and stories, including:

- Steeper Means Faster
- Horizontal Means Stopped
- Negative Slope Means Backward Motion
- Simultaneity (when do A and B's graphs cross?)
- The Difference Between Parallel Graphs and Coincident Graphs

There are also opportunities to examine issues of how realistic are the models? For example, consider continuity of change. Could any physical object move the way the animations do? (It's an interesting contrast where objects with stable mass never move with discontinuous velocity, but other, say money quantities, almost <u>always</u> have discontinuous rate-changes.) The *CBR* activities in Lesson 1 provide direct contact with these issues as students (among many other activities) attempt approximations of "corners."

9. Relating Graphs to Stories: After running and discussing a few story-graph pairs, an interesting twist is to turn off the identifiers (making the graphs anonymous) and pick a small set of graphs and dots to display, say 3-4. The teacher will then ask one student, whose work is displayed, to read their story and ask the class to figure out which graph and dot goes with the story. The resulting discussions will help the students in deepening their graph interpretation skills while simultaneously giving additional students the floor, especially those with graphs and stories that might not be very original or distinct.

The teacher will say: **I will run the animation and your job is to figure out which graph and dot goes with the story.**

Given that the animations were created from story-scripts that were given to the students by other groups, ask if there are any differences between the original graphs that accompanied the original story-scripts.

Feel free to repeat the animation or stepping and encourage discussion till a well-reasoned consensus has developed regarding the fit.

Instructions for the students:

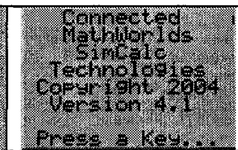
- In your groups, create an exciting sack rack story-script for your own race with A which ends in tie, and create a Position vs Time Graph for B that makes your race happen.

Write your description below.

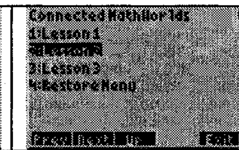
Creating a Position vs Time Graph to model your exciting sack race.

- You'll open MathWorlds from the APPS menu and choose the 1st of 2 activities – "Sack Races."

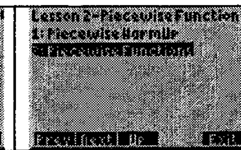
Choose MthWrlds



Press Enter Key



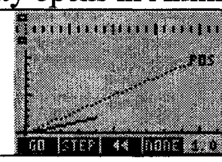
Select this item

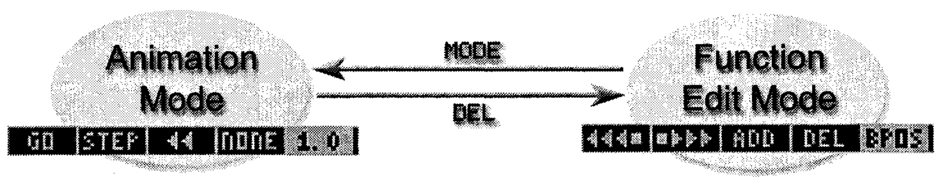


Select this item

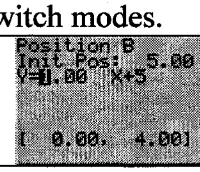
Select this item

This activity opens in Animation Mode.





You switch to Function Edit Mode to make changes to your function. Notice that the SoftKey labels change when you switch modes.



You'll use the blue Cursor Keys and HotSpots, shown above, to edit your graph.

1. Press the MODE or DEL Key to switch to Function Edit Mode.
   Make an exciting Position graph for B by adding and adjusting segments on B's Position graph.
   Scaling: Note that the scale of the "world" is now 1 m, so A travels 20 m in 10 sec. The vertical scale of the Position graph is no longer 1, but 2 m.
   Because the vertical scaling is now 2 m, there are times when you **ADD** a new segment that is 1 sec wide your new segment will look horizontal. However, when you stretch it to the right to 2 sec in width, you see that it slopes upward by 1 tic mark, which stands for 2m. Hence to make it flat, you need to drag it down by one tic mark.

2. Be careful not to extend your segments too far or add so many segments that your graph extends far off the screen. If this happens, then you will be adding, deleting or adjusting segments that are out of sight and it will seem like nothing is happening! You might then need to delete segments (using the **DEL** SoftKey, NOT the DEL hardkey) till you get back to something you can see! At least for now, try to keep most of your graph and motion on the screen.

3. You can adjust segments to the left of the last one by moving the Hotspot from segment to segment—use the two left-most SoftKeys that look like arrows, which move the HotSpot left or right.

4. When you think you have the Position-graph you want, return to Animation Mode to try out your race. To re-adjust it, go back to Function Edit Mode.

5. As you are testing and finalizing your race, write your script in the form of a list of descriptions, one for each segment of your graph. Draw your graph on paper to accompany your written story. It may be helpful to label your list using letters A, B, C, etc.

Sketch of the graph:

6. Bring your exciting sack race story-script to one of the other groups so that they could create a Position vs Time Graph to match it.

7. Create a Position vs Time Graph for the exciting sack race story-script that was delivered to you from another group. (Everyone in the group should create this Position vs Time Graph.)

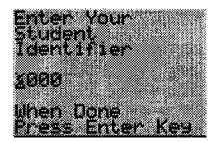
Sketch of the graph:

8. When you have the motion and function you want Pause MathWorlds so your teacher can collect this function.

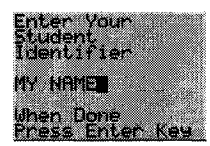
PAUSING MATHWORLDS TO
HAVE YOUR FUNCTION COLLECTED

Make sure that your calculator is connected to a Hub WITH the black wire firmly plugged in at each end.

Pressing the 2nd Key followed by the STO Key takes you this screen where you enter your identifier:

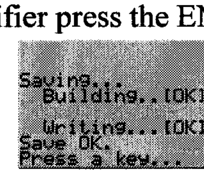


Now use the ALPHA Keys to enter your name, or your **Group Number-Count-Off Number** (like **0304** for **Group Number 3, Count-Off Number 4**) followed by your name, as directed by your teacher.



*Using the ALPHA Keys to enter letters is easier if you press 2nd Key followed by the **ALPHA** Key. This keeps the ALPHA Keys active till you press the **ALPHA** Key again.*

When you have entered your identifier press the ENTER Key. You'll see this screen:



Press the ENTER Key to <u>Pause</u> MathWorlds. The calculator will return to its Home Screen, ready for your teacher to collect your function.

KEEP THE FOLLOWING IN MIND

1. **DO NOT RUN MATHWORLDS WHILE YOUR FUNCTION IS BEING COLLECTED.** MathWorlds prevents network communication when it is running!
2. **DO NOT EXIT MATHWORLDS.** Exiting, rather than Pausing, will cause MathWorlds to delete the data it needs to Resume processing your work. If you Exit MathWorlds you will start at the Main Menu and you'll have to open the activity again.
3. **Leave the calculator in Pause, at least until you see the word SUCCESS appear on the display.**

Lesson 3: Summarizing Individual and Group Understanding

Activity 1: Distance vs. Time Graphs

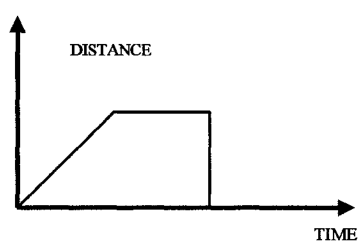
Instructions for the teacher:

- Have the students complete this activity.
- Lead a discussion to validate the correct answer and to provide rationale for not choosing the distracters.

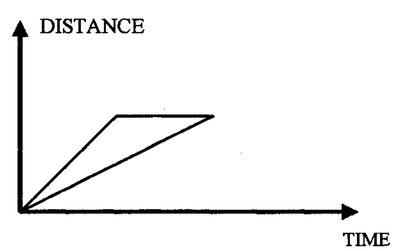
Instructions for the students:
Circle the Distance vs. Time Graph That Goes with This Motion (If there's more than one, circle them all.)

Henri walks away from the zero-mark, stops for awhile, and then returns to his starting point.

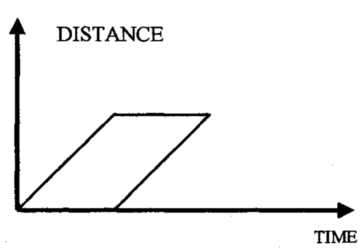
(a)



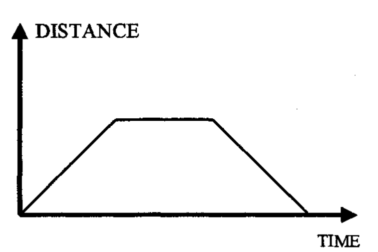
(b)



(c)



(d)



Lesson 3: Summarizing Individual and Group Understanding

Activity 2: Santa's Having a Bad Day.

Instructions for the teacher:

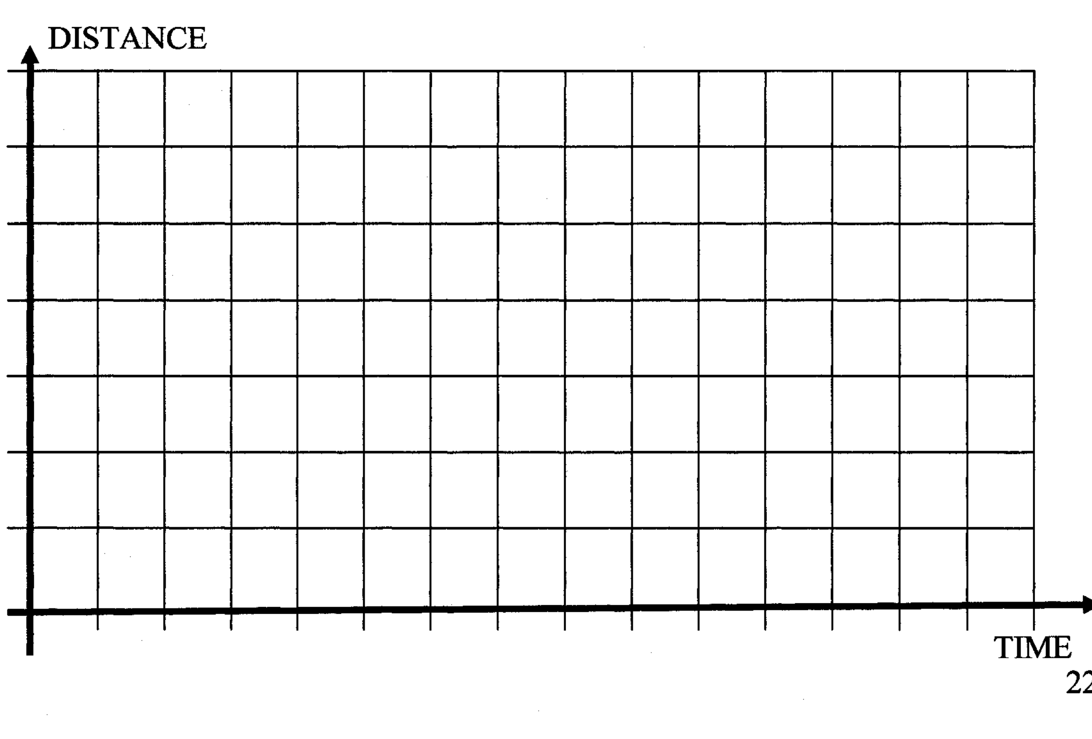
- Have the students complete this activity.
- Lead a discussion to validate, defend, and/or refine their solutions.

Instructions for the students:
Things are tough up at the North Pole. No snow, the reindeer broke out of their pen. Elves quit.

Santa made a 7-part trip as described below in A to G.

On the Distance vs. Time axis system below, graph his trip using segments. Label the segments with the letters A to G so we can see which part of your graph goes with which part of his trip.

A. Santa heads off on foot walking slowly with his heavy bag.
B. After awhile, he decides to drop the bag and rush back to his starting point to get his wagon-bike.
C. He jumps on the bike & heads out at a fast pace to pick up his bag.
D. He stops for 1 second to toss the bag in the wagon.
E. He then continues at his same fast pace till his front tire blows out.
F. He stops for one second to jump off the bike and grab the bag.
G. Finally, he heads off at his same slow walking pace again for the rest of his journey.



Lesson 3: Summarizing Individual and Group Understanding

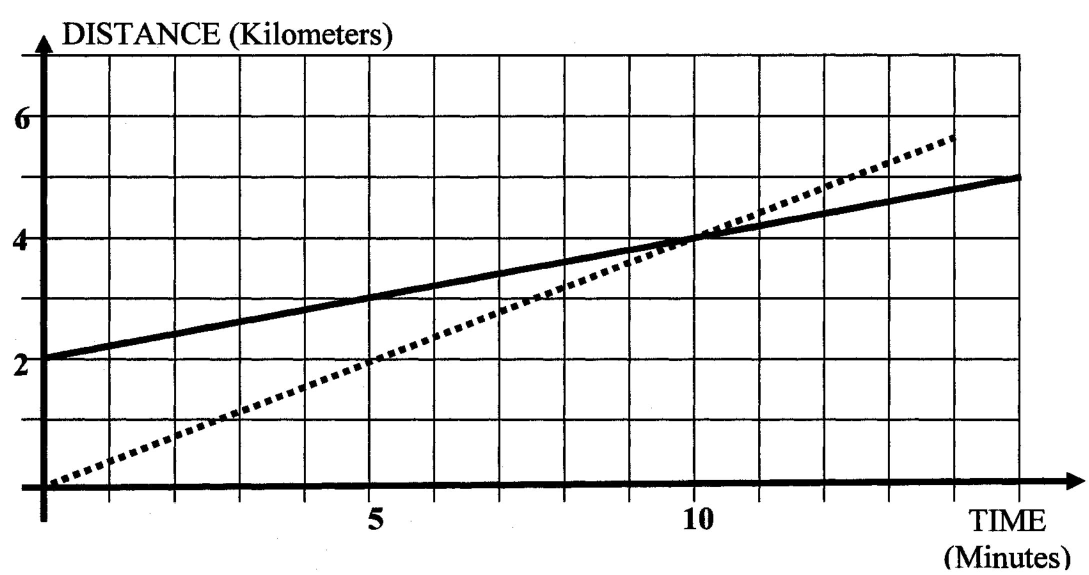
Activity 3: Comparing Bikers A and B

Instructions for the teacher:
- Have the students complete this activity.
- Lead a discussion to validate, defend, and/or refine their solutions.

Instructions for the students:
Below are graphs of two bikers who start a trip at the same time. **A** has the dotted graph and **B** has the solid graph. Use these graphs to answer the questions below. Time is in minutes and distance is in km from the starting line, which is at zero km.

1. Where does **A** start? ______________________________
2. When does **A** finish his trip? ______________________________
3. Where does **B** finish his trip? ______________________________
4. Which biker travels faster? ____Explain your answer:__________________
5. Which biker has traveled farther at 5 minutes? ___ Explain your answer:_____ ____________________
6. Who is ahead at 5 minutes?_____________
7. Is there a time when the 2 bikers are the same dist. from the starting line? Y N
   a. If your answer to #6 is Yes, what is that time? _______
   b. If your answer to #6 is Yes, what is the distance? _______

Which biker traveled the greater distance over its entire trip? _______



Appendix 3 - Certification of ethical acceptability