COMPUTER GENERATION OF EQUIVALENT NETWORKS


ABSTRACT

Computer techniques are presented for generating equivalent networks by linearly transforming nodal parameter matrices in an iterative manner. The techniques are restricted primarily to passive, reciprocal networks with one or two kinds of elements. It is shown that the transformation matrix can be determined by inspection of the parameter matrices so that "growth" of new elements can be controlled and negative elements avoided. The method is applied to generate equivalent networks with reduced element spread or with reduced sensitivity to component tolerances as well as to eliminate undesirable elements from an initial design.

The results presented here represent extensions to the theory of continuously equivalent networks which was formulated recently by J.D. Schoeffler.

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by

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A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Master of Engineering.

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August, 1967.
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# TABLE OF CONTENTS

## ABSTRACT

<table>
<thead>
<tr>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
</tr>
</tbody>
</table>

## ACKNOWLEDGEMENTS

<table>
<thead>
<tr>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ii</td>
</tr>
</tbody>
</table>

## TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>INTRODUCTION</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

## CHAPTER II CONTINUOUSLY EQUIVALENT NETWORKS

| 2.1 | Differential Equations of Transmission |
| 2.2 | The Concept of Row Comparison |
| 2.3 | Constraints on B Due to Row Comparison |
| 2.4 | Secondary Growth |
| 2.5 | Application to Network Optimization |
| 2.6 | Summary of Chapter II |

<table>
<thead>
<tr>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>12</td>
</tr>
<tr>
<td>16</td>
</tr>
<tr>
<td>19</td>
</tr>
<tr>
<td>28</td>
</tr>
</tbody>
</table>

## CHAPTER III INCREMENTAL EQUIVALENCE

| 3.1 | Equations for Incremental Equivalence |
| 3.2 | Calculation of the Values of Elements of B Matrix |
| 3.3 | Illustrative Examples |
| 3.4 | Summary of Chapter III |

<table>
<thead>
<tr>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
</tr>
<tr>
<td>32</td>
</tr>
<tr>
<td>33</td>
</tr>
<tr>
<td>49</td>
</tr>
</tbody>
</table>

## CHAPTER IV SENSITIVITY MINIMIZATION

| 4.1 | Derivation of Sensitivity Equations |
| 4.2 | Analysis of Adjoint Equations of Sensitivity Minimization |
| 4.2.1 | Invariance of the Sum of Sensitivities of Each Element Kind |
| 4.2.2 | Changes in Element Values and Changes in Their Sensitivity |
| 4.2.3 | Sensitivities of New Elements |
| 4.2.4 | Conditions for Invariance of Element Sensitivities |
| 4.3 | Effect of Constrained B on the Sensitivity |
| 4.4 | Problems Associated with Network Minimization |
| 4.4.1 | Method of Computing Elements of q (0) |
| 4.4.2 | Stability of Adjoint Equations |
| 4.5 | Computation of Sensitivity |
| 4.6 | Computer Program |
| 4.7 | Illustrative Examples |
| 4.7.1 | Reduction of Pole Sensitivity in RC Network |
| 4.7.2 | Reduction of Classical (Bode) Sensitivity in Terminated LC Network |
| 4.8 | Summary of Chapter IV |

<table>
<thead>
<tr>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
</tr>
<tr>
<td>55</td>
</tr>
<tr>
<td>55</td>
</tr>
<tr>
<td>56</td>
</tr>
<tr>
<td>58</td>
</tr>
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<td>58</td>
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<td>60</td>
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<td>61</td>
</tr>
<tr>
<td>63</td>
</tr>
<tr>
<td>63</td>
</tr>
<tr>
<td>65</td>
</tr>
<tr>
<td>68</td>
</tr>
<tr>
<td>69</td>
</tr>
<tr>
<td>72</td>
</tr>
<tr>
<td>74</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

Network synthesis techniques often yield networks which are not fully satisfactory for practical realization. Element values may have large spread, or the network may be unduly sensitive to the changes in element values. These difficulties may force the designer to repeat the synthesis many times until a more satisfactory design is obtained. Alternatively, one can regard the output of a synthesis procedure as merely a starting point and obtain a more suitable network by generating equivalent networks.

The generation of equivalent networks by linear transformation was first proposed by Cauer\(^1\) in 1929 and independently by Howitt\(^2\) in 1931. Both Cauer and Howitt applied this transformation to the loop impedance matrices. It can also be applied to the nodal admittance matrix, which has the advantage of easier identification of network elements. In the latter case the transformation is given by

\[
Y' = A^\dagger Y A
\]  

(1.1)

where \(A\) is \(n \times n\) nonsingular constant transformation matrix, \(Y\) is \(n \times n\) nodal admittance matrix of the original network, and \(Y'\) is the matrix of the new network.

If the transformation matrix \(A\) is properly selected, the new network obtained by this transformation will have the same driving and transfer impedance as the original network. To retain such invariance it is necessary for the \(m^{th}\) and \(n^{th}\) rows of \(A\) to be unit vectors if the port nodes are selected as \(m\) and \(n\).
The transformation described by equation (1.1) must be applied in discrete steps. Because of this "one-shot" nature of the transformation, it is practically impossible to employ this technique to progressively optimize such empirical quantities as element spread or sensitivity; moreover, it is also very difficult to select the elements of A so that a transformerless realization without negative elements can be obtained.

An important contribution which circumvents these difficulties has recently been made by Schoeffler who suggested to replace the one-shot transformation in equation (1.1) by a set of differential equations so that the transformation becomes continuous within a given vector space. For this purpose it is possible to imagine an incremental transformation which in the limit becomes a continuous transformation. For this purpose let

\[ A = I + B \Delta x \]

where \( I \) is an identity matrix and \( \Delta x \) is an increment of an independent variable \( x \), and where the absolute value of elements of \( B \) is bounded by unity. Upon substituting in equation (1.1) and passing to the limit as \( \Delta x \to 0 \), the equation becomes

\[ \dot{Y} = B^T Y + YB \]  \hspace{1cm} (1.2)

where \( \dot{Y} \) represents \( \frac{dY}{dx} \). Equation (1.2) is Schoeffler's equation of continuous equivalence. With equation (1.2) it is possible to generate equivalent networks iteratively in conjunction with the minimization of a performance criterion.

A similar approach has also been advocated by Calahan who used the concept of continuous equivalence in connection with Bashkow's state matrix A to generate networks with lossy elements from lossless networks. Calahan also proposed a
nonlinear transformation which can be used for generating equivalent networks. In this case the coefficients of the network are arrived at iteratively from topological formulae. It is assumed that the designer knows the network structure and has an estimate of the element values.

In this thesis the application of the Schoeffler's concept of continuous equivalence is investigated in two element-kind networks. The elements of the network are considered linear, passive, reciprocal, and it is sought to generate new networks which can be realized without mutual inductances. The performance criteria considered are reduction of element spread and sensitivity to parameter variations. It is assumed that the number of nodes and the definition of variables does not change in the transformation process.

The second chapter reviews briefly Schoeffler's equations and deals with the problem of restricting the equivalent network to contain a controlled number of positive elements without mutual coupling. For this purpose a relationship between the elements of the transformation matrix and of the elements of the nodal admittance matrix is established. The occurrence of "secondary growth" is discussed and the method of selecting a transformation matrix so that element spread is reduced with a limited number of new elements is shown by means of an example.

In the third chapter, a method is discussed for generating equivalent networks in a series of finite steps. It is shown that this method can be used to advantage when it is required to generate or eliminate an element in a particular locality. Attention is directed mainly towards the simplification of the linear transformation
technique through the selective reduction of the transformation matrix.

The fourth chapter deals with the generation of equivalent networks with reduced classical and root sensitivity. The relationship between changes in element values and changes in the corresponding sensitivity coefficients is discussed. The effect of constraints imposed on the transformation matrix by the sensitivity minimization procedure is demonstrated and reduction of sensitivity is shown by means of two examples.
CHAPTER II
CONTINUOUSLY EQUIVALENT NETWORKS

2.1 Differential Equations Of Transformation

For the purpose of the study of continuously equivalent networks it is necessary to examine in detail the matrix differential equation

\[ \dot{Y} = B^T Y + Y B \]  

(2.1)

Since the nodal admittance matrix can be expressed as

\[ \dot{Y} = Y_g + s Y_c + \frac{Y_r}{s} \]  

(2.2)

where \( Y_g, Y_c, \) and \( Y_r \) are the nodal conductance, capacitance and reciprocal inductance matrices and since the elements of \( Y \) are linear combinations of network elements, Equation (2.1) can be expressed as

\[ \begin{align*}
\dot{g} &= M_g \ g \\
\dot{c} &= M_c \ c \\
\dot{r} &= M_r \ r
\end{align*} \]  

(2.3)

where \( c, g \) and \( r \) are column vectors of the various capacitive, inductive, and conductive elements respectively. If the network contains \( p \) nodes, the dimension of these vectors is \( p (p - 1)/2 \) since each element is connected to exactly two nodes. If two nodes are not directly connected a branch of zero value can be thought of as joining
these nodes. Clearly the elements of $M_g$, $M_c$, and $M_r$ are linear combinations of the elements of the $B$ matrix. The solution of these equations therefore requires only the integration of differential equations with constant coefficients which can be easily carried out on a computer. The initial values of $c$, $g$ and $f$ are given by the admittances of the original network, many of which are generally zero. If the differential equation is replaced by the difference equation a new network will be formed which differs from the original one by only a small amount. Some of the elements which were originally zero may have now a small non-zero value, which depending on the sign of their derivative – and thus on the selection of $b$'s – may be positive or negative. The introduction of new elements is in the remainder of the text usually referred to as "growing" new elements. Prevention of the growth of negative elements is one of the major difficulties in generating equivalent networks. The network is therefore regarded as full graph in which initially some of the branches have zero admittance.

Because all network elements are clearly displayed in the off-diagonal entries of the indefinite nodal admittance matrix, it is not necessary to go through the process of actually forming the various $M$ matrices. It is possible to work with the off-diagonal entries of the indefinite nodal admittance matrix and the $B$ matrix in order to formulate any constraints required for a given transformation. The advantage is that it permits rapid visualization of the changes taking place in the network. Some caution must be exercised, however, since an off-diagonal element $y_{ij}$ of $Y$, the indefinite matrix, represents the negative of the admittance connected between nodes $i$ and $j$. 
2.2 The Concept of Row Comparison

A crucial task associated with generating equivalent networks continuously is to select the elements of the transformation matrix B so that the following conditions are satisfied:

1. Certain network functions remain invariant.
2. Elements of new network are positive.
3. Number of new elements in the new network is controlled.

In order to maintain the desired invariance it is necessary that all elements of certain rows of B equal zero. Determination of these rows for the numbering system of port nodes shown below is as follows (p is the reference node):

Case (a) Three Terminal Network

![Three Terminal Network](image)

FIGURE 2.1. THREE TERMINAL NETWORK.

If input only is to be kept invariant only row 1 has to be kept zero. If input, output and transfer functions are to remain invariant rows 1 and p - 1 have to be zero.
If input, output and transfer functions are to remain unchanged during the transformation, then rows 1, 2 and \( p - 1 \) must be zero. Should the reference node be other than one of the port nodes, then the corresponding row of \( B \) must also be zero.

Once the desired port invariance requirements are met, it is necessary to establish the relationship between the remaining elements of \( B \) and those of \( Y \) and \( \dot{Y} \) such that the latter two constraints are satisfied by the transformation. These constraints are somewhat related in the sense that both require control of the growth of new elements and can therefore be treated simultaneously. Since whatever constraints exist apply to individual parameter matrices, the notation adopted here is to omit the subscript which identifies the parameter matrix.

In order to establish the relationship between elements of the matrices, it is necessary to write out equations for them explicitly. If the element is not connected to
the reference node \( p \), the equation of the element of \( Y \) in the location \( mn \) is given by

\[
\hat{y}_{mn} = \sum_{i=1}^{p-1} y_{in} b_{im} + \sum_{i=1}^{p-1} y_{mi} b_{in}
\]  

(2.4)

where some of the rows of \( B \) are zero to preserve the desired invariance. Once again note that \( y_{st} \) is numerically equal to the element connected between the nodes \( s, t \) but with opposite sign. Similarly, any diagonal element \( y_{ss} \) is the sum of all elements connected to the node \( s \) and is generally a positive number.

The equations for elements connected to the reference node \( p \) can be obtained easily from the indefinite nodal matrix \( Y \). The equation of one such element in the location \( mp \) can be written as

\[
\hat{y}_{mp} = \sum_{i=1}^{p-1} y_{pi} b_{im} - \sum_{i=1}^{p-1} y_{mi} \sum_{j=1}^{p-1} b_{ij}
\]  

(2.5)

The linear nature of these equations makes it worthwhile to investigate the effect of element of \( B \), say \( b_{qr} \), on the generated network. The aggregate effect of the various elements of \( B \) can then be obtained by superimposing the individual effects. Considering Equation (2.4) first, it is apparent that since \( m \neq n \) any element of \( B \) can appear only in one or the other summation on the right hand side of (2.4). Assume that element \( b_{qr} \) (this represents a slight change in notation since either \( m \) or \( n \) can be equal to \( r \)), appears in the first summation. Equation (2.4) reduces to

\[
\hat{y}_{mn} = y_{qn} b_{qr}
\]
and because \( Y \) is symmetrical this can be written as

\[
y_{nr} = y_{nq} \cdot b_{qr}
\]

If \( b_{qr} \) appears in the second summation of (2.4) this equation reduces to

\[
y_{mr} = y_{mq} \cdot b_{qr}
\]

and by symmetry

\[
y_{rm} = y_{qm} \cdot b_{qr}
\]

There is no restriction on \( n \) or \( m \) except that \( n, m \neq p \); therefore the effect of \( b_{qr} \) can be stated as follows:

The effect of an element \( b_{qr} \) is to form elements in row* \( r \) of \( Y \) out of the row \( q \) of \( Y \).

Since the location of an element in \( Y \) indicates which element will be subjected to change during the transformation including the growth of new elements, it is a simple matter to determine the effect of any element of \( B \) upon the topology of the network by comparing rows* of the various parameter matrices.

* Because of symmetry of \( \dot{Y} \) and \( Y \) this applies to both columns and rows. Only rows are mentioned to avoid repetition with the understanding that this and similar statements apply to both rows and columns.
Equation (2.5) is valid for elements connected to the reference node. It is somewhat different in the sense that these derivatives are not only functions of the individual elements of B but also, due to the second summation of (2.5), of sums of elements in various rows of B. If \( \sum_{i=1}^{p-1} b_{ij} = 0 \) for all \( i \), then by using the same procedure as in the case of Equation (2.4) it can be shown that the effect of any element of B is the same as for Equation (2.4). Therefore, the row comparison technique can be applied to all off-diagonal elements of \( Y \), and the topology of the new network can be determined solely by this method.

It is useful to point out the different effects of diagonal and off-diagonal elements of B. New branches of the network can be generated only when \( q \neq r \), i.e. due to the off-diagonal elements of B. A diagonal element \( b_{qq} \) cannot cause growth of new elements but it will effect all the branches of the network connected to the node q. It is, therefore, a logical element to choose to bring the sum of elements in a row to zero or any other value when it is so desired since there is no immediate danger of growing negative elements.

When the sum of elements in rows of B is not zero, the row comparison technique is still applicable but it is necessary to consider the additional contributions of these sums to the derivatives of the elements in the row p. It is extremely difficult to find a simple technique similar to row comparison which would assess quickly what additional contributions they represent. It has been found easier to decide so from Equation (2.5) when it was required.
It remains to be determined when a sum of elements in a row of $B$ has to be zero. This requirement exists whenever an admittance matrix of one element type has no entry in the row $p$. An example of such networks are some low pass ladder filters where no inductor is connected to the common node, such as one shown in Figure 2.3. Because there are no entries in the row $p$ of $Y_p$ no new elements can be grown to the node $p$ due to off-diagonal elements of $B$. (Note that the row comparison technique implies that in order to grow a new element to a given node $s$ there has to exist at least one element of the same kind connected to the node $s$). New elements then can be grown only due to the non-zero sums in the rows of $B$. However, it is shown in Appendix A that the sum of all derivatives in the row $p$ of $Y_p$ in such case is zero. It is therefore not possible to grow a new positive element without simultaneously growing a negative one. In such a case it is generally required that the sum of elements in every row of $B$ was zero. The word "generally" is used to indicate that there exist network structures where it is not required that the sum of elements in rows of $B$ was zero even if one element kind is not connected to the reference node. These structures represent a somewhat special case and are shown in Appendix B.

2.3 Constraints on $B$ Due to Row Comparison

The method of row comparison is useful mainly because it predicts where new elements will grow due to any element of $B$ by comparing two rows of the various parameter matrices. It is thus possible to determine very rapidly which element of $B$ should be modified should growth of a negative branch occur. It also permits the growth
of new branches of the network selectively by setting to zero those elements of $B$ which would cause growth of undesired new branches. The procedure of selecting $B$ by row comparison is perhaps best illustrated by an example. Consider the low pass filter shown in Figure 2.3 together with its parameter matrices and $B$ matrix where rows 1 and 5 are zero to preserve port invariance. Since there is no inductor connected to the reference node it is required that $\sum_{j=1}^{p-1} b_{ij} = 0$ for all $i$. Therefore only off-diagonal elements can be chosen independently. By row comparison the effect of $b_{21}$, for example, is to cause growth of a new capacitor and inductor between the nodes, 1 and 3. In order to make these branches positive, $b_{21}$ must also be positive (recall the change of sign in nodal admittance matrix). $\delta_{21}$ and $c_{21}$ will then have a negative derivative because both $\delta_{22}$ and $c_{22}$ are positive, however, since they were initially non-zero they can remain positive quantities. It is instructive to apply this rule to $b_{31}$. In this case two new elements in each parameter matrix will grow. One will appear between the nodes 1 and 3, the other between nodes 2 and 4. However, since the diagonal element is positive and the off-diagonal negative, a negative element must grow unless $b_{31} = 0$. An inspection of other elements of $B$ shows that elements $b_{24}$, $b_{25}$, $b_{31}$, $b_{35}$, $b_{41}$, and $b_{42}$ would cause growth of at least one negative element.

One possible method of dealing with such negative growth is to use algebraic constraint equations and to apply linear programming as suggested by Schoeffler. However,

** Refer to the last paragraph of Section 2.2.
FIGURE 2.3.  
(a) Lowpass Ladder Network;  
(b) Parameter Matrices.  
(c) Transformation Matrix $B$ to Keep Input, Output Invariant.
in many instances it is simpler to set these \( b \)'s to zero and to use only those which do not give simultaneously a positive and negative derivative. The reduced \( B \) matrix for the network shown in Figure 2.3 is then given by:

\[
\begin{array}{cccc}
 b_{21} & b_{22} & \cdots & b_{23} \\
 b_{32} & b_{33} & \cdots & b_{34} \\
 \vdots & \vdots & \ddots & \vdots \\
 b_{43} & b_{44} & \cdots & b_{45} \\
\end{array}
\]

**FIGURE 2.4.** **B** **MATRIX** **REDUCED** **TO** **GROW** **POSITIVE** **ELEMENTS.**

The \( B \) matrix thus selected clearly gives sufficient conditions for the growth of positive elements. (It can be noted that the new elements will form redundant loops as mentioned by Guillemín.\(^9\))

It is apparent how to proceed if it is desired to alter only part of a network. All that is required is to proceed as previously and keep non-zero only those \( b \)'s which affect the pertinent part of the network.
It is evident that since each parameter matrix may impose a different set of constraints on the elements of $B$, the number of constraints and consequently the difficulties in performing the transformation increases with the number of different types of elements. Practically, this limits the method to two element-kind networks.

2.4 Secondary Growth

Secondary growth occurs due to the repeated process of numerical integration. New elements, however small, will usually appear in the parameter matrices after the first integration step. Thus in the second integration step somewhat different matrices are integrated. It is possible that these new elements will cause growth of some additional elements which may be numerically much smaller. This process may continue until a full graph is generated. This type of growth is referred to here as secondary growth. Its occurrence is highly undesirable since it represents an uncontrolled growth of new elements. To demonstrate such growth consider once again the low pass filter of Figure 2.3 (a). If the matrix $B$ is selected as shown in Figure 2.4, so that there is no danger of growing negative branches, then after one integration step the parameter matrices and the corresponding network will be as shown in Figure 2.5. If the integration is continued, additional new elements will grow so that after the next step the network is as shown in Figure 2.6. It is easy to see that after one more integration step one additional inductor and capacitor would grow between the nodes 1 and 5 at which point the secondary growth would stop.
Parameter Matrices of Network Equivalent to that in Figure 2.3 after One Integration Step.

Corresponding Network.
FIGURE 2.6. Network Equivalent to that in Figure 2.3 after Two Integration Steps. Elements between the Nodes 1, 4 and 2, 5 due to Secondary Growth.
The obvious way to stop such growth, if algebraic constraint equations are to be avoided, is to put additional elements of $B$ equal to zero. In this example a suitable choice seems to be to put all $b$'s in the row 3 equal to zero. The network which would result from application of such reduced matrix $B$ would have topology as shown in Figure 2.7.

Secondary growth imposes additional restrictions on the matrix $B$ in that it effectively reduces the number of useful elements that can be made available for optimization purposes.

2.5 Application to Network Optimization

A congruent transformation in its original (algebraic) form can be interpreted geometrically as a one-shot transformation from a point in a vector space, depicting the original network, to another point in the space, representing the new network. It is virtually impossible to select this transformation so that some empirical performance criterion such as element spread is reduced. On the other hand, the continuous transformation can be used for this function since it is a simple matter to minimize a performance criterion with differential equation constraint. It is thus possible to review the selection of the control vector – in this case matrix $B$ – so that the desired optimum is reached progressively. To this end it is useful to regard both the network elements and the elements of the matrix $B$ as the continuous function of the independent time-like variable $x$, so that the transformation can be described as
FIGURE 2.7.  
(a) Equivalent Network with Secondary Growth Eliminated.

(b) B Matrix Reduced to Eliminate Secondary Growth.
\[ y = f(y, b) \]  \hspace{1cm} (2.6)

where \( y = y(x) \) and \( b = b(x) \).

During the actual integration, all \( b \)'s have a constant value, so that an integration routine for linear differential equations with constant coefficients can be used. The dimension of vector \( y \) is potentially \( p(p-1)/2 \) for a network with \( p \) nodes. The word potentially is used to indicate that some of the elements may be zero. Clearly any solution of (2.6) represents a family of equivalent networks.

Since there is no interaction between the equations of the various kinds of elements of the network during the transformation, the problem of optimizing a two element kind network, for example an \( \text{L C} \), can be construed as finding the control vector \( b(x) \) which would minimize a performance criterion under the constraints

\[ \dot{\xi} = f(\xi, b) \]

\[ \delta = f(\xi, b) \] \hspace{1cm} (2.7)

The optimization procedure requires selection of a meaningful criteria. A useful criterion for the reduction of the spread is given by

\[ \phi = \sum_{i=1}^{p(p-1)/2} \left[ w_1 (c_i - \overline{c})^2 + w_2 (\delta_i - \overline{\delta})^2 \right] \] \hspace{1cm} (2.8)

where \( w_1 \) and \( w_2 \) are weighting factors for \( c \) and \( \delta \) respectively, and \( \overline{c} \) and \( \overline{\delta} \) are the desired values of capacitors and inverse inductors respectively. If the method of steepest descent is employed, it is necessary to compute \( \dot{\phi} = \frac{\partial \phi}{\partial x} \). Clearly
\[
\dot{\rho} = 2 \sum_{i=1}^{p(p-1)/2} \left[ w_1 (c_i - \bar{c}) c_i + w_2 \left( c_i - \bar{c} \right) \bar{c} \right]
\]  
(2.9)

Note that Equation (2.9) is linear with respect to the \(b_i\)'s. Therefore \(\dot{\rho}\) can be minimized by assigning the \(b_i\)'s extremal values of appropriate sign.

In connection with the optimization procedure two additional problems must be solved.

1. Determination of suitable values for \(\bar{c}\) and \(\bar{c}'\).

2. Optimization under constraint \(\sum_{i=1}^{p(p-1)/2} b_{ij} = 0\) (if so required).

With reference to the first problem recall that in order to obtain sufficient conditions for the growth of positive elements and to eliminate secondary growth, the number of useful elements of \(B\) is usually considerably reduced. The sign of those off-diagonal elements of \(B\) which would cause growth of new elements is determined by the requirement that the new elements were positive. Therefore it is not possible to determine the values of \(\bar{c}\) and \(\bar{c}'\) arbitrarily. Instead it is necessary to give up the control as to what the final element value will be, and merely reduce the spread of the elements. Several criteria have been tried, such as average value, algebraic mean between the lowest and highest value, or simply \(\bar{c} = \bar{c}' = 0\). Values of \(\bar{c}\) and \(\bar{c}'\) (when non-zero) were periodically re-calculated during the computation and it was found that in most cases they gave very nearly the same results so that there is no special advantage to choose one over the other.
In dealing with the second point, one is greatly aided by the linear nature of Equation (2.9) and by the knowledge that the diagonal element of any row of $B$ can be chosen to bring the sum to zero without affecting the growth of new elements. To demonstrate this point, consider the example of selecting $b$'s under this constraint in the case of elliptic filter shown in Figure 2.3. The $B$ matrix, reduced to eliminate secondary growth, is shown in Figure 2.7. Only the selection for row 2 is shown, since the procedure is the same for any other row. The contribution to the criterion (2.9) due to the elements in the row 2 of $B$ is written as $\dot{\rho}_1$. It can be written as follows:

$$\dot{\rho}_1 = K_1 b_{21} + K_2 b_{22} + K_3 b_{23}$$

Upon substituting

$$b_{22} = -(b_{21} + b_{23})$$

this equation becomes

$$\dot{\rho}_1 = (K_1 - K_2) b_{21} + (K_3 - K_2) b_{23}$$

The optimization under this constraint is therefore very straightforward and is readily implemented on a computer.

A flow chart of a computer program designed to carry out iterative equivalence is shown in Figure 2.8. The program in its present form is intended for experimentation and requires some experience in generating equivalent networks. The control data includes integration step size, number of integrations, performance criteria and how often new values of $B$ should be selected. With reference to the latter it can
READ DATA, FORM NODAL ADMITTANCE MATRIX

SELECT THE B MATRIX AUTOMATICALLY?

YES

SELECT B MATRIX BY ROW COMPARISON, REDUCE IF REQ. TO ELIMINATE SEC. GROWTH

COMPUTE criterion

SELECT NUMERICAL VALUES OF b's

COMPUTE criterion

ANY ELEMENT BECOMING NEGATIVE?

YES

CORRECT APPROPRIATE ELEMENT OF B, criterion

NO

INTEGRATE

ARE ALL INTEGRATIONS COMPLETED?

YES

PRINT ELEMENT VALUES

COMPUTE criterion

IS IMPROVEMENT OF CRITERIA SATISFACTORY?

YES

SELECT NEW b's?

NO

STOP

FIGURE 2.8. FLOW CHART OF THE COMPUTER PROGRAM.
be done after each integration step, after several steps, or not at all. It has been found that the frequency of the selection of b's makes little or no difference as far as a final result is concerned. However, this point merits further investigation.

An option is provided to let the computer select the B matrix, which includes the reduction to prevent the secondary growth, or these may be selected by the operator. Also, it is optional whether or not the network should be subjected to second transformation after the first transformation has been completed. The sign of the derivative of every element is checked and if an element tends to become negative the value of the corresponding b is properly adjusted. This part of the program is very effective in preventing growth of new negative elements. However, if any of the original elements eventually becomes negative, the corrective procedure is such that the elements may oscillate around the zero value. Obviously, the program can be improved in this respect.

Several methods of preventing hunting around the optimum value have been tried, but none of them was fully satisfactory. For example, the step size was reduced when close to the optimum value, but the subsequent improvement was then invariably quite small. Therefore, at present the computation is stopped when the improvement is less than a specified amount. Selection of this minimal improvement requires some computational experience for it depends both on the step size and number of integrations before the improvement of the criteria is checked.

As mentioned earlier, values of b's are constrained to be unity in magnitude in order that the gradient be maximized; hence it is only required to select
the sign of each entry in $B$. If the sign of the coefficient $b_{ij}$ in Equation (2.9) is positive, then the sign of $b_{ij}$ must be negative. Note that only off-diagonal $b$'s appear in Equation (2.9) when $\sum_{i=1}^{p-1} b_{ij} = 0$ since under these circumstances diagonal entries can be replaced by the negative sum of the off-diagonal entries of their respective rows. A Runge-Kutta routine is used to perform the integration.

In dealing with the application of the network equivalence to the reduction of spread, which has been suggested by several authors $^{3,4,13}$, it is obvious that since no a priori knowledge concerning the existence of an available optimum is available, it cannot be guaranteed that a new network with a smaller spread can be obtained. Success also depends very much on the initial selection of $B$. It has been pointed out that some elements of $B$ were put equal to zero to assure the growth of positive elements while others were put equal to zero to prevent the secondary growth. The selection of which elements should be set to zero is of primary importance. Although a general rule is difficult to state, it is clearly preferable to retain those $b$'s which will effect the largest or the smallest elements of the network in the desired manner, i.e. to give them positive or negative derivative, as the case may be. However, in certain problems the spread was reduced optimally if some of the "permissible" $b$'s were also put equal to zero, so that $B$ matrix was restricted more than required by the above procedure. In the example of the low pass ladder filter $^{21}$ shown in Figure 2.9, the largest reduction of spread was achieved when in addition to those $b$'s set to zero in Figure 2.7b, the elements $b_{21}$ and $b_{45}$ were also put to zero, so that the only elements used in the transformation were $b_{22}$, $b_{23}$, $b_{43}$ and $b_{44}$.
FIGURE 2.9.  
(a) Lowpass Ladder Network.
(b) Its Equivalent Network with Reduced Spread of Elements.
Both the original and the final network were analyzed and found in almost exact agreement. The discrepancies were in the fourth digit both for the input and transfer impedance and are more likely due to the round-off error of the four digit print-out than actual error due to integration. The measure of the reduction of the spread was taken as the ratio of the largest to the smallest value of both inductors and capacitors and it is as follows:

<table>
<thead>
<tr>
<th>Original Network</th>
<th>Equivalent Network</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio of Inductors: ( \frac{0.663}{.0472} = 14.3 )</td>
<td>1.766 = 4.88</td>
</tr>
<tr>
<td>Ratio of Capacitors: ( \frac{21.09}{.307} = 70.2 )</td>
<td>2.52 = 11.2</td>
</tr>
</tbody>
</table>

Both types of elements carried the same weight during the integration. The reduction has been achieved at the cost of four additional elements. Whether or not the reduction thus obtained is sufficient to justify this additional cost has to be decided in the final analysis by the network designer.

2.6 Summary of Chapter II

Analysis of the explicit equations for the elements of \( Y \) and \( \tilde{Y} \) has lead to a simple interpretation of the relationship between these elements and those of the transformation matrix \( B \). The principal advantage of this interpretation is that it enables a simple correlation of the effect of any element of \( B \) upon the structure of the network by inspection of the parameter matrices. It yields sufficient conditions for the growth of
new positive elements.

In most larger networks certain elements of $B$ are usually kept equal to zero to prevent growth of negative elements. The $B$ matrix is further restricted to prevent secondary growth, which is caused by the iterative nature of the computation. Thus relatively few elements of $B$ can be left to perform the actual transformation.

Reduction of the spread of the elements can be achieved at the cost of additional elements. The error introduced during the computation was found to be very small, so that the network can be regarded as being truly equivalent. In certain cases the best reduction has been achieved by restricting the transformation matrix more than it is required to assure that new elements will be positive and that secondary growth is eliminated. It seems, therefore, that further studies are necessary in order to determine how the initial $B$ matrix can be selected optimally.
3.1 Equations for Incremental Equivalence.

The method of row comparison which gives sufficient conditions for generating positive new elements can also be used to generate equivalent networks in discrete steps rather than in the continuous manner outlined earlier. The resulting techniques are similar to those of Cauer, Howitt, and others, except that the B matrix is used.

If the transformation matrix is decomposed as

$$ A = I + B $$

and substituted into equation (1.1) the following matrix equation is obtained for the incremental nodal admittance matrix:

$$ \Delta Y = B^T Y + Y B + B^T Y B $$

The corresponding change in an element connected between nodes m and n then becomes

$$ \Delta y_{mn} = \sum_{i=1}^{p-1} b_{im} y_{in} + \sum_{i=1}^{p-1} y_{mi} b_{in} + \sum_{i=1}^{p-1} b_{im} \sum_{j=1}^{p-1} b_{jm} y_{mi} $$

(3.1)

if m and n do not represent the reference node. Further, the change in an element connected to the reference node can be written as

$$ \Delta y_{mp} = \sum_{i=1}^{p-1} y_{pi} b_{im} - \sum_{i=1}^{p-1} y_{mi} \sum_{j=1}^{p-1} b_{ij} - \sum_{i=1}^{p-1} b_{im} \sum_{k=1}^{p-1} y_{ik} \sum_{j=1}^{p-1} b_{ij} $$

(3.2)
The notation is the same as that used previously with continuously equivalent networks, i.e., whenever the general case is discussed the subscript is omitted. The letter $y$ then may mean $g$, $c$, or $\delta$.

An equation such as (3.1) or (3.2) can be written for each network element. These equations relate directly network elements and entries of the transformation matrix. Since it is necessary for network elements to be positive, a set of inequality constraints of the form

$$f_i \left( b_1, b_2, \ldots, b_k \right) \geq 0$$

(3.3)

where $f_i \left( b_1, b_2, \ldots, b_k \right)$ is linear and/or quadratic in the $b$'s can be established immediately. Generating an equivalent network is then equivalent to finding a set of $b$'s which satisfy these inequality constraints. The number of inequalities in (3.3) is generally higher than the number of $b$'s so that the system is overdetermined.

When the $B$ matrix is large this system of inequalities is as difficult to solve as the original techniques of Cauer, etc. However, if there are only a few non-zero $b$'s, the number of quadratic terms is small, and the equations for the increments of network elements are mostly linear in the $b$'s. Then, it is possible to use the row comparison technique in aiding the selection of $b$'s which are compatible with the constraints in (3.3), particularly in relation to the sign of these parameters. Moreover, because the nodal admittance matrix is hyperdominant, the magnitude of the $b$'s will generally be less than one. Thus there exists a range of permissible values of $b$'s.

Family of equivalent networks can be generated by selecting the $b$'s anywhere within this range.
Because the number of the b's is reduced to facilitate the computation, this type of transformation is suited for an application in one particular locality of the network. A typical example is insertion of a capacitor of specific value between two nodes to provide for stray capacitance, elimination of a negative element, and so forth.

3.2 Calculation of the Values of Elements of B Matrix.

The values of the b's are obtained by changing some of the inequality constraints which correspond to the change of element of interest into equality constraints, from which the b's are calculated. It is preferable to select as many equality constraints as there are available b's. Their selection is usually governed by the following considerations:

(a) primary purpose of the transformation (insertion or elimination of a given element etc.)

(b) constraints on the B matrix when required

\[
\begin{align*}
p-1 & \\
\sum_{j=1}^{p-1} b_{ij} &= 0
\end{align*}
\]

(c) other network considerations (keep the number of new elements in the network at minimum etc.)

As long as the values of the b's obtained from these equalities are within the permissible range, the remaining inequalities of (3.3) are automatically satisfied and the resulting network will have no negative elements. The required element values which appear in the equality constraints, particularly under the point (a) above, therefore has to be such that the solution of the equalities yields the b's within the permissible range. Because the number of the b's is small, the solution of the equations requires no
special computational techniques. However, any general rules for the selection and solution of the equality constraints are difficult to state since the permissible range of the b's and the transformation vary from network to network. The technique is best illustrated by examples.

3.3 Illustrative Examples

Example 1.

It is required to eliminate negative capacitor which is across the input nodes 1 and 6 of the ladder network shown in Figure 3.1. This figure also includes the network parameter matrices where only those rows and columns which are effected by the transformation are shown. Using the row comparison technique it can be seen that a possible choice of the b's for the elimination of the negative capacitor are $b_{21}$ and $b_{22}$, subjected to the constraint

$$b_{21} + b_{22} = 0 \quad (3.4)$$

This constraint is required since there is no inductor connected to the reference node. If the elements of the new network are denoted by prime, then $b_{21}$ and $b_{22}$ have to be selected so as to satisfy the following system of inequalities

$$c'_{12} = c_{12} + b_{21}c_{22} + b_{22}c_{12} + b_{22}b_{21}c_{22} \geq 0 \quad (3.5)$$

$$\delta'_{12} = \delta_{12} + b_{21} \delta_{22} + b_{22} \delta_{12} + b_{22}b_{21} \delta_{22} \geq 0$$

$$c'_{13} = b_{21}c_{23} \geq 0$$

$$\delta'_{13} = b_{21} \delta_{23} \geq 0$$

$$c'_{16} = c_{16} + b_{21}c_{26} \geq 0$$
FIGURE 3.1. (a) Low Pass Ladder Filter.
(b) Reduced Parameter Matrices.
\[
\begin{align*}
\gamma'_{23} &= \gamma'_{23} + b_{22}\gamma'_{23} = 0 \\
\gamma'_{26} &= \gamma'_{26} + b_{22}\gamma'_{26} = 0
\end{align*}
\]

The inequalities in equation (3.5) represent a relationship among the elements of the nodal admittance matrices with their sign reversed, so that for example \( c_{22} \) represents the negative sum of all the elements connected to the node 2.

It is now possible to determine the sign of \( b_{21} \) from the requirement that \( c_{13}, \gamma'_{13}, \) and \( c_{16} \) were positive. Clearly \( b_{21} \) has to be positive and due to equation (3.4), \( b_{22} \) has to be negative. The determination of the upper limit of the magnitude of \( b_{21} \) and \( b_{22} \) requires examination when any element, the value of which is reduced by the transformation, becomes zero. By inspection of \( c_{23}, \gamma'_{23}, \) and \( c_{26} \) it can be seen that \( |b_{22}| \leq 1 \). The limits on \( b_{21} \) due to \( c_{12} \) and \( \gamma'_{12} \) can be determined by calculating the values of \( b_{21} \) for which they become zero. These values are \( b_{21} = 1.0 \) and \( .2634 \) for \( c_{12} \), and \( b_{21} = 1.0 \) and \( .6612 \) for \( \gamma'_{12} = 0 \). However, the value \( b_{21} = 1.0 \) (and \( b_{22} = -1.0 \)) is not permissible since in this case the entire row and column 2 of both parameter matrices become zero and the transformation is invalid. (The matrix A of the original Cauer transformation is singular in this case). Thus the upper limit of \( |b_{21}| \leq |b_{22}| \) is .2634.

The lower limit of \( b_{21} \) can be found from the requirement that \( c_{16} \) which was originally negative was at least zero. This gives \( b_{21} = .06003 \). The permissible range of \( b_{21} \) therefore is from \(.06003 \) to \(.2634 \). Selection of any value of \( b_{21} \) within
these limits yields a realizable network. For example \( b_{21} = 0.09902 \) gives the network shown in Figure 3.2. In this case the negative capacitor has been eliminated at the cost of two additional elements. However, no attempt has been made to minimize the number of new elements. The number of new elements can be reduced by using all the available \( b \)'s of the row 2 and by using suitable equality constraints. By row comparison, the available \( b \)'s are \( b_{21}, b_{22}, \) and \( b_{23} \). It is possible to use them either simultaneously or successively. Successive use will be demonstrated here because of greater computational simplicity. In the first step only \( b_{21} \) and \( b_{22} \) will be used as previously, except that \( b_{21} \) is selected at its lower limit so that \( c_{16} = 0 \). The calculation of element values is then carried out as previously. In the next step \( b_{23} \) and \( b_{22} \) are used under the constraint

\[
b_{23} + b_{22} = 0
\]

The inequalities for \( b_{23} \) and \( b_{22} \) now are

\[
\begin{align*}
c_{13}^1 &= c_{13} + b_{23}c_{12} \geq 0 \\
\zeta_{13}^1 &= \zeta_{13} + b_{23}\zeta_{12} \geq 0 \\
c_{23}^2 &= c_{23} + b_{23}c_{22} + b_{22}c_{23} + b_{23}b_{22}c_{22} \geq 0 \\
\zeta_{23}^2 &= \zeta_{23} + b_{23}\zeta_{22} + b_{22}\zeta_{23} + b_{23}b_{22}\zeta_{22} \geq 0 \\
c_{36}^3 &= c_{36} + c_{26}b_{23} \geq 0 \\
c_{12}^4 &= c_{12} + b_{22}c_{12} \geq 0 \\
\zeta_{12}^5 &= \zeta_{12} + b_{22}\zeta_{12} \geq 0 \\
c_{26}^6 &= c_{26} + b_{22}c_{26} \geq 0
\end{align*}
\]
ALTERNATIVE EQUIVALENT NETWORKS OF THAT IN FIGURE 3.1 AFTER REMOVAL OF A NEGATIVE CAPACITOR.
Because no new element is generated, in this case $b_{23}$ may be positive or negative. The limiting values of $b_{23}$ can be found to be $b_{23} = -0.0352$ in which case $\delta_{13}^* = 0$ and $b_{23} = 0.2634$ in which case $c_{23}^1 = 0$. Since it is preferable to reduce the number of inductors $b_{23} = -0.0352$ was used so that $\delta_{13}^* = 0$. The resulting network is shown in Figure 3.3.

If instead of the lower limit of $b_{21}$ the upper limit is used, so that $c_{12}^1 = 0$, and $b_{23}$ is selected to give $\delta_{13}^* = 0$, then the network shown in Figure 3.4 is obtained.

In both these latter cases it was possible to remove the negative capacitor without increasing the number of elements in the network.

Example 2.

It is desired to insert a capacitor of 40 pF across the output nodes of the network shown in Figure 3.5. The part of the network which is of interest for the transformation is shown in Figure 3.6 where the tapped coils have been replaced by their equivalent circuits. The parameter matrices and the B matrix showing the permissible b's are shown in Figure 3.7. In this case there can be used two methods to obtain the required capacitor.

Method 1. Use of off-diagonal b's.

It can be seen from the parameter matrices that the required capacitor can be obtained by first generating a capacitor between the nodes 2 and 9 with $b_{32}$, and then to obtain the capacitor between the nodes 1 and 9 with $b_{21}$. In the final step of the
FIG. 3.5 LADDER NETWORK

This network has been designed by R & D Labs of Northern Electric Co. in Ottawa, who have granted the permission to publish it in this thesis.
FIG. 3.5 LADDER NETWORK

THIS NETWORK HAS BEEN DESIGNED BY R & D LABS OF NORTHERN ELECTRIC CO. IN OTTAWA, AND HAVE GRANTED
THE PERMISSION TO PUBLISH IT IN THIS THESIS.
Figure 3.6. Pertinent section of network in Figure 3.5 with tapped coils removed.
FIGURE 3.7.  
(a), (b) Parameter Matrices of Network in Figure 3.6.  
(c) Transformation Matrix B.
transformation, \( b_{23} \) can be used to reduce the number of elements. The sum of elements in the rows 2 and 3 of B is kept equal to zero during the transformation. This method is therefore quite similar to the one used previously and the description can be brief.

The set of inequalities which has to be satisfied during the first step of the transformation is as follows:

\[
\begin{align*}
c'_{23} &= c_{23} + b_{32}c_{33} + b_{33}c_{32} + b_{32}b_{33}c_{33} \geq 0 \\
\xi'_{23} &= \xi_{23} + b_{32}\xi_{33} + b_{33}\xi_{32} + b_{33}b_{32}\xi_{33} \geq 0 \\
\xi'_{24} &= b_{32}\xi_{34} \geq 0 \\
c'_{25} &= b_{32}c_{35} \geq 0 \\
c'_{26} &= b_{32}c_{36} \geq 0 \\
c'_{27} &= b_{32}c_{37} \geq 0 \\
c'_{29} &= b_{32}c_{39} \geq 0 \\
\xi'_{29} &= b_{32}\xi_{39} \geq 0 \\
\xi'_{34} &= \xi_{34} + b_{33}\xi_{34} \geq 0 \\
c'_{36} &= c_{36} + b_{33}c_{36} \geq 0 \\
c'_{37} &= c_{37} + b_{33}c_{37} \geq 0 \\
c'_{39} &= c_{39} + b_{33}c_{39} \geq 0 \\
\xi'_{39} &= \xi_{39} + b_{33}\xi_{39} \geq 0
\end{align*}
\]
By inspection $b_{32}$ has to be positive and its upper limit is 0.790 in which case $c'_{23}$ becomes zero. This value of $b_{32}$ has been used to reduce the number of elements of the network.

The second step of transformation, where $b_{21}$ is used, is similar and $b_{21}$ is selected to give the required 40 pF capacitor between the nodes 1 and 9. Finally, $c_{13}$ is eliminated with $b_{23}$. The resulting network is shown in Figure 3.8.

It is possible to reduce further the number of elements if the capacitor across the nodes 1 and 9 is made larger than 40 pF by selecting the value of $b_{21}$ such that $\mathbf{B}_{12} = 0$. The transformation otherwise remains the same. The network is shown in Figure 3.9.

It was therefore possible to obtain the required capacitor, however, the total number of elements has increased by 5 inductors and 7 capacitors when the capacitor of 40 pF was obtained exactly, and by 4 inductors and 7 capacitors when the capacitor larger than required was obtained.

Method 2. Use of diagonal elements of $B$.

The use of only diagonal elements of $B$ has the advantage that only elements to the reference node are generated. Recall that a diagonal element $b_{qq}$ changes all the elements connected to the node $q$, and also that it enters into the sum of elements of $B$ in the row $q$ which if not zero can generate new elements to the reference node. A suitable procedure is to use $b_{22}$, the value of which is selected to give the required capacitor across the output nodes 1 and 9. Due to this step a negative capacitor and inductor will be generated from the node 2 to the reference node 9. These two negative elements are eliminated in the next step with $b_{33}$ and the negative increment which occurs due to $c_{33}$.
FIGURE 3.8. NETWORK AFTER INSERTION OF THE REQUIRED CAPACITOR BY OFF-DIAGONAL b's.
FIGURE 3.9. EQUIVALENT NETWORK BY OFF-DIAGONAL b's WITH THE NUMBER OF ELEMENTS MINIMIZED.
and $f_{33}$ is absorbed by $c_{39}$ and $f'_{39}$ respectively. Thus although the inequalities are not compatible in the first step, they are compatible if $b_{22}$ and $b_{33}$ were used simultaneously. However, for simplicity of computation it is preferable to use $b_{22}$ and $b_{33}$ in separate steps.

The relationship between the elements of the original and the network transformed with $b_{22}$ is as follows:

\[
\begin{align*}
c'_1 &= c_{12} + b_{22} c_{12} \\
\delta'_{12} &= \delta_{12} + b_{22} \delta_{12} \\
c'_2 &= c_{23} + b_{22} c_{23} \\
\delta'_{23} &= \delta_{23} + b_{22} \delta_{23} \\
c'_{19} &= -b_{22} c_{12} \\
\delta'_{19} &= -b_{22} \delta_{12} \\
c'_{29} &= -b_{22} c_{22} - b_{22} c_{22} \\
\delta'_{29} &= -b_{22} \delta_{22} - b_{22} \delta_{22} \\
c'_{39} &= c_{39} - b_{22} c_{23} \\
\delta'_{39} &= \delta_{39} - b_{22} \delta_{23}
\end{align*}
\]

Since $c'_{19}$ and $c_{12}$ are known, the required $b_{22}$ can be readily calculated. Clearly, $b_{22}$ has to be negative, and also $c'_{19} \leq c_{12}$ so that all the elements connected to the node 2 remained positive. The only two elements which in this case become negative are $c'_{29}$ and $\delta'_{29}$, due to the sign of $c_{22}$ and $\delta_{22}$. They can be eliminated in the next
step of the transformation with \( b_{33} \). The equations for the transformation with \( b_{33} \) are shown below:

\[
\begin{align*}
\ell'_{23} &= \ell_{23} + b_{33} \ell_{23} \\
\ell'_{34} &= \ell_{34} + b_{33} \ell_{34} \\
c'_{35} &= c_{35} + b_{33} c_{35} \\
c'_{36} &= c_{36} + b_{33} c_{36} \\
c'_{37} &= c_{37} + b_{33} c_{37} \\
c'_{39} &= c_{39} + b_{33} c_{39} - b_{33} c_{33} - b_{33}^2 c_{33} \\
\ell'_{39} &= \ell_{39} + b_{33} \ell_{39} - b_{33} \ell_{33} - b_{33}^2 \ell_{33} \\
c'_{29} &= c_{29} - b_{33} c_{23} \\
\ell'_{29} &= \ell_{29} - b_{33} \ell_{23} \\
\ell'_{49} &= -b_{33} \ell_{34} \\
c'_{59} &= -b_{33} c_{35} \\
c'_{69} &= -b_{33} c_{36} \\
c'_{79} &= -b_{33} c_{37}
\end{align*}
\]

To minimize the number of new elements \( b_{33} \) is selected so that \( c'_{29} = 0 \).

Since the capacitors \( c'_{69} \) and \( c'_{79} \) are .011 \( \text{pF} \) and .006 \( \text{pF} \) respectively they can be considered to be part of stray capacitance of the inductors \( \ell'_{69} \) and \( \ell'_{79} \). The resulting network is shown in Fig. 3.10. The number of elements has been increased by the transfor-
FIGURE 3.10. EQUIVALENT NETWORK USING ONLY DIAGONAL ELEMENTS OF B.
mation by 2 capacitors and 3 inductors.

The second method is more advantageous since fewer elements were required to obtain the desired capacitor. However, it can be used only in those cases where the required new element is connected to the reference node and where both types of elements are connected in parallel from some other node to the reference node. The transformation can be achieved with fewer elements when two nodes are close to each other. Also, both types of elements which have to absorb the negative increment have to be sufficiently large so that they would not become negative.

3.4 Summary of Chapter III

Equivalent networks can be suitably generated in finite increments when the \( B \) matrix contains only few non-zero elements. The row comparison method makes it possible to select the \( b \)'s so that the inequalities relating the elements of the original and transformed network are compatible. However, it is also necessary to find the permissible range of the \( b \)'s, especially if it is desired to reduce the number of elements in the new network. Although this requires to solve the equations for the \( b \)'s for each case separately, the computation is quite simple and presents no difficulty. This method can be seen as a suitable alternative to repeating the synthesis procedure when only a small change of the network is required.
CHAPTER IV

SENSITIVITY MINIMIZATION

4.1 Derivation of Sensitivity Equations

The problem of minimization of sensitivity by means of equivalent networks is a difficult one. It may be viewed as minimizing an invariant function $F$ with respect to changes in network elements. $F$ may be an input or transfer function, or it may be a critical frequency of a network function. To examine this problem in a greater detail, consider the application of continuous equivalence to the minimization of classical (Bode) sensitivity. The sensitivity of the $j$'th element is given by

$$S(F, y_j) = \frac{\partial F}{\partial y_j} \frac{y_j}{F}$$  \hspace{1cm} (4.1)

which for the purpose of this analysis can be written as

$$S(F, y_j) = q_j y_j$$  \hspace{1cm} (4.2)

In this equation $q_j = \frac{1}{F} \frac{\partial F}{\partial y_j}$ is a normalized derivative of $F$ with respect to the $j$'th element of the network and is in general a complex quantity.

Since the number of elements which can potentially exist in a network with $p$ nodes is $p (p-1) / 2$, the number of derivatives is also $p (p-1) / 2$. The sensitivity of $F$ to various network elements can therefore be described in terms of a vector $\mathbf{q}$ and of
vector $\mathbf{y}$, each having the dimension $p(p-1)/2$. Clearly, both the vector $\mathbf{q}$ and the vector $\mathbf{y}$ will be affected by any network transformational process.

A suitable criterion for the reduction of sensitivity is to minimize

$$\phi = \sum_{i=1}^{n} \left| S(F, y_i) \right|^2 \quad (4.3)$$

where $n = p(p-1)/2$, and $\phi$ is evaluated at certain frequency points of interest. Substituting elements of the vectors $\mathbf{q}$ and $\mathbf{y}$, and denoting complex conjugate of $q$ as $q^*$, Equation (4.3) can be written as

$$\phi = \sum_{i=1}^{n} y_i^2 q_i q_i^*$$

If the method of steepest descent is used to minimize $\phi$, it is necessary to minimize (i.e. find maximum negative)

$$\dot{\phi} = 2 \sum_{i=1}^{n} \left[ y_i q_i q_i^* y_i + y_i^2 \text{Re}(q_i q_i^*) \right] \quad (4.4)$$

where

$$\dot{\phi} = \frac{d\phi}{dx}$$

* The sensitivity of an element $y_k$ which is zero is clearly zero, however $q_k = \frac{1}{F} \frac{\partial F}{\partial y_k}$ is generally not zero.
is a derivative with respect to the independent parameter \( x \) in the continuously variable network theory.

In order to minimize the sensitivity by continuously equivalent networks, \( q \) and \( \dot{q} \) must be found at each step of iteration procedure which generates the equivalent network. This is a formidable task even for the largest computers.

The difficulties can be alleviated somewhat by determining how the vector \( q \) can be computed directly from the transformational process. This has been found by Schoeffler\(^{5,6}\) for the classical sensitivity, and has been extended by Blöstein\(^{19}\) to any port invariant function such as root sensitivity.

Since the network function, as viewed from the ports, must remain invariant during the transformation, its derivative with respect to the independent variable \( x \) must be zero.

\[
\dot{F} = \sum_{i=1}^{n} \frac{\partial F}{\partial g_i} \frac{\dot{g}_i}{F} + \sum_{i=1}^{n} \frac{\partial F}{\partial c_i} \frac{\dot{c}_i}{F} + \sum_{i=1}^{n} \frac{\partial F}{\partial \delta_i} \frac{\dot{\delta}_i}{F} = 0
\]

Differentiating this equation with respect to the elements of one kind — for example capacitors — yields

\[
\sum_{i=1}^{n} \frac{\partial^2 F}{\partial c_k \partial g_i} \frac{\dot{g}_i}{F} \frac{\ddot{g}_i}{F} + \sum_{i=1}^{n} \frac{\partial^2 F}{\partial c_k \partial c_i} \frac{\ddot{c}_i}{F} + \sum_{i=1}^{n} \frac{\partial^2 F}{\partial c_k \partial \delta_i} \frac{\ddot{\delta}_i}{F} + \sum_{i=1}^{n} \frac{\partial F}{\partial c_i} \frac{1}{F} \frac{\partial^2 F}{\partial c_k \partial c_i} \frac{\dot{c}_i}{F} + \sum_{i=1}^{n} \frac{\partial F}{\partial \delta_i} \frac{1}{F} \frac{\partial^2 F}{\partial c_k \partial \delta_i} \frac{\dot{\delta}_i}{F} = 0
\]
where \( k = 1, 2, \ldots \), \( n \) and \( g_i, c_i \) and \( \overline{c}_i \) refer to the \( i \)th conductance, capacitance and reciprocal inductance, respectively.

Assuming that the order of differentiation can be interchanged and using

\[
q_{ci} = \frac{\partial F}{\partial c_i} \frac{1}{F}
\]

shows that

\[
\sum_{i=1}^{n} \frac{\partial q_{ck}}{\partial g_i} g_i + \sum_{i=1}^{n} \frac{\partial q_{ck}}{\partial c_i} c_i + \sum_{i=1}^{n} \frac{\partial q_{ck}}{\partial \overline{c}_i} \overline{c}_i = - \sum_{i=1}^{n} q_{ci} \frac{\partial g_i}{\partial c_k}
\]

\[
\sum_{i=1}^{n} q_{ci} \frac{\partial \overline{c}_i}{\partial c_k} - \sum_{i=1}^{n} q_{ci} \frac{\partial \overline{c}_i}{\partial c_k}
\]

(4.5)

It has been pointed out when dealing with the continuously equivalent networks that each element kind is subjected to a transformation independently so that it is possible to write *

\[
\begin{align*}
\dot{c} &= M_c c \\
\dot{g} &= M_g g \\
\dot{\overline{c}} &= M_{\overline{c}} \overline{c}
\end{align*}
\]

* \( M_c \), \( M_g \) and \( M_{\overline{c}} \) are submatrices of Schoeffler's\(^5\) \( M \) matrix and are obtained by eliminating rows and columns of \( M \) which correspond to those capacitances, conductances and inverse inductances respectively which remain zero during the transformation. Clearly, this derivation is also valid for the \( M \) matrix.
Consequently, for capacitors, it follows that

\[ \sum_{i=1}^{n} q_{ci} \frac{\partial \mathbf{g}_i}{\partial c_k} = 0 \]

\[ \sum_{i=1}^{n} q_{ci} \frac{\partial \mathbf{f}_i}{\partial c_k} = 0 \]

\[ \sum_{i=1}^{n} q_{ci} \frac{\partial \mathbf{c}_i}{\partial c_k} = \sum_{i=1}^{n} q_{ci} m_{ik} \]

where \( m_{ik} \) is the \( ik \) th element of \( M_c \).

Equation (4.5) can then be written as

\[ q_{ck}' = - \sum_{i=1}^{n} m_{ik} q_{ci} \]

where \( q_{ck}' \) represents the left side of (4.5). Clearly, this implies

\[ q_{c}' = - M_c^t q_{c} \]

Similarly it follows that

\[ q_{g}' = - M_g^t q_{g} \]

\[ q_{r}' = - M_r^t q_{r} \]

Therefore, \( q \) satisfies the set of differential equations which are adjoint to the differential equations describing \( \chi \).
If $T$ represents any other port invariant property such as the location of a pole or zero, and $q_i$ is re-defined as $\frac{\partial T}{\partial y_i}$ by carrying out a similar derivation, the same set of adjoint equations would be valid.

Hence, to reduce either classical or root sensitivity, it is merely required to calculate the initial values of $\dot{q}$, select the elements of $M$ so as to minimize the criterion in (4.4) and then integrate the differential equations involving $\dot{y}$ and $\dot{q}$ simultaneously. Each integration step produces a new network for which the minimization step is repeated until a minimum is reached.

### 4.2 Analysis of Adjoint Equations of Sensitivity Minimization

The differential equations relating $\dot{q}$ and $\dot{y}$ are useful not only because it is now feasible to reduce the sensitivity by the means of continuous transformation, but also because it is possible to deduce from them relationships between changes of the element values and their respective sensitivities. Some of these relationships are discussed in this section.

#### 4.2.1 Invariance of the Sum of Sensitivities of Each Element Kind

It has been shown by Goursat that the scalar product of the solution of a differential equation and its adjoint is constant, i.e.

$$q^T \cdot \dot{q} = \text{Const}$$
It is evident that since there is a complete separation of variables during the transformation that the scalar product of each element kind remains invariant during the transformation. Since $q^T \cdot g$ represents the sum of sensitivities of conductive elements, it can be stated that the sum of sensitivities of each element kind in an RLC network remains invariant during the transformation.

Consequently, the sensitivities of elements of one kind can be reduced only by the changes - such as growth of new elements - of the elements of the same kind. This property can also be useful as a check of the accuracy of the numerical integration.

4.2.2 Changes in Element Values and Changes in Their Sensitivity

A correlation between the sensitivities of individual elements and changes in their element values can also be found and in some cases this can be useful for the selection of B matrix. It is only necessary to consider the relationship between two elements of $y$ and $g$. The discussion is limited to the real part of $g$; however, it obviously applies to the imaginary part of $g$ as well. Also, $y_i$ refers to the $i$'th element of $g$, $c$, or $\Phi$, and $q_i$ represents the corresponding normalized derivative.

In the equation

$$y_k = m_{ki} y_i$$

the element $y_i$ can be thought of as contributing to the derivative of $y_k$. It follows from
the adjoint equations that

\[ \text{Re} \{ \dot{q}_i \} = -m_{ki} \text{Re} \{ q_k \} \]

It is only necessary for the purpose of this analysis to observe the signs of the variables. If \( m_{ki} \) is positive so that \( \dot{y}_k \) is positive, the absolute value \( \text{Re} \{ q_i \} \) will decrease if \( \text{Re} \{ q_k \} \) and \( \text{Re} \{ q_i \} \) have the same signs, or vice versa if their signs are opposite. Therefore, if an element contributes to the derivative of another one and this contribution is positive, then the absolute value of its real part will decrease if the signs of their real parts are the same.

The practical utility of this relationship is as follows:

(a) The \( B \) matrix should be selected so that the \( y \)'s of particularly sensitive elements do not contribute positively to the derivative of another one if the real parts of their respective \( q \)'s have opposite signs.

(b) If all the \( q \)'s of the network have the same sign the growth of new elements will always reduce their sensitivity. This is true, for example, with respect to pole sensitivities in RC networks, where the \( q \)'s must have the same sign. Consequently, the minimization of sensitivity at one pole frequency minimizes the sensitivity at all pole frequencies.
4.2.3 Sensitivities of New Elements

Some reflection on the equations reveals that even though \( y_k(0) = 0 \)
\( \dot{q}_k \) need not be zero, so that elements not present in the initial network may have a
profound influence on the computation of \( \dot{q} \). It remains to determine whether it is
necessary to find the initial sensitivity of only some or all elements which may potentially
exist in the network.

Consider any element of \( y \), for example \( y_k \), where \( y_k(0) \) need not
be zero. A sufficient condition that \( \dot{y}_k = 0 \) is that row \( k \) of \( M \) have zero elements.
This means that the column \( k \) of \( M^\dagger \) will have only zero elements. Hence, \( q_k \) cannot
enter into computation so that its initial values need not be found. It has to be remembered,
however, that there may exist a situation in which \( \dot{y}_k = 0 \), but \( y_k \) may contribute
to the derivative of another element so that \( \dot{q}_k \neq 0 \). In such case \( q_k \) does not enter
into the computation of other \( q \)'s, but the sensitivity of \( y_k \) changes as discussed in the
Section 4.2.2. To maintain control over the sensitivity of \( y_k \) it is necessary to find
\( q_k(0) \). The following statement summarizes this discussion:

It is not necessary to find the initial values of \( q \)'s for those elements
which remain invariant during the transformation and do not contribute
to the derivative of another element.

4.2.4 Conditions for Invariance of Element Sensitivities

It is a simple matter to determine which element sensitivities remain in-
variant during the transformation. Clearly, if the sensitivity of element \( y_k \) is to
remain invariant, it is sufficient that \( \dot{y}_k \) and \( \dot{q}_k \) remain zero during the transformation. However, if \( \dot{q}_k = 0 \), then the row \( k \) of \( M^\dagger \) will have zero elements. Consequently, the column \( k \) of \( M \) has only zero elements so that \( y_k \) cannot contribute to the derivative of another element of \( y \). Hence, if the derivative of an element remains zero during the transformation and if it does not contribute to the derivative of another element, its sensitivity remains invariant.

In terms of the elements of \( B \) matrix, this means that if \( y_k \) is connected between the nodes \( m \) and \( n \), its sensitivity remains invariant provided rows \( m \) and \( n \) contain only zero elements. In addition those elements in the columns \( m \) and \( n \) of \( B \) which would make \( \dot{y}_{m,n} \neq 0 \) have to be zero.

These conditions are always satisfied for the termination elements of a two-port network, so that the sensitivity with respect to termination elements always remains invariant as the two-port is transformed. This is a particular case of a more general result due to Blostein\(^{19}\).

### 4.3 Effect of Constrained \( B \) on the Sensitivity

A constraint that was frequently applied during the transformation was

\[
\sum_{j=1}^{p-1} b_{ij} = 0, \quad i = 1, 2, \ldots, p - 1.
\]

As discussed in Chapter II, diagonal elements of \( B \) were used to satisfy this constraint since such elements cannot induce growth of new
elements. It is of interest to determine what effect such a selection has on sensitivity.

For this purpose note that the inability to grow new elements has to be retained when the diagonal elements in \( B \) appear in \( M \). Consequently, they must appear on the main diagonal of \( M \) and nowhere else. Hence, it is only necessary to find what effect the diagonal elements of \( M \) have on sensitivity.

Consider the derivative

\[
\dot{S}(F, y_i) = y_i \dot{q}_i + y_i \ddot{q}_i
\]

expanded in terms of \( M \), \( y \), and \( q \). It follows immediately from the differential equations of \( y \) and \( q \) that \( S(F, y_i) \) does not contain terms from the main diagonal of \( M \), so that diagonal elements of \( B \) have no effect on sensitivity.

4.4 Problems Associated with Network Minimization

The minimization of sensitivity by continuously equivalent networks requires the determination of the relevant elements of \( q(0) \). These include \( q \)'s for elements which are initially zero but which will be grown during the transformation. One possible approach is to use topological formulae. However, this method is not computationally feasible for larger networks. A more suitable method for finding \( q(0) \) is outlined in the Section 4.4.1.
The problem of stability of adjoint equations is dealt with in the Section 4.4.2. It is shown that stability depends on the characteristic values of the M matrix so that either \( y \) or \( q \) equations are unstable.

4.4.1 Method of Computing Elements of \( q(0) \)

A simple method of generating \( q(0) \) when dealing with the classical sensitivity can be developed from the nodal impedance matrix. Consider a nodal impedance matrix \( Z = Y^{-1} \) where \( Y \) represents a nodal admittance matrix. Clearly,

\[
YZ = I
\]

where \( I \) is the identity matrix. If this product is differentiated with respect to any element of the original network \( y_{rs} \) connected between the nodes \( r \) and \( s \),

\[
\frac{\partial Y}{\partial y_{rs}} Z + Y \frac{\partial Z}{\partial y_{rs}} = 0
\]

so that

\[
\frac{\partial Z}{\partial y_{rs}} = - Z \frac{\partial Y}{\partial y_{rs}} Z
\]

The derivative \( \frac{\partial Y}{\partial y_{rs}} \) is a matrix where the only non-zero elements are +1 in the location \( rr \) and \( ss \), and -1 in the locations \( rs \) and \( sr \). Expanding this matrix for the network shown in Figure 4.1 shows that for an impedance transfer function
This technique can easily be extended to other types of network functions and to networks without a common node (see Reference 23). Thus the initial \( q \) values can be obtained by very simple manipulation of the nodal impedance matrix.

When dealing with root sensitivity the above technique is not valid and one must resort to other methods of analysis.
4.4.2 Stability of Adjoint Equations

The propagation of error in a system of adjoint equations can be understood from the general solution of such a system which is given by

\[ \gamma(x) = \gamma(0) e^{Mx} \]
\[ q(x) = q(0) e^{-M^T x} \]

The propagation of error will depend upon the initial error of \( \gamma(0) \) and \( q(0) \), and on the characteristic values of \( M \). If the initial errors in \( \gamma(0) \) and \( q(0) \) are \( \epsilon_\gamma \) and \( \epsilon_q \), respectively, then the solution of these equations becomes

\[ \gamma(x) = (\gamma(0) + \epsilon_\gamma) e^{Mx} \]
\[ q(x) = (q(0) + \epsilon_q) e^{-M^T x} \]

Since \( M \) is not generally self-adjoint some characteristic values of either \( M \) or \( -M^T \) will have positive real parts and errors will propagate in the solution of one or the other set of equations. Consequently care must be exercised to minimize \( \epsilon_\gamma \) and \( \epsilon_q \) by computing the initial conditions with sufficient accuracy, usually to about eight places.

4.5 Computation of Sensitivity

Practical considerations in the computation of \( q(x) \) indicate that one or the other of the following alternatives should be taken. Either the rules for finding the permissible elements of the \( B \) matrix - particularly the rule of row comparison - should
be interpreted in terms of elements of the $M$ matrix and used for the sensitivity minimization, or alternatively, an algorithm should be found to compute the elements of $q$ in terms of elements of $B$ directly. The second alternative has been adopted because of the high requirements the $M$ matrix places on the memory of the computer. For example, a network with 20 nodes would require storage of $20 \times (20-1)/2 \times 20 \times (20-1)/2$ or 36100 memory locations for the entries of $M$.

Therefore, for the purpose of the computation of the elements of $q(x)$ and $\dot{q}(x)$, two square, symmetrical matrices, $Q$ and $\dot{Q}$, were formed. Their size was the same as that of the nodal admittance matrices. Since the $q$'s are generally complex, two such matrices were required for each element kind; one for the real part, and one for the imaginary part. In this formulation, an element of $\text{Re} \{Q_c\}$ in the location $sr$ or $rs$ is the real part of $q$ of the capacitor between the nodes $r$ and $s$. Similarly, an element of $\text{Re} \{\dot{Q}_c\}$ is the derivative of the real part of $q$ of the capacitor connected between the nodes $r$ and $s$. In this manner, the elements of the $Q$ and $\dot{Q}$ matrices are just as easily identified with the elements of the network as are the elements of the various $Y$ and $\dot{Y}$ matrices. The computation of sensitivities of various elements merely requires multiplication of the corresponding off-diagonal elements of the $Y$ and $Q$ matrices with the appropriate change of sign. The change of sign is required because of the negative sign of the off-diagonal elements of the $Y$ matrices.

Since the diagonal elements of $Q$ and $\dot{Q}$ were not relevant for these purposes, they were set to zero.
A formula from which the elements of the \( \hat{Q} \) matrices can be obtained is derived in Appendix C. It is given by

\[
q_{m,n} = \sum_{i=1}^{p} (q_{ni} - q_{mi}) b_{mi} + \sum_{i=1}^{p} (q_{mi} - q_{ni}) b_{ni} + \\
\sum_{i=1}^{p-1} \sum_{j=1}^{p-1} (q_{np} - q_{mp}) (\sum_{i=1}^{p} b_{mi} - \sum_{i=1}^{p} b_{ni})
\]

where \( q_{ii} = q_{ii} = 0 \) \( i = 1, 2, \ldots, p \)

This equation is valid for all elements of \( \hat{Q} \) including those connected to the reference node \( p \). For a two element type network there are required four \( \hat{Q} \) matrices and four \( \hat{Q} \) matrices so that for a 20 node network only \( 8 \times 20 \times 20 = 3200 \) memory locations are required. This compares very favourably with the previously stated memory requirement for the \( M \) matrix.

4.6. **Computer Program**

The flow chart of a computer program used for sensitivity minimization is shown in the Figure 42. It is written in Fortran IV for a I.B.M. 7044 machine. It has been obtained by appropriate modification of the program used for the reduction of the spread of element values. The network size is limited to 20 nodes and it can accommodate only two kinds of elements.
READ DATA, FORM NODAL ADMITTANCE & Q MATRICES

SELECT THE B MATRIX AUTOMATICALLY?

SELECT B MATRIX BY ROW COMPARISON

COMPUTE $\xi \{ s_i \}^2$

SELECT NUMERICAL VALUES OF b's

COMPUTE $\xi$, $\phi$, $\zeta$

ANY ELEMENT BECOMING NEGATIVE?

INTEGRATE Y'S AND q'S

ARE ALL INTEGRATIONS COMPLETED?

PRINT ELEMENT VALUES, SENSITIVITIES, COMPUTE $\xi \{ s_i \}^2$

IMPROVEMENT OF CRITERIA SATISFACTORY?

SELECT NEW b's?

STOP

READ B MATRIX

CORRECT APPROPRIATE ELEMENT OF B, $\xi$, $\phi$, $\zeta$

FIGURE 4.2. FLOW CHART OF COMPUTER PROGRAM.
The network is described to the computer by assigning a number to each node. The port nodes have to be specified as shown in Chapter II. The input consists of the node numbers between which the element is connected, element values, and their initial q's. From this information, the nodal and Q matrices are formed. Also required are the integration step size and various options such as number of integrations before each print-out etc. The B matrix can be specified either by the operator or it can be determined automatically by the computer by means of the row comparison method. In the latter case a check is made for the secondary growth. If it does occur, pertinent entries of the B matrix are eliminated on the basis of which b's would contribute less to the reduction of sensitivity. No check for the secondary growth is made when the B matrix is selected by the operator. The option of selecting the elements of B by operator has been made available because of the difficulty to arrange the program to make all the logical decisions which are required to take advantage of various special cases, and also to permit experimentation.

The numerical values of the b's are determined by setting each b in turn equal to +1 and checking its contributions to the criterion in (4.4) for sign. If the contribution is positive the sign of b is reversed. If the sum of elements in any row of B is required to be zero, the diagonal element is made equal to the negative sum of the off-diagonal elements.

The sign of the elements and their derivatives is checked to prevent reduction of sensitivity by growing negative elements. If any element is negative or zero the appropriate b is changed so that its derivative becomes either positive or zero as the
case may require. This check is made after each selection of the numerical values of the b's. Consequently, if the b's are kept constant throughout the computation, growth of the negative elements may occur, mainly through the contribution of the diagonal elements of the Y matrices. If the check is made after a number of integrations some elements may oscillate between positive and negative values. This method is very effective however in preventing negative growth of those elements which were initially zero.

The integration is carried out by the Runge-Kutta method, where the step size is determined by the operator. As in the program for the reduction of the spread of element values, it is possible to specify how often the intermediate results are printed out. With each print-out the sum of sensitivities of each element kind is also given to serve as a check of the accuracy of the integration. (Note that the sum of sensitivities of each element kind is invariant during the transformation).

The computation stops when either the specified number of iterations has been completed, or when the improvement of the criteria is less than a specified quantity, usually 1 or 2 %, in order to prevent oscillation about the optimum point.

4.7 Illustrative Examples

Various aspects of reduction of sensitivity by means of the continuously equivalent network have been discussed. It now remains to demonstrate the practical
applications of this method by means of examples. Two examples are shown. One deals
with the reduction of pole sensitivity in an RC network and the other shows reduction of
classical sensitivity in an LC network.

4.7.1 Reduction of Pole Sensitivity in RC Network

The original network is shown in Figure 4.3. Inspection of the
element matrices for this network (not shown) indicates that since there are no resistors
connected to the reference node 5, the B matrix has to be constrained so that
\[ \sum_{i=1}^{4} b_{ii} = 0 \text{ for all } i. \]
The permissible b's can be found by row comparison and they
are \( b_{22}, b_{24}, b_{33}, \) and \( b_{34} \). There is no secondary growth due to these b's.
Using the row comparison method, new elements which will be grown are \( g_{14}, g_{34}, \)
and \( c_{34} \). Since all the elements of the original network are effected by the trans­
formation, it is necessary to find the initial q's of all existing elements plus those which
will be grown during the transformation. Topological formulae were used to compute
these initial q values to about eight digits of accuracy.

The reduction of sensitivity has been carried out for the pole closest to
the origin, which was located at \( s = -0.122829 \). The performance criterion was
\[ \rho = \sum_{i=1}^{n} S(F, y_i) \]^{2} and this was reduced from \( 0.2354 \cdot 10^{-1} \) to \( 0.13086 \cdot 10^{-1} \)
i.e., by approximately 44%. Potential error due to the integration error of adjoint
equations was checked by printing out sums of sensitivities of each element kind during
FIGURE 4.3. RC LADDER – ORIGINAL NETWORK.

FIGURE 4.4. RC LADDER – FINAL NETWORK.
the integration. This was found to agree to the sixth digit. A more significant check was made after the computation was completed by calculating the element sensitivities of the final network. It was found that the largest error was located in the third significant digit for the sensitivity of $g_{12}$ where the reduction of sensitivity was largest from $0.2407 \cdot 10^{-2}$ to $0.5480 \cdot 10^{-5}$. Since the final value of this element is small in comparison with others it had a negligible effect upon the sum of sensitivities. The error in sensitivities of other elements was smaller and lay between the fourth and sixth digit. The polynomial of the final network was also computed as a further accuracy check. It was found that all coefficients agreed exactly with those of the initial network. Consequently, the poles remained invariant during the transformation.

It is interesting to note that the sensitivities of other poles also have been reduced and therefore reduction of one pole sensitivity has led to the reduction of all pole sensitivities. This result has been anticipated by the discussion in the Section 4.2.2.

Experiments with the program have shown that there was a negligible difference in the final network when the $b$'s were selected continuously, or when they were kept constant throughout the integration. It is felt, however, that this would not be true generally.
4.7.2 Reduction of Classical (Bode) Sensitivity in Terminated LC Network

In this example, the technique was applied to reduce the classical sensitivity of an elliptic low pass ladder filter \( b_1 \) terminated in unit source and load resistors. This network is shown in Figure 4.5. The reduction has been carried out at the frequency \( \omega = 0.85 \) radians close to the edge of the pass band. The \( B \) matrix consisted of \( b_{21}, b_{22}, b_{23}, b_{43}, b_{44}, \) and \( b_{45} \). Since there is no inductor connected to the reference node 6, it is required that the \( b \)'s in each row add to zero. All elements of row 3 were set to zero to prevent secondary growth. The new elements grown during the transformation were \( c_{13}, c_{15}, c_{35}, \) and \( c_{35} \). Since all elements of the network were affected by the transformation, it was necessary to find initial \( q \)'s for all elements of the original network and of the elements grown during the transformation. The \( q \)'s were obtained by using the matrix inversion technique alluded to earlier, and were given with eight digit accuracy. During the integration the sum of the real and imaginary parts of the sensitivities was printed out as the check on the accuracy of integration. The comparison of these sums in the original and final network showed agreement to six digits for both the real and imaginary parts. The sensitivities of the elements of the final network were also checked individually using the matrix inversion scheme. The comparison with the output of the integration showed that disagreement was at most in the sixth digit and therefore the reduction is sufficiently accurate for all practical purposes. The improvement of the criteria \( \phi = \sum_{i=1}^{n} S(F, \gamma_i)^2 \) was from 5.8 to 4.09, i.e. by 29%. The final network is shown in Figure 4.6.
Reduction of Classical (Bode) Sensitivity in Terminated LC Network

In this example, the technique was applied to reduce the classical sensitivity of an elliptic low pass ladder filter terminated in unit source and load resistors. This network is shown in Figure 4.5. The reduction has been carried out at the frequency \( \omega = 0.85 \) radians close to the edge of the pass band. The \( B \) matrix consisted of \( b_{21}, b_{22}, b_{23}, b_{43}, b_{44}, \) and \( b_{45} \). Since there is no inductor connected to the reference node 6, it is required that the \( b \)'s in each row add to zero. All elements of row 3 were set to zero to prevent secondary growth. The new elements grown during the transformation were \( \delta_{13}, c_{13}, \delta_{35}, \) and \( c_{35} \). Since all elements of the network were affected by the transformation, it was necessary to find initial \( q \)'s for all elements of the original network and of the elements grown during the transformation. The \( q \)'s were obtained by using the matrix inversion technique alluded to earlier, and were given with eight digit accuracy. During the integration the sum of the real and imaginary parts of the sensitivities was printed out as the check on the accuracy of integration. The comparison of these sums in the original and final network showed agreement to six digits for both the real and imaginary parts. The sensitivities of the elements of the final network were also checked individually using the matrix inversion scheme. The comparison with the output of the integration showed that disagreement was at most in the sixth digit and therefore the reduction is sufficiently accurate for all practical purposes. The improvement of the criteria \( \rho = \sum_{i=1}^{n} S(F, y_i)^2 \) was from 5.8 to 4.09, i.e., by 29%. The final network is shown in Figure 4.6.
FIGURE 4.5. LC LOW PASS LADDER - ORIGINAL NETWORK.

FIGURE 4.6. LC LOW PASS LADDER - FINAL NETWORK.
In general, experimental results revealed that it is necessary to find the initial values of the $q's$ to about eight digits to ensure the reliability of the results, especially when many integration steps are required.

4.8 **Summary of Chapter IV**

The method of reducing both the root and classical sensitivity by continuously equivalent networks has been discussed and demonstrated by examples. It is shown that it can be used effectively; however, it requires that initial values of the $q's$ be obtained with sufficient accuracy to overcome the propagation of error during the integration.
CHAPTER V

SUMMARY AND CONCLUSIONS

In comparison with the original algebraic transformations of Cauer\(^1\), \(^{18}\) and others\(^2\), \(^3\), \(^7\), continuously equivalent networks have several distinct advantages.

First, it enables an entire network to be continually transformed so as to progressively minimize a performance criterion based on practical design requirements. Secondly, the linearity with respect to the elements of the transformation matrix \(B\) makes it possible to consider the effect of individual \(b\)'s separately and then superimpose their effect. In this way, it is possible to detect by a very simple method those \(b\)'s which cause the growth of negative elements. The method can also be used to prevent the secondary growth by keeping suitable \(b\)'s equal to zero. The advantages of the procedure are simplicity and computer adaptability. The possibility of using algebraic constraint equations instead of the row comparison technique does not appear to offer significant advantages for the two element type network with respect to yielding a transformation matrix with more degrees of freedom, particularly since different types of constraint equations would be required for each new network. On the other hand, the number of constraints on the \(B\) matrix increases markedly with the number of types of network elements, so that the technique becomes unsuitable for optimization of a performance criterion in networks with more than two kinds of elements.

The incremental network equivalence outlined in Chapter III can be looked upon as an attempt to use the concept of row comparison with algebraic equations. To maintain as much linearity in these equations as possible, the number of the \(b\)'s must be
very small for any transformation step. The method is useful, therefore, for altering one particular locality of a network at a time. Although this limits its usefulness in comparison with the continuous equivalence, it nevertheless has merit for certain practical applications such as eliminating negative elements. There have been instances, however, where it is not possible to eliminate a negative element by this method. For example, it cannot eliminate the negative element in a Brune section. This task appears to be impossible when the number of nodes is left unchanged and it seems necessary to increase the number of nodes by suitably augmenting the parameter matrices. Some work in this respect has been done by Baker, who outlined a method of eliminating a negative element for certain types of networks by augmenting their loop impedance matrices. No attempt has been made here to interpret his method for nodal admittance matrices, or to generalize the class of networks that can be handled. It seems, however, that due to its practical implications, the method of augmenting parameter matrices deserves further study. Evidently, it could be applicable for both incremental and continuous equivalence.

The use of continuously equivalent networks for the reduction of sensitivity is greatly facilitated by the ability to generate differential equations for \( q(x) \). These equations are also useful for gaining insight into the correlation between changes of element values and their respective sensitivities. The practical disadvantage of minimizing classical sensitivity at one frequency is that nothing is known about the sensitivities at other frequencies. The suggestion of reducing the sensitivity at several selected frequency points simultaneously is feasible, but not very attractive where larger networks are concerned. It seems more practical to reduce the root sensitivity of a dominant singularity or a linear combination of dominant singularities.
A shortcoming of this purely numerical technique is the absence of a theoretical lower bound of the criteria. Although in the examples shown the sensitivity has been reduced, it is not known if further - perhaps significant - improvement of the criteria was possible.

The potential instability of the numerical integration of the adjoint equations of $x$ and $q$ must also be considered. Although in the examples shown the instability was not as severe as anticipated, it is nevertheless required to compute the initial values of $q$ with much greater accuracy than is easily obtainable. Since the stability depends on the characteristic values of the $M$ matrix which are not known during the transformation, the possibility of obtaining entirely wrong results does exist and careful accuracy checks are always necessary.

In view of the fact that the propagation of error appears to be confined to the $q$'s, it seems this difficulty could be alleviated, at least for classical sensitivity, by calculating new $q$'s after each transformation step by a matrix inversion method rather than by integration. This possibility was not explored in this thesis, and warrants further investigation.

Although the generation of continuously equivalent networks by linear transformation is an useful tool, it is nevertheless quite restrictive. For example, it yields networks which are exactly equivalent when often networks equivalent within a gain constant would be acceptable. It also retains the number of nodes unchanged, unless the area of augmentation of the parameter matrices is more thoroughly explored. It seems that other types of transformations, perhaps of a non-linear character, should be developed. The search for such transformations represents a fruitful area of network research.
APPENDIX A

When the indefinite matrix $Y$ of a network with $p$ nodes, where $p$ is the reference node, has no element in row $p$, a new element to the reference node can be grown only if some rows of $B$ do not sum to zero. Denote $\sum_{i=1}^{p-1} b_{i} = K_i$, where $i = 1, 2, \ldots, p-1$. Then the derivatives for the elements in the row $p$ are,

$$
\begin{align*}
\dot{y}_{p,1} &= -y_{11} K_1 - y_{12} K_2 - y_{13} K_3 - \cdots - y_{1,p-1} K_{p-1} \\
\dot{y}_{p,2} &= -y_{21} K_1 - y_{22} K_2 - y_{23} K_3 - \cdots - y_{2,p-1} K_{p-1} \\
\dot{y}_{p,3} &= -y_{31} K_1 - y_{32} K_2 - y_{33} K_3 - \cdots - y_{3,p-1} K_{p-1} \\
&\vdots \\
\dot{y}_{p,p-1} &= -y_{p-1,1} K_1 - y_{p-1,2} K_2 - \cdots - y_{p-1,p-1} K_{p-1}
\end{align*}
$$

Since any $y_{ii} = -y_{1i} - y_{2i} - y_{3i} - \cdots - y_{p-1,i}$, these equations can be written as

$$
\begin{align*}
\dot{y}_{p,1} &= -y_{12} (K_2 - K_1) - y_{13} (K_3 - K_1) - \cdots - y_{1,p-1} (K_{p-1} - K_1) \\
\dot{y}_{p,2} &= -y_{12} (K_1 - K_2) - y_{23} (K_3 - K_2) - \cdots - y_{2,p-1} (K_{p-1} - K_2) \\
\dot{y}_{p,3} &= -y_{13} (K_1 - K_3) - y_{23} (K_2 - K_3) - \cdots - y_{3,p-1} (K_{p-1} - K_3) \\
&\vdots \\
\dot{y}_{p,p-1} &= -y_{1,p-1} (K_1 - K_{p-1}) - y_{2,p-1} (K_2 - K_{p-1}) - y_{3,p-1} (K_3 - K_{p-1})
\end{align*}
$$
Clearly

\[ \dot{y}_{p,1} + \dot{y}_{p,2} + \dot{y}_{p,3} + \ldots + \dot{y}_{p,p-1} = 0 \]

Therefore it is not possible to obtain a positive derivative without simultaneously obtaining a negative one so that it is not possible to grow only positive elements to the reference node. The derivatives \( \dot{y}_{p,1}, \dot{y}_{p,2}, \) etc. will be zero if

\[ K_1 = K_2 = K_3 = \ldots = K_{p-1} \]

Since due to the port invariance requirements some of the rows of \( B \) have to be zero, it is generally required that

\[ K_1 = K_2 = K_3 = \ldots = K_{p-1} = 0 \]
APPENDIX B

A special class of networks where it is not required that \( \sum_{j=1}^{p-1} b_{ij} = 0 \) in order to prevent the growth of negative elements to the reference node is depicted by example in Fig. B1. The important feature of this network is that not more than one inductor is connected to any node of the network.

This means that if such an inductor is connected between the nodes \( m \) and \( n \), \( \delta_{mm} = \delta_{nn} = -\delta_{mn} = -\delta_{nm} \). Thus in the example of Fig. B.1,
\[
\begin{align*}
\delta_{11} &= \delta_{22} = -\delta_{12} = -\delta_{21} , \\
\delta_{33} &= \delta_{44} = -\delta_{34} = -\delta_{43} , \text{ etc.}
\end{align*}
\]

The conditions under which no new inductors will be grown to the reference node \( \delta \) can be obtained by examining the equations for the derivatives in row \( \delta \). For example,
\[
\begin{align*}
\dot{\delta}_{38} &= \delta_{33} \sum_{j=1}^{p-1} b_{3j} + \delta_{34} \sum_{j=1}^{p-1} b_{4j} , \\
\dot{\delta}_{48} &= \delta_{43} \sum_{j=1}^{p-1} b_{3j} + \delta_{44} \sum_{j=1}^{p-1} b_{4j} .
\end{align*}
\]

It is evident from \( \delta_{0} \) and from equations (2.4) and (2.5) that no other elements of \( B \) can contribute to \( \dot{\delta}_{38} \) and \( \dot{\delta}_{48} \).

Since \( \delta_{33} = \delta_{44} = -\delta_{34} = -\delta_{43} \) it follows immediately that
\[
\begin{align*}
\dot{\delta}_{38} &= \dot{\delta}_{48} = 0 \text{ if } \\
\sum_{j=1}^{p-1} b_{3j} &= \sum_{j=1}^{p-1} b_{4j} .
\end{align*}
\]
FIGURE B.1. LOW PASS LADDER NETWORK.
The same conclusion for other pairs of rows of \( B \) follows and the extension to other networks of this type is self-evident.

In most cases the sign of sums of elements of various pairs of rows of \( B \) cannot be selected freely, but it is restricted by the requirement that only positive new capacitors grow to the reference node.

When the \( B \) matrix consists of diagonal elements only, this transformation becomes in effect the "differential equivalent" of the algebraic procedure sometimes called the "internal impedance level transformation"\(^24\).
APPENDIX C

The formation of an algorithm for calculation of the elements of the $\hat{Q}$ matrix can be done quite mechanically from the adjoint equations of $\hat{y}$ and $q$. It requires to determine how the b's enter into any column of $M$ and then to re-arrange this column to become a row of $-M^\dagger$. This is equivalent to finding the contributions of any element $y_{mn}$ to the elements of $\hat{\chi}$ such as

$$\hat{\chi}_{rn} = b_{mr} y_{mn} \quad (1.C)$$

and then to re-arrange the equation to read

$$\hat{q}_{mn} = -b_{mr} q_{rn} \quad (2.C)$$

When all the b's are considered, the right hand side of $\hat{q}_{mn}$ will be a sum of terms which are obtained by the interchanges outlined by (1.C) and (2.C).

The b's in the column $n$ of $M$ appear due to the off-diagonal element $y_{mn}$ and also due to the diagonal elements $y_{mm}$ and $y_{nn}$ since

$$y_{mm} = -y_{1m} - y_{2m} - \cdots - y_{mn} \cdots$$

In addition, the effect of sums of elements in the row of $B$ has also to be considered. Hence $y_{mn}$ can contribute to the derivatives of various elements of $\hat{\chi}$ as follows:
\[ y_{1n} = (b_{m1} - b_{n1}) \gamma_{mn} \]
\[ y_{2n} = (b_{m2} - b_{n2}) \gamma_{mn} \]
\[ \vdots \]
\[ y_{mn} = (b_{mm} - b_{nm}) \gamma_{mn} \]
\[ \vdots \]
\[ y_{pn} = \left( -\sum_{i=1}^{p-1} b_{ni} + \sum_{i=1}^{p-1} b_{ni} \right) \gamma_{mn} \]
\[ y_{1m} = (b_{n1} - b_{m1}) \gamma_{mn} \]
\[ y_{2m} = (b_{n2} - b_{m2}) \gamma_{mn} \]
\[ \vdots \]
\[ y_{nm} = (b_{nn} - b_{mn}) \gamma_{mn} \]
\[ \vdots \]
\[ y_{pm} = \left( -\sum_{i=1}^{p-1} b_{ni} + \sum_{i=1}^{p-1} b_{mi} \right) \gamma_{mn} \]

Carrying the interchange as shown by (1.C) and (2.C) and making use
of the symmetry of the $Q$ matrix, $q_{mn}$ is given by

$$
q_{mn} = (b_{m1} - b_{n1}) q_{n1} + (b_{m2} - b_{n2}) q_{n2} + (b_{m3} - b_{n3}) q_{n3} + \ldots \nonumber \\
+ (\sum_{i=1}^{p-1} b_{ni} - \sum_{i=1}^{p-1} b_{mi}) q_{np} 
$$

$$
+ (b_{n1} - b_{m1}) q_{m1} + (b_{n2} - b_{m2}) q_{m2} + (b_{n3} - b_{m3}) q_{m3} + \ldots \nonumber \\
+ (\sum_{i=1}^{p-1} b_{ni} - \sum_{i=1}^{p-1} b_{mi}) q_{mp}
$$

For the purpose of checking the contribution of various $b$'s to the criterion (4.4) it is preferable to re-arrange this equation in terms of individual $b$'s rather than their differences, so that

$$
q_{mn} = \sum_{i=1}^{p-1} (q_{ni} - q_{mi}) b_{mi} + \sum_{i=1}^{p-1} (q_{ni} - q_{mi}) b_{ni} 
$$

$$
+ (q_{np} - q_{mp}) (\sum_{i=1}^{p-1} b_{ni} - \sum_{i=1}^{p-1} b_{mi})
$$

where $q_{ii} = 0$ for all $i$. 
REFERENCES


