The Laplacian and its Eigenfunctions over Self-Similar Sets

Karim Elmallakh
Department of Mathematics and Statistics
McGill University, Montreal
March 2022

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of Master of Science in Mathematics

Abstract

In this work, we review some of the major steps in the construction of the Laplacian over general self-similar fractal sets, and study properties of its spectrum and eigenfunctions of the constructed formulation. After a detailed exposition of self-similar sets and their characterisation, functional analytic and measure-theoretic techniques are used to develop a viable Laplacian over self-similar fractal sets equipped with a regular harmonic structure. An exposition of the properties of eigenfunctions and a Weyl's-type law is presented for the corresponding eigenvalues. The work concludes with a discussion of the heat kernel, and possible methods of heat kernel embeddings of post-critically finite fractals into Hilbert spaces, emulating the general techniques over manifolds.

Abrégé

Dans ce thèse, nous présontons les étapes importants dans la construction de la Laplacian sur des ensembles fractales auto-sembable et nous étudions les valeurs propres et fonctions propres de ce opérateur différentiel. Après une exposition détaillée sur des ensembles fractale auto-similaire et leurs caractéerisation, des techniques d'analyse fonctionelle et la théorie de la mesure sont utilisées pour développer un Laplacian viable équipé avec une structure harmonique régulière. Un exposition de les propriétés des les fonctions et valeurs propres, et une analogue de la théorème de Weyl sur des asymptotique des valeurs propres de la Laplacian sur $\mathbb{R}^n$ est presentée. Ce thèse conclut avec une discussion de la noyau de la chaleur de la Laplacian et des homéomorphismes injectifs entre des fractals auto-sembable et un espace Hilbert avec des fonctionnes propres.

Acknowledgements

Firstly, I would like to thank the contributions of my supervisors Prof. Dmitry Jakobson and Prof. Gantumur Tsogtsgerel for their incredibly valuable guidance, support, and resources throughout the writing of this thesis and the completion of this degree.

Furthermore, I would like to thank my dear friend Pauline Lelandais, without whom I have absolutely no clue how I would have made it through this degree, and written this thesis. Thank you for listening to my ideas, brainstorming, and supporting me these past two years.

I would also like to give a special mention to my friend Vibha Rao who was also there to support me and listen to my mathematical ideas and beyond throughout the pandemic. I am forever grateful to you.

Lastly, I would like to thank the Department of Mathematics and Statistics at McGill University for providing me with the financial support to pursue this degree and write this thesis, as well as providing me with academic and administrative support along the way.

Table of Contents

Chapter 1

Introduction

Fractals are a ubiquotous occurrence in mathematics, and have many applications to various fields of research, including geography, physics, biology, and many others; a lot of natural structures exhibit fractal geometry, from neuroplasticity and neural geometry to the length of coastlines. This makes studying diffusion processes over such sets incredibly useful, but from a purely mathematical perspective, the study of analysis on fractals leads to a very rich theory linking various branches of mathematics including graph theory, differential geometry, real and complex analysis, and probability. In this work, we begin in Chapter 2 by constructing a specific class of fractal sets known as self-similar fractals, and discuss notions of measure and integration over such sets in order to build a theory of differential operators, and specifically an analogue of the classical Laplacian over such sets.

In chapter 3, we begin investigating the architecture required to build the Laplacian over self-similar sets by viewing them as limits of graph networks equipped with the appropriate Dirichlet forms, giving rise to a resistance metric. We further discuss the relations between these Dirichlet forms and use functional analytic and measure-theoretic techniques to relate these bilinear Dirichlet forms to our previously constructed notion of measure. We use this to construct a non-negative self-adjoint operator analogous to the classical Laplacian over these sets. Chapter 4 then expands on chapter 3 by explicitly defining the Dirichlet and Neumann Laplacians as self-adjoint operators, and following the groundbreaking work of Kigami in [7] and [8] to define a pointwise formulation, known as the Kigami Laplacian, and develop a Gauss-Green formula.

The fourth chapter of this work then proves the existence of eigenvalues and eigenfunctions of the Kigami Laplacian, and investigates some of their basic properties. We also state and prove an important theorem presented by Kigami and Lapidus in [9] regarding the asymptotics of the Laplacian eigenvalues.

We conclude this work with the fifth chapter, where we discuss an application of eigenfunctions to the construction of the heat kernel. It is first introduced formally, and we prove its existence and preliminary continuity and differentiability properties. We then use the heat kernel to construct an embedding of a post-critically finite self-similar set into $\ell^2(\mathbb{R})$, emulating the usual techniques used to construct heat kernel embeddings of manifolds in differential geometry presented by Berard, Besson, and Gallot in [3]. We also discuss the differences between the manifold and fractal cases in the conclusion, and the shortcomings of heat kernel embeddings of fractals as opposed to manifolds, and briefly propose further avenues of research into constructing metric-preserving embeddings of fractals into Hilbert-spaces.

Chapter 2

Preliminaries

We begin with an exposition of the basics of self-similar and post-critically finite (p.c.f) fractals, including their construction and how one can begin to study analysis on such structures; specifically, we review the construction of the Laplacian in this context. The following work is an adaptation of the work of Kigami and Strichartz from [8] and [12].

2.1 Construction of Self-Similar Sets

To begin our construction, we take (X, d) to be a metric space, and $C(X)$ the space of all compact non-empty subsets of X equipped with the Hausdorff metric.

Definition 2.1.1. *For $A, B \in C(X)$, we define the Hausdorff distance between A and B to be*

$$\delta(A, B) = \inf \left\{ r > 0 \middle| \bigcup_{y \in A} B_r(y) \supseteq B, \bigcup_{y \in B} B_r(y) \supseteq A \right\}$$

where $B_r(y)$ is the ball of radius r centered at y.

It can be easily verified that Hausdorff distance is well-defined over $C(X)$ and is indeed a metric; furthermore, it is fairly well known that equipped with this metric, if (X, d) is complete, then $C(X)$ is complete.

Definition 2.1.2. *Let (X, d) be a metric space, and $f : X \to X$ a Lipschitz continuous map with Lipschitz constant $0 < r < 1$. Then, f is referred to as a contraction with contraction ratio r.*

Given this definition, the Banach Fixed Point Theorem establishes the existence and uniqueness of a fixed point for a contraction mapping f. In other words, for $f : X \to X$ a contraction and (X, d) a complete metric space, there exists a unique $x \in X$ such that $f(x) = x$. The goal of this section is to establish the existence of self similar sets using the Banach Fixed Point Theorem applied to the complete metric space $(C(X), \delta)$. To this end, we require the following lemmas:

Lemma 2.1.1. *Let $A_1, A_2, B_1, B_2 \in C(X)$. Then,*

$$\delta(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{\delta(A_1, B_1), \delta(A_2, B_2)\}.$$

Proof. Let $r > \max\{\delta(A_1, B_1), \delta(A_2, B_2)\}$, by definition of $\delta(A_i, B_i)$ for $i = 1, 2$, we have that

$$\bigcup_{y \in A_i} B_r(y) \supseteq B_i,$$

which implies

$$\bigcup_{y \in A_1 \cup A_2} B_r(y) \supseteq B_1 \cup B_2.$$

By symmetry of the metric δ, we have a similar result with the roles of A_i and B_i interchanged, demonstrating that $r \geq \delta(A_1 \cup A_2, B_1 \cup B_2)$, thus proving the lemma. □

Remark 2.1.1. *By induction, Lemma 1 extends to finite unions of sets in $C(X)$. More precisely,*

$$\delta(\cup_{i=1}^{n} A_i, \cup_{i=1}^{n} B_i) \leq \max_{1 \leq i \leq n}\{\delta(A_i, B_i)\}.$$

When referencing this lemma in the proof of further results in this section, we refer to this more general formulation.

Lemma 2.1.2. *Let (X, d) be a metric space, and $f : X \to X$ a contraction with contraction ratio r. Then for all $A, B \in C(X)$, $\delta(f(A), f(B)) \leq r\delta(A, B)$.*

Proof. For $A, B \in C(X)$, let $s := \delta(A, B)$. By definition of δ

$$\bigcup_{y \in A} B_s(y) \supseteq B, \text{ and } \bigcup_{y \in B} B_s(y) \supseteq A.$$

It follows that for $x \in B$ and $f(x) \in f(B)$, by the above containment we have that

$$f(x) \in f\left(\bigcup_{y\in A} B_s(y)\right) = \bigcup_{y\in A} f(B_s(y)),$$

And so there exists a $y \in A$ such that $f(x) \in f(B_s(y))$. Then, since f is a contraction with contraction ratio $0 < r < 1$, we obtain the following string of inequalities

$$d(f(x), f(y)) \leq rd(x, y) \leq rs < s.$$

So $f(x) \in \bigcup_{f(y)\in f(A)} B_{rs}(y)$, demonstrating that $f(B) \subseteq \bigcup_{f(y)\in f(A)} B_{rs}(y)$. By symmetry, we have the analogous result $f(A) \subseteq \bigcup_{f(y)\in f(B)} B_{rs}(y)$. Then, by definition of the Hausdorff metric as the infimum, we get that $\delta(f(A), f(B)) \leq rs = r\delta(A, B)$, as desired. □

Now that we have established how the Hausdorff metric acts on images of contractions, we are prepared to begin our construction of self-similar sets.

Theorem 2.1.3. *Let (X, d) be a complete metric space and $\{f_i\}_{i=1}^n$ be a finite set of contractions from X to itself, with contraction ratios $\{r_i\}_{i=1}^n$ respectively. Then there exists a unique $K \in C(X)$ called the self similar set with respect to $\{f_i\}_{i=1}^n$, satisfying*

$$K = \bigcup_{i=1}^n f_i(K).$$

Proof. Consider $C(X)$ equipped with the Hausdorff metric δ, and define the function

$$F : C(X) \to C(X)$$
$$F(A) = \bigcup_{i=1}^n f_i(A).$$

It is clear that F is well-defined and is indeed a map from $C(X)$ to itself, as finite unions of compact sets are compact.And by Lemmas 1 and 2, we have that for $A, B \in C(X)$

$$\delta(F(A), F(B)) \leq \max_{1\leq i\leq n} \delta(f_i(A), f_i(B)) \leq \max_{1\leq i\leq n} r_i\delta(A, B)$$

Letting $r := \max_{1\leq i\leq n} r_i$, the above string of inequalities demonstrates that $\delta(F(A), F(B)) \leq r\delta(A, B)$. Thus, F is a contraction with contraction ratio r, and by the Banach Fixed Point Theorem, we have that there exists a unique $K \in C(X)$ such that $K = F(K)$. In other words,

$$K = \bigcup_{i=1}^{n} f_i(K)$$

as desired. □

2.2 Characterisation via Shift Spaces and Iterated Function Systems

Given a self-similar set K with associated contractions $\{f_i\}_{i=1}^n$, we now wish to see what happens as we arbitrarily take compositions of each of the f_i for $1 \leq i \leq n$. We first introduce some necessary notation.

Definition 2.2.1. *For $m, n \in \mathbb{N}$, we define the set of words of length m to be the Cartesian product of the set $\{1, 2, ..., n\}$ m times; i.e.:*

$$W_n^m = \{1, 2, ..., n\}^m = \{w_1 w_2 w_3 ... w_m | w_i \in \{1, 2, ..., n\}\}$$

Furthermore, we define the empty word to be $W_0^m = \{\emptyset\}$, and $W_*^n = \bigcup_{m=1}^{\infty} W_n^m$. In settings where there can be no ambiguity, for ease of notation we will refer to W_n^m as W_n, and W_*^n as W_*.

Definition 2.2.2. *We define the shift space of n symbols to be*

$$\Sigma^n := \{\omega_1 \omega_2 \omega_3 ... \mid \omega_i \in \{1, 2, 3, ..., n\}\}$$

Likewise, for simplicity, we will refer to this space as Σ when the context is clear.

Given a shift space Σ (with n symbols), it is clear that both appending a symbol from $\{1, 2, ..., n\}$ to a word $\omega = \omega_1 \omega_2 \omega_3 ... \in \Sigma$, and removing a symbol from ω yields another word in Σ respectively. More formally, for $k \in \{1, 2, ..., n\}$, we can define the two maps

$$\sigma_k : \Sigma \to \Sigma$$
$$\sigma_k(\omega) = k\omega_1 \omega_2 \omega_3 ...$$

and

$$\sigma : \Sigma \to \Sigma$$
$$\sigma(\omega) = \omega_2\omega_3\omega_4...$$

The latter map, which is of more significance for our purposes, is referred to as the **shift map**. Now given this framework, for an arbitrary set of contractions $\{f_i\}_{i=1}^n$, we can consider arbitrary compositions of the f_i using words $w = w_1w_2...w_m \in W_*$ by defining $f_w = f_{w_1} \circ f_{w_2} \circ f_{w_3} \circ ... \circ f_{w_m}$, and the associated image spaces $K_w = f_w(K)$. We will refer to a set $K_w \subseteq K$ as a **cell of level m**. Furthermore, by defining the appropriate metric on the shift space Σ, we can realise it as a self-similar set, so that given a self-similar set K with contractions $\{f_i\}_{i=1}^n$, we can establish a one-to-one correspondence between K and its shift space Σ. A set is called **post-critically finite** if for all $1 \leq i, j \leq n$, we have that $f_i(K) \cap f_j(K)$ is non-empty and finite, in addition to the symbolic representations of these points (in terms of words of length m) are periodic. We now prove some important theorems characterising self-similar sets via shift spaces, as done in [8].

Theorem 2.2.1. *Let $\omega, \tau \in \Sigma$ with $\omega \neq \tau$, and $r \in (0, 1)$. Then, the map*

$$\delta_r : \Sigma \times \Sigma \to \mathbb{R}$$
$$\delta_r(\omega, \tau) = \begin{cases} r^{s(\omega,\tau)} & \omega \neq \tau \\ 0 & \omega = \tau \end{cases}$$

where $s(\omega, \tau) := \min_{m \in \mathbb{N}}\{\omega_m \neq \tau_m\} - 1$ defines a metric on Σ, and (Σ, δ_r) is a compact metric space. Furthermore, with respect to this metric, the maps σ_k for $k \in \{1, 2, 3, ..., n\}$ are contractions with contraction ratio r, and Σ is the self similar set with respect to $\{\sigma_k\}_{k=1}^n$.

Proof. We begin by establishing that δ_r is indeed a metric on Σ. For $\omega, \tau \in \Sigma$, it is clear that $\delta_r(\omega, \tau)$ is non-negative and is 0 if and only if $\omega = \tau$; similarly, symmetry trivially follows from the definition of $s(\omega, \tau)$. So, it suffices to prove the triangle inequality.

We first note that $s(\omega, \tau)$ is clearly a natural number, and as a quantity measures how far along both words we need to go before the symbols of both words start to differ. More precisely, given $n = s(\omega, \tau)$, we get that for all $1 \leq i \leq n$, $\omega_i = \tau_i$ and $\omega_{n+1} \neq \tau_{n+1}$.

So it follows that for $\omega, \tau, \kappa \in \Sigma$, $\min\{s(\omega,\kappa), s(\tau,\kappa)\} \leq s(\omega,\tau)$. In other words, the index at which all three start to differ, is less than the index at which any two individually start to differ.

Therefore, for ω, τ, κ distinct words, we have that

$$\delta_r(\omega,\kappa) \leq r^{\min\{s(\omega,\kappa),s(\tau,\kappa)\}} \leq r^{s(\omega,\kappa)} + r^{s(\tau,\kappa)} = \delta_r(\omega,\tau) + \delta_r(\tau,\kappa)$$

As desired. To prove compactness of (Σ, δ_r), let $m \in \mathbb{N}$, and take a sequence $\{\omega^n\}_{n=1}^{\infty} \subseteq \Sigma$, and consider the sets

$$S_m = \{n \in \mathbb{N} \mid (\omega^n)_i = \tau_i \text{ for all } 1 \leq i \leq m \text{ and } \tau \in \Sigma\}$$

So for fixed $\tau \in \Sigma$, S_m is the set of all indices where ω^n agrees with τ. If we can prove that S_m is infinite for each m, that would imply that there are infinitely many ω^n that agree with tau at the first m terms. This would then allow us to extract a subsequence that converges to τ. We proceed with a proof of this claim by induction.

For the base case of $m = 1$, there are trivially infinitely many words that share a symbol at the first index. Since there are only finitely many symbols and infintely many ω^n, by the pigeonhole principle, we get that S_1 is infinite.

Now assume that for a fixed k, S_k is infinite. As in the base case, we have that there will be infinitely many ω^n that agree at the symbol in position $k+1$, and by the inductive hypothesis, there are infinitely many ω^n that agree with a $\tau \in \Sigma$ at the first k symbols. Combining both of these facts together, we can find a $\tau^* \in \Sigma$ such that there are infinitely many ω^n that agree at the first $k+1$ symbols, as desired. Then, since each S_m is infinite, we can extract a subsequence of $\{\omega^n\}$ of consecutively agreeing terms at each $(\omega^n)_m$ that converges to a $\tau \in \Sigma$ as $n \to \infty$, thus demonstrating that Σ is compact with respect to the topology induced by the metric δ_r.

Lastly, it remains to prove that for $1 \leq k \leq n$, σ_k is a contraction, and that Σ is a self similar set with respect to $\{\sigma_k\}_{k=1}^{n}$. . For $\omega, \tau \in \Sigma$, with $\omega = \omega_1\omega_2\omega_3...$ and $\tau = \tau_1\tau_2\tau_3...$ we have that

$$\delta_r(\sigma_k(\omega), \sigma_k(\tau)) = r^{s(\sigma_k(\omega),\sigma_k(\tau))} = r^{s(k\omega_1\omega_2\omega_3...,k\tau_1\tau_2\tau_3...)} = r^{s(\omega,\tau)+1} = r\delta_r(\omega,\tau).$$

Therefore, for all $1 \leq k \leq n$, $\delta_r(\sigma_k(\omega), \sigma_k(\tau)) \leq r\delta_r(\omega, \tau)$, so that each σ_k is a contraction (with equality), with contraction ratio r. And since, for all $\omega \in \Sigma$, there exists $1 \leq k \leq n$ such that $\omega_1 = k$, we get that $\omega \in \bigcup_{k=1}^n \sigma_k(\Sigma)$. Trivially $\bigcup_{k=1}^n \sigma_k(\Sigma) \supseteq \Sigma$, so that

$$\Sigma = \bigcup_{k=1}^n \sigma_k(\Sigma).$$

This completes the proof of the theorem. □

Now that we've realised the shift space Σ as a self-similar set with respect to the appropriate contractions, it remains to establish a correspondence between an arbitrary self-similar set K and its shift space Σ. A self-similar set characterised and expressed in this way is referred to as an **iterated function system**. This correspondence is established in the following theorem:

Theorem 2.2.2. *Let (X, d) be a complete metric space, and $K \subseteq X$ be the self-similar set with respect to the contractions $\{f_i\}_{i=1}^n$ with corresponding contraction ratios $\{r_i\}_{i=1}^n$. Then, for $\omega = \omega_1\omega_2\omega_3... \in \Sigma$, the map defined by*

$$\pi : \Sigma \to K$$
$$\pi(\omega) = \bigcap_{m=1}^\infty K_{\omega_1\omega_2...\omega_m}$$

is a well-defined, continuous, surjective map, and for all $1 \leq i \leq n$, $\pi \circ \sigma_i = f_i \circ \pi$.

Proof. We begin by proving that π is a well-defined map, which amounts to showing that the set $\bigcap_{m=1}^\infty K_{\omega_1\omega_2...\omega_m}$ contains only one point. Since K is a self-similar set, we get that for all $1 \leq i \leq n$, $f_i(K) \subseteq K$. This implies that we have the following inclusions of image spaces

$$K_{\omega_1\omega_2...\omega_m\omega_{m+1}} = f_{\omega_1\omega_2...\omega_m}(f_{\omega_{m+1}}(K)) \subseteq f_{\omega_1\omega_2...\omega_m}(K) = K_{\omega_1\omega_2...\omega_m}$$

This demonstrates that the sequence of sets $\{K_{\omega_1\omega_2...\omega_m}\}_{m=1}^\infty$ is decreasing, and since each f_i is continuous, and K is compact (as an element of $C(X)$), we get that $\{K_{\omega_1\omega_2...\omega_m}\}_{m=1}^\infty$ are compact, and thus $\bigcap_{m=1}^\infty K_{\omega_1\omega_2...\omega_m}$ is non-empty and compact.

Now, to prove that $\pi(\omega)$ is well-defined, let $r := \max_{1\leq i\leq n} r_i$ and $w = \omega_1\omega_2\omega_3...\omega_m \in W_m$. Then,

$$\operatorname{diam}(K_w) = \sup_{x,y \in K} d(f(x), f(y)) \leq r^m \sup_{x,y \in K} d(x,y) = r^m \operatorname{diam}(K)$$

So that as $m \to \infty$, we get that $\operatorname{diam}(\bigcap_{m=1}^{\infty} K_{\omega_1\omega_2\ldots\omega_m}) = 0$, and so $K_{\omega_1\omega_2\ldots\omega_m}$ contains only one point, and thus $\pi(\omega)$ is well-defined.

To prove continuity, let $\omega, \tau \in \Sigma$ and $R > 0$ with $\omega = \omega_1\omega_2\ldots\omega_m\ldots$ and $\tau = \tau_1\tau_2\ldots\tau_m\ldots$ such that $\delta_R(\omega,\tau) \leq R^m$. In other words, ω and τ agree up to the first m symbols, and hence $K_{\omega_1\omega_2\ldots\omega_m} = K_{\tau_1\tau_2\ldots\tau_m}$. This implies $\pi(\omega), \pi(\tau)$ exist in the same set $K_{\omega_1\omega_2\ldots\omega_m}$, and $d(\pi(\omega),\pi(\tau)) \leq r^m\operatorname{diam}(K)$, which tends to 0 as $m \to \infty$ and so can be made arbitrarily small, proving continuity.

Now, for $\omega \in \sigma$ and $1 \leq i \leq n$ we have that

$$\pi(\sigma_i(\omega)) = \bigcap_{m=1}^{\infty} K_{i\omega_1\omega_2\ldots\omega_m} = \bigcap_{m=1}^{\infty} f_i(K_{\omega_1\omega_2\ldots\omega_m}) = f_i(\pi(\omega))$$

demonstrating that $\pi \circ \sigma_i = f_i \circ \pi$. And lastly, by Theorem 2.2.1, we have that $\Sigma = \bigcup_{k=1}^{n} \sigma_k(\Sigma)$, so that

$$\pi(\Sigma) = \pi(\bigcup_{k=1}^{n} \sigma_k(\Sigma)) = \bigcup_{k=1}^{n} \pi(\sigma_k(\Sigma)) = \bigcup_{k=1}^{n} f_k(\pi(\Sigma))$$

This demonstrates that $\pi(\Sigma)$ is a self-similar set with respect to $\{f_k\}_{k=1}^{n}$, as it is a compact non-empty subset, and by uniqueness of self-similar sets, we must have that π is surjective, with $\pi(\Sigma) = K$. □

2.3 Measure and Integration

Now that we have constructed a self-similar set K, and characterised it as an iterated function system with respect to contractions $\{f_k\}_{k=1}^{n}$, to define differential operators on such spaces, we need a notion of measure and integration. This is usually done by constructing a regular probability measure over K, often referred to as a **self-similar measure**.

We follow Kigami's construction [8], although various parts of this construction can also be found in [12]. To construct such a measure μ, for a cell $C \subseteq K$, we require four main conditions:

- **[Positivity]** For $C \neq \emptyset$, $\mu(C) > 0$,
- $\mu(K) = 1$,
- **[Finite Additivity]** Given $\{C_k\}_{k=1}^n \subseteq K$ almost disjoint cells (disjoint cells with the exception of possibly intersecting at their boundaries - which could be a finite set of points), we have

$$\mu\left(\bigcup_{k=1}^n C_k\right) = \sum_{k=1}^n \mu(C_k)$$

- **[Continuity]** For $\{C_k\}_{k=1}^\infty \subseteq K$ decreasing sets, then

$$\mu\left(\bigcap_{k=1}^\infty C_k\right) = \lim_{k\to\infty} \mu(C_k)$$

To construct such a measure μ, we assign weight 1 to all of K, and then inductively assign weights to the following cells of level m. To this end, for K self-similar with respect to contractions $\{f_k\}_{k=1}^n$, we choose a set of weights $\{\mu_k\}_{k=1}^n$ such that

$$\sum_{k=1}^n \mu_k = 1$$

Then, for $w = w_1 w_2 ... w_m \in W_m$, and a cell $f_w(K) \subseteq K$, we define

$$\mu(f_w(K)) := \prod_{k=1}^m \mu_{w_k}$$

It is easily seen that the collection of finite unions of cells (of a self-similar set) form an algebra, and μ inductively constructed satisfying the conditions above forms a pre-measure. So that by the Caratheodory Extension Theorem, we can extend μ uniquely to form a measure over all of K. It is important to note, however, that while such a measure μ is unique, we can construct various measures depending on how we choose to inductively assign our weights on the algebra of cells of level m. The special case were for all $1 \leq i, j \leq n$ with $i \neq j$, $\mu_i = \mu_j$ is referred to as the **standard measure over** K.

Now that we have constructed a measure over K, for cells $f_w(K)$ of level m, and $x_w \in f_w(K)$, we define the integral of a continuous function over K, like so:

$$\int_K F d\mu := \lim_{m\to\infty} \sum_{|w|=m} F(x_w)\mu(f_w(K))$$

Note that this integral is well-defined and finite by virtue of F being continuous and K being compact. This guarantees that F achieves a (finite) maximum and minimum over K, and is further uniformly continuous. This makes the value of our integral independent of the choice of points x_w.

Chapter 3

Construction of the Laplacian over Self-Similar Sets

Now that we have established the necessary preliminaries and context in which we are operating and explored the general construction and characterisation of our domains, in this chapter we introduce the required machinery to construct differential operators, and specifically, the Laplacian, in a way that is analogous to its interpretation over smooth domains. This is generally done by considering graph approximations of our self-similar sets, and defining the appropriate bilinear energy to create a weak formulation of our Laplacian. This weak or variational formulation is then used to develop a pointwise formulation, which we show possesses all the familiar properties of the Laplacian over smooth surfaces. Furthermore, we show that the domain of such an operator is non-trivial. As in the previous chapter, the following work is also adapted from [8].

3.1 Graph Approximations of Self-Similar Sets

Now that we have constructed self-similar sets, an alternative formulation in terms of graphs proves to be helpful in defining differential operators via difference quotients. We create this graph formulation inductively. Let K be a self-similar set with respect to the contractions $\{f_i\}_{i=1}^n$. For each $m \in \mathbb{N}$, we construct graphs Γ_m with associated vertex and edge sets (V_m, E_m); we call each Γ_m the **graph approximation of level m**.

We begin by defining the vertex sets, V_m. Let V_0 be a (finite) set of boundary points of K (note that since K is compact, we can always define such a finite set by choosing a point from every finite-subcover of an arbitrary open cover of K, but it will be different from the topological boundary ∂K).

To define V_m, let $w = w_1 w_2 ... w_m \in W_m$ be a word of length m, and as in the previous chapter, we write

$$f_w = f_{w_1} \circ f_{w_2} \circ ... \circ f_{w_m}$$

And define V_m inductively, like so

$$V_m = \bigcup_{i=1}^m f_i(V_{m-1}),$$

where the index $i \in \{1, 2, ..., n\}$. We can obtain a more explicit formulation of V_m in terms of the boundary points of K using the above recurrence relation and our formulation of K via shift spaces. We can write V_{m-1} using our recurrence

$$V_{m-1} = \bigcup_{j=1}^{m-1} f_j(V_{m-2})$$

And substituting this into our defining recurrence relation, we get

$$V_m = \bigcup_{i=1}^m f_i(\bigcup_{j=1}^{m-1} f_j(V_{m-2})) = \bigcup_{i=1}^m \bigcup_{j=1}^{m-1} f_i(f_j(V_{m-2}))$$

We can then reformulate this in terms of words $w \in W_m$, as the indices iterate through each permutation of $1 \leq i, j \leq m$, and continuing recursively as above, we obtain

$$V_m = \bigcup_{w \in W_m} \bigcup_{p \in V_0} f_w(p)$$

where $p \in V_0$ are our (finitely many) boundary points.

3.2 Dirichlet Forms and Laplacians over Finite Graphs

To formulate a weak Laplacian we emulate the general procedure in analysis over smooth surfaces, and we reframe the problem variationally. In other words, we minimise an appropriate energy functional over our graph approximations Γ_m and take a continuum limit to obtain a functional over our self-similar set K that minimises energy.

For a general (finite) graph $G = (V, E)$, we begin by first introducing the linear space $\ell(V)$ as the space of all functions of from the vertex set V into $\mathbb{R}$, where addition and scalar multiplication are defined pointwise. Furthermore, for $u, v \in \ell(V)$, we define the inner product over $\ell(V)$ by

$$\langle u, v \rangle_V = \sum_{p \in V} u(p)v(p)$$

When no ambiguity can occur, we will often drop the subscript V in the notation of the inner product. Given this context, we define our graph energy as follows:

Definition 3.2.1. *Let V be a finite set. A symmetric bilinear form*
$\varepsilon : \ell(V) \times \ell(V) \to \mathbb{R}$, is called a Dirichlet Form on V if it satisfies the following conditions:

(i) For all $u \in \ell(V), \varepsilon(u, u) \geq 0$

(ii) $\varepsilon(u, u) = 0$ if and only if u is constant on V.

*(iii) **(Markov Property)** For any $u \in \ell(V)$, we define $\bar{u} : V \to \mathbb{R}$ by*

$$\bar{u}(p) = \begin{cases} 1 & \text{if } u(p) \geq 1, \\ u(p) & \text{if } 0 < u(p) < 1 \\ 0 & \text{if } u(p) \leq 0 \end{cases}$$

Then, $\varepsilon(\bar{u}, \bar{u}) \leq \varepsilon(u, u)$.

We denote the space of all Dirichlet Forms over V by $\mathcal{DF}(V)$, and the space of all symmetric bilinear forms satisfying (i) and (ii) by $\widetilde{\mathcal{DF}}(V)$.

Definition 3.2.2 (Characteristic Functions and Linear Operators)**.** *Let V be a finite set, and $U \subseteq V$. Then the characteristic function of U is the function*

$$\chi_U^V(p) = \begin{cases} 1 & \text{if } q \in U \\ 0 & \text{otherwise.} \end{cases}$$

In cases where there is lack of ambiguity, for ease of notation, we write χ_U for the characteristic function of U. And in the case where $U = \{p\}$ is a singleton, we write χ_p instead of $\chi_{\{p\}}$.

Furthermore, for a linear operator $H : \ell(V) \to \ell(V)$, for $p, q \in V$, we define

$$H_{pq} = (H\chi_q)(p),$$

and for $u \in \ell(V)$, we have

$$(Hu)(p) = \sum_{q \in V} H_{pq} u(q).$$

In other words, if we know how H acts on χ_q for $q \in V$, we can determine the action of H on all of V by summing over all $q \in V$. Alternatively, to further motivate this definition, it's clear that the set $\{\chi_q\}_{q \in V}$ is a basis for $\ell(V)$, and so for an arbitrary linear operator H, we can study its action by restricting our attention to the basis elements of our space.

Definition 3.2.3 (Laplacian)**.** *Let $H : \ell(V) \to \ell(V)$ be a symmetric linear operator. Then, H is called a Laplacian over V if it satisfies the following conditions:*

(i) H is non-positive definite; i.e.: we have that $\langle u, Hu \rangle_V \leq 0$,

(ii) $H(u) = 0$ over V if and only if u is constant on V,

(iii) $H_{pq} \geq 0$ for all $p, q \in V$ with $p \neq q$.

We denote the collection of Laplacians over $\ell(V)$ by $\mathcal{LA}(V)$, and the space of all symmetric linear operators satisfying the first two conditions in the above definition by $\widetilde{\mathcal{LA}}(V)$.

From definitions 3.2.1 and 3.2.3, we can clearly see that $\mathcal{DF}(V) \subseteq \widetilde{\mathcal{DF}}(V)$, and $\mathcal{LA}(V) \subseteq \widetilde{\mathcal{LA}}(V)$. the following theorem demonstrates that we can establish a one-to-one correspondence between $\mathcal{DF}(V)$ and $\mathcal{LA}(V)$. In other words, for each Laplacian, there exists an associated Dirichlet form, and vice versa. This allows us to construct differential operators, and specifically a Laplacian, analogous to the one over smooth spaces, variationally.

Theorem 3.2.1. *Let $H : \ell(V) \to \ell(V)$ be a symmetric linear operator, and define the quadratic form*

$$\begin{gathered} \varepsilon_H(.,.) : \ell(V) \times \ell(V) \to \mathbb{R} \\ (u, v) \mapsto -\langle u, Hv \rangle. \end{gathered}$$

Now, we define the map

$$\begin{gathered} \pi : \widetilde{\mathcal{LA}}(V) \to \widetilde{\mathcal{DF}}(V) \\ \pi(H) = \varepsilon_H. \end{gathered}$$

Then, π is bijective, with $\pi(\mathcal{LA}(V)) = \mathcal{DF}(V)$.

Proof. To prove injectivity, let H_1 and H_2 be non-positive definite symmetric linear operators satisfying $\pi(H_1) = \pi(H_2)$. Then by definition of π, for $u \in \ell(V)$, we have that

$$\langle u, H_1 u \rangle = \langle u, H_2 u \rangle,$$

which can equivalently be written as

$$\iff \langle u, (H_2 - H_1)u \rangle = \langle u, 0 \rangle.$$

Since $u \in \ell(V)$ is arbitrary, we have that $H_1 - H_2 = 0$ if and only if $H_1 = H_2$.

For surjectivity, let $\varepsilon \in \widetilde{\mathcal{DF}}(V)$. Then, as a symmetric bilinear form, for $u \in \ell(V)$, we have that the map $v \mapsto \varepsilon(u, v)$ is a linear operator from $\ell(V)$ to itself. Then, since $\ell(V)$ is an inner product space of finite dimension, it's isomorphic to $\mathbb{R}^n$, and thus complete.

So by the Riesz-Representation theorem for Hilbert spaces, there exists a linear operator H such that

$$\varepsilon(u,v) = -\langle u, Hv\rangle$$

Then, since $\varepsilon(u,u) \geq 0$, we have that $-\langle u, Hu\rangle$ is non-negative definite, so H is non-positive definite. Furthermore, $\varepsilon(u,u) = 0$ implies u is constant on V, and thus Hu is constant over V. This demonstrates that $H \in \widetilde{\mathcal{LA}}(V)$, and thus π is a bijective map from $\widetilde{\mathcal{LA}}(V)$ to $\widetilde{\mathcal{DF}}(V)$.

It remains to show that $\pi(\mathcal{LA}(V)) = \mathcal{DF}(V)$. We first note that for $p, q \in V$,

$$H_{pq}(u(p) - u(q))^2 = H_{pq}u(p)^2 - 2H_{pq}u(p)u(q) + H_{pq}u(q)^2$$

And summing across all $p, q \in V$, we get the identity

$$\sum_{p,q\in V, p\neq q} H_{pq}(u(p) - u(q))^2 = -2\sum_{p,q\in V} H_{pq}u(p)u(q)$$

Notice that the right hand side is just twice $\varepsilon_H(u,u) = -\langle u, Hu\rangle$, by definition of Hu and the inner product on $\ell(V)$, so

$$\varepsilon_H(u,u) = \tfrac{1}{2}\sum_{p,q\in V} H_{pq}(u(p) - u(q))^2$$

It remains to show that ε_H possesses the Markov property. Recall the function

$$\tilde{u}(p) = \begin{cases} 1 & \text{if } u(p) \geq 1 \\ u(p) & \text{if } 0 < u(p) < 1 \\ 0 & \text{if } u(p) \leq 0. \end{cases}$$

We need to consider various cases. For $p, q \in V$, if $0 < u(p) < 1$ and $0 < u(q) < 1$, then $\tilde{u}(p) = u(p)$ and $\tilde{u}(q) = u(q)$, so that $(\tilde{u}(p) - \tilde{u}(q))^2 = (u(p) - u(q))^2$. If both $u(p), u(q) \geq 1$ or $u(p), u(q) \leq 0$, then $\tilde{u}(p) = \tilde{u}(q) = 1$ or $u(p) = u(q) = 0$, so that $(\tilde{u}(p) - \tilde{u}(q))^2 = 0 \leq (u(p) - u(q))^2$. Given $u(p) \geq 1$ or $u(p) \leq 0$ and $0 < u(q) < 1$, we have that for $c = 0, 1$, $(\tilde{u}(p) - \tilde{u}(q))^2 = (c - u(q))^2 = (u(q) - c)^2 \leq (u(q) - u(p))^2$. Lastly, if $u(p) \geq 1$ and $u(q) \leq 0$, we have that $(\tilde{u}(p) - \tilde{u}(q))^2 = (1 - 0))^2 \leq (1 - u(q))^2 = (u(q) - 1)^2\, leq (u(q) - u(p))^2$.

In all cases, since $-H$ is a Laplacian, we have for all $p, q \in V$ that $H_{pq} \geq 0$ and thus ε_H satisfies the Markov Property, and is ta Dirichlet form. This yields the containment $\pi(\mathcal{LA}(V)) \subseteq \mathcal{DF}(V)$.

We do the converse inclusion via the contrapositive. Let H be a symmetric bilinear form with $H_{pq} < 0$ for some $p \neq q$, and without loss of generality, we can assume that $H_{pq} = -1$. To simplify our notation, let $x = u(p)$, $y = u(q)$, and $z = u(a)$ for $a \neq p \neq q$. Then, we can express $\varepsilon_H(u,u)$ as

$$\varepsilon_H(u,u) = \alpha(x-z)^2 + \beta(y-z)^2 - (x-y)^2$$

for some positive constants α, β, by the non-negative definiteness of ε_H. Then for any function where $x = 1, y < 0$ and $z = 0$, this collapses to

$$\begin{aligned} \varepsilon_H(u,u) &= \alpha + \beta y^2 - (1-y)^2 \\ &= \alpha + \beta y^2 - 1 + 2y - y^2 = \alpha + (\beta - 1)y^2 - 1 + 2y \end{aligned}$$

Likewise, since $y = u(q) < 0$, we have that $\tilde{u}(q) = 0$, so that

$$\varepsilon_H(\tilde{u}, \tilde{u}) = \alpha - 1$$

Then if $\dfrac{-2}{\beta - 1} < y < 0$, we have that $\varepsilon_H(u,u) \leq \varepsilon_H(\tilde{u}, \tilde{u})$, and thus ε_H is not a Dirichlet form, and so $\pi(H) \notin \mathcal{DF}(V)$, as desired. □

3.3 Resistance Networks, Effective Resistance, and Sequences of Discrete Laplacians

Definition 3.3.1. *(Resistance Network and Effective Resistance) Let V be a finite set, and $H \in \mathcal{LA}(V)$. Then, the pair (V, H) is called a resistance network. Furthermore, for $H \in \widetilde{\mathcal{LA}(V)}$, we define the effective resistance as*

$$R_H(p,q) := \max_{\substack{u \in \ell(V) \\ \varepsilon_H(u,u) \neq 0}} \left\{ \frac{|u(p) - u(q)|^2}{\varepsilon_H(u,u)} \right\}$$

Note that the effective resistance can be equivalently defined as, as is done in [8]

$$R_H(p,q) = \left(\min_{\substack{u \in \ell(V) \\ u(p)=1, u(q)=0}} \varepsilon_H(u,u) \right)^{-1}$$

The proof of this equivalence is fairly involved, and not fairly enlightening for our purposes, and thus we omit it here. The full details proving this equivalence can be found in Chapter 2.1 of [8].

The importance of the effective resistance is two-fold: it defines a metric over V, and it gives us an upper bound on the value of $|u(p) - u(q)|^2$ for $u \in \ell(V)$. For the latter (and rather simpler) point, we notice that for $u \in \ell(V)$ with $\varepsilon_H(u,u) \neq 0$,

$$\frac{|u(p) - u(q)|^2}{\varepsilon_H(u,u)} \leq R_H(p,q),$$

implies that

$$|u(p) - u(q)|^2 \leq R_H(p,q)\varepsilon_H(u,u). \tag{3.1}$$

In regards to the former point, we notice two things, the first which we prove in this section, and the second which is instrumental in constructing solutions to the eigenvalue problem. Firstly, $\sqrt{R_H}$ defines a metric over V for any symmetric linear operator in $\widetilde{\mathcal{LA}(V)}$; secondly, returning to our setting of self-similar sets with a sequence of finite graph approximations, $\sqrt{R_H}$ turns the union of these approximations into a Hilbert space. In light of this, after appropriately defining the fractal Laplacian, we can apply the spectral theorem to get a set of functions solving the eigenvalue problem.

Proposition 3.3.1. *Let $H \in \widetilde{\mathcal{LA}(V)}$, and R_H be its associated effective resistance over V. Then, $\sqrt{R_H}$ defines a metric over V.*

Proof. It is clear from the definition of R_H that it is symmetric and that its takes value 0 when $p = q$ for $p, q \in V$. It remains to show the triangle inequality, and that given $R_H(p,q) = 0$, we have $p = q$.

For the latter claim, let $R_H(p,q) = 0$, and let $u \in \ell(V)$ achieve the maximum in the definition of R_H. Then,

$$0 \le |\chi_p - \chi_q| \le R_H(p,q)\varepsilon_H(u,u) = 0.$$

Equivalently, $\chi_p = \chi_q$ and so $p = q$. Now, for the triangle inequality, given $p, q, r \in V$, we have that

$$\frac{|u(p) - u(q)|}{\sqrt{\varepsilon_H(u,u)}} \le \frac{|u(p) - u(r)|}{\sqrt{\varepsilon_H(u,u)}} + \frac{|u(r) - u(q)|}{\sqrt{\varepsilon_H(u,u)}}$$

Taking the maximum across all $u \in \ell(V)$ with $\varepsilon_H(u,u) \neq 0$, we get that

$$\sqrt{R_H(p,q)} \le \sqrt{R_H(p,r)} + \sqrt{R_H(r,q)}$$

This demonstrates that $\sqrt{R_H}$ is a metric over V. □

We are now prepared to look at sequences of graph approximations, and sequences of resistance networks.

Definition 3.3.2. *(Compatible sequence of resistance networks)*

Let V_m be a finite set for all $m \in \mathbb{N}$, and $H_m \in \widetilde{\mathcal{LA}}(V_m)$. Then, $\mathcal{S} := \{(V_m, H_m)\}_{m=1}^{\infty}$ is called a compatible sequence (of resistance networks) if $V_m \subseteq V_{m+1}$, and $H_{m+1} = H_m$ on V_m. This is often denoted symbollically as $(V_m, H_m) \le (V_{m+1}, H_{m+1})$.

Furthermore, we define the set V_* as the union of all the V_m; i.e.:

$$V_* = \bigcup_{m=1}^{\infty} V_m$$

Before we proceed, we elaborate on some notation for the sake of clarity. As in the definition of $\ell(V)$ for a set V with finite cardinality, we define $\ell(V_*)$ to be all real-valued maps on V_*. We further define the set

$$\mathcal{F}(S) := \left\{ u \in \ell(V_*) \middle| \lim_{m\to\infty} \varepsilon_{H_m}(u|_{V_m}, u|_{V_m}) < \infty \right\}$$

Throughout the rest of this work, for a Laplacian (or symmetric linear operator) H_m over V_m, we will often refer to $\varepsilon_H(u,u)$ as the *energy* of u over V_m with respect to H_m, and we refer to a function $u \in \mathcal{F}(S)$ as a function with *finite (Dirichlet) energy over V_*.*

Definition 3.3.3. *(Bilinear Energy and Effective Resistance over V_*)*

Let $S = (V_m, H_m)_{m=1}^{\infty}$ be a compatible sequence of resistance networks. Then, for $u, v \in \mathcal{F}(S)$, we define the bilinear form

$$\varepsilon_S : \ell(V_*) \times \ell(V_*) \to \mathbb{R}$$
$$\varepsilon_S(u, v) = \lim_{m\to\infty} \varepsilon_{H_m}(u|_{V_m}, v|_{V_m})$$

which has associated Dirichlet form

$$\varepsilon_S(u, u) := \lim_{m\to\infty} \varepsilon_{H_m}(u|_{V_m}, u|_{V_m})$$

Furthermore, for $p, q \in V_*$, we define the effective resistance associated with S by

$$R_S(p, q) := R_{H_m}(p, q)$$

for sufficiently large m such that $p, q \in H_m$.

Proposition 3.3.1 demonstrates the following characterisation of the effective resistance $R_S(.,.)$, which we often call the **resistance form.**

Proposition 3.3.2. *Let $p, q \in V_*$. Then,*

$$R_S(p, q) = \max_{\substack{u\in\mathcal{F}(S)\\ \varepsilon_S(u,u)\neq 0}} \left\{ \frac{|u(p) - u(q)|^2}{\varepsilon_S(u, u)} \right\}$$

*Furthermore, $\sqrt{R_S(p, q)}$ is a metric over V_* (often called the **resistance metric**), and for any $p, q \in V_*$ we have the following estimate*

$$|u(p) - u(q)|^2 \leq R_S(p, q)\varepsilon_S(u, u)$$

Remark 3.3.1. *The above estimate demonstrates that $\mathcal{F}(S) \subseteq C(V_*, R_S^{1/2})$, the space of continuous functions over V_* equipped with the metric $\sqrt{R_S}$. Furthermore, if V_* is the collection of all finite graph approximations of a self-similar set K, we have that $u \in \mathcal{F}(S)$ is bounded and uniformly continuous, since K is compact (by construction).*

Proof (of Proposition 3.3.2). Let $m \in \mathbb{N}$ be sufficiently large such that $p, q \in V_m$. Then, by definition,

$$R_S(p,q) = R_{H_m}(p,q) = \max_{\substack{u \in \ell(V_m) \\ \varepsilon_{H_m}(u,u) \neq 0}} \left\{ \frac{|u(p) - u(q)|^2}{\varepsilon_{H_m(u,u)}} \right\}$$

Then, since S is a compatible sequence, we can take $m \to \infty$ to get the desired result. Furthermore, by Proposition 3.3.1, it follows that $\sqrt{R_S}$ is a metric over V_*, and that we have the above estimate on $|u(p) - u(q)|^2$. □

Now, given a compatible sequence $S = \{(V_m, H_m)\}_{m=1}^{\infty}$, we want to turn $\mathcal{F}(S)$ into a Hilbert space. It turns out that this amounts to defining the appropriate equivalence relation on $\mathcal{F}(S)$; since $\varepsilon_S(u,v) = 0$ if and only if $u - v$ is constant over V_*, this provides us with a viable candidate for such a relation. Turning $\mathcal{F}(S)$ into a Hilbert space is then incredibly useful, since if we can construct an appropriate symmetric self-adjoint operator (that has the same properties as the classical Laplacian), then we can apply the spectral theorem to get solutions to the eigenvalue problem, which is ultimately our goal.

However, before we can prove this we need the following lemma, which relates $\ell(V_m)$ to $\mathcal{F}(S)$.

Lemma 3.3.1. *Let $u \in \ell(V_m)$. Then there exists a linear map*

$$h_m : \ell(V_m) \to \mathcal{F}(S),$$

such that $h_m(u)|_{V_m} = u$, and

$$\begin{aligned} \varepsilon_{H_m}(u,u) &= \varepsilon_S(h_m(u), h_m(u)) \\ &= \min_{v \in \mathcal{F}(S), v|_{V_m} = u} \varepsilon_S(v,v). \end{aligned}$$

Remark 3.3.2. *We further note that this lemma implies that given a nested sequence of sets $V_m \subseteq V_{m+1}$, and a function $u \in \ell(V_m)$, we can construct a Dirichlet form ε_{m+1} on V_{m+1} that agrees with ε_m on V_m and minimises the value of ε_{m+1} on V_{m+1}. In other words, we can construct a sequence of Dirichlet forms over $\{V_m\}_{m=1}^{\infty}$ that minimises the values of each ε_m on V_m. This is sometimes referred to as a harmonic extension of u.*

Proof of Lemma 3.3.1. See Chapter 2.2 (pg. 51-52) of [8]. □

Theorem 3.3.2. *Let $u, v \in \mathcal{F}(S)$, and define the equivalence relation $u \sim v$ if and only if $u - v$ is constant on V_*. Then ε_S is a positive definite symmetric form on $\mathcal{F}(S)/\sim$ that defines an inner product, and induces a Hilbert space structure.*

Furthermore, if for all $m \in \mathbb{N}, H_m \in \mathcal{LA}(V)$, then ε_S satisfies the Markov property, and is thus a Dirichlet form over $\mathcal{F}(S)$.

Proof. We first note that the latter statement follows from Theorem 3.2.1 and the fact that $S = \{(V_m, H_m)\}_{m=1}^{\infty}$ is a compatible sequence. It remains to prove that $\sim$ is indeed an equivalence relation on $\mathcal{F}(S)$, and that ε_S is an inner produce that turns $\mathcal{F}(S)/\sim$ into a Hilbert space.

It's clear that $\sim$ is reflexive and symmetric; it remains to prove transitivity. Let $u, v, w \in \mathcal{F}(S)$ with $u \sim v$ and $v \sim w$. Then, there exist constants $K, L \in \mathbb{R}$ such that $u - v = K$ and $v - w = L$. Then, $u - w = u - v + v - w = K + L$, which is constant. Hence, $u \sim w$, and $\sim$ is transitive.

Now, from the definition of ε_S, it is clear that it is a bilinear form and that ε_S is a symmetric quadratic form, and thus always non-negative. Furthermore, if $u \sim v$, then for all $m \in \mathbb{N}$ $H_m(u - v) = 0$, and thus $H_m(u) = H_m(v)$. So, by definition of ε_{H_m}

$$\varepsilon_{H_m}(u, v) = -\langle u, H_m v\rangle = -\langle u, H_m u\rangle = \varepsilon_{H_m}(u, u)$$

and likewise, by symmetry of H_m

$$\varepsilon_{H_m}(u, v) = \varepsilon_{H_m}(v, v)$$

So that $\varepsilon_{H_m}(u, u) = \varepsilon_{H_m}(v, v)$. Taking $m \to \infty$ we get that $\varepsilon_S(u, u) = \varepsilon_S(v, v)$ given $u \sim v$. Therefore, ε_S is a well-defined positive definite, symmetric bilinear form on $\mathcal{F}(S)/\sim$. This makes it an inner product over $\mathcal{S}/\sim$. It remains to prove that this space is complete.

To do this, for arbitrary $p \in V_*$, define the space $\mathcal{F}_p := \{u \in \mathcal{F}(S) | u(p) = 0\}$. Then, $\mathcal{F}/\sim$ is isomorphic to $\mathcal{F}_p$ since $\mathcal{F}(S)$ is the set of equivalence classes of functions that are constant over V_*, and without loss of generality, we can restrict our attention to the equivalence class of functions whose difference is 0. So, if we can show that $\mathcal{F}_p$ equipped with ε_S is a Hilbert space, then $\mathcal{F}(S)/\sim$ is a Hilbert space.

Let $\{v_n\}_{n=1}^{\infty} \subseteq \mathcal{F}_p$ be a Cauchy sequence, and define $v_n^m := h_m(v_n|_{V_m})$, where h_m is the linear operator as defined in Lemma 3.3.1 (where $p \in V_m$ for sufficiently large m). Then, by its characterisation as the minimser of ε_S, we have that

$$\varepsilon_S(v_k^m - v_l^m, v_k^m - v_l^m) \leq \varepsilon_S(v_k - v_l, v_k - v_l)$$

Thus, ε_S is an inner product on $\mathcal{F}_p \cap \ell(V_m)$, where we identify $\ell(V_m)$ with the restriction $h_m(\ell(V_m))$for $u \in \ell(V_*)$. Then, since $\mathcal{F}_p \cap \ell(V_m)$ is a finite dimensional Hilbert space, it is complete and thus there exists a function $v^m \in \mathcal{F}_p \cap \ell(V_m)$ such that $v_n^m \to v^m$ as $n \to \infty$. Furthermore, we have that $v^{m+1}|_{V_m} = v^m$, and thus there exists $v \in \ell(V_*)$ with $v|_{V_m} = v^m$.

Define $C = \sup_{n \in \mathbb{N}} \varepsilon_S(v_n, v_n) < \infty$, since for all $n \in \mathbb{N}, v_n \in \mathcal{F}(S)$, and ε_S is a bounded linear operator. Then,

$$\varepsilon_S(v^m, v^m) \leq \sup_{n,m \in \mathbb{N}} \varepsilon_S(v_n^m, v_n^m) = C < \infty.$$

So that $v \in \mathcal{F}(S)$. Now, we are ready to show that $v_n \to v$. Let $\epsilon > 0$, and since $\{v_n\}_{n=1}^{\infty}$ is Cauchy, let $n \in \mathbb{N}$ such that for all $k > n$

$$\varepsilon_S(v_n - v_k, v_n - v_k) < \tfrac{\epsilon}{3}$$

Since $\{v_n^m\}_{n=1}^{\infty}$ converges to v^m, and as $m \to \infty$ $v_n^m \to v_n$, and $v^m \to v$ there exists an $m \in \mathbb{N}$ such that

$$|\varepsilon_S(v_n - v, v_n - v) - \varepsilon_S(v_n^m - v^m, v_n^m - v^m)| < \tfrac{\epsilon}{3}$$

And given that h_m minimises ε_S, we have

$$\varepsilon_S(v_n^m - v_k^m, v_n^m - v_k^m) \leq \varepsilon_S(v_n - v_k, v_n - v_k) < \tfrac{\epsilon}{3}$$

So that,

$$
\begin{aligned}
|\varepsilon_S(v_n - v, v_n - v)| &= |\varepsilon_S(v_n - v, v_n - v) - \varepsilon_S(v_n^m - v^m, v_n^m - v^m) + \varepsilon_S(v_n^m - v^m, v_n^m - v^m)| \\
&\leq |\varepsilon_S(v_n - v, v_n - v) - \varepsilon_S(v_n^m - v^m, v_n^m - v^m)| + |\varepsilon_S(v_n^m - v^m, v_n^m - v^m)| \\
&< \tfrac{\epsilon}{3} + |\varepsilon_S(v_n^m - v^m, v_n^m - v^m)|
\end{aligned}
$$

Likewise, by our second estimate

$$
\begin{aligned}
|\varepsilon_S(v_n^m - v^m, v_n^m - v^m)| &= |\varepsilon_S(v_n^m - v^m, v_n^m - v^m) - \varepsilon_S(v_n^m - v_k^m, v_n^m - v_k^m) + \varepsilon_S(v_n^m - v_k^m, v_n^m - v_k^m)| \\
&\leq |\varepsilon_S(v_n^m - v^m, v_n^m - v^m) - \varepsilon_S(v_n^m - v_k^m, v_n^m - v_k^m)| + |\varepsilon_S(v_n^m - v_k^m, v_n^m - v_k^m)| \\
&< \tfrac{\epsilon}{3} + |\varepsilon_S(v_n^m - v_k^m, v_n^m - v_k^m)|
\end{aligned}
$$

And finally, using our last estimate, we have that

$$
\varepsilon_S(v_n^m - v_k^m, v_n^m - v_k^m) \leq \varepsilon_S(v_n - v_k, v_n - v_k) < \tfrac{\epsilon}{3}
$$

So in summary,

$$
|\varepsilon_S(v_n - v, v_n - v)| < \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon
$$

and so $\{v_n\}_{n=1}^{\infty}$ converges to $v \in \mathcal{F}_p$ as desired. □

3.4 Dirichlet Forms, Laplacians, and Measure

The content of the previous section lays the groundwork for the development of the Laplacian over self-similar sets. However, there are a few minor details to smooth out. Given a self-similar set K and a compatible sequence of graphs $\{V_m, H_m\}_{m=1}^{\infty}$ with $V_m \subseteq V_{m+1}$ and $V_* = \cup_{m=1}^{\infty} V_m$, it may not always be the case that $V_* = K$; furthermore, V_* is countable given V_m finite), and thus V_* equipped with $\sqrt{R_S}$ (its resistance metric) may not be complete. To circumvent both of these issues, we take the completion of $(V_*, \sqrt{R_S})$, which turns out to correspond exactly with K if S is a compatible sequence of Dirichlet forms (i.e.: they satisfy the Markov Property). For further discussion regarding these details, see Chapter 2.3 of [8].

Furthermore, it is important to note that in section 3.3, we built a resistance form and its associated resistance metric using a compatible sequence of (finite) graph Laplacians.

We can then consider the set of all such forms (respectively metrics) over V_* in greater generality. It can be shown that there exists a bijective mapping between resistance forms and resistance networks as done in [6] and [8]. Then, given this framework, in conjunction with a a notion of measure and integration, we can construct our Laplacian. In a sense, this can be viewed as a ”weak” formulation of our Laplacian. In the next section, we develop a more direct characterisation of our Laplacian that corresponds with the standard classical Laplacian.

Throughout this section, we assume that a resistance form $(\varepsilon_S, \mathcal{F}(\mathcal{S}))$, and associated metric $\sqrt{R_S}$ has been constructed from a compatible sequence S, using the methods of section 3.3. We will drop the subscript S when no confusion can arise.

Theorem 3.4.1. *Let X be a set, and $(\varepsilon, \mathcal{F})$ be a resistance form over X with associated resistance metric R. Suppose that $(X, R^{1/2})$ is a separable space equipped with a σ-finite Borel measure μ, and for $u, v \in L^2(X \cap \mathcal{F})$ define*

$$\varepsilon_1(u, v) := \varepsilon(u, v) + \int_X u(x)v(x)\mu(dx)$$

Then ε_1 is an inner product on $L^2(X \cap \mathcal{F})$ that induces a Hilbert space structure.

Moreover, if μ is a finite measure, and $\int_X R(p, p_*)\mu(dp) < \infty$ for some $p_* \in X$, then the identitiy map

$$I : L^2(X \cap \mathcal{F}, \varepsilon_1) \to L^2(X, \mu)$$

is a compact operator, where $L^2(X, \mu)$ is equipped with its standard L^2 norm.

Proof. We only prove the former part of the theorem. The compactness of the identity, while an important result, isn't incredibly significant for the purpose of this work. The details of the proof can be found in [8].

Let $u, v \in L^2(X \cap \mathcal{F})$. We first show that $\varepsilon_1(.,.)$ is an inner product on $L^2(X \cap \mathcal{F})$. Firstly, it is well defined by Hölder's inequality, and the fact that $u, v \in \mathcal{F}$, and it is clearly bilinear. Now, consider

$$\varepsilon_1(u,u) = \varepsilon(u,u) + \int_X u^2 d\mu$$

Then, this quantity is always non-negative since ε is a Dirichlet form on $\mathcal{F}$, and given equality

$$\varepsilon_1(u,u) = \varepsilon(u,u) + \int_X u^2 d\mu = 0,$$

we have

$$\varepsilon(u,u) = -\int_X u^2 d\mu.$$

The right hand side of the above equality is always non-positive, since $\int_X u^2 d\mu \geq 0$. But $\varepsilon(u,u) \geq 0$ as a Dirichlet form, and thus

$$\varepsilon(u,u) = -\int_X u^2 = 0,$$

which implies that

$$\int_X u^2 = 0.$$

Therefore, $u = 0$ on $X \cap \mathcal{F}$. If $u = 0$ then we clearly have that $\varepsilon_1(u,u) = 0$. This proves that ε_1 is an inner product, and it remains to show that $L^2(X \cap \mathcal{F})$ is complete.

Let $\{u_n\}_{n=1}^{\infty} \subseteq L^2(X \cap \mathcal{F})$ be a Cauchy sequence with respect to the norm induced by ε_1. Then, for $p \in X$ define the sequence $v_n := u_n - u_n(p)$, and it is readily observable that for all $n \in \mathbb{N}$ we have $v_n(p) = 0$, and so $v_n \in \mathcal{F}_p$. By Theorem 3.3.2, we have that $\mathcal{F}_p$ is complete (with respect to ε), and thus there exists $v \in \mathcal{F}_p$ such that

$$\varepsilon(v_n - v, v_n - v) \to 0$$

Recall the defining property of a resistance form: For $p, q \in X$

$$R(p,q) := \sup_{u \in \mathcal{F}, u \neq 0} \left\{ \frac{|u(p) - u(q)|^2}{\varepsilon(u,u)} \right\}$$

From which we can deduce that for $u = v_n - v$

$$|v_n(p) - v(p) - v_n(q) + v(q)|^2 \leq R(p,q)\varepsilon(v_n - v, v_n - v)$$

But since $v_n, v \in \mathcal{F}_p$, we have that for all $n \in \mathbb{N}, v_n(p) = v(p) = 0$, so that the above inequality becomes

$$|v_n(q) - v(q)|^2 \leq R(p,q)\varepsilon(v_n - v, v_v - v)$$

so that $v_n \to v$ pointwise. Now, since X is σ-finite, for each $m \in \mathbb{N}$ there exists a set $K_m \subseteq X$ such that K_m is bounded and has finite measure, with

$$\bigcup_{m=1}^{\infty} K_m = X.$$

Since K_m is bounded, we have that on each K_m

$$\int_{K_m} |v_n - v|^2 d\mu \leq \int_{K_m} R(p,q)\varepsilon(v_n - v, v_n - v)$$
$$= \mu(K_m)R(p,q)\varepsilon(v_n - v, v_n - v) \to 0,$$

as $n \to \infty$, so that $v_n \to v$ in $L^2(K_m, \mu)$.

A priori, we don't know if $\{u_n\}_{n=1}^{\infty}$ is bounded over X, and so we restrict our attention to K_m, being bounded subsets of X. However, because $\{u_n\}_{n=1}^{\infty}$ is Cauchy in $L^2(K_m, \mu)$ and given K_m bounded,

$$\int_{K_m} |u_n - u_m|^2 d\mu \leq \varepsilon(u_n - u_m, u_n - u_m) + \int_{K_m} |u_n - u_m|^2 d\mu$$
$$= \varepsilon_1(u_n - u_m, u_n - u_m) \to 0$$

since $\{u_n\}_{n=1}^{\infty}$ is Cauchy with respect to ε_1. Hence, $\{u_n\}_{n=1}^{\infty}$ is Cauchy with respect to the $L^2(K_m, \mu)$ norm.

Now, by definition of v_n, we have that $u_n(p) = (u_n - v_n)|_{K_m} \in \mathbb{R}$, and since u_n and v_n are Cauchy in $L^2(K_m, \mu)$, it follows that $\{u_n(p)\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$, and thus converges to some $c \in \mathbb{R}$ as $n \to \infty$. Then, since $v_n \to v$ with respect to ε, and we have $u_n = v_n + u_n(p)$, we get that $u_n \to u = v + c$ with respect to ε. In other words, we have that over K_m

$$\varepsilon(u_n - u, u_n - u) \to 0 \text{ as } n \to \infty.$$

This further implies that

$$\int_{K_m} |u_n - u|^2 d\mu \leq \varepsilon(u_n - u, u_n - u) + \int_{K_m} |u_n - u|^2 d\mu$$
$$= \varepsilon_1(u_n - u, u_n - u)$$

Now, recall that $v_n \to v$ in $L^2(K_m, \mu)$, and $u_n(p)$ and c are constants (with $u_n(p) \to c$ in $\mathbb{R}$), so that in $L^2(K_m, \mu)$

$$\|u_n - u\|_{L^2(K_m,\mu)} = \|v_n + u_n(p) - v - c\|_{L^2}$$
$$\leq \|v_n - v\|_{L^2(K_m,\mu)} + \left(\int_{K_m} |u_n(p) - c|^2\right)^{1/2}$$
$$= \|v_n - v\|_{L^2(K_m,\mu)} + \sqrt{\mu(K_m)}|u_n(p) - c|$$

Both the above quantities now go to 0, so we have that $u_n \to u$ in $L^2(K_m, \mu)$, and thus combined with the fact that $u_n \to u$ with respect to ε, we have that $u_n \to u$ with respect to ε_1 over K_m.

Finally, it remains to show that $u_n \to u$ in $L^2(X, \mu)$. By the completeness of $L^2(X, \mu)$, there exists a $\tilde{u}$ such that $u_n \to \tilde{u}$ in $L^2(X, \mu)$. However, for all $m \in \mathbb{N}$, by uniqueness of the limit, we must have the $u = \tilde{u}$ over K_m, and since $\cup_{m=1}^{\infty} K_m = X$, we get that $u = \tilde{u}$ over X. Hence $u_n \to u$ in $L^2(X, \mu)$, and since $u_n \to u$ with respect to ε, we must have that $u_n \to u$ with respect to ε_1 over all of X and thus $L^2(X, \mu) \cap \mathcal{F}$ is complete (and thus a Hilbert space) with respect to inner product ε_1. □

Now, we use the following theorem from functional analysis, as presented in Appendix B of [8] to construct our desired Laplacian.

Theorem 3.4.2. *Let Q be a non-negative quadratic form on a real and separable Hilbert Space $\mathcal{H}$ (with inner product $\langle ., . \rangle$) with dense domain Dom(Q). Then, the following are equivalent*

(1) *Dom(Q) = Dom($H^{1/2}$), and $Q = Q_H$ for some non-negative self-adjoint operator H on $\mathcal{H}$, where $Q_H := \langle H^{1/2} f, H^{1/2} g \rangle$ for all $f, g \in$ Dom($H^{1/2}$)*

(2) *Given $Q_* := Q(f, g) + \langle f, g \rangle$ for any $f, g \in$ Dom(Q). Then, $(\mathrm{Dom}(Q), Q_*)$ is a Hilbert space.*

where $H^{1/2}$ is the unique nonnegative self-adjoint operator such that $(H^{1/2})^2 = H$.

Theorems 3.4.1 and 3.4.2 now provide the perfect setting to construct our desired Laplacian. This is summarised in the following theorem:

Theorem 3.4.3. *Let $(X, R^{1/2})$ be a separable Hilbert space, equipped with a σ-finite Borel measure μ, and let $(\varepsilon, \mathcal{F})$ be a resistance form over X with associated resistance metric R. Then, if $L^2(X,\mu) \cap \mathcal{F}$ is dense in $L^2(X,\mu)$ with respect to the L^2-norm, there exists a non-negative self-adjoint operator H on $L^2(X,\mu)$ such that Dom$(H^{1/2}) = \mathcal{F}$, and for all $u, v \in \mathcal{F}$, we have $\varepsilon(u,v) = \langle H^{1/2}u, H^{1/2}u\rangle$. Moreover, if $\mu(X) < \infty$ and there exists $p_* \in X$ such that $\int_X R(p,p_*)\mu(dp) < \infty$, then H has compact resolvent.*

Proof. Let $\mathcal{H} = L^2(X,\mu)$, and $Q := \varepsilon(.,.)$, and $\text{Dom}(Q) = \mathcal{F}$ in Theorem 3.4.2. Then by Theorem 3.4.1, we have that ε_1 turns $L^2(X,\mu) \cap \mathcal{F}$ into a Hilbert space, and so by Theorem 3.4.2, we get the existence of the desired self-adjoint operator H with $Q = Q_H = \langle H^{1/2}u, H^{1/2}u\rangle$. □

Chapter 4

Pointwise Formulation of the Laplacian and the Gauss-Green Formula

The previous chapter provides the necessary framework to build the Laplacian over a self-similar set, and in fact we constructed a variational formulation in terms of graph energies and self-similar measures. Given a self similar set $K = \bigcup_{1 \leq i \leq n} f_i(K)$ with respect to contractions $\{f_i\}_{i=1}^{n}$, we take $\{\Gamma_m\}_{m=1}^{\infty}$ to be a sequence of graph approximations of K with vertex sets $\{V_m\}_{m=1}^{\infty}$. By convention, we take V_0 to be the graph approximation of the boundary of K. In this context, and in light of the variational formulation of Chapter 3 and finalised here, we begin by defining the concept of a harmonic structure and regular harmonic structure, its relation to compatible sequences of Laplacians H_m over V_m, and the resistance metric ε over $V_* = \bigcup_{m=1}^{\infty} V_m$. We also establish necessary and sufficient conditions for when $\bar{V_*} = K$, where $\bar{V_*}$ is the completion of V_* with the respect to the effective resistance metric (and K is endowed with its original metric), and use this to define the Dirichlet and Neumann Laplacian over K, denoted $-H_D$ and $-H_N$ respectively.

In the latter part of this Chapter, we introduce the pointwise formulation of the Laplacian, known as the Kigami Laplacian, and investigate some of its properties, culminating in an analogue of the Gauss-Green formula for the Kigami Laplacian over K, and we state a theorem that asserts that the pointwise formulation corresponds to our constructed variational formulations H_D and H_N with the appropriate boundary conditions. Most results reviewed in this chapter are obtained from [7].

4.1 Harmonic Structures and the Effective Resistance Topology

We begin by introducing the concept of a harmonic structure, as defined by Kigami in [7].

Definition 4.1.1. *Let $D \in \mathcal{LA}(V_0)$ and for all $1 \leq i \leq n$ let $r_i > 0$, and define the vector $\boldsymbol{r} = (r_1, r_2, ..., r_n)$. Then, for all $u, v \in \ell(V_m)$, we define $\varepsilon_m \in \mathcal{DF}(V_m)$ by*

$$\varepsilon^{(m)}(u,v) := \sum_{w \in W_m} \frac{1}{r_w} \varepsilon_D(u \circ f_w, v \circ f_w)$$

where $w = w_1 w_2 ... w_m$, and $r_w := r_{w_1} r_{w_2} ... r_{w_m}$. For ease of notation, for the remainder of this chapter, we write $\varepsilon_m := \varepsilon^{(m)}$.

Remark 4.1.1. *Note that by Theorem 3.2.1, we have that for all $m \in \mathbb{N}$, there exists a corresponding Laplacian $H_m \in \mathcal{LA}(V_m)$ for each $\varepsilon^{(m)}$, since each of the V_m are finite sets. An explicit characterisation can be formulated for H_m given the identity $\varepsilon_m = \varepsilon_{H_m} = \langle u, H_m v \rangle$ for $u, v \in \ell(V_m)$.*

More explicitly, it can be shown that for $w \in W_m$

$$H_m = \sum_{w \in W_m} \frac{1}{r_w} R_w^T D R_w$$

where $R_w : \ell(V_m) \to \ell(V_0))$ is a linear map defined by $R_w(u) := u \circ f_w$, and R_w^T is its transpose [8].

Now, we note that for all $m \in \mathbb{N}$, ε^{m+1} can be expressed inductively, like so

$$\varepsilon_{m+1}(u,v) = \sum_{i=1}^{n} \frac{1}{r_i} \varepsilon_m(u \circ f_i, v \circ f_i) \tag{4.1}$$

since for a fixed word in $w = w_1w_2...w_m \in W_m$, we have $\tilde{w} = ww_{m+1} \in W_{m+1}$ for $w_{m+1} \in \{1, 2, ..., n\}$. In light of this, we can view the sequence $\{(V_m, H_m)\}_{m=1}^{\infty}$ as a self-similar sequence, since the each term in the sequence is dependent on the preceding term, and each V_m is the vertex set of an m'th level approximation of a self-similar set.

Definition 4.1.2 (Harmonic Structure and Regular Harmonic Structure)**.** *Let $D \in \mathcal{LA}(V_0)$ and $\boldsymbol{r} \in \mathbb{R}^n$) be as in Definition 4.1.1. Then, the pair $(D, \boldsymbol{r})$ is called a* ***harmonic structure*** *if $\{(V_m, H_m)\}_{m=0}^{\infty}$ is a compatible sequence, where $H_0 := D$.*

If in addition, we have that $0 < r_i < 1$ for all $r_i \in \boldsymbol{r}$, then $(D, \boldsymbol{r})$ is called a ***regular harmonic structure****.*

So far, we have defined what a harmonic structure is, and Definition 4.1.1 can be thought of as a graph energy over the vertex sets V_m. However, both of these definitions, especially Definition 4.1.1, seem fairly unmotivated, and it is unclear how such a sequence can be constructed from a sequence of graph approximations of a self-similar set K. In practice, the operator D is often chosen differently according to the properties of the fractal we are considering. This is often done by analysing the edge sets E_m of the m'th level graph approximations, and a harmonic structure can then be constructed via its inductive formulation by aiming to fix the energy ε_m on V_m, and minimising the increase in ε_m on V_{m+1}. The process of extending the ε_m in this manner is often times referred to as a harmonic extension.

At this stage in the construction, all of the relevant structures are defined over finite sets, and so all of our operators are finite dimensional, and thus expressible as matrices in $\mathbb{R}^{m\times m}$. Thus, the process of constructing a harmonic extension can be reduced to a minimisation problem in $\mathbb{R}^n$, where we try to minimise the graph energy given our choice of r_i for all $1 \leq i \leq n$. For more details regarding this process, and more explicit examples, see [12]

The following proposition makes this process more rigorous, and provides a necessary and sufficient condition for the pair $(D, \mathbf{r})$ to be a harmonic structure.

Furthermore, if we define the operator

$$\mathcal{R}_{\mathrm{r}} : \mathcal{LA}(V_0) \to \mathcal{LA}(V_0)$$
$$\mathcal{R}_{\mathbf{r}}(D) = H_1|_{V_0}$$

called the **renormalisation operator**. By the following proposition, we have that D is a harmonic structure if and only if it is a fixed point of $\mathcal{R}_{\mathbf{r}}$. Furthermore, for $\lambda > 0$ and $\alpha > 0$, by definition of H_1 given in Definition 4.1.1

$$\mathrm{R}_{\lambda\mathrm{r}}(\alpha D) = \sum_{w \in W_1} \frac{1}{\lambda r_w}[R_w^T(\alpha D)R_w]|_{V_0}$$
$$= \frac{\alpha}{\lambda} H_1|_{V_0}$$
$$= \frac{\alpha}{\lambda} R_{\mathrm{r}(D)}$$

So if D is an eigenvector of $\mathcal{R}_{\mathrm{r}}$ (i.e.: $\mathcal{R}$r(D) = λD), the preceding calculation demonstrates that D is a fixed point of $\mathcal{R}_{\lambda\mathrm{r}}(D)$. This reduces the existence of a harmonic structure to (equivalently) a fixed point or eigenvalue problem over the renormalisation operator. This however, remains an open problem for self-similar sets in general; however, Lindstrøm's paper [11] provides a solution to the existence problem over nested fractals, self-similar fractals satisfying additional symmetry, connectivity, and nesting properties amongst the cells f_w for $w \in W_m$. Examples of such sets include the Sierpinski Triangle, Koch curve, amongst other self-similar fractals. By constructing Brownian motion over such fractals, and proving they are strong Markov processes with continuous sample paths, it can be shown that the renormalisation map has a fixed point, given the existence of a harmonic structure over K [11].

Proposition 4.1.1. *Let K be a self-similar set with defining contractions $\{f_i\}_{i=1}^n$, and let V_0 and V_1 be the (graph) boundary and vertex set of a 1-level approximation of K. Then, for $D \in \mathcal{LA}(V_0)$, and $\boldsymbol{r}$ = $(r_1, r_2, ..., r_n)$, $(D, \boldsymbol{r})$ is a harmonic structure if and only if $(V_0, D) \leq (V_1, H_1)$, where $H_1 \in \mathcal{LA}(V_1)$ is the linear operator associated with the Dirichlet form ε_1.*

Remark 4.1.2. *Essentially, Proposition 4.1.1 allows us to focus our construction of a harmonic structure by creating a harmonic extension from V_0 to V_1.*

Proof (of Proposition 4.1.1). If $(D, \mathbf{r})$ is a harmonic structure, we trivially have that $\{(V_m, H_m)\}_{m=0}^{\infty}$ is a compatible sequence of r-networks, and so $(V_0, D) \leq (V_1, H_1)$.

We prove the converse by induction. Assume that $(V_0, D) \leq (V_1, H_1)$, then this takes care of the base case, and it remains to show that given $(V_{m-1}, H_{m-1}) \leq (V_m, H_m)$, then we have that $(V_m, H_m) \leq (V_{m+1}, H_{m+1})$.

So let $\{f_i\}_{i=1}^{n}$ be the defining contractions of K, and take $u \in \ell(V_m)$. Using an analogue of the identity from Lemma 3.3.1, and by the induction hypothesis $(V_{m-1}, H_{m-1}) \leq (V_m, H_m)$

$$\varepsilon_{m-1}(u \circ f_i, u \circ f_i) = \min_{\substack{v \in \ell(V_{m+1}) \\ v|_{V_m} = u}} \varepsilon_m(v \circ f_i, v \circ f_i) \tag{4.2}$$

Now, by equations (4.1) and (4.2), we have

$$\begin{aligned} \varepsilon_m(u, u) &= \sum_{i=1}^{n} \frac{1}{r_i} \varepsilon_{m-1}(u \circ f_i, u \circ f_i) \\ &= \sum_{i=1}^{n} \frac{1}{r_i} \min_{\substack{v \in \ell(V_{m+1}) \\ v|_{V_m} = u}} \varepsilon_m(v \circ f_i, v \circ f_i) \\ &= \min_{\substack{v \in \ell(V_{m+1}) \\ v|_{V_m} = u}} \sum_{i=1}^{n} \frac{1}{r_i} \varepsilon_m(v \circ f_i, v \circ f_i) \\ &= \min_{\substack{v \in \ell(V_{m+1}) \\ v|_{V_m} = u}} \varepsilon_{m+1}(v, v) \end{aligned}$$

Therefore, $(V_m, H_m) \leq (V_{m+1}, H_{m+1})$, which proves the inductive hypothesis. □

Before we can continue to construct a pointwise formulation of the Laplacian over a self-similar set, we need to deal with a minor detail we have circumvented thus far. Putting together the results of Chapter 2 and the previous section, given a harmonic structure (D, r), and its induced compatible sequence $\{(V_m, H_m)\}_{m=0}^{\infty}$, we have that there exists $(\varepsilon, \mathcal{F})$ and R an effective resistance form and effective resistance metric respectively over K.

We mentioned at the beginning of section 3.4 that V_* is not necessarily equal to K, and so to circumvent this issue, we consider the completion of V_*, which we denote here by Ω (where the completion is taken with respect to the topology induced by R). It can be easily shown that $(\varepsilon, \mathcal{F})$ and R are an effective resistance form and resistance metric over Ω respectively. However, it may be that $\Omega \neq K$ in general, and so we need to check that the topology over Ω induced by the effective resistance metric is equivalent to the original topology over K. The following theorem, which we do not prove here, provides a necessary and sufficient condition for this to be the case.

Theorem 4.1.1. *Let K be a self-similar set, and (D, r) be a harmonic structure with induced compatible sequence $\{(V_m, H_m)\}_{m=0}^{\infty}$ approximating K. Let $V_* = \cup_{m=0}^{\infty} V_m$ and Ω the completion of V_* with respect to effective resistance metric R induced by $(V_m, H_m)\}_{m=0}^{\infty}$, and $\Omega = \overline{V_*^R}$, the completion of V_* with respect to R. Then, the following are equivalent.*

(1) $\Omega = K$

(2) (Ω, R) *is compact*

(3) (Ω, R) *is bounded*

(4) for any $u \in \mathcal{F}, \sup_{p \in \Omega} |u(p)| < \infty$

(5) (D, r) *is a regular harmonic structure*

Proof. See Kigami Chapter 3.3, pg. 85. in [8]. □

In light of Theorem 4.1.1., for the remainder of this chapter, for a self-similar set K with harmonic structure $(D, \mathbf{r})$, we will assume that $(D, \mathbf{r})$ is regular and its induced effective resistance $(\varepsilon, \mathcal{F})$ is a regular Dirichlet form. This allows us to bypass any idiosyncrasies of the metric topology induced by the effective resistance metric R, as mentioned earlier. For details on defining the Laplacian in the event that $(D, \mathbf{r})$ is not a regular harmonic structure, see Chapter 3.4 of [8].

Furthermore, to create a pointwise formulation of the Laplacian, and analyse the spectrum and associated eigenfunctions over K, we need to define the appropriate boundary conditions. The following theorem and its corollary summarise all of this information.

Theorem 4.1.2. *Let K be a self-similar set with respect to the contractions $\{f_i\}_{i=1}^n$, and take $(D, \mathbf{r})$ to be a regular harmonic structure over K. If μ is a self-similar measure over K, then the resistance form $(\varepsilon, \mathcal{F})$ is a local Dirichlet form on $L^2(K, \mu)$, and the corresponding self-adjoint operator H_N on $L^2(K, \mu)$ has compact resolvent.*

Proof. Since $(D, \mathbf{r})$ is a regular harmonic structure, we have that $\Omega = K$ by Theorem 4.1.1, where Ω is the completion of V_* with respect to the resistance metric R. By Theorem 3.3.2, we have that ε is a local Dirichlet form over $\mathcal{F}$, and by Theorem 3.4.1, we have that it is an inner product on $L^2(K, \mu)$ that induces a Hilbert space structure. Thus, invoking Theorem 3.4.3, we have that there exists a unique non-negative self-adjoint operator H_N on $L^2(K, \mu)$ with compact resolvent satisfying $\varepsilon(u, v) = \int_K fv d\mu$ for $f \in \mathcal{F}$ for any $v \in \mathcal{F}$, and so $f = H_N u$. □

This non-negative self-adjoint operator H_N is the operator associated with the Neumann Laplacian over K, hence the subscript N; the operator $-H_N$ is the Neumann Laplacian. The following corollary defines the Dirichlet Laplacian.

Corollary 4.1.2.1. *Deine the set $\mathcal{F}_0 := \{u \in \mathcal{F} \mid u|_{V_0} = 0\}$. Then under the conditions of Theorem 4.3.1, we have that $(\varepsilon, \mathcal{F})$ is a local regular Dirichlet form on $L^2(K, \mu)$. The corresponding non-negative self-adjoint operator H_D on $L^2(K, \mu)$ has compact resolvent.*

Proof. This follows by Theorem 4.2.1 over $K \setminus V_0$. □

4.2 The Pointwise Laplacian and Neumann Derivatives

In order to define a pointwise formulation of H_N (and H_D respectively), it remains to prove that there exist functions $u \in \mathcal{F}$ satisfying $-H_N u = 0$ (respectively, $-H_D = 0$), which we call harmonic functions. This is because the pointwise formulation of the Laplacian depends on the existence of such functions and is in terms of specific harmonic functions over K. For the purposes of this thesis, we assume the existence of such functions; a thorough exposition of the existence of harmonic functions over can be found in chapter 3.2 of [8], but we quote some definitions and results important to defining the pointwise formulation here. For the remainder of this thesis, we take K to be equipped with a regular harmonic structure $(D, \mathbf{r})$, which has an associated compatible sequence $\{(V_m, H_m)\}_{m=0}^{\infty}$ and associated regular Dirichlet form $(\varepsilon, \mathcal{F})$, where recall that $\mathcal{F}$ is the domain of ε. This section once again predominantly follows the results of [7], where most of these results were presented.

Definition 4.2.1. *Let $p \in V_0$, and let $x \in K \setminus V_0$. Then, we define the function ψ_p to be the harmonic function that $\psi_p|_{V_0} = \chi_p^{V_0}$.*

Furthermore, for arbitrary $w \in W_m$, if $u \circ f_w$ is harmonic for a function u over K, and $u|_{V_m} = \chi_p^{V_m}$. Then we call u the m-harmonic function with boundary value $\chi_p^{V_m}$, and we write $u = \psi_p^m$

Remark 4.2.1. *More generally, an m-harmonic function u is a function over K such that $u \circ f_w$ is harmonic. It can be shown that any m-harmonic function can be written as a linear combination of $\{\psi_p^m\}_{p \in V_m}$, and the piecewise harmonic functions over V_* form a basis of $\ell(V_*)$. [8]*

Furthermore, it can be shown that any m-harmonic function u satisfies $\varepsilon(u, f) = 0$ for any $f \in \mathcal{F}$ given $f|_{V_m} = 0$. This justifies our use of the term harmonic in this context.

We are finally ready to present the piecewise formulation of the Laplacian, which was defined by Kigami in [7].

Definition 4.2.2 (The Pointwise or Kigami Laplacian). *Let μ be a self-similar measure, and $f \in C(K)$. If there exists $u \in C(K)$ satisfying*

$$\lim_{m \to \infty} \max_{p \in V_m \setminus V_0} \left| \frac{1}{\mu_{m,p}} (H_m) u(p) - f(p) \right| = 0 \tag{4.3}$$

where $\mu_{m,p} := \displaystyle\int_K \psi_p^m d\mu$. If such a u exists, we write $\Delta_\mu u = f$, where Δ_μ is called the Laplacian associated with measure μ and harmonic structure $(D, \mathbf{r})$. We often denote the domain of Δ_μ by $\mathcal{D}_\mu$.

Remark 4.2.2. *Note that for $f = 0$, by the existence of harmonic functions over K, we have that there exists $u \in C(K)$ satisfying $\Delta_\mu u = 0$ and thus $\mathcal{D}_\mu$ is non-trivial. It can also be shown that $\Delta_\mu : \mathcal{D}_\mu \to C(K)$ is a linear map.*

Remark 4.2.3. *It is not imperative to take μ to be a self-similar measure and (D, r) to be a regular harmonic structure. In practice, this is often the case, but the above definition given and regular Borel probability measure and any harmonic structure over K. See [7] and [8].*

Remark 4.2.4. *For $a, b \in \mathbb{R}$, with $a < b$, if μ a self similar measure with weights $\mu_i = \frac{1}{2}$ for $i = 1, 2$, we have that Δ_μ corresponds to the classical Laplacian over $[a, b]$.*

We now define the Neumann derivative of a function f over K that is in the domain of the Laplacian.

Definition 4.2.3. *Let $f \in \mathcal{D}_\mu$ and $p \in V_0$. The Neumann derivative, of f at p, denoted $(df)_p$, is defined as*

$$(df)_p = \lim_{m \to \infty} -(H_m f)(p)$$

The following theorem establishes that the above pointwise formulation corresponds to the variational Dirichlet and Neumann Laplacians constructed in Chapter 3. The proof of this theorem requires the construction of Green's functions over a self-similar set; however, this is outside the scope of this thesis, and thus we omit it here. It was first proved by Fukishima and Shima in [2] for the special case of the standard Laplacian over the Sierpinski Gasket, and the more general case is presented in [7] and [8]. It is important to note that Kigami's proof in [8] p.c.f self-similar fractals that are equipped with a possibly non-regular harmonic structure.

Theorem 4.2.1. *Let μ be a self-similar measure, and $(D, \boldsymbol{r})$ a regular harmonic structure over K a self-similar set. Define*

$$\mathcal{D}_{D,\mu} = \{u \in \mathcal{D}_\mu \mid u|_{V_0} = 0\}$$
$$\mathcal{D}_{N,\mu} = \{u \in \mathcal{D}_\mu \mid (du)_p = 0 \text{ for } p \in V_0\}$$

Then, $\mathcal{D}_{D,\mu} = \mathrm{Dom}(H_D) \cap \mathcal{D}_\mu$, with $H_D = -\Delta_\mu$ on $\mathcal{D}_{D,\mu}$, and $\mathcal{D}_{N,\mu} = \mathrm{Dom}(H_N) \cap \mathcal{D}_\mu$ with $H_N = -\Delta_\mu$ on $\mathcal{D}_{N,\mu}$.

Proof. See Chapter 3.7 of [8]. □

The following is also an important corollary of this theorem, also found in [8], which we also don't prove here. The specific case of the Sierpinski Gasket was also discussed in [2]. It essentially stating that we can treat the operators H_b (for $b = N, D$) and the pointwise Laplacian interchangeably.

Corollary 4.2.1.1. *For $b \in \{N, D\}$, if $f \in Dom(H_b)$ and $H_b f \in C(K)$, then $f \in Dom(H_b) \cap \mathcal{D}_\mu$.*

4.3 The Gauss-Green Formula

We now establish an analogue of the Gauss-Green formula for the Kigami Laplacian as presented in [7]. We begin by proving a few preliminary lemmas.

Lemma 4.3.1. *Let $v \in C(K)$, and $u \in \mathcal{D}_\mu$. Then,*

$$\lim_{m\to\infty} \sum_{p \in V_m \setminus V_0} v(p) H_m u(p) = \int_K v \Delta_\mu u d\mu$$

Proof. Define $f_m(x) = \displaystyle\sum_{p \in V_m} v(p) \frac{H_m u(p) \psi_p^m(x)}{\mu_{m,p}}$. Then, by definition of $\Delta_\mu u$, we have that $f_m \to v\Delta_\mu u$ pointwise as $m \to \infty$. Now, given $A \subseteq K \setminus V_0$ a compact subset, we have that all of the components within the sum are continuous functions over A and thus achieve a maximum over A, so that f_m is bounded from above by $c|v||H_m u||\psi_p^m|$, which are continuous and harmonic functions respectively, and thus integrable; here c denotes the cardinality of the finite set $V_m \setminus V_0$. Thus, applying the generalised dominated convergence theorem, we have that

$$\int_K f_m d\mu \to \int v \Delta_\mu$$

In particular, we have that

$$\int_K f_m(x)d\mu(x) = \int_K \sum_{p\in V_m} v(p)\frac{H_m u(p)\psi_p^m(x)}{\mu_{m,p}}$$
$$= \sum_{p\in V_m} v(p)H_m u(p)\frac{\int_K \psi_p^m(x)d\mu(x)}{\mu_{m,p}}$$
$$= \sum_{p\in V_m} v(p)H_m u(p)$$

where the latter inequality follows by the definition of $\mu_{m,p}$ given in Definition 4.3.2. Therefore, as $m \to \infty$ we have that

$$\sum_{p\in V_m} v(p)H_m u(p) \to \int_K v\Delta_\mu u d\mu$$

□

Lemma 4.3.2. *Let $f \in \mathcal{D}_\mu$ and $p \in V_0$, then*

$$\lim_{m\to\infty} -(H_m f)(p) = -(Df)(p) + \int_K \psi_p \Delta_\mu f d\mu$$

Proof. See Proof of *Lemma 3.7.5* in [8]. □

Note that by Lemma 4.3.2, the Neumann derivative is well-defined for $f \in \mathcal{D}_\mu$. We now prove an analogous Gauss-Green formula for the Laplacian over self-similar K.

Lemma 4.3.3. *Let $v \in C(K)$ and $u \in \mathcal{D}_\mu$. Then,*

$$\lim_{m\to\infty} \varepsilon_m(v,u) = \sum_{p\in V_0} v(p)(du)_p - \int_K v\Delta_\mu u d\mu$$

Proof. By definition of $\varepsilon_m(v,u)$, we have that

$$\varepsilon_m(v,u) = -\sum_{p\in V_0} v(p)(H_m u)(p) - \sum_{p\in V_m\setminus V_0} v(p)(H_m u)(p) \tag{4.4}$$

where we split the sum across the disjoint union of the two sets V_0 and $V_m \setminus V_0$. Then, by Lemma 4.3.1, we have

$$\lim_{m\to\infty} \varepsilon(v,u) = \sum_{p\in V_m\setminus V_0} v(p)(H_m u)(p) = \int_K v\Delta_\mu u d\mu$$

combining this with the definition of $(du)_p$ (Definition 4.3.2), and taking the limit as m tends to infinity, we get the desired the result. □

Theorem 4.3.4 (Gauss-Green Formula). *For a compatible sequence $\{V_m, H_m\}_{m=0}^{\infty}$ associated with a regular harmonic structure (D,* **r***) we have that the domain of ε is contained in $\mathcal{D}_\mu$. For $u \in \mathcal{D}_\mu$ and $v \in \mathcal{F}$ we have that*

$$\varepsilon(v, u) = \sum_{p \in V_0} v(p)(du)_p - \int_K v \Delta_\mu u d\mu$$

If in addition, we have that $v \in \mathcal{D}_\mu$

$$\sum_{p \in V_0} (v(p)(du)_p - u(p)(dv)_p) = \int_K (v\Delta_\mu u - u\Delta_\mu v) d\mu$$

Proof. By Lemma 4.3.2, we have that

$$\lim_{m \to \infty} \varepsilon_m(u, v) = \sum_{p \in V_0} v(p)(du)_p - \int_K v \Delta_\mu u d\mu$$

Furthermore, we have that $(du)_p$ is well-defined, and hence everything on the right hand side of the above equation is finite. So by definition of $\varepsilon(u, v)$, we have that $u \in \mathcal{D}_\mu$ and we get the desired equality.

To prove the latter part of the theorem, let $u, v \in \mathcal{D}_\mu$. Note that we can write

$$\varepsilon_{H_m}(u, v) = \sum_{p,q \in V_m} (H_m)_{pq}(u(p) - u(q))(v(p) - v(q)))$$

where it is clear that ε_{H_m} is symmetric, and thus ε is as well. Thus, applying Lemma 4.3.2 to $\varepsilon(u, v)$ and $\varepsilon(v, u)$ respectively, we have that

$$\varepsilon(v, u) = \sum_{p \in V_0} v(p)(du)_p - \int_K v \Delta_\mu u d\mu,$$

and

$$\varepsilon(u, v) = \sum_{p \in V_0} u(p)(dv)_p - \int_K u \Delta_\mu v d\mu.$$

Subtracting both equations, we obtain

$$\sum_{p \in V_0} (v(p)(du)_p - u(p)(du)_p) = \int_K (v \Delta_\mu u - u \Delta_\mu v) d\mu.$$

□

Chapter 5

Eigenvalues of the Laplacian and Weyl's Law over Self-Similar Domains

In this chapter, we establish the existence of eigenvalues and eigenfunctions associated with the Kigami Laplacian Δ_μ over a connected p.c.f self-similar set K with respect to a regular harmonic structure $(D, \mathbf{r})$ and self-similar measure μ. We establish the existence of both Dirichlet and Neumann eigenvalues (and eigenfunctions), where the boundary conditions are as defined in Theorem 4.2.1 of the previous chapter. We also introduce an eigenvalue-counting function and study its asymptotics, stating a Weyl-type law originally proved by Kigami and Lapidus in [9].

5.1 Eigenvalues and Eigenfunctions of the Laplacian

We begin by defining the Dirichlet and Neumann eigenvalues of Δ_μ.

Definition 5.1.1. *Let K be a p.c.f. self-similar structure equipped with regular harmonic structure $(D, \boldsymbol{r})$ and self-similar measure μ. We define the space of Neumann eigenfunctions associated with eigenvalue λ by*

$$E_N(\lambda) := \{\varphi \in \mathcal{D}_{N,\mu} \mid \Delta_\mu \varphi = -\lambda\varphi\}$$

Furthermore, we define the Dirichlet eigenfcuntions associated with eigenvalue λ *by*

$$E_D(\lambda) := \{\varphi \in \mathcal{D}_{D,\mu} \mid \Delta_\mu \varphi = -\lambda\varphi\}$$

Note that by linearity of Δ_μ, $E_N(\lambda)$ and $E_D(\lambda)$ equipped with pointwise addition of functions and scalar multiplication are vector spaces over $\mathbb{R}$. If their respective dimensions are non-trivial, then λ and φ are referred to as non-trivial Neumann/Dirichlet eigenvalues and eigenfunctions respectively. Furthermore, by Theorem 4.2.1, they coincide with those of the non-negative self-adjoint operators H_N and H_D constructed in Theorem 4.1.2 (and Corollary 4.1.2.1 respectively). For ease of notation, we let $b = D, N$ denote the Dirichlet and Neumann boundary conditions respectively

Proposition 5.1.1. *Let* K *be a p.c.f. self-similar set equipped with regular harmonic structure* $(D, \boldsymbol{r})$*, where* $\boldsymbol{r} = (r_1, r_2, ..., r_n)$*, and self-similar measure* μ *with weights* $\{\mu_i\}_{i=1}^n$*. Consider the non-negative self-adjoint operators* H_b *for* $b = N, D$ *with regular Dirichlet forms* $(\varepsilon, \mathcal{F})$ *and* $(\varepsilon, \mathcal{F}_0)$ *respectively, as defined in Theorem 4.1.2 and Corollary 4.1.2.1.*

If for all $1 \leq i \leq n$ *we have that* $r_i\mu_i < 1$*, the following are equivalent*

(1) $\varphi \in \text{Dom}(H_b)$ and $H_b\varphi = \lambda\varphi$

(2) $\varphi \in \mathcal{F}$ (or $\mathcal{F}_0$), and $\varepsilon(\varphi, u) = \lambda(\varphi, u)_\mu$ where $(.,.)_\mu$ is the inner product over $L^2(K, \mu)$

(3) $\varphi \in E_b(\lambda)$

The preceding statements are furthermore equivalent given Dirichlet boundary conditions as well (replacing instances of H_N *with* H_D*,* $\mathcal{F}$ *with* $\mathcal{F}_0$*, and* $E_N(\lambda)$ *with* $E_D(\lambda)$ *respectively.*

Proof. The proof of this proposition requires the use of Green's functions over K and their properties, the details of which can be found in [7] and as previously mentioned, this is outside the scope of this thesis. The proof can be found in Kigami's [8]. □

Given the established correspondence between the Dirichlet and Neumann eigenvalues associated with Δ_μ and those of H_D and H_N respectively, we have the following theorem

Theorem 5.1.1. *Let Δ_μ be the Laplacian over a connected pcf self-similar set K. Then there exist $\lambda \in \mathbb{R}$ and $u \in L^2(K,\mu)$ non-trivial solutions to the equation*

$$\Delta_\mu u = -\lambda u \tag{5.1}$$

for boundary conditions $b = D, N$. Furthermore, there exist countably many $\{\lambda_k^b\}_{k=1}^\infty$ and $\{\varphi_k^b\}_{n=1}^\infty \in E_b(\lambda_k^b)$ satisfying (5.1), with

$$0 \leq \lambda_1^b \leq \lambda_2^b \leq ... \leq \lambda_k^b \leq \lambda_{k+1}^b \leq ...$$

with the only limit point at $+\infty$, and $\{\varphi_k^b\}_{k=1}^\infty$ form a complete orthonormal basis of $L^2(K,\mu)$.

Proof. By Theorem 4.2.1, we have that Δ_μ corresponds with the non-negative self-adjoint operators H_b. In addition, by Theorem 4.1.2 and Corollary 4.1.2.1, we have that H_b have compact resolvents. Thus, invoking the spectral theorem for compact self-adjoint operators, we have that there exist countably many $\{\lambda_n^b\}_{n=1}^\infty \subseteq \mathbb{R}$ and $\{\varphi_n^b\}_{n=1}^\infty \subseteq L^2(K,\mu)$ satisfying the properties stated in the theorem, and by Proposition 5.1.1, all such $\varphi_n^b \in E(\lambda_n^b)$ and thus satisfy (5.1). □

5.2 The Eigenvalue-Counting Function and Weyl's Law

Now that we have the existence of eigenvalues, we can introduce the eigenvalue counting function, and study its asymptotics. The eigenvalue-counting function and its asymptotics was studied by Lapidus in [10], and the result presented here is adapted from [9], where the result is presented as it is here.

Definition 5.2.1. *We define the eigenvalue-counting function for boundary condition $b \in \{D, N\}$ and μ a self-similar measure over K, $\rho_b(.,\mu) : \mathbb{R} \to \mathbb{R}$ by $\rho_b(x,\mu) := \sum_{\lambda \leq x} \dim(E_b(\lambda))$.*

Proposition 5.2.1. *$\rho_b(x,\mu)$ has the following properties*

(1) $\rho_b(x,\mu) = \max_{k \in \mathbb{N}} \{\lambda_k^b \leq x\}$

(2) $\rho_b(x,\mu) \to \infty$ as $x \to \infty$

Proof. Fix $x \in \mathbb{R}$. Then if each eigenvalue has multiplicity 1, the equality in (1) is clear. However, if for $k \in \mathbb{N}$ has multiplicity $m \geq 1$, then $\dim(E_b(\lambda_k^b)) \geq 1$. Note that by the spectral theorem, we have that the only accumulation point of $\{\lambda_k^b\}_{k=1}^{\infty}$ is $+\infty$ so that no λ_k^b has infinite multiplicity.

Without loss of generality, we can assume that k is the last such occurrence of this eigenvalue in the list with $\lambda_k^b \leq x$, and that all smaller eigenvalues have multiplicity 1, so that $\dim(E_b(\lambda_{k-m}^b)) = \dim(E_b(\lambda_{k-m+1}^b)) = ... = \dim(E_b(\lambda_k^b))$, and $\dim(E_b(\lambda_n^b)) = 1$ for all $n \leq k-m$. But $\rho_b(x,\mu) = \sum_{\lambda \leq x} \dim(E_b(\lambda))$ where we do not repeat eigenvalues with higher multiplicity in the index of the sum, and thus no over-counting due to multiplicity takes place .

Since $\dim(E_b(\lambda_{k-m}^b)) = \dim(E_b(\lambda_{k-m+1}^b)) = ... = \dim(E_b(\lambda_k^b))$, we have that $\dim(E_b(\lambda_k^b)) = k-(k-m)+1 = m+1$, and thus

$$\rho_b(x,\mu) = k - m - 1 + m + 1 = k = \max_{k \in \mathbb{N}} \{\lambda_k^b \leq x\}$$

proving (1). Using (1) in conjunction with the fact that the eigenvalues only accumulate at $+\infty$, it is then clear that $\rho_b(x,\mu) \to \infty$ as m gets arbitrarily large. □

Now that we have established the elementary properties of $\rho_b(x,\mu)$, we want to more precisely pin down its asymptotics, and prove an analogous Weyl's Theorem for bounded domains in $\mathbb{R}^n$. This is usually stated like so, as adapted from [4]

Theorem 5.2.1 (Weyl's Theorem in $\mathbb{R}^n$). *Consider the standard Dirichlet eigenvalue problem of the standard Laplacian $-\Delta$ over $\Omega \subseteq \mathbb{R}^n$ bounded. Then, given $\rho(x) := |\{i \in \mathbb{N} \mid \lambda_i \leq x\}|$, where $|A|$ here denotes the cardinality of a set A, we have that as $x \to \infty$*

$$\rho(x) \sim \frac{1}{(2\pi)^n} m(B(0,1)) m(\Omega) x^{n/2} + o(x^{n/2})$$

where $B(0,1)$ is the unit ball in $\mathbb{R}^n$ and $m(.)$ denotes $n-$dimensional Lebesgue measure.

For a proof of this theorem, we refer the reader to Lapidus [10].

Although to establish such a theorem for the Kigami Laplacian over K, given both Dirichlet and Neumann boundary conditions, we require the following results, as seen in [13]

Theorem 5.2.2 (Feller's Renewal Theorem). *Let $t_* > 0$, and $f : \mathbb{R} \to R$ a measurable function such that $f(t) = 0$ for all $t < t_*$, and satisfies*

$$f(t) = \sum_{i=1}^{n} f(t - \alpha_i)p_i + u(t) \tag{5.2}$$

where for all $1 \leq i \leq n$, $\alpha_i > 0$, $_i > 0$ with $\sum_{i=1}^{n} p_i = 1$, and $u : \mathbb{R} \to \mathbb{R}$ is a non-negative Riemann integrable function over $\mathbb{R}$ and $u(t) = 0$ for all $t < t_*$. Then, the following statements hold

(1) **Lattice Case:** If there exists $T > 0$ such that $\alpha_i = m_i T$ for $\{m_i\}_{i=1}^{n}$ relatively prime positive integers, then there exists a T-periodic function $G(t)$ given by

$$G(t) = \left(\sum_{i}^{n} m_i p_i \right)^{-1} \sum_{i \in \mathbb{Z}} u(t + iT)$$

satisfying $|f(t) - G(t)| \to 0$ as $t \to \infty$.

(2) **Non-Lattice Case:** If $\sum_{i=1}^{n} \mathbb{Z}\alpha_i$ is a dense additive subgroup of $\mathbb{R}$, then $\lim_{t\to\infty} f(t)$ exists and is given by

$$\lim_{t\to\infty} f(t) = \left(\sum \alpha_i p_i \right)^{-1} \int_{\mathbb{R}} u(t)dt$$

Lemma 5.2.3. *Let μ be a self-similar measure over K with weights $\{\mu_i\}_{i=1}^{n}$, and $(D, \boldsymbol{r})$ a regular harmonic structure with $\boldsymbol{r} = (r_1, r_2, ..., r_n)$.Consider the eigenvalue counting functions $\rho_b(x, \mu)$ for $b \in \{D, N\}$. Then, there exist positive constants $M_1, M_2 > 0$ such that*

$$\rho_D(x,\mu) - M_1 \leq \sum_{i=1}^{n} \rho_D(r_i\mu_i x,\mu) \leq \rho_D(x,\mu) \leq \rho_N(x,\mu) \leq \rho_D(x,\mu) + M_2 \tag{5.3}$$

Proof. See *Lemma 2.3* in [9]. □

The proof of the main theorem is an application of Feller's Renewal Theorem, with the appropriate assumptions being met by virtue of the second and third inequalities in (5.3); the significance of Lemma 5.2.3 also lies in showing that we have an analogous Weyl's law for both Dirichlet and Neumann boundary conditions. We now state and prove the theorem here.

Theorem 5.2.4 (Weyl's Law for the eigenvalue-counting function of Δ_μ). *Let μ be a self-similar measure with weights $\{\mu_i\}_{i=1}^n$, and $(D, \boldsymbol{r})$ a regular harmonic structure with $\boldsymbol{r} = (r_1, r_2, ..., r_n)$. Assume that for all $1 \leq i \leq n$ we have $r_i\mu_i < 1$, and d_S is the (unique) real number, called the spectral dimension/exponent of K, satisfying*

$$\sum_{i=1}^{n} \gamma_i^{d_S} = 1$$

for $\gamma_i := \sqrt{r_i\mu_i}$. Then, for $b = N, D$ we have that

$$0 < \liminf_{x\to\infty} \frac{\rho_b(x,\mu)}{x^{d_S/2}} \leq \limsup_{x\to\infty} \frac{\rho_b(x,\mu)}{x^{d_S/2}} < \infty$$

Moreover, the following statements hold:

(1) **Lattice Case:** If $\sum_{i=1}^{n} \mathbb{Z}\log(\gamma_i)$ is a discrete subgroup of $\mathbb{R}$ with generator $T > 0$. Then, there exists a right-continuous T-periodic function G independent of the boundary conditions satisfying

$$0 < \inf_{x\in\mathbb{R}} G(x) \leq \sup_{x\in\mathbb{R}} G(x) < \infty$$

and

$$\rho_b(x,\mu) = (G(\log(x/2)) + o(1))x^{d_S/2}$$

where $o(1) \to 0$ as $x \to \infty$.

(2) **Non-Lattice Case:** If $\sum_{i=1}^{n} \mathbb{Z} \log(\gamma_i)$ is a dense subgroup of $\mathbb{R}$, then there exists a Riemann integrable function R over $\mathbb{R}$ such that

$$\lim_{m\to\infty} \frac{\rho_b(x,\mu)}{x^{d_S/2}} = \left(\sum_{i=1}^{n} -\gamma_i^{d_S} \log(\gamma_i) \right)^{-1} \int_{\mathbb{R}} e^{-d_S/2} R(e^{2t}) dt$$

In particular, we have that the limit exists, is finite, and independent of any boundary conditions.

Proof. It suffices to prove the theorem for the Dirichlet case, and we note that the Neumann case follows immediately by the latter inequality involving the Neumann eigenvalue-counting function in Lemma 5.2.3. For ease of notation, we write $\rho_D(x) := \rho_D(x,\mu)$.

We begin by defining the $\mathbb{R}$-valued function

$$R(x) := \rho_D(x) - \sum_{i=1}^{n} \rho_D(\mu_i r_i x),$$

and thus have the identity,

$$\rho_D(x) = \sum_{i=1}^{n} \rho_D(r_i \mu_i x) + R(x). \tag{5.4}$$

Now, further define the pair of $\mathbb{R}$-valued functions

$$f(t) := e^{-d_S t} \rho_D(e^{2t}),$$
$$u(t) := e^{-d_S t} R(e^{2t}),$$

Parametrising $x = e^{2t}$ and multiplying identity (5.4) by $e^{-d_S t}$, by definition of $f(t)$ and $u(t)$, we have

$$f(t) = \sum_{i=1}^{n} e^{-d_S t} \rho_D(r_i \mu_i e^{2t}) + u(t)$$

Now, note that we can write $r_i \mu_i = e^{2 \ln \sqrt{r_i \mu_i}} = e^{2 \ln \gamma_i}$, so that the previous equation becomes

$$
\begin{aligned}
f(t) &= \sum_{i=1}^{n} e^{-d_S t}\rho_D(e^{2(\ln\gamma_i + t)}) + u(t) \\
&= \sum_{i=1}^{n} e^{d_S \ln\gamma_i} e^{-d_S(t+\ln\gamma_i)}\rho_D(e^{2(\ln\gamma_i + t)}) + u(t) \\
&= \sum_{i=1}^{n} \gamma_i^{d_S} f(t + \ln\gamma_i) + u(t)
\end{aligned}
$$

Now, for all $1 \leq i \leq n$ set $p_i := \gamma_i^{d_S}$ and $\alpha_i := -\ln\gamma_i$. Then since $r_i\mu_i < 1$, we have that $\alpha_i > 0$, and $\sum_i^n p_i = 1$, by definition of the spectral exponent, so

$$
f(t) = \sum_{i=1}^{n} p_i f(t - \alpha_i) + u(t)
$$

which is the required form for f to satisfy Feller's Renewal Theorem. It remains to show that $u(t) = f(t) = 0$ for all $t < t_*$ for some $t_* \in \mathbb{R}$, and that $u(t)$ is indeed integrable.

To this end, consider the first eigenvalue, λ_1, of H_D. Then, since H_D is positive definite, $\lambda_1 > 0$, and thus $\rho_D(x) = 0$ for all $x < \lambda_1$. Taking $t_* < \frac{1}{2}\ln\lambda_1$, we obtain $f(t) = 0$ for all $t \leq t_*$; by identity (5.4), this also demonstrates that $R(t) = 0$, and so $u(t) = 0$. Furthermore, by Lemma 5.2.3, we have there exists a constant $M > 0$ such that for all $x \in \mathbb{R}$

$$
0 \leq \rho_D(x) - \sum_{i=1}^{n} \rho_D(r_i\mu_i x) \leq M.
$$

So $|R(t)| \leq M$ for all $t \in \mathbb{R}$, and

$$
|u(t)| \leq M e^{-d_S t}.
$$

This demonstrates that u is bounded from above a by a Riemann integrable function. In addition, ρ_D is also clearly a step function as an eigenvalue counting function (part (1) of Proposition 5.1.1), and thus has at most countably many discontinuities. Hence, u is Riemann integrable, and f satisfies all of the required properties of Feller's Renewal theorem, and we can conclude that in the non-lattice case, $\lim f(t)$ exists as $t \to \infty$, with

$$
f(t) \to \frac{\displaystyle\int_{\mathbb{R}} e^{-d_S t} R(e^{2t})}{\displaystyle\sum_{i=1}^{n} (-\ln\gamma_i)\gamma_i^{d_s}}. \tag{5.5}
$$

In the lattice case, taking $T > 0$ as the generator of the additive group $\sum_{i=1}^{n} \mathbb{Z} \ln \gamma_i$, we have that there exists a T-periodic function G satisfying

$$G(t) = \frac{\sum_{i\in\mathbb{Z}} e^{-d_S(t+iT)} R(e^{2(t+iT)})}{\sum_{i=1}^{n} (\ln \gamma_i \gamma_i^{d_s})}$$

with $f(t) - G(t) \to 0$ as $t \to \infty$.

In both cases, note that by setting $t = \ln x/2$, we get that $f(\ln x/2) = x^{-d_S/2}\rho_D(x)$, so that our limits in t can be translated to limits in x, and we are indeed investigating the behaviour of $x^{-d_s/2}\rho_D(x)$. In the non-lattice case, we have that the limit exists, and is given by (5.5), and we are done.

In the non-lattice case, since $f(t) - G(t) \to 0$ as $t \to \infty$, we have that

$$e^{-d_S t}\rho_D(e^{2t}) = G(t) + o(1)$$

Taking $t = \ln x/2$ again, we get that

$$\rho_D(x) = (G(\ln x/2) + o(1)))x^{d_S/2}$$

where $o(1) \to 0$ as $x \to \infty$, as desired. Furthermore, since $R(x) \leq M$ for all $x \in \mathbb{R}$, and is positive for all $t < t_*$ as previously established, we have that

$$0 < \inf G(x) \leq \sup G(x) < \infty$$

Furthermore, it is clear to see that R is right-continuous in as a (finite) linear combination of step functions, and by boundedness of $|R(x)|$, we can apply the dominated convergence theorem to get that G is right-continuous as well. Lastly, by Lemma 5.2.3, we have that there exists a $C > 0$

$$\rho_D(x) \leq \rho_N(x) \leq \rho_D(x) + C$$

and so the theorem immediately follows for ρ_N with the exact same expressions for G and limit point of f in the lattice and non-lattice cases respectively, and thus the asymptotics of $x^{-d_S/2}\rho_D(x)$, and properties of G are independent of any boundary conditions. This completes the proof of the theorem.

□

Chapter 6

The Heat Kernel and Eigenfunction Embeddings

In this chapter, we define the Dirichlet and Neumann heat kernels associated with the Dirichlet and Neumann Laplacians respectively over a p.c.f self-similar set K. We establish some of their basic properties, including continuity, differentiability, and their relation to the heat equation over K.The main results of this chapter are adapted from [8], although it is important to note that similar estimates were obtained by Barlow in [1] via probabilistic methods. We then discuss the use of the heat kernel in embedding manifolds into the Hilbert space ℓ^2 as done in [3], and extend the result to embedding self-similar fractals into Hilbert spaces, and discuss how the situation over fractals differs from that over manifolds. For this chapter, we once again take K equipped with a self-similar measure, and regular harmonic structure $(D, \mathbf{r})$.

6.1 The Heat Kernel

To construct the heat kernel, we consider the sequence of eigenvalues $\{\lambda_n^b\}_{n=1}^{\infty}$ and their associated eigenfunctions $\{\varphi_n^b\}_{n=1}^{\infty} \subseteq E_b(\lambda_n^b)$ for boundary conditions $b \in \{N, D\}$. By Theorem 5.1.1, we have that the eigenvalues are increasing and tend to infinity, and that

$\{\varphi_n^b\}_{n=1}^{\infty}$ form a complete orthonormal basis of $L^2(K, \mu)$ for a self-similar measure μ. In light of this, we can formally define the heat kernel.

Definition 6.1.1 (Heat Kernel). *For $b \in \{D, N\}$, we define the heat kernel as the function*

$$p_b(t, x, y) : (0, \infty) \times K^2 \to \mathbb{R}$$
$$p_b(t, x, y) = \sum_{n=1}^{\infty} e^{-t\lambda_n^b} \varphi_n^b(x) \varphi_n^b(y)$$

A priori, it is unclear whether the previous infinite sum is indeed convergent, and thus whether the heat kernel is well-defined. However, as it turns out, the heat kernel is indeed well defined, and in fact C^1 on $[T, \infty]$ for any $T > 0$. To prove this, we require the following lemmas, the first of which can be found in [1].

Lemma 6.1.1. *Let d_S be the spectral dimension of K, and consider the sequence of eigenvalues $\{\lambda_n^b\}_{n=1}^{\infty}$. There for any $n \geq 2$ there exists constants $c_1, c_2 > 0$ such that*

$$c_1 n^{2/d_S} \leq \lambda_n^b \leq c_2 n^{2/d_S}$$

Proof. By our analogue for Weyl's theorem (Theorem 5.2.4 in the previous chapter), we have that the limsup and liminf of $x^{-d_S/2}\rho_b(x)$ exist, and thus there exist constants $C, D > 0$ such that for $x > 0$ sufficiently large

$$Cx^{d_s/2} \leq \rho_b(x) \leq Dx^{d_S/2} \tag{6.1}$$

Now, by part (1) of Proposition 5.2.1 and applying the previous inequality, we have that $n \leq \rho_b(\lambda_n^b) \leq D\lambda_n^b$. In particular, rearranging and defining $c_1 := \frac{1}{D^{2/d_S}}$ this we get that

$$c_1 n^{2/d_S} \leq \lambda_n^b$$

This demonstrates the first inequality. For the latter inequality, let $\epsilon > 0$, and notice that by (6.1), we have that

$$C(\lambda_n^b - \epsilon)^{d_S/2} \leq \rho_b(\lambda_n^b) \leq n$$

where the latter inequality follows from Proposition 5.2.1 part (1) and the fact that $\lambda_n^b - \epsilon < \lambda_n^b$. Since this holds for $\epsilon > 0$ arbitrary, we can take $\epsilon \to 0$ to get that

$$C(\lambda_n^b)^{ds/2} \leq n$$

Rearranging and defining $c_2 := \frac{1}{C^{2/d_S}}$, we get that

$$\lambda_n^b \leq c_2 n^{2/d_S}$$

Combining both inequalities yields the desired result. □

Lemma 6.1.2. *For any $\alpha, \beta > 0$ and any $T > 0$, then $\sum_{n=1}^{\infty} n^{\alpha} e^{-n^{\beta} t}$ is uniformly convergent on $[T, \infty)$.*

Proof. By the power series expansion of e^x, we have that

$$n^{-\alpha} e^{n^{\beta} t} = \sum_{k=1}^{\infty} \frac{n^{\beta k - \alpha} t^k}{k!}$$

So let $m \in \mathbb{N}$ be the first index such that $\beta k - \alpha > 1$. Then, by Taylor's remainder theorem, we have the following upper bound on $n^{-\alpha} e^{n^{\beta} t}$

$$n^{-\alpha} e^{n^{\beta} t} \leq \frac{n^{\beta m - \alpha} t^m}{m!}$$

Taking reciprocals, we get that

$$n^{\alpha} e^{n^{-\beta} t} \leq \frac{n^{-(\beta(m+1)-\alpha)}(m+1)!}{t^{m+1}} \leq \frac{n^{-(\beta(m+1)-\alpha)}(m+1)!}{T^{m+1}}$$

for all $t \leq T$, which we note is a bound independent of t. Furthermore, $\sum_{n=1}^{\infty} n^{-(\beta(m+1)-\alpha)} < \infty$ since $\beta(m+1) - \alpha > 1$, so that $\sum_{n=1}^{\infty} n^{\alpha} e^{n^{\beta} t}$ converges uniformly in t. □

Moreover, in order to prove continuity of the heat kernel, we need the following estimate on the L^{∞} norm of eigenfunctions. Specifically, if $\Delta_{\mu} \varphi = -\lambda \varphi$ for $\lambda > 0$, then $\varphi \in L^{\infty}(K)$, and we have the following estimate

$$||\varphi||_{\infty} \leq c \lambda^{\alpha/2} ||\varphi||_2 \tag{6.2}$$

where $\alpha := \max_{1\leq i\leq n} \dfrac{\ln \mu_i}{\ln \mu_i r_i}$, $||.||_\infty$ and $||.||_2$ refer to the norms $L^\infty(K)$ and $L^2(K,\mu)$ respectively, and $c > 0$ is a constant independent of λ and φ. This estimate appeared for the first time in Chapter 4.5 of [8], along with the other results in the book's chapter.

We are now ready to prove some significant properties of the heat kernel.

Theorem 6.1.3. *Let $p_b(t,x,y)$ denote the heat kernel with respect to boundary condition b, and for $(t,x) \in (0,\infty) \times K$ define the function $p_b^{(t,x)} : K \to \mathbb{R}$ by $p_b^{(t,x)}(y) = p_b(t,x,y)$. Then the following statements hold*

(1) p_b is continuous on $(0,\infty) \times K^2$

(2) For all $(t,x) \in (0,\infty) \times K$, $p_b^{(t,x)} \in$Dom$(H_b) \cap \mathcal{D}_\mu$

(3) For all $(x,y) \in K^2$ fixed, $p_b(t,x,y) \in C^1((0,\infty))$

(4) Given any $(t,x,y) \in (0,\infty) \times K^2$

$$\frac{\partial p_b(t,x,y)}{\partial t} = (\Delta_\mu p_b^{(t,x)})(y)$$

(5) For $s,t > 0$ and $x,y \in K$

$$p_b(s+t,x,y) = \int_K p_b(s,x,x')p_b(s,x',y)d\mu(x')$$

Proof. We prove each of the assertions of the theorem in order. For claim (1), by inequality (6.2), we have that for all $x,y \in K$, there exits $c > 0$

$$|\varphi(x)||\varphi(y)| \leq c^2(\lambda_n^b)^\alpha ||\varphi_n^b||_2^2$$

where $\alpha := \max_{1\leq i\leq n} \frac{\ln \mu_i}{\ln r_i \mu_i}$. Now, by Lemma 6.1.1, we can extend this inequality using our bounds on λ_n^b to

$$|\varphi(x)||\varphi(y)| \leq C^2 n^{2\alpha/d_S}$$

where $C := c\sqrt{c_2}||\varphi_n^b||_2$ and c_2 is the constant from Lemma 6.1.1. Furthermore, applying the lemma again have that there exists $c_1 > 0$ such that $c_1 n^{2/d_S} \leq \lambda_n^b$ so that for all $t > 0$ we have $e^{-\lambda_n^b t} \leq e^{-c_1 n^{2/d_S} t}$. Hence

$$\sum_{n=1}^{\infty} e^{-\lambda_n^b t}|\varphi_n^b(x)||\varphi_n^b(y)| \leq C^2 \sum_{n=1}^{\infty} n^{2\alpha/d_S} e^{-c_1 n^{2/d_S} t}$$

Applying Lemma 6.1.2, we get that the above sum on the right hand side converges uniformly on $[T, \infty)$ for all $T > 0$, and thus $p_b(t, x, y)$ in continuous over $(0, \infty) \times K^2$ (where we note that any solution u to $\Delta_\mu u = f$ is continuous by Definition 4.2.2).

To prove assertion (2), we use that the eigenfunctions form a basis of $L^2(K, \mu)$ to write $f = \sum_{i=1}^{n} a_n \varphi_n^b$. Then, we get that $f \in \mathrm{Dom}(H_b)$ if and only if $\sum_{n=1}^{\infty} |a_n \lambda_n^b|^2 < \infty$, since $H_b(\varphi_n^b) = \lambda_n^b \varphi_n^b$. Now, computing $H_b(p_b^{(t,x)}(y))$, we have that

$$\begin{aligned} H_b(p_b^{(t,x)}(y)) &= H_b(\textstyle\sum_{n=1}^{\infty} e^{-\lambda_n^b t}\varphi_n^b(x)\varphi_n^b(y)) \\ &= \sum_{n=1}^{\infty} e^{-\lambda_n^b t}\varphi_n^b(x) H_b(\varphi_n^b(y)) \\ &= \sum_{n=1}^{\infty} \lambda_n^b e^{-\lambda_n^b t}\varphi_n^b(x)\varphi_n^b(y) \end{aligned} \tag{6.3}$$

where we used the equality $H_b(\varphi_n^b(y)) = \lambda_n^b \varphi_n^b(y)$ in the last step. Therefore, applying our previous reasoning to $p_b^{(t,x)}(y)$, we have that $p_b^{(t,x)}(y) \in \mathrm{Dom}(H_b)$ if and only if $\sum_{n=1}^{\infty} (\lambda_n^b e^{-\lambda_n^b t} \varphi_n^b(y))^2 < \infty$.

Using the estimate from Lemma 6.1.1 and our estimates in the proof of part (1) of the theorem, we have that

$$\sum_{n=1}^{\infty} (\lambda_n^b e_n^{-\lambda_n^b t b}(y))^2 \leq K \sum_{n=1}^{\infty} n^{4/d_S + 2\alpha/d_S} e^{-4/d_S c_1 t}$$

for constant $K > 0$ and c_1 as in Lemma 6.1.1, and by Lemma 6.1.2, we have that the sum on the right hand side of the above equation converges uniformly on $[T, \infty)$ for $T > 0$, and thus $p_b^{(t,x)}(y) \in \mathrm{Dom}(H_b)$ with $H_b(p_b^{(t,x)}(y))$ given by (6.3). This further shows that $H_b^{(t,x)}(y)$ is continuous and thus by Corollary 4.2.1.1, $p_b^{(t,x)} \in \mathrm{Dom}(H_b) \cap \mathcal{D}_\mu$.

We now proceed to proving claim (3), that $p_b(t,x,y) \in C^1((0,\infty))$ as a function of t. Since the infinite sum in (6.3) converges uniformly, we have that for $t_1 < t_2$ in $(0,\infty)$

$$\int_{t_1}^{t_2} -(H_b(p_b^{(t,x)}))(y)dt = \int_{t_1}^{t_2} \sum_{n=1}^{\infty} -\lambda_n^b e^{-\lambda_n^b t} \varphi_n^b(x)\varphi_n^b(y)dt$$
$$\sum_{n=1}^{\infty} \int_{t_1}^{t_2} -\lambda_n^b e^{-\lambda_n^b t} \varphi_n^b(x)\varphi_n^b(y)dt = p_b(t_2,x,y) - p_b(t_1,x,y)$$

Thus demonstrating that $p_b(t,x,y) \in C^1((0,\infty))$ with respect to t, as desired. Combining the previous integral identity $p_b(t,x,y)$ with equation (6.3), we then immediately get assertion (4) that

$$\frac{\partial p_b(t,x,y)}{\partial t} = (\Delta_\mu p_b^{(t,x)})(y)$$

Lastly, it remains to prove that $p_b(s+t,x,y) = \int_K p_b(s,x,x')p_b(s,x',y)d\mu(x')$. So consider

$$\begin{aligned}
&\int_K p_b(s,x,x')p_b(s,x',y)d\mu(x') \\
&= \int_K \left[\sum_{n=1}^{\infty} e^{-\lambda_n^b t}\varphi_n^b(x)\varphi_n^b(x')\right]\left[\sum_{n=1}^{\infty} e^{-\lambda_n^b s}\varphi_n^b(x')\varphi_n^b(y)\right] d\mu(x') \\
&= \int_K \sum_{n=1}^{\infty}\sum_{m=1}^{n} e^{-\lambda_m^b t}\varphi_n^b(x)\varphi_n^b(x')e^{-\lambda_{n-m+1}^b s}\varphi_{n-m+1}^b(x')\varphi_{n-m+1}^b(y)d\mu(x') \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{n} e^{-\lambda_n^b t - s\lambda_{n-m+1}^b}\varphi_n^b(x)\varphi_{n-m+1}^b(y)\int_K \varphi_n^b(x')\varphi_{n-m+1}^b(x')d\mu(x')
\end{aligned}$$

where the last equality follows by the fact that the sum in the definition in p_b converges uniformly. Now, since $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal sequence, we have that $\int_K \varphi_n^b(x')\varphi_{n-m+1}^b(x')d\mu(x') = 1$ if and only if $n+1 = 2m$ and is 0 otherwise. Therefore, upon reindexing $k = 2m = n+1$ the previous expression collapses to

$$\sum_{k=1}^{\infty} e^{-(t+s)\lambda_k^b}\varphi_k^b(x)\varphi_k^b(y) = p_b(t+s,x,y)$$

And thus, we have the desired equality, completing the proof of the last assertion of the theorem. □

6.2 Heat Kernel Embeddings into Hilbert Spaces

In the final section of this thesis, we turn to a common application of the heat kernel in the theory over manifolds. Given a smooth n-dimensional Riemannian manifold M, we have that for each $t \in (0, \infty)$ there exists and embedding $\psi_t : M \to H$ for a Hilbert space H, and that the pull-back metric $(\psi_t)^*$ of the embedding is asymptotic to the original metric g on M as $t \to 0$ [3] Originally proven by Berrard, Besson, and Gallot in [3], we present here an adaptation of the original construction for manifolds to construct an embedding of a self-similar set K into a Hilbert space, notably $\ell^2(\mathbb{R})$.

Theorem 6.2.1. *Let (K, d) be a post-critically finite self-similar set with metric d and let $\{^b_n\}_{n=1}^\infty$ be a basis of eigenfunctions of Δ_μ. Then, we define the family of maps for $t > 0$ with respect to this basis*

$$\psi_t^b : K \to \ell^2(\mathbb{R})$$
$$\psi_t^b(x) = \sqrt{2}(4\pi)^{d_S/4} t^{((d_S+2)/4)} \{e^{-\lambda_n^b t/2} \varphi_n^b(x)\}_{n=1}^\infty$$

For any $t > 0$, we have that ψ_t^b is an embedding of K into $\ell^2(\mathbb{R})$.

Proof. We first show that for $t > 0$ and a fixed eigenfunction basis $\{\varphi_n^b\}_{n=1}^\infty$, the map ψ_t^b is continuous. To prove this, it suffices to show that

$$\Psi_t^b : K \to \ell^2(\mathbb{R})$$
$$\Psi_t^b(x) = \{e^{-\lambda_n^b t/2} \varphi_n^b(x)\}_{n=1}^\infty$$

is continuous in $(t, x) \in (0, \infty) \times K$. To this end, take a sequence $\{(t_n, x_n)\}_{n=1}^\infty \subseteq (0, \infty) \times K$ converging to $(t, x) \in (0, \infty) \times K$ as $n \to \infty$. Then,

$$||\Psi_{t_n}^b(x_n) - \Psi_t(x)||_{\ell^2}^2 = \sum_{n=1}^\infty |e^{-\lambda_n^b t_n/2} \varphi_n^b(x_n) - e^{-\lambda_n^b t/2} \varphi_n^b(x)|^2$$
$$= \sum_{n=1}^\infty |e^{-\lambda_n^b t_n} \varphi_n^b(x_n)^2 - 2e^{-\lambda_n^b (t_n+t)/2} \varphi_n^b(x_n)\varphi_n^b(x) + e^{-\lambda_n^b t \varphi_n^b(x)^2}$$
$$= p_b(t_n, x_n, x_n) - 2p_b(\tfrac{t_n+t}{2}, x_n, x) + p_b(t, x, x)$$

But since p_b is continous over $(0,\infty)\times K^2$, we have that taking the limit of the latter expression as $(t_n, x_n) \to (t, x)$, we get that $||\Psi^b_{t_n}(x_n) - \Psi^b_t(x)||^2_{\ell^2} \to 0$ as $n \to \infty$, demonstrating continuity.

It remains to show that $\Psi^b_t(x)$ is injective. For injectivity, take $x, y \in K$ and let $\Psi^b_t(x) = \Psi^b_t(y)$. Then, for each $n \in \mathbb{N}$

$$e^{-\lambda^b_n t}\varphi^b_n(x) = e^{-\lambda^b_n t}\varphi^b_n(y)$$
$$\varphi^b_n(x) = \varphi^b_n(y)$$
$$x = y$$

since $\{\varphi^b_n\}^\infty_{n=1}$ is a basis, and thus φ^b_n is injective. Both steps demonstrate that Ψ^b_t is an injective homeomorphism, and thus an embedding of K into $\ell^2(\mathbb{R})$. □

Remark 6.2.1. *It is important to note that we can modify Ψ^b_t into an embedding in the unit sphere S^∞ in $\ell^2(\mathbb{R})$ by defining the family of maps*

$$K^b_t : K \to S^\infty$$
$$K^b_t(x) = \frac{1}{\sum_{n=1}^\infty e^{-\lambda^b_n t/2}(\varphi^b_n(x))^2}\{e^{-\lambda^b_n t/2}\varphi^b_n(x)\}^\infty_{n=1}$$

This is adapted from a similar map found in [3].

However, in regards to the pull-back metric, it was proven by Kajino in [5] that as $t \to 0$ the limit of $t^{d_S/2}p_b(t, x, x)$ does not exist over affine nested fractals. This suggests that even though we can construct an embedding of K into ℓ^2 when (K, d) is equipped with a regular harmonic structure and self-similar measure (where d is the original metric on K), the asymptotics of the pull-back metric do not behave similarly to the original metric as $t \to 0$ due to the highly variational nature of the heat kernel near 0. More explicitly, as stated in [5]

Theorem 6.2.2. *Let K be an affine nested fractal with boundary V_0 having greater than 3 vertices. Then the limit*

$$\lim_{t\to 0} t^{d_S/2}p_b(t, x, x)$$

does not exist for any $x \in K \setminus A$ for $A \subseteq K$ a Borel set of measure 0 with respect to a self-similar measure μ containing V_0.

Kajino also focused explicitly on the Sierpinski gasket, and its generalisations the $d-$dimensional level l Sierpinski gasket or the N-polygasket. This is unfortunate, given that nested fractals, of which the Sierpinski gasket is included, exhibit more regularity than post-critically finite fractals or even more general self-similar fractals. This suggests that a completely alternate method is needed to construct a family of embeddings Ψ_t that have more regular asymptotics as $t \to 0$. Perhaps naively this could be achieved by taking a convolution of $p_b(t, x, y)$ with an appropriate smooth function or integrating $p_b(t, x, y)$ around a small neighbourhood of $x \in K$, although more investigation is required before any concrete conclusions can be made.

Chapter 7

Conclusion

Starting with the basic infrastructure of self-similar sets, this work provides a self-contained exposition of self-similar sets and their architecture, as well as the construction of the Dirichlet and Neumann Laplacians as self-adjoint operators over post-critically finite self-similar sets. A detailed exposition of the work of Kigami from [8] and [7] is then explored in regards to creating a pointwise formulation of an analogous classical Laplacian on self-similar sets. The work of Kigami and Lapidus [9] on the spectrum of the Laplacian and the formulation of a Weyl's type law of asymptotics of the Kigami Laplacian is then reviewed. This then leads into a formulation of the heat kernel, and its asymptotics, and a review of the work of Berard, Besson and Gallot in [3] on heat kernel embeddings of manifolds is adapted to the fractal case, and we discuss its shortcomings.

A naturally arising research question from this work then is how one would construct an embedding of a self-similar set K into a Hilbert space (one with relatively more structure), in a way that preserves the original metric over K. There seems to be little work done in this respect, although the work of Kajino [5] does demonstrate the turbulent nature of the variation of the heat kernel near 0, thus suggesting alternative approaches are requiring, meriting this proposed research direction.

Furthermore, stepping back to the construction of the Laplacian, an obvious gap in the current literature pertains to the existence of harmonic structures over self-similar sets. The work of Lindstrøm [11] demonstrates their existence over nested fractals, but given that a lot of the current theory hinges on the existence of harmonic structures, and in particular regular harmonic structures over fractal sets, this would be a crucial area of study, including study of the renormalisation operator presented in Chapter 3.

In addition to this, the author of this thesis is further interested in how one would go about constructing analogous Laplacians for a broader class of fractals, beyond those that are post-critically finite. Given the ubiquity of fractal structures in mathematics and other fields, the study of analysis and diffusion processes over such sets would further merit this as a fruitful area of research. Given the work of Lindstrøm [11], as well as that of Fukushima and Shima [11] in this domain that is more probabilistic in nature, perhaps this would be a more fruitful avenue by which to tackle the questions proposed here.

The author of this thesis hopes that this thesis was successful in being a relatively self-contained exposition on the construction of the Laplacian, its spectrum, and eigenfunctions over self-similar sets and their applications by detailing the already existing strong theoretical results currently available. The author further endeavours that this thesis ignites further research and interest in analysis on fractals, and how to further expand the existing theory given its detailed review of the promising results currently available and outlining the wealth of possible questions that naturally arise from them.

Bibliography

[1] M.T. Barlow. Diffusion on fractals, 1998.

[2] Shima T. Fukushima, M. On a spectral analysis for the sierpinski gasket. *Potential Anal 1*, page 1–35, 1992.

[3] Besson G. Bérard P. Gallot, S. Embedding riemannian manifolds by their heat kernel. *Geometric and functional analysis*, 4(4):373–398, 1994.

[4] D. S. Grebenkov and B.-T. Nguyen. Geometrical structure of laplacian eigenfunctions. *SIAM Review*, 55(4):601–667, 2013.

[5] Naotaka Kajino. On diagonal oscillation of the heat kernels on post-critically finite self-similar fractals. *Probability Theory and Related Fields, Springer*, pages 51–74, 06 2013.

[6] J. Kigami. Harmonic calculus on limits of networks and its application to dendrites. *Journal of Functional Analysis*, 128(1):48–86, 1995.

[7] Jun Kigami. Harmonic calculus on p.c.f. self-similar sets. *Transactions of the American Mathematical Society*, 335(2):721–755, 1993.

[8] Jun Kigami. *Analysis on Fractals*. Cambridge Tracts in Mathematics. Cambridge University Press, 2001.

[9] Jun Kigami and Michel Lapidus. Weyl's problem for the spectral distribution of laplacians on p.c.f. self-similar fractals. *Communications in Mathematical Physics*, 158:93–125, 05 1993.

[10] Michel L. Lapidus. Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the weyl-berry conjecture. *Transactions of the American Mathematical Society*, 325(2):465–529, 1991.

[11] Tom Lindstrøm. Brownian motion on nested fractals. *Memoirs of the American Mathematical Society*, 420, 01 1990.

[12] Robert S. Strichartz. *Differential Equations on Fractals: A Tutorial*. Princeton University Press, 2018.

[13] Feller W. *An Introduction to Probability Theory and Its Applications.* Wiley;, third ed. edition, 1968.