Nonperturbative Regulators for Supersymmetric Theories in 3 and 4 Dimensions

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DEDICATION

This dissertation is dedicated to my parents, to my siblings and to the Sweaters family.

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I am very much indebted to my advisor Guy D. Moore for his seemingly endless patience, near constant availability and unfailing (hopefully) confidence in my work. Special thanks also to the rest of the physics faculty and my fellow students, especially those with the dubious honor of sharing my office space, for providing a comfortable environment in which to conduct research. Special thanks also to the people of the physics department at the University of Chicago who were kind enough to house me temporarily while I did much of this work. Special thanks to Argonne National Lab - and Dr. Rao Kotamarthi specifically - for providing me a work space during the last month while I did much of the writing for this dissertation. Very special thanks to the secretarial staff and network administrators of the department at McGill for answering all my questions with patience. I'd like to thank my friends and family of course. Finally, thanks to my amazing girlfriend for being an amazing girlfriend and for marrying me and becoming my amazing wife, even though I have so much student loan debt.

ABSTRACT

We show that all fundamental barriers to simulations of various supersymmetric field theories in 3 and 4 dimensions with a lattice regulator can be removed with known and established methods and provide detailed procedures to accomplish this end for $\mathcal{N}$=1 Super-Yang-Mills theory in 3 dimensions and $\mathcal{N}$=4 Super-Yang-Mills theory in 4 dimensions. We also describe generalizations to various other 3 and 4D theories with varying levels of detail where appropriate and analyze a novel new approach to lattice supersymmetry: discretization of a particular topological twist of Super-Yang-Mills in 2 and 4 dimensions.

ABRÉGÉ

Nous montrons que les obstacles fondamentaux des simulations de différents domaines théoretiqe de champs supersymétriques de 3 et 4 dimensions avec Régularisation sur réseau peuvent être surmonté avec méthodes établies, et nous fournissons des procèdures dètaillées pour le théorie $\mathcal{N}$=1 Super-Yang-Mills en 3 dimensions et $\mathcal{N}$=4 Super-Yang-Mills en 4 dimensions. Nous décrivons, avec diffèrents niveaux de détail, les généralisations à divers autres thèories de 3 et 4D, et nous analysons un approche de réseau supersymétrie: discrétisation d'une torsion topologique de Super-Yang-Mills dans 2 et 4 dimensions.

STATEMENT OF ORIGINALITY

This dissertation is based on work performed over the course of the last 3 years. Much of section 1.2 is taken from an unpublished review of lattice gauge theory on which I collaborated with my advisor Guy D. Moore. Chapter 2 is based on a paper published in collaboration with Guy D. Moore [1]. Much of the original idea for this project was due to Guy D. Moore, though I fleshed out most details and did most of the writing. Chapter 3 is all material original to this dissertation. Chapter 4 is based on work that was done in collaboration with Guy D. Moore and Joel Giedt over the last year. Nearly all the writing is original to the author; the manuscript is a heavily expanded version of the one submitted for publication [2] and so bares little resemblance to that work. The ideas for this work were inspired through conversations amongst the collaboration, though I was responsible for fleshing out much of the detail. Any calculations not originally performed by the author were also conducted independently as a check on that work.

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CHAPTER 1
Introduction

1.1 Continuum Supersymmetry

Supersymmetry (SUSY) has been postulated to be a key element of beyond-the-standard-model physics for over 30 years. Since the first applications of graded lie algebras ([3], an early review) to the first SUSY extensions of the Poincaré group [4, 5] and their applications to gauge theories [6, 7] - both in the early 70s - to the proposal of so-called soft SUSY breaking (described in more detail below) and the construction of the minimaly supersymmetric extension of the standard model (MSSM) [8] in the early 80s, supersymmetry has fascinated and surprised physicists with its phenomenological depth, its usefulness and its theoretical simplicity. It is often relied upon as an assumption in more ambitious theories of quantum gravity (such as string theory) and may in fact be a necessity for the consistent formulation of some or all of these theories. More importantly, as the low energy manifestation of one or another consistent theory of quantum gravity, it represents perhaps the first available experimental signature of these theories. Observation of a manifestation of one or another supersymmetric extension to the standard model in reality would provide the high energy physics community with the first wholly new and unambiguous evidence that we're "on the right track" in our quest to unify relativity with the standard model. Indeed, its been stated (somewhat anecdotally perhaps) that the highest percentage of physicists (by about 3 to 1 over its closest rival) believe that supersymmetry will be the next discovery on the march toward higher and higher energy scales in collider experiments, and a key element in the final theory of quantum gravity.

Supersymmetry is also very important to phenomenologists. The lightest supersymmetric partner (LSP) is by far considered to be the best candidate for a weakly interacting massive particle (WIMP) [9] which is, in turn, the best candidate for explaining the apparent gravitational overabundance of matter in the universe (this is often referred to as the dark matter problem).

At the same time, supersymmetry is more often than not the major theoretical component in proposed solutions to the infamous hierarchy problems of cosmology and particle physics. Any quantum theory that purports to explain low energy particle physics from some more fundamental (and thus higher energy scale) set of principles than those that comprise the standard model must necessarily face a problem of finely tuning quantum loop corrections so that parameters of the standard model, particularly the mass of the Higgs boson, are kept low enough to allow for the theoretical mechanisms of electroweak symmetry breaking, a cornerstone of the standard model, to work appropriately. Supersymmetry is similarly invoked often to try and mediate the larger, and perhaps more infamous, hierarchy problem of the cosmological constant in cosmology. This cosmic energy density should be a direct signature of the fundamental theory of quantum gravity (whatever it may be), and so one naively expects its scale to by set by the Planck mass, which gives an estimate of the density around $10^{120}(eV)^4$, a number that people love to quote for its sheer absurdity. Supersymmetry might ameliorate the lion's share of this fine tuning as well, though there is significantly more mystery and ambiguity surrounding this side of the fine tuned renormalizations problem (or hierarchy problem).

For a review of nonperturbative phenomena in 4-D theories see [10]. Extra symmetries and holomorphy in SUSY greatly constrain the dynamics of these theories, allowing for the study of non-perturbative phenomena. Chiral symmetry breaking [11] and confinement are examples of interesting non-perturbative phenomena for which exact statements can be made in SUSY theories. From this we can hope to learn things applicable to real life QCD where the non-perturbative dynamics are far too complicated to make exact statements.

One of the most compelling motivations for finding and implementing a nonperturbative regulator for supersymmetric theories (and thus allowing for the nonperturbative determination of correlation functions) is the study of certain theoretical dualities (or dual descriptions) that appear most often in these theories. These include early proposals of dualities between branches of particular 3D SUSY theories called mirror symmetries [12, 13, 14, 15, 16, 17]. These are potentially useful theoretical tools, and their study might shed light on other dual descriptions. Since the conjecture by Juan Maldecena of a correspondence (same thing as a duality or dual description) between strongly coupled $\mathcal{N}$=4 SYM theory, a conformal gauge field theory or CFT, and weakly coupled gravity on an AdS (Anti de Sitter space) background [18, 19, 20, 21], this field is a (probably the most) rapidly progressing cutting edge field in high energy theory. Moreover, gravitational duals may exist for all or most quantum field theories, and exhaustive study of these dualities may even provide (or provide clues to) the necessary parameter simplifications and theoretical mechanisms to uncover the TOE (theory of everything). Lattice simulations of the gauge theory side of these dualities are such an attractive opportunity that researchers have even tried to construct gravitational duals of lattice gauge theories themselves [22] or simulate the gauge duals of specific configurations of zero dimensional (D0) branes in type II string theory [23, 24] (this is doable with D0-branes because the dual description is zero dimensional, *i.e.* quantum mechanics).

As if this wasn't enough to guarantee that supersymmetry retain the title of darling theory of physics and physicists in the coming decades, applications for it have been found accross the discipline in statistical field theory, quantum mechanics [25], and elsewhere.

1.1.1 The basic physics

For Dirac matrix conventions - including conventions on charge conjugation, Majorana spinors, Chiral symmetry, etc. - see section 4.1, where we derive the continuum action and SUSY transformations of 4D $\mathcal{N}$=4 supersymmetry by compactifying the 10D $\mathcal{N}$=1 theory on a 6D toroid. Unless explicitly stated, our conventions follow those of Weinberg [26].

Supersymmetry in its most basic form is simply a symmetry that exchanges fermions and bosons. In order to accomplish this exchange between particles whose spins differ by $\frac{1}{2}$ the generators of the SUSY must themselves be anti-commuting and spin-$\frac{1}{2}$. This implies that the generators must transform non-trivially under the action of the Poincaré group. Supersymmetry must then be a non-trivial extension of the Poincaré algebra, i.e. a new space-time symmetry. The no-go theorem of Coleman and Mandula, which we will not prove here but is described nicely in [26], indicates that there can be no mixing between internal and space-time symmetries when the generators of the symmetries all satisfy a Lie algebra. Supersymmetry evades this constraint because the generators instead satisfy a graded (or super) Lie algebra.

The most instructive way to view the effects of the SUSY extension of the standard Poincaré algebra is to treat the SUSY generators, say Q and $\bar{Q}$ schematically, as translation operators along some new fermionic spatial directions labelled by Grassman valued coordinates θ and $\bar{\theta}$. One can then construct a field theory on this new space-time as a theory of *superfields*, fields that live in both the standard d bosonic dimensions and the newly introduced fermionic ones. The finite SUSY transformation of these fields are then written in direct analogy to standard translations such that a general translation in the new theory is $\exp\Big(i\theta Q + i\bar{Q}\bar{\theta} - ix_\mu P^\mu\Big)$. One can therefore interpret SUSY as a generalization of the usual space-time to include fermionic, as well as the usual bosonic, spatial dimensions.

To be more precise, the Poincaré algebra

$$\begin{aligned}
\Big[P^\mu, P^\nu\Big] &= 0\,, \\
\Big[J^{\mu\nu}, J^{\alpha\beta}\Big] &= i\Big(\eta^{\nu\beta}J^{\mu\alpha} + \eta^{\nu\beta}J^{\mu\alpha} - \eta^{\nu\beta}J^{\mu\alpha} - \eta^{\nu\beta}J^{\mu\alpha}\Big)\,, \\
\Big[J^{\mu\nu}, P^\alpha\Big] &= i\Big(\eta^{\mu\alpha}P^\nu - \eta^{\nu\alpha}P^\mu\Big)\,,
\end{aligned} \tag{1.1}$$

(the algebra of translations, generated by P^μ, and of the Lorentz transformations, generated by $J^{\mu\nu}$) is extended with

$$\begin{aligned}
\Big[Q, J^{\mu\nu}\Big] &= \sigma^{\mu\nu}Q \\
\Big\{Q, \bar{Q}\Big\} &= 2\,\gamma{\cdot}P
\end{aligned} \tag{1.2}$$

(both Q and $\bar{Q}$ commute with P).

Supersymmetry requires that particles and their super-partners, particles with all the same quantum numbers (including mass) except for spin, be gathered together in multiplets much the same as any internal symmetry. These multiplets must contain the same number of bosonic and fermionic degrees of freedom. Schematically the multiplets are constructed in the same familiar way that one would solve the quantization of angular momentum with raising and lowering operators. For some multiplet there will exist a state with maximum helicity, say λ_{max}, such that $\bar{Q}|\lambda_{max}\rangle = 0$. The helicity raising operator, $\bar{Q}$, is directly proportional to the SUSY generator, thus justifying the notation. Starting from this we can use the lowering operator, say Q, to obtain every other state in the multiplet by

$$Q|\lambda_{max}\rangle \propto |\lambda_{max} - \frac{1}{2}\rangle \,. \tag{1.3}$$

For simple SUSY (N=1) that's the whole story since

$$\{Q, Q\} = 0 \longrightarrow Q^2 = 0 \,. \tag{1.4}$$

So if we act on the state with maximum helicity twice, we get zero. In extended SUSY (N>1) there are N sets of SUSY generators so that N half-integer steps can be made.

For the phenomenology of global supersymmetry with massless particles, it is only necessary to consider multiplets with helicity less than or equal to one, such that two kinds of multiplets can be constructed. Where a multiplet is not CPT self-conjugate, we are obliged to combine it with its conjugate multiplet to construct a CPT invariant theory (since CPT flips helicity). The first is the vector multiplet (plus CPT conjugate) which, for simple SUSY, contains a gauge boson of helicity ± 1 and a two component Weyl fermion (the gaugino) with helicities $\pm\frac{1}{2}$. Since the gaugino resides in the same super-multiplet as a gauge boson, it must transform under the adjoint representation of the gauge group. The field theory of a vector multiplet is called super Yang-Mills (SYM).

The other possibility is a multiplet with maximum helicity 1/2 and its CPT conjugate. Its particle content is thus a two component Weyl fermion and two real scalar fields that we can combine into one complex scalar. We will refer to this construction as a matter multiplet since it can be made to transform under the fundamental representation of a symmetry group, and is thus necessary to incorporate matter into a SUSY theory. In general, for SUSY theories with any value of N, multiplets consist of helicities $\{\lambda_{max}, \ldots, \lambda_{max} - N/2\}$.

Since SUSY is a symmetry that exchanges bosons and fermions, it is not surprising that there should be some relationships between the parameters in a SUSY Lagrangian. The superspace formulation allows for the construction of SUSY Lagrangians in a simple way and provides a simple understanding of this fact. We will give only a brief explanation of how to extract SUSY Lagrangians from the superspace formalism; more detailed descriptions and derivations can be found in any number of places including [26, 27].

The basic object in the formalism is the superpotential $f(\Phi)$, which is a function of the superfields. Superfields can be expanded in terms of bosonic and fermionic functions of the regular space-time coordinates, which naturally gives rise to the multiplet structure discussed above. We can thus hope to construct actions in d bosonic dimensions with phenomenological implications by taking some combination of superfields in the full space and integrating out the dependence on the fermionic coordinates. In practice this can be accomplished quite simply.

As an example we will take the superpotential

$$f(\Phi) = \sum_{ijk} \lambda_{ijk} \Phi_i \Phi_j \Phi_k \,. \tag{1.5}$$

Here we take the indices to be of some arbitrary label that distinguishes the fields. The Yukawa and scalar interactions for the components of these superfields can then be found to be (I use the notation of 4 component Dirac and Majorana spinors throughout this work)

$$\mathcal{L}_{int} = \left(\sum_{ij} \left(\frac{\partial^2 f(\phi)}{\partial\phi_i \partial\phi_j} \right) \bar{\psi}^i_L \psi^j_L + \text{h.c.} \right) + \sum_i \left| \frac{\partial f(\phi)}{\partial\phi_i} \right|^2 , \tag{1.6}$$

where we have now used ϕ instead of Φ to indicate that the superfields in the functional derivative should be replaced by the scalar fields defined in the bosonic space-time coordinates. It is easy to see that this gives scalar self couplings with coefficients $\sim\lambda^2$ and Yukawa couplings $\sim\lambda$. I will take this as motivation for the equality of couplings and move on now to the phenomenological implications.

1.1.2 Cancellation of divergences

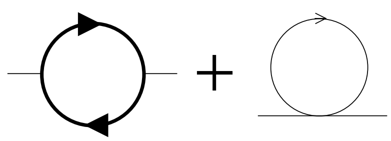


Figure 1–1: A simple example of divergence cancellation in a SUSY theory. The heavy lines with heavy arrows are fermionic, the light lines with light arrows are scalars.

Let's consider the contribution to the inverse 2-point function of a scalar field evaluated at zero external momentum in a theory of scalars and fermions to get a feeling for how supersymmetry ensures the cancellation of problematic divergences in quantum field theory. Fig. (1–1) shows the two diagrams that contribute to the scalar self-energy at one loop in a theory of fermions and scalars. The first diagram contributes

$$\Pi_\phi(\text{fermion}) = -n_f \lambda_y^2 \int_p \text{Tr} \left(\frac{(-i\not{p}) P_L (-i\not{p}) P_R}{p^2 p^2} \right) . \tag{1.7}$$

Here $P_{L,R}$ are the left and right chiral projectors, $\frac{1}{2}(1\pm\gamma_5)$. In 4-D this is a quadratic UV divergence. The second diagram in Fig. (1–1) is also UV divergent:

$$\Pi_\phi(\text{scalar}) = -2n_s\lambda_s^2 \int_p \frac{1}{p^2}\,. \tag{1.8}$$

Adding these together we find

$$\Pi_\phi(\text{tot}) = \Big(2n_f\lambda_y^2 - 2n_s\lambda_s^2\Big) \int_p \frac{1}{p^2}\,. \tag{1.9}$$

In a SUSY theory then, where the number of fermions equals the number of scalars ($n_f{=}n_s$) and the couplings are equal, the quadratic divergences exactly cancel.

1.1.3 SUSY Ward identities in the continuum

The expectation value of an operator in a QFT with Euclidean signature is formally given by the functional integral

$$\langle\mathcal{O}\rangle = Z^{-1}\int \mathcal{D}[X]\mathcal{O}\; e^{-S[X]}\,, \tag{1.10}$$

with X here representing the set of all local field variables in the theory and Z the partition function. Since we expect observables to be invariant under infinitesimal SUSY variations, we can write

$$\delta_\xi\langle\mathcal{O}\rangle = Z^{-1}\int \mathcal{D}[X]\;\delta_\xi\Big(\mathcal{O}\; e^{-S[X]}\Big) = 0 \tag{1.11}$$

where we have used that both the path integral measure and partition function are invariant under the SUSY transformation. Since the variation in the action is a total divergence (of the supercurrent), we have that

$$\delta_\xi\langle\mathcal{O}\rangle = \langle\delta_\xi\mathcal{O}\rangle - \langle(\partial_\mu S_\mu)\mathcal{O}\rangle = 0\,. \tag{1.12}$$

If we further restrict ourselves to operators localized outside the region of the local infinitesimal SUSY transformation ($x \neq y$) we have $\delta_\xi \mathcal{O} = 0$ and thus

$$\langle \partial{\cdot}S_x \, \mathcal{O}_y \rangle = 0 \, . \tag{1.13}$$

1.1.4 Spontaneous SUSY breaking

It is still uncertain whether the experimental energy levels attained by the Large Hadron Collider (LHC) will reach that required to see any evidence of supersymmetry in the data that will begin pouring in over the next few years. The situation is uncertain because we lack a robust understanding of the mechanisms of spontaneous supersymmetry breaking and are thus unable to say too much about the low energy spectrum of the theory. Part of the reason for this is that many of the most interesting models of spontaneous SUSY breaking rely on nonperturbative mechanisms and so they are difficult to study analytically.

The MSSM contains 120 tunable parameters, nearly all of which are mass and trilinear coupling parameters that are forbidden from the theory only by the SUSY itself and so can be generated radiatively by one or another generic UV SUSY breaking mode (presumed to come from some higher energy dynamics). These terms all break SUSY *softly*, meaning all SUSY breaking parameters appear in the Lagrangian with positive mass dimension (only relevant and *not* marginal terms are generated) Without this restriction the ability of supersymmetry to solve the Higgs hierarchy problem is somewhat restricted or alltogether ruined (depending on your prospective on the hierarchy problem).

Why is it so hard to break SUSY? One of the reasons is that nonrenormalization theorems [28, 29] imply that if supersymmetry is not broken at tree level, then it won't be broken dynamically to any order in a loop expansion.

That's a pretty powerful statement; we can certainly arrange for SUSY to be broken at tree level, but then the scale of the SUSY breaking is set by dimensionful parameters exlicitly in the action, and those are presumed to be at or near the Planck scale. SUSY may very well be broken at these high energies; if so however, it would fail to solve the most important problems in its repertoire: it wouldn't solve the hierarchy problems, it wouldn't give us a good WIMP and it wouldn't give us information about itself or an eventual TOE through observations at the LHC (or any other collider we can imagine being built for a very long time).

In 1981 Witten discovered a new possible source of low energy dynamical SUSY breaking [30] in the vacuum effects of certain nonperturbative phenomena. The potential vacua of the theory may very well be supersymmetric (meaning SUSY is *not* spontaneously broken by the vacuum state) at tree level, and indeed to any loop order in a perturbative expansion (as required by nonrenormalization theorems), with SUSY broken by instantons, purely nonperturbative objects in quantum field theory. Agood review is [31].

The difficulty with Witten's proposal is precisely the fact that the mechanism is nonperturbative and so can not be studied in the usual ways. Some methods have been devised for analysing the potential of SUSY breaking in a particular theory, such as by calculation of the supersymmetric (or Witten) index. Witten has managed to do this calculation in three [32] and four [33] dimensions, for many interesting theories (see Chapter 2 for more details), and has conjectured the existence of dynamical nonperturbative SUSY breaking in some of these theories, but no conclusive evidence has yet been found.

What if we could solve these theories in a nonperturbative way? Enter lattice gauge theory.

1.2 Lattice Gauge Theory

A number of introductions to lattice gauge theory exist; for instance, there is a very useful book by H. Rothe [34], which explains most of the concepts and technology. There are also shorter papers by a number of authors: [35] is an old but clear review of the ideas of lattice gauge theory well suited to individuals with a background in statistical mechanics. Other reviews of LGT, numerical LGT techniques and specific LGT concepts in general and in QCD in particular include [36, 37, 38, 39, 40, 41, 42]. The discussion in this section is mistly in 4D, so that comparisons to the bulk of lattice literature is straightforward. The generalization of results to other dimensions is mostly trivial.

The idea of a lattice calculation is to look at some field theory problem which can be formulated in terms of a Euclidean path integral, say,

$$\frac{1}{Z}\int \mathcal{D}\Phi\ \mathcal{O}_1(\Phi)\mathcal{O}_2(\Phi)\, e^{-\int d^4x \mathcal{L}_{\mathrm{E}}(\Phi)}\,, \tag{1.14}$$

and then actually "do the integral," for instance by some numerical Monte-Carlo procedure. This is an interesting thing to do when the action is so non-Gaussian that analytic techniques, such as perturbation theory, are not reliable. It allows us to compute from first principles nonperturbative phenomena. For the current discussion we will talk about scalar Φ^4 theory, to avoid issues about gauge invariance and special complications associated with fermions. We will return to these special complications, since, in fact, the only reason that lattice techniques are interesting is that QCD is asymptotically free, and this actually demands that we discuss gauge invariance.

The problem is that $\mathcal{D}\Phi$ really means an integral over the value of Φ, for every point in space. This is an infinite number of integrals. There is also a serious issue of renormalization that is being swept under the rug in the above discussion, which has to be dealt with.

The first step is to reduce the amount of work potentially needed, by making the spacetime considered finite. Replacing $\int_{-\infty}^{\infty} d\tau$ with $\int_0^{N/m} d\tau$, with $N \gg 1$ and m the mass gap, is good enough because any Euclidean correlation functions that we'll be interested in will be exponentially suppressed outside this region. The same is true of the spatial extent; compactifying space with $L \gg 1/m$ leaves only exponentially small sensitivity to L, provided that

1. we choose a space with no boundaries (otherwise the regions near boundaries are messed up by boundary effects),
2. we choose a space with zero (metric) curvature (otherwise there is something local which tells you the space is finite).

a torus (or twisted torus) is an ideal choice. We make it now.

Next is the radical step; we replace continuous space with a lattice, so there are only a finite number of points, and therefore a finite number of integrations to be performed. So the integral we want turns into

$$\int \mathcal{D}\Phi \Rightarrow \int \prod_{x_1=[a,\ldots,L];\; x_2=\ldots} d\Phi(x_1,x_2,x_3,\tau)\,, \tag{1.15}$$

which is a finite number $(L/a)^3(\tau/a)$ of integrations. That means it is at least conceptually possible to do the integrals by some Monte-Carlo procedure. The next few sections will be devoted to the subtleties and difficulties of deriving the appropriate tree level lattice action to study a particular continuum quantum field theory.

1.2.1 The basics of discretization

The choice of a lattice action is not at all unique. Any local action which reduces to the continuum action in the limit $a \to 0$ can be used. It is easy to translate ultralocal potential terms to the lattice by simply exchanging the continuum fields for lattice variables (fields defined only at the lattice sites) and treating the parameters as bare parameters

$$\int d^4x V(\Phi(x)) \Rightarrow a^4 \sum_x V_{\rm bare}(\Phi(x)) \,. \tag{1.16}$$

Derivatives are slightly more tricky because we cannot of course take infinitesimal differences on a discrete space. Derivatives will therefore become finite differences on the lattice. How we construct these differences depends on the specific theory and its symmetries. The simplest solution is

$$\frac{1}{2}(\nabla\Phi)^2(x) \quad \Rightarrow \quad \frac{1}{2}\sum_\mu \left[\frac{\Phi(x+a\hat{\mu}) - \Phi(x)}{a}\right]^2 \,, \tag{1.17}$$

$$\int_x \quad \Rightarrow \quad a^2 \sum_{x,\mu} \left[\Phi^2(x) - \Phi(x+a\hat{\mu})\Phi(x)\right] \,. \tag{1.18}$$

In the second expression we have re-arranged some terms, which is equivalent in the continuum to integrating by parts. The term with $\Phi(x+a\hat{\mu})\Phi(x)$ is often called a “hopping term.”

The right IR effective theory is the most general theory we can write down, with the field content and symmetries of the lattice theory. The parameters of that effective theory must then be determined by a matching calculation. The symmetries are

- Discrete $\Phi \to -\Phi$ symmetry; Φ always appears in even powers in any operator in the Lagrangian.
- Discrete translation symmetry
- Hypercubic point symmetry

Discrete translation symmetry is what we have instead of continuous translation symmetry. Rather than ensuring momentum conservation, it ensures momentum conservation up to $2\pi/a$ (*Umklapp*). This is good enough since the IR effective theory only deals with small momenta $p \ll 1/a$; in practice discrete translation symmetry is just as good as full translation symmetry. However, we will have to remember this *Umklapp* possibility when we think about fermions on the lattice.

Hypercubic symmetry is that subgroup of $O(4)$ which takes the hypercubic lattice (points with all integer coordinates) to itself. That is the same as permutations of the (x, y, z, τ) axes, with possible sign flips on each axis. The group therefore has $4!\,2^4 = 384$ (in general, $D!\,2^D$) elements. It is NOT $O(4)$ (Euclidean rotation) invariance. Since it is a smaller invariance it less severely constrains what operators can appear.

Is it good enough?

To answer this, let's list some operators and see if they are allowed under hypercubic symmetry.

$$(\partial_1\Phi)^2 \qquad \text{no} \qquad \text{breaks } (x_1 \leftrightarrow x_2) \tag{1.19}$$

$$\sum_\mu (\partial_\mu\Phi)^2 \qquad \text{yes} \tag{1.20}$$

$$\sum_\mu (\partial_\mu\Phi) \qquad \text{no} \qquad \text{breaks } (x_1 \leftrightarrow -x_1) \tag{1.21}$$

$$\sum_\mu (\partial_\mu\Phi)^4 \qquad \text{yes} \tag{1.22}$$

The last of these is something impossible under $O(4)$ symmetry.

Generally,

- $O(4)$: no "hanging" Lorentz indices, Lorentz indices summed in pairs;

- Hypercubic: no "hanging" Lorentz indices, Lorentz indices summed in even numbers (that is, μ can appear 2 times, 4 times, etc.)

There are operators allowed by hypercubic but not $O(4)$ invariance, but they must have at least 4 derivatives; the first such operator is

$$\sum_\mu \Phi \partial_\mu^4 \Phi \tag{1.23}$$

(and others related to it by integration by parts). This is dimension 6, and so nonrenormalizable. (It is actually present at tree level in the derivative operator we constructed; even if it weren't it would get generated radiatively.)

However, it just happens that, when we list all *renormalizable* and hypercubic symmetric operators, that the set of operators we can write also display full $O(4)$ invariance. Therefore, $O(4)$ invariance is recovered in the infrared as an accidental symmetry (rather as parity is recovered as an accidental symmetry of QCD, even though it is absent in the Standard Model). This means that the lattice treatment IS "good enough;" differences between the lattice and continuum values of correlation functions will vanish as the second power of (ap), so the theories are the same in the infrared (which is the most we could have asked for).

We emphasize that renormalization will occur between lattice and continuum effective theories; in particular the mass term additively renormalizes, $m^2_{\rm eff} \sim \lambda/a^2 + m^2_{\rm bare}$. This makes taking the continuum limit at fixed physical m^2 quite challenging. However, such mass generation is protected from occurring additively by symmetries in QCD, so this is one way that treating QCD is actually easier than scalar field theories on the lattice.

1.2.2 The gauge action

First consider the action,

$$S = a^2 \sum_{x,\mu} \Big[2\Phi^\dagger(x)\Phi(x) - \Phi^\dagger(x)\Phi(x+a\hat{\mu}) - \Phi^\dagger(x+a\hat{\mu})\Phi(x) \Big] \, , \qquad (1.24)$$

where we have "integrated by parts" for convenience. This action is invariant under a global symmetry transformation,

$$\Phi \rightarrow G\Phi \ , \ \Phi^\dagger \rightarrow \Phi^\dagger G^\dagger \qquad \text{provided} \qquad G^\dagger = G^{-1} \, . \qquad (1.25)$$

If we now gauge this symmetry, that is, let $G \rightarrow G(x)$, this will no longer be the case. Specifically,

$$\Phi^\dagger(x)\Phi(x+a\hat{\mu}) \rightarrow \Phi^\dagger(x) G^{-1}(x) G(x+a\hat{\mu}) \Phi(x+a\hat{\mu}) \neq \Phi^\dagger(x)\Phi(x+a\hat{\mu}) \, . \quad (1.26)$$

This term is then not gauge invariant. The most obvious and direct solution is to replace the term in the action with

$$\Phi^\dagger(x) \; U \; \Phi(x+a\hat{\mu}) \qquad (1.27)$$

and let $U \rightarrow G(x) \; U \; G^{-1}(x+a\hat{\mu})$ under the action of the gauged symmetry transformation. Since the transformation of the U matrices involves the *local* tranformation properties of two neighboring lattice sites, it is natural to think of them as "living" on the link in between. They are thus referred to as link matrices and labelled by the two sites, $U{=}U(x, x{+}a\hat{\mu})$ or $U_\mu(x)$.

For the gauge theory constructed in this way to be unitary, we must also impose the condition that

$$U^\dagger_\mu(x) = U^{-1}_\mu(x) \, . \qquad (1.28)$$

This implies that U must (for any non-trivial symmetry transformation) have an orientation, so that the inverse operation can be defined on the same link as the action of the link matrix in the opposite direction. We can thus say that $U_\mu(x)$ transports Φ at site $x+\mu$ to site x (by convention) and $U_\mu^\dagger(x)$ transports it back. These elements are enough to present a definite prescription for constructing gauge invariant lattice field theories: fields evaluated at different lattice sites must be connected by links to be gauge invariant.

It now remains to make the link matrices themselves dynamical field variables by constructing an action that will reproduce the familiar field strength term in the continuum limit. This term must be constructed solely from the lattice link variables. In order to retain gauge invariance, the construction must form a closed loop. This is easy to see from the gauge transformation properties. The "elementary plaquette" defined as

$$\mathrm{Tr}\,\Box_{\mu\nu}(x) \equiv \mathrm{Tr}\,U_\nu(x)U_\mu(x+a\hat{\nu})U_\nu^\dagger(x+a\hat{\mu})U_\mu^\dagger(x) \tag{1.29}$$

is the simplest such construction that can be made. Since we will treat the Us as the dynamical gauge field variables, we will have to consider any terms generated by renormalization in the IR theory of interest,

For the rest of this section we will specialize to the case of particular interest to this work and to the study of lattice gauge theories more generally, SU(N). The link matrices will then be elements of some representation of SU(N), which we will write in the standard way:

$$U_\mu(x) = e^{iaA_\mu} \qquad \text{with} \qquad A_\mu = t^A A_\mu^A\,. \tag{1.30}$$

The t^A are the generators of SU(N) in some representation and satisfy the standard Lie algebra. aA_μ^A parametrizes the expansion of the link matrices in the basis described by the t^A. The A_μ carry a vector index and are the analog of the continuum gauge fields. On the lattice, the expansion of the exponential in powers of the argument can no longer be made to truncate at $1{+}iaA_\mu$ by considering arbitrarily small a. To retain the unitarity of the link matrices, we must treat the full series expansion of the exponential, with higher order terms supressed by powers of the small but finite lattice spacing. The terms generated in this way will be completely irrelevant in the IR, but will play an important role in the maintenance of gauge invariance in the UV, where they can be very large.

At leading order in this expansion the plaquette in Eq. (1.29) is just 1. At next order

$$\begin{aligned} \Box_{\mu\nu} &\simeq 1 + ia\Big(A_\nu(x{+}\hat\nu/2) + A_\mu(x{+}\hat\nu{+}\hat\mu/2) - A_\nu(x{+}\hat\nu/2{+}\hat\mu) - A_\mu(x{+}\hat\mu/2)\Big) \\ &\simeq 1 + ia^2(\partial_\nu A_\mu - \partial_\mu A_\nu)\,. \end{aligned} \tag{1.31}$$

If we label A fields as living at the middles of links and $\Box_{\mu\nu}$ as living at the center of the plaquette, the first corrections to this expression are order a^4, that is, are cubic in derivatives.

Since the last term cancelled at $O(a)$, we have to go one order higher, where there are 6 contributions from pairs of A fields and also contributions from A^2 arising from a single link;

$$\begin{aligned} \Box_{\mu\nu} &\simeq \text{(above)} - a^2\Big(A_\mu^2 + A_\nu^2 + A_\nu A_\mu - A_\nu A_\nu - A_\nu A_\mu - A_\mu A_\nu - A_\mu A_\mu + A_\nu A_\mu\Big) \\ &= \text{(above)} - a^2\Big[A_\nu\,,\,A_\mu\Big]\,, \end{aligned} \tag{1.32}$$

which fills out the field strength. Therefore,

$$\Box_{\mu\nu} \simeq 1 - ia^2 F^a_{\mu\nu} T^a\,. \tag{1.33}$$

We also know that $\Box_{\mu\nu}$ is unitary; therefore the $-ia^2 F$ term ensures that there is also a $-a^4 F^2$ term,

$$\Box_{\mu\nu} \simeq 1 - ia^2 F^a_{\mu\nu} T^a - \frac{a^4}{2} F^a_{\mu\nu} F^b_{\mu\nu} T^a T^b + \dots\,, \tag{1.34}$$

and so its trace contains $-a^4 F^2/4$. We can get the standard gauge action by summing over the traces of plaquettes,

$$\sum_{x,\mu>\nu} [N_\mathrm{c} - \mathrm{Tr}\,\Box_{\mu\nu}(x)] \;\simeq\; \int d^4x \frac{1}{8} F^a_{\mu\nu} F^a_{\mu\nu}\,. \tag{1.35}$$

Lattice people always write the action with prefactor β/N_c, β plays the part of the inverse gauge coupling. We see that at tree level,

$$S = \frac{\beta}{N_\mathrm{c}} \sum_{x,\mu>\nu} (N_\mathrm{c} - \mathrm{Tr}\,\Box_{\mu\nu}(x)) \quad \Rightarrow \quad \beta = \frac{2N_\mathrm{c}}{g_0^2}\,. \tag{1.36}$$

This action is called the Wilson action and is the simplest available. It gets corrections to F^2 at dimension 6, that is, $F_{\mu\nu} D^2_\mu F_{\mu\nu}$; this correction is also not $O(4)$ invariant. One can make more sophisticated actions, such as one involving 1×2 boxes rather than just squares;

$$S_\mathrm{Symanzik}: \quad \sum_{\mu>\nu} \mathrm{Tr}\,\Box_{\mu\nu} \to \frac{5}{3} \sum_{\mu>\nu} \mathrm{Tr}\,\Box_{\mu\nu} - \frac{1}{12} \sum_{\mu>\nu} \left(\mathrm{Tr}\,\boxed{}_{\mu\nu} + \mathrm{Tr}\,\boxed{}_{\nu\mu}\right)\,. \tag{1.37}$$

This action has no tree level dimension 6 operators and gives improved rotational invariance convergence.

In Chapter 2 we use the Sheikholeslami-Wohlert (SW) improved fermion action [43] to improve the convergence of the spectral properties of the fermion matrix to $\mathrm{O}(a^2)$. This is described in section 2.4.1, though the idea is very similar to the Symanzik improvement described above. It is not technically necessary to include an improved bosonic action with this implementation, since the continuum limit is already approached at $\mathrm{O}(a^2)$ for the simplest case of the Wilson action.

Before we construct a lattice action and interesting measurables, we must determine what the integration measure in the path integral is. Obviously we should make the replacement,

$$\mathcal{D}A_\mu \;\Rightarrow\; \prod_{x,\mu} dU_\mu(x)\,. \tag{1.38}$$

What does $dU_\mu(x)$ mean, though? If we think of $U_\mu(x)$ as generated by its Lie algebra elements,

$$U_\mu(x) = \exp(iaT^a A^a_\mu) \;: \qquad dU_\mu(x) \stackrel{?}{=} \prod_a dA^a_\mu(x)\,? \tag{1.39}$$

This is wrong; this measure is not gauge invariant. A gauge transformation, say, at x, must leave the measure $dU_\mu(x)$ (and the measure $dU_\mu(x{-}\hat\mu)$) invariant. What that means is that we need a measure for the group manifold $\mathrm{SU}(N_\mathrm{c})$ which is preserved under "rotation" of the group manifold by right or left group multiplication.

It turns out that such a measure generally exists for compact, simple gauge groups, and it is unique up to a dilatation (which corresponds to an uninteresting overall multiplicative factor in the partition function). The measure is called the Haar measure. It can be generated as follows. Choose an orthonormal basis for the Lie algebra T^a. The infinitesimal volume element at the origin is $T^1 \wedge T^2 \wedge T^3 \cdots$. This means that we assign $\delta^{N_\mathrm{c}^2-1}$ measure to the box containing all points of form $g \in \{1 + i\delta \sum_a T^a e^a; e^a \in [0,1]\}$. Then, the measure near the point g' is that we assign the same volume to the set of points g for which $g(g')^{-1}$ lies in that box.

For an example, consider SU(2). For any $\vec{\theta}$ of length 2π, $\exp(i\vec{\theta}\cdot\vec{T})$ is the $-\mathbf{1}$ element of the group. (Recall that $T^a = \tau^a/2$ with τ^a the Pauli matrices; $\exp i\pi\tau^a = -\mathbf{1}$ for each a.) Obviously, then, the measure for such large $\vec{\theta}$ should approach zero. The correct measure turns out to be

$$dU_\mu(x) = \prod_{a=1,2,3} J(A^2)dA^a_\mu(x)\,, \qquad J(A^2) = \frac{2(1-\cos|A|)}{|A|^2}\,, \tag{1.40}$$

up to uninteresting multiplicative rescaling.

The integration $\prod dU_\mu(x)$ obviously overcounts physical configurations; in fact it integrates uniformly over all possible gauge orbits. There is no problem with doing this, though; because the $U_\mu(x)$ and the $\Lambda(x)$ reside on compact spaces, this amounts to a finite overall multiplicative factor in the partition function, which is of no physical consequence. This is the lattice's answer to how to fix the gauge: Don't fix the gauge. Integrate over all possible gauge field configurations, including redundantly integrating over a configuration and its gauge copies. With the lattice implementation this space of redundant integrations, $\Pi_x\Lambda(x)$, is compact and of finite volume, so no harm is done.

Building a lattice gauge theory with these components and this integration measure, and using a gauge invariant action, automatically ensures *exact* gauge invariance of the IR effective theory, which is strong enough to ensure that only desired dimension 4 operators can appear.

1.2.3 Perturbation theory

The lattice is a nonperturbative, well defined regulation. (In fact, it is the ONLY nonperturbatively defined regulation scheme we know.) We can study perturbation theory in this regulation scheme, both for its own sake and as a way to do matching calculations between the lattice and the continuum. This is especially important for lattice supersymmetry.

What are the fields and Feynman rules? we will outline the procedure for getting them, but not give too great detail. H. Rothe's book [34] contains quite a complete account. The diagrams are collected in Appendix A.

First we must choose a definition for the gauge field, based on the link. We want $U \simeq 1 + iaT^A A^A_\mu$, but this is not a unitary matrix; how do we deal with the higher order in A piece? Two natural choices would be,

$$\begin{aligned} U_\mu &= \exp(iaT^A A^A_\mu)\,, \\ \text{or} \qquad U_\mu - U^*_\mu &= 2iaT^A A^A_\mu\,. \end{aligned} \tag{1.41}$$

The latter is more convenient for going from lattice U matrices to A fields. The former is more convenient for perturbation theory and people usually use it.

As usual you have to fix the gauge. This will introduce ghosts, which are not TOO different from normal, and we will not discuss them further except to say that we have listed the two lowest order ghost gauge vertices in appendix A. The interesting things are that

1. inverse propagators and vertices have trigonometric momentum dependence, and
2. there are extra, unexpected vertices.

These two are related to each other and are inevitable if one does gauge theory on a lattice.

Consider the scalar action

$$\frac{1}{2}(D_\mu\Phi)^2 \to \frac{a^2}{2}\sum_{x,\mu}\Big(2\Phi^\dagger(x)\Phi(x) - \Phi^\dagger(x)U_\mu(x)\Phi(x{+}\hat\mu) - \Phi^\dagger(x{+}\hat\mu)U^\dagger_\mu(x)\Phi(x)\Big)\,. \tag{1.42}$$

First expand it to zero order in A to find the inverse propagator. Go to momentum space by replacing

$$\Phi(x) = \int_{\mathrm{BZ}} \frac{d^4p}{(2\pi)^4} e^{ip\cdot x}\Phi(p)\,, \tag{1.43}$$

where $\int_{\mathrm{BZ}}$ means the integral only extends over the Brillouin zone, $p_1 \in (-\pi/a, \pi/a]$ and similarly for the other 3 components. Note that the zone is a hypercube, another reflection of the breaking of $O(4)$ invariance. From now on we will use the notation

$$\int_p \equiv \int_{\mathrm{BZ}} \frac{d^4p}{(2\pi)^4}\,. \tag{1.44}$$

Substituting and performing the sum over the lattice sites yields

$$\begin{aligned} \mathcal{L} &= \int_{pk} \Phi_p^\dagger \sum_{x,\mu} e^{i(k-p)\cdot x}\Big[1 + e^{i(k-p)\cdot\mu} - e^{-ip\cdot\mu} - e^{ik\cdot\mu}\Big]\Phi_k \\ &= \int_p \Phi_p^\dagger \sum_\mu \Big[2 - 2\cos(p_\mu)\Big]\Phi_p\,, \end{aligned} \tag{1.45}$$

from which we can read off the inverse propagator

$$\Delta^{-1}(p) = \sum_\mu \frac{2}{a^2}(1-\cos ap_\mu) = \sum_\mu \frac{4}{a^2}\sin^2\frac{ap_\mu}{2} \equiv \tilde{p}^2\,. \tag{1.46}$$

The low ap limit of the propagator is p^2 but there are a^2p^4 corrections; for hard momenta the propagator is significantly different, as it must be because it is smooth and periodic. Non-smooth behavior can only be achieved by nonlocal actions, and would carry other costs.

The linear in A term, that is, the three point vertex, is determined by expanding U to linear order in A. Write A as living at the middle of the link, as is natural; the term in the action is

$$\frac{a^2}{2}\sum_{x,\mu} iaA_\mu^A(x{+}\hat{\mu}/2)\Big(-\Phi^\dagger(x)T^A\Phi(x{+}\hat{\mu}) + \Phi^\dagger(x{+}\hat{\mu})T^A\Phi(x)\Big) \tag{1.47}$$

which, denoting the Φ momentum as k and the $\Phi^\dagger$ momentum as p (both incoming to the vertex), gives a Feynman rule

$$T^A \frac{2}{a} \sin \frac{(k-p)_\mu a}{2} \,. \tag{1.48}$$

This differs from the continuum vertex by the substitution of $(2/a)\sin[(k-p)_\mu a/2]$ for $(k-p)_\mu$. These are equivalent at small momentum and differ at $O(a^2)$. The quadratic in A term is also peculiar; since $\Phi^\dagger$ and Φ are evaluated one site off from each other, it behaves as

$$\{T^A\,,\, T^B\} g_{\mu\nu} \cos \frac{(k-p)_\mu a}{2} \tag{1.49}$$

whereas usually the cosine term would be 1.

One may continue with the expansion; at A^3 order there is a vertex with no continuum analog, which however has an explicit a^2 suppression. It is only needed in relatively high loop calculations; in fact there are infinitely many vertices but as they have ever more lines, they are only needed at ever higher orders in the loop expansion. The forms of all these terms can actually be determined completely by looking at the inverse propagator and insisting that the derivatives there be covariant derivatives.

The gauge field action can be expanded similarly, though it is more complicated. The inverse propagator is

$$G^{-1}_{\mu\nu} = g_{\mu\nu}\tilde{k}^2 - \tilde{k}_\mu \tilde{k}_\nu \,, \qquad \tilde{k}_\mu \equiv \frac{2}{a} \sin \frac{k_\mu a}{2} \,. \tag{1.50}$$

The cubic interaction term looks like its normal form but with the substitutions

$$(k_1 - k_2)_\mu \delta_{\nu\lambda} \rightarrow (\widetilde{k_1 - k_2})_\mu \cos \frac{(k_3)_\nu a}{2} \delta_{\nu\lambda} \quad \text{etc.} \tag{1.51}$$

The 4 gluon piece looks horrible. Besides the type of complication already encountered, multiplying the expected piece $\Gamma^{ABCD} \propto f^{ABE} f^{CDE}(\ldots)+$ permutations, it has a completely unexpected *symmetric* in external indices piece, of form

$$\begin{aligned}
\Gamma^{ABCD}_{\mu\nu\lambda\rho}(p,q,r,s) \quad = \quad & \text{(Antisymmetric piece)} \; + \\
& \frac{g^2}{12}\left\{ \frac{2}{3}\delta_{AB}\delta_{CD} + d_{ABE}d_{CDE} + (AC,BD) + (AD,BC) \right\} \\
& \times(\tilde{p}\,\tilde{q}\,\tilde{r}\,\tilde{s}) \text{ with various Lorentz structure} \,. \qquad (1.52)
\end{aligned}$$

This extra piece comes about because $\Box \simeq \exp(-ia^2 F_{\mu\nu})$; at a^8 level there is an F^4 term with the T indices totally symmetric as in the above, which looks like four A fields and four derivatives. The full feynman rule, generalized to the case of arbitrary gauge group, is given in Eq. A.1. This term is totally irrelevant as an IR effective operator, but in the ultraviolet it is very large.

One last complication comes from the non-trivial integration measure discussed in the previous section. First we generalize to arbitrary gauge group, rewrite the measure as

$$DU = e^{-S_{\rm measure}[A]} \, DA \qquad (1.53)$$

with

$$S_{\rm measure} = -\,{\rm Tr}\, T^A T^B \sum_{x,\mu} {\rm Tr} \ln\left(\frac{2(1-\cos A_\mu)}{(A_\mu)^2} \right) \text{ and } \quad A_\mu \equiv \sum_A F^A A^A_\mu , \qquad (1.54)$$

and then we taylor expand cos and ln to get,

$$S_{\rm measure} = \frac{1}{2} A^A_\mu A^B_\nu \times \left(\frac{1}{6} \mathcal{T}_F \delta_{\mu\nu}\, {\rm Tr}\, F^A F^B \right) \; + \; O(A^4) \,, \qquad (1.55)$$

which gives a feynman rule

$$= -\frac{1}{6}\mathcal{T}_F \mathcal{C}_A \delta_{\mu\nu}\,. \tag{1.56}$$

This diagram looks like a quadratically divergent mass insertion for the gauge boson; it precisely cancels other quadratic divergences that would otherwise spoil the gauge invariance of the theory (see the example calculation of section 1.2.7).

Its important to note that all the contributions from S_{measure} will enter as 1-loop corrections to the analagous vertices. If we absorb the coupling g_0 in Eq. 1.36 into the definition of the gauge field, we see that the 2-pt measure insertion picks up an overall g_0^2 relative to the gauge field propogator – and so enters as a 1-loop effect. The $O(A^4)$ term in the measure will then have a g_0^4 suppression and so will enter as a 1-loop correction to the 4-pt gauge field vertex.

1.2.4 Tadpole improvement

Such large but very UV interactions lead to significant "unexpected" renormalizations. This means that, for instance, the matching between the lattice coupling and the MOM scheme coupling doesn't give $\mu_{\text{MOM}} \simeq 1/a$ as expected, but

$$g_0^2 = g_{\text{MOM}}^2(\mu = 83.5/a) \text{ at one loop}\,. \tag{1.57}$$

In other words, one must put in a much larger β value than naively expected (make the gauge action "stiffer" than expected) in order to get a given physical coupling constant. This is dominantly due to large renormalizations from "tadpoles" (simple closed loops originating from a single vertex), mostly from the above, symmetrical piece of the vertex (and its higher point brothers).

These large terms are common (approximately proportional) in essentially all operators, and can be easily, approximately, cancelled by "tadpole improvement" as advocated by Lepage and Mackenzie[44]. The idea is that the corrections are due to very UV corrections which make each link U look "shorter" than it is; averaging over UV fields on a compact manifold like $\mathrm{SU}(N_{\mathrm{c}})$ effectively averages with some weight over a patch of the manifold around the "infrared" value of the link. Think of $\mathrm{SU}(N_{\mathrm{c}})$ as an n-sphere (SU(2) actually is the 3-sphere); such averaging gives you a point on the interior of the sphere, that is, it effectively makes U "shorter". To compensate, measure the mean value of $\mathrm{Tr}\,\Box$, and multiply each link by $(\mathrm{Tr}\,\mathbf{1}/\mathrm{Tr}\,\Box)^{1/4}$ (or just multiply terms in the action by this quantity to the power of the number of links involved). This crude prescription comes startlingly close to undoing the large renormalizations and is now widely used under the rubric of "tadpole improvement."

1.2.5 Fermions in lattice gauge theory

The fermionic part of the usual path integral for a single fermion is supposed to be

$$
\begin{aligned}
Z_{\mathrm{fermi}} &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp(-S(\bar{\psi},\psi,A)) \,, \\
S(\bar{\psi},\psi,A) &= M\bar{\psi}\psi + \bar{\psi}D_\mu\gamma^\mu\psi \,.
\end{aligned}
\tag{1.58}
$$

Here the fields $\bar{\psi}$, ψ are Grassmanian 4-component spinors, that is, at each point there are four Grassman variables associated with $\bar{\psi}$ and four associated with ψ.

There is a potential problem we see right away, which is that there is a dimension 3 operator $\bar{\psi}\psi$. When the regularization (lattice) is at some very UV point, what prevents a large $\propto \Lambda \sim 1/a$ coefficient for this operator from being induced? The answer is chiral symmetry, which is completely broken by a mass term:

$$
\begin{aligned}
\psi_m &\Rightarrow (L_{mn} P_L + R_{mn} P_R)\, \psi_n \,, \\
\bar{\psi}_m &\Rightarrow \bar{\psi}_n \left(L^\dagger_{nm} P_R + R^\dagger_{nm} P_L \right) \psi_n \,,
\end{aligned}
\tag{1.59}
$$

with L, R independent $\mathrm{U}(N_{\rm f})$ matrices, $N_{\rm f}$ the number of fermionic flavors, an $P_{L,R}$ the usual left and right handed projection operators. Because $\gamma^\mu P_{L,R} = P_{R,L}\gamma^\mu$, the kinetic term is invariant under the transformation;

$$
\begin{aligned}
\bar{\psi}\gamma^\mu\psi &\Rightarrow \bar{\psi}\left(L^\dagger P_R + R^\dagger P_L \right) \gamma^\mu \left(L P_L + R P_R \right) \psi \\
&= \bar{\psi}\,\gamma^\mu \left(L^\dagger P_L + R^\dagger P_R \right)\left(L P_L + R P_R \right) \psi \\
&= \bar{\psi}\gamma^\mu\psi \,.
\end{aligned}
\tag{1.60}
$$

However the mass term is not invariant,

$$
\begin{aligned}
\bar{\psi}\psi &\Rightarrow \bar{\psi}\left(L^\dagger P_R + R^\dagger P_L \right)\left(L P_L + R P_R \right) \psi \\
&= \bar{\psi}\left(L^\dagger R P_R + R^\dagger L P_L \right) \psi \,.
\end{aligned}
\tag{1.61}
$$

If the masses are unequal, it is not even invariant under the vector ($L = R$) subgroup.

Therefore, provided that *only* the mass term violates this symmetry, it is protected from being radiatively induced; if one puts a mass term into the theory, then while it gets multiplicatively renormalized (that is, $M_{\rm bare} = Z M_{\rm renorm}$), it is not additively renormalized.

What is a lattice implementation of $\bar{\psi}\gamma^\mu D_\mu \psi$? Obviously the difference between D_μ and ∂_μ is that we should parallel transport group indices using the gauge links whenever we compare things at different points. What should ∂_μ mean?

A single finite difference, $\partial_\mu\psi \to \psi(x+a\hat{\mu})-\psi(x)$, breaks the cubic symmetry of the lattice since it takes a difference one way, but not the other. It is also not reflection-Hermitian (the Euclidean analog of Hermitian, where one takes complex conjugates and performs a reflection in some direction; think of the reflection as being in the time direction, and accounting for the fact that e^{-iHt} becomes e^{iHt} and that $\psi^\dagger$ is related to $\bar{\psi}$ by a $\gamma^0_{\rm E}$). When you take its reflection-Hermitian conjugate, $\bar{\psi}$ and ψ change roles and there is a $-$ sign (except for the time component which is unchanged); so

$$\bar{\psi}(x)\gamma^\mu U_\mu(x)\psi(x+\hat{\mu}) \to -\bar{\psi}(x+\hat{\mu})\gamma^\mu U^\dagger_\mu(x)\psi(x)\,. \tag{1.62}$$

The forward difference becomes minus a backwards difference. Failure of reflection Hermiticity means that the lattice theory doesn't correspond to a unitary Minkowski theory (just as a non-Hermitian Minkowski action means a non-unitary theory).

The obvious way around this problem is to take the difference one site off the discrete position x. That is,

$$\partial_\mu\psi \to \frac{\psi(x+a\hat{\mu})-\psi(x-a\hat{\mu})}{2a}\,. \tag{1.63}$$

Applying this to the free fermion action yields

$$\int d^4x\;\bar{\psi}\not{\partial}\psi \to a^4\sum_{x,\mu}\bar{\psi}\,\gamma^\mu\frac{\psi(x+a\hat{\mu})-\psi(x-a\hat{\mu})}{2a}\,, \tag{1.64}$$

and the calculation of the propagator proceeds in a way very similar to Sec. 1.2.3. As before, we obtain a trigonometric function of the momentum; namely

$$S_p^{-1} = \frac{i}{a}\sum_\mu \gamma_\mu\sin(ap_\mu) \equiv i\not{\bar{p}}\,. \tag{1.65}$$

Just as in the case of the scalar field, the fermion propagator directly reproduces the continuum expression at $a \to 0$ plus IR irrelevant high order corrections. However, the spectrum of the theory displays a fatal flaw: the propagator has a pole at $p=0$ as expected but it also has a pole at $p=\pi/a$, i.e. inside (at the edge) of the Brillouin zone, whereas the scalar propagator, $\sin^2(p/2)$, *peaks* at this edge. The IR effective theory being described by this action in the continuum limit is thus a theory of more than one fermion. In d dimensions the theory would describe 2^d fermions at locations (0,0,0), (π/a,0,0), (0,π/a,0), etc .... This is known as the fermion doubling problem. In this thesis we will only be concerned with two of the solutions to the doubling problem, Wilson fermions and Ginsparg-Wilson (GW) fermions. Wilson fermions are described below, wheras GW fermions are described in section 1.2.8.

The idea is to add a high-dimension operator to the action that is non-zero at the edges of the Brillouin zone, thus lifting the degeneracy. The most obvious candidate for such a term, since we know that the scalar propagator has the correct massless spectrum, is

$$\mathcal{L}_W = -\frac{ra}{2}\bar{\psi}\partial^2\psi \,. \tag{1.66}$$

Putting the finite difference version together with Eq. (1.64), we obtain a symmetric free fermion action with no doublers:

$$\begin{aligned} S_{\text{free}} \;=\; & a^4 \sum_{x,\mu} \bar{\psi}(x) \Bigg[\gamma_\mu \frac{\psi(x+a\hat{\mu}) - \psi(x-a\hat{\mu})}{2a} \\ & \qquad + \frac{ar}{2}\; \frac{-\psi(x+a\hat{\mu}) + 2\psi(x) - \psi(x-a\hat{\mu})}{a^2} \Bigg] \,. \end{aligned} \tag{1.67}$$

The value of the Wilson coefficient r is to be chosen by the practitioner (though $r > 0$ is required and $r \leq 1$ is desirable to ensure reflection positivity). I will present results only for $r = 1$ in this thesis, as it is the vastly more common choice. This can then be re-written in a more convenient form as

$$S_{\text{free}} = 4ar \sum_x \bar{\psi}\psi + \frac{a^3}{2} \sum_{x,\mu} \bar{\psi}(x) \Big[(\gamma_\mu - r)\psi(x + a\hat{\mu}) - (\gamma_\mu + r)\psi(x - a\hat{\mu}) \Big] \,. \quad (1.68)$$

If we now calculate the propagator we find

$$S_p = \frac{-i\hat{\not{p}} + M_p}{\hat{p}^2 + M_p^2} \qquad \text{with} \qquad \hat{p}^2 \equiv \frac{1}{a^2} \sum_\mu \sin^2(p_\mu a) \qquad \text{and} \qquad M_p = \frac{ar}{2} \tilde{p}^2 \,. \quad (1.69)$$

$\hat{\not{p}}$ is defined in Eq. (1.65) and M_p is the momentum dependent effective lattice mass induced by the Wilson term.

To see how this term lifts the degeneracy, look at a one-dimensional version of the denominator in Eq. (1.69) with r=1:

$$\begin{aligned} \frac{1}{a^2} \sin^2(ap) + \frac{4}{a^2} \sin^4(ap/2) &= \frac{4}{a^2} \Big[\sin^2(ap/2) \cos^2(ap/2) + \sin^4(ap/2) \Big] \\ &= \frac{4}{a^2} \sin^2(ap/2) \,, \end{aligned} \quad (1.70)$$

which is precisely a single component of the inverse scalar propagator. As was mentioned above, this function peaks at the edges of the Brillouin zone. This motivates the conclusion that large values ($r \simeq 1$) of the Wilson parameter lift the degeneracy more definitely than small ones.

The price of this choice is that chiral symmetry has been broken completely, at the lattice spacing scale. There is no accidental IR symmetry which will save us, any chiral symmetry we get in the IR has to come about by explicit tuning.

The lowest order chiral symmetry violating term we are stuck with, is a mass term $\bar{\psi}\psi$, which will now be generated with coefficient $\sim g^2/a$ even if we don't put it in. To see why, remember the discussion about tadpole corrections to links: averaging over UV fields approximately replaces the term $U_\mu(x)$ with $ZU_\mu^{\mathrm{IR}}(x)$ with $Z \sim 1 - g^2$ a renormalization and U^{IR} the link when only the IR gauge fields are present. This means

$$\begin{aligned} &-U_\mu(x)\psi(x_+) + 2\psi(x) - U_\mu^\dagger(x_-)\psi(x_-) \\ &\qquad \simeq \quad -ZU_\mu^{\mathrm{IR}}(x)\psi(x_+) + 2\psi(x) - Z{U^{\mathrm{IR}}}_\mu^\dagger(x_-)\psi(x_-) \\ &\qquad = \quad Z\left(-U_\mu^{\mathrm{IR}}(x)\psi(x_+) + 2\psi(x) - {U^{\mathrm{IR}}}_\mu^\dagger(x_-)\psi(x_-)\right) + 2(1-Z)\psi(x)\,. \end{aligned} \tag{1.71}$$

The last piece looks like a mass term, which must be compensated for with a negative mass counterterm. Delicate tuning of the counterterm is needed, because a mass is radiatively generated at every loop order and nonperturbatively. This requires, for instance, determining the mass based on correlation lengths (say, the pion to rho mass ratio) and tuning to get the desired mass, separately at each lattice spacing considered. This complication is also the basic obstacle to implementations of supersymmetry in 4 dimensions. It is discussed extensively in Chapter 3 and overcoming it is the primary focus of Chapter 4.

Another serious complication is that, since one is integrating over the links U_μ, occasionally a link configuration appears with unusually small UV fluctuations. Then, the mass squared counterterm (which is put into the Lagrangian, it does not vary as we perform the path integral) will over-compensate and the mass will be negative. Such configurations can have negative determinant. This is a problem both for the algorithm generating gauge field configurations (leading to extra numerical costs in performing Wilson quark simulations at small masses), and to the physical interpretation of what we are doing. This problem seems to be ameliorated in the small a limit, such that the scale where negative eigenvalues set in is $m \sim a\Lambda_{\rm QCD}^2$. Therefore there is no problem in the formal $a \to 0$ limit, but there are problems at values of a which are currently realistic ($a \sim 0.1$ Fermi) and m small enough to be in the chiral limit.

We should note that this problem of exceptional configurations is absent in so-called twisted mass fermions [45] (the bare mass is flavor dependent and chirally twisted). This only works for flavor doublets of fermions (like the up and down quarks for example). The technique has gained serious traction in recent years in the Wilson fermion community (though it can theoretically be applied to any fermion implementation) precisely as a solution to the problem of exceptional configurations.

The other problem with the Wilson action is that violating chiral symmetry introduces unwanted corrections in physical correlation functions at the $O(a)$ (dimension 5 operator) level. This problem can be patched up by much hard work [46, 47]. One must add other dimension 5, chiral symmetry breaking operators, such as [43]

$$\frac{iac_{\rm SW}}{4}\bar{\psi}\sigma_{\mu\nu}G^{\mu\nu}\psi\,, \tag{1.72}$$

to the action, and tune the coefficient $c_{\rm SW}$ to make $O(a)$ effects vanish in physically interesting operators. The tuning is done at the nonperturbative level by varying $c_{\rm SW}$ until axial Ward identities, which would be valid in a chirally symmetric theory, are satisfied for infrared observables.

We use the 3D analog of this improvement in Chapter 2 to improve the spectral properties of the fermion matrix (as mentioned already). In that case, no difficult nonperturbative tuning is required because the theory is superrenormalizable and so O(a) errors are cancelled with only a tree level determination of the coefficient.

1.2.6 Feynman rules with fermions

In Sec. 1.2.3 the basic notions of the perturbative expansion and the fundamental differences in momentum dependence that arise from the fourier analysis in discrete space have been described. The complications that come with the addition of fermions are then largely algebraic (from the inclusion of the Wilson term in the case of Wilson fermions).

Following closely with Sec. 1.2.3 we expand the action

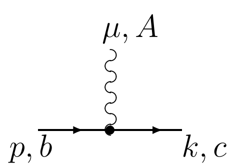
$$S_{\mathrm{F}} = \frac{a^4}{2}\sum_{x,\mu}\bar{\psi}(x)\Big[2r\psi(x) + \big(\gamma_\mu - r\big)U_\mu(x)\psi(x{+}\mu) - \big(\gamma_\mu + r\big)U_\mu^\dagger(x{-}\mu)\psi(x{-}\mu)\Big] \tag{1.73}$$

in powers of A to determine the gauge interactions.

The linear in A piece is

$$\begin{aligned} S_{A\bar{\psi}\psi} = \frac{ia^4}{2}\sum_{x,\mu}\bar{\psi}(x)\Big[\, & \gamma_\mu\Big(A_\mu(x)\psi(x{+}\mu) + A_\mu(x{-}\mu)\psi(x{-}\mu)\big)\Big) \\ & -r\Big(A_\mu(x)\psi(x{+}\mu) - A_\mu(x{-}\mu)\psi(x{-}\mu)\big)\Big)\Big], \end{aligned} \tag{1.74}$$

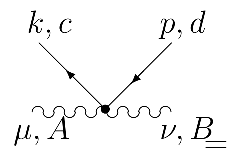
which gives a feynman rule



$$= -T^A_{bc}\left(i\gamma_\mu \cos\frac{a(p{+}k)_\mu}{2} + r\sin\frac{a(p{+}k)_\mu}{2}\right). \tag{1.75}$$

Here b, c take whatever values are appropriate for the representation under which the fermions transform (so b, c is B, C for the adjoint representation and $T^A_{BC} = -if_{ABC}$). This reproduces the continuum vertex in the $a \to 0$ limit plus corrections at $O(a)$.

The quadratic in A piece has no continuum analog; it is simply an artifact of the lattice construction that arises because the expansion of the link matrices must include powers of the gauge field at all orders. One contribution arises from the addition of the Wilson term to remove doublers, so it will take much the same form as its scalar analog (since the Wilson term 'looks like' a scalar kinetic term). This gives a feynman rule



$$= \frac{a}{2}\{T^A, T^B\}_{cd}\,\delta_{\mu\nu}\left(i\gamma_\mu \sin\frac{a(p+k)_\mu}{2} - r\cos\frac{a(p+k)_\mu}{2}\right). \quad (1.76)$$

There are an infinite number of higher point vertices just like in the case of the scalar field – with more and more gauge boson lines – but these are suppressed by large powers of the lattice spacing and are rarely necessary for practical calculations. The vertex vanishes linearly with a in the continuum limit.

1.2.7 Example: vacuum polorization and gauge invariance

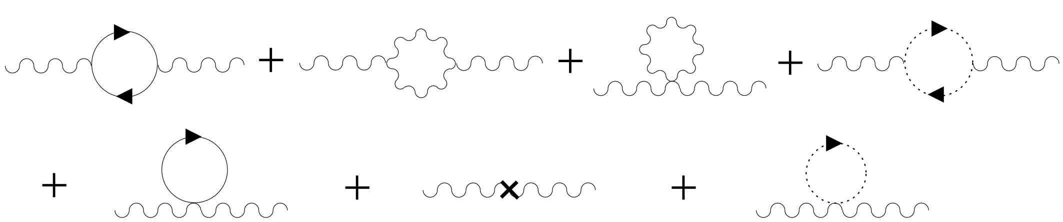
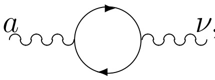


Figure 1–2: Diagrams contributing to the gauge boson mass at one loop in LPT.

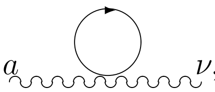
To be precise, we will consider a non-abelian theory with an arbitrary gauge group and arbitrary fermion content. This is discussed somewhat in [34]; the calculation was originally performed by [48]. As in the continuum, the diagrams containing fermion loops may be considered seperately, since the divergences must cancel even without their inclusion. We will not include scalar fields in the example, though their inclusion is quite straight-forward. We calculate only the gauge boson mass correction (gauge boson self energy at zero external momentum, $\Pi^{\mu\nu}(p{=}0)$) since the full contribution is far too complex. The diagrams are shown in Fig. 1–2; the diagrams on the first line exist also in the continuum, while the others are obviously artifacts of the discretization.

Consider the contribution of a single species of fermion in an arbitrary representation:

μ, a  ν, b

$$= -\operatorname{Tr} T^a T^b \int_k \operatorname{Tr}\left[i\gamma_\mu \cos(ak_\mu) + r\sin(ak_\mu)\right] S_k \times \left[i\gamma_\nu \cos(ak_\nu) + r\sin(ak_\nu)\right] S_k$$

and

μ, a  ν, b

$$= -\frac{a}{2}\operatorname{Tr}\{T^a, T^b\} \int_k \delta_{\mu\nu} \operatorname{Tr}\Big[i\gamma_\mu \sin(ak_\mu) - r\cos(ak_\mu)\Big] S_k .$$

The sum will then integrate to zero because we can rewrite it as a total derivative. Taking a cue from the continuum we consider the inverse propagator,

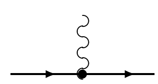
$$S_k^{-1} = i\hat{\not k} + M_k = \frac{1}{a}\sum_\alpha \left[i\gamma_\alpha \sin(ak_\alpha) + r\Big(1 - \cos(ak_\alpha)\Big)\right], \tag{1.77}$$

and take the second derivative of the log (the derivatives are with respect to k)

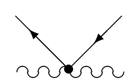
$$\operatorname{Tr} \partial_\nu \partial_\mu \ln S_k^{-1} = \operatorname{Tr}\left[S_k(\partial_\nu\partial_\mu S_k^{-1}) - (\partial_\mu S_k^{-1}) S_k (\partial_\nu S_k^{-1}) S_k\right]. \tag{1.78}$$

It is then easy to see that, since

$$\partial_\mu S_k^{-1} = i\gamma_\mu \cos(ak_\mu) + r\sin(ak_\mu) \;\propto$$



$$\text{and} \quad \partial_\nu\partial_\mu S_k^{-1} = a\delta_{\mu\nu}\Big(-i\gamma_\mu \sin(ak_\mu) + r\cos(ak_\mu)\Big) \;\propto$$



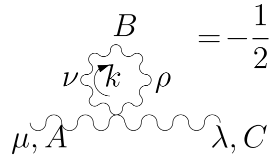
,

we can express the fermion contribution to the gauge boson mass as

$$\Pi^{\mu\nu}_{\text{fermions}} = \operatorname{Tr} T^a T^b \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} \partial_\nu \partial_\mu \operatorname{Tr} \ln S_k^{-1} = 0\,. \tag{1.79}$$

The gauge gauge interactions:

A quick look at the 4 gluon vertex of Eq. (A.1) will remind you why we have chosen to calculate the self-energy in the limit of zero external momenta; in this limit many – indeed most – of the terms in the vertex are zero; specifically, the fully symmetric piece described in 1.2.3 does not contribute at all. With these simplifications, the diagram is actually much easier to calculate than the one with the 3 gluon vertex, so we will start with it:



$$= -\frac{1}{2}\times 2F^{A}_{BE}F^{C}_{EB}\tilde{\Delta}^{\nu\rho}_{k}\Big[\delta_{\mu\lambda}\delta_{\nu\rho}\cos(ak_\mu) - \delta_{\mu\rho}\delta_{\nu\lambda}\cos\frac{ak_\mu}{2}\cos\frac{ak_\lambda}{2} + \frac{1}{6}\delta_{\mu\lambda}\tilde{k}_\mu\left(2\delta_{\mu(\rho}\tilde{k}_{\nu)}\right) - \frac{1}{12}\delta_{\mu\nu}\delta_{\mu\lambda}\delta_{\mu\rho}\tilde{k}^2\Big]$$

The $\frac{1}{2}$ is a symmetry factor determined in the usual way familiar from continuum PT. To simplify this expression we use trig half angle formulas with wild abandon, note that

$$\int \tilde{k}_\mu \tilde{k}_\lambda = \delta_{\mu\lambda}\int \tilde{k}^2_\mu\,,$$

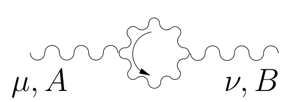
and analyze sums over the gauge boson propogator such as

$$\sum_\nu \tilde{\Delta}^{\mu\nu}_k \tilde{k}_\nu = \frac{1}{\tilde{k}^2}\left(\tilde{k}_\mu - (1-\xi)\tilde{k}_\mu\right) = \xi\frac{\tilde{k}_\mu}{\tilde{k}^2}\,.$$

Finally we arrive at

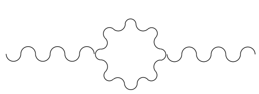
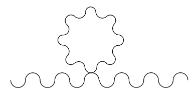
$$-C_A\delta_{AC}\delta_{\mu\lambda}\int_k\left[\frac{2}{\tilde{k}^2} - \frac{7}{6}\frac{\tilde{k}^2_\mu}{\tilde{k}^2} + \frac{\hat{k}^2_\mu}{(\tilde{k}^2)^2} - \frac{1}{12} + \xi\left(\frac{\cos(k_\mu) + \frac{1}{4}\tilde{k}^2_\mu}{\tilde{k}^2} - \frac{\hat{k}^2_\mu}{(\tilde{k}^2)^2}\right)\right]. \quad (1.80)$$

The next diagram is much the same, but the algebra is quite a bit nastier; we will skip the details and just give you the answer:



$$= \quad C_A\delta_{AB}\delta_{\mu\nu}\left[6\frac{\hat{k}^2_\mu}{(\tilde{k}^2)^2} + \xi\left(\frac{\cos^2\frac{k_\mu}{2}}{\tilde{k}^2} - \frac{\hat{k}^2}{(\tilde{k}^2)^2}\right)\right]. \quad (1.81)$$

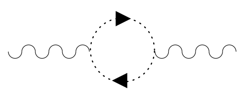
We see immediately that the ξ-dependence cancels between the two diagrams since $\cos(x) = \cos^2(x/2) - \sin^2(x/2)$. Combining these and using the integrals from appendix B we get

 + 

$$= \frac{1}{2} C_A \delta_{AB} \delta_{\mu\nu} \left(\frac{\Sigma}{4\pi} + \frac{1}{8} \right) .$$

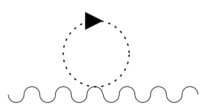
The Ghosts:

The ghost diagrams are much easier. Like in the continuum we have a diagram



$$\begin{aligned} &= -i^2 f_{ACD} f_{BCD} \int \tilde{k}_\mu \cos \frac{k_\mu}{2} \tilde{k}_\nu \cos \frac{k_\nu}{2} \\ &= -C_A \delta_{AB} \delta_{\mu\nu} \left(\frac{1}{2} \frac{\Sigma}{4\pi} - \frac{1}{16} \right) . \end{aligned}$$

and then the one with no continuum analog



$$\begin{aligned} &= -\frac{1}{6} f_{ACD} f_{BCD} \delta_{\mu\nu} \int \tilde{k}_\mu^2 \frac{1}{\tilde{k}^2} \\ &= -\frac{1}{24} C_A \delta_{AB} \delta_{\mu\nu} . \end{aligned}$$

From the measure counterterm, discussed extensively in Sections 1.2.2 and 1.2.3 we get

$$= -\frac{1}{12} C_A \delta_{AB} \delta_{\mu\nu} . \tag{1.82}$$

Adding these up, we see that the contributions cancel and gauge invariance is retained (at least at one loop).

1.2.8 Ginsparg-Wilson fermions

We will skip discussing the well known no-go theorem of Nielsen and Ninomiya [49] which, stated simply, says that one cannot construct a lattice Hamiltonian D satisfying $D\gamma^5 + \gamma^5 D = 0$, because it is described very well throughout the literature. Instead we will proceed with its consequences and the progress that has been made in avoiding its consequences.

Since a chirally symmetric theory in the continuum has this property, $D = D_\mu \gamma^\mu$ and $\gamma^\mu \gamma^5 + \gamma^5 \gamma^\mu = 0$, it looks like we cannot get chiral symmetry on the lattice.

How close can we get? Ginsparg and Wilson asked this in a long-forgotten but now famous paper, in 1982 [50]. They considered starting with a chirally symmetric theory in the continuum, and arriving at a lattice theory by blocking; essentially, if the continuum field is ϕ and the intended lattice field is ψ, we define the lattice field to follow the continuum field in some way,

$$\psi(x) = \int d^4y \, \alpha(y-x)\phi(y) \,, \tag{1.83}$$

with x a lattice site, y a continuous parameter, and α some (unimportant) weighting function, which could be as simple as $\delta(x-y)$ (forcing the lattice thing to equal the continuum one at the same point). Then, considering the integration over ϕ, done in continuous space, one sees how close the induced action for ψ can come to being chirally invariant. Ginsparg and Wilson concluded that the best you could do was

$$D\gamma^5 + \gamma^5 D = aD\gamma^5 D \,. \tag{1.84}$$

For suitably infrared fields (or for small enough a, same thing), this becomes the desired continuum relation.

It turns out that this is much better than just "a discrepancy which vanishes with the lattice spacing." The propagator is the inverse of D,

$$D_{\alpha\gamma}(x,z) S_{\gamma\beta}(z,y) = \delta_{\alpha\beta}\delta^4(x-y) \,; \tag{1.85}$$

acting with S on both sides of Eq. (1.84), we find that it satisfies

$$\gamma^5 S + S\gamma^5(x,y) = a\gamma^5\delta^4(x-y)\,. \tag{1.86}$$

The point is that, not only does this vanish as the lattice spacing is made small; it is also *ultralocal.* At all finite distances, we observe a propagator with exact chiral symmetry. This turns out to be enough to give us all the benefits we want of chiral symmetry, such as the absence of additive mass renormalization.

The first such closed form operator was constructed by Neuberger and Narayanan, (see [51, 52, 53]) and is closely related to an idea developed by David B. Kaplan [54]. A good review of the state of affairs around the time of these developments is [42], though it should be noted that the understanding of the connections between the two techniques were somehwat vague at the time. One Dirac operator which satisfies the Ginsparg-Wilson relation, called (for historical reasons) the "overlap operator," is

$$D = 1 - \frac{A}{[A^\dagger A]^{1/2}}\,, \qquad A \equiv 1 - D_W\,, \tag{1.87}$$

where D_W is the Wilson-Dirac fermion operator discussed extensively in section 1.2.5.

To see the magic of this combination, consider the infrared behavior, and imagine that D_W has gotten an additive mass renormalization that we didn't want, that is,

$$D_W \sim m + \partial\!\!\!/\,, \quad \text{so} \quad A \simeq 1 - m - \partial\!\!\!/\,. \tag{1.88}$$

The derivative only appears in $A^\dagger A$ at quadratic order; if we work to first order in derivatives (remember, we are asking about IR behavior), we find

$A^\dagger A \simeq (1-m)^2$, so

$$\frac{A}{[A^\dagger A]^{1/2}} \simeq \frac{1-m}{1-m} - \frac{1}{1-m}\not{\partial}\,; \qquad 1 - \frac{A}{[A^\dagger A]^{1/2}} \simeq \frac{1}{1-m}\not{\partial}\,. \tag{1.89}$$

The renormalization of the kinetic term is harmless – it can be undone by rescaling ψ. The mass correction, though, has disappeared.

The flip side is that the overlap operator involves the inverse of an operator. It isn't obvious that it will be local. It will *not* be local in the sense of $D(x-y)$ vanishing beyond some finite range; it *will* be local, for vacuum gauge fields, in the sense of $D(x-y)$ vanishing as an exponential of $(x-y)$, so that as the lattice spacing is taken to zero, the operator becomes exactly local. To see that this is true, look at the Fourier transform of D,

$$\tilde{D}(p) = 1 - \frac{1 - i\gamma^\mu \tilde{p}_\mu - \frac{1}{2}\tilde{p}^2}{\sqrt{1 + \frac{1}{2}\sum_{\mu<\nu}(\tilde{p}_\mu)^2(\tilde{p}_\nu)^2}}\,. \tag{1.90}$$

This function is analytic everywhere within the Brillouin zone; therefore its Fourier transform, which is the spatial dependence, shows exponential tails. Exponential locality is sufficient for most purposes.

It follows trivially from the Ginsparg Wilson relation that the lattice action,

$$\bar{\psi} D \psi\,, \tag{1.91}$$

has an exact invariance [55] (a very good and seminal paper; worth reading) under the infinitesimal transformation,

$$\begin{aligned} \delta\bar{\psi} &= \bar{\psi}\gamma^5\,, \\ \delta\psi &= \gamma^5(1-D)\psi \equiv \hat{\gamma}^5\psi\,. \end{aligned} \tag{1.92}$$

This exact invariance is enough to give the desired properties of a chirally symmetric theory, such as the vanishing of additive mass renormalization and of dimension 5, chiral symmetry breaking operators. However, the transformation under this symmetry looks different between ψ and $\bar{\psi}$. Therefore, the *measure* of the path integral is not necessarily invariant under this transformation. In fact, it is not; if we ask about the expectation value of an operator $\mathcal{O}$ under the fermionic part of the action,

$$\langle \mathcal{O} \rangle \equiv \int d\bar{\psi} d\psi \; \mathcal{O} \; e^{-\bar{\psi} D \psi} \,, \tag{1.93}$$

its variation turns out to be

$$\langle \delta \mathcal{O} \rangle = -a \, \mathrm{Tr} \, \{\gamma^5 D\} \langle \mathcal{O} \rangle \,. \tag{1.94}$$

The free $\gamma^5 D$ is traceless; but at nonzero gauge field, it turns out that $\mathrm{Tr}\, \gamma^5 D$ is determined by the index of D, that is, the number of chiral zero modes; and that this equals (twice) the topological number. This is exactly what is demanded by the ABJ anomaly.

We see that "ordinary" chiral transformations of ψ are not a symmetry of the theory; but a modified chiral transformation is an exact symmetry, except under the measure. Its failure under the measure gives exactly the axial (ABJ) anomaly. This is Fujikawa's way of seeing the origin of the anomaly. The last section of [56] contains a history of these same discoveries from the 90's and a more complete list of references.

An exact chiral invariance whets our appetite to write down a chiral theory. Define projection operators,

$$P_{\pm} \equiv \frac{1 \pm \gamma^5}{2} \qquad \hat{P}_{\pm} \equiv \frac{1 \pm \hat{\gamma}^5}{2} \,, \qquad \hat{\gamma}^5 \equiv \gamma^5 (1 - D) \;\; \text{as before} \,. \tag{1.95}$$

The latter is also a projection operator, because

$$\hat{\gamma}^5\hat{\gamma}^5 = \gamma^5(1-D)\gamma^5(1-D) = \gamma^5\gamma^5 - \gamma^5(D\gamma^5 + \gamma^5 D - D\gamma^5 D) = \gamma^5\gamma^5 = 1\,, \tag{1.96}$$

using the Ginsparg Wilson relation in the next to last equality.

Then, since $\bar{\psi}P_+$ only talks to $\hat{P}_-\psi$ and $\bar{\psi}P_-$ only talks to $\hat{P}_+\psi$, why don't we just throw out $\bar{\psi}P_-$ and $\hat{P}_+\psi$ in the integration, and only integrate over $\bar{\psi}P_+$ and $\hat{P}_-\psi$?

That will be fine, *IF* we can figure out what the measure of the path integration should be. The measure depends on the gauge fields in a nontrivial way, because $\hat{P}_\pm$ do, and $P_\pm$ do not; so there is not a cancellation of the gauge field dependence between the $\mathcal{D}\bar{\psi}$ and $\mathcal{D}\psi$ integration measures.

The problem is that the measure has a gauge field dependent phase, which cannot in general be determined in an unambiguous way. A quantum operator ψ is the contraction of a set of Grassman variables c_i with a set of spinors u_i;

$$\psi(x) = \sum_i c_i(x) u_i(x)\,. \tag{1.97}$$

The set of all ψ is a sum over an index α, which ranges over location and spinorial index;

$$\psi = \sum_\alpha c_\alpha u_\alpha\,. \tag{1.98}$$

Under a unitary change of variables,

$$u \to u\,\mathcal{U}^{-1}\,, \quad c \to \mathcal{U}c \tag{1.99}$$

the Grassman determinant $\mathrm{Det}\,D$, will get rotated by the determinant of the unitary transformation,

$$\mathrm{Det}\,D \to \mathrm{Det}\,\mathcal{U}\;\mathrm{Det}\,D\,. \tag{1.100}$$

Therefore there is a phase which depends on our choice of "canonical" spinors u_i in terms of which we write the path integral measure.

Normally we don't worry about this, because, if we perform a rotation on ψ, we should perform the same rotation on $\bar{\psi}$, and the phase will be opposite between them and will cancel. Now, however, we have to perform a unitary transformation on ψ, into the basis where h$\hat{P}_-$ is diagonal; and we do *not* want to perform the same transformation on P_+, since it will *not* be diagonal in that basis. Therefore, we pick up a phase. If we wanted a chiral theory with no gauge couplings, this again would not be a problem, as the phase would be common and would factor out of the path integral. Now, however, the phase is gauge field dependent, since $\hat{\gamma}^5$ is. It is not obvious, whether there is an unambiguous – or even sensible – way to choose this gauge field dependent phase.

If we are perverse, we can view a vectorlike but massless theory as a chiral theory which happens to have an equal number of right and left handed degrees of freedom. We are sure that such theories exist. The key, in such theories, is that this phase is exactly the opposite between the right and left handed species of fermions, and so it cancels between species. This suggests that chiral theories can be constructed, but only if the phases in the definition of the fermion measure cancel between species [57]. Hiroshi Suzuki [58] and Martin Lüscher [59] have shown (independently) that, to all orders in perturbation theory, this is exactly what happens, and that the criterion that the phase ambiguity vanishes between species, is precisely the condition that the theory is free of gauge anomalies. This means that the lattice can be used as an all-orders regularization of chiral theories satisfying anomaly cancellation.

One expects the nonperturbative analysis to be more challenging; for instance, Witten has shown [60] that SU(2) theory with a single chiral fermion in the fundamental representation is anomalous and cannot exist as a theory, even though it has vanishing perturbative anomalies. Lüscher has made some progress towards nonperturbative construction of chiral theories [59]; he has shown the existence of certain abelian chiral theories [61], nonperturbatively. However no results exist yet for nonabelian theories, which are more interesting because of asymptotic freedom. Lüscher has written two more detailed but still readable reviews of these and other important aspects of the construction of chiral theories on the lattice, [57] or [62]. This is an open and very interesting problem in lattice gauge theory.

1.3 Naive lattice supersymmetry and its failings

It would be immeasurably helpful if we were able to test more of the techniques for studying supersymmetric theories and more of the nonperturbative phenomena in supersymmetric theories by "solving" the theories involved in a non-perturbative way. The lattice is the best candidate method, in general, for solving such theories. Unfortunately, the lattice regulator almost inevitably breaks the supersymmetry. The lattice is, after all, a regularization scheme designed to preserve exact gauge symmetry at the expense of manifest Poincaré invariance. Since supersymmetry is a space-time symmetry, i.e. an extension of the Poincaré group, it is of no surprise that it is broken on the lattice. A great deal of work has gone into looking for ways to avoid or ammeliorate this problem (see Chapter 3 for examples and discussions of some of the techniques), some more successfully than others, though no general scheme for lattice implementation of SUSY field theories with broad applicability has yet been presented. A general scheme is precisely the goal of our work, though we have found it more useful and more enlightening to instead construct specific implementations of the most difficult and complex of a particular class of theories (or simply those theories whose difficulties are unique) and provide straightforward generalizations wherever possible.

In this dissertation we give explicit and detailed instructions for how to make such constructions in two sample theories, both of which are of intense theoretical interest to field theorists, 3D $\mathcal{N}$=1 SYM in Chapter 2 and 4D $\mathcal{N}$=4 SYM in Chapter 4, describe how to extend our previous results for 3D $\mathcal{N}$=2 super QCD to various other gauge theories in 3D in section 2.2 and we discuss other 4D theories briefly in Chapter 3. These constructions all have the distinguished benefit of being on relatively well established theoretical footings throughout.

CHAPTER 2
3D Supersymmetry on the Lattice

One way around the difficulties of discretizing SUSY on the lattice is to consider minimally supersymmetric Yang-Mills theory, which contains only gauge fields and fermions. The theory is then automatically (accidentally) supersymmetric provided one can correctly implement the fermions. Recent advances in fermion implementations [54, 63] have made it possible to achieve this program in 4 dimensions. A basic review of this material and extensive references for those interested in more detailed treatments can be found in Sec. 3.2.

In this chapter we instead consider 3-dimensional minimally supersymmetric Yang-Mills theory ($\mathcal{N}$=1 supersymmetry, with two real supercharges). This theory is free of scalars and so correct implementation of the fermions again yields the right supersymmetric IR limit "accidentally." However, the implementation of the fermions is quite intricate, since one must impose a Majorana condition, and the implementation is further complicated by phases arising both from the fermionic determinant and from a Chern-Simons (CS) term, which is possible (and we will show, required) in this theory.

The goal of this chapter is to give a recipe for studying $\mathcal{N}$=1 SUSY in three dimensions with a Chern-Simons term on the lattice. This theory is believed to display very interesting nonperturbative properties that make it a prime target for simulation. In particular, Witten has conjectured [32] that the theory either preserves or spontaneously breaks supersymmetry, depending on the value of the Chern-Simons term. A lattice study of the theory would then constitute the first test (to our knowledge) of a nonperturbative supersymmetry breaking mechanism.

In section 2.1 we discuss superrenormalizability in quantum field theories with $D \leq 3$ and explain briefly how it enables us to develop lattice regulated SUSY theories (i.e. write lattice actions with supersymmetric continuum limits). Much of this work is drawn from our initial studies of $\mathcal{N}=2$ lattice supersymmetry [64]. In section 2.3 we will review the continuum action of the theory, the necessity of a Chern-Simons term, and the anomaly condition which fixes the Chern-Simons term to take certain half-integer values. We will also present a proof, apparently unrecognized before, that the theory with vanishing Chern-Simons term has a vanishing partition function and is therefore not well defined. In section 2.4 we will describe the discretization process, showing that the magnitude of the fermion determinant can be included using a rooted 3-D (2 component) Wilson-Dirac fermion with SW improvement and mass counterterm. We then show how to extract the Chern-Simons phase and the phase of the properly regularized rooted determinant. In section 2.2 we briefly discuss generalizations of this work (for example [64]) to 3D theories with extended supersymmetry. We leave the simulation itself for a future work; the goal here is to show that such simulations can be done and to provide the required tools.

2.1 Superrenormalizability in $D \leq 3$

No one is interested in the results of a lattice calculation *per se*. After all, a lattice "field theory" is actually just a statistical model, not a true field theory. The reason that the lattice technique can teach us something about field theory, is that in the *infrared*, the correct effective description of a lattice theory is as a quantum field theory. What one must do is to ensure that the infrared behavior of the lattice theory coincides with the continuum quantum field theory of interest, or at least that it does so in the small lattice spacing limit, so that the behavior of the field theory can be probed by making a zero lattice spacing extrapolation.

It is easy to write down a lattice gauge theory which, at tree level, will look in the infrared like the theory of interest[1] . The problem is that, in the UV (at the lattice spacing scale), the lattice theory typically does not have the full symmetries of the theory we are interested in. Generally it is possible to formulate lattice theories so that they have exact gauge and (hyper)cubic symmetries. However, under supersymmetry, the variation of a fermionic field can involve the derivative of a bosonic field; and since derivatives become finite differences on the lattice, supersymmetry will generically be badly broken at the lattice spacing scale. Furthermore, even if we construct the lattice theory to satisfy supersymmetric relations in the infrared, radiative effects involving UV (SUSY breaking) modes will typically communicate those effects to the infrared modes of interest.

The IR effective theory is not the tree level theory. Rather, it is the theory one obtains, by writing down the most general continuum quantum field theory consistent with the field content and symmetries of the lattice, and performing a matching calculation between the lattice theory and that continuum effective

[1] In four dimensions this statement is true of vector-like theories, but serious complications arise if one wants a chiral theory, that is, a theory with two component spinors in a representation which is not real or pseudoreal and which are not balanced by an equal number of spinors of opposite handedness in the same representation, as we discussed in section 1.2.5.

theory, to determine what the actual parameters of the IR effective theory are. For instance, if we made a tree level lattice implementation of the Wess-Zumino model (which in 3 dimensions exhibits N=2 supersymmetry, that is, it has 4 real supersymmetry generators),

$$\mathcal{L}_{\rm bare} = \partial_\mu \Phi^* \partial^\mu \Phi + \bar{\psi} \partial\!\!\!/ \psi + \Big(\lambda \Phi \psi^\top e \psi + {\rm h.c.} \Big) + \lambda^2 \Big(\Phi^\dagger \Phi \Big)^2 \,, \tag{2.1}$$

with Φ a complex scalar and ψ a two component spinor, then we would generically recover an infrared theory where all terms permissible with this field content were present;

$$\begin{aligned} \mathcal{L}_{\rm IR} \;\; = \;\; & Z_\phi \partial_\mu \Phi^* \partial^\mu \Phi + Z_\psi \bar{\psi} \partial\!\!\!/ \psi + \Big(\lambda_y \Phi \psi^\top e \psi + {\rm h.c.} \Big) + \lambda_s^2 \Big(\Phi^* \Phi \Big)^2 \\ & + m_\phi^2 \Phi^* \Phi + m_\psi \bar{\psi} \psi + \; ({\rm High \; Dim.}) \,. \end{aligned} \tag{2.2}$$

Here Z_ϕ and Z_ψ represent the difference in field normalization between the lattice and continuum fields; they can be removed by a field rescaling, but we must keep them in mind when we compare lattice correlation functions with their continuum counterparts. The point is that the IR behavior typically involves radiatively generated terms which do not respect the intended supersymmetry. In particular one does not expect $m_\psi^2 = m_\phi^2$.

In 4 dimensions this problem is severe. The SUSY violating, radiatively induced terms appear at all orders in perturbation theory, with coefficients, at high order, which are only suppressed with respect to the lower order coefficients by powers of a dimensionless coupling. Further, additive scalar mass renormalizations are divergently large at *every loop order*. That is, in 4 dimensions, the contributions to the mass squared parameter are of order

$$\delta m_\phi^2 \text{ at} \qquad 1 \text{ loop: } \lambda^2/a^2 \,; \quad 2 \text{ loops: } \lambda^4/a^2 \,; \quad 3 \text{ loops: } \lambda^6/a^2 \,; \; \ldots \,, \tag{2.3}$$

where a is the lattice spacing. Every such coefficient is problematic; a severe non-perturbative tuning is needed to remove them. It is not at all clear how to perform such a tuning; generally we can only perform non-perturbative tunings in lattice gauge theories if we have one exact conservation law or Ward identity per tuning required.

The beauty of 3D is that the theory is generally super-renormalizable. Consequently, the UV is very weakly coupled; specifically, as the lattice spacing is taken to zero, the coupling at the scale of the lattice spacing falls linearly with lattice spacing a. This means that, while the SUSY breaking nature of the UV regulator radiatively induces SUSY breaking effects in the IR, the matching calculation which determines them converges very quickly. At each loop order, we determine the matching of parameters to one more power of the lattice spacing a. For instance, in the above model, if we compute the mass squared for the scalar field, generated by UV physics, the contributions at different orders in the loop-wise expansion are again of order λ^2, λ^4, λ^6, ... But λ^2 has mass dimension 1. Since the matching calculation involves only UV physics, the only scale which can balance the explicit powers of mass is the lattice spacing scale. (The infrared contribution cancels between fermionic and bosonic loops, precisely because we have arranged the interactions to respect supersymmetry.) Therefore, the terms in the loop-wise expansion are of order

$$\delta m_\phi^2 \text{ at } \qquad 1 \text{ loop: } \lambda^2/a\,; \quad 2 \text{ loops: } \lambda^4\,; \quad 3 \text{ loops: } a\lambda^6\,; \; \ldots \,. \tag{2.4}$$

The one and two loop contributions are significant and must be removed by an appropriate counter-term. However, three and higher loop effects vanish in the $a \to 0$ limit, and so can be neglected.

Fermion mass tunings are even easier,

$$\delta m_\psi \text{ at} \qquad 1 \text{ loop: } \lambda^2\,; \quad 2 \text{ loops: } a\lambda^4\,; \quad 3 \text{ loops: } a^2\lambda^6\,; \; \ldots\,. \tag{2.5}$$

Only perturbative tuning to one loop is required to remove all SUSY violating effects up to O(a). For the scalar self-coupling λ_s^2, the one loop correction is already $O(a\lambda^4)$, and so a tree level treatment is already sufficient.

It is then clear that only a finite loop order is needed before all remaining corrections are suppressed by powers of a. It is therefore feasible to perform the matching calculation to the requisite order analytically, and to tune the lattice theory based on the *purely analytic* result of this perturbative matching calculation, to ensure that the IR effective theory satisfies all relations implied by SUSY up to $O(a)$ corrections.

The example theory we have used for this discussion, the 3D Wess-Zumino model, contains most all the complications and difficulties associated with this tuning already because it contains scalar fields. We have performed this matching calculation in the past for theories with scalars, including the $\mathcal{N}$=2 Wess-Zumino model and $\mathcal{N}$=2 SU(N_c) SYM theory, both with arbitrary matter supermultiplets (fields transforming in the fundamental representation of the SU(N_c) and their superpartners) [64]. This work required a complex but straightforward determination of self-energies in lattice perturbation theory to one loop for the fermions and two loops for the scalars.

In this chapter we will examine minimal supersymmetry in 3D. This theory contains no scalar fields, so it is technically possible to remove all SUSY violating effects up to O(a) with only a relatively simple one loop calculation of the gaugino self-energy in lattice perturbation theory. We have chosen to take the implementation one step further however by improving the Wilson action up to $O(a^2)$ with a Sheikholeslami-Wohlert (SW) term [43] (described below). It is thus necessary to tune the fermion mass to two loops to eliminate contributions up to $O(a^2)$. Though this calculation can certainly be performed analytically, we advocate a very simple off-line nonperturbative determination of the appropriate tuning factor in what follows. To remove all O(a) errors from the lattice theory, it is also necessary to perform fermionic wave function and field strength renormalizations to one loop in perturbation theory, which we have also done below.

We should note that it has been known for some time in the context of minimal supersymmetry in 4D (see section 3.2 for more details) that it is possible to protect a lattice theory from the appearence of spurious additive corrections to fermion masses by implementing the fermions with a remnant of chiral symmetry with either the domain wall or overlap techniques. This technique can be (and has been) implemented with Yang-Mills theory + a Majorana fermion transforming in the adjoint representation of the gauge group and the result is that the effective IR description is accidentally Super-Yang-Mills theory (with the bare mass of the fermion set to zero). These same techniques are available in 3 dimensions with the same basic result, however the large increase in computational power required to implement these techniques, combined with the relative ease of tuning fermion masses in a 3D Wilson fermion implementation, makes these techniques much less attractive.

We should note also that these arguments work even better for lattice implementations of theories in $D<3$. A 2D theory analagous to the one described above would require even less tuning to have a supersymmetric IR description; $[\lambda]=1$ in 2D, so the only $O(a^0)$ SUSY violating effect in the theory is a one loop renormalization of the scalar mass.

2.2 Extended supersymmetry in 3D

Numerous SUSY theories in $D<4$ can be implemented with this approach, and it should be possible now to study a huge amount of important physical concepts and phenomena in lower D SUSY field theories theories with current thoeretical technology and relatively unspectacular computing resources. We have already mentioned the $\mathcal{N}=2$ theories in 3D [64], and we see no reason these results should not easily generalize to more extended 3D SUSY theories such as the $N=8$ SYM theory in three dimensions, which has been conjectured by Seiberg to possess a non-trivial IR fixed point (an interacting conformal theory) [65]. We should be able to generalize the results of [64] to the case of $\mathcal{N}=4,6,8$ with a minimum of work in fact, because the bare mass counterterms that we calculated in that work were independent of group structure.

For example, We can construct $\mathcal{N}=4$ SYM in 3D as $\mathcal{N}=2$ SYM with a single extra Chiral supermultiplet of scalars and fermions transforming in the adjoint representation of the gauge group, provided that we choose the couplings appropriately to satisfy the extra supersymmetry (or R-symmetry) transformations. In this case the necessary mass counterterms that must be added to tune the lattice theory to its SUSY limit are (in the notation of [64])

$$\begin{aligned}
\delta M &= g^2\mathcal{C}_A\frac{C_{gf}+2C_{yf}}{4\pi} \\
\delta m^2 &= -2g^2\mathcal{C}_A\frac{C_{ys}}{4\pi a}+g^4\left\{\mathcal{C}_A^2\frac{C_{234}-\frac{5}{18}\Sigma^2}{16\pi^2}-\frac{4}{3}\mathcal{T}_F\mathcal{C}_A(\mathcal{C}_F-\frac{1}{6}\mathcal{C}_A)\frac{4\pi\Sigma}{16\pi^2}\right\} \\
\text{where}\quad & C_{234}=C_{g2}^{fund}+C_{g3}^{fund}+C_{g4}^{fund}=C_{g2}^{adj}+C_{g3}^{adj}+C_{g4}^{adj}. \qquad (2.6)
\end{aligned}$$

These counterterms are constructed from those of [64]) by $n_f=1$, $n_a=0$, $\lambda=0$, and $\mathcal{T}_F,\mathcal{C}_F\rightarrow\mathcal{C}_A$ (except in the last term of the scalar mass counterterm where the group structure comes largely from the fully symmetric pure lattice artifact piece of the 4-pt gauge boson vertex). Under these exchanges, the scalar counterterms are the same (as expected since now all fields transform in the same representation of the gauge group and are rotated amongst each other by the R-symmetry). Some care must be taken with the fermionic counterterms, but the result is nevertheless straightforward. The equality of the counterterms after these exchanges is dependent upon the relation in line three of the above Eq. (2.6). We had not realized this equality in our previous work, but it can be easily checked from the results of our numerical integrations to be well within the stated error bars.

The N=8 (16 real supercharge) theory can be constructed similarly as the N=2 SYM theory with 3 matter hyper-multiplets of the N=2 theory (matter content: 4 Majorana fermions, a gauge boston, and 7 real scalar fields) all transforming under the adjoint representation of the gauge group. A lattice has recently been constructed in [66] to study this theory by a very different technique. Gerneralizations to other 3D SUSY theories should be straightforward; though we have not examined each theory in detail, we believe that numerical studies of *any* 3D SUSY theory (modulo the existence of a fundamental hinderance to lattice implementation such as a gauge anomaly) could be undertaken immediately.

2.3 3D $\mathcal{N}$=1 SYM in the Continuum

The field content of the theory consists of a gauge field and a 2-component, Majorana fermion in the adjoint representation (gaugino). In 4-dimensional notation, the gaugino is a 2-component Weyl fermion which has been further reduced from 2 complex to 2 real components by the application of a Majorana condition (possible in 3 dimensions). The *Euclidean* action is

$$S = \frac{1}{g^2}\int d^3x\left(\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \frac{1}{2}\bar{\psi}_a(\not{D}\psi)_a\right). \tag{2.7}$$

With these conventions the eigenvalues of the Dirac operator are all pure imaginary. Because the fermion is in a real representation of the gauge group, the fermionic operator possesses a doubled spectrum; if ψ_λ is an eigenvector of $\not{D}$ with eigenvalue $i\lambda$, then $\epsilon\psi^*_\lambda$ (where $\epsilon = -i\sigma_2$) is also an eigenvector with the same eigenvalue;

$$\begin{aligned}
\sigma_\mu D_\mu \psi_\lambda &= i\lambda\psi_\lambda \\
\sigma^*_\mu D_\mu \psi^*_\lambda &= -i\lambda\psi^*_\lambda \\
\epsilon\sigma^*_\mu D_\mu \psi^*_\lambda &= -i\epsilon\lambda\psi^*_\lambda \\
\sigma_\mu D_\mu (\epsilon\psi^*_\lambda) &= i\lambda(\epsilon\psi^*_\lambda)\,.
\end{aligned} \tag{2.8}$$

The Majorana condition consists of taking only one of these degenerate sets of eigenvalues to define the fermion contribution to the path integral. That is, in the path integral replace

$$\det \not{D} = \prod_i \lambda_i \quad \rightarrow \quad \sqrt{\det \not{D}} = \prod_i{}' \lambda_i \tag{2.9}$$

where $\prod_i{}'$ is defined by taking only one eigenvalue from each degenerate pair.

We can add a Chern-Simons term to this action in 3D provided we include an appropriate mass term for the fermions so that SUSY is retained:

$$\begin{aligned}
S_{CS} &= -\frac{ik}{16\pi}\int_x \epsilon_{\mu\nu\rho}\Big(F^a_{\mu\nu}A^a_\rho - \frac{1}{3}f_{abc}A^a_\mu A^b_\nu A^c_\rho\Big) + \frac{1}{2}\frac{k}{4\pi}\int_x \bar{\psi}_a\psi_a \\
&\equiv -2\pi i k N_{cs} + \frac{1}{2}\frac{m}{g^2}\int_x \bar{\psi}\psi\,, \qquad m = \frac{g^2 k}{4\pi}\,.
\end{aligned} \tag{2.10}$$

Here k is the level of the CS theory or CS coupling. It is straightforward to check that the action is indeed invariant under the SUSY transformations

$$\begin{aligned}
&\delta A^a_\mu = \bar{\alpha}\sigma_\mu\psi_a\,, \quad \delta F^a_{\mu\nu} = \bar{\alpha}(\sigma_\nu\partial_\mu - \sigma_\mu\partial_\nu)\psi_a \\
\text{and} \qquad &\delta\psi_a = -\frac{i}{2}\epsilon_{\mu\nu\rho}F^a_{\mu\nu}\sigma_\rho\alpha\,,
\end{aligned} \tag{2.11}$$

with α the Grassman valued Majorana spinor parameterizing the SUSY transformation. Our convention for Majorana spinors is $\bar{s} = s^{\top}\epsilon$.

It has long been known [67, 32] that - for gauge group $\mathrm{SU}(N_c)$ - k must equal $N_c/2$ modulo an integer to avoid a gauge anomaly so that, in particular, the theory with odd N_c is ill defined for vanishing CS term. In Sec. 2.3.1 we review this argument and present a similarly motivated argument for the theory with even N_c that implies that these theories are also ill defined for vanishing k.

2.3.1 Anomalies in 3D SU(N_c) SYM

In [60] Witten gave the first example of a theory with a rooted determinant that is sick with a global gauge anomaly, namely 4D SU(2) gauge theory with an odd number of left-handed fermion doublets. The problem with this theory is that it is impossible to define the fermionic determinant over the space of gauge connections (gauge fields modulo all gauge transformations) such that it is both continuous and single-valued.

We will review the 3D analog, formulated by Redlich [67] for gauge group $\mathrm{SU}(N_c)$, which is slightly simpler and is of immediate importance for the current discussion. The general issue is that there are phase ambiguities in performing Grassman integrations. This is a problem in writing a path integral unless the phase ambiguity can be reduced to a single gauge-field independent phase, which factors out from the partition function and cancels in determining any correlation function. Therefore we pick some gauge field A_0, (chosen so that the Dirac operator has no zero eigenvalues) and call its contribution to the path integral real and positive. Then we determine the sign for any other configuration A by insisting on continuity of $\det \not{D}$ along a path from A_0

to A. This is possible for any gauge field because our group manifold is path connected.

The procedure is sketched in Fig. 2–1: we watch the low lying eigenvalues of the spectrum as the path parameter t is varied from 0 to 1 and count the number of eigenvalue pairs that change sign from positive to negative. This gives the relative sign of the two determinants in the path integral.

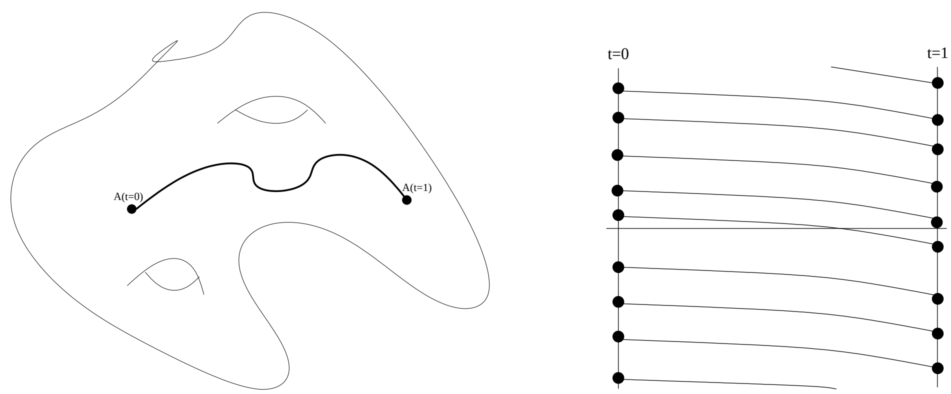


Figure 2–1: A path in configuration space and its associated eigenvalue flow

This prescription is unique unless the number of eigenvalue zero-crossings depends on the path. This can happen because the space of connections is multiply connected; if two paths from A_0 to A form a noncontractible loop, there is no guarantee that they lead to the same sign choice for det $\not{D}(A)$.

Call the space of 3D gauge connections X. It contains noncontractible loops if the third homotopy group π^3 of the gauge group is nontrivial. (It is the third homotopy group because we are in 3 dimensions.) This is the case for all SU(N_c), for which $\pi^3 = \mathcal{Z}$. Consider a nontrivial loop from A_0 back to A_0. This path is effectively a 4D gauge field configuration, where the space is $S_1\times$ 3D space and the 4D gauge fields are $A_M = (A^\mu, A_4 = 0)$. The path is noncontractible if the instanton number of this gauge field configuration is nonzero. The four dimensional Weyl determinant has a number of zero modes

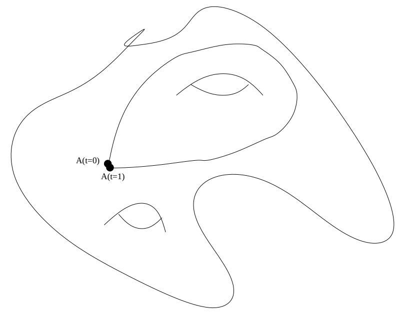


Figure 2–2: A closed loop in configuration space with a non-trivial winding number

determined by the Atiyah-Singer theorem [68, 69, 70]; for the fundamental representation this is 1 and for the adjoint representation this is $2N_\mathrm{c}$. The 4D zeros correspond to zero crossings, and therefore sign flips, of the 3D fermionic determinant. If the number of sign flips in traversing the loop is odd, then the definition of the determinant cannot be both continuous, nontrivial, and single valued.

For our case this is relevant because we want the square root of the Weyl determinant; the $2N_\mathrm{c}$ zero crossings become N_c zero crossings when we choose one from each pair of eigenvalues, and this leads to a sign flip if N_c is odd.

There is an additional sign if the theory is defined with a nonzero Chern-Simons term. The CS term picks up a factor of $2\pi\nu$ in traversing a path of instanton number ν, so the path integral picks up an overall factor of

$$(-1)^{\nu N_\mathrm{c}} \exp(i2\pi\nu k)\,. \tag{2.12}$$

This implies that, in order to avoid a gauge anomaly, $k{=}N_\mathrm{c}/2$ modulo an integer. Only certain (half-integer) values of the Chern-Simons term are allowed, and in particular, the theory with N_c odd and vanishing CS term is ill defined.

We will now show that the supersymmetric theory with N_c even and Chern-Simons coefficient $k = 0$ (and therefore fermion mass of zero) is also anomalous, a point which to our knowledge has not been noticed before. Consider a configuration $A(x)$ and its parity dual $A'(x) = -A(-x)$. We claim that these give canceling contributions to the partition function if the Chern-Simons term is absent and the fermion mass is zero. They clearly have the same bosonic action, so we must show only that their rooted fermion determinants are equal and opposite. We define the sign of the rooted determinant for configuration $A(x)$ to be positive and connect them with a path from $A(x)$ to the trivial vacuum and from the trivial vacuum to $A'(x)$ via the parity dual of this path (see again Fig. 2–1).

The Dirac operator for the trivial vacuum configuration has (N_{c}^2-1) pairs of zero eigenvalues (we implicitly work on a torus with standard boundary conditions). There are then (N_{c}^2-1) pairs of eigenvalues that cross zero at the vacuum configuration. Furthermore, if n_+ pairs cross zero somewhere on the path *between* $A(x)$ and the vacuum, than the number of pairs n_- that cross zero between the vacuum and $A'(x)$ will be the same. The total number of pairs which change sign in going from $A(x)$ to $A'(x)$ is therefore $(N_{\mathrm{c}}^2-1)+2n_+$. This is odd, so the fermion rooted determinant flips sign and the configurations give canceling contributions to the partition function, which vanishes identically. An example of the eigenvalue flow for gauge group SU(2) is shown in Fig. 2–3 for the case m=0.

This problem is avoided at nonzero Chern-Simons term because $A(x)$ and $A'(x)$ have opposite Chern-Simons number and so enter the partition function with opposite phase, rather than canceling. Similarly, at nonzero fermion mass the eigenvalues of the Dirac operator are complex and introduce nonzero phases which are opposite between the configuration and its parity dual.

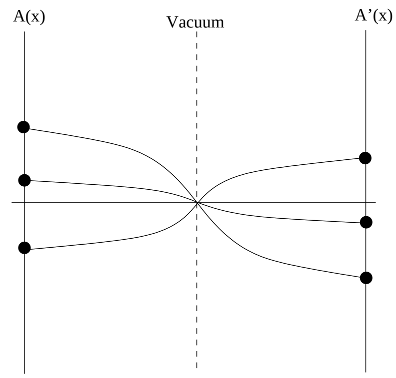


Figure 2–3: Flow of almost zero mode eigenvalues of the Dirac operator as the configuration is varied between parity conjugate gauge fields on a path through the vacuum.

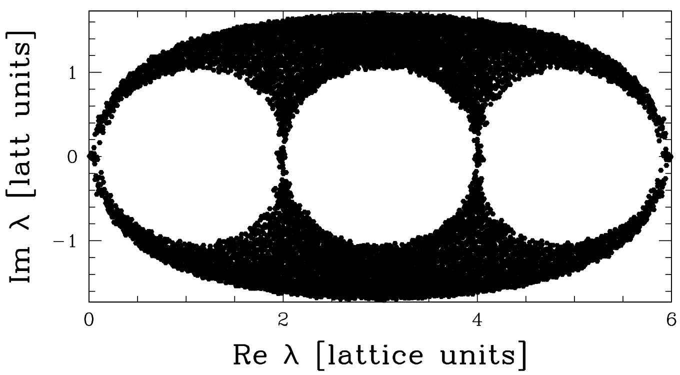


Figure 2–4: Eigenvalues for 20 Wilson fermion configurations in a 8^3 box at $g^2a = 0.5$, illustrating the "return loops" in the complex plane.

2.3.2 Regularization dependence of determinant

It is necessary to clarify what is meant by the "continuum" fermionic determinant. The issue is that continuum theories are always defined as limits of discrete (or otherwise regularized) theories and in a discrete theory with volume regularization there are a finite number of eigenvalues for the Dirac equation. In traversing a closed loop with nonzero instanton number, we just saw that a nonzero number of eigenvalues cross from negative to positive value. Since the final configuration is the same as the starting one, it has the same spectrum. Since there are a finite number of negative eigenvalues, there must be some compensating flow of positive eigenvalues to negative, somewhere in the complex plane. In other words, eigenvalues must "return" somewhere in the complex plane. We illustrate this for Wilson fermions in Fig. 2–4. The figure superimposes the eigenvalue spectra, in the complex plane, of 20 quenched gauge field configurations in an 8^3 box with lattice spacing $g^2 a = 0.5$. Each dot is a pair of eigenvalues (the pairing of the spectrum discussed in the last section occurs for both the Wilson and overlap lattice implementations of the Dirac operator). The spectrum of eigenvalues for $\not{D}$ parallels the imaginary axis near zero but bends out into the complex plane for large eigenvalues $\lambda \sim 1/a$ and forms a loop, so eigenvalues moving from negative to positive values "push" eigenvalues around the loop to reappear at negative values. These extra vanishing-imaginary-part eigenvalues are the lattice fermion doublers which have been pushed out into the complex plane by the Wilson term; for the Wilson action in 3 dimensions there are actually 3 extra places where eigenvalues cross zero imaginary part, corresponding to $(\pi, 0, 0)$, $(\pi, \pi, 0)$, and $(\pi, \pi\, \pi)$ type doublers.

The problem is that this "return loop" will contribute to the partition function even in the continuum limit. Because it represents deeply UV physics, its contribution to the partition function must be representable in an effective IR description in terms of local effective operators. The 3D theory has only one marginal operator, which is the Chern-Simons term. Therefore the return loop can (and will) induce a Chern-Simons term, but will not otherwise change the infrared description (in the small a limit). It is easy to see that the size of this Chern-Simons term is fixed by the rate of spectral flow near zero; the number of eigenvalues which circle around the loop must be the number needed to refill the negative eigenvalues when the spectrum flows upwards.

The sign of this extra Chern-Simons term depends on whether the return loop is at positive or negative imaginary part, which is a regularization detail. The desired continuum theory is the one in which this regularization effect has been removed by a Chern-Simons number counterterm.

2.4 The Discretization

This section describes the discretization of $\mathcal{N}=1$ SYM with a Chern-Simons term in 3D. The theory contains nontrivial phases; that is, different gauge field configurations contribute to the partition function with different complex phase as well as different magnitude. The plan is to treat this using the Edinburgh method; one studies the theory on the lattice by building a Markov chain sample weighted by the magnitude of the action $\left|\exp(-S)\sqrt{\det \not{D}}\right|$ and then includes the phase as part of the observable. Phase cancellation reduces the statistical power in a volume dependent way. Therefore our implementation is on the same footing as finite chemical potential simulations in QCD; they work in principle, but whether they work in practice depends on how severe the phase cancellation problem turns out to be.

The implementation consists of two parts; the real part of the bosonic action and magnitude of the fermionic determinant, and the phase. The real bosonic action is completely standard. We describe the magnitude of the determinant first, then the Chern-Simons part of the phase, then the phase in the determinant.

2.4.1 Fermion Implementation

Though there may be some advantages to using an overlap fermion implementation to do the simulation, we believe that the numerical simplicity of the Wilson implementation makes it a much more sensible choice. The usual problems with Wilson fermions are less severe than in 4 dimensions; chiral symmetry is a non-issue because there is no such thing in 3 dimensions, and the additive renormalization of the mass is well behaved because the theory is super-renormalizable. One can easily work at a fine lattice spacing where the problem of exceptional configurations is well under control (note that we are only interested in the theory at finite fermion mass, since as we just argued the massless theory is anomalous).

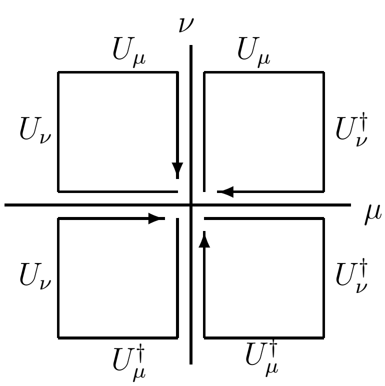
The fermionic action reads

$$\begin{aligned} S_W &= a^4 \sum_x \bar{\psi}_x \left(m_0 + \frac{3r}{a} \right) \psi_x \\ &\quad - a^4 \sum_{x,\mu} \bar{\psi}_x \frac{(r - \sigma_\mu) U_\mu(x) \psi_x + (r + \sigma_\mu) U_\mu^\dagger(x-\mu) \psi_{x-\mu}}{2a} . \end{aligned} \tag{2.13}$$

We can improve the convergence of the spectrum to the continuum limit from $O(a)$ to $O(a^2)$ via the 3D analog of the Sheikholeslami-Wohlert term [43] (SW) term

$$S_{sw} = a^4 \sum_{x,\mu,\nu} \frac{r c_{sw}}{16} \bar{\psi}_x [\sigma_\mu, \sigma_\nu] \Big(P_{\mu\nu}(x) - P_{\mu\nu}^\dagger(x) \Big) \psi_x \tag{2.14}$$

Here $P_{\mu\nu}(x)$ is the *average* of the 4 plaquettes in the $\mu\nu$ plane as shown in Fig. 2–5. This improvement is probably necessary to implement simulations at reasonable lattice spacings.



$$= \frac{1}{4} \mathrm{Tr} \Big(U_\mu^\dagger(x) U_\nu^\dagger(x+\mu) U_\mu(x+\nu) U_\nu(x) + \ldots \Big)$$

Figure 2–5: Clover field strength for the Sheikholeslami-Wohlert improvement term.

Since the 3D theory is superrenormalizable we need only the tree level determination of the SW coefficient, c_{sw}=1, to remove $O(a)$ corrections other than mass renormalizations. Much of the complication of the improvement in 4D is thus avoided in the 3D version.

The improvement to the spectral properties is quite dramatic, though it parallels closely the improvement in 4D so we refer the interested reader to, for example, [71] for analysis of 4D Dirac operator spectra with improved and unimproved actions in a situation which is fairly analogous to the cases we consider in 3D. Fig. 2–6 shows an example of the improvement to the physical branch of the spectrum for 20 configurations on a 14^3 lattice with $g^2 a = 0.5$. Improvement "squeezes" the eigenvalues toward the solid line–a parabola incorporating the "bending" in the complex plane present in the tree level Dirac operator due to the k dependence of the Wilson term. The spectra shown are shifted appropriately so that both represent a fermion determinant for the case of zero physical mass.

The disadvantage of Wilson fermions is that we must tune the fermion mass to remove an additive correction. To consistently improve the theory to eliminate all $O(a)$ errors we need to do this at the two loop level (see [64] for a discussion). Here we present the calculation of the one-loop mass counterterm; in practice both one and two loop counterterms can be determined quite easily numerically by analyzing the low-lying eigenvalues in the Dirac operator spectrum at a few different lattice spacings.

The SW term modifies only the $\bar{\psi}A\psi$ vertex. Following closely the 4D treatment of [72] we determine the new three point vertex to be (with $\bar{\psi}(p)$,

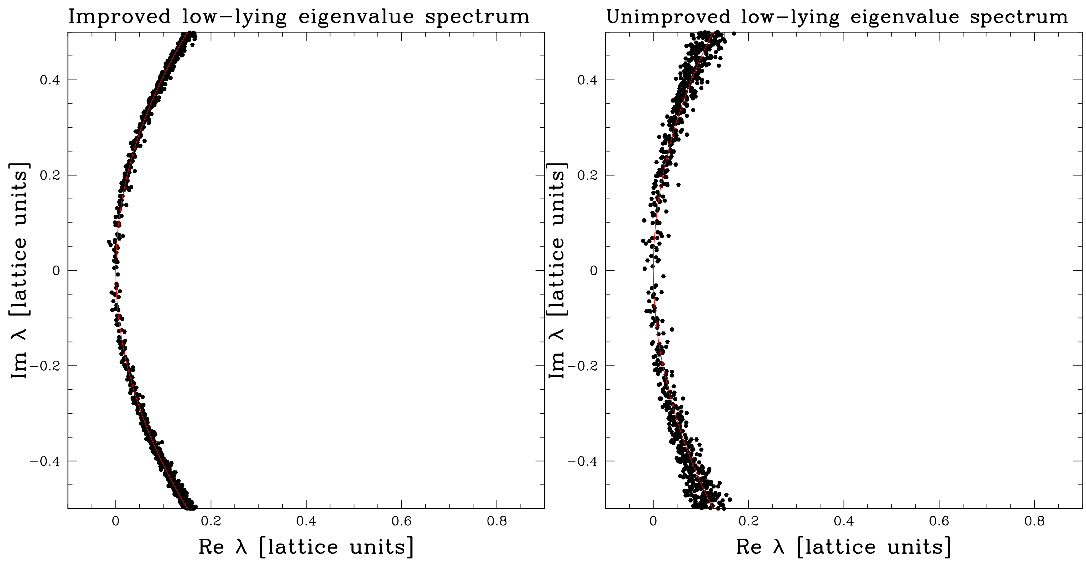


Figure 2–6: Physical branch of the spectrum of the Wilson-Dirac operator; 20 superimposed configurations (left) with and (right) without improvement (14^3 lattices with $g^2a = 0.5$)

$\psi(p')$ and $A(k{=}p'{-}p)$)

$$\Big(V_\mu\Big)^a_{bc}(p,p') = -gT^a_{bc}\left(i\sigma_\mu \cos\frac{(p{+}p')_\mu}{2} + \frac{r}{2}\widetilde{(p{+}p')}_\mu + \frac{c_{sw}r}{4}[\hat{\not{k}},\sigma_\mu]\cos\frac{k_\mu}{2}\right). \quad (2.15)$$

The one-loop bare mass counterterm required to tune the theory to its SUSY limit, for Wilson coefficient $r = 1$, is

$$\delta m = \frac{g^2 C_A}{4\pi}(-2.3260) = \frac{C_A^2}{2\pi\beta}(-2.3260)\,, \quad (2.16)$$

with C_A the first Casimir of the adjoint representation of the gauge group - $C_A{=}N_\mathrm{c}$ for SU(N_c).

To eliminate all $O(a)$ errors it is also necessary to compute a fermionic wave-function renormalization at 1-loop (which can be interpreted as an $O(a)$ multiplicative renormalization of the mass) and a 1-loop renormalization of the field strength (or equivalently of g^2, or equivalently of the lattice spacing). The contribution of bosonic loops to the gauge coupling renormalization was computed in [73], and the fermionic contribution is not difficult. In the same notation as that previous work,

$$\mathcal{L}_{\rm latt}(x) = \frac{2}{Z_g g^2 a^4}\sum_{i<j}{\rm Tr}(1-P_{ij}(x)) + Z_\psi \bar{\psi}\left(D_W + D_{sw} + \frac{Z_m}{Z_\psi}m + \frac{\delta m}{Z_\psi}\right)\psi, \tag{2.17}$$

where it was determined that

$$Z_g^{\rm bos.} = 1 - \frac{ag^2}{4\pi}\left(\frac{2\pi}{9N_{\rm c}}(2N_{\rm c}^2-3) + N_{\rm c}\left(\frac{37\xi}{12}-\frac{\pi}{9}\right)\right) \text{ with } \xi{=}0.152859325, \tag{2.18}$$

we have

$$\frac{Z_m}{Z_\psi} = 1 - \frac{ag^2N_{\rm c}}{4\pi}e_m \qquad \text{with} \quad e_m = 2.75066732(8) \tag{2.19}$$

$$\text{and} \quad Z_g^{\rm total} = Z_g^{\rm bos.} - \frac{ag^2N_{\rm c}}{4\pi}e_g \quad \text{with} \quad e_g = 0.204254488(7). \tag{2.20}$$

To treat $N_{\rm f}$ fundamental representation fermions rather than one adjoint with a Majorana condition, replace $N_{\rm c} \to C_{\rm f}$ in the expression for Z_m/Z_ϕ and change $N_{\rm c} \to N_{\rm f}$ in the expression for Z_g (the 1/2 trace normalization group theory factor cancels a factor of 2 from not imposing the Majorana condition).

As already mentioned, δm is easily measured nonperturbatively by - for example - looking at the lowest lying eigenvalues of the Dirac operator for a range of β values and fitting to find what value of δm is needed to get the zero imaginary part eigenvalues to have vanishing real part as well. To tune out the O(a) error analytically would require a 2-loop determination of the fermion self energy, which is much more difficult.

The numerical implementation of the magnitude of the determinant is achieved by including $(\det \mathcal{D}^\dagger\mathcal{D})^{1/4}$ in the path integral, which can be accomplished by conventional pseudofermion techniques. There should be no issues with locality (we believe) because the operator's spectrum is doubled, so the fourth root is actually well defined (up to a phase).

2.4.2 Evaluation of Chern-Simons number

Here we detail how to determine the Chern-Simons number of a 3D lattice configuration. Our approach is borrowed from an investigation of the electroweak sphaleron rate by one of us [74], but we summarize it here for completeness.

The literal definition of Chern-Simons number $N_{\rm CS}$ in terms of the integral of $F\wedge A - \frac{1}{3}A\wedge A\wedge A$, given in Eq. (2.10), is too gauge dependent and lattice spacing sensitive to be of much use, so we use instead the following (gauge invariant) properties of Chern-Simons number, which can be taken as a definition of $N_{\rm CS}$, modulo an integer:

- $N_{\rm CS}$ for the vacuum is 0 (modulo an integer); and
- the $N_{\rm CS}$ difference between two configurations is (modulo an integer) equal to the integral of $\epsilon^{\mu\nu\alpha\beta}\,{\rm Tr}\,F_{\mu\nu}F_{\alpha\beta}/32\pi^2$ along a path through configuration space connecting those configurations.

The second point requires some clarification. As previously discussed, given two 3D configurations $A^i_1(\mathbf{x})$, $A^i_2(\mathbf{x})$, one can find a path connecting them through gauge field configuration space; that is, one can find $A^i(\mathbf{x},\tau)$ with $\tau\in[0,1]$ an affine parameter and $A^i(\mathbf{x},0)=A^i_1(\mathbf{x})$, $A^i(\mathbf{x},1)=A^i_2(\mathbf{x})$. We may think of the path as a 4-dimensional gauge field configuration, with τ as the fourth coordinate, $D_0\equiv D_\tau$, and $F_{0i}=i[D_0,D_i]$. The $N_{\rm CS}$ difference between two configurations is the integral

$$N_{\rm CS}(A_2)-N_{\rm CS}(A_1)=\int d^3x\int_0^1 d\tau\epsilon_{ijk}\frac{{\rm Tr}\,F_{0i}F_{jk}}{8\pi^2}\,. \tag{2.21}$$

We have not specified how A_0 is to be chosen along the τ direction, but this turns out not to matter; a different choice leads to a change in F_{0i} of $D_i\delta A_0$, but the D_i can be integrated by parts onto F_{jk} and $\epsilon_{ijk}D_iF_{jk}$ vanishes by the

Bianchi identity. The idea is then to define $N_{\rm CS}$ for a configuration (modulo an integer) by finding a path from that configuration to the vacuum and integrating $F\tilde{F}$ along that path.

The complication in using this approach to define $N_{\rm CS}$ on the lattice is that there is no lattice definition of the field strength $F_{\mu\nu}$ which satisfies the Bianchi identity; therefore the procedure is ambiguous. Further, no specific prescription for choosing A_0 generically leads to an integer $N_{\rm CS}$ around a closed loop. There are mathematically rigorous [75, 76, 77] and numerically implementable [78, 79] methods to find the integer value around closed loops, but these are not helpful here, since we really want the value on a path with distinct beginning and end points. The trick instead is to note that the problems with lattice implementations of $F\tilde{F}$ arise when the fields are "coarse" (plaquettes far from identity, most of excitations at the lattice spacing scale). We can define $N_{\rm CS}$ uniquely by choosing any unique prescription for a path from a configuration to the vacuum. We can ensure that the result is as close as possible to the continuum meaning of $N_{\rm CS}$ if our unique prescription is one which quickly smooths out the lattice-spacing scale fluctuations in the gauge field configuration. The early "smoothing out" part of the path then contributes a small UV "lattice artifact" contribution to $N_{\rm CS}$ and the remaining path gives a contribution which closely resembles the continuum value of $N_{\rm CS}$ for this configuration.

A good choice is the "cooling path" or gradient decent under the energy,

$$D_\tau A^i(x,\tau) = -\frac{\partial H[A(\mathbf{x},\tau)]}{\partial A^i(x,\tau)}\,, \qquad H[A] \equiv \int d^3y\, \frac{1}{2}\,{\rm Tr}\, F_{ij}^2[A(\mathbf{y},\tau)]\,. \qquad (2.22)$$

We use a Symanzik [80, 81] or "rectangle-improved" definition of H and of the field strength appearing in $F\tilde{F}$, as described in [74], which also shows extensive tests of the approach.

To summarize, the procedure is to determine the Chern-Simons number of a lattice configuration by integrating $F\tilde{F}$ along the "cooling" or gradient-decent path through configurations to the vacuum. The procedure is unique and gives an answer which is continuous over the space of gauge field configurations except at a "sphaleron" separatrix, where it is discontinuous by (almost exactly) an integer. If Markov-chain configurations are tightly enough sampled, one can determine the integer part by continuity.

On fine lattices this definition of Chern-Simons number should correctly reproduce the continuum notion up to corrections suppressed by two powers of the lattice spacing.

A straightforward alternative method to implement integer Chern-Simons numbers would be to evaluate the phase in the determinant of a fundamental representation Wilson fermion with a negative mass of order the lattice spacing (so the origin of the complex plane is inside the leftmost "circle" in Fig. 2–4). This method would be numerically much less efficient, since the numerical effort in taking a determinant rises as the third power of the number of lattice points or a^9, while the approach presented only grows worse as a^5 and can be reduced to a^3 through the careful use of blocking [74].

2.4.3 Fermionic phase

Since in practice only the *magnitude* of the rooted fermion determinant can be included dynamically in a lattice simulation, we must still describe a prescription for assigning to the Wilson-Dirac operator a phase for each field configuration in order to complete the prescription for the discretization of the theory.

One approach to do this would be to explicitly evaluate the (complex) determinant of $\mathcal{D}$ for each configuration in the Markov chain and then halve the phase, fixing the sign ambiguity by using a tightly sampled Markov chain and demanding continuity. Then one must subtract the Chern-Simons phase contribution from the "return loop" discussed in Subsection 2.3.2. This approach is correct but numerically expensive.

After correcting for the Chern-Simons term induced by the "return loop" this phase is dominated by the contributions of the low lying eigenvalues, so we can develop a more efficient procedure by focusing on determining the phase arising from these eigenvalues. Since this point is key to our approach we should explain it in a little detail. Look again at Fig. 2–6. In the continuum limit the eigenvalues will lie, not on an arc, but on a straight line with real part m and imaginary part set by the eigenvalue under the $\not{D}$ operator (for the free theory, by k). At large eigenvalue the weak-coupling approximation is valid (since the theory is super-renormalizable this is true by a power of the eigenvalue). The density of eigenvalues therefore scales with the free theory density of states, $k^2 dk$. An eigenvalue's phase difference from $\pm\frac{\pi}{2}$ is $\tan^{-1}(m/k)$, so naively the phase arising from large eigenvalues could be large. What is important, though, is the phase difference configuration by configuration, and this becomes small, essentially because the large eigenvalues do not change very much as a function of the gauge field. To see this, note that the large eigenvalues represent short-range physics. The influence on the effective IR behavior can be expanded as Wilsonian renormalization of effective IR operators. The lowest order parity-odd operator is the Chern-Simons term; all others are higher by at least two powers of derivatives and therefore suppressed by at least g^4/k^2. Since we are allowing $O(a^2)$ errors, the only operator we need to incorporate correctly from the large k eigenvalues is therefore their contribution to the Chern-Simons term. Therefore our strategy will be to include low eigenvalues' phases explicitly and to determine the phase contribution of large eigenvalues in terms of their contribution to an effective Chern-Simons term.

We can extract the smallest M eigenvalues using the Arnoldi method at much less numerical effort than is required to determine the full determinant. Each such eigenvalue λ_i of the Wilson-Dirac operator with renormalized mass m takes a value in the complex plane, (r_i, i_i). In the continuum limit the real parts r_i always equal m. At finite lattice spacing there will be $O(g^4a^2)$ and $O(i^2a^2)$ deviations in the real part. (There would be $O(g^2a)$ deviations arising from the dimension 5 operator $\bar{\psi}\sigma_{\mu\nu}F^{\mu\nu}\psi$ had we not included the SW term.) We "clean" the low lying eigenvalues by projecting them to the $r = m$ axis, see Fig. 2–7. Each eigenvalue then contributes a phase

$$\phi_i = \tan^{-1}\left(\frac{\sqrt{(r_i - m)^2 + i_i^2}}{m}\right) . \tag{2.23}$$

This amounts to projecting the physical branch of the Wilson-Dirac spectrum onto the axis of the continuum spectrum and then calculating the phase as sketched in Fig. 2–7. The idea is to incorporate the phase of all eigenvalues for which the angle ϕ_i lies in a range $[-\phi_{\max}, \phi_{\max}]$. In the $a \to 0$ limit we must take $\phi_{\max} \to \frac{\pi}{2}$. This requires more eigenvalues at finer lattice spacing; this can be made more efficient by using the shifted Arnoldi method.

The Markov evolution of the gauge fields U_μ will move around the eigenvalues so that eigenvalues regularly move in and out of the "window" in which we explicitly include them. When an eigenvalue goes above or below $\phi_{\max}$, the

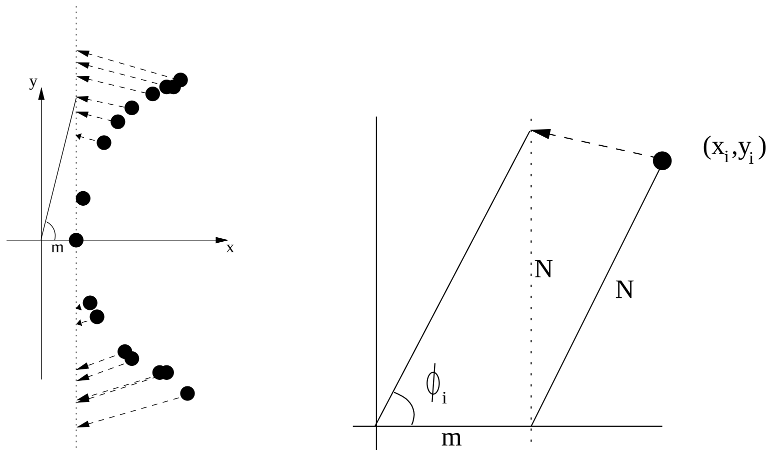


Figure 2–7: Sketch of the projection for the phase prescription.

phase we determine will abruptly change by $\mp\phi_{\rm max}$. Therefore each configuration along the Markov chain must be reasonably close to the last, so that these phases can be determined by continuity. The change in phase between neighboring configurations $U_{1,2}$ contributed by all eigenvalues lying above $\phi_{\rm max}$ *and* below $-\phi_{\rm max}$ is simply an effective Chern-Simons term, as discussed above. The size of the contribution can be determined by considering the amount of spectral flow due to a changing Chern-Simons number, and is well approximated by $2(\frac{\pi}{2} - \phi_{\rm max})N_{\rm c}\,(N_{\rm CS}(U_2) - N_{\rm CS}(U_1))$. We have confirmed this in quenched simulations, for instance by analyzing the $\phi_{\rm max}$ dependence of the procedure and seeing that this contribution ensures independence on this artificial parameter. If we choose the vacuum with, for example, $N_{\rm CS} = 0$ to have zero phase, then this prescription uniquely determines a phase for all configurations.

2.4.4 The sign (or phase) problem

Now that the appropriate implementations of the fermionic field content and the Chern-Simons term have been detailed there remains no fundamental barrier to the simulation of the theory and so we turn our attention to an important technical issue of the simulation.

The lattice simulation consists of replacing the path integral by a sum over a finite set of link field configurations that are distributed with a probability given by the Boltzmann factor for the theory, $|\exp(-S[U])|$. For complex action, the ensemble average for an observable $\mathcal{O}$ will be

$$\langle\mathcal{O}\rangle \approx \frac{\sum_{i=1}^{N}\mathcal{O}(U_i)e^{i\phi_i}}{\sum_{i=1}^{N}e^{i\phi_i}}\,, \tag{2.24}$$

where $\phi = \arg\exp(-S[U])$. Obviously this leads to cancellations between link configurations from the sampling, and thus to a reduction in statistics. That is, for a sample of N independent configurations, the error in the numerator scales as $\sqrt{N}$; but the denominator will be smaller than N, so the error in the operator will not be $1/\sqrt{N}$. This problem could be eliminated by performing "phase quenching" on the theory, but this does uncontrolled damage to the theory which in our case we believe is severe. Therefore we must face this phase cancellation issue.

To determine how bad the phase cancellations will be, we start by looking at the theory with very large fermion mass, so that the effect of integrating out the fermion is well approximated by a shift to the Chern-Simons coefficient, $k \to k - N_{\rm c}/2$. In large volumes, we expect that N_{cs} will be Gaussian distributed around zero. The degree of phase cancellation is determined by how badly our determination of 1 in the denominator of Eq. (2.24) is "suppressed": all measurables must be scaled by the result of the partition function which we evaluate as $\langle 1\rangle$ with the average replaced by a sum on configurations with phases. Fluctuations go as $1/\sqrt{N}$ with N the number of configurations (unless each configuration contributes better statistics–this depends on the measurable). The point is that these fluctuations are to be compared with an average value which suffers phase cancellation. We estimate this by doing the (Gaussian) integral over N_{cs} to find how big the partition function actually is:

$$\langle 1 \rangle = \int \frac{dN_{cs}}{\sqrt{2\pi\chi}} \exp(-i2\pi k N_{cs}) \exp(-N_{cs}^2/2\chi) \tag{2.25}$$

with χ the variance of N_{cs} and k the effective Chern-Simons coefficient. The integral gives

$$\exp(-2\pi^2 k^2 \chi) \,. \tag{2.26}$$

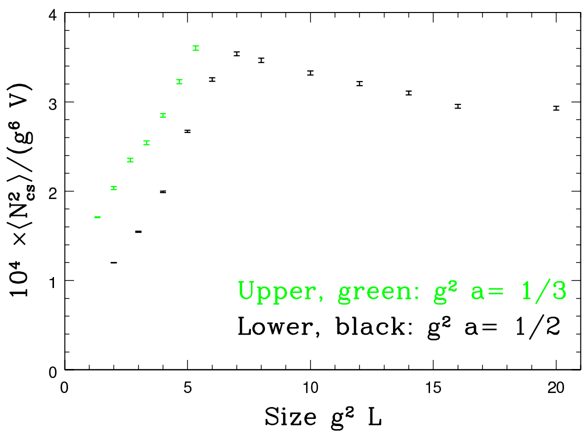


Figure 2–8: (color online) Variance of $N_{\rm CS}$ as a function of volume, for two lattice spacings, in quenched SU(2) gauge theory and using the definition of $N_{\rm CS}$ presented in the text.

Fig. 2–8 shows the dependence of χ/V on the volume of the lattice, in quenched simulations. As the figure shows, going to a finer lattice induces a constant shift in χ, and the volume dependence becomes weak for boxes larger than about $8/g^2$. At this value, on a "reasonably fine" lattice of $g^2 a = 0.5$ [$\beta = 8$], the $N_{\rm CS}$ variance is about .175, leading to a sign problem induced loss of statistical power of order $\exp(-2\pi^2 k^2 \times .175) = 1/30$ for $k = 1$. This volume is therefore at the limit of practicality. Statistical power falls exponentially if we try to make the volume any larger. However, a box of length $8/g^2$ is very large. The theory has a mass gap and the large volume limit should be approached rapidly in a box a few times longer than the longest correlation length. For the SU(2) theory, the lightest glueball mass is $m_{0^{++}} \simeq 1.66g^2$ [82] and the inverse correlation length involved in the Debye mass is $\simeq 1.14g^2$ [83]. These both suggest that the dominant physics is on quite short scales $\sim 1/g^2$, though the string tension suggests a longer correlation length $1/\sqrt{\sigma} \simeq 3/g^2$ [82]. Therefore this volume is probably sufficient to effectively achieve the continuum limit.

The sign problem grows more severe at large k, as indicated by Eq. (2.26). Fortunately the mass scales with k, so we can reduce the volume as $V \propto 1/k^3$ in the large k limit. We must also tighten the lattice spacing a to keep ka fixed, and since $\langle N_{\rm CS}^2 \rangle$ turns out to have a linear UV divergence (which causes the lattice spacing dependence observed in Fig. 2–8), this means that χ will scale as $V/a \sim k^{-2}$. In the large k limit the severity of the sign problem therefore approaches a finite limit.

Fig. 2–8 is based on quenched configurations. When we include the effects of dynamical fermions, we expect the situation at small k to improve for two reasons. The first is that inclusion of the magnitude of the determinant in the Boltzmann factor suppresses configurations with large $N_{\rm CS}$ (an effect we have observed in preliminary quenched simulations). This is not surprising since we know that Dirac operators for configurations with half-integer $N_{\rm CS}$ (sphalerons) have zero eigenvalues. We thus expect this suppression to be more effective at smaller fermion masses. We will not attempt to quantify this statement further.

The second reason is that the phase contributed by the determinant partially cancels against the phase from the Chern-Simons term. For large fermion mass, for example, we know from continuum methods that integrating out the fermion gives a shift in k or, equivalently, a contribution to the partition function $\exp(i\pi N_c N_{CS})$. Furthermore, we have confirmed for the case of N_c=2 Wilson-Dirac fermions – with the prescription described above for assigning determinant phases – that this is approximately the phase of the rooted fermion determinant for the relatively small masses (k=1, 2, 3) of interest here. An example is shown in Fig. 2–9 for k=1. In this case the cancellation of the phase in the partition function is nearly exact, and the phase problem disappears. Finally we remark that, in the algorithm for determining N_{CS}, one can begin the integration of $F\tilde{F}$ after a short amount of cooling, eliminating the contribution from the most UV modes. This eliminates a lattice artifact "noise" contribution to χ but leaves that from interesting IR physics intact.

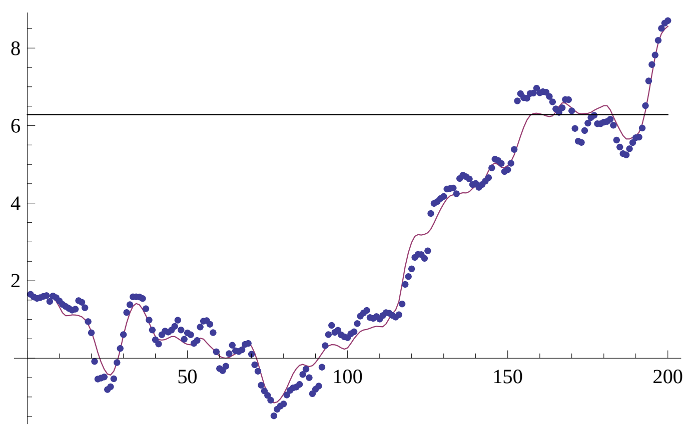


Figure 2–9: Evolution of the phase of the rooted fermion determinant over a Markov chain of length 200. The continuous line is $2\pi N_{cs}$. The system transitions through a sphaleron and subsequently fluctuates around the topologically distinct vacuum at 2π.

2.5 Conclusion

Three dimensional minimally supersymmetric gauge theory can be implemented on the lattice (as can extended supersymmetry). As we showed, this theory necessarily involves complex phases; the theory with vanishing Chern-Simons term and massless fermions is anomalous because the path integral is odd under parity transformations and the partition function therefore vanishes identically.

Our implementation has concentrated on numerical efficiency. Lattice spacing corrections should first occur at $O(a^2)$. Fermions need only be implemented using the clover-improved Wilson method, not the much more expensive overlap method. The Chern-Simons phase can be determined without reference to fermionic operators, and the phase in the rooted Dirac determinant is determined by finding only the low lying eigenvalues using the Arnoldi method, rather than by taking the full determinant. This efficiency is important because the theory suffers from a sign problem which will make it difficult to take the large volume limit. We are guardedly optimistic that the sign problem will not be as severe as might be feared. First, the phase of the fermionic determinant and of the Chern-Simons term are of opposite sign and partially cancel. Second, when the Chern-Simons term is large, the theory is massive and the volume requirement should be reduced.

Now that an implementable method has been presented, it would be very interesting to study 3D $\mathcal{N}=1$ SYM on the lattice. In particular it would greatly improve our insight into nonperturbative supersymmetry breaking if we could study and (presumably) verify Witten's conjectures regarding spontaneous SUSY breaking in this theory [32]. It would also be interesting to study extended SUSY theories nonperturbatively in 3D, both for their own phenomenology and because we can learn many things through these relatively simple and numerically inexpensive simulations that would provide valuable insights into techniques and phenomenology for lattice studies of 4D SUSY field theories, which require vastly more resources. We now turn our attention to these 4D theories.

CHAPTER 3
4D Supersymmetry on the Lattice

3.1 Some succeed while others fail

It is often assumed that the study of supersymmetry on the lattice will be difficult or impossible in four dimensions because multiple relevant/marginal operators that violate SUSY are allowed by lattice symmetries (see section 1.3). This is partially correct, as we will explore in the following sections.

In Section 3.2 we will discuss some of the work on $\mathcal{N}$=1 Super-Yang-Mills. In this case SUSY arises accidentally in the continuum limit, once the appropriate fermion implementation has been established. In Secs. 3.3 and 3.4, we will contrast this rather elegant result with two examples of theories that have yet to find satisfactory lattice implementations, $\mathcal{N}$=1 Wess-Zumino theory, and $\mathcal{N}$=2 Super-Yang-Mills theory. In both cases it is the presence of scalar fields and Yukawa terms in the model that hamstring the approach, and both offer interesting lessons that we will apply to $\mathcal{N}$=4 Super-Yang-Mills in the next chapter.

A rather large literature exists on approaches to discretizing SUSY theories in ways very much unlike conventional lattice action transcriptions. The basic idea in each approach is to try and find creative discretizations that preserve some (usually small) amount of the supersymmetry directly in the lattice action, so that some (or perhaps even all) of the fine tunings can be made unnecessary. Two of these approaches, the "orbifold" prescription [66, 84, 85, 86] and the “twisted SUSY” prescription [87, 88, 89, 90, 91] have gained some popularity and momentum in recent years. These efforts have shown some results already in low-D supersymmetric theories [92], and the hope is that they can be successfully extended to 4D [91]. Only a highly restricted set of theories can be implemented with these techniques thus far however ($\mathcal{N}$=D=2, 4 and some of the respective dimensional reductions of these), and there are many unresolved issues in general and with the 4D actions in particular. We will discuss the twisted formulation and some difficulties of the technique in some detail in section 3.5. We take the perspective that many different avenues of study will be necessary to understand the vast and rich nonperturbative phenomenology of supersymmetric field theory, and so none should be ignored. A good review of these efforts is [93].

3.2 $\mathcal{N} = 1$ SYM, with and without GW fermions

Pure $\mathcal{N}$=1 Super-Yang-Mills [94] is a notable exception to the major difficulties of 4D lattice supersymmetry. In this case, the only SUSY-violating operator that is generated in the effective action unsuppressed by the lattice spacing is a gaugino mass term. Two ways to implement this theory on the lattice have been persued heavily in the last decade. The continuum action of $\mathcal{N} = 1$ SUSY is

$$\mathcal{L} = -\frac{1}{4}F^2 - \frac{1}{2}\bar{\psi}\partial\!\!\!/\psi\,. \tag{3.1}$$

The first method was proposed by Curci and Veneziano as early as 1987 [95], and requires only well established tools of lattice simulations. It was popularly assumed before this that SUSY and the lattice could simply not be reconciled at all, barring the development of some sort of novel new theoretical technique. By simply implementing Yang-Mills theory and an adjoint Majorana gaugino both with the usual Wilson actions however, and then tuning the gaugino mass by some nonperturbative mechanism (such as those described in section 4.3) to zero, supersymmetry can be restored in the continuum limit. The idea is precisely the same as we employed in the last chapter to implement minimal supersymmetry in 3D and it has been employed in 4D simulations with limited to moderate success over the last ten years [96, 97, 98, 99, 100, 101, 102, 103], mostly by the DESY-Munster-Roma collaboration (or subsets). The dissertation of Roland Peetz [104] gives a good review of much of these efforts.

The other option is to implement an exact lattice chiral symmetry in the form of Ginsparg-Wilson (GW) fermions (see section 1.2.8). This prevents additive renormalization of the gluino mass in the continuum limit, and hence the only SUSY-violating operator is forbidden in that limit by setting the bare mass to zero [105, 106, 107, 108, 109, 110, 111]. It is the latter technique which appears to hold the most promise for now; 'See especially [111] for exciting new results that are just being minted. We will use this technique in the following chapter to implement the $\mathcal{N}$=4 theory.

3.3 When scalars attack: $\mathcal{N} = 1$ WZ theory in 4D

The $\mathcal{N}$=1 Wess-Zumino model is the simplest SUSY theory one can write down, in terms of both field content and interactions. The field content is a single complex scalar field and a single Majorana fermion. The interaction terms are ultralocal and so are easily transcribed to a bare lattice action. The continuum action is simply

$$\mathcal{L} = |\partial_\mu \Phi|^2 + \bar{\psi}\partial\!\!\!/\psi + \lambda\Phi\bar{\psi}\psi + \lambda^2(\Phi^*\Phi)^2 \,. \tag{3.2}$$

Given these facts, we might expect that a lattice implementation of this theory would be possible. The problem arises because there are no symmetries in the theory to protect against generation of scalar masses from quantum effects except for the very supersymmetry that is broken by the lattice. The only reason that it is possible to implement minimally superysmmetric Yang-Mills theory on the lattice is because fermions can also be protected from obtaining a mass by Chiral symmetries (and gauge bosons are protected by gauge symmetries of course). Chiral symmetry is also broken on the lattice, but the breaking is less severe and its possible to arrange to retain a piece of it (enough of it) using GW fermions or to tune the lattice theory to its Chiral limit while simultaneously removing the lattice. While there are still difficulties associated with simulations at very small fermion masses (see section 1.2.5 for example) we view these as mostly technical.

There is no such secondary symmetry for scalar fields. Large quantum corrections to the scalar mass are generated by UV SUSY breaking modes from the lattice spacing scale appearing at every loop order (see section 2.1). The only option left to implement this theory is numerically expensive non-perturbative tunings of the scalar mass within the simulation itself. We describe tunings like this in great detail in section 4.3, where we have proposed their usefulness in the implementation of 4D $\mathcal{N}$=4 SYM theory on the lattice. This technique would almost certainly work also to simulate the massless Wess-Zumino model; if we consider the mass as a deformation of the SUSY theory, there should be a well defined phase transition in the theory as we vary the mass through zero, allowing relatively easy nonperturbative tuning using something like the method of Binder cumulants, described in section 4.3. Implementation of the massive Wess-Zumino model is not so clear. Besides the addition of a mass term for the scalar and fermion, the massive theory also contains a scalar cubic term (because there is now a quadratic term in the superpotential), so that the behavior is less obvious. Since the limiting factor in the field of lattice supersymmetry will continue to be numerical resources and cost for the near future, we do not expect Wess-Zumino models to garner enough interest to be implemented on the lattice, so that these technical challenges are of limited importance.

This theory has one other problem that is less well known: the incompatibility of lattice fermions with the Majorana decomoposition in the presence of Yukawa terms. This fact is not at all surprising with hindsight since the Yukawas are Chiral, they couple the left and right handed components of the fermions independently, and Chiral symmetry is broken on the lattice. We may have hoped that GW fermions would work unmodified in the presence of Yukawas, but this is not quite the case. This difficulty can be resolved by adding a set of auxilliary (nondynamical) fermion fields to the action, though we save the details for section 4.2.

3.4 Extended SUSY: $\mathcal{N} = 2$ SYM in 4D

In much the same spirit as section 2.2, the action of $\mathcal{N}$=2 SYM can be constructed as $\mathcal{N}$=1 SYM with an extra Chiral supermultiplet transforming in the adjoint representation tacked on and the couplings chosen to satisfy the supersymmetries

$$\mathcal{L} = \frac{1}{g^2}\Big\{ -\frac{1}{4}F^2 - \frac{1}{2}\bar{\chi}_A(\not{D}\chi)_A - |D_\mu\Phi|^2 - \frac{1}{2}\bar{\psi}_A(\not{D}\psi)_A$$

$$-\sqrt{2}f^{ABC}\left(\bar{\chi}_A P_L \psi_B \Phi_C^* - \bar{\psi}_A P_R \chi_B \Phi_C\right) + \frac{1}{2}\left(f^{ABC}\Phi_B^*\Phi_C\right)^2\Bigg\}. \quad (3.3)$$

Obviously this theory suffers from the same difficulties as the Wess-Zumino model. The presence of a scalar field means some form of nonperturbative tuning would be necessary to tune this theory to a SUSY continuum limit and the presence of the Yukawa terms mean that auxilliary fermion fields must be added to implement the Majorana condition consistently.

What about the fact that there are two fermion fields in the theory? Does that add extra complications to the continuum limit. The answer turns out to be no and the reason is the following.

Extended SUSY theories have multiple sets of SUSY generators that are connected by a symmetry, called R-symmetry (which we already discussed in section 4.1). This symmetry survives in the action as a field space symmetry that rotates the fermions between each other and the scalars between each other in such a way that the action is invariant. In the case where we consider $\mathcal{N}$-extended SUSY algebras, this R-symmetry group is precisely SU($\mathcal{N}$). The complex scalar field Φ is thus a singlet of SU(2). The two adjoint representation Majorana fermions transform in the fundamental **2** representation of SU(2) ($\simeq$ SO(3), which is simply generated by the usual Pauli matrices). Provided the lattice implementation retains this symmetry, or at least some (hopefully large) subset of it, the UV renormalizations of the IR theory will be greatly restricted. This picture is slightly modified in the presence of the auxilliary fields (see section 4.2), but the results are the same.

How many tunings would then be required to tune a bare lattice action to this SUSY continuum theory, besides of course the scalar and, if necessary, fermion mass terms? For gauge group SU($N \leq 3$), it turns out that the Yukawa and quartic terms in the bare actions are the unique R-invariant forms of these terms. They will get renormalizations that will require tuning of course, but that's it. For SU($N\geq4$) there is one other scalar quartic term that must be tuned. This is explained in more detail in section 4.3. Assuming that we are using GW fermions to simulate the theory, which is probably necessary so that all tunings are of purely bosonic terms and can be done offline (we can tune the Yukawa term by rescaling the scalar kinetic term), this means that we have either 3 or 4 nonperturbative tunings that we must perform to simulate the theory.

What if we somehow broke the R-symmetry in the lattice formulation? How different would this counting procedure be? Very. Without the unbroken R-symmetry, a huge variety of scalar quadratic and quartic terms would be generated independently in the IR theory and they would all have to be nonperturbatively tuned away. This would be impossible, and it is one of the reasons that we are wary of the twisted SUSY lattice approach proposed by Simon Catterall, which mixes up the spatial and R-symmetry rotation groups and uses the resulting twisted variables in the action instead of the usual ones.

3.5 Twisted SUSY on the lattice

3.5.1 Twisted field variables

Catterall's twisted approach only applies to theories in D dimensions where the number of continuum supercharges is a multiple of 2^D. All we will need to know about the technique for the analysis herein is that the action is written in terms of twisted field variables (see [113] for a well known example of topological twisting in SYM theory). Catterall's twisting can be thought of as a mixing between the internal R-symmetry rotation group and the spatial rotation group. We define a new spatial rotation group

$$SO(D)' = \mathrm{diag}(SO(D) \times SO(D)_R)\,, \tag{3.4}$$

and then we evaluate the behavior of the SUSY generators under this new group, and use that as our basis for translating fields to the lattice. The limitation to $\mathcal{N} = D = 2, 4$ is now clear, since otherwise the R-symmetry group would not be appropriate for the mixing. It turns out that one of the SUSY generators transforms trivially under this new rotation group so that we might hope that the resulting field theory can be transcribed to the lattice without breaking this one supersymmetry.

The full expansion of the SUSY generators in this new twisted basis is

$$q = QI + Q_\mu \gamma_\mu + Q_{\mu\nu}\gamma_{\mu\nu} + \dots\,. \tag{3.5}$$

Since the generators now have general tensorial Lorentz structure, the fields of the resulting field theory, both scalar and fermion, will as well. In general then the scalars, $(\phi, A_\mu, B_{\mu\nu}, W_{\mu\nu\rho}, \bar{\phi}(=\phi^\dagger))$, and fermionic, $(\eta, \psi_\mu, \chi_{\mu\nu}, \theta_{\mu\nu\rho}, \kappa_{1234})$, field variables of the 4D theory (for the 2D theory we keep only the first three) transform nontrivially under space rotations and must therefore be translated to the lattice as site, link, plaquette or hypercube fields. Though this leads to rather peculiar looking actions, it is really no more than an exotic change of variables.

It turns out that the fermionic fields that result from this technique can be precisely described *in the continuum* by a Kähler-Dirac (KD) fermion. For our purposes we can think of the KD fermion as much the same as the staggered fermion: multiple degenerate fermions composed into a single fermionic object that gets spread out on the lattice. It is really just a matrix valued fermion field that satisfies, instead of the usual Dirac equation, the Käler-Dirac equation

$$(d - d^\dagger)\Psi = 0\,, \tag{3.6}$$

where d is the exterior derivative and $d^\dagger$ its adjoint.

We will first treat the *much* simpler case of $\mathcal{N}$=D=2 twisted lattice SUSY [90], since numerical studies of this theory [112, 92] have already shown some results, and have verified that the full supersymmetry is restored in the naive IR limit (the corrections vanish as some power of the lattice spacing).

3.5.2 $\mathcal{N}$=D=2 twisted lattice SUSY

The continuum action of the theory after integrating out the B field is

$$\begin{aligned} S = \beta\,\mathrm{Tr}\int d^2x\Big(&\frac{1}{4}[\phi,\bar\phi]^2 - \frac{1}{4}\eta[\phi,\eta] - F_{12}^2 - D_\mu\phi D_\mu\bar\phi - \chi_{12}[\phi,\chi_{12}] \\ &-2\chi_{12}(D_1\psi_2 - D_2\psi_1) - 2\psi_\mu D_\mu\eta/2 + \psi[\bar\phi,\psi_\mu]\Big)\,. \end{aligned} \tag{3.7}$$

In order to implement the fermionic variables as link or plaquette fields, we must define a conjugate field living on the same link. This complexification, or doubling of the field content, means that the group is promoted from $U(N)$ to $GL(N,C)$. It also means that we will have to modify the action a bit, by switching some of the fields for their conjugates and other changes. In order to target N=D=$2,4$ SYM theory, the final path integral is constrained to follow a contour along which all the fields are real (except ϕ and $\bar\phi$, which are taken to be conjugate to one another). For the fermions, this is accomplished by replacing the fermion determinant in the bosonic effective action for the Pfaffian. For the bosonic sector, the projection to the appropriate contour must be made in the action by setting ImU_μ=0, Im$W_{\mu\nu\rho}$=0, and $\phi^A = (\bar\phi^A)^*$. Note that we use anti-Hermitian generators of the gauge group in this section so that the results can be compared more easily to Catterall's work.

Gauge invariance must, of course, be maintained in the bosonic action after the reality condition is enforced. We believe this is the case, though its less than obvious. The perspective of the twisted practitioners is that the doubling of the bosonic degrees of freedom is simply an unfortunate SUSY consequence of the need to double the fermionic degrees of freedom in order to define the KD fermion on the lattice and that the doubling is "natural" since links can support pairs of fields corresponding to orientation.

That the Pfaffian of the fermion operator in the complexified theory reproduces the weight in the bosonic path integral appropriate for a real KD fermion can be easily seen as follows: we define

$$\eta_{a\bar{a}} = \eta^A T^A_{a\bar{a}}\,,$$

such that the η^A must be complex valued. The decomposition of the lattice KD fermion is then

$$\begin{aligned}
\Psi^\dagger M \Psi &= \left(\Psi_R^\top - i\Psi_I^\top\right)^A (T^A)^\dagger M \left(\Psi_R + i\Psi_I\right)^B (T^B) \\
&= \Psi_R M \Psi_R + \Psi_I M \Psi_I + i\left(\Psi_R^\top M \Psi_I - \Psi_I^\top M \Psi_R\right) \\
&= \Psi_R M \Psi_R + \Psi_I M \Psi_I \quad \text{provided} \quad M^\top = -M\,.
\end{aligned}$$

This is indeed the case for the KD operator constructed of twisted variables.

In two dimensions it is possible to show directly that the lattice action does indeed have a SUSY IR limit (up to possible O(a) errors) without fine tunings. This is not at all surprising of course, since even the naive lattice transcriptions of SUSY actions in 2D have SUSY descriptions in the IR with corrections suppressed by the lattice spacing. We have not attempted the calculation to higher order. Here we show the calculation of the propogator as a warm up to 4D.

The fermionic lattice action in component form is

$$S_f = \beta\,\mathrm{Tr}\,\sum_x \left(-\chi_{12}^\dagger (D_1^+\psi_2 - D_2^+\psi_1) - \psi_\mu^\dagger D_\mu^+ \frac{\eta}{2} + \text{h.c.} \right).$$

Here $\psi_{1,2}$ are vector valued fermionic field variables in the $1,2$ direction and so live on links $\psi_{1,2} = \psi(x + \mu_{1,2}/2)$.

The fermionic action in KD form is $\Psi^\dagger M_0 \Psi$, with

$$\Psi = \begin{pmatrix} \eta/2 \\ \chi_{12} \\ \psi_1 \\ \psi_2 \end{pmatrix}, \quad M_0 = \begin{pmatrix} 0 & K \\ -K^\dagger & 0 \end{pmatrix} \quad \text{and } K = \begin{pmatrix} -D_1^- & -D_2^- \\ D_2^+ & -D_1^+ \end{pmatrix}.$$

Then, with $(D^\pm)^\dagger = (D^\pm)^\top = -D^\mp$,

$$K^\dagger = \begin{pmatrix} -D_1^+ & D_2^- \\ -D_2^+ & -D_1^- \end{pmatrix}.$$

M is then antisymmetric as required (and trivially anti-Hermitian), The relationship with the component action is then

$$\mathcal{L}_f \;=\; \Psi^\dagger M_0 \Psi = -\eta^\dagger D_\mu^- \psi_\mu - \psi_\mu^\dagger D_\mu^+ \eta - \chi_{12}^\dagger D_{[1}^+ \psi_{2]} - \psi_{[2}^\dagger D_{1]}^- \chi_{12}\,.$$

Changing to momentum space in the usual way yields

$$\begin{aligned}
&= -\eta^\dagger \psi_\mu \left(e^{ip_\mu/2} - e^{-ip_\mu/2} \right) - \psi_\mu^\dagger \eta e^{-ip_\mu/2} \left(e^{ip_\mu} - 1 \right) \\
&\quad -\chi_{12}^\dagger e^{-i\frac{p_1+p_2}{2}} \left[\psi_2 \left(e^{i(p_1 + \frac{p_2}{2})} - e^{i\frac{p_2}{2}} \right) - \psi_1 \left(e^{i(p_2+\frac{p_1}{2})} + e^{i\frac{p_1}{2}} \right) \right] \\
&\quad -\left[\psi_2^\dagger e^{-i\frac{p_2}{2}} \left(e^{i\frac{p_1+p_2}{2}} - e^{i\frac{p_2-p_1}{2}} \right) - \psi_1^\dagger e^{-i\frac{p_1}{2}} \left(e^{i\frac{p_1+p_2}{2}} - e^{i\frac{p_1-p_2}{2}} \right) \right] \chi_{12} \\
&= \left(\eta^\dagger \psi_\mu + \psi_\mu^\dagger \eta \right)(-i\tilde{p}_\mu) + \left(\chi_{12}^\dagger \psi_2 + \psi_2^\dagger \chi_{12} \right)(-i\tilde{p}_1) + \left(\chi_{12}^\dagger \psi_1 + \psi_1^\dagger \chi_{12} \right)(i\tilde{p}_2)\,.
\end{aligned}$$

This we can then easily rewrite in the KD notation as

$$\mathcal{L}_f = \Psi^\dagger \begin{bmatrix} \begin{matrix} & \\ & 0 & \\ & \end{matrix} & \begin{matrix} -i\tilde{p}_1 & -i\tilde{p}_2 \\ i\tilde{p}_2 & -i\tilde{p}_1 \end{matrix} \\ \begin{matrix} -i\tilde{p}_1 & i\tilde{p}_2 \\ -i\tilde{p}_2 & -i\tilde{p}_1 \end{matrix} & \begin{matrix} & \\ & 0 & \\ & \end{matrix} \end{bmatrix} \Psi \,.$$

This matrix is again anti-hermitian, as it should be. $|M_0|^2 = \tilde{p}^2$ so the inverse is easy to find and the propogator is simply

$$S = -M_0^{-1} = \frac{M_0}{\tilde{p}^2} \,.$$

3.5.3 $N = D = 4$ Twisted SUSY

We will omit many of the details of the formulation as they are mostly the same as the 2D theory. The details can be found in [91]. The continuum bosonic action is actually precisely the same as the well known Marcus twisted action of [113], after replacing the fields $W_{\mu\nu\rho}$ by their dual field W_σ. It turns out that this duality transformation further complicates the implementation of gauge invariance on the lattice, so it is not a possible simplification.

We will write all actions as they are to be simulated so that the derivation of the feynman rules is as transparent as possible. The path integral that would be performed numerically is

$$\int \mathcal{D}A\mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}W \det{}^{\frac{1}{2}}(M)\; e^{-S_B} \,.$$

The bosonic action and the fermion determinant are treated seperately below.

The antisymmetric W and θ fields are most easily treated perturbatively as 4 seperate scalar degrees of freedom. For ease of writing it makes sense to label these fields by the *missing direction*:

$$W_{123} \equiv w_4 \quad W_{124} \equiv w_3\,, \quad W_{134} \equiv w_2\,, \quad W_{234} \equiv w_1\,,$$

and similarly for the θ fields. We have used lower case latin letters for this missing direction label to stress the fact that it is simply a shorthand. It is not the same as the Hodge-like duality transformation that is used to transform Catterall's twisted continuum action into the twisted action due to Marcus. That duality transformation is not compatible with lattice gauge invariance.

The purely bosonic action reads

$$\begin{aligned} S_{\mathrm{B}} \;=\; & \sum_x \mathrm{Tr}\left[\frac{1}{2}F^2 + |D_\mu^+\phi|^2 + \sum_{i=1..4}|D_i^+ w_i|^2 + \sum_{i\neq j}|D_i^- w_j|^2 \right.\\ & \left. +\frac{1}{4}[\phi,\phi^\dagger][\phi,\phi^\dagger] + \sum_{i<j}[w_i,w_j][w_i,w_j]^\dagger + \sum_i[\phi,w_i^\dagger][w_i,\phi^\dagger]\right]. \end{aligned}$$

Similar to the 'missing direction' notation for the W and θ fields, we write $\kappa_{1234} \equiv \kappa$. This notation makes it clear that κ is a Lorentz scalar living at the lattice sites. The notation is less clear in the case of the W and θ fields. These fields are three index antisymmetric tensor fields and so live at the center of a cube, $w_1 \equiv W_{234} = W(x+\mu_2/2+\mu_3/2+\mu_4/2)$ as shown in Fig. 3–1 (the W field here is $W_{\nu\lambda\rho}$, but the ν direction is projected out so that it looks like it lives on a face).

The fermionic action can be written in KD form much the same way as the 2D case. It is

$$\mathcal{L}_f = \Psi^\dagger M \Psi\,, \quad M = \begin{pmatrix} 0 & K \\ -K^\dagger & 0 \end{pmatrix} \quad \text{and}$$

$$\Psi = \left(\frac{\eta}{2}, \chi_{12}, \chi_{13}, \chi_{14}, \chi_{23}, \chi_{24}, \chi_{34}, \frac{\kappa}{2}, \psi_1, \psi_2, \psi_3, \psi_4, \theta_4, \theta_3, \theta_2, \theta_1\right)$$

and the free fermion part of the fermion matrix is then given by

$$K_0 = \begin{pmatrix} -D_1^- & -D_2^- & -D_3^- & -D_4^- & 0 & 0 & 0 & 0 \\ D_2^+ & -D_1^+ & 0 & 0 & -D_3^- & -D_4^- & 0 & 0 \\ D_3^+ & 0 & -D_1^+ & 0 & D_2^- & 0 & -D_4^- & 0 \\ D_4^+ & 0 & 0 & -D_1^+ & 0 & D_2^- & D_3^- & 0 \\ 0 & D_3^+ & -D_2^+ & 0 & -D_1^- & 0 & 0 & -D_4^- \\ 0 & D_4^+ & 0 & -D_2^+ & 0 & -D_1^- & 0 & D_3^- \\ 0 & 0 & D_4^+ & -D_3^+ & 0 & 0 & -D_1^- & -D_2^- \\ 0 & 0 & 0 & 0 & D_4^+ & -D_3^+ & D_2^+ & -D_1^+ \end{pmatrix}$$

By analogy to the 2D case, we can see that this will give the desired doubler free determinant.

The Yukawa terms involving the W field are

$$\begin{aligned} \mathcal{L}_{\Psi w} \;=\;& \theta_i^\dagger [w_i, \frac{\eta}{2}] + \frac{\kappa^\dagger}{2}\Big([\psi_1, w_1] - [\psi_2, w_2] + [\psi_3, w_3] - [\psi_4, w_4]\Big) \\ &+\chi_{12}^\dagger\Big([\theta_{[1}, w_{2]}] + [\psi^\dagger_{(3}, w_{4)}]\Big) - \chi_{13}^\dagger\Big([\theta_{[1}, w_{3]}] + [\psi^\dagger_{[2}, w_{4]}]\Big) \\ &+\chi_{14}^\dagger\Big([\theta_{[1}, w_{4]}] - [\psi^\dagger_{(2}, w_{3)}]\Big) + \chi_{23}^\dagger\Big([\theta_{[2}, w_{3]}] + [\psi^\dagger_{(1}, w_{4)}]\Big) \\ &-\chi_{24}^\dagger\Big([\theta_{[2}, w_{4]}] - [\psi^\dagger_{[1}, w_{3]}]\Big) + \chi_{34}^\dagger\Big([\theta_{[3}, w_{4]}] + [\psi^\dagger_{(1}, w_{2)}]\Big) + \text{h.c.}\,. \end{aligned}$$

As explained in [91], $\psi_\mu^\dagger$ must be replaced by $U_\mu^\dagger \psi_\mu U_\mu^\dagger$ for the action to be written in terms of the KD fermion. This introduces SUSY breaking operators that look like $Aw\Psi^\dagger\Psi$. The new vertex will contribute to mass renormalizations at two loops and 3-point vertex renormalizations at one loop. We will not attempt to determine the precise form of these terms, since the calculation is rather involved and the lattice action prescription (specifically the reality projection) is ambiguous enough as to make the precise feynman rules unclear.

The $[\theta, w]$ terms must be modified as well in order to retain gauge invariance. The easiest way to do so is to insert an ordered path of link matrices,

$$[\theta_{\mu\lambda\rho}, W_{\nu\lambda\rho}] \to [\theta_{\mu\lambda\rho}, W_{\nu\lambda\rho}] U_\lambda^\dagger(x+\lambda+2\rho) U_\lambda^\dagger(x+2\rho) U_\rho^\dagger(x+\rho) U_\rho^\dagger(x)\,. \qquad (3.8)$$

This is depicted diagramatically in Fig. 3–1, where the notational discrepencies are clear, though unfortunate ($l \to \lambda$, $p \to \rho$, $O \to \theta$, $X \to \chi$, etc). This will

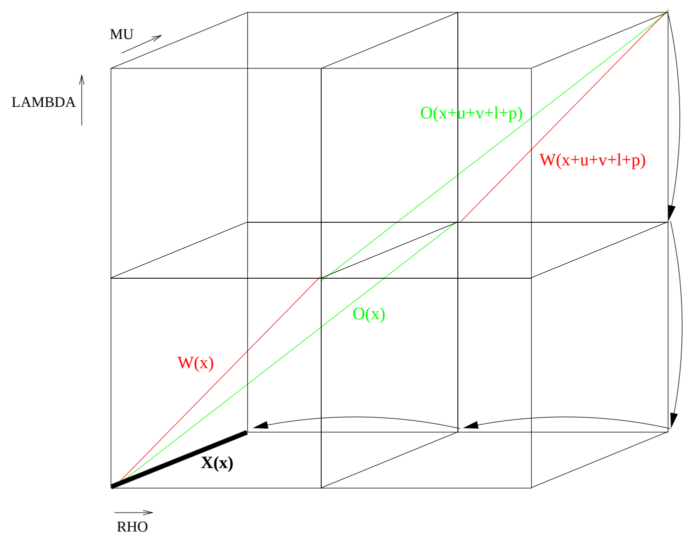


Figure 3–1: A diagram of one of the SUSY breaking terms in the lattice twisted $\mathcal{N}$=4 SYM theory. The arrows represent gauge link fields that must be inserted into the term coupling the fermionic components θ (here labelled O) and χ (labelled X) to the three-form scalar field W in order to retain gauge invariance in the lattice action.

introduce more SUSY violating terms of the form $Aw\Psi^\dagger\Psi$.

We will not attempt to write the position space expressions in matrix KD form since they are rather formidable. We expand the fields in terms of the *anti*hermitian generators of gauge group in the fundamental representation, $X = X^A T^A$. The traces are then done using the relations of Appendix C. The expressions are then much simpler to express in KD form. The group and momentum dependence of the expressions in k-space takes the form

$$Y_i^A \equiv Y_{\mu\nu\lambda}^A = \frac{\mathcal{T}_F}{2}\left(i\widetilde{k_{\mu\nu\lambda}}D^A + 2\cos(k_{\mu\nu\lambda})F^A\right) \text{ with } \quad \widetilde{k_{\mu\nu\lambda}} \equiv 2\sin(k_{\mu\nu\lambda})\,, \tag{3.9}$$

where i is the label corresponding to the missing spacetime index. We have used the notation $k_{\mu\nu\lambda} \equiv k_\mu + k_\nu + k_\lambda$ extensively.

We then write the terms as $\sum_i w_i^A \Psi^\dagger(k) V_i Y_i^A \Psi$, etc.... The V_i are $16{\times}16$, each has 7 nonzero entries in the upper right quadrant corresponding to the $w\chi\psi$, $w\chi\theta$, and $w\kappa\psi$ terms, and one nonzero entry amongst the last four elements of the first column corresponding to the $w\theta\eta$ term. We thus quote the results in terms of an $8{\times}8$ matrix and a 4 component column vector. The momentum space expressions for the $w\chi\theta$ terms has a phase remnant, $\phi_{ab} \equiv e^{ik_{ab}}$. The results can be found in Appendix D. It is then straightforward to determine the feynman rules for the Yukawas. The purely bosonic feynman rules are much easier to derive and can be determined in a similar way, except of course for the fact that their are still ambiguities in the lattice action due to the reality projection.

We have included these results because we feel it is an important and unfinished avenue of research to check the severity of SUSY violations in this approach, and we very much hope that someone more adventurous than ourselves will continue this work, once the appropriate clarifications of the lattice action have been made. Our concern is again amplified by the fact that it is not clear that R-symmetry is retained in this formulation (it may be broken by the addition of these extra terms for example, if it was exact to begin with) and so the number of SUSY breaking terms that can appear in the IR theory could be very large. These would all require nonperturbative tuning, and that would certainly make simulations of the theory unmanageable

3.6 Conclusions

Theoretically all (non-anomalous) SUSY theories can be implemented in a lattice framework, but its not always easy. Important nonperturbative simulations for 4D $\mathcal{N}$=1 are logging processor cycles as we speak, and this is really the success story of lattice SUSY. 4D theories with scalar fields hold a monumental difficulty in that they require nonperturbative tunings to reach their SUSY limit in the IR. For theories without substantial theoretical import, such as (probably) the $\mathcal{N}$=1 Wess-Zumino model, this will probably mean that significant numerical results will have to wait until computing resources are much (*MUCH*) cheaper. $\mathcal{N}$-extended SYM theories in 4D should hold enough interest to physicists to see results in much less time, though we will argue in the next chapter that these efforts should be focused on $\mathcal{N}$=4, not $\mathcal{N}$=2. Novel new ways to transcribe SUSY actions to the lattice might end up being viable alternatives to the more conventional approaches, but we don't see how they could be superior to the (rather straightforward and well established) tuning approach in superrenormalizable low-D theories, and are skeptical of the advantages of such a proposal in 4D. Certainly if an approach could be found that eliminated completely the need for expensive nonperturbative tunings to reach the IR theory, it would be a great advantage, but considering that all the theories described in this dissertation can be implemented with the tuning approach, without new and questionable techniques and with computing resources that are starting to become available to such studies, we think this is a much more productive avenue.

CHAPTER 4
$\mathcal{N} = 4$ Supersymmetry in 4D

$\mathcal{N}$=4 SYM has scalars and Yukawa couplings, and so various masses and couplings will receive divergent corrections in the continuum limit and must be (nonperturbatively) fine-tuned away. Hence lattice $\mathcal{N}$=4 SYM has always seemed impractical by this fine-tuning approach. [1]

We show here that this is not the case. Using GW fermions the four gluinos can be kept massless and the $SU(4)_R$ symmetry in the lattice theory can be preserved, which greatly restricts the form of the renormalizations. This leaves a scalar mass, two quartic couplings, and a Yukawa coupling to tune. The Yukawa coupling can be tuned by rescaling the scalar kinetic term[2] ; so all tunings can be done by adjusting bosonic terms in the action. This allows the tunings to be done by the "Ferrenberg-Swendsen method" [114, 115], exploring a wide swath of coupling constant space "offline" from the results of a single Monte-Carlo simulation. The parameter range available with good statistics can be enlarged using multicanonical techniques [116, 117]. Thus we arrive at the encouraging result that all fine-tuning can be performed through an

[1] Other approaches recently suggested include those of Refs. [91, 66], involving "orbifold" or "twisted SUSY" lattices.

[2] This is obvious because the Yukawa coupling strength y can always be absorbed into a redefinition of the scalar fields $\phi \to \phi/y$, causing it to reappear in the scalar kinetic term.

"offline" analysis, *i.e.*, new simulations and fermion matrix inversions are not required.

In section 4.1 we derive the continuum action and supercurrent from dimensional reduction of the familiar 10D $\mathcal{N}=1$ action and SUSY transformations on a 6D torus. In Sec. 1.1.3 we derived the familiar continuum Ward identities for comparison with the lattice derivation. In section 4.2 we give the discretization and describe those subtleties of the prescription that appear at the level of the bare action. In section 4.3 we study the renormalized theory and determine the (finite) set of parameters that will require fine tuning to ensure the theory approaches its SUSY limit as the lattice regulator is removed. We then describe our approach to the tuning, which can be summarized roughly as follows: by artificially weighting the Boltzman factor so that the ensemble generated in the Markov chain overlaps strongly with cannonical ensembles at *multiple* parameter values, we are able to select (i.e. tune) the parameter after the simulation (i.e. offline) to the SUSY point. This procedure easily generalizes to an arbitrary number of parameters. The rest of section 4.3 is devoted to finding a reasonably clean observable for each tuning parameter. We will argue that two relatively simple conditions can be extracted from the effective potential, with the remainder chosen variously from the set of SUSY Ward ids. Section 4.4 collects our outlook and conclusions.

The continuum field content is $SU(N_c)$ YM theory with an $SU(4)_R$ internal symmetry; there are four Majorana fermions whose left handed components transform in the fundamental **4** representation of $SU(4)_R$ and 6 real scalars in the antisymmetric tensor representation. The six real scalars will be expressed with a single index ϕ_m, m=1. . .6, or composed into $SU(4)_R$ Weyl matrices:

$\phi_{ij}{=}\phi_m\hat{\sigma}_{m,ij}$ and $\phi^{ij}{=}\phi_m\hat{\bar{\sigma}}^{ij}_m$, where $\hat{\sigma}$'s are just $SU(4)_R$ Clebsch-Gordon coefficients involved in $\mathbf{4}^*{\ni}\,\mathbf{6}{\otimes}\mathbf{4}$.

$$S = \frac{1}{g^2}\mathrm{Tr}\Big\{\frac{1}{2}F^2 + |D_\mu\phi_m|^2 + \bar{\psi}_i \not{D}\psi_i \\ +\sqrt{2}\bar{\psi}_i\left(\phi^{ij}P_L{-}(\phi^{ij})^*P_R\right)\psi_j + [\phi_m,\phi_n][\phi_m,\phi_n]\Big\}. \quad (4.1)$$

Again, this action is derived in section 4.1, though this is a somewhat more compact and convenient form for our purposes.

4.1 Continuum action of $\mathcal{N} = D = 4$ SYM

$\mathcal{N}$=4 is called maximal supersymmetry because 4 sets of SUSY generators Q_{1-4} is the most that can be accomodated in a theory (that does not contain spin 2 gravitons). This can be easily seen from the results of section 1.1.1, since 4 Q's can generate 4 half-integer helicity steps. This implies that only vector multiplets (λ_{max}=1) can be constructed with $\mathcal{N}$=4 and that the multiplet is self conjugate under CPT; we see immediately that this theory is a Yang-Mills theory, and that the scalar and fermionic degrees of freedom (d.o.f.) and their interactions are uniquely determined by the gauge d.o.f. (since the spin zero states $\sim|0\rangle$ and the spin $\frac{1}{2}$ states $\sim|\pm\frac{1}{2}\rangle$ can all be obtained as some function of Qs acting on the gauge boson state $|\pm 1\rangle$).

The continuum action of the maximally supersymmetric theory is most easily derived by dimensional reduction on a 6-torus from the 10D $\mathcal{N} = 1$ theory. This also gives an intuitive interpretation of where the SU(4) ($\simeq SO(6)$) internal R-symmetry in the $\mathcal{N}$=4 theory comes from. In the above description using helicity raising and lowering operators, the SU(4) symmetry survives in the action because it is a symmetry of the supersymmetry generators the spinor Qs transform in the fundamental $\mathbf{4}$ of SU(4) and the $\bar{Q}$s transform in the antifundimental $\bar{\mathbf{4}}$.

The 10D theory has a single 32 component Majorana-Weyl fermion and a single massless 10D gauge boson (8 real fermionic and 8 real bosonic degrees of freedom) with action

$$S = \int d^{10}X \Big(-\frac{1}{4} F^A_{MN} F^{MN}_A - \frac{1}{2} \bar{\Psi}_A \Gamma^M (D_M \Psi)_A \Big) \,, \tag{4.2}$$

where $\bar{\Psi} = \Psi^\top C_{10}$, and the 10D reality condition is

$$C_{10} \Psi = i \Gamma_0 \Psi^* \,. \tag{4.3}$$

This action is invariant under

$$\begin{aligned} \delta A^A_M &= \bar{\xi}_{10} \Gamma_M \Psi^A \,, \\ \delta \Psi^A &= -\frac{1}{4} F^A_{MN} [\Gamma^M, \Gamma^N] \xi_{10} \,, \end{aligned} \tag{4.4}$$

where ξ parametrizes the SUSY transform. The action that results from the reduction on the 6-torus will retain the SO(6) symmetry from the reduced dimensions as an internal symmetry of the action and will be invariant under the reduced versions of the 10D SUSY transformations. We choose a basis of gamma matrices

$$\begin{aligned} &\Gamma_M = \{ \mathbf{1}_8 \otimes \gamma_\mu, \hat{\gamma}_m \otimes \gamma_5 \} \,, \qquad \text{with} \quad \hat{\gamma}_m = \begin{pmatrix} 0 & \hat{\sigma}_m \\ \hat{\bar{\sigma}}_m & 0 \end{pmatrix} , \\ &\hat{\sigma} = \{ -i\mathbf{1}, i\gamma_5 \vec{\gamma}, \gamma_5 \gamma_0, \gamma_5 \} \quad \text{and} \quad \hat{\bar{\sigma}} = \{ +i\mathbf{1}, \hat{\sigma}_{5,\ldots,9} \} \,, \end{aligned} \tag{4.5}$$

so that

$$\gamma_5 = -i\gamma^0 \ldots \gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \quad \text{and} \quad \hat{\gamma}_{10} = -i\hat{\gamma}_4 \ldots \hat{\gamma}_9 = \begin{pmatrix} \mathbf{1}_4 & 0 \\ 0 & -\mathbf{1}_4 \end{pmatrix} , \tag{4.6}$$

and the charge conjugation matrices are

$$C_4 \equiv -\beta\gamma^2 = \begin{pmatrix} -e & 0 \\ 0 & e \end{pmatrix} = \hat{\epsilon} \quad \text{and} \quad C_6 = -i\hat{\gamma}_1\hat{\gamma}_3\hat{\gamma}_5 = \begin{pmatrix} 0 & \hat{\epsilon}^{\,i}_{\ j} \\ -\hat{\epsilon}_i^{\ j} & 0 \end{pmatrix}. \quad (4.7)$$

Here $\hat{\epsilon}$ acts on the internal R-symmetry indices after compactification; that it takes the same matrix form as C_4 is a coincidence of our choice of $\{\hat{\gamma}\}$ and our convention for C. $\Gamma_{11} = \hat{\gamma}_{10} \otimes \gamma_5$, $C_{10} = C_6 \otimes C_4$ and the Majorana-Weyl conditions in 10D,

$$\Gamma_{11}\Psi = \Psi\,, \quad \text{and} \quad C_{10}\Psi = i\Gamma_0\Psi^*\,, \tag{4.8}$$

allows us to parametrize the 10D spinor as

$$\Psi = \begin{pmatrix} \psi_{iL} \\ \hat{\epsilon}^{\,i}_{\ j}\psi^j_R \end{pmatrix}, \tag{4.9}$$

where $\psi_{iL} = \{\psi_{1L}, \psi_{2L}, \psi_{3L}, \psi_{4L}\}$ transforms as a vector under the internal SU(4) (=SO(6)) after the reduction and ψ^i_R as a contragradient vector. L,R refer to the standard 4D left and right chiral projections of the spinor (here taken as $\gamma_5\psi_L = \psi_L$, $\gamma_5\psi_R = -\psi_R$). The reality condition on the 10D spinor is then satisfied provided that the 4D spinor components satisfy the (SU(4) invariant) reality condition

$$\begin{aligned} \beta\psi^*_{iL} &= -C_4\psi^i_R\,. \\ \text{or} \quad \bar{\psi}_{iL} &= \left(\psi^i_R\right)^\top C_4\,. \end{aligned} \tag{4.10}$$

The second relation in Eq. (4.10) is formally the same in either Minkowski or Euclidean space and is thus sometimes more convenient.

In order to write the action so that the SU(4) R symmetry is manifest, we compose the 3 complex scalars - constructed from the 6 real degrees of freedom of the gauge boson in the reduced directions - into an antisymmetric SU(4) tensor,

$$\phi^{ij} \equiv \frac{1}{\sqrt{2}}\,\hat{\epsilon}^{\,i}_{\;k}\Big(\sum_m A_m\hat{\bar{\sigma}}_m\Big)^{kj} = \begin{pmatrix} 0 & \Phi_1^* & \Phi_3 & -\Phi_2 \\ & 0 & -\Phi_2^* & -\Phi_3^* \\ & & 0 & \Phi_1 \\ & & & 0 \end{pmatrix} \tag{4.11}$$

$$\text{and} \quad \phi_{ij} \equiv \frac{1}{\sqrt{2}}\Big(\sum_m A_m\hat{\sigma}_m\Big)_{ik}\,\hat{\epsilon}^{\,k}_{\;j} = (\phi^{ij})^*\,, \tag{4.12}$$

which satisfies an SU(4) reality condition, $(\phi^{ij})^* = \frac{1}{2}\epsilon_{ijkl}\phi^{kl}$.

We use the notation of [118] in 4D except where explicitly noted. The 4D action in terms of these variables is then

$$S = \int_x \Bigg[\frac{1}{4}(F^A_{\mu\nu})^2 + \frac{1}{2}(D_\mu\phi^{ij})_A(D^\mu\phi_{ij})_A - \frac{1}{2}\Big(\psi^{i\top}_{AR}C_4(\not{D}\psi_{iL})_A + \psi^\top_{iAL}C_4(\not{D}\psi^i_R)_A\Big)$$
$$+\frac{\sqrt{2}}{2}f^{ABC}\Big(\phi^A_{ij}\Big(\psi^{i\top}_{BR}C_4\psi^j_{CR}\Big) - \phi^{ij}_A\Big(\psi^\top_{iBL}C_4\psi_{jCL}\Big)\Big) + \frac{1}{8}\Big|f^{ABC}\phi^{ij}_B\phi^{kl}_C\Big|^2\Bigg], \tag{4.13}$$

which, along with the above reality conditions on the fermionic and scalar variables, is Hermitian.

The reduced SUSY transformations are

$$\begin{aligned}
\delta_\xi A^A_\mu &= -\xi^{i\top}_R\epsilon\gamma_\mu\psi_{iAL} + \xi^\top_{iL}\epsilon\gamma_\mu\psi^i_{AR} \\
\delta_\xi\phi^{ij}_A &= \sqrt{2}\Big(\xi^{i\top}_R\epsilon\psi^j_{AR} - \frac{1}{2}\epsilon^{ijkl}\xi^\top_{kL}\epsilon\psi_{lAL}\Big)\,, \\
\delta_\xi\psi_{iAL} &= -\frac{1}{2}F_A\!\cdot\!\sigma\;\xi_{iL} + \sqrt{2}\gamma^\mu\Big(D_\mu\phi_{ij}\Big)_A\xi^j_R + f^{ABC}\phi^B_{ij}\phi^{jk}_C\xi_{kL}\,, \\
\delta_\xi\psi^i_{AR} &= -\frac{1}{2}F_A\!\cdot\!\sigma\;\xi^i_R + \sqrt{2}\gamma^\mu\Big(D_\mu\phi^{ij}\Big)_A\xi_{jL} + f^{ABC}\phi^{ij}_B\phi^C_{jk}\xi^k_R\,,
\end{aligned} \tag{4.14}$$

where $F\cdot\sigma = F_{\mu\nu}\sigma^{\mu\nu}$ and $\sigma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu,\gamma^\nu]$. In order to obtain a purely bosonic path integral that can be integrated numerically by lattice techniques, we must write the fermionic action in terms of Dirac variables with $P_L\psi_i{=}\psi_{iL}$

and $P_R\psi_i{=}\psi_R^i$

$$S_d = \int_x -\bar{\psi}_{iA}(\not{D}\psi_i)_A - \sqrt{2}\, f^{ABC}\bar{\psi}_{iB}\Big(\phi_A^{ij}P_L - \phi_{ij}^A P_R\Big)\psi_{j\,C}\,, \tag{4.15}$$

which then decomposes into two independent copies of the fermionic action in Eq. (4.13) under the standard Majorana decomposition

$$\psi = \frac{1}{\sqrt{2}}\Big(\chi + i\eta\Big)\,, \qquad \bar{\psi} = \frac{1}{\sqrt{2}}\Big(\chi^\top C - i\eta^\top C\Big)\,, \tag{4.16}$$

along with the appropriate R-symmetry invariant chiral decomposition, $P_L\chi_i = \chi_{iL}$ and $P_R\chi_i{=}\chi_R^i$. Notice also that the reality condition of Eq. (4.10) is then implied by the paramterization of the Majorana spinor,

$$\chi_i \equiv \begin{pmatrix} \alpha_i \\ e\alpha_i^* \end{pmatrix} = \chi_{iL} + \chi_R^i \quad \Longrightarrow \quad \chi_R^i = \begin{pmatrix} 0 \\ e\alpha_i^* \end{pmatrix} = C_4\beta\chi_{iL}^*\,. \tag{4.17}$$

This proves as usual that the fermionic determinant resulting from Eq. (4.15) is the sqaure of a Pfaffian and that the appropriate factor in the path integral over the Majorana valued field can be included as the root of the determinant. This is described in more detail in Section 4.2.1.

In terms of these variables, the action of the R-symmetry (supressing SU(4) indices for the moment) is

$$\psi \to \Big(UP_L + U^*P_R\Big)\psi\,, \quad \bar{\psi} \to \bar{\psi}\Big(U^\top P_L + U^\dagger P_R\Big) \quad \text{and} \quad \phi \to U^*\phi U^\dagger, \tag{4.18}$$

where U is the SU(4) rotation matrix in the fundamental representation. Note that the R-symmetry is (not surprisingly) a Chiral symmetry. The supersymmetry transformations also generalize straightforwardly, however, since they fail to hold anyway for the lattice action, we will have little use of them in what follows.

In order to compare the derivation in this section most easily to the continuum literature, we have reduced to 4 dimensions with a Minkowski signature. By a naive Wick rotation, the only formal change to the equations of this section is that summations on space-time indices go from 1 to 4 with $\gamma_4{=}\beta$. Instead of hermiticity, the action now displays the analogous Euclidean symmetry called Hermitian reflectivity. We choose the action of this symmetry to be Hermitian conjugation along with a full space parity inversion, as is standard. The usefulness of the second expression of Eq. (4.10) is now clear; with Euclidean signature, $\bar{\psi}{=}\psi^\dagger$ instead of $\psi^\dagger\beta$ and all further relations follow simply from adding a parity inversion to the action of † (since parity changes the sign of γ_5 and so flips handedness $L{\leftrightarrow}R$).

In [119] they point out that this Wick rotation is not technically consistent because the R-symmetry group should also be Wick rotated from SU(4) (=SO(6)) to SO(5, 1) (the symmetry group of the internal space in the case where we reduce from the 10D $\mathcal{N}{=}1$ theory to 4 dimensions with a Euclidean signature) and this should affect the reality conditions for ψ and ϕ^{ij}. This is not expected to make any difference in actual calculations (on the lattice or otherwise). One simply treats all fields as complex without reality conditions and since the action is holomorphic it depends only on the fields and not their conjugates. The path integral, and specifically the Grassman integration, is the same [120].

4.1.1 The supercurrent

The supercurrent is also most easily calculated by dimensional reduction from the 10D supercurrent in the $\mathcal{N}{=}1$ theory, which takes the simple form

$$S_P = \frac{1}{4}[\Gamma_M, \Gamma_N]F^A_{MN}\Gamma_P\Psi_A\,. \tag{4.19}$$

The 4D $\mathcal{N}$=4 supercurrent is then

$$S^{(x)}_{i\mu R} = \frac{1}{2} F_A \cdot \sigma \gamma_\mu \psi_{iAL} + \sqrt{2} \not{D} \phi^A_{ij} \gamma_\mu \psi^j_{AR} - f^{ABC} \phi^B_{ij} \phi^{jk}_C \gamma_\mu \psi_{kAL} \,, \qquad (4.20)$$

and in terms of Dirac fermions

$$S^{(x)}_{i\mu} = \left\{ \frac{1}{2} F_A \cdot \sigma \, \delta_{ij} + \sqrt{2} \not{D} \left(\phi^A_{ij} P_L + \phi^{ij}_A P_R \right) - f_{ABC} \left(\phi^B_{ik} \phi^{kj}_C P_R + \phi^{ik}_B \phi^C_{kj} P_L \right) \right\} \gamma_\mu \psi_{jA} . \qquad (4.21)$$

This is also the easiest way to determine the set of relevant and marginal operators that have the same quantum numbers as the supercurrent and can therefore mix with it under renormalization when SUSY is broken (as it is on the lattice). In the $\mathcal{N}$=1 theory there is only one such term, the mixing current (since it mixes with the supercurrent),

$$T_M = F^A_{MN} \Gamma_N \Psi_A \,. \qquad (4.22)$$

This implies a mixing current in the 4D $\mathcal{N}$=4 theory like

$$T^{(x)}_{i\mu} = \left\{ \gamma_\nu F^A_{\mu\nu} \delta_{ij} - \sqrt{2} D_\mu \left(\phi^{ij} P_L + \phi_{ij} P_R \right)_A \right\} \psi_{jA} \,. \qquad (4.23)$$

4.2 Lattice Action

The $SU(4) \simeq SO(6)$ preserving bosonic lattice action is a trivial transcription from the continuum. Of course we must allow for generic coefficients and non-SUSY terms, so that the SUSY-restoring counterterms can be tuned. In our case these are entirely scalar terms. The precise type of Ginsparg-Wilson [50] fermion to be used, be they domain wall [54] or overlap [63], is not important for the considerations here; we need only note that it satisfies the Ginsparg-Wilson relation

$$\{\gamma_5, D\} = RaD\gamma_5 D \,, \qquad (4.24)$$

and "γ_5-hermiticity"

$$D^\dagger = \gamma_5 D \gamma_5 \,. \tag{4.25}$$

Eq. (4.24) allows us to write a Lagrangian density

$$\mathcal{L} = \bar{\psi} D \psi \,, \tag{4.26}$$

that is invariant under a lattice modified chiral transformation with

$$\delta\psi = i\varepsilon\hat{\gamma}_5\psi \,, \qquad \delta\bar{\psi} = i\varepsilon\bar{\psi}\gamma_5 \,, \tag{4.27}$$

where $\hat{\gamma}_5 = \gamma_5(1-RaD)$ squares to one and so defines a set of projection operators, $\hat{P}_{L/R} = \frac{1}{2}(1 \pm \hat{\gamma}_5)$, that act on ψ. This reduces to the usual chiral transformation and chiral projectors in the naive continuum limit.

Unfortunately, using the naive prescription for associating continuum and lattice variables in the presence of Yukawa terms, leads to a set of lattice actions which are inconsistent with either the lattice chiral symmetry or the Majorana decomposition [121].

In order to write Yukawa terms that are consistent with the lattice chiral symmetry and the Majorana decomposition, we follow the original formulation of Lüscher's exact lattice chiral invariance [55], which was applied to the Majorana case in [122]. For each fermion ψ_i in the theory we associate an auxiliary fermion field Ψ_i and generalize the action of Eq. (4.13) as

$$S = a^4 \sum_{x,i,j} \bar{\psi}_i D \psi_i - \frac{\omega}{aR}\bar{\Psi}_i\Psi_i + (\bar{\psi}+\bar{\Psi})_i^B \, Y_{BC}^{ij} \, (\psi+\Psi)_j^C \tag{4.28}$$

where the Yukawas can now be written as simply the naive ultralocal lattice version of the Yukawas in Eq. (4.15), $Y_{BC}^{ij} \equiv \sqrt{2} f_{ABC} \left(\phi_A^{ij} P_L - (\phi_A^{ij})^* P_R\right)$. The factor of ω/R has been inserted for later convenience.

This action is invariant under a lattice chiral transformation of the form

$$\begin{aligned}
\delta\psi &= i\varepsilon\left(A\psi + C\Psi\right), \quad & \delta\Psi &= i\varepsilon\left(F\psi + H\Psi\right), \\
\delta\bar{\psi} &= i\varepsilon\left(\bar{\psi}B + \bar{\Psi}E\right), \quad & \delta\bar{\Psi} &= i\varepsilon\left(\bar{\psi}G + \bar{\Psi}I\right), \\
\delta\phi &= -i\varepsilon\left(T^*\phi + \phi T\right), \quad & \delta\phi^* &= i\varepsilon\left(T\phi^* + \phi^* T^*\right),
\end{aligned} \tag{4.29}$$

where $A \to I$ are matrices determined below. The transformation rule for the scalar field is just the naive lattice version of the continuum transformation - the infinitesimal version of Eq. (4.18) with $U = \exp(i\varepsilon T)$. This form is necessary to avoid the no-go theorem of [121]. The invariance of the Yukawa terms under this transformation now implies that the sums $\psi + \Psi$ and $\bar{\psi} + \bar{\Psi}$ must transform like a continuum fermion field

$$\begin{aligned}
\delta(\psi + \Psi)/i\varepsilon &= (A+F)\psi + (C+H)\Psi = \left(TP_L - T^*P_R\right)(\psi + \Psi) \\
\delta(\bar{\psi} + \bar{\Psi})/i\varepsilon &= \bar{\psi}(B+G) + \bar{\Psi}(E+I) = (\bar{\psi} + \bar{\Psi})\left(T^*P_L - TP_R\right)
\end{aligned} \tag{4.30}$$

Invariance of the quadratic fermion terms implies another four relations, including the usual $BD = -DA$ for which the general Ginsparg-Wilson type solution is

$$A = T\hat{P}_L - T^*\hat{P}_R \quad \text{and} \quad B = T^*\bar{P}_L - T\bar{P}_R\,. \tag{4.31}$$

The transformation is now fully and uniquely specified by the choice of lattice projectors, $\hat{P}$ and $\bar{P}$. The requirement is only that these projectors reduce to the continuum ones in the naive limit. Two conventions are used in the literature which we will call the non-symmetric and symmetric choices. The non-symmetric choice is

$$\hat{P}_{L/R} = \frac{1}{2}(1 \pm \hat{\gamma}_5) \quad \text{and} \quad \bar{P}_{L/R} = P_{L/R}\,, \tag{4.32}$$

which is what we used in Eq. (4.27). This is the most common choice in the literature because of the convenient fact that $\hat{\gamma}_5$ squares to 1 and so $\hat{P}$ is a set of projectors in the usual sense: $\hat{P}^2_{L/R} = \hat{P}_{L/R}$ and $\hat{P}_L\hat{P}_R = 0$. The symmetric choice (which Lüscher uses in [55]) is

$$\hat{P}_{L/R} = \frac{1}{2}\Big(1 \pm \gamma_5(1 - \frac{Ra}{2}D)\Big) \quad \text{and} \quad \bar{P}_{L/R} = \frac{1}{2}\Big(1 \pm (1 - \frac{Ra}{2}D)\gamma_5\Big)\,. \tag{4.33}$$

Interestingly, each of these choices leads to a consistent set of transformation for only one value of the GW parameter R. This constraint may be removed from the GW relation at the cost of transferring it to the action by using $-\frac{\omega}{aR}\bar{\Psi}\Psi$ as the quadratic term for the auxilliary field as we did above.

For convenience, define the continuum form of the fermion transformation matrix to be $TP_L - T^*P_R \equiv \mathbf{u}$. For the non-symmetric solution we have a set of transformations (here $\hat{P}_{L/R} = \frac{1}{2}(1 \pm \hat{\gamma}_5)$ as usual)

$$\begin{aligned}
&\delta\psi/i\varepsilon = (T\hat{P}_L - T^*\hat{P}_R)\psi\,, && \delta\Psi/i\varepsilon = \tfrac{Ra}{\omega}(T{+}T^*)\gamma_5 D\psi + \mathbf{u}\Psi\,,\\
&\delta\bar{\psi}/i\varepsilon = \bar{\psi}\,\mathbf{u}^* + \bar{\Psi}(T{+}T^*)\gamma_5\,, && \delta\bar{\Psi}/i\varepsilon = -\bar{\Psi}\,\mathbf{u}\,,\\
&\delta\phi/i\varepsilon = -T^*\phi - \phi T\,, && \delta\phi^*/i\varepsilon = T\phi^* + \phi^* T^*\,,
\end{aligned} \tag{4.34}$$

which is consistent only for $\omega = 2$.

The set of transformations for the symmetric solution is only consistent for ω=1,

$$\begin{aligned}
&\delta\psi/i\varepsilon = \hat{\mathbf{u}}\psi + \tfrac{1}{2}(T{+}T^*)\gamma_5\Psi\,, && \delta\Psi/i\varepsilon = \tfrac{Ra}{2}(T{+}T^*)\gamma_5 D\psi + \tfrac{1}{2}(T{-}T^*)\Psi\,,\\
&\delta\bar{\psi}/i\varepsilon = \bar{\psi}\bar{\mathbf{u}}^* + \bar{\Psi}\tfrac{1}{2}(T{+}T^*)\gamma_5\,, && \delta\bar{\Psi}/i\varepsilon = \bar{\psi}\tfrac{Ra}{2}(T{+}T^*)D\gamma_5 - \bar{\Psi}\tfrac{1}{2}(T{-}T^*)\,,\\
&\delta\phi/i\varepsilon = -T^*\phi - \phi T && \delta\phi^*/i\varepsilon = T\phi^* + \phi^* T^*\,,
\end{aligned} \tag{4.35}$$

where $\hat{P}$ and $\bar{P}$ are given by Eq. (4.33) and $\hat{\mathbf{u}}$ and $\bar{\mathbf{u}}^*$ are the same as above with the projectors replaced by their appropriate lattice counterparts.

Under these transformations, the sum $(\psi+\Psi)$ indeed transforms like a field in the continuum theory, hence the Yukawas need not be modified on the lattice with any higher derivative terms. This is essential for a consistent formulation of the Majorana projection, Eq. (4.16). The requirements on the charge conjugation matrix are straightforward,

$$C^\top = -C\,, \qquad (CD)^\top = -CD\,, \qquad (CY)^\top = -CY\,. \tag{4.36}$$

4.2.1 Square root prescription

Consider the theory at vanishing Yukawa terms. Each fermion species is decoupled and contributes a factor of $\sqrt{\det D}$ to the bosonic effective action.[3] The square root prescription is the following.

The determinant is defined as the product of the eigenvalues of the Dirac operator,

$$D\psi_n = \lambda_n \psi_n \quad \longrightarrow \quad \det D = \prod_n \lambda_n\,. \tag{4.37}$$

[3] We take the prospective that the 'real' (fully regulated and unambiguous) definition of the Dirac fermion contribution to the path integral is precisely the appropriate limit of some regulated Dirac determinant. In this treatment, chirality and Majorana conditions are rephrased as prescriptions for finding roots of this determinant. We mention this only to point out that this rooting is necessary in any regularization scheme for Majorana fermions and is totally well defined because of eigenvalue pairing. Therefore the square root procedure yields a local theory, as opposed to what occurs for instance when rooting a staggered action with taste splitting, where locality and unitarity are at best recovered only in the continuum limit.

This determinant is real since

$$(\det D)^* = \det D^\dagger = \det \gamma_5 D \gamma_5 = (\det \gamma_5)^2 \det D = \det D\,. \tag{4.38}$$

Furthermore, given $D\psi_n = \lambda_n \psi_n$ and γ_5-hermiticity, we have

$$D(\gamma_5 \psi^\dagger) = \lambda^* \gamma_5 \psi^\dagger\,, \tag{4.39}$$

which implies that the spectrum of D has a conjugate pairing (for every eigenstate ψ of D with eigenvalue λ there is an eigenstate $\gamma_5\psi^\dagger$ with eigenvalue λ^*).

Now, writing the eigenvalues as $\lambda_n = l_n \exp(i\phi_n)$, the phases cancel and we have

$$\det D = \prod_n{}' \, l_n^2\,, \tag{4.40}$$

where $\prod'$ is defined to include only those eigenvalues with positive (or negative) imaginary part. The square root prescription is then simply

$$\sqrt{\det D} = \prod_n{}' \, l_n\,. \tag{4.41}$$

This prescription generalizes to the Yukawa theory in the straightforward way. Writing the $\mathrm{SU}(4)_R$ structure in vector form Eq. (4.28) becomes

$$S = a^4 \sum_x [\bar{\psi}_1 \cdots \quad \bar{\Psi}_1 \cdots]\, \mathbf{M} \begin{bmatrix} \psi_1 \\ \vdots \\ \Psi_1 \\ \vdots \end{bmatrix}\;;\;\; \mathbf{M} = \begin{bmatrix} \mathbf{D}+\mathbf{Y} & \mathbf{Y} \\ \mathbf{Y} & -\frac{\omega}{aR}+\mathbf{Y} \end{bmatrix}_{BC} . \tag{4.42}$$

Here $\mathbf{Y}_{BC}$ is just the Yukawas of Eq. (4.15) in matrix form, $\mathbf{D}_{BC}=\mathbf{1}_4 \otimes D\,\delta_{BC}$, etc. . .

The action of hermitian reflection on individual components of the Yukawas is $(\Phi_{1A}P_L)^\dagger = \Phi^*_{1A}P_R$ and thus, as a matrix operation, $\mathbf{Y}^\dagger{=}\mathbf{Y}$. γ_5-hermiticity generalizes in a straightforward way

$$\mathbf{M}^\dagger = (\mathbf{1}_8 \otimes \gamma_5)\, \mathbf{M}\, (\mathbf{1}_8 \otimes \gamma_5)\,, \tag{4.43}$$

and the square root prescription thus generalizes as well.

4.3 Tuning to $\mathcal{N}$=4

4.3.1 Counterterms

We are interested in tuning the lattice action such that the effective infrared description is $\mathcal{N}$=4 SYM. Generically there is a nontrivial matching between lattice and effective IR theories and all relevant or marginal terms consistent with lattice symmetries will appear in the infrared, except at special points in bare parameter space. We can arrive at the desired special point (*i.e.*, $\mathcal{N}$=4 SYM) by introducing the SUSY-violating operators into the bare action and fine-tuning counterterms. These counterterms fall into three categories: a scalar mass term, a Yukawa term, and two or four scalar quartic terms, depending on the number of colors for the gauge group, restricted here to $SU(N_c)$.

As mentioned in the Introduction, the tuning of the Yukawa term, schematically $y\overline{\psi}\phi\psi$, is accomplished through a rescaling of the scalars. Thus let $y = Z_y y_r$ and then note that $\phi \to \phi/Z_y$ fixes the Yukawa coupling to the critical value y_r, whereas the kinetic term becomes $(Z_\phi/Z_y^2)|D_\mu\phi|^2$, where Z_ϕ is an arbitrary constant in front of the scalar kinetic term. (This rescaling also changes the potential terms, but they are being tuned anyway.) Thus fine-tuning Z_ϕ in the bare action is equivalent to fine-tuning the strength of the Yukawa coupling y. Another way to see this is that the coupling strength y

can be eliminated from the Yukawa interaction term in favor of a noncanonical kinetic term $(1/y^2)|D_\mu\phi|^2$, through a rescaling $\phi \to \phi/y$. It is just a matter of where one chooses to place the bare action parameters.

4.3.2 Tr_R and unique quartic invariants

We will need several properties of the trace over the SU(4) R indices. In terms of our previous notation this trace is of course

$$\mathrm{Tr}_R\phi_1\phi_2\phi_3\phi_4 = \phi_1^{ij}\phi^2_{jk}\phi_3^{kl}\phi^4_{li}\,.$$

Its easy to show that

$$4\,\mathrm{Tr}_R\phi_A\phi_B\phi_C\phi_D = \mathrm{Tr}_R\{\phi_A,\phi_B\}\{\phi_C,\phi_D\} + \mathrm{Tr}_R[\phi_A,\phi_B][\phi_C,\phi_D]\,,$$

because the trace of the cross terms vanish. Also,

$$\{\phi_A,\phi_B\}_{ij} = \frac{1}{4}\,\mathrm{Tr}_R(\phi_A\phi_B)\;\mathbf{1}_{ij}\,, \tag{4.44}$$

and

$$\mathrm{Tr}_R[\phi_A,\phi_B][\phi_C,\phi_D] = \mathrm{Tr}_R(\phi_A\phi_D)\,\mathrm{Tr}_R(\phi_B\phi_C) - \Big(c \leftrightarrow d\Big). \tag{4.45}$$

We may now write the R trace of 4 ϕ's as

$$\begin{aligned}4\,\mathrm{Tr}_R\phi_A\phi_B\phi_C\phi_D = {} & \mathrm{Tr}_R(\phi_A\phi_B)\,\mathrm{Tr}_R(\phi_C\phi_D) - \mathrm{Tr}_R(\phi_A\phi_C)\,\mathrm{Tr}_R(\phi_B\phi_D)\\ & + \mathrm{Tr}_R(\phi_A\phi_D)\,\mathrm{Tr}_R(\phi_B\phi_C)\,.\end{aligned} \tag{4.46}$$

As for the gauge group traces: in SU(N<4), we only have one trace to take, $\mathrm{Tr}_G\,\phi\phi = \frac{1}{2}\phi_A\phi_A$. We can then write 4 potentially distinct invariant terms,

$$\mathrm{Tr}_R(\phi_A\phi_B)\,\mathrm{Tr}_R(\phi_A\phi_B)\,, \qquad \mathrm{Tr}_R(\phi_A\phi_A)\,\mathrm{Tr}_R(\phi_B\phi_B)\,,$$

$$\mathrm{Tr}_R(\phi_A\phi_B\phi_A\phi_B) \qquad \text{and} \qquad \mathrm{Tr}_R(\phi_A\phi_A\phi_B\phi_B)\,.$$

By applying the appropriate contractions to Eq. (4.46) we find

$$\begin{aligned} 4\,\mathrm{Tr}_R(\phi_A\phi_B\phi_A\phi_B) &= 2\,\mathrm{Tr}_R(\phi_A\phi_B)\,\mathrm{Tr}_R(\phi_A\phi_B) - \mathrm{Tr}_R(\phi_A\phi_A)\,\mathrm{Tr}_R(\phi_B\phi_B) \\ 4\,\mathrm{Tr}_R(\phi_A\phi_A\phi_B\phi_B) &= \mathrm{Tr}_R(\phi_A\phi_A)\,\mathrm{Tr}_R(\phi_B\phi_B)\,, \end{aligned} \tag{4.47}$$

so there are 2 unique gauge and R-symmetry invariant quartic terms and thus only 2 quartic couplings that will require tuning.

In SU(N>3) we have one more way to take the trace and we can make a similar argument as the above. For convenience define

$$\mathrm{Tr}_G\,\phi\phi\phi\phi \equiv \phi_A\phi_B\phi_C\phi_D\,C_{ABCD}$$

so that the four possible terms are

$$\begin{aligned} T_1 &= \mathrm{Tr}\,(\phi_A\phi_B)\,\mathrm{Tr}\,(\phi_C\phi_D)\,C_{ABCD}\,, \\ T_2 &= \mathrm{Tr}\,(\phi_A\phi_C)\,\mathrm{Tr}\,(\phi_B\phi_D)\,C_{ABCD}\,, \\ S_1 &= \mathrm{Tr}\,(\phi_A\phi_B\phi_C\phi_D)\,C_{ABCD}\,, \\ S_2 &= \mathrm{Tr}\,(\phi_A\phi_C\phi_B\phi_D)\,C_{ABCD}\,. \end{aligned} \tag{4.48}$$

Using again Eq. (4.46) and the invariance of C_{ABCD} under index cycling yields

$$4S_1 = 2T_1 - T_2 \quad \text{and} \quad 4S_2 = T_2\,, \tag{4.49}$$

precisely as before. We thus have two more gauge and R-symmetry invariant quartic combinations in SU(N>3), implying a total of 4 quartic couplings that will require tuning.

4.3.3 Reweighting

Suppose we perform a Monte Carlo simulation at one value m_1 of the scalar mass m, so that the configurations sample the distribution determined by the action $S = S_{m=0} + \frac{1}{2}\int m_1^2\phi^2$. Following the "Ferrenberg-Swendsen reweighting" method [114, 115] one can use the following "reweighting identity" to compute the expectation value of an operator O for the distribution with a different mass m_2:

$$\langle\mathcal{O}\rangle = \frac{\sum_C \mathcal{O}_C e^{-\frac{1}{2}(m_2^2-m_1^2)\int\phi_C^2}}{\sum_C e^{-\frac{1}{2}(m_2^2-m_1^2)\int\phi_C^2}}. \tag{4.50}$$

There is a limited regime of utility to this technique, due to the so-called "overlap problem." For instance, if the exponential in (4.50) is large where the simulated distribution has little weight, a finite sampling will have large errors. The mismatch of the distributions gets worse as the number of lattice sites increases, because the exponent is extensive.

A way to ameliorate the overlap problem, which has been found to work in other contexts, is "multicanonical reweighting" [116]. One replaces S with $S + W[O_1, O_2, \ldots]$, where $W[O_1, O_2, \ldots]$ is a carefully chosen function of some small set of observables (in our case W will be a function of $\int\phi^2$, the distinct $\int\phi^4$'s, and $\int(D\phi)^2$). The expectation value of an observable in the distribution corresponding to S is:

$$\langle\mathcal{O}\rangle = \frac{\sum_C \mathcal{O}_C \, e^{W[O_1^C,\ldots]}}{\sum_C e^{W[O_1^C,\ldots]}} \tag{4.51}$$

The function W produces a weighted average over a continuum of canonical ensembles, some of which will have a good overlap with the distribution that one is reweighting to. The challenge is to design a W such that sampling is flattened over the range of observables one is interested in.

For instance, in studying first order phase transitions (e.g., [117]), one chooses O_1 to be the order parameter of the transition; in a model with a scalar field, typically $O_1 = \int\phi^2$. One tunes W to cancel the nonperturbative effective potential for this operator, so that the Monte-Carlo simulation samples evenly in $\int \phi^2$. This enhances statistics for configurations intermediate between the phases. In the mass scan example above, one has

$$\langle \mathcal{O} \rangle = \frac{\sum_C \mathcal{O}_C e^{W[\int \phi^2] - \frac{1}{2}(m_2^2 - m_1^2)\int \phi_C^2}}{\sum_C e^{W[\int \phi^2] - \frac{1}{2}(m_2^2 - m_1^2)\int \phi_C^2}}. \tag{4.52}$$

Now wherever the exponential happens to be at its maximum, a large number of configurations will be generated, due to the flat distribution w.r.t. $\int \phi^2$. What we will describe below in Section 4.3.4 is how this should be extended to the $\mathcal{N}$=4 case.

Two approaches to engineering a good function W exist: (1) a bootstrap method [123] that iterates between Monte Carlo simulation and adjusting W, and (2) optimizing W w.r.t. its parameters, in a small volume, and then using step-scaling to extrapolate to a good estimate for W in larger volumes. For instance one can start with 4^4 and 6^4 volumes, where statistics accumulate rapidly and unreweighted simulations still cover broad parameter ranges.

4.3.4 Tuning with the effective potential

In our case the reweighting function W will depend on the four bosonic contributions to the action, $\int\phi^2$, $\int(D\phi)^2$, $\int\phi_1^4$ and $\int\phi_2^4$ (the two quadratic and two independent quartic operators you can form from the scalars, integrated over space). By sampling with the weight

$$\rho = \mathcal{D}[A,\phi]\,\det[D]\; e^{-S[A,\phi\,;\,m_0^2,\ldots]}\; e^{-W[\int\phi^2,\ldots]}\,, \tag{4.53}$$

and you can reproduce the ensemble at some particular set of values for m^2, Z_ϕ, λ and λ', via

$$Z = \textstyle\sum_c e^{+W[\ldots]}\; e^{-[(m^2-m_0^2)\phi_c^2 + (Z_\phi - Z_\phi^0)(D\phi)_c^2 + \ldots]}\,, \tag{4.54}$$

with W chosen so that the sample has a reasonable number of configurations for all values of $\int\phi^2$, $\int(D\phi)^2$, $\int\phi^4$ and $\int(\phi^2)^2$ within some interesting range.

Now, define the gauge invariant effective potential in finite volume as follows:

$$e^{-\Omega\, V[A^2]} = \sum_c e^{+W[\ldots]}\; e^{-[(m^2-m_0^2)\phi_c^2\ldots]} \times \delta(A^2 - \frac{\phi_c^2}{\Omega}) \tag{4.55}$$

with Ω the 4-volume (so that A^2 represents the mean value of the squared scalar field). It is easy to check that varying m^2 changes $V[A^2]$ by adding a linear component. Therefore, measuring $V[A^2]$ immediately determines how $\langle\phi^2\rangle$ and $F = \ln Z$ vary as a function of m^2. One easily generalizes to the effective potential as a function of all four bosonic operators, which gives a quick way to explore the effect of varying parameters.

It remains to specify how to tune the parameters. Consider $\mathcal{N}=4$ SYM with an added $a^2(\,\mathrm{Tr}\;\phi_m\phi_m)^3$ term in the potential, but deformed by mass and quartic interactions. The SUSY point is a second order phase transition point; for negative quartic deformation there is a first order transition as m^2 is varied (the system jumps from a massive state about the origin to a massive state with a large VEV because the quartic term bends down). For positive quartic deformation there is no phase transition. The optimal value of m^2 (fixing other parameters) is the point of maximal susceptibility $\langle(\int_x \phi^2)^2\rangle - (\langle\int_x \phi^2\rangle)^2$. Determining this point in finite volume leads to $\mathcal{O}(1/L^2)$ errors in the determined value of m^2, which can be improved by scaling over multiple volumes. Finding the flat quartic term which gives second-order behavior should also be possible; it has been successfully achieved in the context of the electroweak phase transition [124]. Therefore it should be possible to use the phase diagram to tune at least two parameters. Note that we needed to add a lattice-size ϕ^6 term; this is harmless but it raises the issue that the flat direction actually means that unbroken $\mathcal{N}$=4 SYM is not well behaved in finite volume; the moduli are not fixed and the partition function diverges because of the integral over the infinite moduli space. Therefore it will always be necessary to break SUSY somehow. We advocate doing so via twisted boundary conditions; for instance, instead of periodic boundary conditions we can add a rotation by angle Θ to all fermionic fields in one direction. The choice $\Theta{=}\pi$ is the maximal global breaking of SUSY and corresponds to treating the thermal ensemble; intermediate values of Θ break SUSY by smaller amounts. This lifts the moduli degeneracy without any *local* SUSY breaking; the effects of Θ are only visible in correlations at the scale of the lattice size, which is anyway contaminated by being in finite volume.

For the case $N_c > 3$ the effective potential should show multiple flat directions in the space of quartic operators; only one quartic direction (some linear combination of the input quartics, due to mixing) should rise steeply. Therefore we expect it should be possible to tune the "extra" quartic operators in the case $N_c > 3$, leaving only one quartic and the Yukawa coupling/wave function to tune.

4.3.5 Tuning with SUSY Ward identities

If supersymmetry is exact then the (R-symmetry **4**) supercurrent $S_{\mu,i}$ is conserved, so $\langle \partial^\mu S_{\mu,i}(x)\mathcal{O}(y)\rangle$ vanishes at $x{\neq}y$ for all local operators $\mathcal{O}$. We can use this property to measure whether we are at the SUSY point in parameter space, and therefore to tune parameters to find the SUSY point. The technique has been pioneered in $\mathcal{N}$=1 SUSY with Wilson fermions by the DESY-Münster group [99]; here we discuss the extension to $\mathcal{N}$=4 SYM.

The supercurrent $S_{\mu,i}$ is a linear combination of three dimension-7/2 operators. It is easy to find lattice operators which reproduce these continuum operators, at tree level and with contamination from higher dimension operators. The choices are not unique and at the nonperturbative level each lattice operator will mix with all continuum operators in the same symmetry channel. Different choices of lattice operator will reproduce the continuum operator with different normalization, mixings, and $\mathcal{O}(a)$ suppressed higher dimension contamination. Hence we express the operators $\mathcal{O}_{\mu,i}$ in a continuum language, and leave the particulars of lattice transcription (which amounts to various "improvements" w.r.t. $\mathcal{O}(a)$ discretization errors) for detailed studies. Our intention here is to lay out the methodology.

In their analysis of the $\mathcal{N}$=1 SYM case, the DESY-Münster group found two dimension-7/2 operators, the supercurrent S_μ and another fermionic current T_μ. These mix in the lattice-continuum matching and so one must write down two lattice operators with undetermined coefficients in order to find something which corresponds purely to S_μ (plus $\mathcal{O}(a)$ dimension-9/2 contamination). In the present $\mathcal{N}$=4 case there are 5 dimension-7/2 operators which we will name $\mathcal{O}^{1..5}_{\mu,i}$, and the renormalized $\mathcal{N}$=4 supercurrent will, in all generality, take the form:

$$\begin{aligned}
S^{\text{ren.}}_{\mu,i} = & \Big\{ Z_1 \frac{1}{2} F_A \cdot \sigma \, \delta_{ij} + Z_2 \sqrt{2} \not{D} \Big(\phi_{ij} P_L + \phi^{ij} P_R \Big)_A \\
& \qquad - Z_3 f^{ABC} \Big(\phi^B_{ik} \phi^{kj}_C P_R + \phi^{ik}_B \phi^C_{kj} P_L \Big) \Big\} \gamma_\mu \psi_{jA} \\
& + \Big\{ Z_4 \gamma_\nu F^A_{\mu\nu} \, \delta_{ij} - Z_5 \sqrt{2} D_\mu \Big(\phi^{ij} P_L + \phi_{ij} P_R \Big)_A \Big\} \psi_{jA} + \mathcal{O}(a) \\
\equiv & \; Z_n \mathcal{O}^n_{\mu,i} + \mathcal{O}(a) \,,
\end{aligned} \tag{4.56}$$

where the terms on the righthand side are bare (lattice) operators. Note that: (1) at tree level the supercurrent corresponds to $Z_1 = Z_2 = Z_3 = 1$ and

$Z_4 = Z_5 = 0$; (2) the renormalization constants Z_n are universal w.r.t. the index i due to the $SU(4)_R$ symmetry preserved by the lattice.

We can tune to the supersymmetric point by varying parameters to force correlation functions of this lattice-implemented $\partial_\mu S_{\mu,i}$ to vanish up to $\mathcal{O}(a)$ corrections. Specifically, to tune two parameters we need to choose 6 operators $\mathcal{O}^n_{\mu,i}$ in the same symmetry channel as $S_{\mu,i}$ (otherwise the correlation function vanishes automatically). The natural choice is $\mathcal{O}^{1,\ldots 5}_{\mu,i}$ plus one dimension-9/2 operator $\mathcal{O}^6_{\mu,i}$. One then measures the matrix of correlation functions

$$M^{mn}(t) \equiv \int d^3\vec{x} \langle \mathcal{O}^{m\dagger}_{0,i}(t,\vec{x}) \mathcal{O}^n_{0,i}(0,0)\rangle \tag{4.57}$$

whose t derivative is the correlation function between $\partial_\mu \mathcal{O}^m_{\mu,i}$ and $\mathcal{O}^n(0,i)$ at vanishing spatial momentum. Since the operators involved are dimension-7/2 we generically expect the elements of $M^{mn}(t)$ to decay as t^{-7}. At the supersymmetric point and for the right choices of Z_m, $Z_m M^{mn}$ decays as at^{-8} for all n. We can fix the undetermined ratios $Z_{2\ldots 5}/Z_1$ by enforcing that this holds for $n = 1\ldots 4$. Forcing that it hold for $n = 5, 6$ gives two conditions which can be used to check whether we are at the SUSY point–tuning to the SUSY point is tuning for $Z_m M^{m5} \sim at^{-8}$ and $Z_m M^{m6} \sim at^{-8}$. Actually since one of the operators is dimension-9/2 we must force one linear combination $Z_m M^{mn} c_n$ to vanish as $a^2 t^{-9}$.

We do not see an obstacle to using this procedure to tune more parameters, if it proves too difficult to tune some of the vanishing quartic couplings via the potential method. Therefore in principle the tuning to the SUSY point can be done by any mixture of the Ward identity method and the effective potential method.

4.3.6 Final tally

We have seen that for $N_\mathrm{c} = 2, 3$ colors, there are four fine-tunings in the action. For $N_\mathrm{c} > 3$ colors there are six. In addition, one must fix the four relative renormalization constants in the supercurrent. All but one of the scalar potential counterterms can be fixed by matching the effective potential, as determined by the multicanonical simulation, to the target theory scalar potential $\mathrm{Tr}[\phi_m, \phi_n][\phi_m, \phi_n]$. The overall strength of this term cannot be determined from the effective potential, because it will be expressed in terms of the bare operators in our approach.

This leaves just six fine-tunings for all number of colors N_c: one fine-tuning of the bare kinetic coefficient for the scalar, one overall scalar potential coefficient, and the four relative supercurrent coefficients. Thus a total of six Ward identities must be measured well enough to distinguish their simultaneous minimum w.r.t. $Z_\phi, Z_1/Z_2, \ldots, Z_4/Z_5$.

4.4 Conclusions

By preserving the $SU(4)_R$ symmetry of the target theory, the number of counterterms that must be fine-tuned is greatly reduced. This can be done by implementing overlap fermions, with the chirality of the Yukawa couplings implemented with auxiliary fermions, extending the method of Kikukawa [122]. Because counterterm fine-tuning can be isolated to the purely bosonic sector, it can all be done off-line, *i.e.*, without the expense of fermion matrix inversions (the bottleneck for all dynamical fermion simulations). This is a great advantage, because a very large number of points in the bare action parameter space will have to be scanned in order to find the $\mathcal{N}$=4 SYM point. Finally, we have explained how the overlap problem can be alleviated by taking a multicanonical approach, flattening the distributions that will be scanned over.

Now we breifly discuss the limitations and challenges of this proposal. The main limitation is that, since $\mathcal{N}$=4 SYM is conformal, the continuum limit is not a weak coupling limit. Our proposal should work at weak coupling, where one knows that the infrared description will be in terms of the same degrees of freedom as one puts on the lattice. But there is no guarantee that one can find lattice parameters which correspond to strongly coupled continuum theories.

The principal challenge is that the method requires GW fermions, which are numerically expensive–especially in a theory such as this one, with massless particles and the corresponding critical slowing down. It will be a challenge to generate enough configurations to measure quantities with sufficient accuracy to determine the SUSY point. Fermions are necessarily involved in the correlation functions of the supercurrent, so storage of propagators during the simulations will be essential to performing the fine-tuning w.r.t. Ward identities. The storage and computing resources that will be required will be substantial, but we believe that for small lattice volumes (say 8^4) the exploratory studies that need to be done can be performed in the near term.

For example, a first computation that needs to be done is to fix the multicanonical reweighting function. Small lattices $(4^4, 6^4)$ should suffice to get a rough idea of how to proceed in further studies $(8^4, 12^4)$. Obviously early stages of such work will be very much technical studies of the lattice theory. Continuum results will take much longer. Nevertheless, the beginnings of first principles nonperturbative study of $\mathcal{N}$=4 SYM are not so far off, we believe, if the current proposal is pursued with some dedication and adequate resources. We hope to report on further progress in that direction in the near future.

CHAPTER 5
Conclusion

In this thesis we have presented a robust technique for implementing supersymmetric field theories with a lattice regulator. We have given some important and very explicit examples of the necessary techniques and shown how nearly every supersymmetric field theory can be implemented on the lattice with minimal extensions of these techniques, which are currently available to, and well established within, the lattice community. Both $\mathcal{N}$=1 in 3D and $\mathcal{N}$=4 in 4D could be near first principle nonperturbative results from lattice studies. We have argued that all (or nearly all) the subtleties of lattice SUSY discretizations that have made simulations so difficult over the last several decades, are contained in one or another of these two theories. It thus seems reasonable to say that we have given a very general and robust prescription for studying SUSY on the lattice that can be extended to *any* supersymmetric theory, provided the gauge symmetry is not anomalous (if an anomalous theory could be implemented on the lattice than it would be mathematically well defined nonpertubatively and this would be a contradiction). Numerical study of even the most complicated and expensive of these proposals, $\mathcal{N}$=4 in 4D, is within reach, or nearly so. We stand to learn a truly vast amount about field theory, about supersymmetry, about QCD and - hopefully - about nature through first principles nonperturbative study of supersymmetric field theories.

APPENDIX A
Lattice Feynman Rules

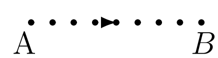
Definitions:

$$\hat{p}_\mu \equiv \frac{1}{a}\sin(ap_\mu), \qquad \hat{\not{p}} \equiv \frac{1}{a}\sum_\mu \gamma_\mu \sin(ap_\mu),$$

$$\tilde{p}_\mu \equiv \frac{2}{a}\sin\frac{ap_\mu}{2}, \qquad \tilde{p}^2 \equiv \frac{4}{a^2}\sum_\mu \sin\frac{ap_\mu}{2}, \qquad M_p \equiv \frac{ar}{2}\tilde{p}^2\,.$$

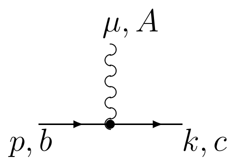
Propagators: (gauge indices are supressed where no confusion is possible)

 $= \dfrac{-i\hat{\not{p}} + M_p}{\hat{p}^2 + M_p^2}\,,$ $= \dfrac{1}{\tilde{p}^2} \equiv \tilde{\Delta}_p\,,$

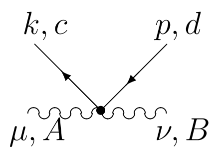
μ  ν $= \dfrac{1}{\tilde{p}^2}\left(g_{\mu\nu} - (1-\xi)\dfrac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}\right),$ 

$= \dfrac{1}{\tilde{p}^2}\delta_{AB}$

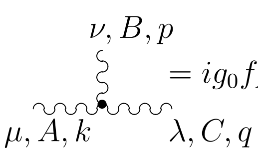
Vertex Rules (*all momenta incoming to vertex*, $\delta(\sum p_i)$ implied)**:**



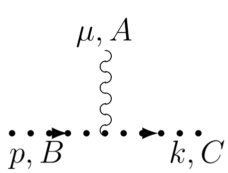
$$= -g_0 T^A_{bc}\left(i\gamma_\mu \cos\frac{a(p-k)_\mu}{2} + r\sin\frac{a(p-k)_\mu}{2}\right)$$



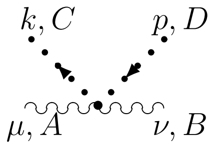
$$= \frac{a}{2}g_0^2\{T^A, T^B\}_{cd}\,\delta_{\mu\nu}\left(i\gamma_\mu \sin\frac{a(p-k)_\mu}{2} - r\cos\frac{a(p-k)_\mu}{2}\right)$$



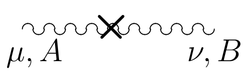
$$= ig_0 f_{ABC}\left(\delta_{\nu\lambda}(\widetilde{q-p})_\mu \cos\frac{k_\nu}{2} + \delta_{\lambda\mu}(\widetilde{k-q})_\nu \cos\frac{p_\lambda}{2} + \delta_{\mu\nu}(\widetilde{p-k})_\lambda \cos\frac{q_\mu}{2}\right)$$



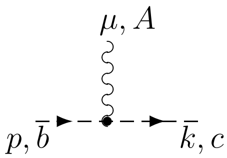
$$= ig_0 f_{ABC}\,\tilde{k}_\mu \cos\frac{p_\mu}{2}$$



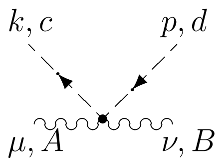
$$= -\frac{1}{12}a^2 g_0^2 \{F^C, F^D\}_{AB}\,\delta_{\mu\nu}\,\tilde{k}_\mu \tilde{p}_\mu\,, \qquad (F^C_{AE} \equiv -i f_{CAE})$$



$$= -\frac{g_0^2 \mathcal{T}_F \mathcal{C}_A}{6a^2}\,\delta_{\mu\nu}\,.$$

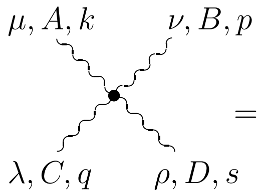


$$= g_0 T^A_{bc} \widetilde{(p-k)}_\mu$$



$$= g_0^2 \{T^A, T^B\}_{cd} \delta_{\mu\nu} \cos\frac{a(p-k)_\mu}{2}$$

The four point gluon vertex (reproduced from [34] and generalized to arbitrary gauge group) is



$$
\begin{aligned}
= & -g_0^2 \sum_E f_{ABE} f_{CDE} \left\{ \begin{array}{l}
\delta_{\mu\lambda}\delta_{\nu\rho}\Big[\cos\frac{(p-s)_\mu}{2}\cos\frac{(k-q)_\nu}{2} - \frac{1}{12}\tilde{k}_\nu\tilde{p}_\mu\tilde{q}_\nu\tilde{s}_\mu\Big] \\
-\delta_{\mu\rho}\delta_{\nu\lambda}\Big[\cos\frac{(p-q)_\mu}{2}\cos\frac{(k-s)_\nu}{2} - \frac{1}{12}\tilde{k}_\nu\tilde{p}_\mu\tilde{q}_\mu\tilde{s}_\nu\Big] \\
+\frac{1}{6}\delta_{\nu\lambda}\delta_{\nu\rho}(\widetilde{s-q})_\mu\tilde{k}_\nu\cos\frac{p_\mu}{2} - \frac{1}{6}\delta_{\mu\lambda}\delta_{\mu\rho}(\widetilde{s-q})_\nu\tilde{p}_\mu\cos\frac{k_\nu}{2} \\
+\frac{1}{6}\delta_{\mu\nu}\delta_{\mu\rho}(\widetilde{p-k})_\lambda\tilde{q}_\rho\cos\frac{s_\lambda}{2} - \frac{1}{6}\delta_{\mu\nu}\delta_{\mu\lambda}(\widetilde{p-k})_\rho\tilde{s}_\lambda\cos\frac{q_\rho}{2} \\
+\frac{1}{12}\delta_{\mu\nu}\delta_{\mu\lambda}\delta_{\mu\rho}\sum_\sigma(\widetilde{p-k})_\sigma(\widetilde{s-q})_\sigma
\end{array} \right\} \\
& -\Big([\nu, B, p] \leftrightarrow [\lambda, C, q]\Big) \\
& -\Big([\nu, B, p] \leftrightarrow [\rho, D, s]\Big) \\
& +2g_0^2 \,\mathrm{Tr}\, T^{(A}T^{B}T^{C}T^{D)} \left\{ \begin{array}{l}
\delta_{\mu\nu}\delta_{\mu\lambda}\delta_{\mu\rho}\sum_\sigma\tilde{k}_\sigma\tilde{p}_\sigma\tilde{q}_\sigma\tilde{s}_\sigma - \delta_{\mu\nu}\delta_{\mu\lambda}\tilde{k}_\rho\tilde{p}_\rho\tilde{q}_\rho\tilde{s}_\mu \\
-\delta_{\mu\nu}\delta_{\mu\rho}\tilde{k}_\lambda\tilde{p}_\lambda\tilde{s}_\lambda\tilde{q}_\mu - \delta_{\mu\lambda}\delta_{\mu\rho}\tilde{k}_\nu\tilde{q}_\nu\tilde{s}_\nu\tilde{p}_\mu - \delta_{\nu\lambda}\delta_{\nu\rho}\tilde{p}_\mu\tilde{q}_\mu\tilde{s}_\mu\tilde{k}_\nu \\
+\delta_{\mu\nu}\delta_{\lambda\rho}\tilde{k}_\lambda\tilde{p}_\lambda\tilde{s}_\mu\tilde{q}_\mu + \delta_{\mu\lambda}\delta_{\nu\rho}\tilde{k}_\nu\tilde{q}_\nu\tilde{p}_\mu\tilde{s}_\mu + \delta_{\mu\rho}\delta_{\nu\lambda}\tilde{k}_\nu\tilde{s}_\nu\tilde{p}_\mu\tilde{q}_\mu
\end{array} \right\}
\end{aligned}
$$

where the notation in the trace on group matrices is symmetrization in all four indices (hence $\frac{1}{24}\,\mathrm{Tr}\,(T^AT^BT^CT^D + T^BT^AT^CT^D + \ldots)$).

APPENDIX B
Some integrals for LPT

In this appendix we ennumerate a few of the integrals that come up very frequently in LPT calculations. We set $a = 1$ everywhere in this section. Define:

$$\int \frac{1}{\tilde{k}^2} \equiv \int_{-\pi}^{\pi} \frac{d^dk}{(2\pi)^d} \frac{1}{4\sum_\mu \sin^2(k_\mu/2)} \equiv \frac{\Sigma}{4\pi} \qquad \text{and} \qquad \int \frac{1}{(\tilde{k}^2)^2} \equiv \frac{\xi}{4\pi} \,. \qquad \text{(B.1)}$$

We also need that $\int 1 = 1$ and we assume $d \geq 2$.

Now, since

$$\int \frac{\tilde{k}_1^2}{\tilde{k}^2} = \int \frac{\tilde{k}_2^2}{\tilde{k}^2}$$

(by cubic invariance), we have

$$\int \frac{\tilde{k}_1^2}{\tilde{k}^2} = \frac{1}{d}\,, \qquad \int \frac{\tilde{k}_1^2}{(\tilde{k}^2)^2} = \frac{1}{d}\frac{\Sigma}{4\pi}\,, \quad \text{and} \qquad \int \frac{\tilde{k}_1^2}{(\tilde{k}^2)^3} = \frac{1}{d}\frac{\xi}{4\pi}\,. \qquad \text{(B.2)}$$

We will also need (recall that $\hat{k} \equiv \sin(k)$ and $\partial_1 \equiv \partial/\partial k_1$)

$$\int \frac{\hat{k}_1^2}{(\tilde{k}^2)^2} = \frac{1}{2}\int \frac{\cos(k_1)}{\tilde{k}^2} - \frac{1}{4}\int \partial_1^2(\ln \tilde{k}^2) = \frac{1}{2}\frac{\Sigma}{4\pi} - \frac{1}{4d}\,, \qquad \text{(B.3)}$$

where the last step follows from the fact that $\partial_1 \ln \tilde{k}^2$ vanishes at $k_1 = \pm\pi$.

Now

$$\begin{aligned} \int \frac{\tilde{k}_1^4}{(\tilde{k}^2)^2} &= 4\int \frac{\tilde{k}_1^2}{(\tilde{k}^2)^2} - 4\int \frac{\hat{k}_1^2}{(\tilde{k}^2)^2} \\ &= \frac{1}{d} - \frac{\Sigma}{4\pi}\Big(2 - \frac{4}{d}\Big)\,, \end{aligned} \qquad \text{(B.4)}$$

and

$$\int \frac{\tilde{k}_1^4}{(\tilde{k}^2)^2} = \frac{1}{d} \int \frac{(\tilde{k}^2)^2 - d(d-1)\tilde{k}_1^2 \tilde{k}_2^2}{(\tilde{k}^2)^2}\,, \tag{B.5}$$

so we also have

$$\int \frac{\tilde{k}_1^2 \tilde{k}_2^2}{(\tilde{k}^2)^2} = \frac{2d-4}{d(d-1)} \frac{\Sigma}{4\pi}\,. \tag{B.6}$$

These are all we have needed here, but just for fun:

$$\int \frac{\tilde{k}_1^4}{(\tilde{k}^2)^3} = \frac{1}{2d}\frac{\Sigma}{4\pi} + \frac{d-4}{d(d-1)}\frac{\xi}{4\pi}\,, \tag{B.7}$$

$$\text{and} \quad \int \frac{\tilde{k}_1^2 \tilde{k}_2^2}{(\tilde{k}^2)^3} = \frac{1}{2d(d-1)}\frac{\Sigma}{4\pi} + \frac{d-4}{d(d-1)}\frac{\xi}{4\pi}\,. \tag{B.8}$$

The specific values of Σ and ξ will depend on the dimension of course. Σ has a UV log divergence for d=2 and is finite in all greater dimensions. The naive degree of divergence of ξ is $4-d$; that is, it has a UV log divergence in d=4.

APPENDIX C
Group Relations

The following relations for the matrices of the anti-Hermitian basis of the fundamental representation of the gauge group were used in the calculation. This is not an exhaustive list of useful relations. Since the idea is to reproduce the adjoint representation as a product of two fundamental representations, the relations will be expressed in terms of the generator of the adjoint representation, $F^A_{BC} \equiv -f^{ABC}$, and the completely symmetric structure functions, $D^A_{BC} \equiv -id^{ABC}$. d_F is the dimension of the fundamental representation. $\mathcal{T}_F$ and $\mathcal{C}_F$ are the first and second Casimirs of the fundamental representation of the group. $\mathcal{T}_F$ is also known as the trace normalization and is usually chosen to be $\frac{1}{2}$. $\mathcal{C}_A$ is the first and second Casimir of the adjoint representation (and is also the same as the dimension of the representation).

We have

$$[T^A, T^B] = f^{ABC}T^C \quad \text{and} \quad \{T^A, T^B\} = \frac{-1}{d_F}\delta^{AB} - id^{ABC}T^C \,. \tag{C.1}$$

The trace relations are

$$\begin{aligned}
\mathrm{Tr}\, T^A T^B &= -\mathcal{T}_F \delta^{AB} \,, \\
\mathrm{Tr}\, T^A T^B T^C &= \frac{\mathcal{T}_F}{2}\left(D^A - F^A\right)_{BC} \,, \\
\mathrm{Tr}\, T^A T^B T^C T^D &= \frac{1}{4d_F}\delta^{AB}\delta^{CD} - \frac{\mathcal{T}_F}{4}\left(\left(D^A - F^A\right)\left(D^C + F^C\right)\right)_{BD} ,
\end{aligned} \tag{C.2}$$

and another useful relation is

$$T^B T^A T^B = \Big(\frac{1}{2}\mathcal{C}_A - \mathcal{C}_F\Big) T^A \,. \tag{C.3}$$

To calculate diagrams we will also need

$$\operatorname{Tr} F^A F^B = -\mathcal{C}_A \delta^{AB} \quad \text{and} \quad \operatorname{Tr} D^A D^B = \Big(\frac{1}{d_F \mathcal{T}_F} + \mathcal{C}_A - 4\mathcal{C}_F\Big)\delta^{AB} \,. \tag{C.4}$$

APPENDIX D
Momentum space expressions for 4D lattice twisted SUSY

We write the terms as $\sum_i w_i^A \Psi^\dagger(k) V_i Y_i^A \Psi$, etc.... The V_i are 16×16, each has 7 nonzero entries in the upper right quadrant corresponding to the $w\chi\psi$, $w\chi\theta$, and $w\kappa\psi$ terms, and one nonzero entry amongst the last four elements of the first column corresponding to the $w\theta\eta$ term. I will thus quote the results in terms of an 8×8 matrix and a 4 component column vector. The momentum space expressions for the $w\chi\theta$ terms has a phase remnant, $\phi_{ab} \equiv e^{ik_{ab}}$.

$$v_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad u_4 = \begin{bmatrix} 0 & & & & & & & \\ & 0 & 1 & & & & & \\ & -1 & 0 & & & & & \\ & & & 0 & & & & \phi_{23} \\ 1 & & & & 0 & & & \\ & & & & & 0 & -\phi_{13} & \\ & & & & & \phi_{12} & 0 & \\ & & & -1 & & & & 0 \end{bmatrix}$$

$$
\begin{aligned}
v_3 &= \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix}
0 & & & & & & & \\
& 0 & & 1 & & & & \\
& & 0 & & & & & -\phi_{24} \\
& -1 & & 0 & & & & \\
& & & & 0 & & \phi_{14} & \\
1 & & & & & 0 & & \\
& & & & -\phi_{12} & & 0 & \\
& & 1 & & & & & 0
\end{bmatrix} \\
v_2 &= \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix}
0 & & & & & & & \\
& 0 & & & & & & \phi_{34} \\
& & 0 & 1 & & & & \\
& & -1 & 0 & & & & \\
& & & & 0 & -\phi_{14} & & \\
& & & & \phi_{13} & 0 & & \\
1 & & & & & & 0 & \\
& -1 & & & & & & 0
\end{bmatrix} \\
v_1 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad u_1 = \begin{bmatrix}
0 & & & & & & & \\
& 0 & & & & & -\phi_{34} & \\
& & 0 & & & \phi_{24} & & \\
& & & 0 & -\phi_{23} & & & \\
& & & 1 & 0 & & & \\
& & -1 & & & 0 & & \\
& 1 & & & & & 0 & \\
1 & & & & & & & 0
\end{bmatrix}
\end{aligned} \tag{D.1}
$$

References

[1] J. W. Elliott and G. D. Moore, *3d n = 1 sym chern-simons theory on the lattice*, arXiv:0708.3214 [hep-lat].

[2] J. W. Elliott, J. Giedt and G. D. Moore, *Lattice four-dimensional N=4 SYM is practical*, 0806.0013.

[3] L. Corwin, Y. Ne'eman and S. Sternberg, *GRADED LIE ALGEBRAS IN MATHEMATICS AND PHYSICS (BOSE-FERMI SYMMETRY)*, *Rev. Mod. Phys.* **47** (1975) 573.

[4] Y. A. Golfand and E. P. Likhtman, *Extension of the Algebra of Poincare Group Generators and Violation of p Invariance*, *JETP Lett.* **13** (1971) 323–326.

[5] D. V. Volkov and V. P. Akulov, *Is the Neutrino a Goldstone Particle?*, *Phys. Lett.* **B46** (1973) 109–110.

[6] J. Wess and B. Zumino, *A Lagrangian Model Invariant Under Supergauge Transformations*, *Phys. Lett.* **B49** (1974) 52.

[7] J. Wess and B. Zumino, *Supergauge Transformations in Four-Dimensions*, *Nucl. Phys.* **B70** (1974) 39–50.

[8] S. Dimopoulos and H. Georgi, *Softly Broken Supersymmetry and SU(5)*, *Nucl. Phys.* **B193** (1981) 150.

[9] G. Jungman, M. Kamionkowski and K. Griest, *Supersymmetric dark matter*, *Phys. Rept.* **267** (1996) 195–373 [hep-ph/9506380].

[10] K. A. Intriligator and N. Seiberg, *Lectures on supersymmetric gauge theories and electric- magnetic duality*, *Nucl. Phys. Proc. Suppl.* **45BC** (1996) 1–28 [hep-th/9509066].

[11] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*, *Nucl. Phys.* **B431** (1994) 484–550 [hep-th/9408099].

[12] K. A. Intriligator and N. Seiberg, *Mirror symmetry in three dimensional gauge theories*, *Phys. Lett.* **B387** (1996) 513–519 [hep-th/9607207].

[13] A. Kapustin and M. J. Strassler, *On mirror symmetry in three dimensional Abelian gauge theories*, *JHEP* **04** (1999) 021 [hep-th/9902033].

[14] O. Aharony, A. Hanany, K. A. Intriligator, N. Seiberg and M. J. Strassler, *Aspects of $N = 2$ supersymmetric gauge theories in three dimensions*, *Nucl. Phys.* **B499** (1997) 67–99 [hep-th/9703110].

[15] J. de Boer, K. Hori, Y. Oz and Z. Yin, *Branes and mirror symmetry in $N = 2$ supersymmetric gauge theories in three dimensions*, *Nucl. Phys.* **B502** (1997) 107–124 [hep-th/9702154].

[16] J. de Boer, K. Hori and Y. Oz, *Dynamics of $N = 2$ supersymmetric gauge theories in three dimensions*, *Nucl. Phys.* **B500** (1997) 163–191 [hep-th/9703100].

[17] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles, and three- dimensional gauge dynamics*, *Nucl. Phys.* **B492** (1997) 152–190 [hep-th/9611230].

[18] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252 [hep-th/9711200].

[19] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291 [hep-th/9802150].

[20] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B428** (1998) 105–114 [hep-th/9802109].

[21] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183–386 [hep-th/9905111].

[22] S. Hellerman, *Lattice gauge theories have gravitational duals*, hep-th/0207226.

[23] S. Catterall and T. Wiseman, *Towards lattice simulation of the gauge theory duals to black holes and hot strings*, *JHEP* **12** (2007) 104 [0706.3518].

[24] S. Catterall and T. Wiseman, *Black hole thermodynamics from simulations of lattice Yang-Mills theory*, 0803.4273.

[25] F. Cooper, A. Khare and U. Sukhatme, *Supersymmetry and quantum mechanics*, *Phys. Rept.* **251** (1995) 267–385 [hep-th/9405029].

[26] S. Weinberg, *The quantum theory of fields. Vol. 3: Supersymmetry*, . Cambridge, UK: Univ. Pr. (2000) 419 p.

[27] M. Drees, *An introduction to supersymmetry*, `hep-ph/9611409`.

[28] S. Ferrara, J. Iliopoulos and B. Zumino, *Supergauge Invariance and the Gell-Mann - Low Eigenvalue*, *Nucl. Phys.* **B77** (1974) 413.

[29] M. T. Grisaru, W. Siegel and M. Rocek, *Improved Methods for Supergraphs*, *Nucl. Phys.* **B159** (1979) 429.

[30] E. Witten, *Dynamical Breaking of Supersymmetry*, *Nucl. Phys.* **B188** (1981) 513.

[31] Y. Shadmi and Y. Shirman, *Dynamical supersymmetry breaking*, *Rev. Mod. Phys.* **72** (2000) 25–64 [`hep-th/9907225`].

[32] E. Witten, *Supersymmetric index of three-dimensional gauge theory*, `hep-th/9903005`.

[33] E. Witten, *Supersymmetric index in four-dimensional gauge theories*, *Adv. Theor. Math. Phys.* **5** (2002) 841–907 [`hep-th/0006010`].

[34] H. J. Rothe, *Lattice gauge theories: An Introduction*, *World Sci. Lect. Notes Phys.* **74** (2005) 1–605.

[35] J. B. Kogut, *An Introduction to Lattice Gauge Theory and Spin Systems*, *Rev. Mod. Phys.* **51** (1979) 659.

[36] J. B. Kogut, *A Review of the Lattice Gauge Theory Approach to Quantum Chromodynamics*, *Rev. Mod. Phys.* **55** (1983) 775.

[37] J. M. Drouffe and C. Itzykson, *Lattice Gauge Fields*, *Phys. Rept.* **38** (1978) 133–175.

[38] L. P. Kadanoff, *Lectures on the Application of Renormalization Group Techniques to Quarks and Strings*, *Rev. Mod. Phys.* **49** (1977) 267–296.

[39] M. Creutz, L. Jacobs and C. Rebbi, *Monte Carlo Computations in Lattice Gauge Theories*, *Phys. Rept.* **95** (1983) 201.

[40] J.-M. Drouffe and J.-B. Zuber, *Strong Coupling and Mean Field Methods in Lattice Gauge Theories*, *Phys. Rept.* **102** (1983) 1.

[41] E. Seiler, *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, *Lect. Notes Phys.* **159** (1982) 1–192.

[42] Y. Shamir, *Lattice Chiral Fermions*, *Nucl. Phys. Proc. Suppl.* **47** (1996) 212–227 [hep-lat/9509023].

[43] B. Sheikholeslami and R. Wohlert, *Improved continuum limit lattice action for qcd with wilson fermions*, *Nucl. Phys.* **B259** (1985) 572.

[44] M. G. Alford, W. Dimm, G. P. Lepage, G. Hockney and P. B. Mackenzie, *Lattice QCD on small computers*, *Phys. Lett.* **B361** (1995) 87–94 [hep-lat/9507010].

[45] **Alpha** Collaboration, R. Frezzotti, P. A. Grassi, S. Sint and P. Weisz, *Lattice QCD with a chirally twisted mass term*, *JHEP* **08** (2001) 058 [hep-lat/0101001].

[46] M. Luscher, S. Sint, R. Sommer and H. Wittig, *Non-perturbative determination of the axial current normalization constant in O(a) improved lattice QCD*, *Nucl. Phys.* **B491** (1997) 344–364 [hep-lat/9611015].

[47] M. Luscher *et. al.*, *Some new results in O(a) improved lattice QCD*, *Nucl. Phys. Proc. Suppl.* **53** (1997) 905–913 [hep-lat/9608049].

[48] H. Kawai, R. Nakayama and K. Seo, *Comparison of the Lattice Lambda Parameter with the Continuum Lambda Parameter in Massless QCD*, *Nucl. Phys.* **B189** (1981) 40.

[49] H. B. Nielsen and M. Ninomiya, *No Go Theorem for Regularizing Chiral Fermions*, *Phys. Lett.* **B105** (1981) 219.

[50] P. H. Ginsparg and K. G. Wilson, *A Remnant of Chiral Symmetry on the Lattice*, *Phys. Rev.* **D25** (1982) 2649.

[51] R. Narayanan and H. Neuberger, *Infinitely many regulator fields for chiral fermions*, *Phys. Lett.* **B302** (1993) 62–69 [hep-lat/9212019].

[52] R. Narayanan and H. Neuberger, *Chiral determinant as an overlap of two vacua*, *Nucl. Phys.* **B412** (1994) 574–606 [hep-lat/9307006].

[53] R. Narayanan and H. Neuberger, *Chiral fermions on the lattice*, *Phys. Rev. Lett.* **71** (1993) 3251–3254 [hep-lat/9308011].

[54] D. B. Kaplan, *A Method for simulating chiral fermions on the lattice*, *Phys. Lett.* **B288** (1992) 342–347 [hep-lat/9206013].

[55] M. Luscher, *Exact chiral symmetry on the lattice and the ginsparg-wilson relation*, *Phys. Lett.* **B428** (1998) 342–345 [hep-lat/9802011].

[56] S. R. Sharpe, *Progress in lattice gauge theory*, hep-lat/9811006.

[57] M. Luscher, *Chiral gauge theories on the lattice with exact gauge invariance*, *Nucl. Phys. Proc. Suppl.* **83** (2000) 34–43 [hep-lat/9909150].

[58] H. Suzuki, *Anomaly cancellation condition in lattice gauge theory*, *Nucl. Phys.* **B585** (2000) 471–513 [hep-lat/0002009].

[59] M. Luscher, *Lattice regularization of chiral gauge theories to all orders of perturbation theory*, *JHEP* **06** (2000) 028 [hep-lat/0006014].

[60] E. Witten, *An su(2) anomaly*, *Phys. Lett.* **B117** (1982) 324–328.

[61] M. Luscher, *Abelian chiral gauge theories on the lattice with exact gauge invariance*, *Nucl. Phys.* **B549** (1999) 295–334 [hep-lat/9811032].

[62] M. Luscher, *Chiral gauge theories revisited*, hep-th/0102028.

[63] H. Neuberger, *Exactly massless quarks on the lattice*, *Phys. Lett.* **B417** (1998) 141–144 [hep-lat/9707022].

[64] J. W. Elliott and G. D. Moore, *Three dimensional $n = 2$ supersymmetry on the lattice*, *PoS* **LAT2005** (2006) 245 [hep-lat/0509032].

[65] N. Seiberg, *Notes on theories with 16 supercharges*, *Nucl. Phys. Proc. Suppl.* **67** (1998) 158–171 [hep-th/9705117].

[66] D. B. Kaplan and M. Unsal, *A Euclidean lattice construction of supersymmetric Yang- Mills theories with sixteen supercharges*, *JHEP* **09** (2005) 042 [hep-lat/0503039].

[67] A. N. Redlich, *Parity violation and gauge noninvariance of the effective gauge field action in three-dimensions*, *Phys. Rev.* **D29** (1984) 2366–2374.

[68] M. F. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian Geometry 3*, *Math. Proc. Cambridge Phil. Soc.* **79** (1980) 71.

[69] M. F. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian geometry 2*, *Math. Proc. Cambridge Phil. Soc.* **78** (1976) 405.

[70] M. F. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian Geometry 1*, *Math. Proc. Cambridge Phil. Soc.* **77** (1975) 43.

[71] C. Gattringer and I. Hip, *Clover improvement, spectrum and atiyah-singer index theorem for the dirac operator on the lattice*, *Nucl. Phys.* **B541** (1999) 305–319 [hep-lat/9806032].

[72] S. Capitani, *Lattice perturbation theory*, *Phys. Rept.* **382** (2003) 113–302 [hep-lat/0211036].

[73] G. D. Moore, *O(a) errors in 3-d su(n) higgs theories*, *Nucl. Phys.* **B523** (1998) 569–593 [hep-lat/9709053].

[74] G. D. Moore, *Measuring the broken phase sphaleron rate nonperturbatively*, *Phys. Rev.* **D59** (1999) 014503 [hep-ph/9805264].

[75] M. Luscher, *Topology of lattice gauge fields*, *Commun. Math. Phys.* **85** (1982) 39.

[76] P. Woit, *Topology and lattice gauge fields*, *Nucl. Phys.* **B262** (1985) 284.

[77] A. Phillips and D. Stone, *Lattice gauge fields, principal bundles and the calculation of topological charge*, *Commun. Math. Phys.* **103** (1986) 599–636.

[78] de Forcrand, Philippe and Garcia Perez, Margarita and Hetrick, James E. and Stamatescu, Ion-Olimpiu, *Topology of full QCD*, *Nucl. Phys. Proc. Suppl.* **63** (1998) 549–551 [hep-lat/9710001].

[79] T. A. DeGrand, A. Hasenfratz and T. G. Kovacs, *Topological structure in the su(2) vacuum*, *Nucl. Phys.* **B505** (1997) 417–441 [hep-lat/9705009].

[80] K. Symanzik, *Continuum limit and improved action in lattice theories. 1. principles and phi**4 theory*, *Nucl. Phys.* **B226** (1983) 187.

[81] K. Symanzik, *Continuum limit and improved action in lattice theories. 2. o(n) nonlinear sigma model in perturbation theory*, *Nucl. Phys.* **B226** (1983) 205.

[82] M. J. Teper, *Su(n) gauge theories in 2+1 dimensions*, *Phys. Rev.* **D59** (1999) 014512 [hep-lat/9804008].

[83] M. Laine and O. Philipsen, *The non-perturbative qcd debye mass from a wilson line operator*, *Phys. Lett.* **B459** (1999) 259–264 [hep-lat/9905004].

[84] D. B. Kaplan, E. Katz and M. Unsal, *Supersymmetry on a spatial lattice*, *JHEP* **05** (2003) 037 [hep-lat/0206019].

[85] A. G. Cohen, D. B. Kaplan, E. Katz and M. Unsal, *Supersymmetry on a euclidean spacetime lattice. ii: Target theories with eight supercharges*, *JHEP* **12** (2003) 031 [hep-lat/0307012].

[86] A. G. Cohen, D. B. Kaplan, E. Katz and M. Unsal, *Supersymmetry on a euclidean spacetime lattice. i: A target theory with four supercharges*, *JHEP* **08** (2003) 024 [hep-lat/0302017].

[87] F. Sugino, *A lattice formulation of super Yang-Mills theories with exact supersymmetry*, *JHEP* **01** (2004) 015 [hep-lat/0311021].

[88] F. Sugino, *Super Yang-Mills theories on the two-dimensional lattice with exact supersymmetry*, *JHEP* **03** (2004) 067 [hep-lat/0401017].

[89] F. Sugino, *Various super Yang-Mills theories with exact supersymmetry on the lattice*, *JHEP* **01** (2005) 016 [hep-lat/0410035].

[90] S. Catterall, *A geometrical approach to $n = 2$ super yang-mills theory on the two dimensional lattice*, *JHEP* **11** (2004) 006 [hep-lat/0410052].

[91] S. Catterall, *Lattice formulation of $N = 4$ super Yang-Mills theory*, *JHEP* **06** (2005) 027 [hep-lat/0503036].

[92] S. Catterall, *Simulations of $N = 2$ super Yang-Mills theory in two dimensions*, *JHEP* **03** (2006) 032 [hep-lat/0602004].

[93] J. Giedt, *Deconstruction and other approaches to supersymmetric lattice field theories*, *Int. J. Mod. Phys.* **A21** (2006) 3039–3094 [hep-lat/0602007].

[94] S. Ferrara and B. Zumino, *Supergauge Invariant Yang-Mills Theories*, *Nucl. Phys.* **B79** (1974) 413.

[95] G. Curci and G. Veneziano, *Supersymmetry and the Lattice: A Reconciliation?*, *Nucl. Phys.* **B292** (1987) 555.

[96] A. Donini, M. Guagnelli, P. Hernandez and A. Vladikas, *Towards $n = 1$ super-yang-mills on the lattice*, *Nucl. Phys.* **B523** (1998) 529–552 [hep-lat/9710065].

[97] **DESY-Munster** Collaboration, R. Kirchner, S. Luckmann, I. Montvay, K. Spanderen and J. Westphalen, *Numerical simulation of dynamical gluinos: Experience with a multi-bosonic algorithm and first results*, *Nucl. Phys. Proc. Suppl.* **73** (1999) 828–833 [hep-lat/9808024].

[98] **DESY-Munster** Collaboration, I. Campos *et. al.*, *Monte Carlo simulation of SU(2) Yang-Mills theory with light gluinos*, *Eur. Phys. J.* **C11** (1999) 507–527 [hep-lat/9903014].

[99] **DESY-Munster-Roma** Collaboration, F. Farchioni *et. al.*, *The supersymmetric Ward identities on the lattice*, *Eur. Phys. J.* **C23** (2002) 719–734 [hep-lat/0111008].

[100] I. Montvay, *Supersymmetric Yang-Mills theory on the lattice*, *Int. J. Mod. Phys.* **A17** (2002) 2377–2412 [hep-lat/0112007].

[101] R. Peetz, F. Farchioni, C. Gebert and G. Munster, *Spectrum of SU(2) SUSY Yang-Mills theory with a light gluino*, *Nucl. Phys. Proc. Suppl.* **119** (2003) 912–914 [hep-lat/0209065].

[102] F. Farchioni, G. Muenster and R. Peetz, *The volume source technique for flavor singlets: A second look*, *Eur. Phys. J.* **C38** (2004) 329–334 [hep-lat/0404004].

[103] F. Farchioni and R. Peetz, *The low-lying mass spectrum of the $N = 1$ SU(2) SUSY Yang- Mills theory with Wilson fermions*, *Eur. Phys. J.* **C39** (2005) 87–94 [hep-lat/0407036].

[104] R. Peetz, *Spectrum of* $\mathcal{N}$=1 *Super yang Mills Theory on the Lattice with a Light Gluino*, *Doctoral dissertation, University of Münster, Germany* (2003) [http://deposit.ddb.de/cgi-bin/dokserv?idn=97018249X].

[105] N. Maru and J. Nishimura, *Lattice formulation of supersymmetric Yang-Mills theories without fine-tuning*, *Int. J. Mod. Phys.* **A13** (1998) 2841–2856 [hep-th/9705152].

[106] H. Neuberger, *Vector like gauge theories with almost massless fermions on the lattice*, *Phys. Rev.* **D57** (1998) 5417–5433 [hep-lat/9710089].

[107] J. Nishimura, *Four-dimensional $n = 1$ supersymmetric yang-mills theory on the lattice without fine-tuning*, *Phys. Lett.* **B406** (1997) 215–218 [hep-lat/9701013].

[108] D. B. Kaplan and M. Schmaltz, *Supersymmetric Yang-Mills theories from domain wall fermions*, *Chin. J. Phys.* **38** (2000) 543–550 [hep-lat/0002030].

[109] G. T. Fleming, J. B. Kogut and P. M. Vranas, *Super Yang-Mills on the lattice with domain wall fermions*, *Phys. Rev.* **D64** (2001) 034510 [hep-lat/0008009].

[110] G. T. Fleming, *Lattice supersymmetry with domain wall fermions*, *Int. J. Mod. Phys.* **A16S1C** (2001) 1207–1209 [hep-lat/0012016].

[111] J. Giedt, R. Brower, S. Catterall, G. T. Fleming and P. Vranas, *Gluinos condensing at the CCNI: 4096 CPUs weigh in*, 0807.2032.

[112] S. Catterall, *On the restoration of supersymmetry in twisted two-dimensional lattice Yang-Mills theory*, *JHEP* **04** (2007) 015 [hep-lat/0612008].

[113] N. Marcus, *The Other topological twisting of N=4 Yang-Mills*, *Nucl. Phys.* **B452** (1995) 331–345 [hep-th/9506002].

[114] M. Falcioni, E. Marinari, M. L. Paciello, G. Parisi and B. Taglienti, *Complex Zeros in the Partition Function of the Four-dimensional SU(2) Lattice Gauge Model*, *Phys. Lett.* **B108** (1982) 331–332.

[115] A. M. Ferrenberg and R. H. Swendsen, *New Monte Carlo Technique for Studying Phase Transitions*, *Phys. Rev. Lett.* **61** (1988) 2635–2638.

[116] B. Baumann, *Noncanonical Path and Surface Simulation*, *Nucl. Phys.* **B285** (1987) 391.

[117] K. Kajantie, M. Laine, K. Rummukainen and M. E. Shaposhnikov, *The Electroweak Phase Transition: A Non-Perturbative Analysis*, *Nucl. Phys.* **B466** (1996) 189–258 [hep-lat/9510020].

[118] S. Weinberg, *The quantum theory of fields. Vol. 3: Supersymmetry*, . Cambridge, UK: Univ. Pr. (2000) 419 p.

[119] A. V. Belitsky, S. Vandoren and P. van Nieuwenhuizen, *Instantons, euclidean supersymmetry and wick rotations*, *Phys. Lett.* **B477** (2000) 335–340 [hep-th/0001010].

[120] P. van Nieuwenhuizen and A. Waldron, *On euclidean spinors and wick rotations*, *Phys. Lett.* **B389** (1996) 29–36 [hep-th/9608174].

[121] K. Fujikawa, M. Ishibashi and H. Suzuki, *Ginsparg-wilson operators and a no-go theorem*, *Phys. Lett.* **B538** (2002) 197–201 [hep-lat/0202017].

[122] Y. Kikukawa and H. Suzuki, *A local formulation of lattice wess-zumino model with exact u(1)r symmetry*, *JHEP* **02** (2005) 012 [hep-lat/0412042].

[123] G. D. Moore and K. Rummukainen, *Electroweak bubble nucleation, nonperturbatively*, *Phys. Rev.* **D63** (2001) 045002 [hep-ph/0009132].

[124] K. Rummukainen, M. Tsypin, K. Kajantie, M. Laine and M. E. Shaposhnikov, *The universality class of the electroweak theory*, *Nucl. Phys.* **B532** (1998) 283–314 [hep-lat/9805013].