Ramanujan Graphs

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ABSTRACT

This thesis reviews some of the major results in the study of expander graphs. In particular this thesis will provide proofs of the Cheeger inequality and of the Alon-Boppana lower bound, the later leading naturally to study of Ramanujan graphs. The relationship between expander graphs and covering spaces will be explored, leading to a generalized notion of Ramanujan graphs. Connections between the matching polynomial and characteristic polynomial of a graph will be demonstrated and these connections will be applied in our presentation of a recent result of Marcus, Spielman and Srivastava which shows there exists Ramanujan families of all degrees.

Several well known constructions of expander graphs will be described throughout this exposition, including a variant of the first explicitly constructed family of expander graphs introduced by Margulis in 1975. Some time will also be spent describing the first construction of families of Ramanujan graphs given by Lubotzky, Phillips, Sarnak in 1988. Throughout this review the reader will be exposed to some beautiful connections between expander graphs and other areas of mathematics including number theory, group theory, graph theory and basic linear algebra. This exposition hopes to serve as an accessible and interesting introduction to the known theory of expander graphs.

ABRÉGÉ

Cette thèse examine certains résultats principaux dans l'étude des graphes expanseurs. En particulier, on présente les preuves de l'inégalité de Cheeger et du théorème d'Alon-Boppana, ce dernier nous amène naturellement à l'étude des graphes de Ramanujan. Nous allons expliquer les relations entre les graphes expanseurs et leurs revêtements et définir une version généralisée de la notion de graphes de Ramanujan. On montrera en détail comment le polynôme caractéristique et le polynôme de couplage d'un graphe sont reliés. On profite de ces liens pour présenter un résultat récent de Marcus, Spielman et Srivastava qui affirme l'existence de familles de graphes de Ramanujan de tous degrés.

Plusieurs constructions bien connues de familles de graphes expanseurs sont explicitées dans cette thèse, y compris une variante de la première construction introduite par Margulis en 1975. Nous décrivons aussi la première construction de familles de graphes de Ramanujan introduite en 1988 par Lubotzky, Philips et Sarnak. Tout au long de ce travail, nous montrons comment l'étude des graphes expanseurs combine magnifiquement de nombreux domaines mathématiques, y compris la combinatoire, la théorie des représentations, la théorie des groupes et la théorie des nombres. Cette thèse se veut être une introduction accessible à l'étude des graphes expanseurs.

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Chapter 0
Introduction

This thesis explores the topic of expander graphs, with a focus on Ramanujan graphs. Expander graphs are a very rich and beautiful object of study, one which touches and combines many different fields of mathematics. In this thesis connections will be made between expander graphs and group theory, representation theory, combinatorics, number theory and linear algebra. Although this thesis views expanders as a worthwhile object of study on their own, it should be mentioned that the applications of expander graphs are widespread. Applications to error correcting codes and probabilistic algorithms will be introduced in chapter 1, but expander graphs can be applied to numerous other topics of study including communication networks, cryptography, sorting networks, and problems in pure mathematics. For a more thorough discussion of the applications of expander graphs the reader is referred to [13] and [20] and references therein. The remainder of this introduction provides a breakdown of the contents of this thesis.

The first chapter begins with a review of some basic graph theoretic definitions. Expander graphs will then be introduced from a spectral perspective in terms of the eigenvalues of the adjacency operator. In this thesis expander graphs will be viewed primarily from this approach, however an equivalent combinatorial definition will also be presented in Chapter 1. The famous Cheeger inequality will be proven, verifying the equivalence of the spectral and combinatorial definitions. Random walks on expander graphs will also be introduced and, as previously stated, some applications of expanders will be touched on in order to motivate further study of the subject. Chapter 1 concludes by reviewing basic representation theoretic definitions, introducing the notion of property (T), and using these concepts to describe a variation of the first explicitly constructed family of expanders given by Margulis in 1975 [27].

In Chapter 2 the following question will be addressed: "What is the largest spectral gap that a family of expander graphs can satisfy'?' This question is answered with a proof of the Alon-Boppana theorem which provides an asymptotic bound on the separation between the two largest eigenvalues. The first explicit construction of an expander family to satisfy this optimal bound was published in 1988 by Lubotzky, Phillips and Sarnak [24]. Their construction provided asymptotically optimal families of $k+1$-regular graphs for all prime numbers k. Their proof uses some number theoretic notions and results, in particular it relies on the Ramanujan conjecture. The use of the Ramanujan conjecture explains why Lubotzky et al. call these graphs “Ramanujan”. In 1992 Morgenstern provided similar constructions of families of $k+1$-regular Ramanujan graphs for all prime powers k [28]. The question of whether or not there exists families of Ramanujan graphs for all degrees greater than 2 was posed in Lubotzky’s book on Expander graphs [23]. A partial answer to this conjecture will be reviewed in the final chapter of this thesis but Chapter 2 will focus on presenting the construction of Lubotzky et al. Enough background information is provided for the reader to follow the arguments and observe the richness of the constructions without being slowed down by all of the details.

The focus of Chapter 3 will be on the relationship between expander graphs and covering spaces. The concept of a covering space will be reviewed so no prior training in this subject is required. The study of a graph’s universal covering space will lead to Greenberg’s generalized definition of Ramanujan graphs (see [25] and references therein). This definition extends the previous definition to include irregular graphs. This chapter will finish with a review of Lubotzky and Nagnibeda’s paper which shows that not all irregular uniform trees cover a family of Ramanujan graphs [25].

Chapters 4 and 5 will be related to a recent result of Marcus, Spielman and Srivastava on Ramanujan graph constructions. Before their work is detailed in Chapter 5, time is spent in Chapter 4 exploring the theory these authors use on matching polynomials. This theory is beautiful in its own right, and it is for this reason that an entire chapter is devoted to this topic.

The final chapter in this thesis conveniently ties together many of the previously developed results in order to present the recent breakthrough by Marcus, Spielman and Srivastava. Their result serves as a partial answer to the previously mentioned conjecture on the existence of regular Ramanujan families of all degrees $\geqslant 3$. The authors show that at least in the case of bipartite graphs, such families do exist [29]. They provide a method for constructing these graphs, and their proof builds on the theory developed in Chapters 3 and 4. Some theory on interlacing polynomials will also be introduced and applied for a proof of their result.

Chapter 1
Introduction to Expander Graphs

This section will cover some basic theory of expander graphs. We will begin by reviewing some basic graph theoretic definitions. The adjacency operator will then be introduced and some of its spectral properties will be explored. Expander graphs will then be defined from a spectral perspective and the spectra of some well known graphs will be presented. A definition of expander graphs from a combinatorial perspective will also be introduced, and the equivalence of the two definitions will be proven. Finally some time will be spent introducing a variation of the first family of expanders by applying property (T).

All graphs in this chapter are assumed to be simple unless otherwise stated. This chapter follows some of the material from lectures 1 and 2 in Tao's blog post [35] and [36].

1.1 Review of relevant terminology

An (undirected) *graph* is a pair $G = (V, E)$ where V is a set called the *vertex set* and E is a collection of (unordered) pairs $\{v, w\}$ of elements of V called the *edge set*. A *path* refers to a sequence of consecutive edges in G. A *closed* path is one that starts and ends at the same vertex. The graph $G = (V, E)$ is called *connected* if there is a path between every pair of vertices in V. G is a *finite graph* if $|V|$ is finite. A graph $G = (V, E)$ is *simple* if its edges are undirected, if every edge consists of two distinct vertices, and if no pair of vertices is repeated in the edge set.

The notation $v \sim w$ will be used to signify that $\{v, w\} \in E$. The notations $V(G)$ and $E(G)$ will sometimes be used instead of V and E respectively to stress their dependence on G.

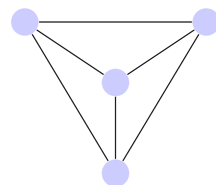
Definition 1.1.1. A *cycle* in a simple graph is a closed path that has no repeated edges. A simple graph is called a *tree* if it contains no cycles and a *forest* is the disjoint union of one or more trees.

Definition 1.1.2. A graph is *bipartite* if its vertex set can be partitioned into two sets (U, W) so that every edge connects a vertex in U to a vertex in W. We write $G = (U, W, E)$.

Definition 1.1.3. For a graph $G = (V, E)$ the *degree of a vertex* $v \in V$ is the number of edges containing v. If each vertex has degree k, where $k \in \mathbb{N}$ then our graph is *k-regular*. In this case we call k the *degree of the graph* G.

Definition 1.1.4. A graph is *complete* if E consists of all $\binom{n}{2}$ unordered pairs. A complete graph on n vertices is $(n-1)$-regular.

Example 1.1.5. *The complete graph on* 4 *vertices:*



Definition 1.1.6. For a graph $G = (V, E)$ with V countable, define $\ell^2(V)$ to be the collection of square summable complex functions:

$$\ell^2(V) = \{f : V \to \mathbb{C} : \sum_{v \in V} |f(v)|^2 < \infty\}$$

This space forms a Hilbert space with inner product, $\langle f, g \rangle = \sum_{x \in V} f(x)\overline{g(x)}$.

1.2 Expanders and Expander Families

Expander graphs will now be defined from a spectral perspective in terms of their eigenvalues. Later they will be introduced from a more combinatorial perspective.

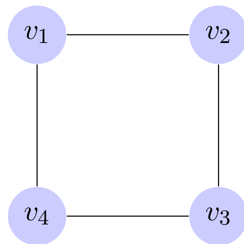
Definition 1.2.1. Given a finite graph $G = (V, E)$, the *adjacency operator* $A : \ell^2(V) \to \ell^2(V)$ on the space of functions $f : V \to \mathbb{C}$ is defined:

$$A(f(v)) = \sum_{w \in V:\ \{v,w\} \in E} f(w) \tag{1.1}$$

Let $v_1,\ v_2,\ \ldots\ v_n$ be an enumeration of V. With respect to this enumeration A can be expressed by an $n \times n$ matrix called the *adjacency matrix* of G, whose entries are:

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j \\ 0 & \text{otherwise} \end{cases}$$

Example 1.2.2. *The following is an example of a finite graph and its associated adjacency matrix.*



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Since the graphs under consideration are undirected graphs, A is a real symmetric matrix. The spectral theorem thus implies that A has n real eigenvalues, $\lambda_0 \geqslant \lambda_1 \geqslant \ldots \geqslant \lambda_{n-1}$, called the *spectrum* of G. Some basic results about the spectrum of a graph G will now be explored.

Lemma 1.2.3. *Let G be a finite graph with average degree k. Then $\lambda_0 \geqslant k$ with equality iff G is k-regular, and if G is k-regular graph then $\lambda_{n-1} \geqslant -k$.*

Proof. Let $|V(G)| = n$ and label the vertices v_i for $i = 1, \ldots, n$. Let k_i denote the degree of vertex v_i. Let $1 \in \ell^2(V)$ denote the constant function sending every element of $V(G)$ to 1. Applying the Rayleigh Quotient[1] yields:

$$\lambda_0 \geqslant \frac{\langle 1, A1 \rangle}{||1||} = \frac{\sum_{i=1}^{n} k_i}{n} = k$$

where the last equality follows from the assumption that the average degree of a vertex in G is k. Observe that 1 is an eigenvector of A exactly when the graph G is k regular. Indeed,

$$A(1(v_i)) = \sum_{j:v_i \sim v_j} 1(v_j) = k_i = k_i 1(v_i) \tag{1.2}$$

This computation shows that in order for k_i to be an eigenvalue of A, k_i must be constant for all vertices v_i. Hence k is only an eigenvalue of G when G is k-regular. Since the Rayleigh Quotient $R(A; x)$ equals λ_0 only when x is the eigenvector associated with λ_0 and since 1 is not an eigenvector of G when G is irregular, it follows that the inequality in (1.2) is strict

[1] Given a finite Hermitian matrix A and a nonzero vector x on a complex inner product space, the Rayleigh Quotient is defined: $R(A, x) := \frac{\langle Ax, x \rangle}{||x||^2} = \frac{x^T A x}{x^T x}$. Since the eigenvectors of A form an orthogonal basis of the inner product space it is easy to see that the Rayleigh Quotient is bounded above and below by the largest and smallest eigenvalues of A respectively.

for this case. It remains to show that $|\lambda_i| \leqslant k$ for all $i = 0, \ldots, n-1$. Let $f,\ g \in \ell^2(V)$ be two normalized functions. Then,

$$\begin{aligned}
|\langle Af, g\rangle| &= \left| \sum_{i=1}^{n} \left(\sum_{j:v_i\sim v_j} f(v_j) \right) \overline{g(v_i)} \right| \\
&= \left| \sum_{i,j:v_i\sim v_j} f(v_j)\overline{g(v_i)} \right| \\
&\leqslant \frac{1}{2} \sum_{i,j:v_i\sim v_j} |f(v_j)|^2 + |g(v_i)|^2 \\
&= \frac{1}{2} \sum_{i=1}^{n} \left(\sum_{j:v_i\sim v_j} |f(v_j)|^2 + |g(v_i)|^2 \right) \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j:v_i\sim v_j} |f(v_j)|^2 + \frac{1}{2} k \sum_{i=1}^{n} |g(v_i)|^2 \\
&= \frac{1}{2} k \sum_{j} |f(v_j)|^2 + \frac{1}{2} k \sum_{i} |g(v_i)|^2 = k
\end{aligned}$$

This means that if x is an eigenvector of A with norm 1 , and λ is its corresponding eigenvalue, then $k \geqslant |\langle Ax, x\rangle| = \lambda\langle x, x\rangle| = |\lambda| \cdot ||x||^2$ and the desired result is achieved. $\square$

Proposition 1.2.4. *Let $k \geqslant 1$ and $G = (V, E)$ be a finite k-regular graph. Then, a) $\lambda_1 = k$ if and only if G is not connected, and*

b) $\lambda_{n-1} = -k$ if and only if G contains a non-empty bipartite graph as a connected component.

Proof. a) "$\Longrightarrow$": Assume $\lambda_1 = k$. Let x be the eigenvector corresponding to λ_1. Since λ_1 is real it can be assumed without loss of generality that x is real. Define $w = x + c1$ where 1 is the constant vector and c is some constant ensuring that all entries of w are greater than 0. Let w_i denote the ith entry of the vector w. Without loss of generality assume w_1 is the largest entry of w (otherwise re-index the vertices). Then $\sum_i a_{1i}w_i = kw_1$. It follows that $w_i = w_1$ whenever $v_1 \sim v_i$. Let $r = |\{i : w_i = w_1\}|$. Then $r \geqslant k$ (since G is k-regular) and $r \neq n$ since x and 1 are orthogonal. Re-index the vertices of our graph so that the first r entries of w_i are equal to w_1. Then $w_{r+1} < w_1$. The adjacency matrix corresponding to our re-indexed vertices is of the form:

$$\left(\begin{array}{c|c} & 0 \\ \hline 0 & \end{array}\right)$$

It follows that our graph is disconnected.

"$\Longleftarrow$": Assume G is not connected. Let $m \geqslant 2$ be the number of connected components of G and let $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_m$ be the corresponding number of vertices in each connected component. Re-index the vertices of our graph so that v_1 to v_{r_1} are in the first connected component, v_{r_1+1} to $v_{r_1+1+r_2}$ are in the second component, etc. The adjacency matrix corresponding to this labelling of vertices is a block diagonal with m blocks. Each block corresponds to a connected component of the graph. Since each connected component of the graph is itself a k-regular graph each block will have an eigenvalue equal to k. Recall that the set of eigenvalues of a block diagonal matrix equals the union of the eigenvalues of its blocks. Since $m \geqslant 2$ it follows that $\lambda_1 = k$.

b) "$\implies$": Assume $\lambda_{n-1} = -k$. Then $Ax = -kx$ for some real eigenvector x. Assume that G is connected (otherwise consider only the component of G whose adjacency matrix contains the eigenvalue $-k$). Without loss of generality we let x_1 be the largest component of x_i in absolute value (otherwise re-index the vertices of G). Assume that x_1 is positive, otherwise multiply x by -1. Then:

$$0 = \sum_{j=1}^{n} a_{1j}x_j + kx_1 = \sum_{j:v_1 \sim v_j} x_j + kx_1 = \sum_{j:v_1 \sim v_j} (x_j + x_1) \tag{1.3}$$

By assumption $-x_j \leqslant x_1$ for all j. Equation (1.3) thus implies $x_j = -x_1$ for $j : v_1 \sim v_j$. Repeating the same argument for each vertex adjacent to v_1, and then for each vertex adjacent to a vertex adjacent to v_1, and so on, it can be deduced that $x_i = \pm x_1$ for all i. It follows that G contains no cycles of odd length. Indeed, a cycle of odd length starting at v_i would imply that x_i is simultaneously $\pm x_1$. This is a contradiction since $x_1 > 0$. It follows that G is bipartite.

"$\Longleftarrow$" (proof from [1]): The following stronger result will be proven: If $G = (U, V, E)$ is a bipartite graph then its eigenvalues are symmetric about zero. Let U have k vertices and V have $n-k$ vertices. Label the vertices of G so that $\{v_1, v_2, \dots v_k\} \in U$ and $\{v_{k+1}, \dots, v_n\} \in V$. The graph's corresponding adjacency matrix, A, is of the form:

$$\begin{array}{c} \text{U} \rightarrow \\ \text{V} \rightarrow \end{array} \begin{pmatrix} 0 & M \\ M^T & 0 \end{pmatrix}$$

Suppose that λ is an eigenvalue of A with corresponding eigenvector

$$x = (x_1, \dots, x_k, x_{k+1}, \dots x_n)^T.$$

Then $-\lambda$ is an eigenvalue of A with associated eigenvector

$$y = (x_1, \dots, x_k, -x_{k+1}, \dots, -x_n)^T.$$

Indeed, for $1 \leqslant i \leqslant k$ we have:

$$\sum_{j=1}^{n} a_{ij} y_j = \sum_{j=k+1}^{n} a_{ij} y_j = -\sum_{j=k+1}^{n} a_{ij} x_j = -\lambda x_i = -\lambda y_i$$

and for $k+1 \leqslant i \leqslant n$ we have:

$$\sum_{j=1}^{n} a_{ij} y_j = \sum_{j=1}^{k} a_{ij} y_j = \sum_{j=1}^{k} a_{ij} x_i = \lambda x_i = -\lambda y_i$$

This verifies that the eigenvalues of G are symmetric about 0. It follows that if G is a k-regular graph containing a bipartite connected component then $-k$ is an eigenvalue of G. □

Definition 1.2.5. For a k-regular graph G call eigenvalues $\lambda_0 = k$, and $\lambda_{n-1} = -k$ if G is bipartite, *trivial* eigenvalues of G.

Definition 1.2.6. Let $G = (V, E)$ be a k-regular graph on n vertices with spectrum $\lambda_0 \geqslant \lambda_1 \geqslant \dots \geqslant \lambda_{n-1}$. G is called an $(n, k, \epsilon)-$*expander* if $\lambda_1 \leqslant (1-\epsilon)k$ for some $\epsilon > 0$.

Proposition 1.2.4 implies that every finite k-regular graph is an (n, k, ϵ)-expander for some ϵ. For this reason it is important to keep track of ϵ and it is often more meaningful to consider sequences of expander graphs.

Definition 1.2.7. A sequence of finite k-regular graphs $\{G_i\}_{i \in \mathbb{N}}$ whose vertex set tends to infinity is called an *expander family* if there exists an $\epsilon > 0$ satisfying $\lambda_1(G_i) \leqslant (1-\epsilon)k$ for all i.

An equivalent definition of expander graphs can be formulated in terms of the eigenvalues of the *graph Laplacian*: $\Delta := kI - A$. Notice that if λ is an eigenvalue of A then $k - \lambda$ is an eigenvalue of the Laplacian with the same corresponding eigenvector. The second largest eigenvalue of the adjacency operator thus corresponds to the smallest non-zero eigenvalue of the Laplacian. This value, $k - \lambda_1$, is called the *spectral gap*. The Laplacian will be discussed in more detail in section 1.3

Example 1.2.8. *For $n \geqslant 3$ let C_n denote the 2-regular graph whose vertex set is the cyclic group $\mathbb{Z}/n\mathbb{Z}$ and whose edge set consists of pairs $\{x, x+1\}$ for all $x \in \mathbb{Z}/n\mathbb{Z}$. These graphs are referred to as* cycle graphs. *Note that each C_n is individually an expander since it is connected. It is shown here, however, that the collection does not form an expander family. Claim: The spectrum of C_n is given by: $2\cos(2\pi j/n)$ for j = 0 .... $n-1$.*

Proof. Label the vertices beginning with v_0 so that $v_r \sim v_{r+1}$ and $v_r \sim v_{r-1} \pmod{n}$ for all $r = 0, \ldots, n-1$. Then λ is an eigenvalue of A corresponding to eigenvector x iff each coordinate x_r of x satisfies $x_{r-1} + x_{r+1} = \lambda x_r$. Let $x^{(j)}$ be the eigenvector corresponding to λ_j. Consider $x_r^{(j)} = \cos(\frac{2\pi r j}{n})$. Then applying the trigonometric identity $\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$ with $x = \frac{2\pi(r-1)j}{n}$ and $y = \frac{2\pi(r+1)j}{n}$ yields:

$$\begin{aligned}
x_{r-1}^{(j)} + x_{r+1}^{(j)} &= \cos\left(\frac{2\pi(r-1)j}{n}\right) + \cos\left(\frac{2\pi(r+1)j}{n}\right) \\
&= 2\cos\left(\frac{\pi(r-1)j}{n} + \frac{\pi(r+1)j}{n}\right)\cos\left(\frac{\pi(r-1)j}{n} - \frac{\pi(r+1)j}{n}\right) \\
&= 2\cos\left(\frac{2\pi r j}{n}\right)\cos\left(\frac{-2\pi j}{n}\right) \\
&= 2\cos\left(\frac{2\pi j}{n}\right) x_r^{(j)}
\end{aligned}$$

□

For $j = 1, \ldots, n-1$, observe that $2\cos\left(\frac{2\pi j}{n}\right) = 2\cos\left(\frac{2\pi(n-j)}{n}\right) \leqslant 2\cos\left(\frac{2\pi}{n}\right)$ and hence $\lambda_1 = 2\cos\left(\frac{2\pi}{n}\right)$. Since λ_1 increases to 2 as $n \to \infty$ there is no ϵ which makes K_n an ϵ-expander for all n. It follows that despite being connected and of constant degree, these graphs do not form an expander family.

Example 1.2.9. *Let K_n denote the complete graph on n vertices with spectrum $\lambda_0 \geqslant \lambda_1 \geqslant \ldots \geqslant \lambda_{n-1}$. Claim: $\lambda_2(K_n) = \ldots = \lambda_n(K_n) = -1$*

Proof. Let A_n denote the adjacency matrix of K_n which has entries:

$$a_{ij} = \begin{cases} 0 & \text{if } i = j \\ \\ 1 & \text{otherwise} \end{cases}$$

Let B denote the $n \times n$ matrix filled with all ones. Then $\lambda_2(B_n) = \ldots = \lambda_n(B_n) = 0$ and $\lambda_1 = n$. If λ is an eigenvalue of B_n with corresponding eigenvector x then $A_n x = (B_n - I_n)x = B_n x - x = \lambda x - x = (\lambda - 1)x$ and hence $\lambda_2(A_n) = \ldots = \lambda_n(A_n) = -1$ and $\lambda_1(A_n) = n - 1$. □

This claim implies that complete graphs are good expanders. Indeed, K_n is an $\left(1 + \frac{1}{n-1}\right)$-expander. They do not, however, form a family of expanders since the degree of K_n increases with n.

1.3 The Discrete Cheeger Constant

This section will begin by revisiting the graph Laplacian mentioned in section 1.2. The Laplacian will be defined in an alternate way more analogous to the classical Laplacian on manifolds. This chapter will then introduce expander graphs from a combinatorial perspective in terms of the Cheeger constant. Finally, the equivalence of the spectral definition and combinatorial definition of expanders will be shown using the Laplacian.

In order to provide an alternate definition of the discrete Laplacian, some machinery will first be built. Begin by choosing an arbitrary orientation on the edges of a graph. (It will be seen later that our choice of orientation has no effect on our definition).

Definition 1.3.1. For $f \in \ell^2(V)$ define the *simplicial co-boundary operator* for graphs $d : \ell^2(V) \to \ell^2(E)$ by

$$df(e) = f(e^+) + f(e^-)$$

where e^- is the origin of the oriented edge e and e^+ is the extremity.

Notice that the Laplacian can be defined in terms of the boundary operators as $\Delta = d^* d$, where d^* is the adjoint of d. [2] Indeed, following Davidoff et al. in [5], let $\delta : V \times E \to$

[2] Recall that the adjoint of an operator d on a linear space V is defined to be the operator d^* satisfying $\langle du, v\rangle = \langle u, d^* v\rangle$ for all u, v in V.

$\{-1, 0, 1\}$ be a function defined:

$$\delta(x,e) = \begin{cases} 1 & \text{if } x = e^+, \\ -1 & \text{if } x = e^-, \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

Then for $f \in \ell^2(V)$, $df(e) = \sum_{x \in V} \delta(x,e) f(x)$ and for $g(x) \in \ell^2(E)$, $d^*g(x) = \sum_{e \in E} \delta(x,e) g(e)$. So,

$$d^*d(f(e)) = \sum_{e \in E} \delta(x,e) \left(\sum_{y \in V} \delta(y,e) f(y) \right) = kf(x) - \sum_{y \sim x} f(y)\,, \tag{1.5}$$

where k is the degree of each vertex. From this calculation it is also clear that the Laplacian does not depend on the chosen orientation.

Definition 1.3.2. For a graph $G = (V, E)$ with $F \subseteq V$, the *boundary of* F, denoted ∂F, is the set of edges connecting a vertex in F to a vertex in V/F.

Definition 1.3.3. For a finite k-regular graph $G = (V, E)$, let $h(G)$ denote the *discrete Cheeger constant* [3] defined by:

$$h(G) = \min \left\{ \frac{|\partial F|}{|F|} : F \subseteq V, 0 < |F| \leqslant \frac{n}{2} \right\}$$

The discrete Cheeger constant can be thought of as the discrete analogue of the Cheeger constant of a compact Riemannian manifold. Since this thesis restricts its attention to graphs, the word “discrete" will sometimes be dropped and $h(G)$ will simply be called the “Cheeger constant ".

Notice that $h(G)$ is positive exactly when G is connected. Definition 1.2.7 of an expander family is thus equivalent to the following:

A sequence of finite k-regular graphs $\{G_i\}_{i \in \mathbb{N}}$ whose vertex set tends to infinity for which there exists an $\epsilon > 0$ satisfying $h(G_i) \geqslant \epsilon$ for all i.

This combinatorial perspective of expander graphs gives a little intuition behind the connectivity properties of expander graphs, and the name “expander". Observe that if F

[3] Some authors call the Cheeger constant the isoparemetric constant or the edge expansion ratio.

is the subset of vertices that realizes $h(G)$ then any other subset $H \subset V$ of size $|F|$ is connected to at least $|F|h(G)$ vertices outside of H. It could be said that sets of size $|F|$ “expand" by a factor of $h(G)$.

Example 1.3.4. *One can show using the Cheeger constant that the cycle graphs presented in example 1.2.8 do not form a family of expanders. Indeed, for $C_n = (V_n, E_n)$ define $F_n \subset V_n$ by $F_n = \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Then $|\partial F_n| = 2$ and $|F_n| = \lfloor \frac{n}{2} \rfloor$ and $h(C_n)$ tends to zero as n goes to infinity.*

The following proposition presents a relationship between the spectrum of a graph and the graph's discrete Cheeger constant, and in turn it verifies that the two definitions of expander families are in fact equivalent.

Proposition 1.3.5. *For a finite k-regular graph $G = (V, E)$ let $k - \lambda_1$ denote the smallest non-trivial eigenvalue of the Laplacian $\Delta(G)$. Then:*

$$\frac{k - \lambda_1}{2} \leqslant h(G) \leqslant \sqrt{2k(k - \lambda_1)}$$

Proof. **The lower bound:** (adapted from section 4.5 of [13]) Recall for a finite $\Delta(G)$ is 0 with corresponding eigenfunction the constant function. Let $S \subset V(G)$ be the subset which realizes $h(G)$, and let $\bar{S}$ denote the compliment of S. Consider the function f defined by $f = |\bar{S}|1_S - |S|1_{\bar{S}}$. Since this function is orthogonal to the constant function, the following inequality involving the Rayleigh Quotient holds:

$$k - \lambda_1 \leqslant \frac{\langle \Delta f, f \rangle}{\|f\|^2}$$

The denominator of this quotient is evaluated as follows:

$$\begin{aligned} \|f\|^2 = \sum_{x \in V} f(x)^2 = \sum_{x \in S} |\bar{S}|^2 + \sum_{x \in \bar{S}} |S|^2 &= |S||\bar{S}|^2 + |\bar{S}||S|^2 \\ &= |S||\bar{S}|(|\bar{S}| + |S|) = n|S||\bar{S}| \end{aligned}$$

and the numerator evaluates to:

$$\langle \Delta f, f \rangle = \langle (kI - A)f, f \rangle = k\langle f, f \rangle - \langle Af, f \rangle = kn|S||\bar{S}| - \left(\sum_{x_i \sim x_j} f(x_i) f(x_j) \right)$$

For two subsets $F, G \subseteq V$ let $E(F, G)$ denote the set of edges connecting a vertex of F to a vertex of G. Using this notation the previous line can be expressed as:

$$\begin{aligned}
kn|S||\bar{S}| &- \Big[2|E(\bar{S}, \bar{S})||S|^2 + 2|E(S, S)||\bar{S}|^2 - 2|\partial S||S||\bar{S}|\Big] \\
&= kn|S||\bar{S}| - \Big[((k|\bar{S}| - |\partial S|)|S|^2 + (k|S| - |\partial S|)|\bar{S}|^2 - 2|\partial S||S||\bar{S}|\Big] \\
&= kn|S||\bar{S}| - \Big[nk|S||\bar{S}| - |\partial S|(|S|^2 + |\bar{S}|^2 + 2|S||\bar{S}|)\Big] \\
&= n^2|\partial S|
\end{aligned}$$

which shows $\langle \Delta f, f \rangle = n^2|\partial(S)|$. Substituting these calculations back into the Rayleigh Quotient yields:

$$k - \lambda_1 \leqslant \frac{n|\partial S|}{|S||\bar{S}|} = \frac{n}{|\bar{S}|} h(G) \leqslant 2h(G)$$

where the last inequality follows from the observation that $|\bar{S}| \geqslant \frac{n}{2}$.

The upper bound: (adapated from [5]): Fix an arbitrary orientation on the edges of G. Let g be an eigenfunction of Δ associated with the eigenvalue $k - \lambda_1$. Define $f(x) = \max\{g(x), 0\}$ and set:

$$B_f = \sum_{e \in E} |f(e^+)^2 - f(e^-)^2|$$

In order to establish the desired results, the following bounds on B_f will be verified:

1. $B_f \leqslant \sqrt{2k(k - \lambda_1)}||f||_2^2$
2. $B_f \geqslant h(G)||f||_2$

Inequality 1. is established first. Let $\beta_r > \beta_{r-1} > \ldots > \beta_1 > \beta_0$ denote the values of f. Set $L_i = \{x \in V : f(x) \geqslant \beta_i\}$. Let $i(e)$ and $j(e)$ be such that for an edge $e \in E$, f takes on values $\beta_{i(e)}$ and $\beta_j(e)$ at the endpoints of e. Index in such a way that $\beta_{i(e)} > \beta_{j(e)}$.

With this notation B_f can be expressed as:

$$\begin{aligned}
B_f &= \sum_{e\in E}(\beta^2_{i(e)} - \beta^2_{j(e)}) \\
&= \sum_{e\in E}(\beta^2_{i(e)} - \beta^2_{i(e)-1} + \beta^2_{i(e)-1} - \beta^2_{i(e)-2} + \ldots + \beta^2_{j(e)+1} - \beta^2_{j(e)}) \\
&= \sum_{e\in E}((\beta^2_{i(e)} - \beta^2_{i(e)-1}) + (\beta^2_{i(e)-1} - \beta^2_{i(e)-2}) + \ldots + (\beta^2_{j(e)+1} - \beta^2_{j(e)})) \\
&= \sum_{e\in E}\sum_{\ell=j(e)+1}^{i(e)}(\beta^2_\ell - \beta^2_{\ell-1})
\end{aligned}$$

Notice that every time an edge connects a vertex x with $f(x) = \beta_{i(e)}$ to a vertex y with $f(y) = \beta_{j(e)} < \beta_{i(e)}$ then $(\beta^2_\ell - \beta^2_{\ell-1})$ will appear in the summation for all ℓ satisfying $j(e)+1 \leqslant \ell \leqslant i(e)$. In other words, $(\beta^2_\ell - \beta^2_{\ell-1})$ will appear in the summation whenever $e \in \partial L_\ell$. Applying this observation B_f is re-written as:

$$\begin{aligned}
B_f &= \sum_{i=1}^{r}|\partial L_i|(\beta^2_{i(e)} - \beta^2_{i(e)-1}) \\
&\geqslant h(X)\sum_{i=1}^{r}|L_i|(\beta^2_i - \beta^2_{i-1}) = h(X)\left[|L_r|\beta^2_r + \sum_{i=1}^{r-1}|L_i/L_{i+1}|\beta^2_i\right] = h(X)||f||^2_2
\end{aligned}$$

where the first inequality follows from the definition of the Cheeger constant and the last equality follows from the observation that $L_i/L_{i(e)-1}$ is exactly the set of vertices $x \in V$ satisfying $f(x) = \beta_{i(e)}$.

Inequality 2. is now established. The Cauchy-Swartz Inequality and the inequality $2(a^2+b^2) \geqslant (a+b)^2$ are applied to obtain:

$$\begin{aligned}
B_f &= \sum_{e\in E}|f(e^+) + f(e^-)| \cdot |f(e^+) - f(e^-)| \\
&\leqslant \left(\sum_{e\in E}(f(e^+) + f(e^-))^2\right)^{1/2}\left(\sum_{e\in E}(f(e^+) - f(e^-))^2\right)^{1/2} \\
&\leqslant \sqrt{2}\left(\sum_{e\in E}f(e^+)^2 + f(e^-)^2\right)^{1/2}||df||_2 \\
&= \sqrt{2}k\left(\sum_{x\in V}f(x)^2\right)^{1/2}||df||_2 \\
&= \sqrt{2k}||f||_2||df||_2
\end{aligned} \tag{1.6}$$

The term $||df||_2$ will now be estimated in terms of $k-\lambda_1$. Set $V^+ = \{x \in V : g(x) > 0\}$. For $x \in V^+$ one obtains:

$$\begin{aligned}(\Delta f)(x) &= kf(x) - \sum_{y\in V} A_{xy} f(y) \\ &= kg(x) - \sum_{y\in V^+} A_{xy} g(y) \\ &\leqslant kg(x) - \sum_{y\in V} A_{xy} g(y) = (k-\lambda_1)g(x)\end{aligned}$$

Applying this estimate yields:

$$\begin{aligned}||df||_2 = \sqrt{< \Delta f, f >} &= \sqrt{\sum_{x\in V^+} (\Delta f)(x) g(x)} \\ &\leqslant \sqrt{\sum_{x\in V^+} (k-\lambda_1) g(x)^2} \leqslant \sqrt{(k-\lambda_1)}||f||_2 \qquad (1.7)\end{aligned}$$

Substituting (1.7) into (1.6) establishes the second inequality.

Both inequalities have now been established, and they can be combined to yield:

$$h(X)||f||_2^2 \leqslant \sqrt{2k(k-\lambda_1)}||f||_2^2$$

Cancelling the $||f||_2^2$ term on each side yields the desired lower bound. □

1.4 Random Walks on Expanders

This section discusses the behaviour of random walks on expander graphs. Random walks on graphs are characterized in part by their spectrum, and so it is no surprise that random walks are used in the study of expander graphs. There is much that can be said about topic, but only an introduction is presented here. The reader is referred to [13] and references therein for more information.

Definition 1.4.1. A *walk* on a graph $G = (V, E)$ is a sequence of finite or infinite vertices $v_1, v_2, \ldots \in V$ where v_{i+1} is adjacent to v_i for all i. Such a sequence is called a *random walk* if each v_{i+1} is selected uniformly at random from the neighbours of v_i, independently for each i. The initial vertex, v_1, is chosen based on a given initial probability distribution, π_1.

A random walk on a graph $G = (V, E)$ will induce a sequence of probability distributions on V, $\pi_2, \pi_3, \ldots$ where $\pi_i(t) = \mathrm{Prob}(v_i = t)$ for all $t \in V$. If G is a finite, regular, connected graph that is not bipartite[4] then the sequence of distributions is known to converge to the uniform distribution on V (see [26] for a proof). What is special about expander graphs is that they converge very *quickly* to the uniform distribution.

Definition 1.4.2. For a k-regular graph G with adjacency matrix A, the *normalized adjacency matrix* of G is defined by $\hat{A} = \frac{1}{k}A$. Observe that if $\hat{\lambda}_0 \geqslant \ldots \geqslant \hat{\lambda}_{n-1}$ are the eigenvalues of $\hat{A}$ then $\hat{\lambda}_0 = 1$ and if G is an (n, k, ϵ)-expander then all non-trivial eigenvalues of G are bounded above by ϵ.

Let $G = (V, E)$ be a finite graph with n vertices. A probability distribution π over V can be expressed as a column vector in $\mathbb{R}^V$ with the added property that all coordinates be non-negative and that $\sum_{v \in V} \pi(v) = 1$. For example, the uniform distribution, μ, can be expressed by $\mu = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)^T$. Consider the probability distributions of the vertices chosen in a random walk on G. Notice that if the initial vertex is chosen with respect to distribution vector π_1 then the probability distribution of the second vertex will be given by $\hat{A}\pi_1$. More generally if π_1 is the initial distribution vector for a random walk $v_1, v_2, \ldots$ then for all $i \geqslant 1$, $\hat{A}^i\pi_1$ will represent the probability distribution of the $(i+1)$-th vertex.

Proposition 1.4.3. *Let G be an (n, k, ϵ)-expander graph. Let μ denote the uniform distribution of V. Then for any initial probability distribution vector π on the vertex set V, and for every positive integer i, we have:*

$$\|\hat{A}^i\pi - \mu\|_2 \leqslant \epsilon^i$$

Proof. It suffices to verify the result for the case where $i = 1$, namely that $\|\hat{A}\pi - \mu\|_2 \leqslant \epsilon$. From there, induction can naturally be applied to extend the result to the case where $i > 1$. To show $\|\hat{A}\pi - \mu\|_2 \leqslant \epsilon$, observe that the uniform distribution μ is invariant under the

[4] The non-bipartite condition can be removed if one considers a *lazy random walk* on G. This is similar to the random walk except that it allows for v_{i+1} to equal v_i.

action of $\hat{A}$ and that $\pi - \mu$ is orthogonal to u. These two observations yield:

$$\|\hat{A}\pi - \mu\|_2 = \|\hat{A}(\pi - \mu)\|_2 \leqslant \epsilon\|\pi - \mu\|_2.$$

Since π is a probability distribution $\epsilon\|\pi - \mu\|_2 \leqslant \epsilon$ and we are done. □

This result shows that in some sense a random walk of length t on an expander graph resembles an sample of t independently chosen vertices from the uniform distribution. Although it has only been shown that the sequence of distributions converges to the uniform distribution exponentially in the ℓ^2-norm, similar results are known to hold with other norms as well. The reader is referred to [13] and references therein for more details on this subject.

Again, notice that this theory gives some insight into the choice of the term "expander" to describe expander graphs. Indeed, let C be a complete graph with a loop at every vertex. Then in a random walk on C every vertex in the walk (except possibly the first) is chosen uniformly at random. Since random walks on expander graphs can be used to approximate a random sample of independently chosen vertices from the uniform distribution, one could say that from the point of view of random walks, expander graphs "expand" to resemble complete graphs.

1.5 Applications of Expander Graphs

Before studying explicit constructions of expander families, some time is taken now to motivate this study. This section will give the reader a very brief introduction to two applications of expander graphs: efficient error reduction in probabilistic algorithms and error correcting codes in communication channels. The notes of Hoory, Linial and Wigderson in [13] are given as the primary resource for this section, along with [16] for the section on probabilistic algorithms.

1.5.1 efficient error reduction in probabilistic algorithms

Probabilistic algorithms, also called randomized algorithms, incorporate a degree of randomness into their behaviour. In addition to the input data, probabilistic algorithms require additonal input in the form of random bits to help drive the random choices they make.

The output of a probabilistic algorithm can vary even over runs with fixed data input. Probabilistic algorithms are useful because they will perform well on average, over all choices of random input.

One of the earlier problems for which probabilistic algorithms were employed was the problem of primality testing: Given a d-bit integer, x, determine whether or not x is prime. Given the number x and a string, r, of d random bits, a probabilistic algorithm computes a boolean function, $f(x,r)$, which behaves as follows: If x is prime then $f(x,r)$ will return the value of 1 for all choices of r and if x is composite then $f(x,r)$ will return the value of 1 with probability $\frac{1}{2}$. This means that if x is not prime then the algorithm fails with probability $\frac{1}{2}$.

Suppose it was necessary to reduce this failure probability below a given threshold. This could easily be accomplished by repeating the experiment a sufficient number of times with r chosen independently at random each time. If the experiment was repeated m times, the probability of failure would be effectively reduced to $\left(\frac{1}{2}\right)^m$. The downside of this method is that every time the process is repeated, another d random bits are required. Is it possible to reduce the probability of failure in a more efficient manner? This section will discuss how expander graphs can play a role in answering this question. Before doing so these types of problems will be defined in a more general framework.

The algorithms mentioned in this primality testing problem are specific examples of the more general class RP of Randomized Polynomial-Time algorithms. Define $\{0,1\}^*$ to be the set of all finite binary strings. A language $L \subseteq \{0,1\}^*$ is in the class RP if there exists a polynomial time probabilistic algorithm which can determine membership in L. When given a binary string, x, of length d and a randomly selected $r \in \{0,1\}^d$ a probabilistic algorithm will compute a boolean function, $A(x,r)$, to indicate whether or not $x \in L$. If $x \in L$ then $A(x,r) = 1$ for all r and otherwise $A(x,r) = 1$ with probability p. As in the primality problem, it is desired to reduce the probability of failure below a certain threshold, ϵ, in the most efficient way possible.

As previously mentioned, expander graphs can be used to solve this problem. Choose an explicit $(2^d, k, \alpha)$-graph G with $V(G) = \{0,1\}^d$. For a fixed k-bit string $x \notin L$ let B_x denote the set of vertices in G for which we will obtain an erroneous result upon input x.

In other words, B_x consists of all the vertices $v \in V$ for which $A(x, v) = 1$. Observe that in order to decrease the algorithm's probability of failure, the probability of sampling a vertex in B_x must increase. The aim is to do so while utilizing the least amount of random bits as possible. Instead of choosing each vertex independently at random every time the experiment is repeated, consider a random walk on the vertices G. This method requires an initial vertex $v_0 \in V$ to be chosen uniformly at random. From this vertex, the algorithm specifies a random walk on G, $v_1, \ldots, v_m$, and returns $A(x, v_i)$ for all $i = 0, \ldots m$. The initial choice of vertex v_0 required d bits, and each step in the walk requires $\log_2(k)$ random bits in order to decide between the k neighbouring vertices. Thus, this algorithm requires a total of $d + (m - 1)\log_2(k)$ random bits. Provided α is sufficiently smaller than ϵ, this algorithm is known to reduce the probability of failure exponentially. In comparison, the naive method of repeating the experiment m times independently reduces the probability of failure exponentially, but requires dm random bits.

The result mentioned above can be made sense of intuitively by referring back to the discussion in the previous section about random walks on expander graphs. It was shown that the probability distributions induced by a random walk on an expander graph converge rapidly to the uniform distribution. So, despite not having been independently chosen, it makes sense that a collection of vertices obtained through random walk would approximate an independent random sample of vertices, and this method clearly requires less random bits. Of course, many details are missing from this discussion, but the aim here was simply to provide an introduction to this topic as motivation for the study of expander graphs. The reader is referred to sections 1 and 3 in [13] and references therein for a more thorough investigation of this theory.

1.5.2 error correcting codes in communication channels

Another useful application of expanders is in the area of error correcting in communication. One of the main challenges in communication is disturbance which distorts messages as they pass through a communication channel. This means that the message that is sent differs from the message that is received. To account for this distortion the sender can send the message along with additional redundant information in hopes that the receiver can detect errors and determine the message that was originally sent.

This process is abstracted mathematically as follows: Adhering to conventions, the sender in our model is called Alice, and the receiver is called Bob. Suppose Alice wants to send an m-bit string $a \in \{0,1\}^m$ to Bob. A bijective map is specified, $\phi : M \subseteq \{0,1\}^m \to C \subset \{0,1\}^n$ where $n > m$. Here, C is called the *code* and the elements of C are called *codewords*. The code is linear if the set C is a linear subspace of $\{0,1\}^n$. Instead of sending Bob the raw m-bit message a, Alice can send Bob the n-bit string $\phi(a)$. Bob would then receive an element $b \in \{0,1\}^n$ which may not equal $\phi(a)$. To try to decode the message Bob receives he could measure the hamming distance, d_H, between b and all of the codewords. The hamming distance between two elements in C is defined to be the number of digits which differ between them. Once Bob finds the codeword c that minimizes the hamming distance to b, Bob will compute $\phi^{-1}(c)$ to retrieve the original message that Alice sent. This process is referred to as error correcting, and the code C can be referred to as an *error correcting code*.

Definition 1.5.1. The *rate*, R, of a code, measures the code's efficiency and is defined:

$$R = \frac{\log_2 |C|}{n}$$

The *normalized distance*, D, measures how much error a code can effectively manage, and is defined:

$$D = \frac{\min_{c_1 \neq c_2 \in C} d_H(c_1, c_2)}{n}$$

Good codes are ones which have a maximum distance with respect to their rate. Although there exist many constructions of error correcting codes which utilize expander graphs, this exposition will describe only one of them. Following Hoory, et al. in [13] and Sipster and Spielman's in [32], a construction of error correcting codes from unbalanced bipartite expander graphs will be presented.

Definition 1.5.2. A bipartite graph $G = (V, W, E)$ is a (c, d, ϵ, δ)-expander if every non-empty subset $S \subset |V|$ with $|S| < \epsilon|V|$ satisfies $\frac{|\partial(S)|}{|S|} > \delta$.

Observe that this definition resembles the combinatorial perspective of expander graphs in which expanders are viewed as graphs whose Cheeger constant is bounded away from zero. The difference here is that expansion is only considered on one side of the graph. The construction we present makes use of a specific type of unbalanced expander referred to as “magical graphs" by Hoory et al. in [13].

Definition 1.5.3. A bipartite graph $G = (L, R, E)$ is a (n, m, d)-*magical graph* if $|L| = m$ and $|R| = m$, every vertex in L has degree d, and the following properties hold:

- $|\partial(S)| \geqslant \frac{5d}{8}|S|$ for every $S \subseteq L$ with $|S| \leqslant \frac{n}{10d}$
- $|\partial(S)| \geqslant |S|$ for every $S \subseteq L$ with $\frac{n}{10d} < |S| \leqslant \frac{n}{2}$

Magical graphs will now be used to construct error correcting codes with $R \geqslant \frac{n}{4}$ and $D > \frac{n}{10d}$. Let $G = (R, L, E)$ be a $(n, \frac{3n}{4}, d)$- magical graph. Notice that the adjacency matrix of G is of the form:

$$\left(\begin{array}{c|c} 0 & A \\ \hline A^T & 0 \end{array}\right)$$

Let A denote the non-zero block whose ij-th entry is 1 if the i-th vertex of R is adjacent to the j-th vertex in L and 0 otherwise. Define C as the kernel of A with respect to arithmetic in $\mathbb{F}_2$:

$$C = \{x \in \{0, 1\}^n : Ax = 0\}$$

It is clear that $rank(A) \leqslant \frac{3n}{4}$ and hence the dimension of C is $\geqslant \frac{n}{4}$. Since the coordinates of vectors in C are all either 0 or 1, it follows that $|C| \geqslant 2^{\frac{n}{4}}$, and thus $R \geqslant \frac{n}{4}$.

The following claim will be needed in order to show that $D > \frac{1}{10d}$:

Claim 1.5.4. *Let G be the the magical graph G defined above. For every nonempty subset $S \subset L$ with $|S| \leqslant \frac{n}{10d}$, there exists a vertex $u \in R$ with exactly one neighbour in S.*

Proof. Fix a subset $S \subset L$ with $|S| \leqslant \frac{n}{10d}$. Let $e(S, \partial(S))$ denote the number of edges between S and $\partial(S)$. Since G is bipartite and since the vertices in L are d-regular, it follows that $e(S, \partial(S)) = d|S|$. Since G is a magical graph one has by hypothesis that $|\partial(S)| \geqslant \frac{5d}{8}|S|$. Dividing $e(S, \partial(S))$ by $\frac{5d}{8}|S|$ yeilds an upper bound of $\frac{8}{5}$ on the average number of neighbours that a vertex in $\partial(S)$ has in S. Since $\frac{8}{5} < 2$ and since every vertex in $\partial(S)$ has at least one neighbour in S, there must be at vertex in $\partial(S)$ with exactly one neighbour in S. ∎

The bound $D > \frac{1}{10d}$ will now be verified. Proceed by searching for the minimum distance between any pair of codewords. Since C is linear it suffices to find the minimal distance between any codeword and the 0 vector. Assume by way of contradiction that there exists a codeword $x \in C$ with less than or equal to $\frac{n}{10d}$ non-zero coordinates. Consider the subset S defined, $S = \{j \in L : x_j = 1\}$. Since $|S| \leqslant \frac{n}{10d}$ claim 1.5.4 guarantees the existence of a vertex $i \in R$ for which $|\partial(i) \cap S| = 1$. This implies that the ijth entry of A is 1. Since the jth entry of x is 1 by definition, it follows that the i coordinate of Ax is 1 contradicting the assumption that $x \in ker(A)$. It follows that the error correcting code generated by the bipartite expander graph G has rate $\geqslant \frac{n}{4}$ and has normalized distance $> \frac{1}{10d}$, as claimed.

There are other ways to construct error correcting code with explicitly defined expander graphs. Some other constructions utilize regular (not necessarily bipartite) expander graphs. The reader is referred to Speilman's [33] for more information on such constructions.

1.6 Kazhdan Property (T)

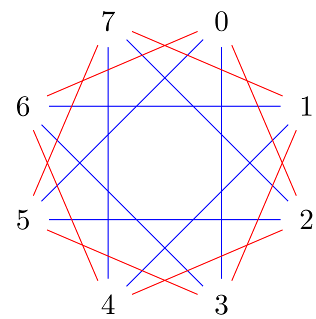
In the previous section two applications of expander graphs were presented. Both applications required explicit graph constructions. Thus far this exposition has only demonstrated collections of graphs which do *not* form expander families. This chapter presents an example of one that does. The first explicit family of expanders was discovered by Margulis in 1975. Margulis's constructions involve the Cayley graphs of groups satisfying a property known as *Kazhdan's Property (T)*. Following Lubotzky in [23], this section will introduce a variation of Margulis's construction, presented by Alon and Milman in [2]. We begin by familiarizing the reader with the necessary terminology before demonstrating how Kazhdan's Property (T) relates to graph expansion and introducing Alon and Milman's construction. This section follows Lubotzky in [23] and Tao in [36].

Definition 1.6.1. Let G be a group and let S be a symmetric subset of G that does not contain the identity. (Here, the term "symmetric" means that $s \in S \implies s^{-1} \in S$.) The *Cayley Graph* of G with respect to S, denoted $Cay(G, S)$, is constructed as follows: The vertices of $Cay(G, S)$ are the elements of G and two vertices $x, y \in G$ are adjacent iff $x = ys$ for some $s \in S$ (since S is symmetric it is equivalent to say $x \sim y$ iff $y = xs$ for some $s \in S$). Thus $Cay(G, S)$ is a $|S|$-regular graph on $|G|$ vertices, and it is connected iff S is a generating set of G.

Definition 1.6.2. A graph $G = (V, E)$ is *vertex transitive* if for any pair of vertices $x, y \in V$ there is an automorphism $f : V \to V$ satisfying $f(x) = y$.

Observe that all Cayley graphs are vertex transitive. Indeed, any group G acts transitively on itself (or in this case its Cayley graph) via right multiplication. This fact will be of use in subsequent chapters.

Example 1.6.3. $Cay(\mathbb{Z}/8\mathbb{Z}, \{2, 3, 5, 6\})$*: The Cayley graph of* $\mathbb{Z}/8\mathbb{Z}$ *generated by* $\{2, 3, 5, 6\}$*. Here, the group operation is addition. Red edges represent addition by* 2 *and* 6 *while blue edges represent addition by* 3 *and* 5.



Before defining Kazhdan's Property (T) some basic concepts from representation theory and group theory are required:

Definition 1.6.4. A *topological group* is a group equipped with a topology such that the group's multiplication and inverse functions are continuous. Any group can be equipped with the discrete topology, as continuity is trivially satisfied under this topology.

Although much of this theory can be defined in a more general context, in order to avoid any unnecessary technical subtleties, the contents of this chapter will restrict its focus on discrete groups which are countable.

Definition 1.6.5. Recall that a *unitary operator* on a Hilbert space is a linear operator that preserves the inner product. For a Hilbert space H, let $U(H)$ denote the space of unitary operators on H. A *unitary representation* of a countable discrete group, G, is a pair (H, ρ) where ρ is a homomorphism, $\rho : G \to U(H)$.

Example 1.6.6. *The* right regular representation *of a countable discrete group G consists of the linear space of all square summable complex functions $f : G \to \mathbb{C}$ and the homomorphism r defined by $(r(g)(f))(x) = f(xg)$.*

Definition 1.6.7. An *invariant vector* of a representation (H, ρ) is a nonzero vector $v \in H$ satisfying $\rho(g)(v) = v$ for all $g \in G$. A subspace $W \subset H$ is called an *invariant subspace* if $\rho(g)(w) \in W$ for all $w \in W$.

Definition 1.6.8. If $\rho : G \to U(H)$ is a unitary representation of a countable discrete group G and W is a closed invariant subspace of V then a *subrepresentation* of ρ can be obtained by restricting: $\rho|_W : G \to U(W)$ defined: $\rho|_W(g)w := \rho(g)w$ for all $g \in G$ and $w \in W$.

Definition 1.6.9. An *irreducible representation* is one which has no non-trivial invariant subspaces.

Property (T) is now ready to be defined. There are several equivalent ways in which one can define this property. Although it was originally defined in terms of the Fell Topology, this exposition follows Tao in [36] and uses the following definition.

Definition 1.6.10. Let $\rho : G \to U(H)$ be a unitary representation of a countable discrete group G and let S be a finite subset of G. The *Kazhdan constant*, $Kaz(G, S, \rho)$, associated with S and ρ is the supremum of all $\epsilon \geqslant 0$ which satisfies the following inequality for all non-zero $v \in H$:

$$\sup_{s \in S} ||\rho(s)v - v|| \geqslant \epsilon||v||$$

Define,

$$Kaz(G, S) = \inf_{\rho} Kaz(G, S, \rho)$$

where ρ ranges over all unitary representations of G with no nontrivial invariant vectors. A group G is said to have *Kazhdan property (T)* (or simply property (T)) if there exists some finite set S for which $Kaz(G, S) > 0$.

In other words, if a countable discrete group has property (T) then there exists a non empty finite set S and some $\epsilon > 0$ so that for every unitary representation with no non-trivial invariant vectors $\rho : G \to U(H)$ and for every $v \in H$ there is an $s \in S$ satisfying $||\rho(s)v - v|| \geqslant \epsilon||v||$.

Remark. The above definition does not require S to be a generating set of G. It is worth mentioning, though, that any countable discrete group with property (T) is finitely generated (see Remark 3 in [36]). This result and the following proposition imply that in order to determine whether a given group G has property (T), it suffices to first check if it is finitely generated, and if so, to then check $Kaz(G, S)$ for a finite generating set S.

Proposition 1.6.11. *Let G be a finitely generated countable discrete group with finite generating set S. Then G has property (T) iff* $Kaz(G, S) > 0$.

Proof. The inverse direction, in which $Kaz(G, S) > 0$ is assumed, follows immediately from definition 1.6.10. To verify the forward direction, assume by way of contradiction that G has property T, but that $Kaz(G, S) = 0$. This means that there exists a unitary representation of G without nontrivial invariant vectors, call it (ρ, H) for which there exists a nonzero vector $v \in H$ satisfying $||\rho(s)v - v|| \leqslant \hat{\epsilon}||v||$ for all $s \in S$ and any $\hat{\epsilon} > 0$. But by property (T) we know that there exists a finite set C and an $\epsilon > 0$ so that $||\rho(c)v - v|| \leqslant \epsilon||v||$ for some $c \in C$. Since S is a generating set we have:

$$\begin{aligned}||\rho(c)v - v|| &= ||\rho(s_1...s_i)v - v|| \\ &= ||\rho(s_1)...\rho(s_i)v - v|| \\ &= ||\rho(s_1)...\rho(s_{i-1})(\rho(s_i)v - v) + \rho(s_1)...\rho(s_{i-2})(\rho(s_{i-1})v - v) \\ &\quad + \ldots + \rho(s_1)(\rho(s_2)v - v) + (\rho(s_1)v - v)||\end{aligned}$$

Applying the triangle inequality yields:

$$\begin{aligned}||\rho(c)v - v|| \leqslant& ||\rho(s_1)...\rho(s_{i-1})(\rho(s_i)v - v)|| + ||\rho(s_1)...\rho(s_{i-2})(\rho(s_{i-1})v - v)|| \\ &+ \ldots + ||\rho(s_1)(\rho(s_2)v - v)|| + ||(\rho(s_1)v - v)|| \\ \leqslant& i\hat{\epsilon}||v||\end{aligned}$$

Since this holds for any $\hat{\epsilon} > 0$ setting $\hat{\epsilon} = \frac{\epsilon}{i}$ yields a contradiction. □

Theorem 1.6.12. *(Proposition 3.3.1 in [23]) Let G be a finitely generated countable discrete group and let S be a finite symmetric generating set of G. Let $\{N_i\}_{i \in \mathbb{N}}$ be a sequence of finite index normal subgroups of G and let $Q_i : G \to G/N_i$ denote the associated quotient maps. Assume that $|Q_i(S_i)| = |S|$ for all i. If G has property (T) then the family of Cayley graphs, $Cay(G/N_i, Q_i(S))$ forms a family of expanders.*

Proof. ([23]) Fix an $i \in \mathbb{N}$ and let $H = \ell^2(G/N_i)$ be the Hilbert space of complex functions on the finite group G/N_i. Consider the unitary representation $r : G \to U(H)$ given by the action $(r(g)f)(x) = f(xg)$ of G on H. Since the right multiplication action of G on G/N_i is transitive (indeed if $N_i x$ and $N_i y$ are two distinct elements of G/N_i then $x^{-1}y \in G$ takes $N_i x$ to $N_i y$) the only invariant vectors under this representation are the constant functions.

Let H_1 denote the space of constant functions on G/N_i. Then H_1 is an invariant subspace of H whose orthogonal compliment, $H_0 = \{f \in H | \sum_{x \in (G/N_i)} f(x) = 0\}$, is an invariant subspace containing no nontrivial invariant vectors. The subrepresentation $r_{H_0} : G \to U(H_0)$ is a unitary representation with no nontrivial invariant vectors. Property (T) and proposition 1.6.11 are applied to deduce that there exists an $\epsilon > 0$ (dependent only on G and S) such that for every $f \in H_0$ there is an $s \in S$ satisfying $||r_{H_0}(s)f - f|| \geqslant \epsilon ||f||$. Choose a particular edge counting function $f \in H_0$. Fix a subset $A \subset G/N_i$ of cardinality $|A| := a \leqslant \frac{1}{2}|G/N_i|$ and then define f as follows:

$$f(x) = \begin{cases} n - a & \text{if } x \in A \\ -a & \text{if } x \notin A \end{cases}$$

Then

$$||f||^2 = \sum_{x \in A} |n-a|^2 + \sum_{x \in V/A} |-a|^2 = a(n-a)^2 + (n-a)a^2 = an(n-a)$$

and

$$||r_{H_0}(s)f - f||^2 = \sum_{x \in V} |f(xs) - f(x)|^2 = \sum_{x \in A, xs \in V/A} |-n|^2 + \sum_{x \in V/A, xs \in A} |n|^2 = 2n^2 |\partial_s(A)|$$

where $\partial_s(A) \subseteq \partial(A)$ denotes the edges which result from the right multiplication action of s on a vertex. Observe:

$$|\partial(A)| \geqslant |\partial_s(A)| = \frac{||r_{H_0}(s)f - f||^2}{2n^2} \geqslant \frac{\epsilon^2 ||f||^2}{2n^2} = \frac{\epsilon^2 a(n-a)}{2n}$$

Since A was arbitrary, it follows that for any set A with $|A| \leqslant \frac{1}{2}$ we have:

$$\frac{\partial(A)|}{|A|} \geqslant \frac{\epsilon^2 (n - |A|)}{2|A|} \geqslant \frac{\epsilon^2}{4}$$

which yields the following inequality on the Cheeger constant: $h(Cay(G/N_i, S)) \geqslant \frac{\epsilon^2}{4}$ for any $N \in \{N_i\}_{i \in \mathbb{N}}$. □

A relationship between Property (T) and expansion has now been established. The following theorem of Kazhdan is required before presenting a well known example of expanders [17].

Theorem 1.6.13. *$SL_d(Z)$ has property (T) for all $d \neq 3$.*

Example 1.6.14. *This example follows Lubotzky in [23] and presents Alon and Milman's construction in [2]. It is well known that $SL_n(\mathbb{Z})$ can be generated by the set $\{A_n,\ B_n\}$ where A_n and B_n are defined as follows:*

$$A_n = \left(\begin{array}{c|ccc} 0 & & & \\ 0 & & & \\ 0 & & I_{n-1} & \\ \vdots & & & \\ 0 & & & \\ \hline (-1)^{n-1} & 0 & \dots & 0 \end{array}\right) \quad B_n = \left(\begin{array}{cc|ccc} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & I_{n-2} & \\ 0 & 0 & & & \end{array}\right)$$

Let S_n denote the set consisting of A_n, B_n and their inverses. Since the necessary assumptions are satisfied, theorems 1.6.13 and 1.6.12 can be applied to deduce that for a fixed $n \geqslant 3$ the collection of graphs given by $Cay(SL_n(\mathbb{Z}/p\mathbb{Z}), S_n)$ with p running over the set of primes, forms a family of expanders.

Chapter 2
Ramanujan Graphs

This chapter explores the idea of being the “best" possible family of expander graphs. The chapter will begin by proving a result of Alon-Bopanna, which gives an asymptotic upper bound on the spectral gap. The first construction of a family of graphs to satisfy this optimal bound will then be introduced. This construction was given by Lubotzky, Phillips and Sarnak in [24] and it is in this paper that the term “Ramanujan" was first used to describe these graphs.

2.1 Alon-Boppanna Lower Bound

Definition 2.1.1. Let $G = (V, E)$ be a finite connected graph. The *length* of a path in G refers to the number of edges transversed. For any two vertices $u, v \in V$ the *distance* between u and v refers to the length of the shortest path between them. The *diameter* of G is the maximum distance between any pair of vertices in V.

Theorem 2.1.2. *(Alon-Boppana Lower Bound)*
Let G be a finite k-regular graph. Define $\lambda(G)$ to be the absolute value of the largest (in absolute value) non-trivial eigenvalue of G.[1] *Then the following inequality holds:*

$$\lambda(G) \geqslant 2\sqrt{k-1}\left(1 - O\left(\frac{logD}{D}\right)\right)$$ [2]

where D denotes the diameter of G.

The following corollary can easily be obtained by applying the well known fact that the diameter of a k-regular graph on n vertices is $\Omega(log_{k-1}(n))$ [3] (see [13] and references therein):

[1] recall from definition 1.2.5 that the trivial eigenvalues are $\lambda_0 = k$, and $\lambda_{n-1} = -k$ if G is bipartite

[2] See Appendix A for a review of Big Oh notation

[3] Here Ω is used to denote Omega notation. See Appendix A for a quick review of this topic

Corollary 2.1.3. *If $\{G_i\}_{i\in\mathbb{N}}$ is a family of k-regular finite graphs with $|V(G_i)| \to \infty$ as $i \to \infty$, then*

$$\liminf_{i\to\infty} \lambda(G_i) \geqslant 2\sqrt{k-1}$$

Proof of Theorem 2.1.2. (adapted from proof 5.2.2 in [13]) Notice that $\lambda(G) \geqslant |\lambda_1(G)|$ and hence it suffices to show that

$$|\lambda_1(G)| \geqslant 2\sqrt{k-1}\left(1 - O\left(\frac{logD}{D}\right)\right)$$

holds. Let A denote the adjacency matrix of $G \in \{G_i\}$. Note that $\lambda_1(A)$ is just another way to write $\lambda_1(G)$ and this theorem will proceed to use the notation $\lambda_1(A)$. The equality $\lambda_1(A^{2p}) = (\lambda_1(A))^{2p}$ holds for any integer p. A lower bound on $\lambda_1(A^{2p})$ will be given by estimating the Rayleigh Quotient of the following function: Let s and t be vertices of G at a distance D apart and define:

$$f(i) = \begin{cases} 1 & \text{if } i = s \\ -1 & \text{if } i = t \\ 0 & \text{for all other } i \in V(G) \end{cases}$$

Since f is orthogonal to the constant function the Rayleigh Quotient is applied to yield:

$$\begin{aligned} \lambda_1(A)^{2p} \geqslant \frac{fA^{2p}f^t}{||f||^2} &= \frac{(A^{2p})_{ss} + (A^{2p})_{tt} - 2(A^{2p})_{st}}{2} \\ &= \frac{(A^{2p})_{ss} + (A^{2p})_{tt}}{2} \end{aligned} \tag{2.1}$$

where the second equality is obtained by setting $p = \lfloor \frac{D-1}{2} \rfloor$. Let T_k denote the k-regular tree graph. Define the *tree number*, a_{2p}, to be the number of closed paths of length $2p$ in T_k starting and ending at v. The positive terms in the numerator of (2.1) count the number of closed paths of length $2p$ starting and ending at vertex s and t respectively. Each of these terms will be greater than or equal to a_{2p}, and hence $\lambda_1(A)^{2p} \geqslant a_{2p}$ is obtained. A suitable bound for a_{2p} will now be found.

Every closed path in a tree has a corresponding string consisting of characters $\{-1, 1\}$. A step is assigned the character 1 if it results in being further away from the initial vertex, and -1 if it results in being closer to the initial vertex. Every string that can be obtained in this way will satisfy the following conditions:

i - the sum of the characters in the string is 0

ii - the sum of the first i characters is non negative for all $i \leqslant 2p$.

It is well known that the number of strings of length $2p$ satisfying these conditions is equal to the p-th Catalan number: $C_p = \binom{2p}{p}/(p+1)$. Each of these C_p strings corresponds to at least $(k-1)^p$ distinct walks in T_k. So the following is obtained:

$$\lambda_1(A)^{2p} \geqslant a_{2p} \geqslant C_p(k-1)^p$$

Using Sterling's approximation, $n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$, it is straightforward to show that $C_p = \Theta(\frac{4^p}{p^{3/2}})$ and hence $\lambda_1(A)^{2p} \geqslant a_{2p} \geqslant C_p(k-1)^p = \Theta((2\sqrt{k-1})^{2p}p^{-3/2})$. [4] Taking the $2p$-th root yields:

$$|\lambda_1(A)| \geqslant 2\sqrt{k-1}\left(O(p^{-\frac{3}{4p}})\right) = 2\sqrt{k-1}\left(O\left(e^{-3log(p)/4p}\right)\right)$$

Applying the Taylor expansion of e^{-x} [5] and substituting $p = \lfloor\frac{\Delta-1}{2}\rfloor$ yields:

$$|\lambda_1(A)| \geqslant 2\sqrt{k-1}(1 - O(log(\Delta)/\Delta))$$

$\square$

Definition 2.1.4. Let G be a finite k-regular graph and let $\lambda(G)$ denote the absolute value of the largest non-trivial eigenvalue (in absolute value) of G. Following Lubotzky et al. in [24], the graph G will be called *Ramanujan* if $\lambda(G) \leqslant 2\sqrt{k-1}$.

Remark. In definition (1.2.5) we define $-k$ to be a trivial eigenvalue for bipartite graphs. The above definition of a Ramanujan graph thus allows for the possibility of bipartite Ramanujan graphs. In the literature it is sometimes unclear as to wether or not the definition

[4] Here Θ is used to denote big theta notation. See Appendix A for a quick review of this topic.

[5] recall $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$ and so $e^{-3log(k)/4k} = 1 - \frac{3logk}{4k} + \left(\frac{3logk}{4k}\right)^2\frac{1}{2!} - \left(\frac{3logk}{4k}\right)^3\frac{1}{3!} + \ldots = 1 - O\left(\frac{logk}{k}\right)$

allows for this possibility. Regardless, our definition does, and it is consistent with the definition given by Lubotzky et al. in [24].

The Alon-Boppana Theorem shows that Ramnujan graphs are asymptotically the best possible. Its not difficult to construct a Ramanujan graph with a small number of vertices. For instance, the k-regular complete graphs presented in example 1.2.9 are Ramanujan. The challenge, however, lies in constructing a Ramanujan family of graphs. Recall that the complete graphs which are individually Ramanujan graphs do not form a Ramanujan family since they are graphs of varying degrees. Few explicit constructions of Ramanujan families are known.

2.2 Explicit Construction of Ramanujan Graphs

The first explicit constructions of families of Ramanujan graphs were given by Lubotzky, Phillips and Sarnak [24] and separately by Margulis [27] in 1988. For any prime number k, the authors were able to construct sequences of $k+1$-regular Ramanujan graphs from Cayley graphs. This section will investigate the original constructions of families of $k+1$-regular Ramanujan graphs as introduced by Lubotzky et al. To prove that these graphs form a Ramanujan family is highly non-trivial and only a summary of the proof will be presented here. In order to describe these constructions, some number theoretic results and definitions will be needed. The following subsection presents this required background.

The works of Davidoff, Sarnak and Valette in [5] and [31] as well as the original paper by Lubotzky, Phillips and Sarnak [24] are the primary references for this section.

2.2.1 Preliminary definitions and results

Definition 2.2.1. The ring of *intergral quaternions*, denoted $\mathbb{H}(\mathbb{Z})$, is defined:

$$\begin{aligned}\mathbb{H}(\mathbb{Z}) = &\{a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in \mathbb{Z},\ i^2 = j^2 = k^2 = -1,\\ &ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j\}\end{aligned}$$

The word “integral" will often be omitted, and an element of the ring will simply be referred to as a *quaternion*. Let $\alpha = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}(\mathbb{Z})$ be a quaternion. The *conjugate* of α, denoted $\bar{\alpha}$, is defined, $\bar{\alpha} := a_0 - a_1 i - a_2 j - a_3 k$. The *norm* of α, denoted $N(\alpha)$, is defined, $N(\alpha) := \alpha\bar{\alpha} = a_0^2 + a_1^2 + a_2^2 + a_3^2$. If $N(\alpha) = 1$ then α is called a *unit*. There are

8 units in $\mathbb{H}(\mathbb{Z})$, namely, $\pm 1, \pm i, \pm j, \pm k$. If ϵ is a unit then $\epsilon\alpha$ is referred to as an *associate* of the quaternion α. A quaternion $\alpha \in \mathbb{H}(\mathbb{Z})$ is *prime* if it is not a unit and if $\alpha = \beta\gamma \implies$ either β or γ is a unit.

The following elementary number theoretic results will be of use later:

Proposition 2.2.2. *(see Proposition 2.5.2 in [5] for proof) Every quaternion has a factorization into prime quaternions.*

Proposition 2.2.3. *(see Corollary 2.5.10 in [5] for proof) A quaternion ω is prime if and only if $N(\omega)$ is prime.*

Proposition 2.2.4. *(see Theorem 2.1.7 Corollary 2.5.10 in [5] for proof) If $p \equiv 1 \pmod 4$ then -1 is a square in $\mathbb{F}_p$.*

Theorem 2.2.5. *(Jacobi's Theorem - see [5] section 2.3 for proof)*

Let n be an odd positive integer. Let $r_4(n)$ denote the number of integer solutions to the equation $a_0^2 + a_1^2 + a_2^2 + a_3^2 = n$. Then $r_4(n) = 8\sum_{d|n} d$.

Since the graphs of Lubotzky Phillips Sarnak are constructed from Cayley graphs of linear groups, the reader is now reminded of some important linear groups of degree 2.[6]

Definition 2.2.6. The *general linear group*, $\mathrm{GL}_2(q)$, is the group of 2×2 invertible matrices with coefficients in $\mathbb{F}_q$. The *projective linear group*, $\mathrm{PGL}_2(q)$, is the quotient group of $\mathrm{GL}_2(q)$ modulo the set of scalar matrices, $\left\{ \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) : \lambda \text{ in } \mathbb{F}_q^\times \right\}$. The *projective special linear group*, $\mathrm{PSL}_2(q)$, is the subgroup of $\mathrm{PGL}_2(q)$ consisting of matrices with determinant 1.

2.2.2 Constructing $X^{p,q}$

Let p and q be two distinct primes, each congruent to 1 modulo 4 [7] and with q "large enough" with respect to p (more on this later). This section will construct graphs $X^{p,q}$, dependent on the choice of p and q, from Cayley graphs of either $\mathrm{PGL}_2(q)$ or $\mathrm{PSL}_2(q)$ with respect to a subset $S_{p,q}$. The subset $S_{p,q}$ will now be constructed.

[6] These definitions generalize to degree n, but only degree 2 linear groups are needed here.

[7] The assumption that p and q are congruent to 1 modulo 4 may be omitted, but following Sarnak [31] it is included here to allow for a simplified description.

Theorem 2.2.5 implies that there are $8(p+1)$ quaternions of norm p. Since $p \equiv 1 \pmod 4$ each of these quaternions will have exactly one odd coordinate.[8] Let α be a quaternion of norm p. Out of the 8 associates $\epsilon\alpha$ of α exactly one of them satisfies the condition that a_0 is odd and positive. It can be concluded that there are $p+1$ solutions to the $a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$ whose a_0 coordinate is odd and positive. Let S_p denote this set of $p+1$ distinguished solutions.

Since $q \equiv 1 \pmod 4$, Proposition 2.2.4 ensures that there exists some $i \in \mathbb{F}_q$ such that $i^2 \equiv -1 \pmod q$. Define a function $\varphi : S_p \to \mathrm{PGL}_2(q)$ mapping each $\alpha = a_o + a_1 i + a_2 j + a_3 k \in S_p$ to the 2×2 matrix over $\mathbb{F}_q$ defined by,

$$\tilde{\alpha} = \begin{pmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{pmatrix}$$

The set $S_{p,q}$ is defined to be the image set $\varphi(S_p)$ of matrices in $\mathrm{PGL}_2(q)$.[9] Notice that $S_{p,q}$ is symmetric in $\mathrm{PGL}_2(q)$ - indeed, each matrix in $S_{p,q}$ is its own inverse - and hence it forms a legitimate set from which we can construct a Cayley graph. Applying the assumption that q is sufficiently large with respect to p ($q > 2\sqrt{q}$ is satisfactory) and making use of the fact that the elements of $S_{p,q}$ have norm congruent to p modulo q, it can be verified that $|S_{p,q}| = p+1$ (see Lemma 4.2.1 in [5] for more details). The Cayley graphs constructed will therefore be $p+1$-regular. The graph $X^{p,q}$ is defined as follows:

- If p is a square modulo q then $S_{p,q} \subset \mathrm{PSL}_2(q) \subset \mathrm{PGL}_2(q)$. In this case $X^{p,q} := \Gamma(\mathrm{PSL}_2(q), S_{p,q})$ and $X^{p,q}$ is a $(p+1)$-regular graph on $\frac{q(q^2-1)}{2}$ vertices.
- If p is not a square modulo q then $X^{p,q} := \Gamma(\mathrm{PGL}_2(q), S_{p,q})$. Then $X^{p,q}$ is a $(p+1)$-regular graph on $q(q^2-1)$ vertices.

[8] Indeed, if an integer x is even then $x = 2y$ and $x^2 = 4y^2 = 0 \pmod 4$. If x is odd then $x = 2y+1$ and $x^2 = 4y^2 + 4y + 1 = 1 \pmod 4$. Since the sum of all 4 coordinates squared is $1 \pmod 4$, only one coordonate can be odd.

[9] It is straightforward to see that $S_{p,q}$ is a subset of $\mathrm{PGL}_2(q)$: Indeed, for any $\tilde{\alpha} \in S_{p,q}$, $\det(\tilde{\alpha}) \equiv p \pmod q \not\equiv 0 \pmod q$ since p and q are distinct primes. It follows that $\tilde{\alpha}$ is invertible as claimed.

2.2.3 Main Result

The following theorem is a special case of the results proven by Lubotzky, Phillips, Sarnak and Margulis:

Theorem 2.2.7. *Let p,q be distinct odd primes* $\equiv 1 \pmod 4$ *satisfying* $q > 2\sqrt{p}$. *Then* $X^{p,q}$ *are* $(p+1)$*-regular Ramanujan graphs of cardinality* $\frac{q(q^2-1)}{2}$ *if q is square modulo p, and of cardinality* $q(q^2-1)$ *otherwise.*

Regardless of whether or not p is a square modulo q, it is clear that as q tends to infinity the size of the vertex set does also. Thus, by fixing p and constructing $X^{p,q}$ for increasing prime values of q, a family of $(p+1)$-regular Ramanujan graphs is obtained.

A brief summary of the proof that the graphs $X^{p,q}$ are Ramanujan will now be presented. Chapter 3.4 of [31], Chapter 4.4 of [5] as well as the original paper by Lubotzky, Philips, Sarnak [24] are cited as references for this proof.

Main ideas of the proof of Theorem 2.2.7. This proof will begin by verifying that each $X^{p,q}$ is connected. In their original paper, Lubotzky et al. describe two different constructions of $X^{p,q}$ and verifiy that they are isomorphic. One of these constructions has already been described in this exposition. We will use the alternate construction of $X^{p,q}$ to show that these graphs are connected. The alternate construction is described as follows:

Let $\Lambda'(2) = \{\alpha \in \mathbb{H}(\mathbb{Z}) | \alpha \equiv 1 \pmod 2$ and $N(\alpha) = p^v, v \in \mathbb{Z}$, p prime $\equiv 1 \pmod 4\}$. Notice that $\Lambda'(2)$ contains the previously defined set S_p. Define $\Lambda(2)$ to be the group given by $\Lambda(2) = \Lambda'(2)/\sim$ where $\alpha \sim \beta$ whenever $\pm p^{v_1}\alpha = p^{v_2}\beta$ for $v_1,\ v_2 \in \mathbb{Z}$. Let $Q : \Lambda'(2) \to \Lambda(2)$ be the quotient map taking $\alpha \in \Lambda'(2)$ to its equivalence class $[\alpha] \in \Lambda(2)$. Let $\Lambda(2q)$ denote the finite index subgroup of $\Lambda(2)$ consisting of all elements $\alpha \in \Lambda(2)$ satisfying $2q | \alpha - a_0$. The graphs $Cay(\Lambda(2)/\Lambda(2q), Q(S_p))$ are isomorphic to the previous construction of $X^{p,q}$. Thus, to show that $X^{p,q}$ is connected it will suffice to show that $Q(S_p)$ generates $\Lambda(2)/\Lambda(2q)$. The following two results will be used:

Lemma 2.2.8. (Theorem 2.5.13 in [5]) *Let* $m \in \mathbb{N}$ *and let* $\alpha \in \mathbb{H}(\mathbb{Z})$ *with* $N(\alpha) = p^m$. *Then* α *admits a unique factorization* $\alpha = \epsilon p^r w_{m-2r}$ *where* ϵ *is a unit and* w_{m-2r} *is a reduced word of length* $m - 2r$ *over* S_p.

A *reduced word* is one that has no subword of the form $\alpha_i\bar{\alpha}_i$. The *length* of the word is the number of symbols in its expression. The main ideas of the proof of this lemma will now be provided.

Proof. Begin by proving existence. Let $\alpha \in \mathbb{H}(\mathbb{Z})$ with $N(\alpha) = p^m$. Applying Proposition 2.2.2, write $\alpha = \delta_1 \ldots \delta_k$ where each δ_i is prime. Proposition 2.2.3 can then be applied to deduce that $N(\delta_i) = p$ for all i, and thus $k = m$. For each δ_i write $\delta_i = \epsilon_i\gamma_i$ where $\gamma_i \in S_p$ and ϵ_i is a unit. This yields:

$$\alpha = \epsilon_1\gamma_1 \ldots \epsilon_m\gamma_m$$

All units can be moved to the left by replacing each $\gamma\epsilon$ by $\epsilon'\gamma'$ where $\gamma' \in S_p$. The resulting expression is unit followed by a word (not necessarily reduced) of length m. The word is reduced by replacing any occurrences of $\gamma\bar{\gamma}$ by a factor of p. Once the word is completely reduced the expression for α becomes,

$$\alpha = \epsilon p^r \gamma_i \ldots \gamma_j$$

where $\gamma_i \ldots \gamma_j$ is a word over S_p of length $m - 2r$. Existence has now been verified.

To prove uniqueness a simple counting argument is used. Begin by counting the number of expressions of the form $\epsilon p^r w_{m-2r}$ (for various values of r). Count the number of quaternions of norm p^m. One will find that there are the same number of each, thus proving uniqueness. The reader is referred to Theorem 2.5.13 in [5] for more details. $\square$

This result leads immediately to the following corollary:

Corollary 2.2.9. (Lemma 2.5.4 in [31]) *If $\beta \equiv 1 \pmod 2$ and $N(\beta) = p^m$ then β can be expressed uniquely in the form $\beta = \pm p^r w_{m-2r}$ where w_{m-2r} is a reduced word over S_p.*

Corollary 2.2.9 implies that the group $\Lambda(2)$ is freely generated by $Q(S_p)$. Hence $Q(S_p)$ generates $\Lambda(2)/\Lambda(2q)$ and thus $X^{p,q}$ are connected.

The spectrum of these graphs will now be investigated. Begin by generalizing the notion of the adjacency matrix of a graph. For a k-regular graph $G = (V, E)$ define A_r to be the $|V| \times |V|$ matrix with entries:

$$(A_r)_{xy} = \text{the number of non-backtracking paths of length } r \text{ from } x \text{ to } y$$

Observe that A_0 is the identity and A_1 is the usual adjacency operator. A few basic observations and calculations yield the following recursion formulas:

$$A_1^2 = A_2 + kA_0 \ , \quad A_r A_1 = A_1 A_r = A_{r+1} + (k-1)A_{r-1} \tag{2.2}$$

From these recursion formulas one can compute the generating function of the A_r. The following relationship can then be deduced between the Chebychev polynomials of the second kind, $U_m(x) = \frac{\sin((m+1)\arccos(x))}{\sin(\arccos(x))}$, and the A_r's:

$$\sum_{0 \leqslant r \leqslant \frac{m}{2}} A_{m-2r} = (k-1)^{\frac{m}{2}} U_m \left(\frac{A_1}{2\sqrt{k-1}} \right) \tag{2.3}$$

(See Chapter 1.4 in [5] for details on (2.2) and (2.3))

Formula (2.3) will be used to estimate the spectrum of $X^{p,q}$. Let n be the number of vertices of $X^{p,q}$ and let $p+1 = \lambda_0 > \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_{n-1}$ be the eigenvalues of A_1. (The strict inequality between λ_0 and λ_1 follows from $X^{p,q}$ being connected.) For $j = 1, 2, \ldots n-1$ write $\lambda_j = 2\sqrt{p}\,\theta_j$ where $\theta_j \in \mathbb{C}$.

In order to prove that $X^{p,q}$ is Ramanujan, it is required to prove that θ_j is real for certain values of j. More specifically, if $X^{p,q}$ is bipartite then it will be required to show that θ_j is real for $j = 1, \ldots n-2$. If $X^{p,q}$ is not bipartite, then it will be required to show that θ_j is real for all $j = 1 \ldots n-1$. To this end, proceed by calculating the trace of both the left and right hand sides of (2.3). The right hand side yields:

$$p^{m/2}\,\mathrm{Tr}\left(U_m \left(\frac{A_1}{2\sqrt{p}} \right) \right) = p^{m/2} \sum_{j=0}^{n-1} \frac{\sin(m+1)\theta_j}{\sin \theta_j} \tag{2.4}$$

and the left hand yields:

$$\sum_{0 \leqslant r \leqslant m/2} \mathrm{Tr}\,(A_{m-2r}) = \sum_{x \in X^{p,q}} \sum_{0 \leqslant r \leqslant m/2} (A_{m-2r})_{xx} \tag{2.5}$$

Recall that since $X^{p,q}$ is a Cayley graph, it is vertex transitive and thus the value $(A_{m-2r})_{xx}$ will be constant for all vertices $x \in X^{p,q}$. The expression in (2.5) can thus be written as,

$$n \sum_{0 \leqslant r \leqslant m/2} (A_{m-2r})_{ee}$$

where e is the vertex in $X^{p,q}$ corresponding to the identity element of the group $\Lambda(2)/\Lambda(2q)$. For vertices $x, y \in \Lambda(2)$ let $d(x,y)$ denote the distance between x and y in T_{p+1}. Observe,

$$\sum_{0\leqslant r\leqslant m/2} (A_{m-2r})_{ee} = \sum_{0\leqslant r\leqslant m/2} |\{\alpha \in \Lambda(2q) : d(\alpha, e) = m - 2r\}| \tag{2.6}$$

Indeed, $\alpha \in \Lambda(2q)$ iff the corresponding path in $X^{p,q}$ is closed. Furthermore, the length of a non-backtracking path in $X^{p,q}$ is equivalent to the distance in the tree T_{p+1}.

Let $r_Q(p^m)$ denote the number of integer representations of p^m by $Q(x_0, x_1, x_2, x_3) = x_0^2 + 4q^2(x_1^2 + x_2^2 + x_3^2)$. In other words, $r_Q(p^m) = |\{\alpha \in \mathbb{H}(\mathbb{Z}) : 2q|\alpha - a_0, N(\alpha) = p^m\}|$. Applying Corollary 2.2.9 and the fact that length of a reduced word over S_p corresponds to distance in the tree T_{p+1} yields,

$$r_Q(p^m) = 2 \sum_{0\leqslant r\leqslant m/2} |\{\alpha \in \Lambda(2q) : d(\alpha, e) = m - 2r\}|. \tag{2.7}$$

The factor of 2 can be made sense of by observing that for any reduced word w in $\Lambda(2q)$ of length $m - 2r$ both $\gamma_+ = +p^r w$ and $\gamma_- = -p^r w$ contribute to $r_Q(p^m)$.

Combining 2.3, 2.5, 2.4, 2.6 and 2.7 yields,

$$r_Q(p^m) = \frac{2p^{m/2}}{n} \sum_{j=0}^{n-1} \frac{\sin(m+1)\theta_j}{sin\theta_j}$$

The Ramanujan conjecture and its proofs by Eichler and Igusa (see [17] and references therein) yield the following approximation of $r_Q(p^m)$:

$$r_Q(p^m) = C(p^m) + O_\epsilon(p^{m(1/2+\epsilon)}) \text{ as } m \to \infty \ \forall\, \epsilon > 0 \tag{2.8}$$

where $C(p^m)$ is the p^m-th Fourier Coefficient of an Eisenstein series of weight two. When m is even Lubotzky et al. [24] show that

$$C(p^m) = \frac{4}{q(q^2-1)} \frac{p^{m+1}-1}{p-1}$$

Putting all of these calculations together yields that for even m, as $m \to \infty$:

$$\frac{4}{q(q^2-1)} \frac{p^{m+1}-1}{p-1} + O_\epsilon(p^{m(1/2+\epsilon)}) = \frac{2p^{m/2}}{n} \sum_{j=0}^{n-1} \frac{\sin(m+1)\theta_j}{\sin\theta_j} \quad \forall\, \epsilon > 0 \tag{2.9}$$

The term $\frac{4}{q(q^2-1)}\frac{p^{m+1}-1}{p-1}$ on the left hand side of (2.9) is exactly the contribution of the trivial eigenvalues to the right hand side. To see this consider both cases separately:

Case 1: p is a square modulo q. Then $n = \frac{q(q^2-1)}{2}$ and $X^{p,q}$ is not bipartite, so there is only one trivial eigenvalue: $\lambda_0 = p + 1$. One can solve $p + 1 = 2\sqrt{p}\cos\theta_0$ to get $\theta_0 = i\log\sqrt{p}$ (expand the cos function as $e^{i\theta} + e^{-i\theta}/2$ and solve as a quadratic in $e^i\theta$). Thus the contribution of the trivial eigenvalue to the right hand side of (2.9) is:

$$\begin{aligned}\frac{2}{n}p^{m/2}\frac{\sin((m+1)i\log\sqrt{p})}{\sin(i\log\sqrt{p})} &= \frac{4p^{m/2}}{q(q^2-1)}\left(\frac{p^{-m/2}(p^{m+1}-1)}{p-1}\right)\\ &= \frac{4}{q(q^2-1)}\frac{p^{m+1}-1}{p-1}\end{aligned}$$

Case 2: p is not a square modulo q. Then $n = q(q^2-1)$ and $X^{p,q}$ is bipartite, thus we have two trivial eigenvalues: $\lambda_0 = p+1$ and $\lambda_{n-1} = -(p+1)$ which yields $\theta_0 = i\log\sqrt{p}$ and $\theta_{n-1} = \pi + i\log p$. This time n fails to contribute a factor of 2 on the numerator. However, since $m+1$ is even the contribution of each trivial eigenvalue is the same, so again the contribution of the trivial eigenvalues to the right hand side of (2.9) is:

$$\frac{4}{q(q^2-1)}\frac{p^{m+1}-1}{p-1}$$

In both cases, then, this term can be cancelled from both the left and right side of (2.9), and only the contributions of the non-trivial eigenvalues is left.

Only the proof for non bipartite case is continued here, but the remainder of the proof for the bipartite case is analogous. In the non bipartite case, the expression in (2.9) can be written:

$$O_\epsilon(p^{\epsilon m/2}) = \frac{2}{n}\sum_{j=1}^{n-1}\frac{\sin((m+1)\theta_j}{\sin\theta_j} \tag{2.10}$$

It is required to prove that the remaining θ_i are real. Observe that if θ is real then $|\frac{\sin(m+1)\theta}{\sin\theta}| \leqslant m+1$.[10] From this observation, deduce:

$$\left|\frac{2}{n}\sum_{i:\ \theta_i\in\mathbb{R}}\frac{\sin(m+1)\theta_i}{\sin\theta_i}\right| < 2(m+1)$$

Now suppose for a contradiction that some θ_j is not real. Then write $\theta_j = i\mu_j$ or $\theta_j = \pi + i\mu_j$ where $\mu_j \in (0, \log\sqrt{p}]$. In either case, the contribution of this term is:

$$\begin{aligned}\frac{2}{n}\frac{\sin(m+1)\theta_j}{\sin\theta_j} &= \frac{2}{n}\frac{\sin(m+1)i\mu_j}{\sin i\mu_j}\\ &= \frac{2}{n}\frac{i\sinh(m+1)\mu_j}{i\sinh\mu_j}\\ &= \frac{2}{n}\left(\cosh m\mu_j + \frac{\cosh\mu + j\sinh m\mu_j}{sinh\mu_j}\right) > 0\end{aligned}$$

In this form, it is easy to see that for large enough m the contribution made by θ_j is much greater than $2(m+1)$, and hence it is the dominating term in the right hand side of (2.10). Thus, with some θ_j not real, $\frac{2}{n}\sum_{j=1}^{n-1}\frac{\sin((m+1)\theta_j)}{\sin\theta_j}$ tends to infinity with m. For small enough ϵ this contradicts the expression:

$$O_\epsilon(p^{\epsilon m/2}) = \frac{2}{n}\sum_{j=1}^{n-1}\frac{\sin((m+1)\theta_j}{\sin\theta_j}\ .$$

It can be concluded that all θ_j corresponding to non-trivial eigenvalues are real. It follows that $X^{p,q}$ is Ramanujan. □

[10] This can be proven by induction beginning with the base case where $m = 1$. The identity $sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$ is used in the induction step.

Chapter 3
Expanders and Lifts of Graphs

The notion of a covering space turns out to be quite useful in exploring the spectrum of graphs. In this section we will see how covering spaces can be used to construct families of expander graphs, and how covering spaces lead to a natural extension of the definition of expanders to the irregular case. The majority of this chapter follows Chapter 6 of [13]. All other sources are cited throughout.

3.1 Review of Covering Spaces

Definition 3.1.1. Let $G = (V, E)$ be a graph. For a vertex $v \in V$ the *neighbourhood of* v, denoted $N(v)$, is defined $N(v) := \{w \in V : v \sim w\}$.

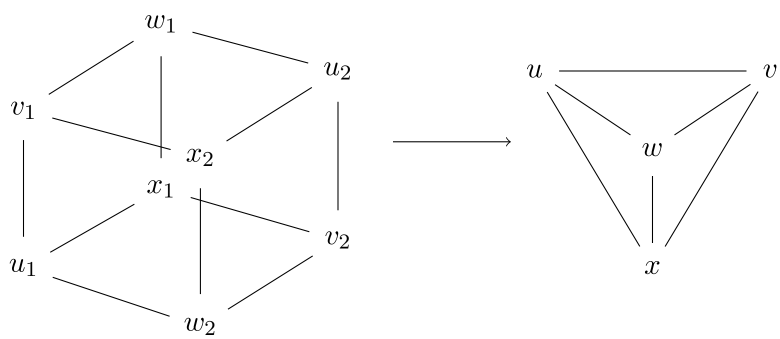
Definition 3.1.2. Let G and H be graphs. A function $f : H \to G$ is a *covering map* if it is surjective and if for every $v \in V(H)$ f maps the neighbour set of v, $N(v)$, bijectively to $N(f(v))$. If there exists a covering map from H to G then H is called a *covering space* or *cover* of G.

Definition 3.1.3. If $f : H \to G$ is a covering map then for $v \in V(G)$ the set $f^{-1}(v)$ is called the *fiber* of v. Similarly if $e \in E(G)$ then $f^{-1}(e)$ is the fibre of e.

Definition 3.1.4. Notice that if $f : H \to G$ is a covering map of a connected graph G then there exists a constant $c \in \mathbb{N}$ so that $|f^{-1}(g)| = c$ for all $g \in G$. The number c is referred to as the *degree* of the map f.

For a connected graph G let $L_n(G)$ denote the set of degree n covering spaces of G. Notice that any graph $H \in L_n(G)$ can be described in the following convenient manner: Begin by labelling the vertices of H so that the fiber of any vertex $v \in V(G)$ consists of vertices labelled $(v, 1), (v, 2), \ldots, (v, n)$. Given such a labelling and given any edge $e = (u, v) \in E(G)$ there is a permutation $\pi_e \in S_n$ so that the set of edges in H between vertices in the fibers of u and v are described by $((u, i), (v, \pi_e(i)))$ for $i \in [n]$. Every element of $L_n(G)$ can be described in this way via an enumeration of the vertices and a collection of permutations. Notice also that an element, J, of $L_n(G)$ can be constructed by setting $V(J) = V(G) \times [n]$ and by choosing a collection of $|E(G)|$ permutations from which the edge set $E(J)$ is assigned.

Example 3.1.5. *The following is an example of a degree 2 cover of K_3. Here $\pi_{(u,v)} = 1$, $\pi_{(v,w)} = 1$, $\pi_{(u,x)} = 1$, $\pi_{(u,w)} = (1, 2)$, $\pi_{(w,x)} = 1$ and $\pi_{(x,v)} = (1, 2)$.*



Notice that in general, elements of $L_n(G)$ need not be connected. Indeed, setting π_e equal to the identity for all edges produces a graph with n separate connected components.

Definition 3.1.6. The *universal cover* of a graph G is a connected covering space of G that has no cycles. For example, the universal cover of K_3 is the infinite tree of degree 3: T_3.

3.2 Old and New Eigenvalues

Proposition 3.2.1. *If H is a covering space of G then every eigenvalue of G is an eigenvalue of H.*

Proof. Let $f : H \to G$ be a covering map of degree d. Suppose y is an eigenvector of A_G with corresponding eigenvalue λ, and let $y(v)$ denote the entry of y corresponding to the vertex $v \in V(G)$. It will be shown that the pullback f^*y is an eigenvector of A_H with corresponding eigenvalue λ. Let $[d]$ denote the set of integers 1 through d. The definition of a covering map implies that for any $v \in V(G)$, $y(v) = y(f(v_i))$ for all $i \in [d]$. Further, if $u \in V(G)$ is adjacent to $v \in V(G)$ in G then for every $i \in [d]$ there is exatly one corresponding $j \in [d]$ such that u_j is adjacent to v_i in H. Thus for any fixed $v_i \in V(H)$

observe,

$$A_H y(f(v_i)) = \sum_{u_j \in V(H): u_j \sim v_i} y(f(u_j))$$
$$= \sum_{u \in V(G): u \sim v} y(u) = A_G y(v) = \lambda y(v) = \lambda y(f(v_i))$$

which verifies that $A_H f^* y = \lambda f^* y$ holds. □

Definition 3.2.2. The eigenvalues and eigenfunctions that H inherits from G are called the *old eigenvalues* and *old eigenvalues* respectively. All other eigenvalues and eigenvectors of H are referred to as *new*.

Proposition 3.2.3. *Let $f : H \to G$ be a finite covering map and let ψ be a new eigenfunction of H. Then $\sum_{f(x)=v} \psi(x) = 0$ for every $v \in V(G)$, ie. ψ sums to zero on every fibre.*

Proof. Since the eigenfunctions of G span the space of real functions on $V(G)$ the old eigenfunctions of H will span the space of real functions on $V(H)$ which are constant on fibres of H. Since distinct eigenfunctions can always be chosen to be mutually orthogonal, any new eigenfunction of H must be orthogonal to all functions that are constant on the fibres. This can only hold when a function sums to zero on each fibre. □

When H is a degree 2 covering space of G the new eigenvalues of H can be calculated explicitly. Before stating this result formally some terminology corresponding to degree 2 covers will be introduced.

Definition 3.2.4. A *signing* of the edges of G is a function $s : E(G) \to \{-1, 1\}$. A *signed adjacency matrix* of G with a signing is the regular adjacency matrix of G but with -1 instead of 1 where the sign of an edge was -1. The signed adjacency matrix is denoted A_s.

Associated with each signing of G is a degree 2 cover where $s(e) = 1$ corresponds to $\pi(e) = 1$ and $s(e) = -1$ corresponds to $\pi(e) = (1, 2)$. In other words, if $\{x, y\} \in E(G)$ then $s(x, y) = 1 \iff \{x_0, y_0\}$ and $\{x_1, y_1\}$ are edges in H and $s(x, y) = -1 \iff \{x_0, y_1\}$ and $\{x_1, y_0\}$ are edges in H.

Example 3.2.5. *The following signed adjacency matrix corresponds to the degree 2 cover in Example 3.1.5:*

$$A_s = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$$

The following proposition was presented as a lemma in Bilu and Linial's 2006 paper [4]. The proof presented here was taken from their paper.

Proposition 3.2.6. *Let H be a degree 2 covering space of G encoded in the signed adjacency matrix A_s. Then the new eigenvalues of H are the eigenvalues of A_s.*

Proof. Let A_H be the adjacency matrix of H. Define A_1 as the adjacency matrix of $(V, s^{-1}(1))$ and A_2 the adjacency matrix of $(V, s^{-1}(-1))$. Then $A = A_1 + A_2$ and $A_s = A_1 - A_2$. Notice that if the rows of the matrix A_H are organized so that rows 1 through n correspond to vertices $V(G) \times 1$ and rows $n+1$ through $2n$ correspond to vertices $V(G) \times 2$ then A_H can be written as:

$$A_H = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}$$

Let h be an eigenvector of A with corresponding eigenvalue μ. Then $h_H := (h, h)^T$ is an eigenvector of A_H with corresponding eigenvalue μ. If g is an eigenvalue of A_s with eigenvalue λ then $g_H := (g, g)^T$ is an eigenvector of A_H with eigenvalue λ. If h_H is constructed for all eigenvectors h of A and if g_H is constructed for all eigenvectors g of A_s then all $2n$ orthogonal eigenvectors of A_H are obtained. □

3.3 Spectrum of Infinite Trees

Recall that for a finite graph G with adjacency matrix A_G the spectrum of G is given by $spec(A_G) = \{\lambda : (A_G - \lambda I) \text{ is not invertible}\}$. Usually one finds $\lambda \in spec(A_G)$ by solving $(A_G - \lambda I)v = 0$. Although the spectrum for an infinite graph T with adjacency operator A_T is defined analogously by $spec(A_T) = \{\lambda : (A_T - \lambda I) \text{ is not invertible}\}$, there are significant differences between the spectra of finite and the spectra of infinite adjacency operators. For instance, observe that for an infinite k-regular graph T, $k \notin spec(A_T)$ since the constant function is not in $l^2(V(T))$.

In general, the spectrum of a bounded self-adjoint operator on an infinite Hilbert space can be considered as the disjoint union of two subsets: the *point spectrum* consisting of all λ such that $(A - \lambda I)$ is not one-to-one, and the *continuous spectrum* for which $(A - \lambda I)$ is not onto (see, for example, chapter 9 in [14]). In the case of the k-regular tree, T, $spec(T)$ is known to be purely continuous (see, for example, [18] or 5.1.2 of [13]).

Definition 3.3.1. The spectral radius of an infinite dimensional operator A, denoted $\rho(A)$, is defined: $\rho(A) := sup\{|\lambda| : \lambda \in spec(A)\}$.

Proposition 3.3.2.

For a self adjoint operator, A_T, one has $\rho(A_T) = \sup_{||x||=1} ||A_T x||$ (see, for example, proposition 9.7 in [14])

Proposition 3.3.3. *The spectrum of* T_k *is* $[-2\sqrt{k-1}, 2\sqrt{k-1}]$.

This theorem will not be fully proven, but the reader is referred to [7] for full details. Instead, this exposition will prove that the spectrum of T_k is fully contained in $[-2\sqrt{k-1}, 2\sqrt{k-1}]$ as shown by Sunada in [34]. This is a much easier result to show. It suffices to verify,

$$|\langle Af, f\rangle| \leqslant 2\sqrt{k-1}||f||^2$$

for all functions $f \in \ell^2(T_k)$. Begin by assigning an orientation to the edges of T_k so that every vertex has one edge pointing towards it and $k-1$ edges pointing away from it. The existence of such an orientation follows from the fact that T_k has no cycles. For an edge oriented from x to y, let $o(e) = x$ and $t(e) = y$ where o is for origin and t is for tail.

Observe,

$$
\begin{aligned}
|\langle Af, f\rangle| = \left|\sum_{x\in V}\sum_{y\in V: y\sim x} f(y)\overline{f(x)}\right| &= \left|\sum_{e\in E}\left[f(t(e))\overline{f(o(e))} + f(o(e))\overline{f(t(e))}\right]\right| \\
&= \left|2\sum_{e\in E}\Re(f(o(e))\overline{f(t(e))})\right| \\
&\leqslant 2\sum_{e\in E}|f(o(e))|\,|f(t(e))| \\
&\leqslant 2\left(\sum_{e\in E}|f(o(e))|^2\right)^{1/2}\left(\sum_{e\in E}|f(t(e))|^2\right)^{1/2}
\end{aligned}
$$

Because of the way that the edges were oriented, every vertex $x \in V$ appears as $o(e)$ exactly once, and as $t(e)$ exactly $k-1$ times. Thus the above line is re-written as,

$$
2\left(\sum_{x\in V}|f(x)|^2\right)^{1/2}\left((k-1)\sum_{x\in V}|f(x)|^2\right)^{1/2} = 2\sqrt{k-1}||f||^2
$$

as desired.

3.4 Generalized Ramanujan Graphs

This section explores the idea of Ramanujan graphs with vertices of non constant degree. The following is a generalization of Theorem 2.1.2, the Alon-Bopanna lower bound.

Theorem 3.4.1. *Let $\{G_i\}$ be a family of graphs covered by the same universal cover T, and such that $|V(G_i)|$ increases to infinity with i. Then $\lambda(G_i) \geqslant \rho(T) - o(1)$ where $\rho(T)$ is the spectral radius previously defined.*

Proof. As in the proof of Theorem 2.1.2, it suffices to show that $|\lambda_1(G)|$ satisfies the desired bound. The following inequality is obtained in exactly the same way as in the proof of 2.1.2:

$$
\lambda_1(A)^{2p} \geqslant \frac{(A^{2p})_{ss} + (A^{2p})_{tt}}{2}
$$

where A^{2p}_{ss} and A^{2p}_{tt} denote the number of closed paths in G of length $2p$ starting and ending at vertex s and t respectively. Let $a^{(s)}_{2k}$ and $a^{(t)}_{2p}$ denote the number of closed paths of length $2p$ in T which start and end at a fiber of s in T and a fiber of t in T respectively. Then $(A^{2p})_{ss} \geqslant a^{(t)}_{2p}$ and $(A^{2p})_{tt} \geqslant a^{(s)}_{2p}$. The following well known fact will be used:

$$
\rho(T) = \limsup_{n\to\infty} a_n^{1/n}
$$

where a_n is the number of closed paths of length n in T starting and ending from any fixed vertex in T (see [25] and references therein). Then

$$(\rho(T) - o(1))^{2p} = a_{2p}^{(s)} = a_{2p}^{(t)}$$

It follows that,

$$\lambda_1(G)^{2p} \geqslant \frac{a_{2k}^{(t)} + a_{2k}^{(s)}}{2} = (\rho(T) - o(1))^{2p} \implies |\lambda_1(G)| \geqslant (\rho(T) - o(1))$$

as claimed. □

This theorem suggests the following definition for a generalized Ramanujan graph:

Definition 3.4.2. A graph G is Ramanujan if $\lambda(G) \leqslant \rho(\tilde{G})$ where $\tilde{G}$ denotes the universal cover of G.

If G is k-regular then $\tilde{G} = T_k$ and $\rho(\tilde{G}) = 2\sqrt{k-1}$ by proposition 3.3.3. This definition thus coincides with the original notion of a Ramanujan graph when the graph under consideration is regular.

3.5 Quotients of uniform infinite trees

The result of Morgenstern mentioned in section 2.2 can be restated in terms of covering spaces as follows: If $k-1$ is a prime power then infinitely many quotients of T_k are Ramanjan graphs. It has been conjectured that the same holds for all $k \geqslant 3$. This conjecture has recently been verified for bipartite graphs, as will be discussed in Chapter 5. Lubotsky and Nagnibeda show that these types of conjectures do not hold in the irregular case. More precicely, they show that not every infinite tree with infinitely many finite quotients cover Ramanugan graphs. In their paper [25] they construct infinite trees which cover infinitely many finite graphs none of which are Ramanujan. This section will review some relevant results and terminology and then will present the proof provided in their paper.

Definition 3.5.1. An infinite tree is called *uniform* if it covers some (and therefore infinitely many) finite graphs.

Proposition 3.5.2. [21] *Any two finite graphs G_1 and G_2 with the same universal covering tree have a common finite cover G.*

The proof of this theorem is omitted here but the reader is referred to the original document for more information [21].

Definition 3.5.3. An *automorphism* of a graph G is an adjacency preserving permutation of $V(G)$. In other words, if two vertices are (non) adjacent then their images under the permutation are also (non) adjacent. The set of automorphisms of a graph forms a group under composition called the *automorphism group*, $Aut(G)$.

Definition 3.5.4. The *fundamental group* of a graph G, denoted $\pi_1(G)$ is a group of equivalence classes of cycles in G which start and end at a fixed vertex v. Two paths are considered equivalent if one can be continuously deformed into the other within the graph G. [1] (See Chapter 1 in [11] for more details.)

Definition 3.5.5. Given a universal covering map $\rho : T \to X$ of a graph X, and path $\alpha \in X$ a *lift* of α is a path $\tilde{\alpha} \in T$ satisfying $\rho(\tilde{\alpha}) = \alpha$.

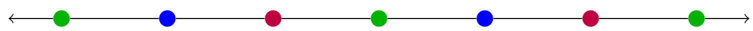
Remark: Some authors use the term “lift" interchangeably with the term “covering space" when graphs are the only mathematical objects under consideration. This exposition follows the more general theory in developed in [11] and uses the distinct definitions provided.

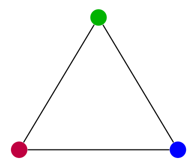
Let $\rho : T \to X$ be the universal covering space of a finite connected graph X. There is a natural action of $\pi_1(X)$ on T in which an element $\alpha \in \pi_1(X)$ translates T by $\tilde{\alpha}$. Since $\alpha \in \pi_1(X)$ each vertex will be permuted to another vertex in the same fiber (or else will be permuted to itself). A fiber preserving permutation necessarily preserves adjacency, and hence the action of $\alpha \in \pi_1(X)$ on T yields an automorphism of T. In fact the set of permutations of T obtained by the action $\pi_1(X) \curvearrowright T$ forms a subgroup of $Aut(T)$. In topology this subgroup is sometimes referred to as the group of “deck transformations" or “covering transformations" of $\rho : T \to X$ since it is exactly the subgroup of transformations which preserve the fibers of the covering map. (See Hatcher’s book on Algebraic Topology [11] for more details.)

[1] The equivalence described here is one that some readers may already be familiar with in a more general setting. It is called “homotopy equivalence".

Given an action $G \curvearrowright T$ the quotient space T/G is obtained by identifying each point $t \in T$ with all of its images $g(t)$ for $g \in G$. [2] Observe that the quotient space resulting from the action of $\pi_1(X)$ on T gives back the graph X.

Example 3.5.6. *Let X be the finite 2-regular graph on 3 vertices (ie. the triangle). The universal cover is an infinite 2-regular tree, T_2. The colours of the vertices below indicate the covering map. Observe that $Aut(T_2) \cong \mathbb{Z} \times \mathbb{Z}_2$ and the action of $\pi_1(X)$ on T_2 embeds as the subgroup $3\mathbb{Z}$. The vertices in the diagram below illustrate the action of $\pi_1(X)$ on T_2, and it is clear that the resulting quotient group is X.*

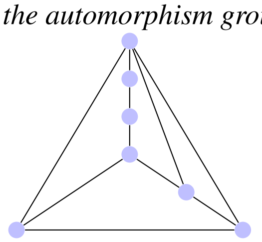




Definition 3.5.7. A graph is called *minimal* if it is equal to the quotient of the universal covering tree $\tilde{X}$ of X by the full automorphism group of $\tilde{X}$, ie. $X = \tilde{X}/\operatorname{Aut}(\tilde{X})$.

Observe that not every uniform tree covers a minimal graph. Indeed, the tree in 3.5.6 does not cover a minimal graph. Observe also that if X is minimal and if X' is another finite quotient of $\tilde{X}$ then X' covers X. Bass and Tits provide an algorithm in [3] to determine whether or not a given finite graph X is minimal. Their result is explained briefly in [25] but is omitted from our discussion.

Example 3.5.8. *The following minimal graph is presented by Bass and Tits in [3]. Observe that the automorphism group of this graph is indeed trivial.*



Lemma 3.5.9. (Cauchy's Interlacing Eigenvalue Theorem) *Let A be an $n \times n$ hermitian matrix and let B be an $n-1 \times n-1$ principal submatrix of A. Let*

[2] The set $\{g(t) : g \in G\}$ is sometimes called the orbit space of t.

$$\lambda_{n-1}(A) \leqslant \lambda_{n-2}(A) \leqslant \ldots \leqslant \lambda_1(A) \leqslant \lambda_0(A) \tag{3.1}$$

denote the eigenvalues of A and

$$\mu_{n-2}(B) \leqslant \mu_{n-3}(B) \leqslant \ldots \leqslant \mu_0(B) \tag{3.2}$$

denote the eigenvalues of B. Then $\lambda_{n-1}(A) \leqslant \mu_{n-2}(B) \leqslant \lambda_{n-2}(A) \leqslant \ldots \lambda_1(A) \leqslant \mu_0(B) \leqslant \lambda_0(A)$.

See Appendix B for a proof of this theorem.

Lemma 3.5.10. [LuNa98] *Let T be an infinite tree with maximum vertex degree k. Then $\rho(T) \leqslant \rho(T_k) = 2\sqrt{k-1}$.*

Proof. Follows from the fact that $\rho(\tilde{X}) = \limsup_{n\to\infty} a_n^{1/n}$ where a_n denotes the number of closed paths of length n in $\tilde{X}$ starting and ending at some fixed vertex. □

The above mentioned result of Lubotzky and Nagnimbeda in [25] is now ready to be stated:

Theorem 3.5.11. *Let X be a minimal graph with a cut vertex x_0. In other words, if vertex x_0 and all edges incident to it are deleted then the resulting graph consists of two disjoint non-empty subgraphs. Call these subgraphs Y and Z. Let k denote the maximum vertex degree in X. Assume that the average degrees of vertices in each Y and Z are strictly greater than $2\sqrt{k-1}$. Then the universal cover $\tilde{X}$ of X is a locally finite uniform tree which covers no Ramanujan graph.*

Proof. The universal cover $\tilde{X}$ of X is a tree with maximum vertex degree k. From Lemma 3.5.10 it follows that $\rho(\tilde{X}) \leqslant \rho(T_k) = 2\sqrt{k-1}$. This proof will proceed to show that X is not Ramanujan by verifying that $\lambda(X) > 2\sqrt{k-1}$. Let $X_0 = Z \cup Y$ be the graph obtained by deleting x_0. The Rayleigh Quotient is used to deduce that $\lambda_0(Y)$ and $\lambda_0(Z)$ are $\geqslant$ the average degrees of Y and Z respectively. Thus $\lambda_0(Y)$ and $\lambda_0(Z)$ are each strictly greater than $2\sqrt{k-1}$ and therefore the two largest eigenvalues of X_0, $\lambda_0(X_0)$ and $\lambda_1(X_0)$, are both greater than $2\sqrt{k-1}$. Applying Cauchy's interlacing eigenvalue theorem yields:

$$\lambda_0(X) \geqslant \lambda_0(X_0)\lambda_1(X) \geqslant \lambda_1(X_0) > 2\sqrt{k-1}$$

and hence

$$\lambda(X) \geqslant |\lambda_1(X)| > 2\sqrt{k-1}$$

It follows that X is not Ramanujan. Let X' be any finite quotient of $\tilde{X}$. Then X is a quotient of X' since X is minimal. This means that any eigenvalue of X is also an eigenvalue of X'. The required result follows. □

It should be mentioned that Lubotzky and Nagnibeda provide explicit examples of graphs which satisfy the conditions of Theorem 3.5.11, and thus whose universal covering trees cover no Ramanujan graphs.

Chapter 4
Matching Polynomials

This section introduces some classic graph polynomials. Particular attention is paid to the matching polynomial of a graph. A relationship between these roots and the graph's spectrum will be developed.

4.1 Introduction

Definition 4.1.1. For a graph G a *matching* in G is a set of edges in G without common vertices.

Definition 4.1.2. For a graph G on n vertices let m_i denote the number of matchings in G with i edges. Set $m_0 = 1$ and define the *bivariate matching polynomial* of G, $M(G; x, y)$, as follows:

$$M(G; x, y) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} m_i(G) x^i y^{n-2i}$$

Definition 4.1.3. Three common matching polynomials are obtained from the bivariate matching polynomial:

- The *generating matching polynomial* [19] :

$$g(G; x) := M(G; x, 1) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} m_i(G) x^i$$

- The *partition matching polynomial* [12]:

$$p(G; x) := M(G; 1, x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} m_i(G) x^{n-2i}$$

- The *acyclic matching polynomial* [19], sometimes simply referred to as the *matching polynomial*:

$$\mu(G; x) := M(G, -1, x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} x^{n-2i} (-1)^i m_i(G)$$

Notice $\mu(G; x) = i^{-n} p(G; ix)$.

Observation 4.1.4. [10] *The acyclic matching polynomial satisfies the following recurrence relation:*

$$\mu(G;x) = x\mu(G-v;x) - \sum_{w:w\sim v} \mu(G-v-w;x)$$

Proof. Let $e \in E(G)$ with endpoints v and w. Let $G-e$ be the graph obtained by deleting edge e from the graph G. It is easy to see that the number of matchings of G of size j satisfies the following recurrence relation: $m_j(G) = m_j(G-e) + m_{j-1}(G-v-w)$. Substituting the right hand side of this recurrence into the definition of $\mu(G,x)$ yields:

$$\begin{aligned}
\mu(G,x) &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (m_i(G-e) + m_{i-1}(G-v-w))(-1)^i x^{n-2i} \\
&= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} m_i(G-e)(-1)^i x^{n-2i} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} m_{i-1}(G-v-w)(-1)^i x^{n-2i} \\
&= \mu(G-e;x) + \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} m_j(G-v-w)(-1)^{j+1} x^{n-2(j+1)} \\
&= \mu(G-e;x) - \mu(G-v-w;x)
\end{aligned}$$

Let $v \in V(G)$ be incident to edges e_1, e_2, …, e_d and let w_1, w_2, …, w_d be the set of neighbours of v labeled such that $e_j = \{v, w_j\}$. Then write,

$$\begin{aligned}
\mu(G;x) &= \mu(G-e_1;x) - \mu(G-v-w_1;x) \\
&= \mu(G-e_1-e_2;x) - \mu(G-v-w_1;x) - \mu(G-v-w_2;x) \\
&= \quad \vdots \\
&= \mu(G-e_1-e_2-\ldots-e_d;x) - \sum_{j=1}^{d} \mu(G-v-w_j;x) \\
&= x\mu(G-v;x) - \sum_{w\sim v} \mu(G-v-w;x)
\end{aligned}$$

□

Lemma 4.1.5. *If a graph G is the disjoint union of the graphs K and L then $M(G;x;y) = M(K;x;y)M(L;x;y)$.*

Proof. Let G be a graph with n vertices that is the disjoint union of two connected components, L with ℓ vertices and K with k vertices. Then,

$$M(L;x;y)M(K;x;y) = \left(\sum_{i=0}^{\lfloor \frac{\ell}{2} \rfloor} x^i m_i(L) y^{\ell-2i}\right) \left(\sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} x^j m_j(L) y^{k-2j}\right)$$

The terms in the above expression can be rearranged to sum over $i+j$ as follows:

$$\begin{aligned}
& (\text{terms in which } i+j=0) \; + \; (\text{terms in which } i+j=1) \\
& + \; \ldots \; + \; \left(\text{terms in which } i+j = \lfloor \frac{\ell}{2} \rfloor + \lfloor \frac{k}{2} \rfloor \right) \\
= \; & \sum_{i+j=0}^{\lfloor \frac{\ell}{2} \rfloor + \lfloor \frac{k}{2} \rfloor} x^{i+j} y^{\ell+k-2(i+j)} \sum_{i=0}^{i+j} m_i(L) m_j(K) \\
= \; & \sum_{r=0}^{\lfloor \frac{\ell}{2} \rfloor + \lfloor \frac{k}{2} \rfloor} x^r y^{\ell+k-2r} \sum_{i=0}^{r} m_i(L) m_{r-i}(K) \\
= \; & \sum_{r=0}^{\lfloor \frac{\ell}{2} \rfloor + \lfloor \frac{k}{2} \rfloor} x^r y^{\ell+k-2r} m_r(G) \\
= \; & \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} x^r y^{n-2r} m_r(G)
\end{aligned}$$

obtaining the required result. $\square$

The following corollary follows immediately from the previous lemma and from the definition of the acyclic matching polynomial.

Corollary 4.1.6. *If a graph G is the disjoint union of the graphs K and L then $\mu(G;x) = \mu(K;x)\mu(L;x)$.*

4.2 Roots of the Acylic Matching Polynomial

Theorem 4.2.1. [12] *For any simple graph G the roots of the acyclic matching polynomial $\mu(G;x)$ are real. Furthermore if $v \in V(G)$ then the roots of $\mu(G,x)$ interlace the roots of $\mu(G-v;x)$.*

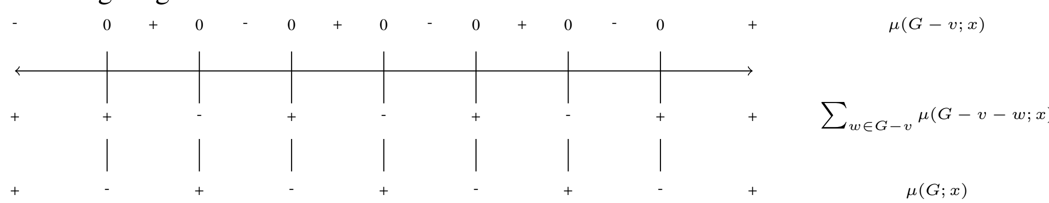
Proof. Begin by verifying that the theorem holds when G is the complete graph on n vertices. Proceed by induction on n. For $n=0$ and $n=1$ the claim is trivially satisfied. For

$n > 1$, consider the recurrence relation from observation 4.1.4:

$$\mu(G;x) = x\mu(G-v;x) - \sum_{w:w\sim v} \mu(G-v-w;x) \tag{4.1}$$

Since G is a complete graph on n vertices, $\mu(G-v;x)$ is a degree $n-1$ polynomial and each $\mu(G-v-w;x)$ is a degree $n-2$ polynomial. Without loss of generality consider the case where n is even. (If n is odd, analogous arguments hold with appropriate sign changes.) Since the leading coefficient of each polynomial is positive, $\mu(G-v;x)$, $\mu(G-v-w;x)$ and $\mu(G;x)$ are all positive for sufficiently large x.

Consider the sign of the right hand side of equation 4.1 as x takes on the values of the zero's of $\mu(G-v;x)$ and as x tends to positive and negative infinity. For each $w \in G-v$ the induction hypothesis implies that the zeros of $\mu(G-v-w;x)$ are real and interlace the zeros of $\mu(G-v;x)$, which are also real. From this information the sign of $\sum_{w\in G-v} \mu(G-v-w;x)$ evaluated at the zeros of $\mu(G-v:x)$ can be deduced. The following diagram:



From this diagram one can conclude that there are n real roots of $\mu(G;x)$ which interlace the zeros of $\mu(G-v;x)$. Since $\mu(G;x)$ is a degree n polynomial this means that all of its roots are real.

It remains to verify the theorem in the case where G is not the complete graph. Consider G to be a weighted graph. By this it is meant that a non-negative value $W(i,j)$ is assigned to every pair of vertices, i, j. $W(i,j) = 0$ is equivalent to $\{i,j\}$ not being an edge in G. The definition of the acyclic matching polynomial is extended to weighted graphs as follows:

$$\mu_W(G;x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{M \text{ is a matching} \\ \text{of size } i}} \prod_{\{v,w\}\in M} W(i,j)(-1)^i x^{n-2i} \tag{4.2}$$

Notice that if all $W(i,j) \in \{0,1\}$ then this reduces to the original matching polynomial for non-weighted graphs. The previously considered recurrence relation for $\mu(G;x)$

extends to the weighted case also:

$$\mu_W(G;x) = x\mu_W(G-v;x) - \sum_{w \in G-v} W(v,w)\mu_W(G-v-w;x)$$

Applying this recurrence relation and analogous arguments as before, it can be deduced that if G is a graph in which $W(i,j)$ is strictly greater than 0 for all $i, j \in V(G)$ then the roots of $\mu_W(G,x)$ are real.

Let G' be a graph on n vertices where edges have weights either 0 or 1. A sequence of weighted graphs on n vertices $\{G_n\}_{n>0}$ will be constructed where $W(i,j) = 1$ if $\{i,j\} \in E(G)$ and $W(i,j) = \frac{1}{n}$ otherwise. Then G_1 is the complete graph on n vertices and G_n tends to G' as $n \to \infty$. It has already been verified that the roots of G_n are real for all $n > 0$. Since the sequence of graphs converges to G' then given any $\delta > 0$ we can find an i such that each coefficients of $\mu_W(G_i;x)$ differ from the corresponding coefficient in $\mu_W(G';x)$ by at most δ. Since a bound on the difference in coefficients between two polynomial functions infers a bound on the difference between roots of the two polynomials, it follows that for any $\epsilon > 0$ there exists an i such that the roots of i are at most ϵ away from the roots of G'. Since the roots of G_i are real, this means that the roots of G' must also be real. □

Definition 4.2.2. Let G be a graph and let $u \in V(G)$. The *path tree* $T(G,u)$ of G at u is defined to be the graph whose vertex set is the set of distinct circuit-less paths in G which start at u. Two vertices are adjacent if one path is a maximal subpath of the other.

Theorem 4.2.3. [9] *Let* $T(G,u)$ *be a path tree of G. Then*

$$\frac{\mu(G;x)}{\mu(G-v;x)} = \frac{\mu(T(G,v);x)}{\mu(T(G,v)-v;x)}$$

and $\mu(G;x)$ *divides* $\mu(T(G,u),x)$.

Proof. Let $|V(G)| = n$ and proceed by induction on n. If $n \leqslant 2$ then G is a tree. In this case the map taking each vertex $w \in V(G)$ onto the unique path joining $v \in V(G)$ to w is an isomorphism from G to $T(G,v)$. Thus for $n \leqslant 2$ the result is immediate. Consider the case where $n > 2$ and assume by induction that the result holds for all graphs with less than

n vertices. Let $v \in V(G)$ and $N(v) = \{w_1, \ldots, w_k\}$. Applying observation 4.1.4 yields:

$$\frac{\mu(G;x)}{\mu(G-v;x)} = x - \sum_{i=1}^{k} \frac{\mu(G-v-w_i;x)}{\mu(G-v;x)} \tag{4.3}$$

$$= x - \sum_{i=1}^{k} \frac{\mu(T(G-v,w_i)-w_i;x)}{\mu(T(G-v,w_i);x)} \tag{4.4}$$

where the (4.4) follows from induction. Let (*) represent $\frac{\mu(G;x)}{\mu(G-v;x)}$. Multiplying both the numerator and denominator of each fraction in (4.4) by $\prod_{j \neq i} \mu(T(G-v,w_j);x)$ yields:

$$(*) = x - \left(\frac{\sum_{i=1}^{k} \mu(T(G-v,w_i)-w_i;x) \prod_{j \neq i} \mu(T(G-v,w_j);x)}{\prod_{j=1}^{k} \mu(T(G-v,w_j);x)} \right) \tag{4.5}$$

Notice that the disjoint union of $T(G-v;w_i) - w_i$ with the graphs $T(G-v;w_i)$ for $i \neq j$ is equal to $T(G,v) - \{v, w_i\}$. Similarly the disjoint union of $T(G-v;w_i)$ for all $\in [k]$ equals $T(G,v) - v$. Hence corollary 4.1.6 is applied to obtain:

$$\frac{\mu(G;x)}{\mu(G-v;x)} = x - \frac{\sum_{i=1}^{k} \mu(T(G,v) - \{v,w_i\};x)}{\mu(T(G,v)-v;x))} \tag{4.6}$$

$$= \frac{\mu(T(G,v))}{\mu(T(G,v)-v)} \tag{4.7}$$

where the last equality follows from observation 4.1.4.

By induction it is also assumed that $\mu(G-v;x)$ divides $\mu(T(G-v,u);x)$ for any $u \in V(G) - \{v\}$. In particular, $\mu(G-v;x)$ divides $\mu(T(G-v,w_i);x)$ for all vertices $w_i \sim v$. Note that $T(G,v)-v$ will be isomorphic to the disjoint union of all graphs $T(G-v,w_i)$, and so if the matching polynomial $\mu(G-v;x)$ divides the matching polynomials $\mu(T(G-v,w_i);x)$, then clearly $\mu(G-v;x)$ divides $\mu(T(G,v)-v;x)$. Then the equation which was just verified:

$$\frac{\mu(G;x)}{\mu(G-v;x)} = \frac{\mu(T(G,v);x)}{\mu(T(G,v)-v;x)} \tag{4.8}$$

implies that $\mu(G;x)$ divides $\mu(T(G,v);x)$. □

The following is a well known result that will be stated without proof:

Proposition 4.2.4. [22] *The characteristic polynomial of a forest coincides with its matching polynomial.*

Observe that the following is an immediate corollary to propositions 4.2.4 and 4.2.3:

Corollary 4.2.5. *Let $T(G,u)$ be a path tree of G. Then the matching polynomial of G divides the characteristic polynomial of $T(G,u)$.*

This theorem implies that the roots of $\mu(G;x)$ have absolute value at most $\rho(T(G,u))$.

Theorem 4.2.6. (Lemma 3.5 in [29]) *Let T be the universal covering tree of a graph G. Then the roots of the matching polynomial of G, $\mu(G;x)$ are bounded in absolute value by the spectral radius of T, $\rho(T)$.*

Proof. Let $v \in V(G)$ and let $T(G,v)$ be the path tree rooted at v. Theorem 4.2.3 guarantees that the roots of $\mu(G;x)$ are bounded in absolute value by

$$||A_{T(G,v)}|| = \sup_{||x||=1} ||A_{T(G,v)}x|| = \sup_{\substack{||y||\,=\,1 \text{ s.t. } A_T y \text{ evalu-} \\ \text{ates to 0 outside of P}}} \leqslant \sup_{||y||=1} ||A_T y||_2 = \rho(T)$$

where the inequality follows from the fact that $T(G,v)$ is an induced subgraph of T and hence the adjacency matrix $A_{T(G,v)}$ is a submatrix of A_T. $\square$

Chapter 5
Bipartite Ramanujan Graphs of All Degrees

5.1 Introduction

As previously discussed, constructing families of Ramanujan graphs has proven to be a highly non-trivial task. Morgenstern was able to construct infinitely many k-regular Ramanujan graphs whenever $k-1$ was a prime power. It has been conjectured that Ramanujan families exist for all $k \geqslant 3$. In its full generality this conjecture remains an open problem. Quite recently, however, Marcus, Spielman and Srivastava confirmed the conjecture for bipartite graphs [29].

Specifically, Marcus et al. succeed in proving that there exists a signing of every d-regular graph G so that the eigenvalues of the associated signed adjacency matrix are all $\leqslant 2\sqrt{k-1}$. Observe that given this result, Proposition 3.2.6 verifies the existence of a covering space of G whose new eigenvalues are all $\leqslant 2\sqrt{k-1}$. Thus, if G were Ramanujan to begin with, then all non-trivial eigenvalues of the covering space would be $\leqslant 2\sqrt{k-1}$. This is not yet enough to conclude that the cover itself would be Ramanujan. Indeed, an upper bound has been obtained on the on the non-trivial eigenvalues, but no lower bound. To solve this problem, the condition that G be bipartite is imposed. With this added assumption, it follows that all new eigenvalues of the covering space are bounded in *absolute value* by $2\sqrt{k-1}$. To see this, recall that covering spaces of bipartite graphs are bipartite and proposition 1.2.4 verifies that the eigenvalues of bipartite graphs are symmetric about zero. So, if G is Ramanujan and bipartite then the result of Marcus et al. verifies the existence of a Ramanujan degree 2 cover, $\hat{G}$, of G. Inductively applying these degree 2 covers yields an infinite sequence of k-regular, bipartite Ramanujan graphs for all $k \geqslant 3$.

The authors were able to prove an analogous result for irregular bipartite Ramanujan graphs as well. Their result in its full generality is stated here:

Theorem 5.1.1. *Let G be a graph with adjacency matrix A and with universal covering tree T. There is a degree 2 covering space $\hat{G}$ of G such that all new eigenvalues of $\hat{G}$ are at most $\rho(T)$.*

The main tools used for the proof of this theorem involve the roots of the matching polynomial and some theory of interlacing families of polynomials. The following two sections will present any remaining results in these areas which are required for the proof, and the final section will bring together these results for a presentation of the proof.

5.2 Application of the Matching Polynomial

Recall from the previous chapter the definition of the acyclic matching polynomial, $\mu(G;x)$. Let G be a graph with edge set $\{e_1,\ldots,e_m\}$. Let $s\in\{\pm1\}^m$ denote a signing of the edge set and let A_s denote the corresponding signed adjacency matrix. Define $f_s(x)$ to be the characteristic polynomial of A_s. Let $\mathbb{E}_{\{s:s\in\{\pm1\}^m\}}$ denote the expected value over all singings s of a random variable. Marcus et al. prove the folling lemma:

Lemma 5.2.1. (Theorem 3.6 in [29]) $\mathbb{E}_{\{s:s\in\{\pm1\}^m\}}[f_s(x)]=\mu(G;x)$

Proof. Let $\operatorname{sym}(A)$ denote the set of permutations of the set A. For $\sigma\in\operatorname{sym}(A)$ let $(-1)^\sigma$ denote the sign of the permutation.

Let G be a graph on n vertices with m edges. Expand the determinant as a sum over the permutations $\sigma\in\operatorname{sym}([n])$ to yield:

$$\mathbb{E}_{\{s:s\in\{\pm1\}^m\}}[\det(xI-A_s)]=\mathbb{E}_{\{s:s\in\{\pm1\}^m\}}\left[\sum_{\sigma\in\operatorname{sym}([n])}(-1)^\sigma\prod_{i=1}^{n}(xI-A_s)_{i,\sigma(i)}\right]$$

Let π denote the part of σ without fixed points. Applying the definition of A_s yields:

$$\begin{aligned}
&\mathbb{E}_{\{s:s\in\{\pm1\}^m\}}[\det(xI-A_s)]\\
&=\mathbb{E}_{\{s:s\in\{\pm1\}^m\}}\left[\sum_{k=0}^{n}x^{n-k}\sum_{\substack{B\subseteq[n]\\|B|=k}}\sum_{\pi\in\operatorname{sym}(B)}(-1)^\pi\prod_{i\in B}(-A_s)_{i,\pi(i)}\right]\\
&=\sum_{k=0}^{n}x^{n-k}\sum_{\substack{B\subseteq[n]\\|B|=k}}\sum_{\pi\in\operatorname{sym}(B)}(-1)^\pi\mathbb{E}_{\{s:s\in\{\pm1\}^m\}}\left[\prod_{i\in B}(-A_s)_{i,\pi(i)}\right]
\end{aligned}$$

Observe that $\mathbb{E}_{\{s:s\in\{\pm1\}^m\}}[(A_s)_{i,j}] = 0$ for all choices of i and j and that $-A_{i,j}$ is independent of $-A_{x,y}$ whenever $\{i,j\} \neq \{x,y\}$. It follows that the only surviving terms are those in which π contains only orbits of size 2.

Since $-A_s$ is symmetric, $\mathbb{E}_{\{s:s\in\{\pm1\}^m\}}[(-A_s)_{ij}(-A_s)_{ji}] = \mathbb{E}_{\{s:s\in\{\pm1\}^m\}}[(-A_s)_{ij}{}^2] = 1$.

The case where π has only orbit size 2 corresponds to a perfect matching of $B \subseteq [n]$. Perfect matchings exists only when $|B|$ is even, and in this case $(-1)^{\pi} = (-1)^{\frac{|B|}{2}}$.

Let M_B denote the collection of perfect matchings on B. Then:

$$\begin{aligned}
\mathbb{E}_{\{s:s\in\{\pm1\}^m\}}[\det(xI - A_s)] &= \sum_{k=0}^{n} x^{n-k} \sum_{\substack{B\subseteq[n] \\ |B|=k}} \sum_{\pi\in M_B} (-1)^{\frac{|B|}{2}} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} x^{n-2k} \sum_{\substack{\pi \text{ is a match-} \\ \text{ing of size k}}} (-1)^k \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} x^{n-2k} (-1)^k m_k \\
&= \mu(G;x).
\end{aligned}$$

The second equality holds since a perfect matching on $2k$ vertices is a matching of size k. $\square$

Recall Theorem 4.2.6 which implies that the roots of the matching polynomial of G are bounded in absolute value by the spectral radius of its universal covering tree. Lemma 5.2.1 has developed an important relationship between the eigenvalues of A_s and the matching polynomial of G. However, on its own Lemma 5.2.1 is not strong enough to prove the existence of the required Ramanujan families.

In order to construct these families through "good" degree 2 covering spaces, it will be shown that there is a signing s in which the largest root of $f_s(x)$ is less than or equal to the largest root of $\mu(G;x)$. Marcus et al. draw upon the theory of interlacing families of polynomials to produce their result. The next section will be devoted to this theory.

5.3 Interlacing Polynomials

Definition 5.3.1. A polynomial $g(x) = \prod_{i=1}^{n-1}(x - \alpha_i)$ *interlaces* a polynomial $f(x) = \prod_{i-1}^{n}(x - \beta_i)$ if $\beta_1 \leqslant \alpha_1 \leqslant \beta_2 \ldots \leqslant \alpha_{n-1} \leqslant \beta_n$. A collection of polynomials $\{f_i\}_{i=1}^{k}$ is said to have a *common interlacing* if there exists a single polynomial g that interlaces each f_i.

Definition 5.3.2. Let $T_1, T_2, \ldots, T_m$ be finite sets. For every m-tuple $t_1, \ldots t_m \in T_1 \times \ldots \times T_m$ let $f_{t_1,\ldots,t_m}$ be a real rooted polynomial with positive leading coefficient. For every partial assignment $t_1, \ldots, t_k \in T_1 \times \ldots \times T_k$ define:

$$f_{t_1,\ldots,t_k} = \sum_{t_{k+1},\ldots t_m \in T_{k+1} \times \ldots \times T_m} f_{t_1,\ldots,t_k,t_{k+1},\ldots,t_m}$$

and define:

$$f_{\emptyset} = \sum_{t_1,\ldots,t_m \in T_1 \times \ldots \times T_m} f_{t_1,\ldots,t_m}$$

A collection $\{f_{t_1,\ldots,t_m}\}_{t_1,\ldots,t_m \in T_1 \times \ldots \times T_m}$ forms an *interlacing family* if for all $1 \leqslant k \leqslant m - 1$ and all $t_1, \ldots, t_k \in T_1 \times \ldots \times T_k$ the set of polynomials $\{f_{t_1,\ldots,t_k,t_{k+1}}\}_{t_{k+1} \in T_{k+1}}$ have a common interlacing.

Lemma 5.3.3. (Theorem 4.4 in [29]) *If $T_1, \ldots, T_m$ are finite sets and $\{f_{t_1,\ldots,t_m}\}$ form an interlacing family of polynomials, then there exists some $t_1, \ldots, t_m \in T_1 \times \ldots \times T_m$ such that the roots of $f_{t_1,\ldots,t_m}$ are bounded above by the largest root of $f_{\emptyset}$.*

Proof. Begin by verifying the following claim: Let $\{f_i\}_{i=1}^{k}$ be real rooted polynomials of constant degree and positive leading coefficients with a common interlacing. Then there exists an i so that the largest root of f_i is at most the largest root of $f_{\emptyset}$.

To verify this claim, let g be a polynomial that interlaces each f_i. Let α_{n-1} denote the largest root of g. It is easy to verify that $f_{\emptyset}$ will have a root, call it β_n which is $\geqslant \alpha_{n-1}$. Since $f_{\emptyset}$ is the sum of the f_i's it is clear that there exists some i such that $f_i(\beta_n) \geqslant 0$. By assumption $f_i(\alpha_{n-1}) \leqslant 0$. It follows that the largest root of f_i is $\leqslant \beta_n$.

This claim is now used to prove the lemma. By assumption it is known that the polynomials $\{f_{t_1}\}_{t_1 \in T_1}$ have a common interlacing, and by definition their sum is $f_{\emptyset}$. The above claim can thus be applied to deduce that there exists an f_{t_1} whose largest root is bounded above by the largest root of $f_\emptyset$. Notice that given any $t_1, \ldots t_k \in T_1 \times \ldots \times T_k$ the polynomials $\{f_{t_1,\ldots t_k, t_{k+1}}\}$ have a common interlacing, and their sum is $f_{t_1,\ldots t_k}$. Applying the claim again verifies the existence of some t_{k+1} so that the largest root of $f_{t_1,\ldots t_k,t_{k+1}}$ is less than the largest root of $f_{t_1\ldots t_k}$. Inductively it can be deduced that there exists some choice $t_1 \ldots t_m$ so that the largest root of $f_{t_1\ldots t_m}$ is less than the largest root of $f_\emptyset$. □

Notice that if the collection $\{f_s\}_{\{s\in\{\pm 1\}^m\}}$ is verified to be an interlacing family, then this would imply that there exists an $s \in \{\pm 1\}^m$ with roots bounded above by the largest root of $f_\emptyset$. By Lemma 5.2.1 the largest root of $f_\emptyset$ coincides with the largest root of the matching polynomial.

In order to prove that $\{f_s\}_{\{s\in\{\pm 1\}^m\}}$ is an interlacing family the following two results (whose proofs are omitted here) will be used.

Proposition 5.3.4. (Proposition 1.35 in [6], and Lemma 4.5 in [29]) *Let f and G be univariate polynomials of degree n such that for all $\lambda \in [0,1]$ $\lambda f + (1-\lambda)g$ has n real roots. Then f and g have a common interlacing.*

Theorem 5.3.5. (Theorem 5.1 in [29]) *Let* $p_1, \ldots, p_m \in [0,1]$. *Then the following polynomial has real roots:*

$$\sum_{s\in\{\pm 1\}^m} \left(\prod_{i:s_i=1} p_i\right)\left(\prod_{i:s_i=-1}(1-p_i)\right) f_s(x)$$

The following result can now be proven:

Theorem 5.3.6. (Theorem 5.2 in [29]) *The collection of characteristic polynomials of the signed adjacency matrices, $\{f_s\}_{s\in\pm 1^m}$, is an interlacing family of polynomials.*

Proof. By Proposition 5.3.4 and by the definition of an interlacing family, it suffices to show that for every partial assignment $(s_1, \ldots, s_k) \in \{\pm 1\}^k$, $0 \leqslant k \leqslant m-1$, and for all $\lambda \in [0,1]$ the polynomial $\lambda f_{s_1,\ldots,s_k,1} + (1-\lambda) f_{s_1,\ldots,s_k,-1}$ has all real roots. Fix a particular partial assignment $(s_1, \ldots, s_k) \in \{\pm 1\}^k$.

Applying theorem 5.3.5 with $p_i = \frac{1-s_i}{2}$ for $0 \leqslant i \leqslant k$, $p_{k+1} = \lambda$ and $p_{k+2} = \ldots = p_m = \frac{1}{2}$ yields the following real rooted polynomial:

$$\begin{aligned}
&\lambda \sum \left(\frac{1}{2}\right)^r f_{s_1,\ldots,s_k,+1,s_{k+2}\ldots,s_m} + (1-\lambda) \sum \left(\frac{1}{2}\right)^r f_{s_1,\ldots,s_k,-1,s_{k+2}\ldots,s_m} \\
&= \left(\frac{1}{2}\right)^r \left(\lambda \sum f_{s_1,\ldots,s_k,+1,s_{k+2}\ldots,s_m} + \ (1-\lambda) \sum f_{s_1,\ldots,s_k,-1,s_{k+2}\ldots,s_m}\right)
\end{aligned}$$

where $r = m - (k+2)$ and each sum is taken over $s_i \in \{\pm 1\}$ $\forall$ $k+2 \leqslant i \leqslant m$. Since multiplying a polynomial by a real scalar leaves its roots unchanged, it is concluded that $\lambda f_{s_1,\ldots,s_k,1} + (1-\lambda) f_{s_1,\ldots,s_k,-1}$ is a real rooted polynomial. □

5.4 Main Result

Sufficient results have now been built up to prove Theorem 5.1.1. Recall the statement of the theorem: *Let G be a graph with adjacency matrix A and with universal covering tree T. Then there is a degree 2 cover of G, $\hat{G}$ so that all new eigenvalues of $\hat{G}$ are at most $\rho(T)$.*

Proof of Theorem 5.1.1. It follows from Theorem 5.3.6 and Lemma 5.3.3 that there exists some signing of G, $s \in \{\pm 1\}^m$, so that the roots of f_s are bounded above by the largest root of $f_\emptyset$. Lemma 5.2.1 implies that the roots of $f_\emptyset$ are equal to the roots of the matching polynomial, $\mu(G; x)$. It follows from Lemma 4.2.6 that the largest roots of $f_{s_1,\ldots,s_m}$ are bounded above by $\rho(T)$. □

As previously explained, this theorem verifies the existence of families of Ramanujan graphs. This will now be proven formally via the following corollary:

Corollary 5.4.1. (Theorem 5.4 in [29]) *For every $k \geqslant 3$ there is an infinite sequence of d-regular bipartite Ramanujan graphs.*

Proof. Let G be the complete bipartite graph of degree k. Since G is Ramanujan it follows from Theorem 5.1.1 and Proposition 3.2.6 that there is a signing of G so that the associated degree 2 cover has non-trivial eigenvalues bounded above by $2\sqrt{k-1}$. Since G is bipartite and Ramanujan, the degree 2 covering space is also bipartite and Ramanujan. The same reasoning is used on this degree 2 covering space to obtain yet another bipartite Ramanujan graph. Continuing to take covering spaces in an inductive manner yields an infinite family of k-regular bipartite Ramanujan graphs. □

Remark. Connectivity of the covering space is not mentioned in our proof, but it follows from Theorem 5.1.1 and Proposition 1.2.4. Indeed, if all new eigenvalues of the covering space are less than or equal to $2\sqrt{k-1}$, then since G was Ramanujan (and thus connected), all non trivial eigenvalues of $\hat{G}$ are less than or equal to $2\sqrt{k-1}$. Then by Proposition 1.2.4, $\hat{G}$ is connected.

Notice that this result can be generalized to the case where G is irregular. Given any finite bipartite Ramanujan graph G, Theorem 5.1.1 can be applied and the arguments analogous to those made above above to construct an infinite family of bipartite Ramanujan graphs via good degree 2 covers.

Chapter 6
Concluding Remarks

Starting from the basics, this thesis has provided a relatively self-contained introduction to the subject of expander graphs and has built up sufficient theory to motivate the result of Marcus, Spielman and Srivastava which confirms the existence of families of bipartite Ramanujan graphs of every degree. The thesis has communicated the significance of their findings and has brought attention to the theory that played a role in their proof. Diverse technical results used by Marcus et al. in their arguments were reviewed, including the theory of matching polynomials and eigenvalues of degree two covering graphs.

Expander graphs are shown to be a very comprehensive area of study. The thesis outlines connections that expander graphs have with numerous other fields in mathematics, including group theory, combinatorics, and number theory. Expander graphs are a very active area of research, and it seems likely that they will continue to be so for some time.

Many questions remain to be answered after Marcus et al.'s breakthrough, including the existence of infinite families of non-bipartite Ramanujan graphs of every degree. Although it is tempting to think that a cleaver argument could easily generalize Marcus et al.'s result to the non-bipartite case, it seems the answer might be more complicated than that. Nevertheless, Marcus et al.'s result has brought us that much closer to a general answer to the question. Another research problem that follows naturally from their paper is that of explicitly defining the "good" two lifts that Marcus et al. have shown to exist. Being able to characterize these lifts in a convenient and simple manner would make it possible to construct the resulting Ramanujan graphs explicitly so they could be used in applications.

In addition to these questions, the author of this thesis is particularly interested in the following questions and areas of research:

- Problem 10.1.1 posed by Lubotzky in [23]: "What is the best [Cheeger constant] one can obtain for a family of k-regular [graphs]". Furthermore, is it possible find an explicit relationship between the discrete Cheeger constant and the spectrum of a graph? Although the Cheeger inequality confirms that the two concepts are related, it fails to reveal how.

- An interesting topic of study is that of the higher dimensional analogues to expander graphs and Ramanujan graphs. This higher dimensional theory is much less developed, and, as far as the author is aware, definitions in higher dimensions have not yet been agreed upon.
- Does there exist a meaningful higher dimensional version of the Cheeger inequality? Currently the author is only aware of the one suggested by Parzanchevski et al. in [30], whose bound turns out to be trivial in higher dimensions. It is clear that this question is not easy to answer.

These are a handful of questions in a field with many possible and worthwhile areas of research. The author hopes that this thesis would be helpful in learning about and developing an appreciation for the exciting subject of Ramanujan graphs.

Appendix A: Big Oh, Big Theta, and Big Omega Notations

All information in this section has been adapted from [8].

Big Oh Notation

A function $g(n)$ is $O(f(n))$ if there exists a real number $c > 0$ and an integer $n_0 > 0$ such that $g(n) \leqslant cf(n)$ for all $n > n_0$. In other words, $g(n)$ is $O(f(n))$ iff the graph of $g(n)$ is always below the graph of $cf(n)$ after n_0.

Big Omega Notation

A function $g(n)$ is $\Omega(f(n))$ if there exists a real number $c > 0$ and an integer $n_0 > 0$ such that $g(n) \geqslant cf(n)$ for all $n > n_0$. For example $5n^2$ is $\Omega(n)$ because $5n^2 \geqslant n$ for all $n \geqslant 1$.

Big Theta Notation

A function $g(n)$ is $\Theta(f(n))$ if there exists two real numbers $c_1 > 0$ and $c_2 > 0$ and an integer $n_0 > 0$ such that $c_1 f(n) \leqslant g(n) \leqslant c_2 f(n)$ for all $n > n_0$.

Appendix B: Proof of Cauchy's Interlacing Eigenvalue Theorem

The following proof was taken from [15].

Proof of 3.5.9. : Without loss of generality assume $A = \begin{pmatrix} a & \bar{y}^* \\ \bar{y} & B \end{pmatrix}$. Let D be the diagonal matrix with diagonal entries the eigenvalues of B. Since B is hermitian there exists a unitary matrix U satisfying $U^*BU = D$. Let $U^*\bar{y} = \bar{z} = (z_2, z_3, \dots, z_n)^T$. The result for the case where the inequalities in (3.1) and (3.2) are strict and where $z_i \neq 0 \;\forall\; i$ will be proven first. Define the unitary matrix $V := \begin{pmatrix} 1 & \bar{0}^* \\ \bar{0} & U \end{pmatrix}$. Then $V^*AV = \begin{pmatrix} a & \bar{z}^* \\ \bar{z} & D \end{pmatrix}$. Let $f(x) = \det(xI - A) = \det(xI - V^*AV)$. Calculate $f(x)$ using the cofactor expansion method and expand along the first row. Upon careful inspection one can see that doing so yields $f(x) = (x-a)(x-\mu_0)\dots(x-\mu_{n-2})\sum_{i=0}^{n-2} f_i(x)$ where $f_i(x) = |z_i|^2(x-\mu_0)\dots(\widehat{x-\mu_i})\dots(x-\mu_{n-2})$ and the hat denotes a deleted term. Notice that if $i \neq j$ then $f_i(u_j) = 0$ and

$$f_i(u_i) = \begin{cases} > 0 & \text{if } i \text{ is even (since there are an even number of negative terms)} \\ < 0 & \text{if } i \text{ is odd (since there are an odd number of negative terms)} \end{cases}$$

$$\implies f(\mu_i) = \begin{cases} < 0 & \text{if } i \text{ is even} \\ > 0 & \text{if } i \text{ is odd} \end{cases} \tag{6.1}$$

Since $f(x)$ is a polynomial of degree n with positive leading coefficient the intermediate value theorem is applied to deduce that $f(x) = 0$ has n real roots (the eigenvalues of A) satisfying $\lambda_{n-1}(A) < \mu_{n-2}(B) < \lambda_{n-2}(A) < \dots < \mu_0(B) < \lambda_0(A)$. It now remains to consider the general case where (3.1) and (3.2) need not be strict and where z_i need not be non-zero. Consider a sequence of positive numbers $\{\epsilon_i\}_{i\in\mathbb{N}}$ tending to zero such that $z_i + \epsilon_k \neq 0$ for all $i = 0, \dots n-2$ and all $k \in \mathbb{N}$, and such that the non zero entries of the following matrix $D(\epsilon_k)$ are distinct for fixed k:

$$D(\epsilon_k) = \begin{pmatrix} \mu_2 + 2\epsilon_k & & \text{\Large 0} \\ & \ddots & \\ \text{\Large 0} & & \mu_n + n\epsilon_k \end{pmatrix} = D + \epsilon_k \text{diag}(2, 3, \ldots n)$$

Define $z(\epsilon_k) = (z_2 + \epsilon_k, z_3 + \epsilon_k, \ldots, z_n + \epsilon_k)$ and $C_k = \begin{pmatrix} a & \overline{z(\epsilon_k)}^* \\ \overline{z(\epsilon_k)} & D(\epsilon_k) \end{pmatrix}$. Let $A_k = V^* C_k V$. Notice that A_k is hermitian and tends to A with k. Using the same arguments as above it can be deduced that $\lambda_{n-1}(A_k) < \mu_{n-2}(B) + n\epsilon_k < \lambda_{n-2}(A_k) < \ldots < \lambda_0(A_k) < \mu_0(B) + 2\epsilon_k < \lambda_1(A_k)$ where $\lambda_i(A_k)$ are the eigenvalues of A_k. The desired result now follows from the observation that $\lambda_i(A_k) \to \lambda_i(A)$ as $k \to \infty$. $\square$

Bibliography

[1] A. S. Asratian. *Bipartite Graphs and their Applications*. Cambridge University Press, 1998.

[2] N. Alon, and V.D. Milman. λ_1*, isoperimetric inequalities for graphs and superconcentrators*. J. Comb. Th. **B 38** (1985),78-88.

[3] H. Bass, and J. Tits. Discreteness criteria for certain tree automorphism groups, Appendix of H. Bass and A. Lubotzky *Tree Lattices*. Progress in Mathematics 176, Birkhäuser, Boston, 2000.

[4] Y. Bilu and N. Linial. Lifts, discrepancy and nearly optimal spectral gap. *Combinatorica*, 26 (2006), pp. 495âĂŞ519.

[5] G. Davidoff, P. Sarnak and A. Valette, *Elementary Number Theory, Group Theory, and Ramanujan Graphs*. Cambridge University Press, New York: 2003. `http://www2.unine.ch/repository/default/content/sites/math/files/shared/documents/articles/valette22-04-02.pdf` (retrieved June 4, 2014)

[6] S. Fisk. *Polynomials, roots, and interlacing.* arXiv:math/0612833, 2008.

[7] J. Friedman. *The spectra of infinite hypertrees*. SIAM J. Comp. **20** (1991) 951-961.

[8] G.Gimel'farb. *Lecture 3: Analysing Complexity of Algorithms, Big Oh, Big Omega, Big Theta Notation*. https://`www.cs.auckland.ac.nz/courses/compsci220s1c/lectures/2013S1C/Part1/220-03.pdf` (retrieved June 4, 2014)

[9] C. D. Godsil. Matchings and walks in graphs. *Journal of Graph Theory*, 5(3): 285-297, 1981.

[10] I. Gutman and H. Hosoya, On the calculation of the acyclic polynomial. *Theoret. Chim. Acta.* 48(1978) 279-286.

[11] A. Hatcher, *Algebraic Topology*. Cambridge University Press. 2002. retrieved online: `http://www.math.cornell.edu/~hatcher/AT/AT.pdf`

[12] O. J. Heilman and E. H. Lieb. Theory of monomer-dimer systems, *Comm Math Phys* 25 (1972), 190âĂŞ232. MR0297280 (45; 6337)

[13] S. Hoory, N. Linial, A. Wigderson. *Expander graphs and their applications*. Bull. Amer. Math. Soc. (N.S.) 43 (2006) 439âĂŞ561 (electronic).

[14] J. Hunter, B. Nachtergaele. *Applied Analysis*. World Scientific. 2001. Retrieved online: `https://www.math.ucdavis.edu/~hunter/book/pdfbook.html`

[15] S. Hwang. Cauchy's interlace theorem for eigenvalues of Hermitian matrices, Amer. Math. Monthly 111 (2004) 157âĂŞ159.

[16] R. M. Karp. An introduction to randomized algorithms. *Discrete Applied Mathematics*. 34 (1991) pp 165- 121. `http://users.eecs.northwestern.edu/~nickle/randAlg/Karp91.pdf`

[17] D. Kazhdan. Connection of the dual space of a group with the structure of its closed subgroups. *Funct. Anal. Appl.* **1** (1967). 63-65.

[18] M. Keller, D. Lenz, S. Warzel. On the spectral theory of trees with finite cone type. Israel J. Math. (2012). doi:10.1007/s11856Ð012-0059Ð3 MR3047064

[19] T. Kotek. Definability of combinatorial functions. PhD thesis, Technion - Israel Institute of Technology, Haifa, Israel, 2012, submitted.

[20] M. Krebs, A. Shaheen. *Expander families and Cayley graphs: A beginners guide*. Oxford University Press. New York, 2011.

[21] F. T. Leighton. Finite common coverings of graphs. *Journal of Combinatorial Theory*. Series B 33 (1982), 231âĂŞ238.

[22] L. Lovász, J. Pelikan. On the eigenvalues of trees. *Periodica Math*. Hung. 3 (1973) 175âĂŞ182. MR0416964 (54; 5026)

[23] A. Lubotzky. *Discrete groups, expanding graphs and invariant measures*. volume 125 of *Progress in Mathematics.* Birkhäuser Verlag, Basel, 1994.

[24] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261âĂŞ 277, 1988.

[25] A. Lubotzky, T. Nagnibeda. Not Every Uniform Tree Covers Ramanujan Graphs. *J. Combin. Theory Ser. B* **74** (1998), 202-212.

[26] A. Madry *CS-621 Theory Gems: Lecture 9*. October 17, 2012. Retreived from `http:`

//thl.epfl.ch/gems/notes/lecture9.pdf on June 11, 2014.

[27] G. A. Margulis. Explicit group theoretical constructions of combinatorial schemes and their application to the design of expanders and concentrators. *Problems of Information Transmission*, 24.1 (1988): 51-60.

[28] M. Morgenstern. Existence and explicit construction of $q+1$-regular Ramanujan graphs for every prime power q *J. Combin. Theory Ser. B* **62**(1) (1994), 44-62.

[29] A. Marcus, D. A. Spielman, N. Srivastava. Interlacing families I: bipartite Ramanujan graphs of all degrees, submitted (2013). http://arxiv.org/abs/1304.4132

[30] O. Parzanchevski, R. Rosenthal, and R.J. Tessler. Isoperimetric inequalities in simplicial complexes. *arXiv preprint arXiv:1207.0638v3*, 2012.

[31] P. Sarnak. *Some applications of modular forms.* Cambridge University Press, New York: 1990.

[32] M. Sipser and D. Spielman. Expander Codes. *IEEE Transactions on Information Theory*, 42(6): 1710- 1722, 1996.

[33] D. Spielman. Constructing Error-Correcting Codes from Expander Graphs. *Emerging Applications of Number Theory*, Springer New York. pp 591-600, 1999. http://link.springer.com/chapter/10.1007%2F978-1-4612-1544-8_25#page-1

[34] T. Sunada. Fundamental groups and Laplacians. *Lecture Notes in Math*. Vol. 1339, Springer-Verlag, N.Y., 1988 pp. 252.

[35] T. Tao. *254B, Notes 1: Basic theory of expander graphs*. http://terrytao.wordpress.com/2011/12/02/245b-notes-1-basic-theory-of-expander-graphs/ (Retrieved Jan 15 2013)

[36] T. Tao, *254B, Notes 2: Cayley graphs and Kazhdan's Property (T)*. http://terrytao.wordpress.com/2011/12/06/254b-notes-2-cayley-graphs-and-kazhdans-property-t/ (retrieved Jan 30 2014)