THE LAGUERRE-SAMUELSON INEQUALITY WITH EXTENSIONS AND APPLICATIONS IN STATISTICS AND MATRIX THEORY

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Much of the biographical information was obtained by visiting the excellent O'Connor-Robertson Internet website [185], while Web access to the databases MathSciNet (for *Mathematical Reviews*) and MATH Database (for *Zentralblatt für Mathematik*) has been of great help in compiling our bibliography. This research was supported in part by a research grant from the Natural Sciences and Engineering Research Council of Canada.

Abstract.

We examine an 1880 theorem of Laguerre concerning polynomials with all real roots and a 1968 inequality of Samuelson for the maximum and minimum deviation from the mean, and establish their equivalence and present several proofs. We also study related inequalities attributed to

- J. M. C. Scott (1936)
- Brunk (1959)
- Boyd (1971) & Hawkins (1971).

Also examined is a 1918 inequality of Szökefalvi-Nagy and some 1935 extensions of Popoviciu concerning the standard deviation and range of a set of real numbers and equivalent inequalities for the internally Studentized range due to K. R. Nair in 1947/1948 and G. W. Thomson in 1955, as well as related bounds on the standard deviation attributed to

- Guterman (1962)
- Mărgăritescu-Vodă (1983)
- Bhatia-Davis (1999).

Extensions and applications in statistics and matrix theory are provided, as well as biographical information and an extensive bibliography.

Résumé.

Nous étudions un théorème de Laguerre (1880) au sujet des polynômes dont toutes les racines sont réelles et une inégalité de Samuelson (1968) au sujet de l'écart maximal et minimal de la moyenne et établissons leur équivalence et présentons plusiers preuves. Nous étudions également des inégalités connexes attribuées à

- J. M. C. Scott (1936)
- Brunk (1959)
- Boyd (1971) et Hawkins (1971).

Sont également étudiées une inégalité de Szőkefalvi-Nagy (1918) ainsi que des extensions de Popoviciu (1935) au sujet de l'écart-type et de l'étendue de nombres réels et des inégalités équivalentes pour l'étendue transformée de Student obtenues par K. R. Nair (1947/1948) et G. W. Thomson (1955). de même que des bornes de l'écart-type s'y rapportant. attribuées à

- Guterman (1962)
- Mărgăritescu-Vodă (1983)
- Bhatia-Davis (1999).

Des extensions et des applications aux statistiques et à la théorie des matrices sont présentées. de même que des informations biographiques et une bibliographie détaillée.

Table of Contents.

1. Introduction and Overview

The purpose of this thesis is to examine an inequality. most commonly attributed to P. A. Samuelson (1968) in the statistical literature. for the maximum and minimum deviation from the mean, given a set of n observations with known standard deviation. As observed by Arnold and Balakrishnan [6] the publication by Samuelson. "How deviant can you be?" in the *Journal of the American Statistical Association* [216] "... spawned a torrent of generalizations. several of which referred to bounds on order statistics. It also spawned a flurry of rediscoveries of earlier notes on these topics. Ultimate priority seems hard to pin down ..."

We will establish the equivalence of Samuelson's Inequality and an 1880 theorem of E. N. Laguerre [109]. virtually unnoticed in the statistical literature. concerning polynomials with all real roots. The bounds provided by Laguerre's Theorem involve the first three coefficients of an n-th degree polynomial while Samuelson's Inequality is in terms of the standard deviation (and the mean) of a set of n real numbers (observations). Several proofs of this Laguerre-Samuelson inequality will be given and the associated literature surveyed.

Related inequalities attributed to H. D. Brunk (1959) [43]. D. M. Hawkins (1971) [86]. A. V. Boyd (1971) [39]. and J. M. C. Scott (1936) [222] will also be examined. Of special note are inequalities due to J. von Szökefalvi Nagy [234] in 1918 and T. Popoviciu [205] in 1935 concerning bounds on the variance and range of a set of real numbers. and equivalent inequalities for the internally Studentized range[1] due to K. R. Nair (1947/1948) [168]. [171] and G. W. Thomson (1955) [244]. These inequalities are also presented with proofs. Other bounds for the variance. due to Bhatia and Davis (1998) [33]. Guterman (1962) [82]. and Mărgăritescu and Vodă (1983) [131] are also studied.

Separate sections are devoted to various extensions and applications of these inequalities in statistics, polynomials. and matrix theory. Also included is some historical and biographical information and an extensive bibliography with over 225 entries.

[1] For a discussion concerning internally Studentized and externally Studentized range. see [57].

2. The Laguerre-Samuelson and Related Inequalities.

2.1. The Laguerre-Samuelson Inequality.

Throughout this thesis $x_1, x_2, \ldots, x_n$ will denote n real numbers with (arithmetic) mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{2.1}$$

and standard deviation (with divisor n):

$$s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} = \sqrt{\frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right)}. \tag{2.2}$$

Then

$$\bar{x} - s\sqrt{n-1} \leq x_j \leq \bar{x} + s\sqrt{n-1} \qquad \text{for all } j = 1, 2, \ldots, n \tag{2.3}$$

or equivalently

$$(x_j - \bar{x})^2 \leq (n-1)s^2 \qquad \text{for all } j = 1, 2, \ldots, n. \tag{2.4}$$

Equality holds in (2.4) if and only if all the x_i other than x_j are equal and so then x_j is either the largest or the smallest of the x_i: equality holds on the left (right) of (2.3) if and only if the $n-1$ largest (smallest) x_i are all equal.

We see, therefore, that given the mean and standard deviation of a set of real numbers, their minimum is bounded below and their maximum bounded above. These bounds are often referred to as "Samuelson's Inequality" in the statistical literature[2] in view of the inequalities established in 1968 by the American economist and Nobel laureate Paul Anthony Samuelson[3] (b. 1915) in the *Journal of the American Statistical Association* [216].

[2] Cf. e.g., Arnold [4], Borwein, Styan and Wolkowicz [37], Chaganty and Vaish [52], Farnum [68], Kabe [97], Mărgăritescu [128], Mathew and Nordström [134], Murty [166], Patel, Kapadia and Owen [194] (p. 263), Puntanen [212] (Example 6.16, pp. 275–276), and Wolkowicz and Styan [256].

[3] For biographical information, see §7.5.

The inequalities (2.3) were (almost certainly first) established in 1880 by the well-known French mathematician Edmond Nicolas Laguerre[4] (1834–1886) in the *Nouvelles Annales de Mathématiques (Paris)* [109]. Laguerre's results were obtained in a completely different notation and context[5].

Laguerre's interest focused on n-th degree polynomials with all roots real. Let $x_1, x_2, \ldots, x_n$ denote the roots, all of which we will assume to be real, of the n-th degree polynomial equation with $n \geq 2$:

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0. \tag{2.5}$$

Since we will assume that this polynomial has degree n we will now suppose, without loss of generality, that

$$a_0 = 1. \tag{2.6}$$

Let

$$t_1 = \sum_{i=1}^{n} x_i \quad \text{and} \quad t_2 = \sum_{i=1}^{n} x_i^2. \tag{2.7}$$

Then

$$a_1 = -\sum_{i=1}^{n} x_i = -t_1 \quad \text{and} \quad a_2 = \sum_{i<j} x_i x_j = \tfrac{1}{2}(t_1^2 - t_2). \tag{2.8}$$

Laguerre [109] proved that

$$-\frac{a_1}{n} - b\sqrt{n-1} \leq x_j \leq -\frac{a_1}{n} + b\sqrt{n-1} \quad \text{for all } j = 1, 2, \ldots, n. \tag{2.9}$$

where

$$b = \sqrt{\frac{(n-1)a_1^2}{n^2} - \frac{2a_2}{n}} = \frac{\sqrt{nt_2 - t_1^2}}{n} \tag{2.10}$$

using (2.8). It follows at once that

$$-\frac{a_1}{n} = \bar{x} \quad \text{and} \quad b = s. \tag{2.11}$$

[4] For biographical information, see §7.1.

[5] While several authors in the mathematical literature refer to Laguerre (cf. e.g., Lupaş [117], Madhava Rao and Sastry [120], Mitrinović [157], pp. 210–211, Popoviciu [205], Sz.-Nagy [234], [235], [236], and Weber [252], pp. 364–371), the only author who we could find in the statistical literature to do so was Rodica-Cristina Vodă [246] in 1983 (in Romanian), who also references Mihăileanu [152].

respectively the mean and the standard deviation defined in (2.1) and (2.2) above, and so the inequalities (2.9) coincide with (2.3).

Laguerre [109], however, did not observe that $-a_1/n$ and b were in fact the mean and standard deviation[6] of the roots x_i: his interest was in obtaining bounds for the roots, whenever they are all real, of an n-th degree polynomial given the first three coefficients—in our formulation the first of these: $a_0 = 1$, cf. (2.6)[7].

In this paper we will, therefore, refer to the inequalities (2.3) or (2.4) as the "Laguerre-Samuelson Inequality".

While "Samuelson's Inequality" is certainly the most popular name for (2.3), the name "Extreme Deviations Inequality" is also used in the (relatively recent) statistical literature[8]: in 1974 Arnold used "extreme deviance" in the title of his paper [4], while "How deviant can you be?" is the title of the seminal paper by Samuelson (1968) [216]; the 1992 survey paper by Olkin [187] is entitled "A matrix formulation on how deviant an observation can be". Much earlier, however, the term "extreme deviate" appears in the title of the 1948 paper by Nair [170] and "extreme observation" in the titles of the papers by Hartley and David (1954) [84] and McKay (1935) [135]. In the hydrology journal *Water Resources Research*, Kirby (1974) [102] uses "standardized maximum deviate".

Wolkowicz and Styan (1988) call (2.3) the "Samuelson-Nair Inequality" in their *Encyclopedia of Statistical Sciences* entry [260], while Arnold and Balakrishnan in their 1989 monograph *Relations, Bounds and Approximations for Order Statistics* [6] present many inequalities related to and including the Laguerre-Samuelson Inequality in their Section 3.2 entitled "Variations on the Samuelson-Scott theme"[9].

In the naming of inequalities (2.3) or (2.4), it is difficult to give proper credit to all discoverers (and rediscoverers) without the name becoming rather cumbersome. Two additional researchers deserve special note for their work with the Laguerre-Samuelson inequality. The Indian statistician Keshavan Raghavan Nair[10] (b. 1910)

[6]The term "standard deviation" was introduced in 1893 (by Karl Pearson (1857–1936) "in a lecture to the Royal Society", cf. Hart [83], p. 626; Stigler [229], p. 328, "although the idea was by then nearly a century old", cf. Abbott [1], p. 105.

[7]Laguerre [109] did not assume that $a_0 = 1$ and so his results involve a_1/a_0 and a_2/a_0 instead of our a_1 and a_2.

[8]Cf. Dwass (1975) [66], O'Reilly (1976) [189], and Quesenberry (1974) [213].

[9]Cf. [6], Theorem 3.3, pp. 45–46, for six proofs of the Laguerre-Samuelson Inequality

[10]For biographical information, see §7.4.

established the Laguerre-Samuelson Inequality (2.3) in his 1947 Ph.D. thesis [168], publishing his proof a year later in 1948 in the *Journal of the Indian Society of Agricultural Statistics* [171], cf. also Nair [176], [177]. J. M. C. Scott[11] established several inequalities (see §2.4 below) on ordered absolute deviations $|x_j - \bar{x}|$ in the Appendix to the 1936 paper [196] by Egon Sharpe Pearson (1895–1980), assisted by C. Chandra Sekar in *Biometrika (London)*: as noted by Arnold and Balakrishnan [6] (Theorem 3.2, p. 44), the Laguerre-Samuelson Inequality is a special case of one of Scott's inequalities.

2.2 The Brunk Inequalities.

Now let us arrange the x_i's in nondecreasing order:

(2.12) $$x_{(1)} = x_{\max} \geq x_{(n-1)} \geq \cdots \geq x_{(2)} \geq x_{\min} = x_{(n)}$$

so that $x_{(j)}$ is the j-th largest. Then

(2.13) $$\bar{x} + \frac{s}{\sqrt{n-1}} \leq x_{\max} = x_{(1)} \leq \bar{x} + s\sqrt{n-1}$$

and

(2.14) $$\bar{x} - s\sqrt{n-1} \leq x_{\min} = x_{(n)} \leq \bar{x} - \frac{s}{\sqrt{n-1}}.$$

The right-hand inequality in (2.13) and the left-hand inequality in (2.14) are the Laguerre-Samuelson inequality (2.3). The left-hand inequality in (2.13) and the right-hand inequality in (2.14) were established (possibly for the first time) in 1959 by Hugh Daniel Brunk (b. 1919), also in the *Journal of the Indian Society of Agricultural Statistics* [43], and so we will refer to them as the "Brunk Inequalities". Unaware of Brunk's results these inequalities were established again by Boyd (1971) [39], Hawkins (1971) [86] and Wolkowicz and Styan (1979) [256], as well as by Lupaş (1977) [117], who considered bounds for the roots of an n-th degree polynomial with all real roots.

[11] From four papers [223], [224], [225], and [226], we infer that J. M. C. Scott was at the Cavendish Laboratory, Cambridge, in the mid-1950s. We have no further biographical information.

Equality holds on the left of (2.13) if and only if equality holds on the left of (2.14) if and only if

$$x_{(1)} = \cdots = x_{(n-1)} \tag{2.15}$$

and then

$$x_{(1)} = \cdots = x_{(n-1)} = \bar{x} + \frac{s}{\sqrt{n-1}} \quad \text{and} \quad x_{(n)} = \bar{x} - s\sqrt{n-1}. \tag{2.16}$$

Equality holds on the right of (2.13) if and only if equality holds on the right of (2.14) if and only if

$$x_{(2)} = \cdots = x_{(n)} \tag{2.17}$$

and then

$$x_{(1)} = \bar{x} + s\sqrt{n-1} \quad \text{and} \quad x_{(2)} = \cdots = x_{(n)} = \bar{x} - \frac{s}{\sqrt{n-1}}. \tag{2.18}$$

2.3 The Boyd-Hawkins Inequalities.

For the k-th largest observation or "order statistic" $x_{(k)}$ we have the following inequalities

$$\bar{x} - s\sqrt{\tfrac{k-1}{n-k+1}} \leq x_{(k)} \leq \bar{x} + s\sqrt{\tfrac{n-k}{k}} \qquad \text{for } k = 2, \ldots, n-1. \tag{2.19}$$

Equality holds on the left of (2.19) if and only if

$$x_{(1)} = \cdots = x_{(k-1)} \quad \text{and} \quad x_{(k)} = \cdots = x_{(n)} \tag{2.20}$$

and then

$$x_{(1)} = \cdots = x_{(k-1)} = \bar{x} + s\sqrt{\tfrac{n-k+1}{k-1}} \quad \text{and} \quad x_{(k)} = \cdots = x_{(n)} = \bar{x} - s\sqrt{\tfrac{k-1}{n-k+1}}.$$

Equality holds on the right of (2.19) if and only if

$$x_{(1)} = \cdots = x_{(k)} \quad \text{and} \quad x_{(k+1)} = \cdots = x_{(n)} \tag{2.21}$$

and then

$$x_{(1)} = \cdots = x_{(k)} = \bar{x} + s\sqrt{\tfrac{n-k}{k}} \quad \text{and} \quad x_{(k+1)} = \cdots = x_{(n)} = \bar{x} - s\sqrt{\tfrac{k}{n-k}}.$$

If we put $k = 1$ in (2.19) then we obtain the same upper bound for $x_{\max} = x_{(1)}$ as in (2.13) but a weaker lower bound. Similarly, if we put $k = n$ in (2.19) then we obtain the same lower bound for $x_{\min} = x_{(n)}$ as in (2.14) but a weaker upper bound.

The inequalities (2.19) were established (possibly for the first time[12]) in 1971 by A. V. Boyd [39] in the *Publikacije Elektrotehničkog Fakulteta Univerziteta u Beogradu. Serija Matematika i Fizika (Belgrade)*[13] (in English) and, also in 1971, by Douglas M. Hawkins [86] in the *Journal of the American Statistical Association*: see also Wolkowicz and Styan [256], [257], [258]. As observed by Arnold and Balakrishnan [6] (p. 49) and Wolkowicz and Styan [256], the inequalities (2.19) are "implicit" in the papers by Mallows and Richter (1969) [124] and Arnold and Groeneveld (1979) [12], while Scott (1936) [222] gives (without proof) the inequality

$$x_{(2)} \leq \bar{x} + s\sqrt{\frac{n-2}{2}}, \tag{2.22}$$

the special case of the upper bound in (2.19) for $k = 2$.

A. Lupaş [117] considered an n-th degree polynomial with all real roots, and in 1977 established (2.19) as well as (2.3) for these roots. Arnold and Groeneveld (1979) [12] give similar bounds to (2.19) for the expected value of the order statistic $X_{(k)}$ taken from a sample of the random variable X, with the mean and standard deviation in (2.19) being replaced by $\mu = \mathsf{E}X$ and $\sigma = \sqrt{\mathsf{var}X}$ respectively. The sample observations need not necessarily be independently and identically distributed, but they are assumed to have common expectation and variance.

We will call (2.19) the "Boyd-Hawkins Inequalities".

2.4. The Scott Inequalities.

The first (explicit) proof of the Laguerre-Samuelson Inequality in the statistical literature was almost certainly that given in 1936 by J. M. C. Scott [222] in the Appendix to the paper by Pearson and Chandra Sekar [196]: the Laguerre-Samuelson Inequality appears there as a special case of (1.19a), the first of three inequalities below, cf. Arnold and Balakrishnan [6], Theorem 3.2, p. 44, where it is observed

[12]Rodica-Cristina Vodă [246], p. 547, comments (in Romanian) that (2.19) "este şi el inclus parţial in rezultatul lui Laguerre" (p. 547) or (in English) "can be partially derived from an old inequality due to Laguerre" (p. 548): no further details are given.

[13]The masthead of this journal also carries the French subtitle: *Publications de la Faculté d'Électrotechnique de l'Université à Belgrade. Série Mathématiques et Physique.*

that "Scott's ingenious constructive proof is apparently the only proof available in the literature."

Let us define the absolute deviations:

$$\delta_i = |x_i - \bar{x}|; \qquad i = 1, \ldots, n. \tag{2.23}$$

and let $\delta_{(i)}$ denote the i-th largest absolute deviation so that

$$\delta_{(n)} \leq \delta_{(n-1)} \leq \cdots \leq \delta_{(1)}. \tag{2.24}$$

Of course the i-th largest absolute deviation $\delta_{(i)}$ will not, in general, be equal to $|x_{(i)} - \bar{x}|$.

Then

$$\delta_{(j)} \leq s\sqrt{\frac{n(n-j)}{j(n-j)+1}} \qquad \text{for } j \text{ odd and } j \neq n. \tag{2.25}$$

$$\delta_{(n)} \leq s\sqrt{\frac{n-1}{n(n+1)}} \qquad \text{for } n \text{ odd}. \tag{2.26}$$

$$\delta_{(j)} \leq s\sqrt{\frac{1}{j}} \qquad \text{for } j \text{ even}. \tag{2.27}$$

We note that $j = 1$ in (2.25) is the Laguerre-Samuelson Inequality (2.4). The inequality (2.26) is, of course, quite different to the Brunk Inequality, cf. (2.14):

$$x_{\min} \leq \bar{x} - \frac{s}{\sqrt{n-1}}. \tag{2.28}$$

Indeed, we obtain equality in (2.26) when $(n-1)/2$ of the x_i are equal to b and all other x_i are equal to $-1/b$, where

$$b = \sqrt{\frac{n+1}{n-1}}. \tag{2.29}$$

On the other hand equality holds in (2.28) if and only if the largest $n-1$ of the x_i are equal.

2.5 The von Szökefalvi Nagy-Popoviciu and Nair-Thomson Inequalities.

We define the range as:

$$r = x_{\max} - x_{\min} = x_{(1)} - x_{(n)}. \tag{2.30}$$

Then the standard deviation and range satisfy the inequality string:

$$\frac{1}{2n}r^2 \le s^2 \le \frac{1}{4}r^2 . \tag{2.31}$$

or equivalently the inequality string:

$$4s^2 \le r^2 \le 2ns^2 . \tag{2.32}$$

As pointed out in 1959 by Brauer and Mewborn [40], the left-hand inequality in (2.31) was established (almost certainly for the first time) in 1918 by Julius von Szökefalvi Nagy [Gyula Szőkefalvi-Nagy][14] (1887–1953) in [234] and the right-hand inequality (probably for the first time) in 1935 by Tiberiu Popoviciu[15] (1906-1975) in [205].

When $n = 2$, the inequality strings (2.31) and (2.32) collapse to equality throughout. For $n \ge 3$, the left-hand inequality in (2.31) and the right-hand inequality in (2.32) are sharp: equality holds if and only if

$$x_{(2)} = \cdots = x_{(n-1)} = \tfrac{1}{2}(x_{(1)} + x_{(n)}) = \tfrac{1}{2}(x_{\max} + x_{\min}). \tag{2.33}$$

The right-hand inequality in (2.31) and the left-hand inequality in (2.32) are sharp only when n is even: equality then holds if and only if

$$x_{(1)} = \cdots = x_{(\frac{1}{2}n)} \quad \text{and} \quad x_{(\frac{1}{2}n+1)} = \cdots = x_{(n)}. \tag{2.34}$$

Popoviciu [205] showed that the right-hand inequality in (2.31) and the left-hand inequality in (2.32) may be strengthened, respectively, to:

$$s^2 \le \frac{n^2-1}{4n^2}r^2 \quad \text{and} \quad \frac{4n^2}{n^2-1}s^2 \le r^2; \qquad \text{with } n \text{ odd} \tag{2.35}$$

equality holds in (2.35) if and only if

$$x_{(1)} = \cdots = x_{(\frac{1}{2}(n-1))} \quad \text{and} \quad x_{(\frac{1}{2}(n-1)+1)} = \cdots = x_{(n)} \tag{2.36}$$

[14] For biographical information, see §7.2.
[15] For biographical information, see §7.3.

or

$$x_{(1)} = \cdots = x_{(\frac{1}{2}(n-1)+1)} \quad \text{and} \quad x_{(\frac{1}{2}(n-1)+2)} = \cdots = x_{(n)}. \tag{2.37}$$

In statistics the ratio

$$q = \frac{r}{s},$$

or its multiple $(1 - n^{-1})^{1/2}q$, is known as the "internally Studentized range", cf. David [57]; see also, e.g., Pearson and Hartley [197][16], and we may rewrite the inequality string (2.32) as

$$2 \leq q = \frac{r}{s} \leq \sqrt{2n} \tag{2.38}$$

and (2.35) as

$$\frac{2n}{\sqrt{n^2 - 1}} \leq q = \frac{r}{s}, \qquad \text{with } n \text{ odd}. \tag{2.39}$$

The right-hand inequality in (2.38) *per se* was established probably for the first time by Keshavan Raghavan Nair (1910–1996) in his 1947 Ph.D. thesis [168] (see also his 1948 paper [171] in the *Journal of the Indian Society of Agricultural Statistics*), while the left-hand inequality in (2.38) and the inequality (2.39) were established by Hugh Daniel Brunk (b. 1919) in his 1959 paper [43] (also in the *Journal of the Indian Society of Agricultural Statistics* following a correspondence with Nair, cf. [175]). These inequalities were also "given" by George William Thomson[17] in his 1955 *Biometrika* paper [244][18], which is frequently cited in the subsequent literature[19].

[16]On p. 89 in [197] it is observed that "This idea of 'Studentizing' the range [dividing by the sample standard deviation] seems to have occurred first to W. S. Gosset ['Student'] himself (see letter of 29 January 1932 quoted by E. S. Pearson [195], p. 245)."

[17]We have no biographical information on George William Thomson, except that in the mid-1950s, c.f. [242], [243], and [244], he was affiliated with the Ethyl Corporation, Detroit, Michigan.

[18]Thomson remarks (on p. 268 of [244]) that "The bounded nature ... of q has not been noted by any of the authors who have investigated this statistic", while in an "Editorial Note" it is observed (in the accompanying footnote on p. 268) that "The existence of these limits has no doubt been noticed by others ... in the correspondence leading to the joint paper by David, Hartley and Pearson [60], the first author [Herbert A. David] gave these limits in a letter of February 1954, but they were omitted in the published paper. [signed] E. S. P[earson, Editor: *Biometrika*].

[19]Cf. e.g., Arnold and Balakrishnan [6], §3.2; Chaganty and Vaish [52]; David [55], p. 190; David [56], and Olkin [187].

We will call (2.31), (2.32) and (2.35) the "von Szőkefalvi Nagy-Popoviciu Inequalities" and (2.38) and (2.39) the "Nair-Thomson Inequalities".

2.6 Upper Bounds for the Variance.

The von Szőkefalvi Nagy-Popoviciu Inequalities (2.31), (2.32) and (2.35) provide these upper bounds for the (sample) variance s^2 of n real numbers:

$$s^2 \le \frac{1}{4} r^2; \qquad \text{with } n \text{ even} \tag{2.40}$$

$$s^2 \le \frac{n^2 - 1}{4n^2} r^2; \qquad \text{with } n \text{ odd.} \tag{2.41}$$

Tighter upper bounds for the variance s^2 are, however, provided by:

$$s^2 \le \frac{1}{n} \sum_{i=1}^{n} (x_i - \tilde{x})^2. \tag{2.42}$$

$$s^2 \le (x_{\max} - \bar{x})(\bar{x} - x_{\min}). \tag{2.43}$$

$$s^2 \le \frac{k(n-k)}{n^2} r^2. \tag{2.44}$$

where $\tilde{x}$ is defined by

$$\tilde{x} = \tfrac{1}{2}(x_{(1)} + x_{(n)}) = \tfrac{1}{2}(x_{\max} + x_{\min}) \tag{2.45}$$

and k is such that $x_{(k)}$ is the smallest x_i greater than or equal to the mean $\bar{x}$. i.e..

$$x_{\max} = x_{(1)} \ge \cdots \ge x_{(k)} \ge \bar{x} \ge x_{(k+1)} \ge \cdots \ge x_{(n)} = x_{\min}. \tag{2.46}$$

H. E. Guterman[20], in a 1962 note in *Technometrics* [82], used (2.42) in the inequality string:

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \le \frac{1}{n}\sum_{i=1}^{n}(x_i - \tilde{x})^2 \le \frac{1}{4}(x_{\max} - x_{\min})^2 = \frac{1}{4}r^2 \tag{2.47}$$

to prove (2.31), and so we will refer to the right-hand side of (2.42) as the "Guterman Upper Bound" for the variance. Equality holds in (2.42) when $\bar{x} = \tilde{x}$.

We note that the right-hand inequality in (2.47) coincides with the inequality (2.43) since

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \tilde{x})^2 \ = \ \frac{1}{4}r^2 + \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 - (x_{\max} - \bar{x})(\bar{x} - x_{\min}). \tag{2.48}$$

The only place where we have found the inequality (2.43) explicitly[21] is in the (as yet unpublished) paper by Bhatia and Davis [33]. We will, therefore, refer to the right-hand side of (2.43) as the "Bhatia-Davis Upper Bound" for the variance.

Equality holds in the inequality (2.43) if and only if j of the x_i's are equal and the other $n - j$ are all equal, i.e.,

$$x_{\max} = x_{(1)} = \cdots = x_{(j)} \ge x_{(j+1)} = \cdots = x_{(n)} = x_{\min}, \tag{2.49}$$

for some $j = 1, 2, \ldots, n - 1$.

The inequality (2.44) was apparently first given explicitly in 1983 by Mărgăritescu[22] and Vodă [131], though it is implicit in Theorem 2 in Brauer and Mewborn (1959) [40]. We will, therefore, refer to the right-hand side of (2.44) as the "Mărgăritescu-Vodă Upper Bound" for the variance. Equality holds in (2.44) if and only if k of the x_i's are equal and the other $n - k$ are all equal, i.e.,

$$x_{\max} = x_{(1)} = \cdots = x_{(k)} \ge \bar{x} \ge x_{(k+1)} = \cdots = x_{(n)} = x_{\min}\,. \tag{2.50}$$

Refer to §4.2 and §4.3 for additional information on these bounds.

[20] We have no biographical information on H. E. Guterman, except that in 1962, cf. [82], he was affiliated with the U. S. Internal Revenue Service.

[21] It really is disguised in the right-hand inequality in (2.47) obtained by Guterman [82].

[22] We have been informed (on 26 April 1999) by Viorel Gh. Vodă that "Dr. Mărgăritescu unfortunately died in 1996—he was probably the best computing-skilled mathematician in Romania."

3. Proofs: The Laguerre-Samuelson and Related Inequalities.

3.1. Proofs of the Laguerre-Samuelson Inequality.

From before, we have the "Laguerre-Samuelson Inequality" (2.3):

$$\bar{x} - s\sqrt{n-1} \leq x_j \leq \bar{x} + s\sqrt{n-1} \qquad \text{for all } j = 1, 2, \ldots, n$$

or equivalently (2.4):

$$(x_j - \bar{x})^2 \leq (n-1)s^2 \qquad \text{for all } j = 1, 2, \ldots, n.$$

We present nine different proofs of this "Laguerre-Samuelson Inequality":

- 3.1.1. Laguerre (1880), Madhava Rao & Sastry (1940), Mitrinović (1970)
- 3.1.2. Thompson (1935)
- 3.1.3. Nair (1947, 1948), Kempthorne (1973), Arnold & Balakrishnan (1989)
- 3.1.4. Arnold (1974), Dwass (1975), Arnold & Balakrishnan (1989)
- 3.1.5. Arnold (1974), O'Reilly (1975, 1976), Arnold & Balakrishnan (1989), Murty (1990)
- 3.1.6. Wolkowicz and Styan (1979, 1980)
- 3.1.7. Smith(1980), Arnold & Balakrishnan (1989)
- 3.1.8. Merikoski and Wolkowicz (1985)
- 3.1.9. Olkin (1992).

Arnold and Balakrishnan [6], pp. 45–46, present six proofs, all of which are mentioned below (§3.1.3–3.1.5, 3.1.7–3.1.9). A further proof of (2.3) using optimization techniques is given in an unpublished research report by Wolkowicz (1985), while Arnold and Balakrishnan [6] imply in their Exercise 7 (p. 62) that (2.3) can also be proved using the arithmetic/geometric mean inequality. As Arnold and Balakrishnan [6] point out (p. 45): "It is instructive to ... consider several alternative proofs. The alternative proofs often suggest different possible extensions ... The Schwarz inequality[23] may be perceived to be lurking in the background of many of the proofs."

[23]Named after [Karl] Hermann Amandus Schwarz (1843–1921) for the inequality he established in 1888 in [221], pp. 343–345; the inequality was established, however, already in 1821 by [Baron] Augustin-Louis Cauchy (1789–1857) in [47], pp. 373–374, and in 1859 by Viktor Yakovlevich Bouniakowsky [Buniakovski, Bunyakovsky] (1804–1899) in [38], pp. 3–4. In this thesis we will call it the Cauchy-Schwarz Inequality, cf. (3.14) below.

3.1.1 Laguerre (1880), Madhava Rao & Sastry (1940), Mitrinović (1970).

Our first proof is that given in 1880 by Edmond Nicolas Laguerre [109], cf. also Madhava Rao and Sastry [120] and Mitrinović [157], pp. 210–211.

For any real scalar u, we have the sum of squares expansion:

$$\sum_{i=1}^{n}(u-x_i)^2 = nu^2 - 2t_1u + t_2 \geq (u-x_j)^2 = u^2 - 2x_ju + x_j^2 \tag{3.1}$$

for any particular x_j, since a sum of squared terms is always greater than or equal to any one of its summands. Here t_1 and t_2 are as in (2.7).

Rearranging (3.1), we see that for any real u,

$$(n-1)u^2 + 2(x_j - t_1)u + (t_2 - x_j^2) \geq 0. \tag{3.2}$$

Since this quadratic function in u is nonnegative, its discriminant must be nonpositive:

$$4(x_j - t_1)^2 - 4(n-1)(t_2 - x_j^2) \leq 0. \tag{3.3}$$

Rearranging and simplifying (3.3) as a quadratic in x_j yields:

$$nx_j^2 - 2t_1x_j + t_1^2 - (n-1)t_2 \leq 0 \tag{3.4}$$

and so x_j must lie in the closed interval $[\alpha_1, \alpha_2]$, where α_1, α_2 are the roots of

$$nx_j^2 - 2t_1x_j + t_1^2 - (n-1)t_2 = 0. \tag{3.5}$$

These roots α_1, α_2 are:

$$\frac{2t_1 \pm \sqrt{4t_1^2 - 4n(t_1^2 - (n-1)t_2)}}{2n} = \frac{-a_1}{n} \pm b\sqrt{n-1} \tag{3.6}$$

using (2.10) and so (2.9) is established. □

We may arrive at the inequality (3.4) more easily, however, cf. Madhava Rao and Sastry [120], since

$$\begin{aligned} -\{nx_j^2 - 2t_1x_j + t_1^2 - (n-1)t_2\} &= (n-1)(t_2 - x_j^2) - (t_1 - x_j)^2 \\ &= (n-1)\sum_{i \neq j} x_i^2 - (\sum_{i \neq j} x_i)^2 \\ &= (n-1)\sum_{i \neq j}(x_i - \hat{x})^2 \geq 0, \end{aligned}$$

cf. (2.2), where

$$\hat{x} = \frac{1}{n-1}\sum_{i \neq j} x_i \tag{3.7}$$

is the "reduced" mean of the $n-1$ roots $x_1, \ldots, x_n$ excluding x_j. □

3.1.2. Thompson (1935).

Almost certainly the first proof in a statistical context is the following proof which is implicit in the 1935 paper of William R. Thompson [241].

Let $\hat{x}$ denote the "reduced" mean of the $n-1$ real numbers $x_1, \ldots, x_n$ excluding x_j, cf. (3.7), and let $\bar{x}$ and s denote the mean and standard deviation, respectively, of all n observations, cf. (2.1) and (2.2). Then

$$\bar{x} - \hat{x} = \frac{1}{n}(x_j - \hat{x}) = \frac{1}{n-1}(x_j - \bar{x}) \tag{3.8}$$

and so

$$\begin{aligned} ns^2 &= \sum_{i=1}^{n}(x_i - \hat{x} + \hat{x} - \bar{x})^2 \\ &= \sum_{i \neq j}(x_i - \hat{x})^2 + (x_j - \hat{x})^2 - n(\hat{x} - \bar{x})^2 \\ &= \sum_{i \neq j}(x_i - \hat{x})^2 + n(n-1)(\hat{x} - \bar{x})^2 \\ &= \sum_{i \neq j}(x_i - \hat{x})^2 + \frac{n}{n-1}(x_j - \bar{x})^2 &(3.9) \\ &\geq \frac{n}{n-1}(x_j - \bar{x})^2. &(3.10) \end{aligned}$$

using (3.8). The inequality (2.4) follows at once.

This proof also shows that equality holds in (2.4) if and only if equality holds in (3.10) and this is so if and only if $x_i = \bar{x}$ for all $i \neq j$. Hence equality holds in (2.4) if and only if all the x_i other than x_j are equal.

Thompson [241] obtains (3.9) explicitly—cf. his (6) on p. 215—but apparently does not obtain the inequality (3.10). Thompson's interest focused on the distribution of the "Studentized deviations" $(x_j - \bar{x})/s$ when the "observations" $x_1, \ldots, x_n$ are independently and identically distributed as a normal random variable with unknown mean and variance. □

3.1.3. Nair (1947/48), Kempthorne (1973), Arnold & Balakrishnan (1989).

We consider the $n \times n$ orthogonal matrix $\mathbf{E} =$

$$\begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & \frac{-(n-2)}{\sqrt{(n-1)(n-2)}} & 0 \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & \frac{1}{\sqrt{n(n-1)}} & \frac{-(n-1)}{\sqrt{n(n-1)}} \end{pmatrix}.$$

the so-called Helmert matrix[24] and let $\mathbf{x} = \{x_i\}$ and $\mathbf{y} = \mathbf{E}\mathbf{x} = \{y_i\}$. Then

$$\sum_{i=1}^{n} x_i^2 = \mathbf{x}'\mathbf{x} = \mathbf{x}'\mathbf{E}'\mathbf{E}\mathbf{x} = \mathbf{y}'\mathbf{y} = \sum_{i=1}^{n} y_i^2 \geq y_1^2 + y_n^2. \tag{3.11}$$

Since

$$y_1 = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \qquad \text{and} \qquad y_n = \sqrt{\frac{n}{n-1}}(\bar{x} - x_n) \tag{3.12}$$

it follows at once from (3.11) that

$$\sum_{i=1}^{n} x_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2 + \frac{n}{n-1}(\bar{x} - x_n)^2. \tag{3.13}$$

[24] Named after Friedrich Robert Helmert (1843-1919) for the matrix he introduced in 1876 [88], cf. also Harville [85], pp. 85-86, Lancaster [110], Read [215], and Stuart and Ord [230], Example 11.3.

If we rearrange the components of the vector $\mathbf{x}$ so that x_j is in the n-th position then, with x_n replaced by x_j, (3.13) becomes (2.4).

Equality holds in (3.13) if and only if equality holds in (3.11) and this is so if and only if $y_2 = \cdots = y_{n-1} = 0$, i.e., all the x_i are equal except for x_n (which we now choose to be x_j).

This is the third proof given by Arnold and Balakrishnan [6], p. 45, and follows that given by K. R. Nair in "a small section of the third part" of his 1947 Ph.D. thesis [168] and published in 1948 [171], and by Oscar Kempthorne in a 1973 "Personal communication" [100] to Barry C. Arnold[25]. □

3.1.4. Arnold (1974), Dwass (1975), Arnold & Balakrishnan (1989).

Barry C. Arnold [4] and Meyer Dwass [66] proved (2.4) using the Cauchy-Schwarz inequality:

$$(\mathbf{a}'\mathbf{b})^2 \leq \mathbf{a}'\mathbf{a} \cdot \mathbf{b}'\mathbf{b} \tag{3.14}$$

for any $n \times 1$ real vectors $\mathbf{a}$ and $\mathbf{b}$. This is the second proof given by Arnold and Balakrishnan [6], p.45. Since $\sum_{i=1}^n (x_i - \bar{x}) = 0$, it follows that

$$x_j - \bar{x} = -\sum_{i \neq j} (x_i - \bar{x}) \tag{3.15}$$

and so

$$\begin{aligned} (x_j - \bar{x})^2 &= \left(\textstyle\sum_{i \neq j} (x_i - \bar{x})\right)^2 \\ &\leq (n-1) \sum_{i \neq j} (x_i - \bar{x})^2 \\ &= (n-1) \textstyle\sum_{i=1}^n (x_i - \bar{x})^2 - (n-1)(x_j - \bar{x})^2 \end{aligned}$$

from (3.14) with the vectors $\mathbf{a} = \{x_i - \bar{x}\}_{i \neq j}$ and $\mathbf{b} = (1, 1, \ldots, 1)'$ both $(n-1) \times 1$. Hence

$$(x_j - \bar{x})^2 \leq \frac{n-1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 = (n-1)s^2$$

[25]Cf. Arnold and Balakrishnan [6], pp. 45 & 158, and Arnold [4] where, in an acknowledgement, it is observed that: "Upon seeing an earlier draft of this note, Oscar Kempthorne supplied me with three of several alternative proofs that he derived for Samuelson's inequality".

from (3.14), and so (2.4) follows immediately. Equality holds if and only if the vectors **a** and **b** are proportional, i.e., all the x_i except for x_j are equal. □

3.1.5. Arnold (1974), O'Reilly (1975,1976), Arnold & Balakrishnan (1989), Murty (1990).

Barry C. Arnold (1974) gave a second proof in [4] which used the "hat" matrix from linear regression analysis: see also O'Reilly [188], [189], and Murty [166].

In the usual full-rank Gauss-Markov linear statistical model

$$\mathsf{E}\mathbf{y} = \mathbf{X}\boldsymbol{\beta}. \tag{3.16}$$

where E denotes (mathematical) expectation and the "model" or "design" matrix $\mathbf{X}$ is $n \times p$ with rank $p < n$. Then it is well known that the $n \times n$ "hat matrix"

$$\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' \tag{3.17}$$

is symmetric and idempotent, and hence nonnegative definite, as is the residual matrix $\mathbf{M} = \mathbf{I} - \mathbf{H}$.

We now let $p = 2$ and $\mathbf{X} = (\mathbf{e} : \mathbf{Cx})$ as in (centered) simple linear regression: here the $n \times 1$ sum vector

$$\mathbf{e} = (1, 1, \ldots, 1)'. \tag{3.18}$$

while the $n \times n$ centering matrix

$$\mathbf{C} = \mathbf{I}_n - \frac{1}{n}\mathbf{e}\mathbf{e}' \tag{3.19}$$

is symmetric and idempotent. Hence

$$\mathbf{H} = \frac{1}{n}\mathbf{e}\mathbf{e}' + \frac{1}{\mathbf{x}'\mathbf{C}\mathbf{x}}\mathbf{C}\mathbf{x}\mathbf{x}'\mathbf{C} = \frac{1}{n}\mathbf{e}\mathbf{e}' + \frac{n}{s^2}\mathbf{C}\mathbf{x}\mathbf{x}'\mathbf{C} \tag{3.20}$$

and so the j-th diagonal element of $\mathbf{M} = \mathbf{I} - \mathbf{H}$:

$$m_{jj} = 1 - \frac{1}{n} - \frac{(x_j - \bar{x})^2}{ns^2} \geq 0. \tag{3.21}$$

since $\mathbf{M}$ is nonnegative definite: the Laguerre-Samuelson Inequality (2.4) follows at once.

Equality holds in (2.4) if and only if equality holds throughout (3.21) and this is so if and only if all the elements in the j-th row (and column) of $\mathbf{M}$ are zero. i.e.. all the x_i except for x_j are equal.

The proof given by O'Reilly [188]. [189]. is similar but uses the model matrix $\mathbf{X} = (\mathbf{e} : \mathbf{x})$ as in uncentered simple linear regession. This O'Reilly proof is the fifth proof of the Laguerre-Samuelson Inequality given by Arnold and Balakrishnan [6]. p. 46. while the Arnold-Murty proof is their fourth. □

3.1.6. Wolkowicz & Styan (1979, 1980).

The proof given by Henry Wolkowicz and George P. H. Styan (1979. 1980) [256]. [258]. cf. also Bancroft [23]. Chaganty [50]. Chaganty and Vaish [51]. [52]. Neudecker and Liu [178]. Puntanen [212] (Example 6.16. pp. 275–276). and Trenkler [245]. essentially uses the following result (Lemma 2.1 in [258]. p. 475):

Lemma 3.1.6 *Let* $\mathbf{w}$ *and* $\mathbf{x}$ *be real nonnull* $n \times 1$ *vectors and let* $\bar{x}$ *and* s *be defined as in* (2.1) *and* (2.2) *above. so that* $\bar{x} = \mathbf{x}'\mathbf{e}/n$ *and* $s^2 = \mathbf{x}'\mathbf{C}\mathbf{x}/n$*. where the centering matrix* $\mathbf{C} = \mathbf{I} - \mathbf{e}\mathbf{e}'/n$ *as in* (3.19)*. with* $\mathbf{e}$ *the* $n \times 1$ *vector of ones. Then*

$$-s\sqrt{n\,\mathbf{w}'\mathbf{C}\mathbf{w}} \le \mathbf{w}'\mathbf{C}\mathbf{x} \le s\sqrt{n\,\mathbf{w}'\mathbf{C}\mathbf{w}}\,. \tag{3.22}$$

Equality holds on the left (right) of (3.22) *if and only if*

$$\mathbf{x} = c\mathbf{w} + d\mathbf{e} \tag{3.23}$$

for some scalars c *and* d *with* $c < 0$ $(c > 0)$.

Proof. The inequality string (3.22) follows at once from the Cauchy-Schwarz Inequality (3.14) with $\mathbf{a} = \mathbf{C}\mathbf{w}$ and $\mathbf{b} = \mathbf{C}\mathbf{x}$. □

If in (3.22) we now substitute

$$\mathbf{w} = \mathbf{e}_j - \mathbf{e}/n = \mathbf{h}_j. \tag{3.24}$$

say. where

$$\mathbf{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)' \tag{3.25}$$

with 1 in the j-th position, then (3.22) becomes (2.3). The equality condition $\mathbf{x} = c\mathbf{w} + d\mathbf{e} = c\mathbf{e}_j + d\mathbf{e}$ shows that equality holds in (2.4) if and only if all the x_i are equal except for x_j. □

3.1.7. Smith (1980), Arnold & Balakrishnan (1989).

Arnold and Balakrishnan [6], p. 46, give the following proof credited to William P. Smith [228], as their sixth (and last) proof of the Laguerre-Samuelson Inequality. This proof is based on the Cantelli Inequality[26], cf. e.g., Patel, Kapadia and Owen [194], p. 51.

Let X denote a random variable with mean 0 and variance 1. Then

$$\text{Prob}(X \leq u) \leq \frac{1}{1+u^2} \quad \text{if } u \leq 0 \tag{3.26}$$

$$\text{Prob}(X \geq u) \leq \frac{1}{1+u^2} \quad \text{if } u \geq 0. \tag{3.27}$$

We now suppose that X is a discrete uniform random variable with

$$\text{Prob}\left(X = \frac{x_i - \bar{x}}{s}\right) = \frac{1}{n} \quad \text{for all } i = 1, \ldots, n. \tag{3.28}$$

Then X has expectation $\mathsf{E}X = 0$ and variance $\mathsf{var}X = 1$.

If we substitute $u = (x_{\min} - \bar{x})/s < 0$ in (3.26) then it becomes

$$\frac{1}{n} \leq 1 \Bigg/ \left\{1 + \left(\frac{x_{\min} - \bar{x}}{s}\right)^2\right\}$$

and so

$$\left(\frac{x_{\min} - \bar{x}}{s}\right)^2 \leq n - 1. \tag{3.29}$$

Substituting $u = (x_{\max} - \bar{x})/s > 0$ in (3.27) gives

$$\frac{1}{n} \leq 1 \Bigg/ \left\{1 + \left(\frac{x_{\max} - \bar{x}}{s}\right)^2\right\}$$

and so

$$\left(\frac{x_{\max} - \bar{x}}{s}\right)^2 \leq n - 1. \tag{3.30}$$

[26]Named after Francesco Paolo Cantelli (1875–1966); for a biographical account see Benzi [31].

Combining (3.29) and (3.30) yields the Laguerre-Samuelson Inequality (2.4). □

3.1.8. Merikoski and Wolkowicz (1985).

J. K. Merikoski and H. Wolkowicz in [150] define a consistent perturbation as one which preserves the first two moments and the ordering of the x_i's, and then state that consistent perturbations must alternate. In other words, if $x_i > x_j > x_k$ and $x_i \uparrow$, $x_i \downarrow$ denote a positive or negative perturbation respectively, then the only consistent perturbations are $x_i \uparrow$, $x_j \downarrow$, $x_k \uparrow$ or $x_i \downarrow$, $x_j \uparrow$, $x_k \downarrow$.

Now, if we consider n numbers $x_1 \geq \cdots \geq x_n$ and suppose that $x_1 > x_i > x_n$, then apply a consistent perturbation $x_1 \uparrow$, $x_i \downarrow$, $x_n \uparrow$ to x_1, all $x_k = x_i$ and all $x_k = x_n$. This perturbation causes x_1 to increase, so $x_1 > x_i > x_n$ cannot maximize x_1. However, x_1 does have a maximum, so we must have $x_2 = \cdots = x_n$. Substituting this scenario into (2.1) and (2.2) gives

$$(x_1 - \bar{x})^2 = (n-1)s^2 \tag{3.31}$$

which is equivalent to (2.4) since $x_1 - \bar{x}$ is maximized.

The Brunk and Boyd-Hawkins inequalities, as well as (4.1) of §4.1, can be proved in a similar fashion. This proof is similar to that given by Samuelson[27] (1968) [216] and Scott (1936) [222]; cf. Arnold and Balakrishnan (1989) [6]. □

3.1.9. Olkin (1992).

Ingram Olkin, in his 1992 survey paper [187], used the following result:

(3.32)
$$c(x_j - \bar{x})^2 \leq \sum_{i=1}^{n}(x_i - \bar{x})^2 \text{ for all } j = 1, \ldots, n \quad \Longleftrightarrow \quad 0 \leq c \leq \frac{n}{n-1}.$$

To prove (3.32) we express both sides of its right-hand side as quadratic forms. Let $\mathbf{x} = (x_1, \ldots, x_n)'$, $\mathbf{e} = (1, \ldots, 1)'$ and where, cf. (3.25), $\mathbf{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)'$ with 1 in the j-th position—all $n \times 1$. We may write

$$x_j - \bar{x} = \mathbf{x}'\mathbf{h}_j \quad \text{with} \quad \mathbf{h}_j = \mathbf{e}_j - \frac{1}{n}\mathbf{e}. \tag{3.33}$$

[27] Samuelson states that "Rigorous proof of the theorem ... is a problem in (non-smooth) concave programming..." [216]

cf. (3.24) above, and so the right-hand side of (3.32) becomes

$$c(x_j - \bar{x})^2 = c\mathbf{x}'\mathbf{h}_j\mathbf{h}_j'\mathbf{x} \leq \mathbf{x}'\mathbf{C}\mathbf{x} = \sum_{i=1}^{n}(x_i - \bar{x})^2, \tag{3.34}$$

where the centering matrix $\mathbf{C}$ is defined as in (3.19). Then (3.34) holds if and only if

$$\mathbf{C} - c\mathbf{h}_j\mathbf{h}_j' = \mathbf{I}_n - \frac{1}{n}\mathbf{e}\mathbf{e}' - c\mathbf{h}_j\mathbf{h}_j' = \mathbf{I}_n - \mathbf{A}\mathbf{A}' \tag{3.35}$$

is nonnegative definite: here $\mathbf{A} = (\mathbf{e}/\sqrt{n} \,:\, \sqrt{c}\,\mathbf{h}_j)$. Since the nonzero eigenvalues of the matrices $\mathbf{AA}'$ and $\mathbf{A}'\mathbf{A}$ coincide, it follows at once that $\mathbf{C} - c\mathbf{h}_j\mathbf{h}_j'$ is nonnegative definite whenever

$$\mathbf{I}_2 - \mathbf{A}'\mathbf{A} = \mathbf{I}_2 - \begin{pmatrix} \mathbf{e}'/\sqrt{n} \\ \sqrt{c}\,\mathbf{h}_j' \end{pmatrix} (\mathbf{e}/\sqrt{n} \,:\, \sqrt{c}\,\mathbf{h}_j) = \begin{pmatrix} 0 & 0 \\ 0 & 1 - c(n-1)/n \end{pmatrix}$$

is nonnegative definite. The result (3.32) follows at once.

Substituting $c = n/(n-1)$ in the right-hand side of (3.32) gives the Laguerre-Samuelson Inequality (2.4).

Some discussion of this proof is given in [23], [50], [51], [178], and [245]—see §5.3 below for additional commentary. □

3.2. Proofs of the Brunk Inequalities.

3.2.1. Brunk (1959).

To prove the "Brunk inequalities" Brunk used the following result ([43], Corollary 1), which we find to be interesting in its own right:

Lemma 3.2.1. *Let the random variable Z be distributed over the closed interval $[0, 1]$ and let p be a nonnegative constant so that $p \leq \operatorname{Prob}(Z = 1)$. Then*

$$p\mathsf{E}Z^2 \leq (\mathsf{E}Z)^2, \tag{3.36}$$

with equality if and only if

$$\operatorname{Prob}(Z = 0) = 1 - p \quad \text{and} \quad \operatorname{Prob}(Z = 1) = p. \tag{3.37}$$

Proof. Since $0 \leq Z \leq 1$ we have $Z^2 \leq Z$ with probability one and so $\mathsf{E}Z^2 \leq \mathsf{E}Z$ and $\text{Prob}(Z = 1) \leq \mathsf{E}Z$. Combining these two inequalities yields

$$p\mathsf{E}Z^2 \leq \text{Prob}(Z = 1) \cdot \mathsf{E}Z \leq (\mathsf{E}Z)^2. \tag{3.38}$$

and (3.36) is established. Equality holds in (3.36) if and only if equality holds throughout (3.38) if and only if $p = \text{Prob}(Z = 1)$ and $Z = Z^2$ with probability one. and so the equality condition (3.37) follows at once. □

To prove the "Brunk inequalities" we now let the random variable X assume each of the n values in (2.12) with probability $1/n$. Then the random variable $Z = (x_{\max} - X)/r$. where the range $r = x_{\max} - x_{\min}$. is distributed over $[0, 1]$. The expectation $\mathsf{E}X = \bar{x}$ and the variance $\mathsf{var}X = s^2$. Hence

$$\mathsf{E}Z^2 = \mathsf{var}Z + (\mathsf{E}Z)^2 = \frac{s^2 + (x_{\max} - \bar{x})^2}{r^2} \tag{3.39}$$

and so from Lemma 3.2.1:

$$\frac{1}{n}\mathsf{E}Z^2 = \frac{s^2 + (x_{\max} - \bar{x})^2}{nr^2} \leq \frac{(x_{\max} - \bar{x})^2}{r^2}. \tag{3.40}$$

which simplifies to

$$s^2 \leq (n - 1)(x_{\max} - \bar{x})^2. \tag{3.41}$$

from which the left-hand inequality in (2.13) follows at once. Equality holds in (3.40) if and only if (3.37) holds and here this becomes (2.15).

To establish the right-hand inequality in (2.14) we repeat the above argument with $Z = (X - x_{\min})/r$. □

3.2.2. Wolkowicz & Styan (1979).

Wolkowicz and Styan [256] provided a completely algebraic (non-statistical) proof of the Brunk inequalities. Since $n(x_{\max} - \bar{x}) = \sum_{i=1}^{n}(x_{\max} - x_i)$ it follows that

$$n^2(x_{\max} - \bar{x})^2 = \left\{\sum_{i=1}^{n}(x_{\max} - x_i)\right\}^2$$

$$= \sum_{i=1}^{n}(x_{\max} - x_i)^2 + \sum_{i \neq i'}(x_{\max} - x_i)(x_{\max} - x_{i'})$$

$$\geq \sum_{i=1}^{n}(x_{\max} - x_i)^2$$

$$= \sum_{i=1}^{n}(x_{\max} - \bar{x} + \bar{x} - x_i)^2 = n\{(x_{\max} - \bar{x})^2 + s^2\}.$$

from which the left-hand inequality in (2.13) follows at once. with equality if and only if $x_{\max} = x_{(1)} = \ldots = x_{(n-1)}$ or (2.15) holds.

If $n^2(\bar{x} - x_{\min})^2$ is expanded similarily, then the right-hand inequality in (2.14) follows at once, with equality if and only if $x_{(2)} = \cdots = x_{(n)} = x_{\min}$ or (2.17) holds. □

3.3. A Proof of the Boyd-Hawkins Inequalities.

3.3.1. Wolkowicz & Styan (1979).

Possibly the simplest proof of (2.19) is that presented in 1979 by Wolkowicz and Styan [256]. We use our Lemma 2.6.1 above, a version of the Cauchy-Schwarz inequality given by Wolkowicz and Styan (Lemma 2.1 in [258], p. 475):

(3.42) $$-s\sqrt{n\,\mathbf{w}'\mathbf{C}\mathbf{w}} \leq \mathbf{w}'\mathbf{C}\mathbf{x} \leq s\sqrt{n\,\mathbf{w}'\mathbf{C}\mathbf{w}}\,,$$

where $\mathbf{w}$ and $\mathbf{x}$ are real nonnull $n \times 1$ vectors and the centering matrix $\mathbf{C} = \mathbf{I} - \mathbf{e}\mathbf{e}'/n$ as in (3.19), with $\mathbf{e}$ the $n \times 1$ vector of ones. Equality holds on the left (right) of (3.22) if and only if

(3.43) $$\mathbf{x} = c\mathbf{w} + d\mathbf{e}$$

for some scalars c and d with $c < 0$ ($c > 0$).

Now let $\mathbf{w} = \sum_{i=k}^{l} \mathbf{e}_i/(l - k + 1)$ and $\mathbf{x} = \{x_{(i)}\}$, where $\mathbf{e}_i$ is defined as in (3.25) above and

(3.44) $$x_{\min} = x_{(n)} \leq x_{(n-1)} \leq \cdots \leq x_{(2)} \leq x_{(1)} = x_{\max}\,.$$

Then $\mathbf{w}'\mathbf{C}\mathbf{x} = \bar{x}_{(k,l)} - \bar{x}$, where the "subsample mean"

(3.45) $$\bar{x}_{(k,l)} = \sum_{i=k}^{l} x_{(i)}/(l - k + 1) \qquad \text{for} \quad 1 \leq k \leq l \leq n.$$

Moreover, $\mathbf{w}'\mathbf{C}\mathbf{w} = (l-k+1)^{-1} - n^{-1}$. Hence (3.42) implies

$$\bar{x} - s\sqrt{\tfrac{k-1}{n-k+1}} \le \bar{x}_{(k,n)} \le \bar{x}_{(k,l)} \le \bar{x}_{(1,l)} \le \bar{x} + s\sqrt{\tfrac{n-l}{l}} \tag{3.46}$$

which, when $l = k$, reduces to

$$\bar{x} - s\sqrt{\tfrac{k-1}{n-k+1}} \le x_{(k)} \le \bar{x} + s\sqrt{\tfrac{n-k}{k}}$$

as in (2.19). From (3.43) we note that equality holds in (2.19) if and only if $\mathbf{x} = c\mathbf{w} + d\mathbf{e}$ for some scalars c and d. The equality conditions for (2.19) follow at once. □

3.4. Proofs of the von Szökefalvi Nagy-Popoviciu Inequalities.

3.4.1. Mărgăritescu & Vodă (1983), (1992), Bhatia & Davis (1999)

The following short and simple proof of the left-hand side of (2.31) was given by Mărgăritescu and Vodă (1983) [131], Gonzacenco, Mărgăritescu and Vodă (1992) [76], and by Bhatia and Davis (1999) [33]:

$$\begin{aligned}\sum_{i=1}^{n}(x_i-\bar{x})^2 &= (x_{\max}-\bar{x})^2 + (x_{\min}-\bar{x})^2 + \sum_{i=2}^{n-1}(x_{(i)}-\bar{x})^2 \\ &= \frac{1}{2}(x_{\max}-x_{\min})^2 + 2(\dot{x}-\bar{x})^2 + (n-2)(x^*-\bar{x})^2 + \sum_{i=2}^{n-1}(x_{(i)}-x^*)^2.\end{aligned}$$

where

$$x^* = \frac{1}{n-2}\sum_{i=2}^{n-1} x_{(i)} = \frac{n\bar{x}-2\dot{x}}{n-2}.$$

Hence

$$\begin{aligned} ns^2 &= \frac{1}{2}(x_{\max}-x_{\min})^2 + \frac{2n}{n-2}(\dot{x}-\bar{x})^2 + \sum_{i=2}^{n-1}(x_i-x^*)^2 \\ &\ge \frac{1}{2}(x_{\max}-x_{\min})^2 + \frac{2n}{n-2}(\dot{x}-\bar{x})^2. \end{aligned}$$

with equality if and only if $x_{(2)} = \cdots = x_{(n-1)}$. Thus

$$ns^2 \ge \frac{1}{2}(x_{\max}-x_{\min})^2,$$

with equality if and only if $x_{(2)} = \cdots = x_{(n-1)}$ and $\tilde{x} = \bar{x}$, i.e., if and only if (2.33) holds. □

3.4.2. Guterman (1962), Bhatia & Davis (1999)

To prove the right-hand inequality in (2.31), H. E. Guterman [82] used the inequality string (2.47):

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \le \frac{1}{n}\sum_{i=1}^{n}(x_i - \tilde{x})^2 \le \frac{1}{4}(x_{\max} - x_{\min})^2 = \frac{1}{4}r^2, \tag{3.47}$$

where

$$\tilde{x} = \tfrac{1}{2}(x_{(1)} + x_{(n)}) = \tfrac{1}{2}(x_{\max} + x_{\min}). \tag{3.48}$$

As Guterman [82] observed, the left-hand inequality in (3.47) follows at once since it is well known that $\sum_{i=1}^{n}(x_i - a)^2$ is a minimum for $a = \bar{x}$.

As mentioned above, the right-hand inequality in (3.47) coincides with the recent inequality due to Bhatia and Davis [33]:

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \le (x_{\max} - \bar{x})(\bar{x} - x_{\min}), \tag{3.49}$$

since

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \tilde{x})^2 = \frac{1}{4}(x_{\max} - x_{\min})^2 + \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 - (x_{\max} - \bar{x})(\bar{x} - x_{\min}).$$

Equality holds in the inequality (3.49), and equivalently in the right-hand inequality in (3.47), if and only if j of the x_i's are equal and the other $n - j$ are all equal, i.e.,

$$x_{\max} = x_{(1)} = \cdots = x_{(j)} \ge x_{(j+1)} = \cdots = x_{(n)} = x_{\min}, \tag{3.50}$$

for some $j = 1, 2, \ldots, n-1$. □

3.4.3. Mărgăritescu & Vodă (1983)

A second proof of the right-hand inequality in (2.31) follows from defining k so that the first k x_i's are greater than or equal to the mean $\bar{x}$ and the other $n - k$ observations are less than or equal to $\bar{x}$, i.e.,

$$x_{\max} = x_{(1)} \ge \cdots \ge x_{(k)} \ge \bar{x} \ge x_{(k+1)} \ge \cdots \ge x_{(n)} = x_{\min}. \tag{3.51}$$

As mentioned in §2.6 above (and proved in §3.5 below), Mărgăritescu and Vodă [131] showed in 1983 that

$$ns^2 \le \frac{k(n-k)}{n} r^2. \tag{3.52}$$

Equality holds in (3.52) if and only if k of the x_i's are equal and the other $n-k$ are all equal, i.e.,

$$x_{\max} = x_{(1)} = \cdots = x_{(k)} \ge \bar{x} \ge x_{(k+1)} = \cdots = x_{(n)} = x_{\min}. \tag{3.53}$$

The right-hand inequality in (2.31) and the left-hand inequality in (2.32) follows from (3.52) by applying the arithmetic-mean / geometric-mean inequality.

If $n = 2h$ is even,

$$ns^2 \le \frac{k(n-k)}{n} r^2 \le \frac{1}{n}\left(\frac{k+n-k}{2}\right)^2 r^2 = \frac{n}{4} r^2, \tag{3.54}$$

with equality in (3.52) if and only if $x_{(1)} = \cdots = x_{(k)}$ and $x_{(k+1)} = \cdots = x_{(n)}$ and equality in (3.54) if and only if $k = n/2 = h$. Together, these equality conditions are identical to those given in (2.34).

If $n = 2h+1$ is odd,

$$ns^2 \le \frac{k(n-k)}{n} r^2 \le \frac{1}{n}\frac{n^2-1}{4} r^2 \tag{3.55}$$

since $(n-2k)^2 - 1 = (n-2k-1)(n-2k+1) \ge 0$, with equality in (3.56) and (3.57) if and only if $x_{(1)} = \cdots = x_{(k)}$ and $x_{(k+1)} = \cdots = x_{(n)}$ and equality in (3.55) if and only if $k = h$ or $k = h+1$. Together, these equality conditions are identical to those given in (2.36) and (2.37). □

In §3.5 below we present a proof of (3.52) which is slightly shorter than that given by Mărgăritescu and Vodă (1983) [131]; we also give four simple proofs of (3.49) in §3.6.

3.5. A Proof of the Mărgăritescu-Vodă Bound.

A proof of (2.44) was given by Mărgăritescu and Vodă (1983) [131] for centered observations with $\bar{x} = 0$. We give an equivalent proof for uncentered observations. If we have order statistics as in (2.12), define k such that the first k order statistics are greater than the mean, and the final $n-k$ order statistics are less than the mean. Then

$$\begin{aligned}
ns^2 &= \sum_{i=1}^{k}(x_{(i)} - \bar{x})^2 + \sum_{j=k+1}^{n}(x_{(j)} - \bar{x})^2 \\
&\le \sum_{i=1}^{k}(x_{(i)} - \bar{x})(x_{\max} - \bar{x}) + \sum_{j=k+1}^{n}(x_{(j)} - \bar{x})(x_{\min} - \bar{x}) \qquad (3.56)\\
&= \sum_{i=1}^{k}(x_{(i)} - \bar{x})(x_{\max} - x_{\min}) \\
&= \left[(n-k)\sum_{i=1}^{k}(x_{(i)} - \bar{x}) + k\sum_{i=1}^{k}(x_{(i)} - \bar{x})\right]\frac{(x_{\max} - x_{\min})}{n} \\
&\le [(n-k)k(x_{\max} - \bar{x}) - k(n-k)(x_{\min} - \bar{x})]\frac{(x_{\max} - x_{\min})}{n} \qquad (3.57)\\
&= \frac{k(n-k)}{n}(x_{\max} - x_{\min})^2 \\
&= \frac{k(n-k)}{n}r^2
\end{aligned}$$

as required. □

3.6. Proofs and Extensions of the Bhatia-Davis Inequality.

The inequality (2.43) probably first appeared implicitly within the right-hand inequality in (2.47) obtained by Guterman [82] in 1962. The right-hand inequality in (2.47), along with (2.48), provides a first proof of (2.43) as mentioned in §2.6 above.

A (second) short and simple (new) proof of (2.43) follows by observing that

$$(x_{\max} - \bar{x})(\bar{x} - x_{\min}) = s^2 + \frac{1}{n}\sum_{i=1}^{n}(x_{\max} - x_i)(x_i - x_{\min}) \ge s^2. \qquad (3.58)$$

Equality holds on the right-hand side of (3.58) under the same conditions as equality in (2.43). □

A third (relatively short and simple new) proof of (2.43). follows from writing

$$\begin{aligned} ns^2 &= \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}(x_i - x_{\max} + x_{\max} - \bar{x})^2 \\ &= \sum_{i=1}^{n}(x_{\max} - x_i)^2 - n(x_{\max} - \bar{x})^2 \\ &\leq (x_{\max} - x_{\min})\sum_{i=1}^{n}(x_{\max} - x_i) - n(x_{\max} - \bar{x})^2 \\ &= n(x_{\max} - \bar{x})(x_{\max} - x_{\min} - x_{\max} + \bar{x}) = n(x_{\max} - \bar{x})(\bar{x} - x_{\min}). \end{aligned}$$

which directly leads to (2.43). □

Bhatia and Davis [33] established several more general versions of the inequality (2.43). Let the random variable X be defined. with probability one. on the interval $[m. M]$. and let the mean $\mathsf{E}X = \mu$ and variance $\mathsf{var}X = \sigma^2$. Then. cf. [33]. we have the following inequality in parallel to the Bhatia-Davis Inequality (2.43):

$$\sigma^2 \leq \sigma^2 + \mathsf{E}(M - X)(X - m) = (M - \mu)(\mu - m). \tag{3.59}$$

Equality holds in (3.59) if and only if the random variable X either equals m or M with probability one. i.e..

$$\mathsf{P}(X = m) + \mathsf{P}(X = M) = 1.$$

The Bhatia-Davis Inequality (2.43) was introduced in [33] as a special case of a more general inequality. If Φ is a positive and unital linear mapping of a C^* algebra $\mathcal{A}$ into a C^* algebra $\mathcal{B}$ and A. $m \leq A \leq M$. is a self-adjoint element of $\mathcal{A}$. then:

$$\Phi(A^2) - (\Phi A)^2 \leq (M - \Phi A)(\Phi A - m). \tag{3.60}$$

If we substitute mathematical expectation E for the positive and unital linear mapping Φ in (3.60) then it becomes (3.59). The proof of (3.60) in [33] is similar to our proof above of (3.59).

On the other hand. if we consider a discrete random variable X in (3.59) such that

$$\mathsf{P}(X = x_i) = p_i; \qquad i = 1, \ldots, n. \tag{3.61}$$

then we obtain a more general inequality than (2.43) involving a weighted variance of n observations $x_1, \ldots, x_n$,

$$s_p^2 = \sum_1^n p_i x_i^2 - \left(\sum_1^n p_i x_i\right)^2 \leq \left(M - \sum_1^n p_i x_i\right)\left(\sum_1^n p_i x_i - m\right) \tag{3.62}$$

where $0 \leq p_i \leq 1$ and $\sum_1^n p_i = 1$. If we set $p_i = 1/n$ for $i = 1, \ldots, n$ in (3.62) then it becomes the Bhatia-Davis Inequality (2.43).

The Bhatia-Davis inequality (3.62) for the weighted variance may also be written in matrix/vector notation:

$$s_p^2 = \mathbf{x}'\mathbf{D}_p\mathbf{x} - \mathbf{x}'\mathbf{p}\mathbf{p}'\mathbf{x} = \mathbf{x}'(\mathbf{D}_p - \mathbf{p}\mathbf{p}')\mathbf{x}, \tag{3.63}$$

where $\mathbf{D}_p$ is a diagonal matrix with diagonal entries p_i. It is interesting to note that $\mathbf{D}_p - \mathbf{p}\mathbf{p}'$ is well-known as the dispersion matrix associated with the multinomial distribution, cf. e.g., Example 15.3 in Stuart and Ord [230].

4. Related Statistical Topics

4.1. An Extension to Differences of Order Statistics

An extension of the right-hand inequality in (2.32) was obtained in 1981 by Fahmy and Proschan [67], who obtained an upper bound for $x_{(p)} - x_{(q)}$, the difference between the p-th and q-th largest of the x_i's, where $1 \leq p < q \leq n$, as follows:

$$x_{(p)} - x_{(q)} \leq s\sqrt{\frac{n(n-q+p+1)}{p(n-q+1)}}. \tag{4.1}$$

Equality holds in (4.1) if and only if the largest p observations are all equal, the smallest q observations are all equal, and the other $n-p-q$ observations are all equal, i.e.,

$$x_{(1)} = \cdots = x_{(p)}; \quad x_{(p+1)} = \cdots = x_{(q-1)}; \quad \text{and} \quad x_{(q)} = \cdots = x_{(n)}. \tag{4.2}$$

and then

$$x_{(p+1)} = \cdots = x_{(q-1)} = \frac{px_{(p)} + (n-q+1)x_{(q)}}{n-q+p+1}. \tag{4.3}$$

When $p = 1$ and $q = n$ then (4.1) reduces to the right-hand inequality in (2.32) and, equivalently, the left-hand inequality in (2.31).

A proof of this bound is given by Arnold and Balakrishnan (1989) [6], who used the fact that since the residual matrix of the full-rank Gauss-Markov linear statistical model is nonnegative definite,

$$\mathbf{a}'[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathbf{a} \geq 0 \tag{4.4}$$

Taking $\mathbf{X} = (\mathbf{e}, \mathbf{Z})$, we have for any $\mathbf{a}$,

$$(\mathbf{a}'\mathbf{Z})\Sigma^{-1}(\mathbf{a}'\mathbf{Z})' \leq n\sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i\right)^2 = n\sum_{i=1}^{n}(a_i - \bar{a})^2, \tag{4.5}$$

where Σ is the population variance-covariance matrix. In the one-dimensional case, this can be rewritten in terms in the ordered $x_{(i)}$'s as

$$\left|\sum_{i=1}^{n} a_i(x_{(i)} - \bar{x})\right| \leq s\sqrt{n\sum_{i=1}^{n}(a_i - \bar{a})^2} \tag{4.6}$$

Returning to $x_{(p)} - x_{(q)}$ where $1 \leq p < q \leq n$, it is clear that

$$x_{(p)} - x_{(q)} = (x_{(p)} - \bar{x}) - (x_{(q)} - \bar{x}) \tag{4.7}$$

$$\leq \frac{1}{p}\sum_{i=1}^{p}(x_{(i)} - \bar{x}) - \frac{1}{n-q+1}\sum_{i=q}^{n}(x_{(i)} - \bar{x}) \tag{4.8}$$

Substituting $a_1 = \cdots = a_p = 1/p$, $a_{p+1} = \cdots = a_{q-1} = 0$, and $a_q = \cdots = a_n = -1/(n-q+1)$ into (4.6) leads immediately to (4.1).

Substituting the case $q = n$ and $p = 1$ into (4.1) gives the right-hand side of (2.32). As observed by both Fahmy and Proschan [67] and Arnold and Balakrishnan (1989) [6], it is not possible to generalize the left-hand side of (2.32) to provide non-trivial lower bounds for the differences of order statistics. A related topic is the ratio of order statistics $x_{(p)}/x_{(q)}$, which was studied from a matrix perspective by Merikoski, Styan and Wolkowicz [144].

As mentioned by Fahmy and Proschan [67], Arnold and Groeneveld [12] in 1979 gave similar bounds for the expected difference of order statistics taken from a sample with common expectation $\mathsf{E}X$ and variance σ^2. Then, for $1 \leq p < q \leq n$ as above,

$$\mathsf{E}[X_{(p)} - X_{(q)}] \leq \sigma\sqrt{\frac{n(n-q+p+1)}{p(n-q+1)}}. \tag{4.9}$$

We are grateful to Herbert A. David for pointing out to us that the "unified approach given in §2 of David [56]" may be used to strengthen the inequality (4.9) to

$$\mathsf{E}[X_{(p)} - X_{(q)}] \leq \sigma\sqrt{\frac{(n-1)(n-q+p+1)}{p(n-q+1)}}. \tag{4.10}$$

4.2. A Comparison of the Bhatia-Davis and Guterman Bounds.

Even though the inequality strings (2.42) and (2.43) are both bounded below by s^2 and above by $\frac{1}{4}r^2$, the values for each upper bound for the variance are, in general, different. For the Guterman inequality (2.42):

$$s^2 \leq \frac{1}{n}\sum_{i=1}^{n}(x_i - \tilde{x})^2 = s^2 + (\tilde{x} - \bar{x})^2 \tag{4.11}$$

while for the Bhatia-Davis inequality (2.43):

$$s^2 \leq (x_{\max} - \bar{x})(\bar{x} - x_{\min}) = \frac{1}{4}r^2 - (\tilde{x} - \bar{x})^2. \tag{4.12}$$

Thus, we can see that the relative tightness of the Guterman and Bhatia-Davis inequalities as upper bounds for the variance (and correspondingly, lower bounds for the range) is a function of the magnitude of the $(\tilde{x} - \bar{x})$ term.

The Guterman inequality as an upper bound for the variance is "better" than the Bhatia-Davis inequality, and correspondingly, the Bhatia-Davis inequality as a lower bound for the range is "better" than the Guterman inequality whenever $\bar{x}$ is "close" to $\tilde{x}$. In the case when $\bar{x}$ is not "close" to $\tilde{x}$, the opposite holds.

$\bar{x}$ is "close" to $\tilde{x}$, for example, when the n numbers are symmetrically distributed and $x_{\max}$ and $x_{\min}$ are virtually the same distance from the mean. $\bar{x}$ is not "close" to $\tilde{x}$ when the n numbers have a skewed distribution and $x_{\max}$ and $x_{\min}$ have significantly different distances from the mean. Equality between the Guterman and Bhatia-Davis bounds obviously occurs when $\tilde{x} = \bar{x}$.

4.3. A Comparison of the Mărgăritescu-Vodă and Bhatia-Davis Bounds.

Similarly to §4.2 above, even though the inequality strings (2.42) and (2.43) are both bounded below by s^2 and above by $r^2/4$, the values for each upper bound for the variance are, in general, different.

For the Mărgăritescu-Vodă inequality (2.44):

$$s^2 \le \frac{k(n-k)}{n^2} r^2 \; = \; \frac{1}{4} r^2 - \frac{(n-2k)^2}{4n^2} r^2 \tag{4.13}$$

while for the Bhatia-Davis inequality (2.43):

$$s^2 \le (x_{\max} - \bar{x})(\bar{x} - x_{\min}) \; = \; \frac{1}{4} r^2 - (\tilde{x} - \bar{x})^2. \tag{4.14}$$

So we have for the Bhatia-Davis bound, denoted β_{BD} and the Mărgăritescu-Vodă bound, denoted β_{MV},

$$\begin{aligned}
\beta_{BD} - \beta_{MV} &= \frac{(n-2k)^2}{4n^2} r^2 - (\bar{x} - \tilde{x})^2 \\
&= \left(\frac{n-2k}{2n} r - (\tilde{x} - \bar{x}) \right) \left(\frac{n-2k}{2n} r + (\tilde{x} - \bar{x}) \right) \\
&= \left(\bar{x} - \frac{k x_{\max} + (n-k) x_{\min}}{n} \right) \left(\frac{(n-k) x_{\max} + k x_{\min}}{n} - \bar{x} \right).
\end{aligned}$$

Now, if we set

$$a = \frac{k x_{\max} + (n-k) x_{\min}}{n} \quad \text{and} \quad b = \frac{(n-k) x_{\max} + k x_{\min}}{n}, \quad \text{then}$$

$$\beta_{\mathrm{BD}} - \beta_{\mathrm{MV}} = (\bar{x} - a)(b - \bar{x}). \tag{4.15}$$

Regardless of whether a or b is greater, the Bhatia-Davis bound will be greater than the Mărgăritescu-Vodă bound whenever $\bar{x}$ lies in the open interval between a and b, so that (4.15) is positive. Thus, the Mărgăritescu-Vodă inequality as an upper bound for the variance is "better" than the Bhatia-Davis inequality, and correspondingly, the Bhatia-Davis inequality as a lower bound for the range is "better" than the Mărgăritescu-Vodă inequality whenever $\bar{x}$ lies in the open interval between a and b. In the case when $\bar{x}$ does not lie in this interval, the opposite holds. Equality between the Bhatia-Davis and the Mărgăritescu-Vodă bound occurs when either

$$\bar{x} = \frac{kx_{\max} + (n-k)x_{\min}}{n} \quad \text{or} \quad \bar{x} = \frac{(n-k)x_{\max} + kx_{\min}}{n} \tag{4.16}$$

One case in which this equality condition is met is when $k = n/2$ and $\tilde{x} = \bar{x}$. In fact, under this condition (2.42), (2.43) and (2.44) are all equal.

A comparison of the Bhatia-Davis and Mărgăritescu-Vodă bounds was also performed via simulation by generating random numbers from several different known distributions and then calculating the Bhatia-Davis bound and the Mărgăritescu-Vodă bound for 1000 trials, each of 100 random numbers. The random number generation was based on algorithms presented by L. Devroye [64] and P. L'Ecuyer [111]. The results are summarized in Table 1 below, in which β_{BD} signifies the Bhatia-Davis bound and β_{MV} signifies the Mărgăritescu-Vodă bound, as before. The most striking result is the complete dominance of the Mărgăritescu-Vodă bound over the Bhatia-Davis bound for observations from the exponential distribution, which is most likely a consequence of the asymmetry of this distribution. These results are still under investigation.

Distribution	Proportion of Trials where $\beta_{MV} > \beta_{BD}$	Proportion of Trials where $\beta_{MV} < \beta_{BD}$
Uniform [0,1]	0.498	0.502
Normal [0,1]	0.709	0.291
Exponential ($\mu = 1$)	1.000	0.000

Table 1: Bound Comparison on Numbers from Known Distributions

4.4. A Discussion of the Mărgăritescu-Vodă k.

The integer k was introduced by Mărgăritescu and Vodă in their 1983 paper [131] as the number of observations greater than or equal to the mean in a sample of n observations, so that $1 \leq k \leq n$. Even earlier in 1977, A. Lupaş [117] introduced an integer k_0, defined similarily, in a non-statistical but related paper exploring bounds for real roots of polynomials. In this paper, k_0 is defined as the ordinal number of real roots greater than or equal to the mean of the n real roots of an n-th degree polynomial, and is involved with a proof of the Laguerre-Samuelson inequality. Despite the fact that this ordinal number of observations above the mean has attracted very little attention in the statistical literature, it is worthy of consideration as an accompaniment to the classical mean, median, and mode of a finite sample.

In the simulation study mentioned in §4.3, in addition to comparisons between the Mărgăritescu-Vodă and Bhatia-Davis bounds for 1000 trials each of 100 random numbers generated from a known distribution, the integer k was also calculated for each of these 1000 trials. From this data, a frequency plot of k, $1 \leq k \leq 100$ is obtained for the 1000 trials from each randomly-generated distribution, and is given in Figures 1–3 below. The most striking contrast is the horizontal shift in the frequency plot of k for the exponentially-distributed numbers, which is most likely a consequence of the asymmetry of this distribution.

Figure 1: Frequency of k in Trials from a Standard Uniform Sample.

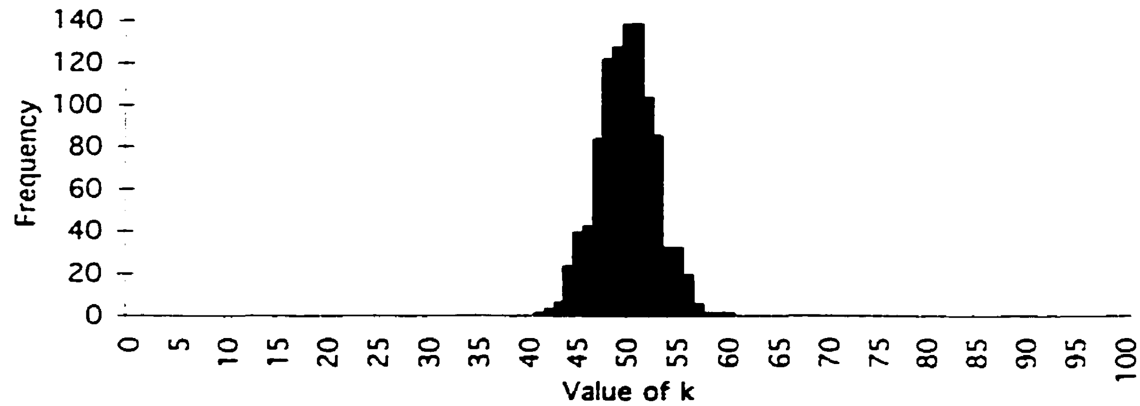


Figure 2: Frequency of k in Trials from a Standard Normal Sample.

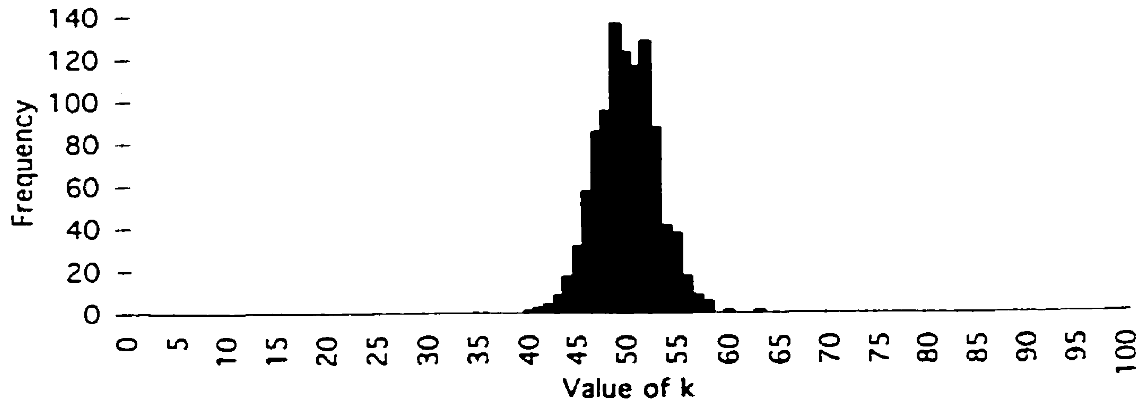
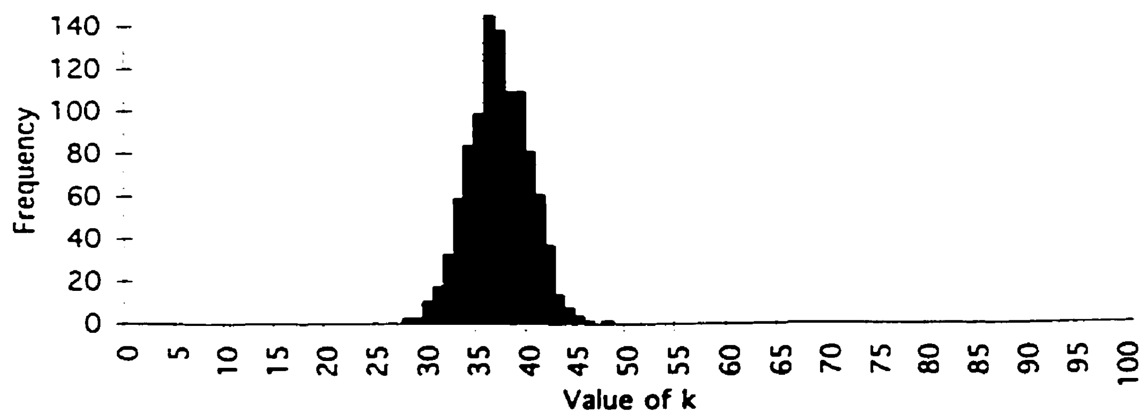


Figure 3: Frequency of k in Trials from an Exponential ($\mu = 1$) Sample.



5. Related Matrix and Polynomial Topics

5.1. Bounds for Real Eigenvalues.

When the real $n \times n$ matrix $\mathbf{A}$ has all its eigenvalues real, e.g., when $\mathbf{A}$ is symmetric, then the Laguerre-Samuelson, Brunk and Boyd-Hawkins inequalities provide bounds for the eigenvalues of $\mathbf{A}$ as observed by Wolkowicz and Styan [258], [259]; see also, e.g., Merikoski [142], Merikoski, Styan and Wolkowicz [144], Merikoski and Wolkowicz [150]. These bounds were rediscovered by Tarazaga [239].

As Mirsky [154] and Brauer and Mewborn [40] pointed out, the mean and variance of the eigenvalues λ_i may be expressed in terms of the trace of $\mathbf{A}$ and the trace of $\mathbf{A}^2$:

$$m = \frac{1}{n}\sum_{i=1}^{n}\lambda_i = \frac{1}{n}\mathrm{tr}\mathbf{A} \tag{5.1}$$

and

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}\lambda_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i\right)^2 = \frac{1}{n}\mathrm{tr}\mathbf{A}^2 - \left(\frac{1}{n}\mathrm{tr}\mathbf{A}\right)^2. \tag{5.2}$$

Then from (2.13) and (2.14) we obtain:

$$m + \frac{s}{\sqrt{n-1}} \leq \lambda_{\max} = \lambda_1 \leq m + s\sqrt{n-1} \tag{5.3}$$

and

$$m - s\sqrt{n-1} \leq \lambda_{\min} = \lambda_n \leq m - \frac{s}{\sqrt{n-1}}. \tag{5.4}$$

while from (2.19):

$$m - s\sqrt{\tfrac{k-1}{n-k+1}} \leq \lambda_k \leq m + s\sqrt{\tfrac{n-k}{k}} \qquad \text{for } k = 2, \ldots, n-1. \tag{5.5}$$

where λ_k is the k-th largest eigenvalue of $\mathbf{A}$, $k = 2, \ldots, n-1$. The question of bounds on eigenvalues when both $\mathrm{tr}\mathbf{A}$ and $\det\mathbf{A}$ are known is discussed in Merikoski and Virtanen [149].

5.2. Bounds for the Spread of Real Eigenvalues.

When the $x_{(i)} = \lambda_i$, the ordered eigenvalues of a matrix $\mathbf{A}$, then the term "spread of a matrix" for $\lambda_1 - \lambda_n$ was introduced[28] in 1956 by Leonid Mirsky[29] (1918–1983) in [154], in which it was also noted that the variance of the λ_i can be expressed as in (5.2) for a symmetric matrix $\mathbf{A}$.

The von Szőkefalvi Nagy-Popoviciu Inequalities (2.32)and (2.31) respectively give the following bounds for the spread in terms of the "variance" of the eigenvalues.

$$\frac{4}{n}\mathrm{tr}\mathbf{A}^2 - \left(\frac{2}{n}\mathrm{tr}\mathbf{A}\right)^2 \leq (\lambda_1 - \lambda_n)^2 \leq 2\mathrm{tr}\mathbf{A}^2 - \frac{2}{n}(\mathrm{tr}\mathbf{A})^2 . \tag{5.6}$$

and for the "variance" of the eigenvalues in terms of the spread.

$$\frac{1}{2n}(\lambda_1 - \lambda_n)^2 \leq \frac{1}{n}\mathrm{tr}\mathbf{A}^2 - \left(\frac{1}{n}\mathrm{tr}\mathbf{A}\right)^2 \leq \frac{1}{4}(\lambda_1 - \lambda_n)^2. \tag{5.7}$$

5.3. Matrix Inequalities Related to the Cauchy-Schwarz Inequality.

Two of our several proofs of the Laguerre-Samuelson inequality were based explicitly on the Cauchy-Schwarz inequality which, as we noted in §3.1, "may be perceived to be lurking in the background of many of the proofs"[30] of the Laguerre-Samuelson inequality, cf. §3.1.4, §3.1.6, and Lemma 3.1.6. Moreover, the discussion in [23], [50], [51], [178], and [245] of the proof given in §3.1.8 is all centered around the Cauchy-Schwarz inequality.

In their 1996 paper Pečarić, Puntanen and Styan [201] presented the following matrix-theoretic extension of the Cauchy-Schwarz inequality: here a g-inverse (generalized inverse) $\mathbf{X}^-$ is any matrix $\mathbf{X}^-$ such that $\mathbf{X}\mathbf{X}^-\mathbf{X} = \mathbf{X}$.

Theorem 5.3.1. *Let* $\mathbf{A}$ *be an* $n \times n$ *symmetric and nonnegative definite matrix with* $\mathbf{A}^{\{p\}}$ *defined as*

$$\mathbf{A}^{\{p\}} = \begin{cases} \mathbf{A}^p; & p = 1, 2, \ldots, \\ \mathbf{P}_\mathbf{A} = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'; & p = 0, \\ (\mathbf{A}^+)^{|p|}; & p = -1, -2, \ldots, \end{cases}$$

where $\mathbf{A}^+$ *is the (unique) Moore-Penrose inverse of* $\mathbf{A}$, *and* $\mathbf{P}_\mathbf{A}$ *denotes the orthogonal projector onto the column space* $C(\mathbf{A})$ *of* $\mathbf{A}$. *Let* $\mathbf{t}$ *and* $\mathbf{u}$ *be* $n \times 1$ *vectors, and*

[28] In their 1943 *Biometrika* paper [197], p. 89, Pearson and Hartley use the term "spread" as an alternate name for "range" at the beginning of their Introduction.

[29] For biographical accounts, see [46],[45], [227].

[30] Arnold and Balakrishnan [6], p. 45.

let h and k be integers. Then

$$(5.8)\qquad (\mathbf{t}'\mathbf{A}^{\{(h+k)/2\}}\mathbf{u})^2 \leq \mathbf{t}'\mathbf{A}^{\{h\}}\mathbf{t} \cdot \mathbf{u}'\mathbf{A}^{\{k\}}\mathbf{u}$$

for $h, k = \ldots, -1, 0, 1, 2, \ldots,$ with equality if and only if

$$(5.9)\qquad \mathbf{At} \propto \mathbf{A}^{\{1+(k-h)/2\}}\mathbf{u}.$$

Several extensions of the Theorem 5.3.1 and some statistical applications are also given in Pečarić, Puntanen and Styan [201].

When $h = 1$ and $k = -1$, then the inequality (5.8) becomes

$$(5.10)\qquad (\mathbf{t}'\mathbf{P}_{\mathbf{A}}\mathbf{u})^2 \leq \mathbf{t}'\mathbf{At} \cdot \mathbf{u}'\mathbf{A}^{+}\mathbf{u}.$$

cf. Bancroft [23].

Equality holds in (5.10) if and only if

$$(5.11)\qquad \mathbf{At} \propto \mathbf{P}_{\mathbf{A}}\mathbf{u}.$$

When $\mathbf{t} = \mathbf{w}$, $\mathbf{u} = \mathbf{x}$ and $\mathbf{A} = \mathbf{C}$, the centering matrix $\mathbf{I}_n - n^{-1}\mathbf{ee}'$ as in (3.19), then $\mathbf{A}^{+} = \mathbf{P}_{\mathbf{A}} = \mathbf{C}$ and (5.10) becomes

$$(5.12)\qquad (\mathbf{w}'\mathbf{Cx})^2 \leq \mathbf{w}'\mathbf{Cw} \cdot \mathbf{x}'\mathbf{Cx}.$$

which is equivalent to (3.22) in Lemma 3.1.6, and the equality condition (5.11) becomes

$$(5.13)\qquad \mathbf{Cw} \propto \mathbf{Cx}.$$

which is equivalent to (3.23) in Lemma 3.1.6[31].

We may also express (5.10) as

$$(5.14)\qquad (\mathbf{t}'\mathbf{u})^2 \leq \mathbf{t}'\mathbf{At} \cdot \mathbf{u}'\mathbf{A}^{-}\mathbf{u} \qquad \text{for all} \quad \mathbf{u} \in \mathcal{C}(\mathbf{A})$$

and for any, and hence for every g-inverse $\mathbf{A}^{-}$, cf. Neudecker and Liu [178]. The quadratic form $\mathbf{u}'\mathbf{A}^{-}\mathbf{u}$ in (5.14) is invariant with respect to the choice of g-inverse

[31]Since $\mathbf{Cw} = k\mathbf{Cx}$ is equivalent to $\mathbf{x} = (1/k)\mathbf{w} + (k\bar{x} - \bar{w})\mathbf{e}$, where $\bar{w} = \mathbf{w}'\mathbf{e}/n$.

$\mathbf{A}^-$ when $\mathbf{u} \in \mathcal{C}(\mathbf{A})$. since then $\mathbf{u} = \mathbf{Av}$ for some $\mathbf{v}$ and so $\mathbf{u}'\mathbf{A}^-\mathbf{u} = \mathbf{v}'\mathbf{AA}^-\mathbf{Av} = \mathbf{v}'\mathbf{Av} = \mathbf{v}'\mathbf{AA}^-\mathbf{Av}$ for any g-inverse $\mathbf{A}^-$. Equality holds in (5.14) if and only if

$$\mathbf{At} \propto \mathbf{u}. \tag{5.15}$$

Chaganty [50] presents (5.14) with the Moore-Penrose inverse $\mathbf{A}^+$ instead of a g-inverse $\mathbf{A}^-$ and observes that equality holds in (5.14) when $\mathbf{t} = \mathbf{A}^+\mathbf{u}$ which. since $\mathbf{u} \in \mathcal{C}(\mathbf{A})$. implies $\mathbf{At} = \mathbf{u}$. cf. (5.15).

Trenkler [245] observes that Baksalary and Kala [18] showed that

$$(\mathbf{t}'\mathbf{u})^2 \leq \alpha\mathbf{t}'\mathbf{At} \qquad \text{for all} \quad \mathbf{u} \in \mathcal{C}(\mathbf{A}) \tag{5.16}$$

provided that then $\mathbf{u}'\mathbf{A}^-\mathbf{u} \leq \alpha$ for any. and hence for every g-inverse $\mathbf{A}^-$.

If we now let $\mathbf{u} = \mathbf{t} \in \mathcal{C}(\mathbf{A})$. then (5.14) becomes

$$(\mathbf{t}'\mathbf{t})^2 \leq \mathbf{t}'\mathbf{At} \cdot \mathbf{t}'\mathbf{A}^-\mathbf{t} \qquad \text{for all} \quad \mathbf{t} \in \mathcal{C}(\mathbf{A}) \tag{5.17}$$

for any. and hence for every g-inverse $\mathbf{A}^-$: when $\mathbf{t} \neq \mathbf{0}$ then equality holds in (5.17) if and only if $\mathbf{t}$ is an eigenvector of $\mathbf{A}$. cf. Lemma 2.1 of Dey and Gupta [65].

5.4. Matrix Inequalities Related to the Sz. Nagy-Popoviciu Inequalities.

Let $\mathbf{z}$ be an $n \times 1$ vector with $\mathbf{z}'\mathbf{z} = 1$. and let $\mathbf{A}$ be an $n \times n$ symmetric matrix. not necessarily nonnegative (or positive) definite. Then the inequality:

$$\mathbf{z}'\mathbf{A}^2\mathbf{z} - (\mathbf{z}'\mathbf{Az})^2 \leq \tfrac{1}{4}(\lambda_1 - \lambda_n)^2. \tag{5.18}$$

was established in 1983 by Styan [231]: here λ_1 and λ_n are. respectively. the largest and the smallest eigenvalues of the symmetric matrix $\mathbf{A}$. Already in 1975. however. Bloomfield and Watson [36] proved that

$$\operatorname{tr}[\mathbf{Z}'\mathbf{A}^2\mathbf{Z} - (\mathbf{Z}'\mathbf{AZ})^2] \leq \frac{1}{4}\sum_{i=1}^{p}(\lambda_i - \lambda_{n-i+1})^2. \tag{5.19}$$

where $\mathbf{Z}$ is an $n \times p$ matrix with $\mathbf{Z}'\mathbf{Z} = \mathbf{I}_p$ and $n \geq 2p$. and where $\lambda_1 \geq \cdots \geq \lambda_n$ are the ordered eigenvalues of the symmetric matrix $\mathbf{A}$. When $p = 1$ then (5.19) reduces to (5.18). In 1996 Jia [94] reestablished (5.19) and referred to (5.18) as the "Styan Inequality" (even in the title of [94]). We note that the matrix $\mathbf{A}$ in (5.18) and (5.19) need not be nonnegative (or positive) definite since both sides of these inequalities remain unchanged if we replace $\mathbf{A}$ by $\mathbf{A} + k\mathbf{I}$, where k is a scalar (positive or negative).

The inequality (5.18) is a generalization of the right-hand inequality in (2.31):

$$s^2 \leq \tfrac{1}{4}(x_{\max} - x_{\min})^2 . \tag{5.20}$$

If we put $\lambda_1 = x_{\max}$, $\lambda_n = x_{\min}$, and $\mathbf{z} = \mathbf{Pe}/\sqrt{n}$, where $\mathbf{e}$ is the $n \times 1$ vector with each element equal to 1, and the $n \times n$ orthogonal matrix $\mathbf{P}$ diagonalizes $\mathbf{A}$ so that the diagonal matrix $\Lambda = \operatorname{diag}\{\lambda_i\} = \mathbf{P}'\mathbf{AP}$, in (5.18) then it becomes (5.20).

Now let $\mathbf{A}$ be an $n \times n$ symmetric positive definite matrix with necessarily positive eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n > 0$, and again let $\mathbf{Z}$ be an $n \times p$ matrix such that $\mathbf{Z}'\mathbf{Z} = \mathbf{I}_p$. Then Liu [112] posed the problem (solved by Bebiano, da Providencia and Li [27] and by Liu [113]): Show that provided $n \geq 2p$:

$$\frac{\operatorname{tr}\mathbf{Z}'\mathbf{A}^2\mathbf{Z}}{\operatorname{tr}(\mathbf{Z}'\mathbf{AZ})^2} \leq \frac{\sum_{j=1}^{p}(\lambda_j + \lambda_{n-j+1})^2}{4\sum_{j=1}^{p}\lambda_j\lambda_{n-j+1}} . \tag{5.21}$$

When $p = 1$ then the inequality (5.21) reduces to the Krasnosel'skiĭ-Kreĭn Inequality:

$$\frac{\mathbf{x}'\mathbf{A}^2\mathbf{x} \cdot \mathbf{x}'\mathbf{x}}{(\mathbf{x}'\mathbf{Ax})^2} \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1\lambda_n} \tag{5.22}$$

due to Krasnosel'skiĭ-Kreĭn [104], cf. Watson, Alpargu and Styan [251].

Another extension of (5.18) was obtained by Liu and Neudecker [114], who showed that:

$$\mathbf{A}_1^2 \odot \mathbf{A}_2^2 - (\mathbf{A}_1 \odot \mathbf{A}_2)^2 \leq_{\mathrm{L}} \tfrac{1}{4}(\lambda_1 - \lambda_n)^2\mathbf{I}_p,$$

where $\odot$ denotes the elementwise Hadamard (or Schur) product and $\leq_{\mathrm{L}}$ the Löwner or nonnegative definite partial ordering. The matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ are $n \times n$ and positive definite with λ_1 and λ_n, respectively, the largest and smallest eigenvalues of the Kronecker product $\mathbf{A}_1 \otimes \mathbf{A}_2$.

A similar extension was given by Pečarić, Puntanen and Styan [201], who proved that

$$\mathbf{T}'\mathbf{A}^2\mathbf{T} - (\mathbf{T}'\mathbf{AT})^2 \leq_{\mathrm{L}} \tfrac{1}{4}(\lambda_1 - \lambda_r)^2\mathbf{T}'\mathbf{AA}^{\dagger}\mathbf{T},$$

where $\mathbf{A}$ is an $n \times n$ nonnegative definite matrix of rank r with r positive eigenvalues $\lambda_1 \geq \cdots \geq \lambda_r > 0$, $\mathbf{A}^{\dagger}$ is the Moore-Penrose inverse of $\mathbf{A}$, and $\mathbf{T}$ is an $n \times k$ matrix such that $\mathbf{AA}^{\dagger}\mathbf{T}$ is a partial isometry, i.e., $\mathbf{T}'\mathbf{AA}^{\dagger}\mathbf{T}$ is (symmetric) idempotent.

For further closely related results see e.g., Baksalary and Puntanen [19], Liu and Neudecker [114], and Mond and Pečarić [163], [165], [164].

5.5. Bounds on Solutions of the Symmetric Equations.

D. S. Mitrinović [156] in 1965 considered bounds for a, b, and c in the following system of equations:

$$\begin{aligned} a+b+c &= p \\ bc+ca+ab &= q \end{aligned}$$

Bounds can easily be derived by considering p and q as the coefficients of a cubic polynomial with real roots a, b, and c. So in Laguerre's formulation, $p = -a_1$, $q = a_2$, and we have corresponding values for the mean and standard deviation of a, b, and c:

$$\bar{x} = \frac{p}{3} \qquad s = \frac{\sqrt{2(p^2-3q)}}{3}.$$

So, based on the Laguerre-Samuelson, Boyd-Hawkins, and Brunk inequalities, we have the following bounds:

$$\begin{aligned} \frac{p}{3} + \frac{\sqrt{p^2-3q}}{3} &\leq \max\{a,b,c\} \leq \frac{p}{3} + 2\frac{\sqrt{p^2-3q}}{3} \\ \frac{p}{3} - \frac{\sqrt{p^2-3q}}{3} &\leq \operatorname{med}\{a,b,c\} \leq \frac{p}{3} + \frac{\sqrt{p^2-3q}}{3} \\ \frac{p}{3} - 2\frac{\sqrt{p^2-3q}}{3} &\leq \min\{a,b,c\} \leq \frac{p}{3} - \frac{\sqrt{p^2-3q}}{3} \end{aligned}$$

These bounds correspond precisely to those given by D. S. Mitrinović [156] in 1965. An example presented by Mitrinović in this paper was proposed by R. L. Goodstein [77] in 1948. Two other special cases of this system were proposed as a problem by C. Ionescu-Ţiu [90] in 1975, which was solved by A. Lupaş [118] in 1984. C. Nicolau, apparently unaware of the work of Laguerre [109], derived [179] identical bounds for the real roots of a cubic polynomial in 1933 and then extended [180] this result to provide bounds, identical to (2.9), for the real roots of an n-th degree polynomial in 1939.

6. Biographical Information

6.1. Edmond Nicolas Laguerre.

Edmond Nicolas Laguerre was born on 9 April 1834 in Bar-le-Duc, France. His mathematical education was the product of several schools, most notably the École Polytechnique in Paris, where he placed a mediocre 46th out of 94 students. This placement was accompanied by the following comments [41]:

> "Cet élève très intelligent aurait pu rester classé dans les premiers de sa promotion, mais n'a pas travaillé. Extrêment dissipé."

In 1854, Laguerre abandoned his studies and accepted a commission in an artillery regiment of the French army. He retired from the military ten years later and returned to the École Polytechnique and was employed as an instructor. His work ethic seems to have improved over the remaining 22 years of his life, as he published 140 papers in the areas of geometry and analysis. Over half of these papers were in geometry, however, this portion of his work is not acknowledged nearly as much as his work in analysis and differential equations. He is especially known for the Laguerre polynomials, which are the solutions of the Laguerre equations. When his health began to fail in 1886, he returned to Bar-le-Duc, where he died on 14 August, 1886. According to [32]: "Laguerre was pictured by his contemporaries as a quiet, gentle man who was passionately devoted to his research, his teaching, and the education of his two daughters."

Copyright release for photograph requested from [185].

6.2. Julius von Szökefalvi Nagy.

The following obituary has been translated from [237][32] by Gabriel Holmes.

Hungarian mathematics has suffered a difficult loss: on October 14, 1953 Gyula (Julius) Szōkefalvi-Nagy passed away after lengthy suffering. Our journal will sorely miss him both as a contributor and as one of our oldest and most hard-working co-workers. He has been correspondent [with us] since 1934, and has been a member of the Hungarian Academy of Science since 1946. Gyula Szōkefalvi-Nagy was born on April 11, 1887 in Erzsébetvaros in Siebenbürgen, and studied from 1905 to 1909 at Koloszvar University, where he quickly received his doctorate in 1909. In the academic year 1911-12 he continued his studies at Gōttingen University, earning a stipend. His academic career came into its own in 1915, when he established algebra and function theory departments at Koloszvar University.

At the same time, he was elected director at the "Marianum" trade-school for girls, where he had worked as a teacher for several years. He remained in this position until 1919 when the Hungarian university left the city of Koloszvar and moved first to Budapest and finally to Szeged. Ten years later, he could reassume his teaching position at the University of Szeged, where his former colleagues still worked. At the same time, he was called upon to succeed Béla Kerékjártó as chairman of the geometry department. Shortly thereafter, however, he was transferred to the University of Koloszvar, which the Hungarian government had recently reopened. Here he continued to work hard at his teaching and scientific work, which were also made difficult by the war and social turmoil.

During this period he wrote a book (in Hungarian) on the theory of geometric constructions. In December 1943, perhaps brought on by the stress associated with this and his administrative duties, he suffered a stroke from which he never spiritually recovered. It permanently crippled half of his body and threatened to return. In September 1945, severely ill but full of unfettered enthusiasm for his work, he returned to Szeged to reassume his teaching position at the university, where his son

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Bela currently works. There he worked until his very last moment. He lectured in the nearest university building and later (due to his worsening condition) at home. He also published 38 more works and prepared an expanded edition of his book on geometric constructions, this time in German.

Gyula Szőkefalvi-Nagy was an enthusiastic and remarkably productive researcher in geometry. In all, he published 148 works, all of which concerned either geometry or geometrically meaningful problems in analysis and classical algebra. His finest research included: geometric and algebraic properties and relationships between algebraic curves and surfaces: curves and surfaces (not necessarily algebraic) of maximal index and maximal class index (one of his most fruitful pieces of research): the geometry of convex regions and bodies: the theory of geometric constructions: and many kinds of questions concerning the distribution of roots of rational and transcendental functions.

We will always cherish the memory of Gyula Szőkefalvi-Nagy and his amazing spirit.

6.3. Tiberiu Popoviciu.

The following obituary has been translated from [208][33] by Alexandru Ghitza.

Professor Tiberiu Popoviciu passed away on October 29, 1975. He was a member of the Faculty of Mathematics of the Babes-Bolyai University in Cluj-Napoca, and a renowned representative of Romanian mathematics. Born on February 16, 1906 in Arad, his exceptional talent became apparent while he was still in high school. He was an active contributor to *Gazeta Matematica* as well as an editor of *Jurnal Matematic* which became a publication known abroad. After he obtained his bachelor degree in Bucharest, he attended the École Normale Superieure in Paris between 1927 and 1930.

This is where he obtained his doctorate in mathematics in 1933 – in his thesis "Sur quelques propriétés des fonctions d'une ou de deux variables réelles", he introduced the notion of a convex function of higher order, which has proved a fundamental tool in the study of approximation of functions.

Back in his natal land, he dedicated all his efforts and talent to the development of mathematical education in Romania and moved to the top of the academic hierarchy in the universities of Cluj, Bucharest and Iasi. In 1946 he settled permanently in Cluj, where he created a prestigious school of numerical analysis, well known and appreciated by the entire mathematical community. At the same time he contributed significantly to the organization of mathematical education, as dean of the Faculty of Mathematics and Physics in Cluj between 1950 and 1953, and as head of the analysis group until a few years before his death.

As a recognition of his extraordinary scientific merit, he was elected correspondent member of the Romanian Academy in 1948, to become a full member in 1963. As such he organized and chaired the Institute for Computing of the Academy in Cluj-Napoca, based on a close collaboration between mathematical research and industry, an idea that Tiberiu Popoviciu supported with energy and competence during his

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entire life.

As a man of science, he contributed to the international renown of Romanian mathematics through his active participation in the international mathematics community as well as the organization, in Cluj-Napoca, of several prestigious scientific activities.

Through his remarkable results, recorded in more than 300 studies and monographs, he appears as a leading specialist in fields like analysis, numerical analysis, algebra, number theory and functional analysis.

Founder of the Romanian school of numerical analysis, he greatly contributed to the subject. He established calculus of divided differences as a pre-differential calculus, proved a general average formula which allows the explicit formulation of the remainder of linear approximation formulas in analysis, and began the study of conservation of shape properties for operators that come up in the theory of best approximation.

A teacher and a scientist, Tiberiu Popoviciu also helped a great deal in the popularization of mathematics, as president of the Cluj subsidiary of the Society for Mathematical Sciences.

For his special contributions, he was awarded the prestigious title of "Emeritus scientist" as well as several awards and medals of the Socialist Republic of Romania.

In Tiberiu Popoviciu, science and education in Romania loses one of its foremost representatives.

6.4. Keshavan Raghavan Nair.

Keshavan Raghavan Nair was born on 13 May 1910, the only son of seven children, in the village of Parur near the southern tip of India. He completed his B.A. (Honours) degree at the Maharaja's Science College in 1932 but had the misfortune of graduating in the midst of the Great Depression. Against the stiff competition of that period, K. R. Nair succeeded in procuring a scholarship for research from the University of Madras that eventually lead to a Master of Science degree in 1936.

The following years spent at the Indian Statistical Institute with colleagues such as P. C. Mahalanobis, R. C. Bose, and C. R. Rao, were described by Nair [177] as "the most fruitful in my research career".

After a seven year term at the Indian Statistical Institure and several additional years under the Government of India, K. R. Nair attended the University of London and completed his Ph.D. degree under the supervision of E. S. Pearson and H. O. Hartley in 1947. Following the completion of his doctorate, Nair returned to India and continued statistical work for the Government of India and then eventually the United Nations.

In 1992 Nair observed in his autobiographical essay "In statistics by design" [177] that

> "On the research side, after almost two decades of hibernation, I found during 1979, to my amused amazement, that an inequality establishing the upper limit of the extreme deviate from a sample mean which I had presented in a small section of the third part of my Ph.D. thesis [168] of 1947 and published in the Journal of the Indian Society of Agricultural Statistics, Vol. I (1948) (see [171]) was rediscovered by the world renowned American Economist, Professor Paul A. Samuelson, Nobel Laureate, two decades later in a paper entitled: "How Deviant Can You Be?" [216] in the Journal of the American Statistical Association, Vol. 63 (1968). ... this paper escaped my attention until 1979 when

I chanced to see a reference to it in an article entitled "Extensions of Samuelson's Inequality" [256] in the "*American Statistician*", Vol. 33 (August 1979 issue). That tempted me to proclaim, through a Letter to the Editor of that Journal in 1980 my precedence over Professor Samuelson in the discovery of this inequality."

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6.5. Paul Anthony Samuelson.

Paul Anthony Samuelson was born on 15 May 1915, though he himself claims that [218] "truly, he was born on the morning of January 2, 1932, at the University of Chicago." He graduated amidst innumerable honours with a B.A. degree in 1935 from the University of Chicago, and then with both a M.A. degree in 1936 and Ph.D. degree in 1941 from Harvard University. During his doctoral work at Harvard, Samuelson wrote his celebrated *Foundations of Economic Analysis.* Tempted away to a position at the Massachusetts Institute of Techonology in 1940, Samuelson was to remain there for his entire career.

He also served as advisor to several government offices, including the U.S. Treasury and the Federal Reserve Board. Among his most prominent accolades were the first John Bates Clark Medal of the American Economic Association in 1947 and the 1970 Nobel Prize in economics.

According to Samuelson's autobiographical essay "Economics in my time" [218]:

> "When I began the study of economics back in 1932 on the University of Chicago Midway, economics was literary economics. A few original spirits—such as Harold Hotelling, Ragnar Frisch and R. G. D. Allen—used mathematical symbols ... Such esoteric animals as matrices were never seen in the social science zoos. At most a few chaste determinants were admitted to our Augean stables. Do I seem to be describing Eden, a paradise to which many would like to return in revulsion against the symbolic puss-pimples that disfigure not only the pages of *Econometrica* but also the *Economic Journal* and the *American Economic Review*?"

In our bibliography below references to *Jahrbuch für die Fortschritte der Mathematik* are denoted by JFM (for reviews published in 1868–1930), *Mathematical Reviews* by MR (for reviews published since 1940), *Zentralblatt für Mathematik* [261] by Zbl (for reviews published since 1931), and to *Current Mathematical Publications* by CMP.

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