OPTIMUM DIGITAL FILTERING OF RANDOM BINARY SIGNALS

by

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ABSTRACT

The problem of nonrecursive digital filtering of stationary random binary signals is considered. The binary signal is fed into a finite length shift register and the waveforms to be analysed are obtained by continuous summation of weighted digits stored in the register.

Expressions for the autocorrelation function and power spectral density of a digitally filtered binary signal are obtained in terms of the weights at the taps of the shift register. A prescribed power spectral density may be approximated by appropriate digital filtering. The approximation criterion used is the mean-square-error between the system function corresponding to the imposed spectral density and that of the digital filter. A spacing parameter and a positioning parameter are defined and the mean-square-error is minimized with respect to these and the weighting parameters for an arbitrary order shift register. The particular cases of a Gaussian and a rectangular or "brickwall" spectral density are illustrated.

An expression is obtained for the probability density function, and the Central Limit Theorem is discussed in terms of equal and binomial coefficient weights. A figure of merit is introduced as an approximation criterion between the probability density of the obtained signal and that of a Gaussian process. Solutions for optimum weights are obtained by numerical computation.

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CHAPTER I

INTRODUCTION

Random and maximum-length pseudo-random binary sequences have recently become useful for simulation[1,2] and testing[3] purposes. Pseudo-random sequences, which may be considered as approximating truly random binary sequences, have been studied extensively in connection with digital communications, radar, sonar, navigation and telemetry[4-7]. The binary aspect of these sequences, their ease of generation and the exactly known statistical properties such as autocorrelation function, power spectral density and probability density function make them particularly useful for digital applications. In order to extend their use for continuous systems, one must modify their properties to be more characteristic of random processes and disturbances occuring in analog systems. Some consideration has been given to this problem in the literature[13, 35] by using analog filtering.

In this thesis we investigate the effects of digital filtering on random and pseudo-random binary sequences. The sequence is fed into a finite length shift register and the signal to be analyzed is obtained by continuous summation of weighted outputs from each stage of the register.

It has recently been shown[4, 10, 11, 12] that some control of the spectral content may be achieved by weighting the outputs of the shift register stages before summation. In Chapter III an arbitrary power spectral density is approximated by the spectral density obtained from a weighted shift register. A mean-square-error criterion is introduced and the weights are optimized to minimize the error. The Gaussian and brickwall spectral density approximations are discussed and nummerical results given.

Kramer[8] and Davies[9] discuss the probability density function of pseudo-random and random binary waveforms obtained by equal summation of the digits stored in the shift register. The probability density is shown to be binomial for random sequences and approximately binomial for pseudo-random sequences. Although the binomial probability density approaches the Gaussian distribution as the length of the register increases indefinately, the approximation is shown not to be optimum for finite length registers. In Chapter IV, we extend the results of Kramer and Davies to arbitrarily weighted registers. The Central Limit Theorem, as it pertains to weighted registers, is discussed and a figure of merit is introduced as an approximation criterion between the probability density of the waveform produced by a weighted register and that of a Gaussian process. Solutions for finite length registers are obtained which optimize the approximation with respect to the weights.

CHAPTER II

DIGITALLY FILTERED RANDOM BINARY SIGNALS

2.0 Introduction

Many of the signals of interest in modern system analysis are non-deterministic in nature. These random or stochastic processes are commonly used for testing and simulation purposes. Most conventional noise generators employ ~~naturally occuring~~ noise sources such as thyratrons and zener diodes. Although these generators are "ideal" in the sense that they possess Gaussian statistics, they have the deficiency of being not very stable, controllable or well-defined.

On the other hand, random binary sequences and, in particular, pseudo-random sequences*, have none of these deficiencies. Although they are not Gaussian, and have a definite power spectral density, proper filtering (either analog or digital) can modify their statistical properties without losing any of the advantages mentioned in the above. We now proceed to discuss the basic properties of random binary signals.

2.1 The Random Binary Process

The random binary sequence**, shown in Figure 2.1, is a continuous stream of statistically independent pulses of width T, amplitude of + 1 or − 1 and no assumed time

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* See Section 2.2

** It should be noted that the process under consideration is not the well known random telegraph wave with random Poisson distributed switching times.

origin.

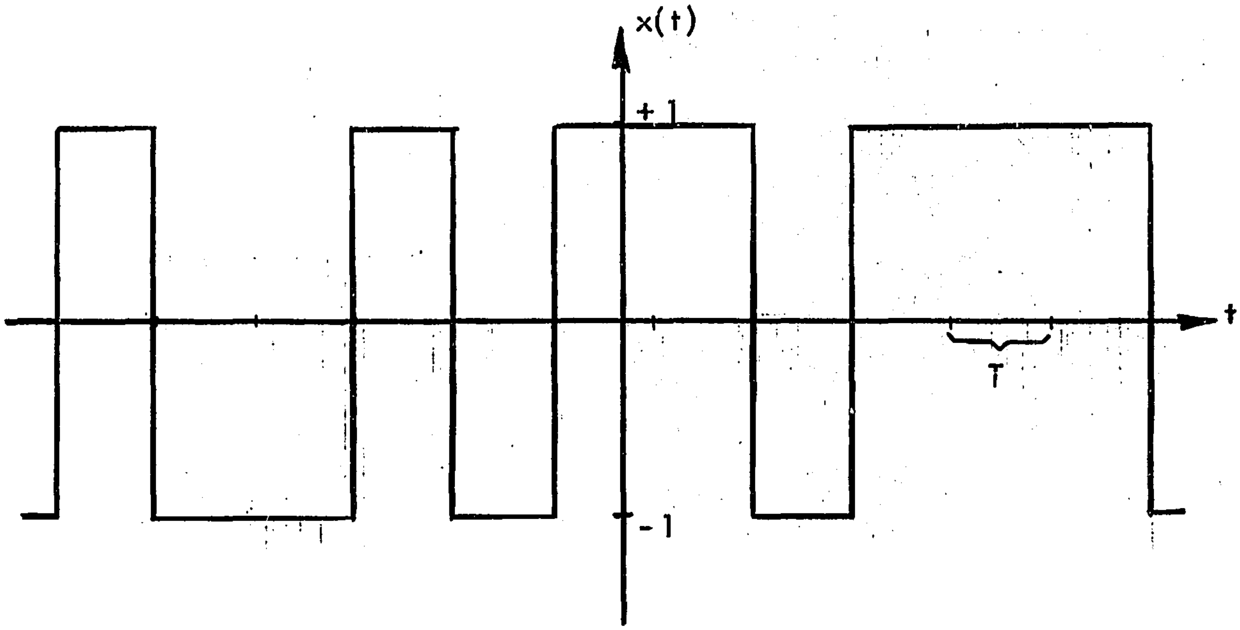


FIGURE 2-1 - RANDOM BINARY SEQUENCE

If the probabilities of the binary sequence x(t) are

$$P\left[x(t) = 1\right] = P\left[x(t) = -1\right] = \frac{1}{2} \qquad (2.1\text{-}0)$$

the process is stationary because the probability is independent of time. Using the ergodic hypothesis which states that for stationary signals, the ensemble averages are equal to time averages, we may write the autocorrelation function as

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x(t+\tau)\, dt \qquad (2.1\text{-}1)$$

If the binary signal x(t) is shifted by τ seconds, $\tau < T$, then both x(t) and $x(t + \tau)$ will have the same sign for time $(T - \tau)/T$, and will have the same or opposite sign with equal probability for all other time. Thus (2.1-1) yields

$$R(\tau) = (T - \tau)/T = 1 - \tau/T \qquad (2.1\text{-}2)$$
$$\text{for} \quad \tau \leq T$$

Due to the symmetry property* of the autocorrelation function, the results of (2.1-2) are also valid for negative τ and hence we may write**

$$R(\tau) = 1 - |\tau|/T \quad \text{for} \quad |\tau| \leq T \qquad (2.1\text{-}3)$$
$$= 0 \quad \text{for} \quad |\tau| > T$$

The power spectral density of the binary signal can be obtained by applying the Wiener-Khintchine relationship***. Hence,

$$S(\omega) = 2 \int_0^T \left(1 - \frac{\tau}{T} \right) \cos(\omega \tau)\, d\tau$$
$$= \frac{2}{T} \left[\frac{1 - \cos(\omega T)}{\omega^2} \right] = T \left\{ \frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}} \right\}^2 \qquad (2.1\text{-}4)$$

The autocorrelation function and power spectral density are shown in Figure 2-2.

---

* The symmetry of the autocorrelation should be noted. Making the change of variable, $s = t + \tau$, we have from (2.1-1)

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(s - \tau)\, x(s)\, ds = R(-\tau)$$

** For a more rigorous derivation of the autocorrelation function of a stationary random binary process, see Rice[32] or Papoulis[17].

***

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau = 2\int_0^{\infty} R(\tau) \cos(\omega\tau)\, d\tau$$

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{j\omega\tau}\, d\omega = \frac{1}{\pi} \int_0^{\infty} S(\omega) \cos(\omega\tau)\, d\omega$$

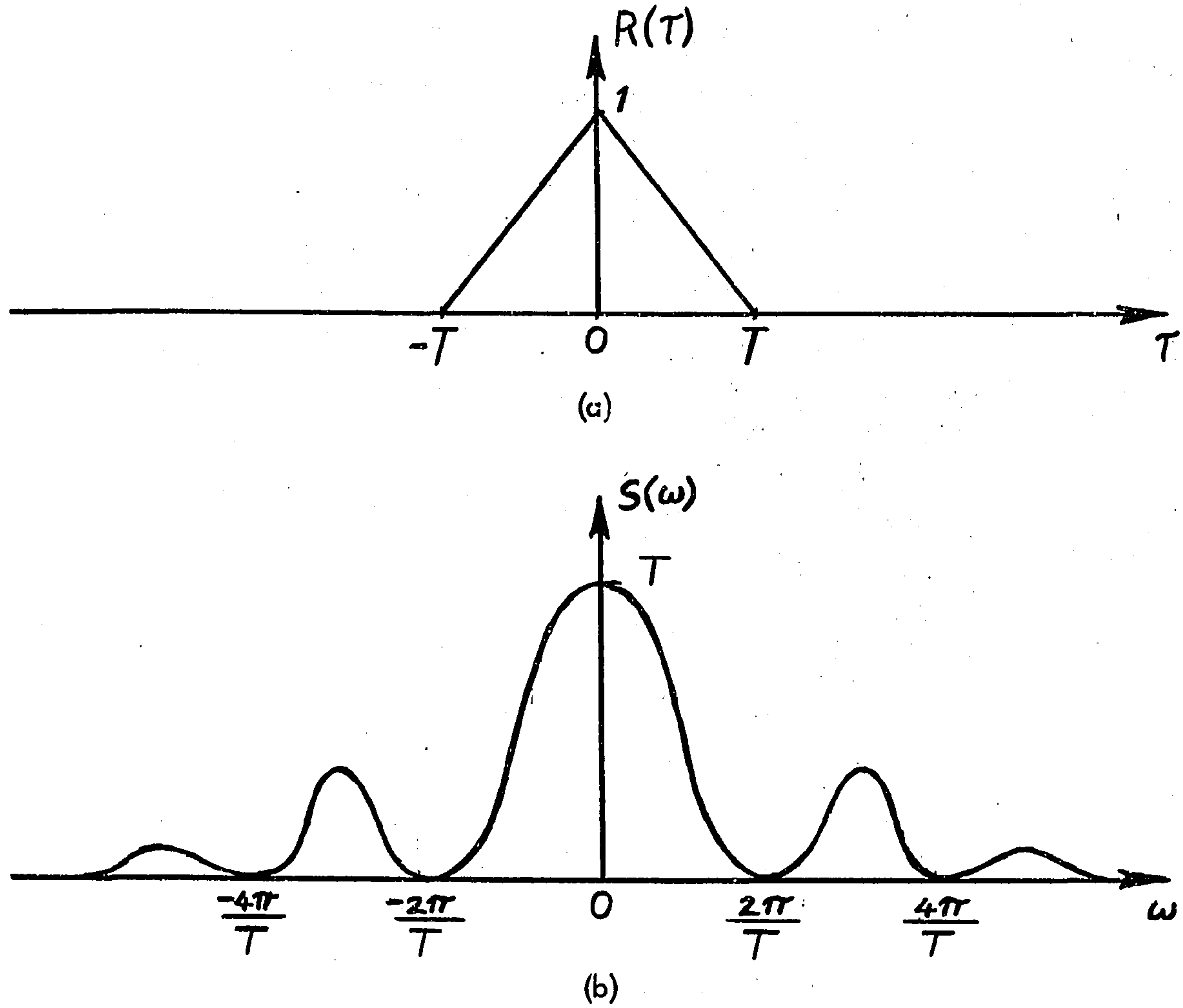
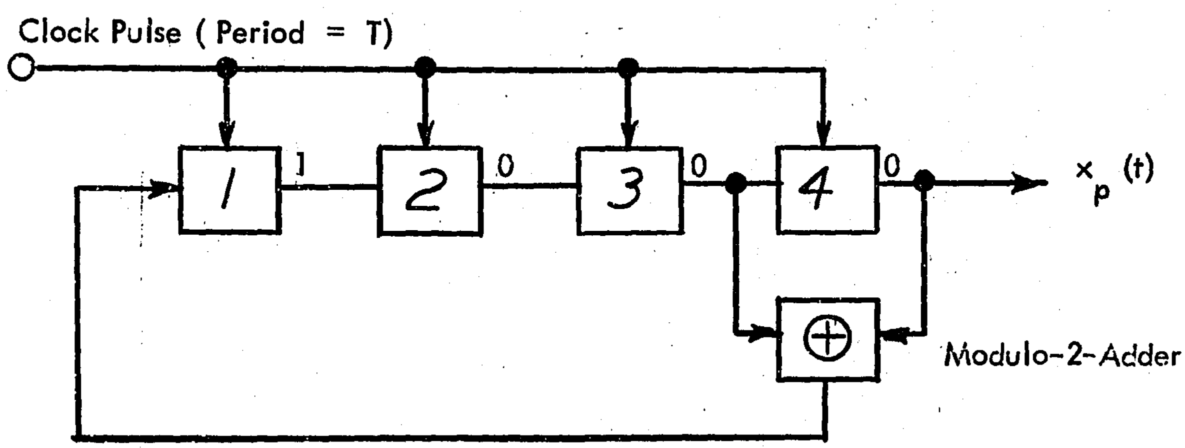


FIGURE 2-2 - AUTOCORRELATION (a) AND POWER SPECTRAL DENSITY (b) OF A STATIONARY RANDOM BINARY SIGNAL.

2.2 Pseudo-Random Sequences

One way of physically realizing a good approximation to a random binary sequence is by the use of m - sequences or maximum - length shift register sequences first described by Huffman[34]. These pseudo-random sequences, although by no means random, possess many of the properties of true random binary noise. As will next be shown, the statistical properties of these sequences (i.e., autocorrelation function, power spectral density and probability distribution) approach those of a random binary sequence as the length of the sequence increases.

Referring to Figure 2-3, an n-bit digital shift register whose output is "scrambled" by suitable digital logic and fed back to recirculate will produce periodic binary sequences. The maximum period length obtainable with "linear" (modulo-2-adder)* feedback is $2^n - 1$ bits. The reason for this is that an n-bit shift register can have 2^n different states, but one of these states will merely reproduce itself (0000 in the circuit of Figure 2-3).



Initial Condition 1000

0100
0010
1001
1100
0110
1011
0101
1010
1101
1110
1111
0111
0011
0001

Period = $2^4 - 1 = 15$

1000
⋮

FIGURE 2-3 - FOUR-STAGE SHIFT-REGISTER CIRCUIT GENERATING A MAXIMUM LENGTH SEQUENCE

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* Modulo-2 addition refers to the logic Exclusive - Or i.e., $(A+B)_{mod\,2} = A \oplus B = A\bar{B} + \bar{A}B$. The corresponding truth table is therefore

A \ B	0	1
0	0	1
1	1	0

Before proceeding further, it should be noted that the levels of the shift register outputs discussed so far have been 0 and 1. This choice of binary digits was used in order to conform to the usual binary notation. To obtain pseudo-random binary noise from the shift register sequence, we have merely to shift the waveform down by 1/2 and multiply by 2. Thus the modified sequence will have two possible values, - 1 and + 1.

As already has been mentioned, the all zero state does not occur in maximum length sequences, and hence in one period of the modified sequence, $x_p(t)$ is + 1 for 2^{n-1} times and - 1 for $2^{n-1} - 1$ times. Thus the probabilities are

$$P\left[x_p(t) = 1\right] = \frac{2^{n-1}}{2^n - 1}$$
$$P\left[x_p(t) = -1\right] = \frac{2^{n-1} - 1}{2^n - 1} \qquad (2.2\text{-}0)$$

Writing (2.2-0) as

$$P\left[x_p(t) = 1\right] = \frac{1}{2 - \frac{1}{2^{n-1}}}$$

and

$$P\left[x_p(t) = -1\right] = \frac{1}{2 - \frac{1}{2^{n-1}}} - \frac{1}{2^n - 1}$$

it is easy to see that

$$\lim_{n \to \infty} P\left[x_p(t) = 1\right] = \lim_{n \to \infty} P\left[x_p(t) = -1\right] = \frac{1}{2} \qquad (2.2\text{-}1)$$

which are the probabilities of the random binary signal.

Korn[1] and Golomb[6] have shown that the autocorrelation function of a pseudo-random binary sequence is

$$R_p(\tau) = 1 - \frac{|\tau - k(2^n - 1)T|}{T}$$

$$\text{for } |\tau - k(2^n - 1)T| \leq T; \; k = 0, \pm 1, \pm 2, \ldots$$

$$= -\frac{1}{2^n - 1} \quad \text{otherwise.} \qquad (2.2\text{-}2)$$

The autocorrelation function is illustrated in Figure 2-4.

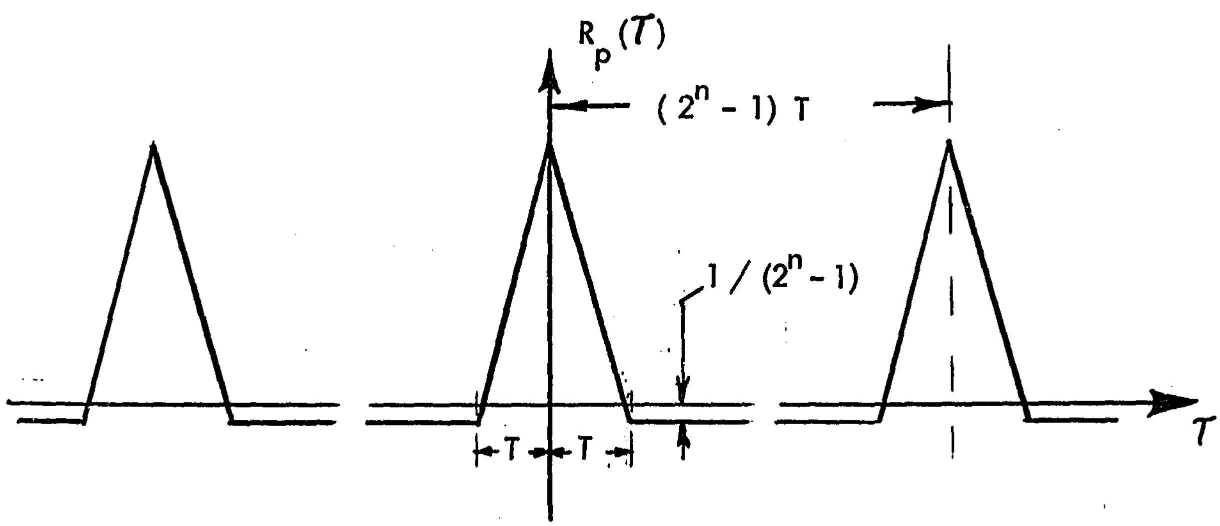

FIGURE 2-4 - AUTOCORRELATION FUNCTION OF A PSEUDO-RANDOM SEQUENCE

It is easily seen that as $n \to \infty$, the autocorrelation function of Figure 2-4 approaches that of Figure 2-2(a). Obtaining the power-spectral density of the pseudo-random sequence in a similar way as for the random binary signal, it can be shown that [1]

$$S_p(\omega) = 2\pi \left(\frac{1}{2^n-1}\right)^2 \left\{ -(2^n-1)\,\delta(\omega) + \sum_{k=-\infty}^{\infty} 2^n \left[\frac{\sin k\pi/(2^n-1)}{k\pi/(2^n-1)}\right]^2 \delta\left[\omega - \frac{2\pi k}{(2^n-1)T}\right] \right\} \qquad (2.2\text{-}3)$$

where $\delta(x)$ is the Dirac delta function.

Thus, due to the periodicity of the autocorrelation function, the power spectral density is a " line " spectrum with spacing $2\pi/(2^n-1)T$ radians and its envelope is similar to Figure 2-2.

In this thesis, unless otherwise mentioned, all results shall pertain directly to stationary random binary signals. In some cases, modifications to these results shall be made in order that they be applicable directly to pseudo-random sequences. Nevertheless, it has been shown in this section that as $n \to \infty$, pseudo-random sequences approach truly random sequence, and consequently all results are useful.

2.3 Digitally Filtered Sequences

Consider a random binary sequence, as defined in section 2.1, which is fed into an n-stage shift register. It is assumed that the binary digits in the register are + 1 and - 1, and also that the clock pulse period of the shift register is T and synchronized to the zero crossings of the binary sequence. If the output of each stage is weighted* by W_k, $k = 1, 2, \ldots n$, and if we sum these weighted outputs continuously, we obtain a new random process y(t) which is related to the input binary process x(t). Since the output

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* It is assumed that the weights may be positive or negative. One way of realizing a negative weight W_k is by weighting the complementary output of the k^{th} stage with a positive weight.

of the k^{th} stage is identical to the input process delayed by kT, the weighted shift register may be looked upon as a tapped-delay-line or in this case a non-recursive digital filter. The digital filter is shown in Figure 2-5.

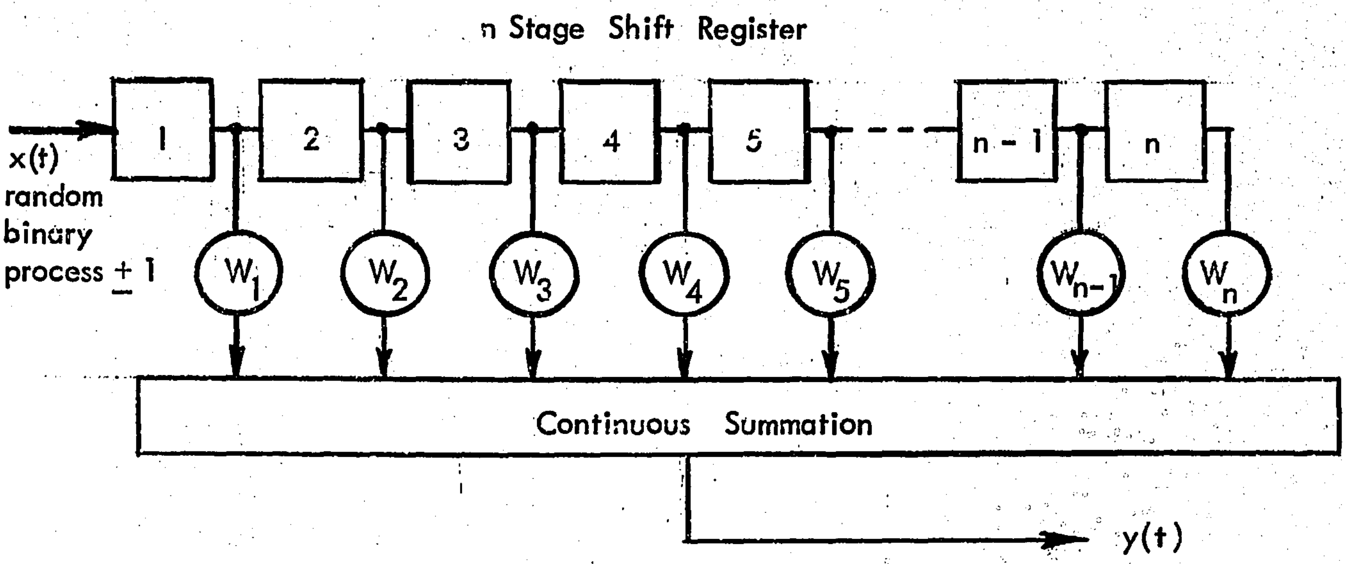

FIGURE 2-5 - CONTINUOUS SUMMATION OF WEIGHTED DIGITS STORED IN A SHIFT REGISTER WITH RANDOM BINARY SEQUENCE INPUT

A similar signal $y_p(t)$ may be obtained using a pseudo-random sequence as the input[10]. In this case, the digital filter may simultaneously be used to make part or all of the shift register which generates the sequence. Care, however, should be taken not to make the length of the digital filter much greater than that of the register generating the pseudo-random sequence, since it has been shown[36, 37] that if this is the case, the amplitude probability densities obtained will differ markedly from those of the random binary sequence.

2.4 Output Autocorrelation Function and Power Spectral Density

Let us now consider the two stage weighted register of Figure 2-6, into which is fed the random binary sequence x(t). The outputs of the first and second stages are delayed versions of the input and are denoted by $x(t - T)$ and $x(t - 2T)$ respectively, where T is the clock period. Weighting the first and second stage output by W_1 and W_2 respectively, we obtain the signals $x_1(t)$ and $x_2(t)$ which we sum continuously to obtain the output y(t).

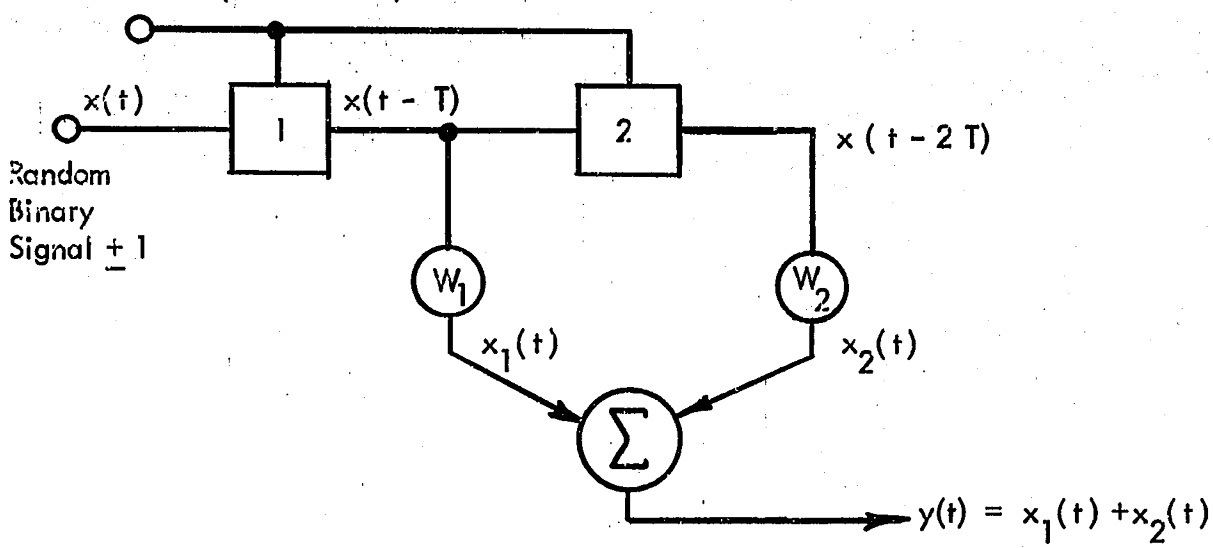

FIGURE 2-6 - 2 STAGE WEIGHTED REGISTER

We may write the autocorrelation function of y(t) as

$$R_{yy}(\tau) = E\left[y(t)\ y(t+\tau)\right] \qquad (2.4\text{-}0)$$

Substituting $x_1(t) + x_2(t)$ for y(t) in (2.4-0), we may write

$$R_{yy}(\tau) = E\left\{\left[x_1(t) + x_2(t)\right]\left[x_1(t+\tau) + x_2(t+\tau)\right]\right\}$$

and after multiplying out the terms, we have

$$R_{yy}(\tau) = E\left[x_1(t)\,x_1(t+\tau)\right] + E\left[x_1(t)\ x_2(t+\tau)\right] + E\left[x_2(t)\,x_1(t+\tau)\right] + E\left[x_2(t)\ x_2(t+\tau)\right]$$

$$= R_{x_1x_1}(\tau) + R_{x_1x_2}(\tau) + R_{x_2x_1}(\tau) + R_{x_2x_2}(\tau) \qquad (2.4\text{-}1)$$

Now, since

$$x_1(t) = W_1 \, x(t - T) \tag{2.4-2}$$

and

$$x_2(t) = W_2 \, x(t - 2T) \tag{2.4-3}$$

we may substitute (2.4-2) and (2.4-3) into (2.4-1) to obtain

$$R_{yy}(\tau) = W_1^2 \, E\left[x(t-T)\,x(t-T+\tau)\right] + W_1 W_2 \, E\left[x(t-T)\,x(t-2T+\tau)\right]$$
$$+ W_2 W_1 \, E\left[x(t-2T)\,x(t-T+\tau)\right] + W_2^2 \, E\left[x(t-2T)\,x(t-2T+\tau)\right] \tag{2.4-4}$$

Defining $R_o(\tau - \lambda)$ as the autocorrelation function of Figure 2-2(a) shifted in τ by λ, we may rewrite (2.4-4) as

$$R_{yy}(\tau) = W_1^2 \, R_o(\tau) + W_1 W_2 \, R_o(\tau - T) + W_2 W_1 \, R_o(\tau + T) + W_2^2 \, R_o(\tau) \tag{2.4-5}$$

Thus we see that the autocorrelation function $R_{yy}(\tau)$ is made up of the four terms

$$R_{x_1x_1}(\tau) = W_1^2 \, R_o(\tau)$$
$$R_{x_1x_2}(\tau) = W_1 W_2 \, R_o(\tau - T)$$
$$R_{x_2x_1}(\tau) = W_2 W_1 \, R_o(\tau + T)$$

and (2.4-6)

$$R_{x_2x_2}(\tau) = W_2^2 \, R_o(\tau)$$

which are shown in Figure 2-7.

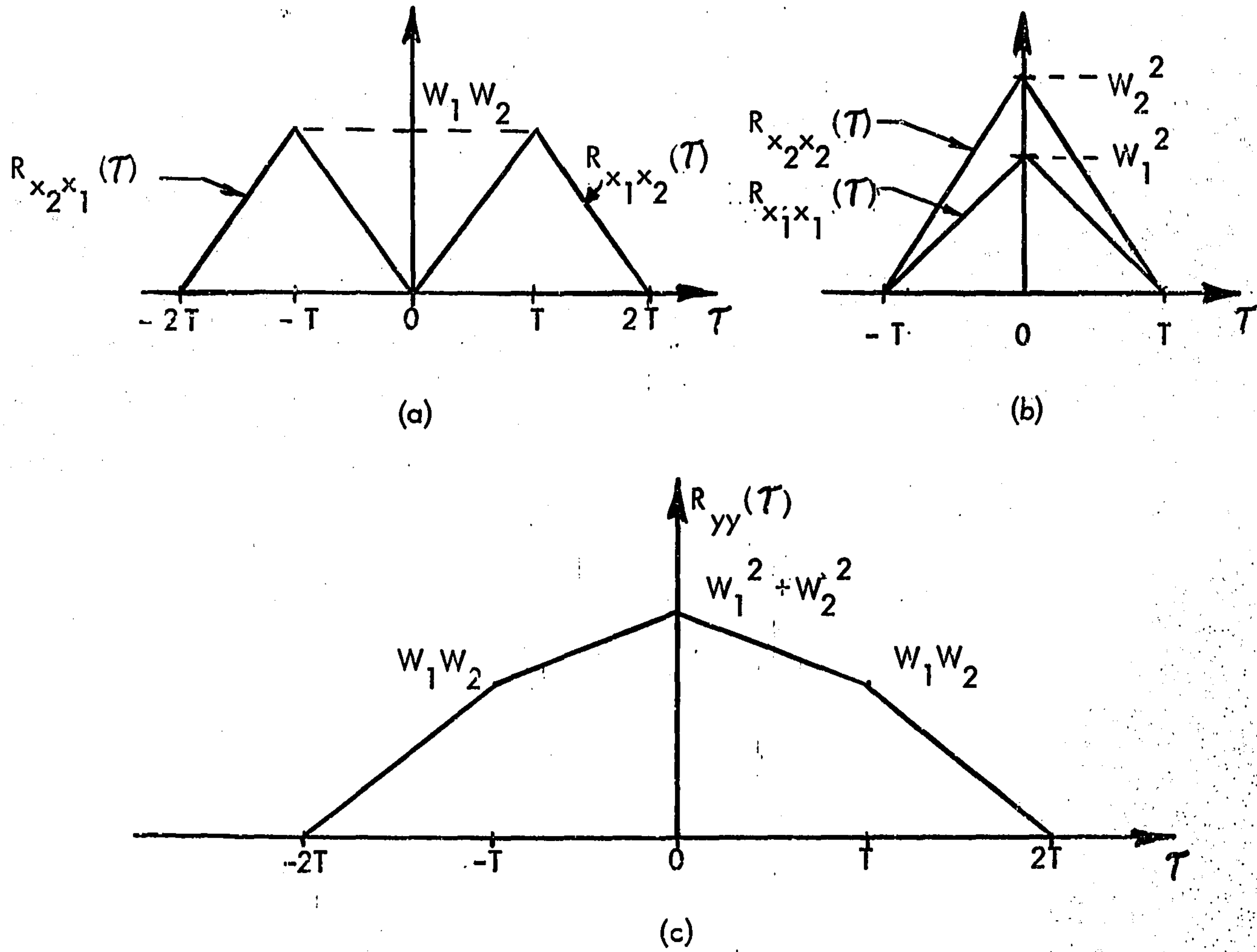

FIGURE 2-7 - OUTPUT AUTOCORRELATION FUNCTION (n = 2)

In general, for a register of length n, we may write

$$R_{yy}(\tau) = E\left\{\left[\sum_{i=1}^{n} x_i(t)\right]\left[\sum_{j=1}^{n} x_j(t+\tau)\right]\right\}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} E\left[x_i(t) \quad x_j(t+\tau)\right]$$

or

$$R_{yy}(\tau) = \sum_{i,j=1}^{n} R_{x_i x_j}(\tau) \qquad (2.4\text{-}7)$$

Now considering the case when $i \leq j$, we may write

$$R_{x_i x_j}(\tau) = W_i W_j R_o \left[\tau + (i - j) T \right] \quad i \leq j \tag{2.4-8}$$

Similarly,

$$\begin{aligned} R_{x_j x_i}(\tau) &= R_{x_i x_j}(-\tau) \\ &= W_i W_j R_o \left[-\tau + (i - j) T \right] \end{aligned} \tag{2.4-9}$$

Since the autocorrelation is an even function, we may write (2.4-9) as

$$R_{x_j x_i}(\tau) = W_j W_i R_o \left[\tau + (j - i) T \right] \tag{2.4-10}$$

Interchanging i and j in (2.4-10) we obtain

$$R_{x_i x_j}(\tau) = W_i W_j R_o \left[\tau + (i - j) T \right] \quad \text{for } i \geq j \tag{2.4-11}$$

which is the same as (2.4-8).

Substituting (2.4-11) and (2.4-8) into (2.4-7), we obtain the following result

$$R_{yy}(\tau) = \sum_{i,j=1}^{n} W_i W_j R_o \left[\tau - (j - i) T \right] \tag{2.4-12}$$

From (2.4-12) it is easily shown that the autocorrelation function $R_{yy}(\tau)$ is a piecewise straight line symmetrical curve as shown in Figure 2-8.

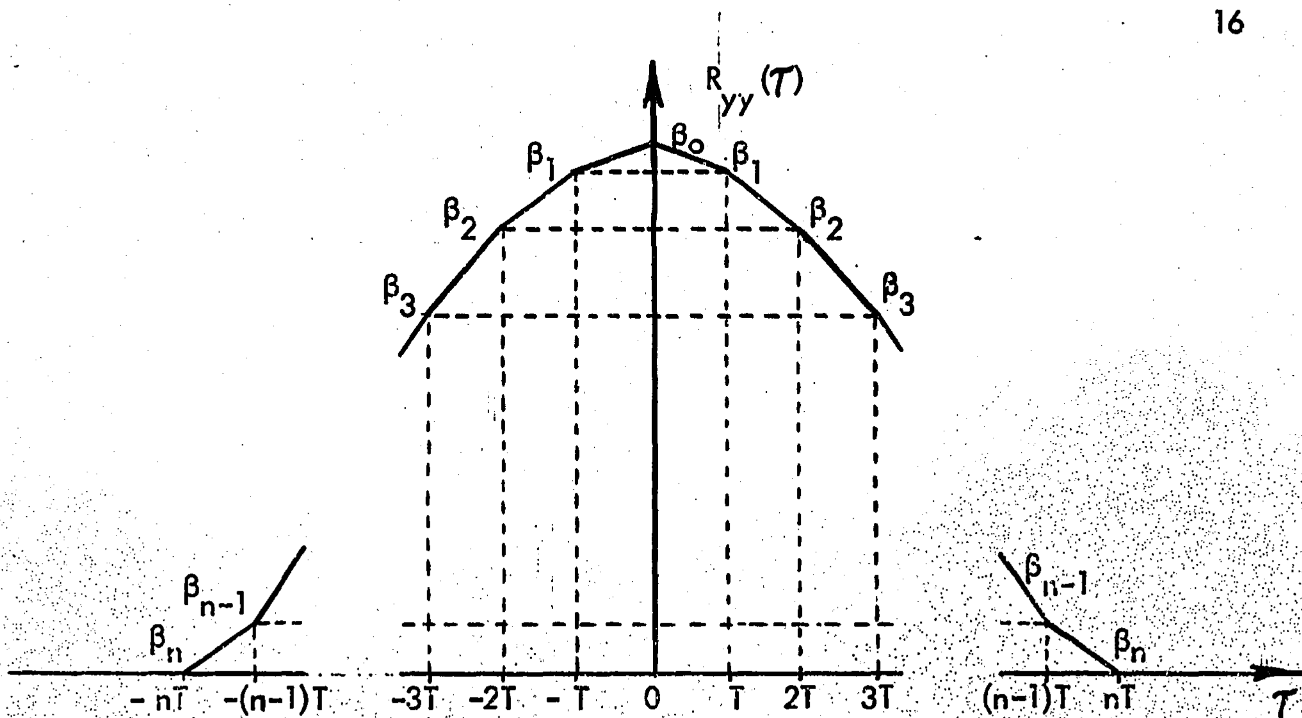

FIGURE 2-8

OUTPUT AUTOCORRELATION FUNCTION - RANDOM BINARY SIGNAL

It can easily be shown that

$$\beta_o = W_1^2 + W_2^2 + \dots + W_n^2$$

$$\beta_1 = W_1 W_2 + W_2 W_3 + \dots + W_{n-1} W_n$$

$$\beta_2 = W_1 W_3 + W_2 W_4 + \dots + W_{n-2} W_n$$

$$- - - - - -$$

or more generally,

$$\beta_k = \sum_{p=1}^{n-k} W_p \, W_{p+k} \qquad k = 0, 1, 2, \dots n-1$$

and (2.4-13)

$$\beta_n = 0$$

By substituting $R_p(\tau)$ (as defined in (2.2-2) and shown in Figure 2-4) for $R_o(\tau)$ in (2.4-12), we obtain the autocorrelation function corresponding to a weighted pseudo-random sequence generator which we write as

$$R_{(p)yy}(\tau) = \sum_{i,j=1}^{n} W_i W_j \; R_p \left[\tau + (i-j) \, T \right] \qquad (2.4\text{-}14)$$

In the same way as for the random binary sequence, it can be shown that the autocorrelation function of (2.4-14) is a periodic piecewise straight line curve as shown in Figure 2-9.

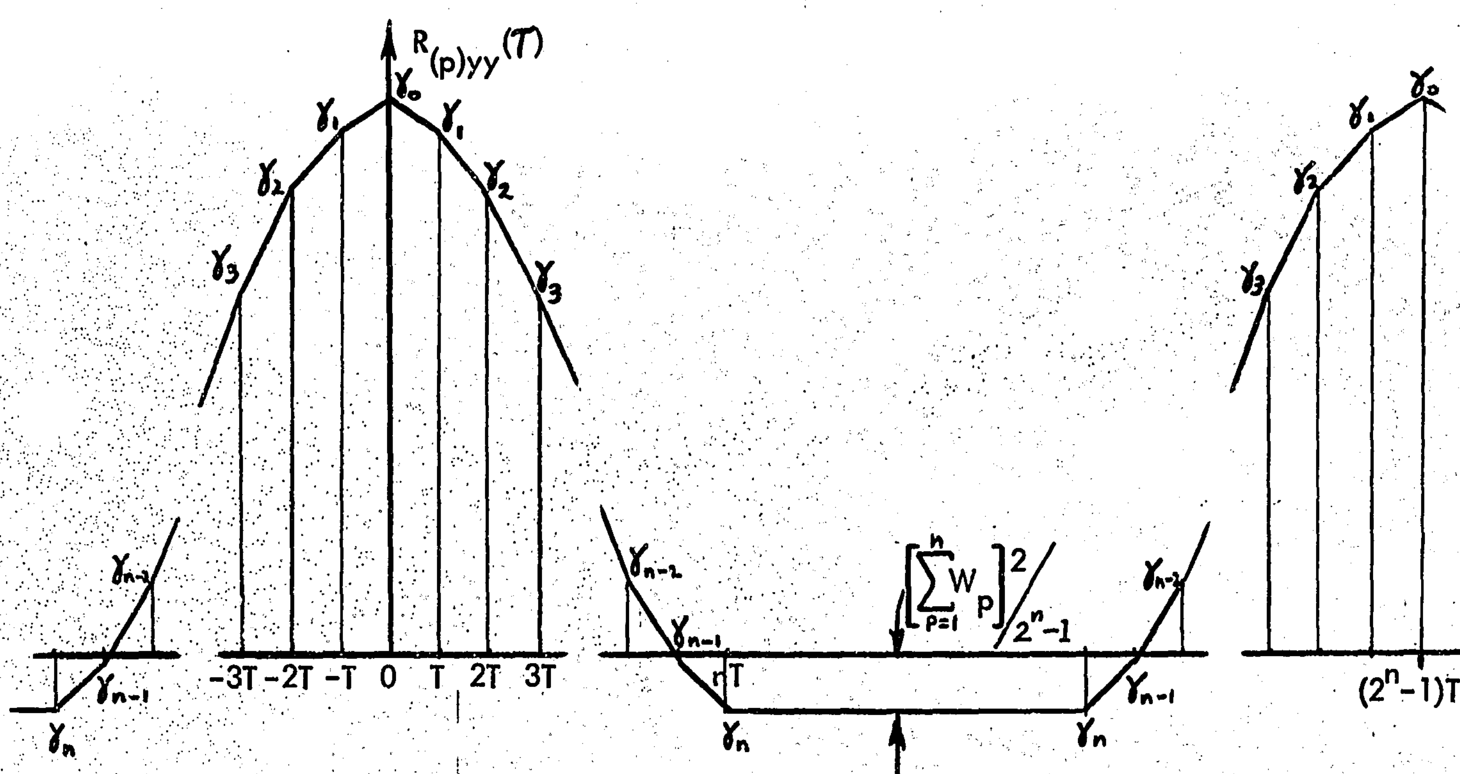

FIGURE 2-9 - OUTPUT AUTOCORRELATION FUNCTION - PSEUDO-RANDOM BINARY SIGNAL

It can also be shown that

$$\gamma_k = \frac{2^n \beta_k - \left[\sum_{p=1}^{n} W_p\right]^2}{2^n - 1} \qquad (2.4\text{-}15)$$

$$k = 0, 1, \ldots . n$$

and β_k is as defined in (2.4-13).

The power spectral density corresponding to a random binary process being fed into an arbitrarily weighted shift register may be obtained by using the Wiener-Khintchine relationship on (2.4-12). However, we shall first evaluate the spectral density in another manner by making the following useful observations.

Let us consider a signal W(t) which is a sequence of pulses separated by a period T ,

$$W(t) = \sum_{k=-\infty}^{\infty} a_k \ \delta(t - kT) \qquad (2.4\text{-}16)$$

where

$$a_k \text{ is either } +1 \text{ or } -1 \qquad (2.4\text{-}17)$$

and satisfies the "uncorrelated" properties *[38]

$$\lim_{n \to \infty} \frac{1}{2n} \sum_{m=-n}^{n} a_m \, a_{k+m} \begin{array}{l} = 1 \ ; \ k = 0 \\ = 0 \ ; \ k \neq 0 \end{array} \qquad (2.4\text{-}18)$$

Papoulis[39] shows that the autocorrelation of W(t) is

$$R_w(\tau) = \frac{1}{T} \, \delta(\tau) \qquad (2.4\text{-}19)$$

from which the power spectral density is flat

$$S_w(\omega) = \frac{1}{T} \qquad (2.4\text{-}20)$$

i.e., W(t) is "white noise".

The signal W(t), its autocorrelation function and spectral density is shown in Figure 2-10.

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* A sequence of uncorrelated numbers satisfying (2.4-17) and (2.4-18) is given by Wiener[38].

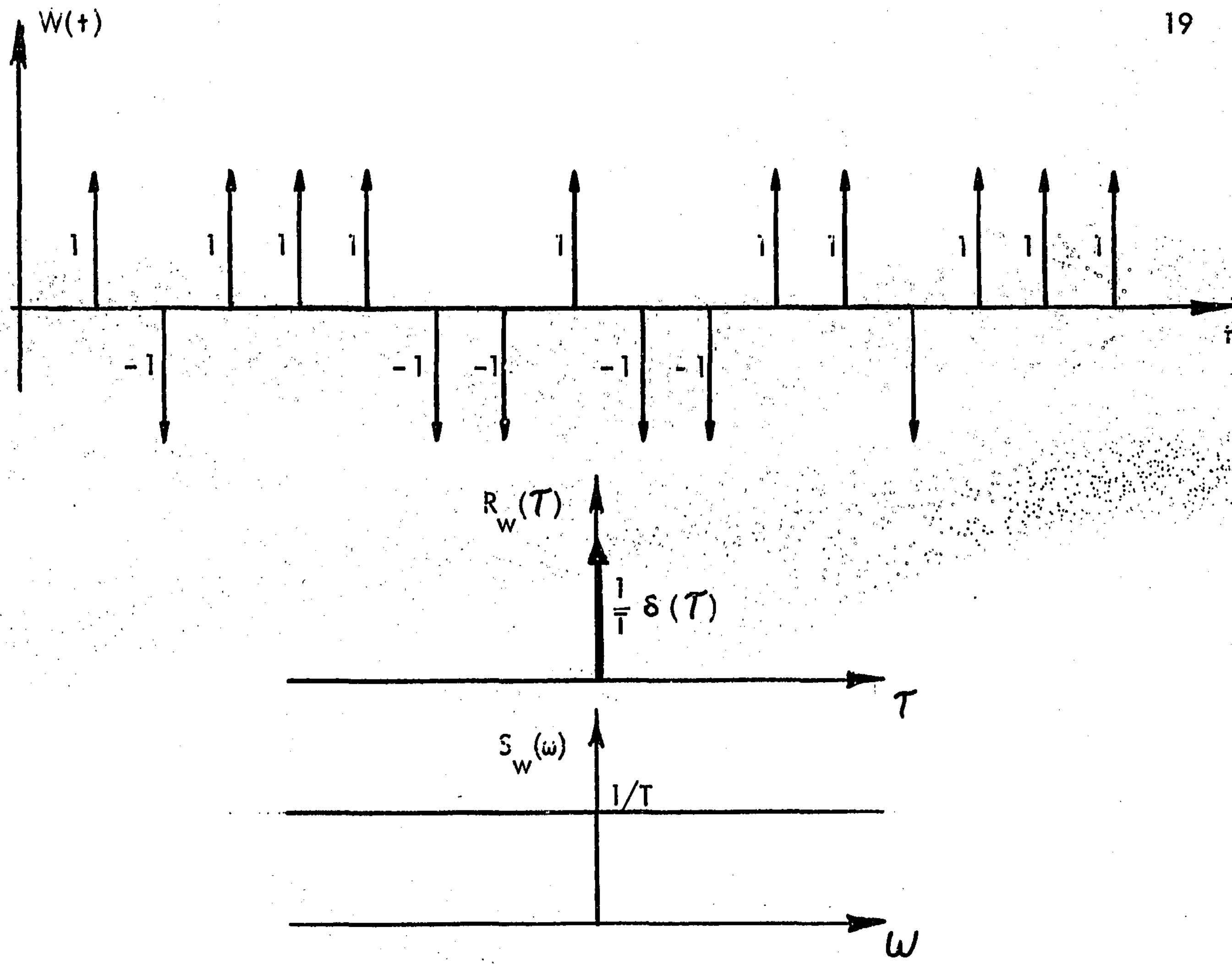

FIGURE 2-10 - SEQUENCE OF UNCORRELATED PULSES, ITS AUTO-CORRELATION FUNCTION AND POWER SPECTRAL DENSITY

Now if W(t) is fed into a zero order hold[40] with impulse response function

$$h_1(t) = H(t) - H(t - T) \qquad (2.4\text{-}21)$$

where H (t) is the Heaviside step function[18, 23], it is easily seen that the output will be the random binary signal x(t) of section 2.1. Thus a random binary signal may be considered as white noise being fed through a system having the impulse response function $h_1(t)$ of (2.4-21).

Making use of the fact [10] that continuous summation of digits stored in a shift register is equivalent to non-recursive digital filtering, the overall impulse response function may be written as

$$\hat{h}(t) = h_1(t) * h_2(t)$$
$$= \int_{-\infty}^{\infty} h_1(\mathcal{T})\, h_2(t-\mathcal{T})\, d\mathcal{T} \qquad (2.4\text{-}22)$$

where $h_2(t)$ is the impulse response function of the digital filter which is simply[41]

$$h_2(t) = \sum_{k=1}^{n} W_k\, \delta\,(t - k\,T) \qquad (2.4\text{-}23)$$

Substituting (2.4-23) and (2.4-21) into (2.4-22) and evaluating the integral, we obtain

$$\hat{h}(t) = \sum_{k=1}^{n} W_k \left\{ H\,(t - k\,T) - H\,(t - [k+1]\,T) \right\} \qquad (2.4\text{-}24)$$

where we have made use of[18]

$$\int_{-\infty}^{\infty} \psi(s)\, \delta(x - s)\, ds = \psi(x)$$

In order to find the system transfer function of the filter, we take the Fourier transform of (2.4-24).

Thus

$$\mathcal{H}_y(\omega) = \int_{-\infty}^{\infty} \hat{h}(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \sum_{k=1}^{n} W_k \left\{ H\left[t - k\,T\right] - H\left[t - (k+1)\,T\right] \right\} e^{-j\omega t}\, dt \qquad (2.4\text{-}25)$$

Taking the summation sign outside of the integral, we may write

$$\mathcal{H}_y(\omega) = \sum_{k=1}^{n} W_k \int_{kT}^{(k+1)T} e^{-j\omega t}\, dt$$

$$= \sum_{k=1}^{n} W_k \left\{ \frac{e^{-j\omega(k+1)T} - e^{-j\omega k T}}{-j\omega} \right\}$$

which can be reduced, with some manipulation, to

$$\mathcal{H}_y(\omega) = \frac{2\sin(\frac{\omega T}{2})}{\omega}\, e^{-j\frac{\omega T}{2}} \sum_{k=1}^{n} W_k\, e^{-j\omega kT}$$

$$= \frac{2\sin(\frac{\omega T}{2})}{\omega}\, e^{-j\frac{\omega T}{2}} \left\{ \sum_{k=1}^{n} W_k \cos(\omega kT) - j \sum_{k=1}^{n} W_k \sin(\omega kT) \right\}$$

(2.4-26)

Now, using the well known result for linear systems namely :

$$S_{yy} = \left| \mathcal{H}_y(\omega) \right|^2 S_w(\omega) \tag{2.4-27}$$

we may substitute (2.4-26) and (2.4-20) into (2.4-27) to obtain the result

$$S_{yy}(\omega) = T \left\{ \frac{\sin(\frac{\omega T}{2})}{(\frac{\omega T}{2})} \right\}^2 \left\{ \left[\sum_{k=1}^{n} W_k \cos(\omega kT) \right]^2 + \left[\sum_{k=1}^{n} W_k \sin(\omega kT) \right]^2 \right\}$$

(2.4-28)

Using a different notation, we may write

$$S_{yy}(\omega) = T \left\{ \frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)} \right\}^2 \left\{ \sum_{k,\,m=1}^{n} W_k W_m \left[\cos(\omega kT)\cos(\omega mT) + \sin(\omega kT)\sin(\omega mT) \right] \right\}$$

which reduces to

$$S_{yy}(\omega) = T \left\{ \frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)} \right\}^2 \left\{ \sum_{k,m=1}^{n} W_k W_m \cos\left[(k-m)\,\omega\, T \right] \right\} \tag{2.4-29}$$

As already has been mentioned, the same result may be obtained by taking the Fourier transform of the autocorrelation function of (2.4-12).

Making use of the translation property of Fourier transforms*, the transform of (2.4-12) may be written as

$$S_{yy}(\omega) = \sum_{k,\,m=1}^{n} W_k W_m \; e^{-j(m-k)\omega T} \; S(\omega) \tag{2.4-30}$$

where $S(\omega)$ is the spectrum of (2.1-4),

and can easily be reduced to (2.4-29). Although the last method of evaluating the power

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* If $\mathfrak{F}\left[f(t)\right] = F(\omega)$, then

$$\mathfrak{F}\left[f(t-a)\right] = e^{-ja\omega}\, F(\omega)$$

spectrum is much more efficient, the usefulness of the first technique shall become evident in Chapter III.

Substituting $S_p(\omega)$ of (2.2-3) for $S(\omega)$ in (2.4-30), the power spectral density corresponding to a weighted pseudo-random noise generator becomes

$$S_{(p)yy} = 2\pi\left(\frac{1}{2^n-1}\right)^2\left\{-(2^n-1)\,\delta(\omega) + \sum_{k=-\infty}^{\infty} 2^n \left[\frac{\sin\frac{k\pi}{2^n-1}}{k\pi/(2^n-1)}\right] \delta\left[\omega - \frac{2\pi k}{(2^n-1)\,T}\right]\right\} \times$$

$$\times \sum_{i,j=1}^{n} W_i W_j \cos\ (i-j)\,\omega T \qquad (2.4\text{-}31)$$

CHAPTER III

SPECTRAL DENSITY APPROXIMATION

3.0 Introduction

In Chapter II, we have evaluated the autocorrelation function, power spectral density, impulse response function and system transfer function resulting from a stationary random binary process being fed through an n-stage arbitrarily weighted shift register. It shall be of interest in this chapter to approximate a prescribed power spectral density with that obtained from the weighted register. The problem remains, however, to find a suitable error criterion so that we may optimize the weights for a finite length register to obtain a good approximation to the imposed spectral density.

In section 2.4, we have shown that a random binary process fed through a weighted shift register is equivalent to passing white-noise of the type shown in Figure 2-10 through a system with system transfer function $\mathcal{H}_y(\omega)$ given by (2.4-26). Since the power spectral density associated with white noise is a constant, we may use the result[39] that given an arbitrary positive function $S(\omega)$, we can find a function $\mathcal{H}(\omega)$ such that

$$K \left| \mathcal{H}(\omega) \right|^2 = S(\omega) \tag{3.0-0}$$

where K is a positive real constant.

Therefore, if K is set to equal $S_w(\omega)$ or $\frac{1}{T}$, and $S(\omega)$ is the prescribed power spectral density, there exists a corresponding system transfer function $\mathcal{H}(\omega)$ which we want to approximate. The approximation criterion discussed in this chapter shall be the mean-square-error between the system function corresponding to the imposed spectral density and that of the weighted register, which we define as

$$I = \int_{-\infty}^{\infty} \left| \mathcal{H}(\omega) - \mathcal{H}_y(n, \omega) \right|^2 d\omega \qquad (3.0\text{-}1)$$

where $\mathcal{H}_y(n, \omega)$ is the system function of (2.4-26). The n is inserted in the parentheses to indicate that $\mathcal{H}_y$ is a function of the register length n.

3.1 Impulse Response Approximation

Since spectra centred about a frequency ω_c may be obtained by modulating a low pass spectrum onto the carrier frequency, we shall restrict our discussion to low-pass system functions resulting in power spectral densities centred about zero-frequency.

Using Parseval's formula, which states that if f(t) and F(ω) are Fourier transform pairs, i.e., if $f(t) \leftrightarrow F(\omega)$, then

$$\int_{-\infty}^{\infty} \left| f(t) \right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| F(\omega) \right|^2 d\omega , \qquad (3.1\text{-}0)$$

and substituting the integrand of (3.0-1) into the right hand side of (3.1-0), we obtain

$$I = \int_{-\infty}^{\infty} \left| \mathcal{H}(\omega) - \mathcal{H}_y(n, \omega) \right|^2 d\omega = 2\pi \int_{-\infty}^{\infty} \left| h(t) - \hat{h}(n, t) \right|^2 dt \qquad (3.1\text{-}1)$$

where $h(t) \leftrightarrow \mathcal{H}(\omega)$ and $\hat{h}(n, t) \leftrightarrow \mathcal{H}_y(n, \omega)$.

From (3.1-1) we see that the approximation is transformed into the time domain and the criterion is therefore the mean-square-error between impulse response functions.

The impulse response function of (2.4-24) is illustrated in Figure 3-1.

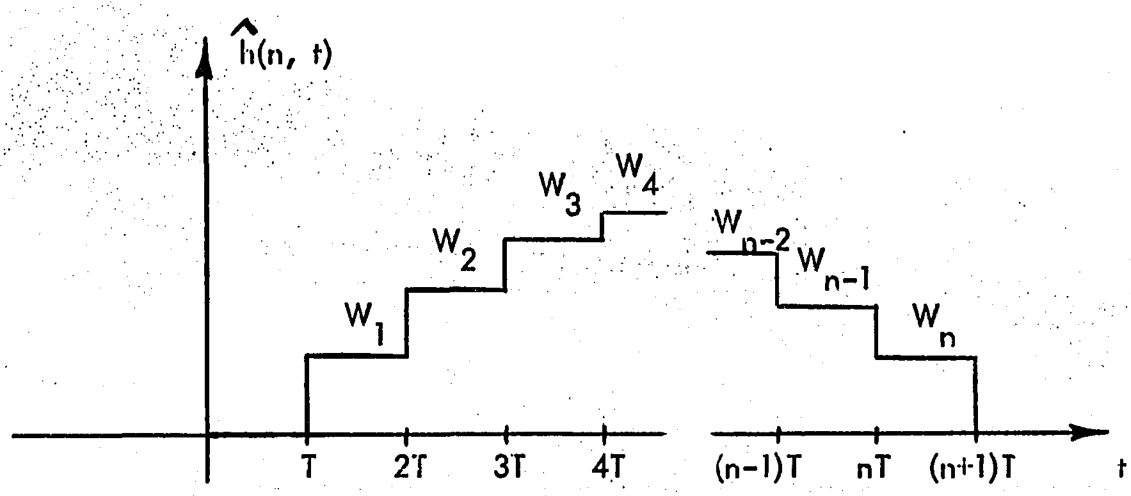

FIGURE 3-1 - IMPULSE RESPONSE FUNCTION CORRESPONDING TO A WEIGHTED SHIFT REGISTER

From Figure 3-1, it may be seen that the problem is reduced to approximating an arbitrary real finite* function of time with an equally-spaced discrete-step function of finite length. The mean-square-error between the two functions are minimized by finding the optimum weight parameters. Two other parameters are optimized, namely the spacing between steps (i.e. the clock pulse period T) and a translation or "starting point " parameter. The

---

* It is meant by finite that the function h(t) is square or Lebesque integrable;

i.e., $$\int_{-\infty}^{\infty} |h(t)|^2 \, dt < \infty$$

latter parameter is equivalent to the delay of the impulse response function. For convenience* we abandon the causality requirement that $\hat{h}(t) = 0$ for $t < 0$, and thus we are free to find the relative position of the two impulse response functions which minimizes the mean-square-error.

3.2 General Minimum Solution

Let f(t) be an arbitrary impulse response function, which is real and Lebeques integrable. Letting s and λ be the spacing and translation parameters respectively, we may write the mean-square-error as

$$I = \int_{-\infty}^{\infty} \left\{ \sum_{k=1}^{n} W_k \left[H(t - ks - \lambda) - H(t - (k+1)s - \lambda) \right] - f(t) \right\}^2 dt \tag{3.2-0}$$

Expanding the squared term in the integrand and making use of

$$H^2(x) = H(x)$$

we may write

$$I = \int_{-\infty}^{\infty} \sum_{k=1}^{n} W_k^2 \left[H(t - ks - \lambda) - H(t - (k+1)s - \lambda) \right] dt$$

$$- 2 \int_{-\infty}^{\infty} \sum_{k=1}^{n} W_k \left[H(t - ks - \lambda) - H(t - (k+1)s - \lambda \right] f(t)\, dt$$

$$+ \int_{-\infty}^{\infty} f(t)^2\, dt \tag{3.2-1}$$

* A delay in the impulse function does not affect the absolute value of the system function, and since we are not concerned with the phase of the output signal in relation to the input, the causality restriction is not required.

which may be written as

$$I = \sum_{k=1}^{n} W_k^2 \int_{ks+\lambda}^{(k+1)s+\lambda} dt - 2\sum_{k=1}^{n} W_k \int_{ks+\lambda}^{(k+1)s+\lambda} f(t)\, dt + \int_{-\infty}^{\infty} f(t)^2\, dt$$

and can be evaluated to yield

$$I = s\sum_{k=1}^{n} W_k^2 - 2\sum_{k=1}^{n} W_k \Big[F\big[(k+1)S+\lambda\big] - F\big[kS+\lambda\big] \Big] + \int_{-\infty}^{\infty} f(t)^2\, dt \tag{3.2-1}$$

where $F(x) = \int_{-\infty}^{x} f(t)\, dt$

Differentiating (3.2-1) with respect to W_k and equating to zero, we obtain the optimum weights

$$W_k = \frac{F\big[(k+1)s+\lambda\big] - F\big[ks+\lambda\big]}{s} \qquad k = 1, 2, \ldots\ldots n \tag{3.2-2}$$

for any spacing s and any translation parameter λ.

Substituting (3.2-2) into (3.2-1), we obtain the minimum mean-square-error for arbitrary s and λ. Thus

$$I = \int_{-\infty}^{\infty} f(t)^2\, dt - \frac{1}{s} \sum_{k=1}^{n} \Big\{ F\big[(k+1)s+\lambda\big] - F\big[ks+\lambda\big] \Big\}^2 \tag{3.2-3}$$

To find the optimum s and λ with the constraint that all the weights are optimized in the manner of (3.2-2), we differentiate (3.2-3) with respect to s to obtain

$$\frac{\partial I}{\partial s} = \frac{1}{s^2} \sum_{k=1}^{n} \left\{ F\left[(k+1)\ s+\lambda\right] - F\left[ks+\lambda\right] \right\}^2$$

$$- \frac{2}{s} \sum_{k=1}^{n} \left\{ F\left[(k+1)s+\lambda\right] - F\left[ks+\lambda\right] \right\} \left\{ (k+1)\ f\left[(k+1)\ s+\lambda\right] - k\ f\left[ks+\lambda\right] \right\}$$

$$\equiv 0 \qquad (3.2\text{-}4)$$

and with respect to λ to obtain

$$\frac{\partial I}{\partial \lambda} = -\frac{2}{s} \sum_{k=1}^{n} \left\{ F\left[(k+1)\ s+\lambda\right] - F\left[ks+\lambda\right] \right\} \left\{ f\left[(k+1)\ s+\lambda\right] - f\left[ks+\lambda\right] \right\}$$

$$\equiv 0 \qquad (3.2\text{-}5)$$

In general we cannot find a closed form solution for s and λ; however, we may find the optimum values using numerical optimization techniques. Direct search methods such as described in Appendix F can be used to minimize (3.2-3) with respect to the parameters s and λ. If, however, f(t) is continuous, we see that the gradient vector comprised of (3.2-4) and (3.2-5) is also continuous and we may use more quickly convergent techniques such as the method of conjugate gradients[42] or that of Fletcher and Powell[43].

3.3 Solution for Even Impulse Response Function

The problem is simplified if we assume the impulse response function f(t) is even. Consider the symmetrical function and the discrete step approximation in Figure 3-2.

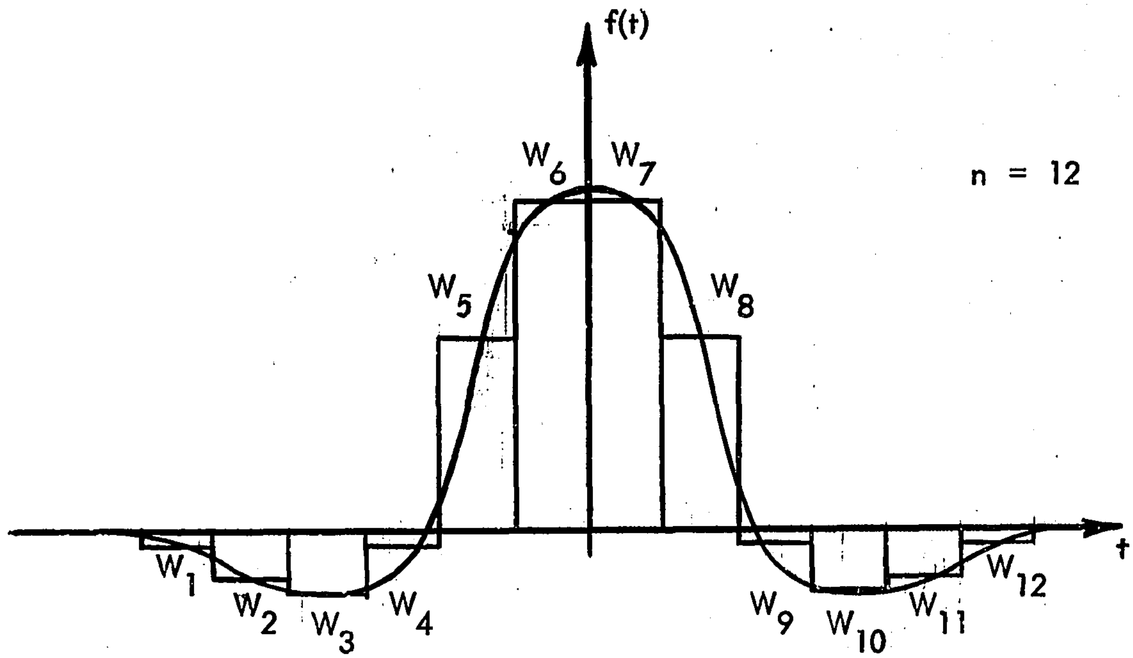

FIGURE 3-2 - EVEN ORDER APPROXIMATION TO IMPULSE RESPONSE FUNCTION

Assuming that the approximation is also an even function of t, we see that the problem of finding the optimum λ of (3.2-3) is eliminated. Two cases shall now be considered.

Case I : n is even

When the length of the shift register is even as in Figure 3-2, due to the symmetry, we need only to evaluate the first or last n/2 weights. Letting $m = \frac{n}{2}$ and renumbering the weights so that

$$W_k^{(new)} = W_{m+k}^{(old)} \qquad k = 1, 2, \ldots\ldots m$$

we can write the mean-square-error as

$$\overline{\epsilon^2_{even}} = 2 \int_0^{\infty} \left\{ \sum_{k=1}^{m} W_k \left[H\left[t - (k-1)s \right] - H\left[t - ks \right] \right] - f(t) \right\}^2 dt \tag{3.3-0}$$

Expanding the integral in the same way as in (3.2-1), we may write

$$\overline{\epsilon^2_{even}} = 2s \sum_{k=1}^{m} W_k^2 - 4 \sum_{k-1}^{m} W_k \left\{ F\left[ks \right] - F\left[(k-1)s \right] \right\} + 2 \int_0^{\infty} f^2(x)\, dx \tag{3.3-1}$$

Differentiating (3.3-1) with respect to W_k, we obtain

$$\frac{\partial \overline{\epsilon^2_{even}}}{\partial W_k} = 4s\, W_k - 4 \left\{ F\left[ks \right] - F\left[(k-1)\, s \right] \right\} \tag{3.3-2}$$

and differentiating again, we obtain

$$\frac{\partial^2 \overline{\epsilon^2_{even}}}{\partial W_k^2} = 4s \tag{3.3-3}$$

Since s, the spacing, is always positive, setting (3.3-2) to zero will result in a minimum of $\overline{\epsilon^2_{even}}$ with respect W_k. We thus obtain

$$W_{k_{opt}} = \frac{F[ks] - F[(k-1)\, s]}{s} \tag{3.3-4}$$

for $k = 1, 2, \ldots\ldots m$

which yields optimum weights for arbitrary s .

Substituting (3.3-4) into (3.3-1), differentiating with respect to s as in (3.2-4) and equating to zero, we obtain the result

$$s_{opt} = \frac{\sum_{k=1}^{m} \left\{ F\left[k s_{opt}\right] - F\left[(k-1) s_{opt}\right] \right\}^2}{2 \sum_{k=1}^{m} \left\{ F\left[k s_{opt}\right] - F\left[(k-1) s_{opt}\right] \right\} \left\{ k f\left[k s_{opt}\right] - (k-1) f\left[(k-1) s_{opt}\right] \right\}} \qquad (3.3\text{-}5)$$

which may be solved for s_{opt} using iterative methods. Thus, to find the optimum weights, we first evaluate s_{opt} using (3.3-5), substitute s_{opt} for s in (3.3-4) and evaluate $W_{k_{opt}}$ using (3.3-4), for k = 1, 2, ... m. The mean-square error will be

$$\overline{\epsilon^2_{even}} = 2 \int_0^{\infty} f^2(x)\, dx - 2 s_{opt} \sum_{k=1}^{m} W_{k_{opt}}^2 \qquad (3.3\text{-}6)$$

<u>Case II : n is odd</u>

Similar results may be obtained when the number of shift register stages is odd. When this is the case, we see that the spacing corresponding to W_1 is $\frac{s}{2}$ as is shown in Figure 3-3 .

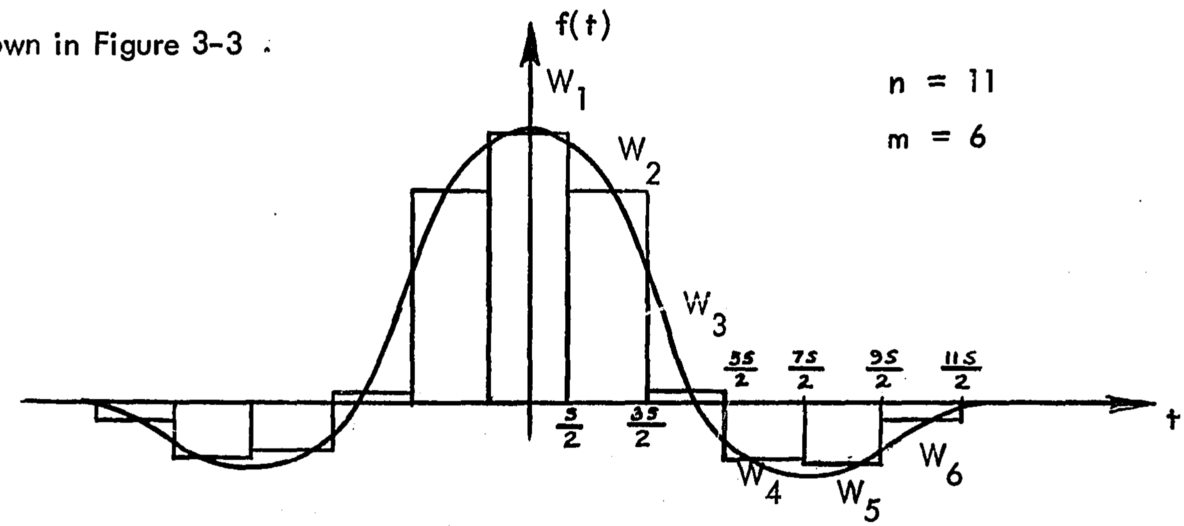

FIGURE 3-3 - ODD ORDER APPROXIMATION TO IMPULSE RESPONSE FUNCTION

Letting $m = \frac{n+1}{2}$, we modify (3.3-0) to be

$$\overline{\epsilon^2_{odd}} = 2\int_0^\infty \left\{ W_1 \left[H(t) - H(t-\frac{s}{2})\right] + \sum_{k=2}^{m} W_k \left[H\left[t - \frac{(2k-3)s}{2}\right] - H\left[t - \frac{(2k-1)s}{2}\right]\right] - f(t) \right\}^2 dt \tag{3.3-7}$$

Proceeding in the same fashion as for Case I, we obtain the results

$$s_{opt} = \frac{2\left[F(\frac{s_{opt}}{2}) - F(o)\right]^2 + \sum_{k=2}^{m}\left\{F\left[\frac{(2k-1)s_{opt}}{2}\right] - F\left[\frac{(2k-3)s_{opt}}{2}\right]\right\}^2}{\left[(\frac{s_{opt}}{2}) - F(o)\right] f(\frac{s_{opt}}{2}) + 2\sum_{k=2}^{m}\left\{F\left[\frac{(2k-1)s_{opt}}{2}\right] - F\left[\frac{(2k-3)s_{opt}}{2}\right]\right\}\left\{\frac{(2k-1)}{2} f\left[\frac{(2k-1)s_{opt}}{2}\right] - \frac{(2k-3)}{2} f\left[\frac{(2k-3)s_{opt}}{2}\right]\right\}} \tag{3.3-8}$$

$$W_{1_{opt}} = \frac{2\ F(\frac{s_{opt}}{2}) - F(o)}{s_{opt}}$$

$$W_{k_{opt}} = \frac{F\left[\frac{(2k-1)}{2} s_{opt}\right] - F\left[\frac{(2k-3)}{2} s_{opt}\right]}{s_{opt}} \tag{3.3-9}$$

$$(k = 2, 3, \ldots\ldots, m)$$

and

$$\overline{\epsilon^2_{odd}} = \int_0^\infty f(x)^2\, dx - \frac{s_{opt}\, W_1^2}{2} - s_{opt} \sum_{k=2}^{m} W^2_{k_{opt}} \tag{3.3-10}$$

3.4 Gaussian and Brickwall Spectral Densities

In a recent note, the author[12] has shown that an approximation to a Gaussian spectral density can be obtained by weighting the shift register with the binomial coefficients. It is of interest in this section to approximate the same power spectral density using the error criterion discussed in this chapter. Defining the spectral density as

$$S(\omega) = e^{-\omega^2} \tag{3.4-0}$$

a choice for $\mathcal{H}(\omega)$ in (3.0-0) may be

$$\mathcal{H}(\omega) = e^{-\frac{\omega^2}{2}} \tag{3.4-1}$$

which is also Gaussian. Taking the inverse Fourier transform of (3.4-1), we obtain the impulse response function

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} e^{j\omega t}\, d\omega = \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} \tag{3.4-2}$$

which is also a Gaussian pulse. Since (3.4-2) is an even function of time, we may make use of the results of section 3.3. Letting

$$Z(t) = \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} \tag{3.4-3}$$

and

$$P(t) = \int_{-\infty}^{t} Z(x)\, dx$$

we may substitute P(x) and Z(x) for F(x) and f(x), respectively, in the equations of section 3.3.

Appendix A consists of a computer printout of the optimum weights, spacing parameter and mean-square-error for even and odd n up to 50. Due to the symmetry of the apporximation, only the last n/2 and (n + 1)/2 weights are listed for even and odd orders n respectively. In Graph 1 the normalized Gaussian system transfer function is illustrated, and compared to the optimum approximations for n = 1, 4, 6, 8 and 48 in Graphs 2, 3, 4, 5 and 6. In Graph 7, the optimum order 50 impulse response approximation to a Gaussian impulse is shown. A plot is made of the mean-square-error, for n up to 50, in Graph 8.

Similar results may be obtained for the brickwall spectral density which may be defined as

$$S(\omega) = 1; \quad |\omega| \leq \pi \qquad (3.4\text{-}5)$$
$$= 0; \quad |\omega| > \pi$$

Thus a convenient choice for $\mathcal{H}(\omega)$ in (3.0-0) may be

$$\mathcal{H}(\omega) = S(\omega) \qquad (3.4\text{-}6)$$

which is the system transfer function of an ideal brickwall filter. The impulse response corresponding to the system transfer function of (3.4-6) may be shown to be[11]

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{H}(\omega)\, e^{j\omega t}\, d\omega = \frac{\sin(t)}{t} \qquad (3.4\text{-}7)$$

In the same way as for the case of a Gaussian spectral density, we let

$$f(x) = \mathrm{si}(x) = \frac{\sin(x)}{x} \qquad (3.4\text{-}8)$$

and

$$F(x) = \mathrm{Si}(x) = \int_{-\infty}^{x} \mathrm{si}(t)\, dt \qquad (3.4\text{-}9)$$

and use section 3.3 to obtain the optimum weights.

In Appendix B, we list the optimum weights, spacing parameter and mean-square-error for even n up to 50. It is interesting to note that the optimum spacing parameter does not decrease uniformly as the order of the approximation is increased*. This is due to the oscillatory nature of the si function of (3.4-8) which gives rise to multiple eigenvalues for equation (3.3-5). This effect can best be seen in Figure 3-4 where the mean-square-error is plotted versus the spacing parameter s . The curves are not drawn to scale in order to emphasize their behaviour as the parameter n is increased.

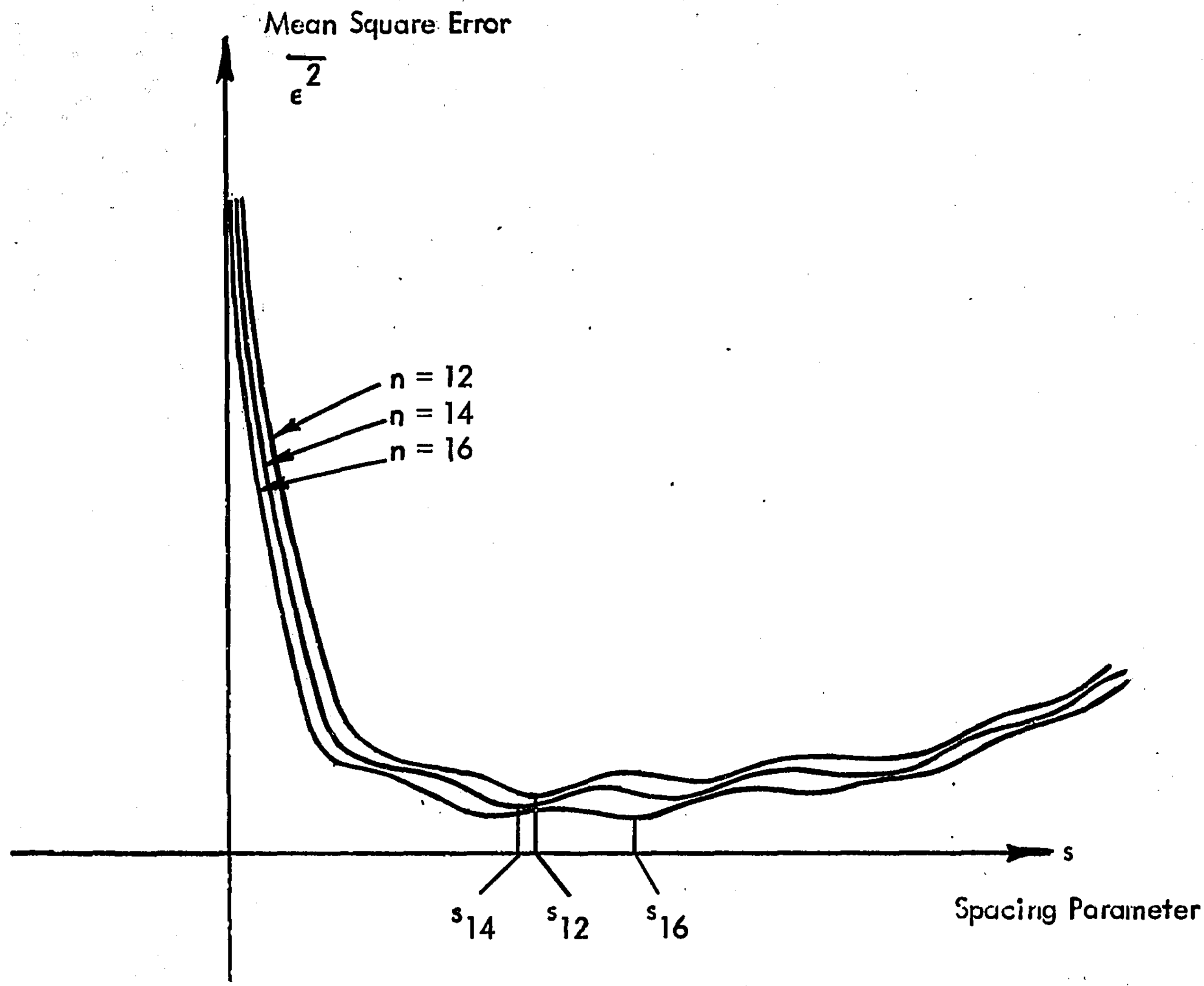

FIGURE 3-4 - BEHAVIOUR OF MEAN-SQUARE-ERROR FOR BRICKWALL SPECTRAL DENSITY APPROXIMATION

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* From Appendix B we see that the optimum spacing parameter increases between orders 6 and 8, 14 and 16, 26 and 28 and between orders 40 and 42.

CHAPTER IV

PROBABILITY DENSITY APPROXIMATION

4.0 Introduction

The probability density function of pseudo-random and random binary waveforms, obtained by equal summation of the digits stored in the shift register, has been studied by Kramer[8] and Davies[9]. The probability density thus obtained is the binomial distribution. Davies extends this result to multilevel sequences, but again, only for the equally weighted case.

In this chapter, we wish to extend these results by finding the output amplitude probability density function of an arbitrarily weighted shift register fed by a random binary sequence. It can be shown that a random process obtained in this way will satisfy the conditions of the Central Limit Theorem, provided that the weights satisfy certain conditions[14, 29]. Thus, although the probability density approaches a normal or Gaussian curve as the number of shift register stages increases indefinately, the problem of finding the best approximation to a normal density still exists when the number of shift register stages is finite. An approximation criterion between the probability density of an arbitrarily weighted shift register and a normal density is introduced, and the weights are optimized to minimize the error.

4.1 Amplitude Probability Density for Arbitrary Weights

We again define the random binary process x(t) by

$$x(t) = \begin{cases} +1 \text{ if binary signal is } 1 \\ -1 \text{ if binary signal is } 0 \end{cases} \Bigg| \quad (N-1)T \leq t < NT \tag{4.1-0}$$

for $N = 0, \pm 1, \pm 2, \pm 3, \pm \ldots\ldots$

and

$$P\left[x(t) = 1\right] = P\left[x(t) = -1\right] = 1/2 \tag{4.1-1}$$

We also denote $x_i(t)$ the random process associated with the event $\{i^{th}$ stage of an n stage shift register being on$\}$

by

$$x_i(t) = \begin{cases} W_i \text{ if the } i^{th} \text{ stage is on} \\ -W_i \text{ if the } i^{th} \text{ stage is off} \end{cases} \Bigg| \quad (n-1)T \leq t < nT \tag{4.1-2}$$

for $n = 0, \pm 1, \pm 2, \pm 3, \pm \ldots\ldots$

and

$$P\left[x_i(t) = W_i\right] = P\left[x_i(t) = -W_i\right] = 1/2 \tag{4.1-3}$$

where W_i is the weight of the i^{th} shift register stage and T is one period of the clock frequency. When t is fixed, we may consider the family of stochastic processes $x_i(t)$, $i = 1, 2, \ldots.n$, as a family of random variables x_i. The probability density $f_i(x)$ of x_i is shown in Figure 4.1. Since x_i is a random variable of discrete type, with only possible values W_i and $-W_i$, we can define the characteristic function[15] of x_i as

$$\phi_i(\omega_i) = E\left\{\exp(j\omega_i x_i)\right\}$$

$$= \exp(j\omega_i W_i) \; P\left[x_i = W_i\right] + \exp(-j\omega_i W_i) \; P\left[x_i = -W_i\right] \tag{4.1-4}$$

Substituting the probabilities of (4.1-3) into (4.1-4) we obtain

$$\phi_i(\omega_i) = \frac{\exp(j\omega_i W_i) + \exp(-j\omega_i W_i)}{2} = \cos(\omega_i W_i) \tag{4.1-5}$$

Since the random variables x_i, $i = 1, 2, \ldots .n$ are independent, their characteristic functions are independent[16], thus

$$\phi(\omega_1, \ldots\ldots\omega_n) = E\left\{\exp\left[j(\omega_1 x_1 + \ldots + \omega_n x_n)\right]\right\}$$

$$= \phi_1(\omega_1) \; \phi_2(\omega_2) \ldots\ldots \phi_n(\omega_n) \tag{4.1-6}$$

where $\phi(\omega_1, \ldots .\omega_n)$ is the joint characteristic function of the random variables x_i.

We now wish to find the characteristic function $\phi_z(\omega)$ of the density $f_z(x)$ corresponding to the random variable

$$z = \sum_{i=1}^{n} x_i \tag{4.1-7}$$

We have from (4.1-6)

$$\phi_z(\omega) = E\left\{\exp(j\omega z)\right\} = E\left\{\exp\left[j\omega(x_1 + x_2 + \ldots + x_n)\right]\right\}$$

$$= \phi(\omega, \omega \ldots \omega) = \phi_1(\omega) \; \phi_2(\omega) \ldots\ldots \phi_n(\omega) \tag{4.1-8}$$

Substituting (4.1-5) into (4.1-8), we can show that

$$\phi_z(\omega) = \frac{1}{2^n} \prod_{k=1}^{n} \left\{ \exp(j W_k \omega) + \exp(-j W_k \omega) \right\}$$

$$= \prod_{k=1}^{n} \cos(W_k \omega) \tag{4.1-9}$$

In order to facilitate the mathematics, we shall now introduce some new notation.

Let $\vec{b}_{k,n}$ be an n dimensional vector, $k = 0, 1, \ldots, 2^n - 1$, whose elements are the ordered digits of the binary number corresponding to k(e.g., $\vec{b}_{7,4} = [0\,1\,1\,1]$). Let $\vec{V}_{k,n}$ be defined as

$$\vec{V}_{k,n} = 2\,\vec{b}_{k,n} - \vec{u}_n \tag{4.1-10}$$

where $\vec{u}_n$ is the n dimensional vector with all elements equal to unity. Thus, we see that $\vec{V}_{k,n}$ is $\vec{b}_{k,n}$ with all zero elements replaced by -1.

Using the above notation and the ordinary definition of the scalar product, we can write (4.1-9) as

$$\phi_z(\omega) = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \exp\left[j\omega \left\langle \vec{V}_{k,n}, \vec{W} \right\rangle \right] \tag{4.1-11}$$

where $\vec{W}$ is the n - dimensional vector with elements W_i, $i = 1, 2, \ldots n$.

To illustrate (4.1-11), consider n = 2. Then,

$$\phi_z(\omega) = \frac{1}{4} \Big\{ \exp\left[j\omega(-W_1 - W_2)\right] + \exp\left[j\omega(-W_1 + W_2)\right] + \exp\left[j\omega(W_1 - W_2)\right] + \exp\left[j\omega(W_1 + W_2)\right] \Big\} \tag{4.1-12}$$

which yields the same result as would (4.1-9).

Thus it may be seen that $\left\langle \vec{V}_{k,n}, \vec{W} \right\rangle$ for all $k = 0, \ldots 2^n - 1$, is the combination of all summations of $\pm W_i$, $i = 1, 2, \ldots n$. It should be noted that for any function G,

$$\sum_{k=0}^{2^n-1} G\left[\langle \vec{V}_{k,n}, \vec{W} \rangle\right] = \sum_{k=0}^{2^n-1} G\left[-\langle \vec{V}_{k,n}, \vec{W} \rangle\right] \qquad (4.1\text{-}13)$$

Using the inversion formula[17], the density $f_z(x)$ of (4.1-7) can be expressed in terms of $\phi_z(\omega)$ by the integral

$$f_z(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_z(\omega)\, e^{-j\omega x}\, d\omega \qquad (4.1\text{-}14)$$

Substituting (4.1-11) into (4.1-14) and using (4.1-13), we obtain

$$f_z(x) = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left\{-j\omega\left[x + \langle \vec{V}_{k,n}, \vec{W} \rangle\right]\right\} d\omega \qquad (4.1\text{-}15)$$

Based on the property[18]

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-j\omega x)\, d\omega = \delta(x) \qquad (4.1\text{-}16)$$

of the Dirac delta function $\delta(x)$, (4.1-15) reduces to

$$f_z(x) = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \delta\left(x + \langle \vec{V}_{k,n}, \vec{W} \rangle\right) \qquad (4.1\text{-}17)$$

The result obtained in (4.1-17) is the amplitude probability density corresponding to a weighted random binary process. To modify this result for n^{th} order maximum length pseudo-random binary sequences, we make the following observations.

In section 2.2 it was shown that in a maximum length pseudo-random binary sequence, the all zero state in the shift register does not occur and that there are only $2^n - 1$ possible states. This undesired state can be shown to correspond to $k = 0$ in (4.1-17). The amplitude probability density corresponding to an m - sequence can therefore be written as

$$f_{zm}(x) = \frac{1}{2^n-1} \sum_{k=1}^{2^n-1} \delta\left(x + \langle \vec{V}_{k,n}, \vec{W} \rangle\right) \qquad (4.1\text{-}18)$$

4.2 The Central Limit Theorem and Equal Weights

The Central - Limit Theorem states that, under certain conditions $f_z(x)$ of (4.1-17) approaches a normal curve as n increases :

$$f_z(x) \simeq \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ - x^2 \Big/ \sum_{k=1}^{n} \sigma_i^2 \right\} \qquad (4.2\text{-}0)$$

where σ_i^2 is the variance of x_i, which can easily be shown to be*

$$\sigma_i^2 = W_i^2$$

If x is properly scaled (i.e., with $1/\sqrt{n}$ so that the limit of the resulting variance is finite), then (4.2-0) becomes an equality for $n \rightarrow \infty$ provided :

(a)

$$\sum_{k=1}^{n} W_i^2 \rightarrow \infty \qquad (4.2\text{-}1)$$

(b) For some $\alpha > 2$

$$\int_{-\infty}^{\infty} x^\alpha f_i(x)\, dx < C = \text{constant} \qquad (4.2\text{-}2)$$

where $f_i(x)$ is the probability density of x_i.

---

* $\sigma_i^2 = E\left\{x_i^2\right\} = W_i^2 P\left[x_i = W_i\right] + (-W_i)^2 P\left[x_i = -W_i\right]$

$= W_i^2 \times 0.5 + (-W_i)^2 \times 0.5 = W_i^2$

These are not the most general conditions for the validity of the theorem. However, they cover a wide range of applications. We note that (4.2-1) is satisfied if $\sigma_i > C' > 0$, where C' is some constant, and this is certainly the case if the random variables x_i have equal variances, i.e., if the weights of all the registers are equal. Similarly, condition (4.2-2) is satisfied because the densities $f_i(x)$ are zero for $|x| > W_i$.

Now for equal weights $W_i = W$, i = 1, 2, ...n, (4.1-9) may be written as

$$\phi_z(\omega) = \cos^n(W\omega) = \frac{(e^{jW\omega} + e^{-jW\omega})^n}{2^n} \tag{4.2-3}$$

and using the binomial theorem, we may write

$$\phi_z(\omega) = \frac{e^{jnW\omega}}{2^n} \sum_{k=0}^{n} \binom{n}{k} e^{-j2W\omega} \tag{4.2-4}$$

where $\binom{n}{k}$ are the binomial coefficients.

Using the inversion formula (4.1-14) and property (4.1-13), we obtain, with some manipulation

$$f_z(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left\{-j\omega\left[W(n-2k)+x\right]\right\} d\omega \tag{4.2-5}$$

and making use of (4.1-16),

we obtain

$$f_z(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \delta\left[W(n-2k)+x\right] \tag{4.2-6}$$

which is a binomial probability density with zero mean.

This result is the same as that obtained by Davies[9], except that the density is centred about zero. This of course, is simply the consequence of defining the off state of the register as − 1 instead of zero.

Using (4.2-6) and the previous discussion, we obtain the well known result that the binomial density approaches the normal or Gaussian density as $n \rightarrow \infty$.

4.3 Binomial Weights

Before considering the problem of finding the weights which correspond to the best approximation to a Gaussian probability density function, it would be of interest to discuss the probability density which results from using the binomial coefficients as the shift register weights. It has been shown[12] that a binomial weighted register produces a power spectral density which approximates a Gaussian, and the approximation becomes exact as the length of the shift register increases indefinately. We shall now consider the resulting probability density.

Defining the weights of an n + 1 stage shift register* as

$$W_k = \binom{n}{k}, \quad k = 0, 1, \ldots\ldots n \qquad (4.3\text{-}0)$$

let us determine if the conditions for the Central Limit Theorem are met. Although we can see that the binomial weights satisfy condition (4.2-1), they do not satisfy (4.2-2). As it was mentioned earlier, however, these are not the most general conditions for the theorem, and we shall now prove the Gaussian convergence of the probability density directly.

Although this result may be obtained using characteristic functions as in (4.1-4), we prefer to use moment generating functions. We define the moment generating function[19] of x_i as

* We consider an n+1 stage register because the binomial expansion of order n has n+1 terms.

$$M_i(\omega_i) = E\left\{\exp \quad \omega_i x_i\right\} \qquad (4.3\text{-}1)$$

Proceeding in the same manner as was done for the characteristic function and substituting the binomial coefficients for the weights, we can show that

$$M_z(\omega) = \prod_{r=0}^{n} \cosh\left[\omega\binom{n}{r}\right] \qquad (4.3\text{-}2)$$

Before proceeding further, we must standardize the random variable, i.e., modify it to have zero mean and unit variance.

Using the moment generating function property[19], namely

$$\left[\frac{d^r \; M_z(\omega)}{d\omega^r}\right]_{\omega=0} = \mu'_r \qquad (4.3\text{-}3)$$

where μ'_r is the r^{th} moment about the origin, we can show that the mean is given by*

$$\mu_z = 0 \qquad (4.3\text{-}4)$$

It can also be shown that the variance is given by**

$$\sigma_z^{\,2} = \sum_{r=0}^{n}\binom{n}{r}^2 \qquad (4.3\text{-}5)$$

---

* $$\mu = \left.\frac{d \prod_{r=0}^{n} \cosh\left[\omega\binom{n}{r}\right]}{d\omega}\right|_{\omega=0} = \left.\sum_{r=0}^{n}\binom{n}{r}\tanh\left[\omega\binom{n}{r}\right]\prod_{k=0}^{n}\cosh\left[\omega\binom{n}{k}\right]\right|_{\omega=0} = 0$$

** $$\sigma_z^{\,2} = \sum_{r=1}^{n+1}\sigma_i^{\,2} = \sum_{r=1}^{n+1}W_i^{\,2} = \sum_{r=0}^{n}\binom{n}{r}^2$$

Making use of the theorem[20] for standardizing a random variable, namely,

$$M_{(x-\mu)/\sigma}(\omega) = \exp\left[-(\mu\omega/\sigma)\right] M_x(\omega/\sigma) \tag{4.3-6}$$

we obtain

$$M_{zs}(\omega) = \prod_{r=0}^{n} \cosh\left[\frac{\omega\binom{n}{r}}{\sigma_z}\right] \tag{4.3-7}$$

Now using the infinite product expansion[21]

$$\cosh(Z) = \prod_{k=1}^{\infty}\left[1 + \frac{4Z^2}{(2k-1)^2\pi^2}\right] \tag{4.3-8}$$

and substituting in (4.3-7), we obtain

$$M_{zs}(\omega) = \prod_{r=0}^{n}\prod_{k=1}^{\infty}\left[1 + \frac{4\omega^2\binom{n}{r}^2}{\sigma_z^2\pi^2(2k-1)^2}\right] \tag{4.3-9}$$

Defining the second moment generating function as

$$\Psi(\omega) = \log M(\omega) \tag{4.3-10}$$

and taking the logarithm on both sides of (4.3-9), we obtain

$$\Psi_{zs}(\omega) = \log \prod_{r=0}^{n}\prod_{k=1}^{\infty}\left\{1 + \frac{4\omega^2\binom{n}{r}^2}{\sigma_z^2\pi^2(2k-1)^2}\right\}$$

$$= \sum_{r=0}^{n}\sum_{k=1}^{\infty}\log\left\{1 + \frac{4\omega^2\binom{n}{r}^2}{\sigma_z^2\pi^2(2k-1)^2}\right\} \tag{4.3-11}$$

Making use of the logarithmic expansion, namely,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots$$

$$= -\sum_{q=1}^{\infty} \frac{(-x)^q}{q} \qquad |x| < 1$$

for sufficiently large* n we may write

$$\Psi_{zs}(\omega) = -\sum_{r=0}^{n} \sum_{k=1}^{\infty} \sum_{q=1}^{\infty} \left[\frac{-4\omega^2 \binom{n}{r}^2}{\sigma_z^2 \pi^2 (2k-1)^2} \right]^q \Big/ q$$

and with some manipulation, we obtain

$$\Psi_{zs}(\omega) = -\sum_{k=1}^{\infty} \sum_{q=1}^{\infty} \frac{(-1)^q \, 4^q \, \omega^{2q} \sum_{r=0}^{n} \binom{n}{r}^{2q}}{q \, \sigma_z^{2q} \, \pi^{2q} (2k-1)^{2q}} \tag{4.3-12}$$

Making use of the expansion for Bernoulli numbers**[22] namely,

$$B_q = \frac{2(2q)!}{\pi^{2q}(2^{2q}-1)} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2q}}$$

with some manipulation we obtain from (4.3-12)

$$\Psi_{zs}(\omega) = \sum_{q=1}^{\infty} \frac{(-1)^{q+1} (2^{2q}-1) B_q \, 4^q \, \omega^{2q} \sum_{r=0}^{n} \binom{n}{r}^{2q}}{2q \, (2q)! \, \sigma_z^{2q}} \tag{4.3-13}$$

---

* It can easily be seen, by substituting $\sum_{r=0}^{n} \binom{n}{r}^2$ for σ_z^2, that

$$x = \frac{4\omega^2 \binom{n}{r}^2}{\sigma_z^2 \pi^2 (2k-1)^2}$$ will approach zero as $n \to \infty$.

Therefore, for sufficiently large n, $|x|$ will be less than 1.

** For example, $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$ etc.

After substituting (4.3-5) into (4.3-13), we may write

$$\Psi_{zs}(\omega) = \sum_{q=1}^{\infty} A_q \, \omega^{2q} \, \frac{\sum_{r=0}^{n} \binom{n}{r}^{2q}}{\left[\sum_{r=0}^{n} \binom{n}{r}^{2} \right]^{q}} \tag{4.3-14}$$

where

$$A_q = \frac{(-1)^{q+1} \, (2^{2q} - 1) \, B_q \, 4^q}{2q \, (2q)!} \tag{4.3-15}$$

Now for $q > 1$, i.e., $q = 2, 3 \ldots\ldots$

$$\lim_{n \to \infty} \sum_{r=0}^{n} \binom{n}{r}^{2q} \Big/ \left[\sum_{r=0}^{n} \binom{n}{r}^{2} \right]^{q} = 0 \tag{4.3-16}$$

and therefore from (4.3-14) — (4.3-16)

$$\lim_{n \to \infty} \Psi(\omega) = A_1 \omega^2 = \frac{\omega^2}{2} \tag{4.3-17}$$

Using the fact that the limit of a logarithm equals the logarithm of the limit (provided that these limits exist), we see, from (4.3-10) that

$$\lim_{n \to \infty} M_{zs}(\omega) = \exp(\omega^2 / 2) \tag{4.3-18}$$

which is the moment generating function of the standard Gaussian density[20].

Referring to the uniqueness theorem[20] according to which a moment generating function, when it exists, determines a unique probability density, we can now say that when $n \to \infty$ the standardized probability density, resulting from using binomial coefficients of order $n - 1$ as weights for a shift register of length n, approaches the standard Gaussian density.

The results of this section, coupled with those of the author[12] reveal that we can use binomial coefficient weights to approximate a signal which is Gaussian both in the frequency and time domain. Such a spectrum is useful for simulation purposes, and particularly for doppler radar simulators.

4.4 Approximation Criterion

As already has been shown in (4.1-7) the probability density resulting from a weighted register is of discrete type. On the other hand, the Gaussian probability density is continuous. This incompatability between the densities implies the ruling out of some of the approximation criteria commonly in use, such as the mean-square-error* or the mean-absolute-error** between the densities.

---

* The mean-square-error is

$$\overline{\epsilon^2} = \int_{-\infty}^{\infty} \left[\frac{1}{2^n} \sum_{k=0}^{2^n-1} \delta\left(x + \langle \vec{V}_{kn}, \vec{W} \rangle\right) - Z(x) \right]^2 dx$$

where Z(x) is the standardized Gaussian curve, i.e.,

$$Z(x) = \exp(-x^2/2) / \sqrt{2\pi} .$$

Expanding the squared integrand gives rise to terms of the form $\delta^2(x + A)$, where A is a constant, and can be shown to have no meaning[23,33].

** The mean absolute error, written as

$$\overline{|\epsilon|} = \int_{-\infty}^{\infty} \left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} \delta\left(x + \langle \vec{V}_{kn}, \vec{W} \rangle\right) - Z(x) \right| dx$$

can be shown to equal 2, irregardless of the weight vector used.

Let us consider the integral of the probability density resulting from a weighted shift register, i.e., the probability distribution function. Taking the integral of (4.1-17), it is easy to show that

$$F_z(x) = \int_{-\infty}^{x} f_z(t)\,dt = \frac{1}{2^n}\sum_{k=0}^{2^n-1} H(x + \langle \vec{V}_{k,n}, \vec{W} \rangle) \tag{4.4-0}$$

where

$$H(x) = \int_{-\infty}^{x} \delta(t)\,dt$$

is the Heaviside step function. Writing the Gaussian distribution* as

$$P(x) = \int_{-\infty}^{x} Z(t)\,dt \tag{4.4-1}$$

where

$$Z(t) = \exp(-t^2/2) / \sqrt{2\pi} \tag{4.4-2}$$

we shall define the error criterion as the mean-square-error between (4.4-0) and (4.4-1). Using as a simple example the two stage shift register with weights 1/2 and 3/4, we illustrate the probability distributions in Figure 4.4.

---

* This function is similar to the well known error function and the following relations may be written :

$$P(x) = \frac{\operatorname{erf}\frac{x}{\sqrt{2}} + 1}{2}$$

$$\operatorname{erf}(x) = 2\,P(x\sqrt{2}) - 1$$

Thus the integral to be evaluated is

$$E(\vec{W}) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2^n} \sum_{k=0}^{2^n-1} H\left(x + \langle \vec{V}_{k,n}, \vec{W} \rangle\right) - P(x) \right\}^2 dx \qquad (4.4\text{-}3)$$

Before we proceed further, let us arrange the 2^n numbers corresponding to $\langle \vec{V}_{k,n}, \vec{W} \rangle$, $k = 0, 1 \ldots\ldots 2^n - 1$, in ascending order, and let us call these numbers η_p, $p = 1, 2, \ldots\ldots 2^n$, where

$$\eta_{2^n} \geq \eta_{2^n-1} \geq \eta_{2^n-2} \geq \cdots\cdots \geq \eta_2 \geq \eta_1 \qquad (4.4\text{-}4)$$

We shall now illustrate η_p by using the example cited before. Let $n = 2$, $W_1 = 1/2$ and $W_2 = 3/4$. Then,

$$\langle \vec{V}_{0,2}, \vec{W} \rangle = (-1 \; -1) \begin{pmatrix} 1/2 \\ 3/4 \end{pmatrix} = -5/4$$

$$\langle \vec{V}_{1,2}, \vec{W} \rangle = (-1 \; +1) \begin{pmatrix} 1/2 \\ 3/4 \end{pmatrix} = +1/4$$

$$\langle \vec{V}_{2,2}, \vec{W} \rangle = (+1 \; -1) \begin{pmatrix} 1/2 \\ 3/4 \end{pmatrix} = -1/4$$

$$\langle \vec{V}_{3,2}, \vec{W} \rangle = (+1 \; +1) \begin{pmatrix} 1/2 \\ 3/4 \end{pmatrix} = +5/4$$

and

$$\eta_1 = -5/4, \; \eta_2 = -1/4, \; \eta_3 = +1/4, \text{ and } \eta_4 = +5/4$$

From the above it may be noted that for any n, k,

$$\eta_k = -\eta_{2^n - k + 1} \qquad (4.4\text{-}5)$$

Thus, we may write the integral (4.4-3) as

$$E(\vec{W}) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2^n} \sum_{k=1}^{2^n} H(x + \eta_k) - P(x) \right\}^2 dx \qquad (4.4\text{-}6)$$

Defining $\eta_o = -\infty$ and $\eta_{2^n+1} = +\infty$, it is easily shown that (4.4-6) is equivalent to

$$E(\vec{W}) = \sum_{k=0}^{2^n} \int_{\eta_k}^{\eta_{k+1}} \left[\frac{k}{2^n} - P(x) \right]^2 dx \qquad (4.4\text{-}7)$$

Using the indefinite integral solved in Appendix C, we may write (4.4-7) as

$$\begin{aligned} E(\vec{W}) = \sum_{k=0}^{n} \Bigg\{ & \frac{k^2}{2^{2n}} \eta_{k+1} + \eta_{k+1} P^2(\eta_{k+1}) + 2P(\eta_{k+1}) Z(\eta_{k+1}) \\ & - \frac{1}{\sqrt{\pi}} P(\sqrt{2}\,\eta_{k+1}) - \frac{k}{2^{n-1}} \eta_{k+1} P(\eta_{k+1}) - \frac{k}{2^{n-1}} Z(\eta_{k+1}) \\ & - \frac{k^2}{2^{2n}} \eta_k - \eta_k P^2(\eta_k) - 2P(\eta_k) Z(\eta_k) \\ & + \frac{1}{\sqrt{\pi}} P(\sqrt{2}\,\eta_k) + \frac{k}{2^{n-1}} \eta_k P(\eta_k) + \frac{k}{2^{n-1}} Z(\eta_k) \Bigg\} \end{aligned} \qquad (4.4\text{-}8)$$

where Z(x) is as defined in (4.4-2).

With some manipulation and cancelling terms*, we may write

$$E(\vec{W}) = \sum_{k=1}^{2^n} \left\{ \eta_k \left[\frac{1-2k}{2^{2n}} + \frac{P(\eta_k)}{2^{n-1}} \right] + \frac{Z(\eta_k)}{2^{n-1}} \right\} - \frac{1}{\sqrt{\pi}} \tag{4.4-9}$$

Making use of property (4.4-5), we can reduce the number of summations in (4.4-9) by half. We write (4.4-9) as

$$E(\vec{W}) = -\frac{1}{\sqrt{\pi}} + \sum_{k=1}^{2^n-1} \left\{ \eta_k \left[\frac{1-2k}{2^{2n}} + \frac{P(\eta_k)}{2^{n-1}} \right] + \frac{Z(\eta_k)}{2^{n-1}} \right\} + \sum_{q=2^{n-1}+1}^{2^n} \eta_k \left[\frac{1-2q}{2^{2n}} + \frac{P(\eta_q)}{2^{n-1}} \right] + \frac{Z(\eta_q)}{2^{n-1}} \Bigg\} \tag{4.4-10}$$

Substituting (4.4-5) and making use of the properties

$$P(-x) = 1 - P(x) \quad \text{and} \quad Z(-x) = Z(x)$$

we may write the last term of (4.4-10) as

$$\sum_{q=2^{n-1}+1}^{2^n} \left\{ \eta_{2^n-q+1} \left[\frac{1-2(2^n-q+1)}{2^{2n}} + \frac{P(\eta_{2^n-q+1})}{2^{n-1}} \right] + \frac{Z(\eta_{2^n-q+1})}{2^{n-1}} \right\}$$

---

* Terms 2, 3, 4, 8, 9 and 10 inside the summation sign of (4.4-8) will appear twice when expanding (once with a plus sign and once with a negative sign) and therefore they will cancel for all except the end terms.

and substituting k for $2^n - q + 1$ we obtain, with some manipulation,

$$E(\vec{W}) = \frac{1}{2^{n-2}} \sum_{k=1}^{2^{n-1}} \left\{ \eta_k \left[\frac{1-2k}{2^{n+1}} + P(\eta_k) \right] + Z(\eta_k) \right\}$$

$$- \frac{1}{\sqrt{\pi}} \qquad (4.4\text{-}11)$$

4.5 Optimum Weights

In this section, we wish to optimize the weights; in other words, we wish to find the shift register weights which minimize (4.4-11). Although we may be able to obtain analytical solutions to the minimization problem, it has been found to be impractical for values of n greater than two (solutions for n equal to 1 and 2 are given in Appendix D and E, respectively). Plots of $E(\vec{W})$ for the 1 and 2 dimensional case are given in Figure 4-5 and 4-6, respectively.

The difficulty in finding analytical solutions for large n can be shown in the following example. Consider n = 4. From (4.4-11) we obtain

$$E(\vec{W}) = \frac{1}{4} \sum_{k=1}^{8} \left\{ \eta_k \left[\frac{1-2k}{32} + P(\eta_k) \right] + Z(\eta_k) \right\} - \frac{1}{\sqrt{\pi}} \qquad (4.5\text{-}1)$$

We shall assume, as in Appendix D and E, that

$$W_4 > W_3 > W_2 > W_1 > 0 \ .$$

This does not impose any restrictions on the weights since, due to the symmetry of (4.4-3), we may interchange any of the elements of the weight vector $\vec{W}$; and also due to (4.1-13), we may multiply any weight element by -1. Thus η_1, η_2 and η_3 are uniquely determined in terms of W_1, W_2, W_3 and W_4, namely :

$$\eta_1 = -W_1 - W_2 - W_3 - W_4$$

$$\eta_2 = +W_1 - W_2 - W_3 - W_4$$

$$\eta_3 = -W_1 + W_2 - W_3 - W_4$$

Now η_4 may either be

$$-W_1 - W_2 + W_3 - W_4 \qquad (4.5\text{-}2)$$

or

$$+W_1 + W_2 - W_3 - W_4 \qquad (4.5\text{-}3)$$

depending on whether W_3 is less or greater than $W_1 + W_2$.

Determining the values of η_k for larger values of k, necessitates more assumptions, (one way or the other), and, in order to find the optimum solution, we must examine all regions of the "weight space" corresponding to the different constraints.

Before proceeding to find the optimum weights numerically, it is interesting to note the non-uniqueness of the solution. Assuming the existence of a solution for order n, where in general, the weights are not equal, i.e.,

$$W_k \neq W_p \qquad \text{for} \begin{cases} p, k = 1, 2, \ldots .n \\ p \neq k \end{cases}$$

due to the symmetry of (4.4-6), there are $2^n \times n!$ solutions. The 2^n factor results from taking all $\pm$ combinations of the weights, and the $n!$ factor results from interchanging elements of the weight vector in all possible ways. From this result we may see that although the weights are optimized with respect to the probability density function, there is still some freedom to interchange the weights in order to obtain $2^n \times n!$ different power spectral densities. We shall now discuss numerical methods for finding the unconstrained optimum weights.

Numerical optimization techniques can be classified into two broad categories : techniques based on conventional mathematical methods and direct search techniques. The mathematical methods use, in some way or another, the gradient of the function to be minimized and hence require the evaluation of the partial derivatives of that function. Although (4.4-11) is continuous, it is not differentiable at all points, and therefore gradient techniques are not appropriate.

The main advantages in using direct search techniques for this application is that they do not require derivatives of the function. Furthermore, all that is required is a way of finding functional values, and there is no need for having the function available in algebraic form.

Three similar direct methods were tried. The method by Hooke and Jeeves[26] (also described briefly in Lavi and Vogl[27], and Wood[28]), and that by Powell[25], were found to be satisfactory, although the convergence rate was extremely dependent on the initial weight parameter estimates. The method exhibiting the best behaviour for this application was one by Flood and Leon[30]. A brief description of this method together with a flow diagram is given in Appendix F. Appendix G contains a short list of optimum weights for orders n up to 10, together with the corresponding probability density error. In Graph 9, we plot the errors of Appendix G versus n . Also included in Graph 9, are similar plots corresponding to equal weights, binomial weights and the Gaussian impulse approximation weights, discussed in Section 3.4. These latter weights (i.e., equal, binomial and Gaussian impulse approximation weights) are normalized in such a way as to minimize the probability density error (as defined in this chapter) with the constraint that the ratios between the weights remains unchanged.

Graph 9 indicates that a great improvement over the equally weighted case is obtained when using the optimum weights to approximate a Gaussian spectral density. The relatively low errors which result when using the Gaussian impulse approximation weights indicates that these weights are better suited for simulating doppler spectra than are the binomial weights discussed in Section 4.3. It is interesting to note that for both the Gaussian impulse approximation weights and the binomial weights, the line joining the probability density errors corresponding to odd orders of n is below that of even n. This result is useful for design purposes, and it implies that odd order approximations are generally advantageous in meeting cost-performance criteria.

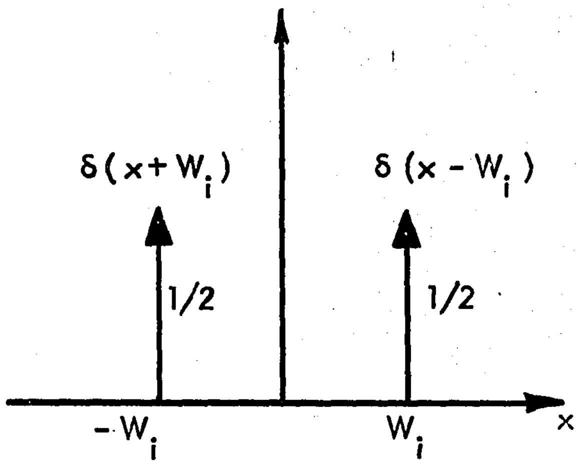


FIGURE 4-1 - PROBABILITY DENSITY FUNCTION FOR ONE WEIGHTED SHIFT REGISTER STAGE

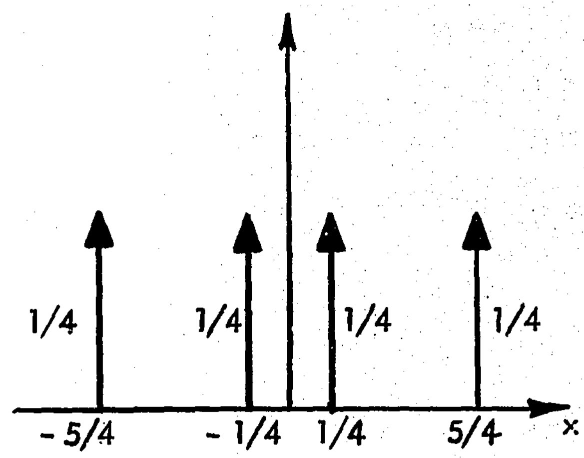


FIGURE 4-2 - PROBABILITY DENSITY, ($n = 2$, $W_1 = 1/2$, $W_2 = 3/4$).

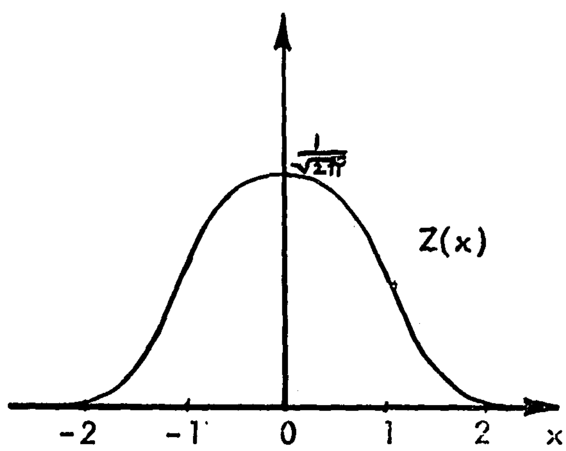


FIGURE 4-3 - GAUSSIAN PROBABILITY DENSITY

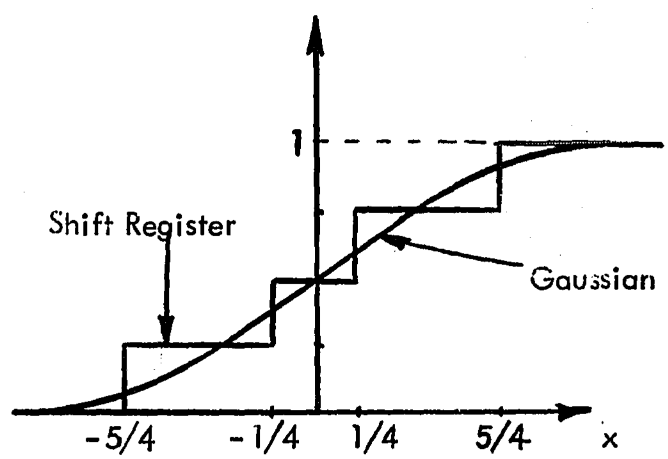


FIGURE 4-4 - GAUSSIAN AND SHIFT REGISTER PROBABILITY DISTRIBUTION

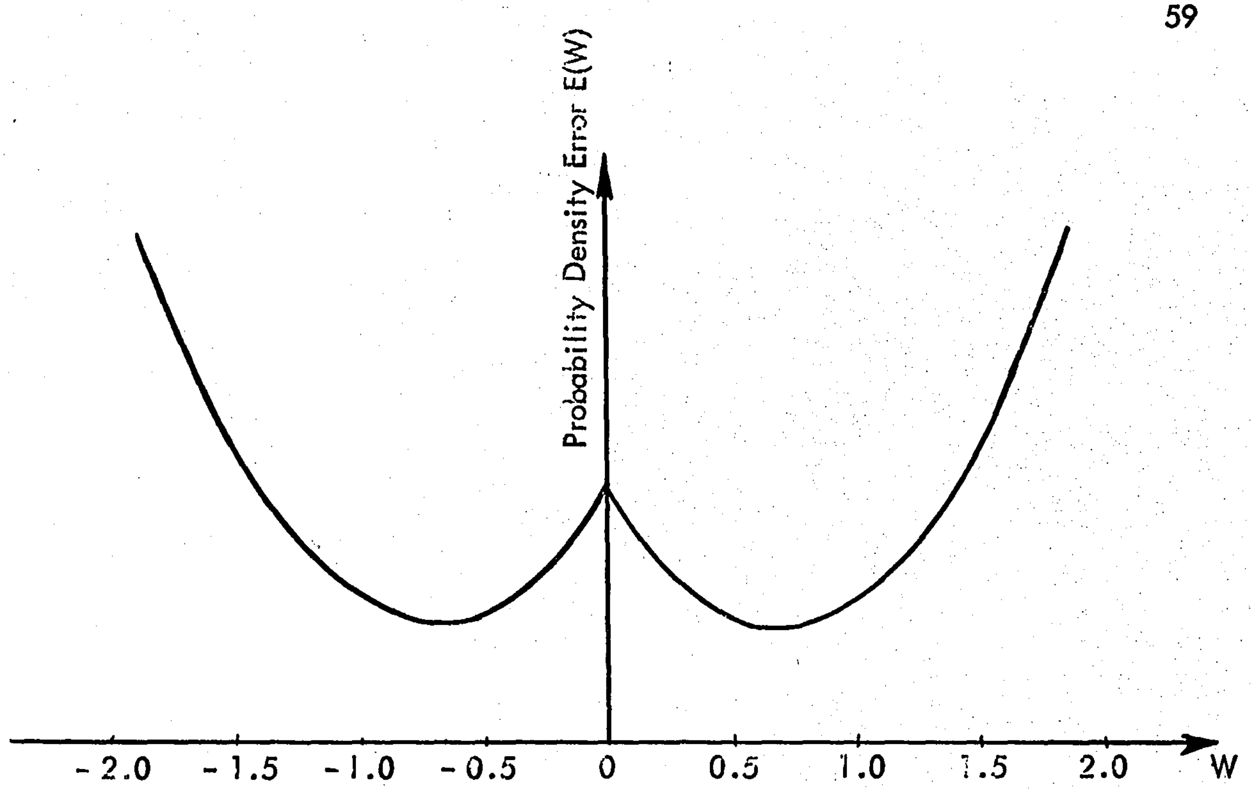


FIGURE 4-5 - PROBABILITY DENSITY ERROR FOR n = 1

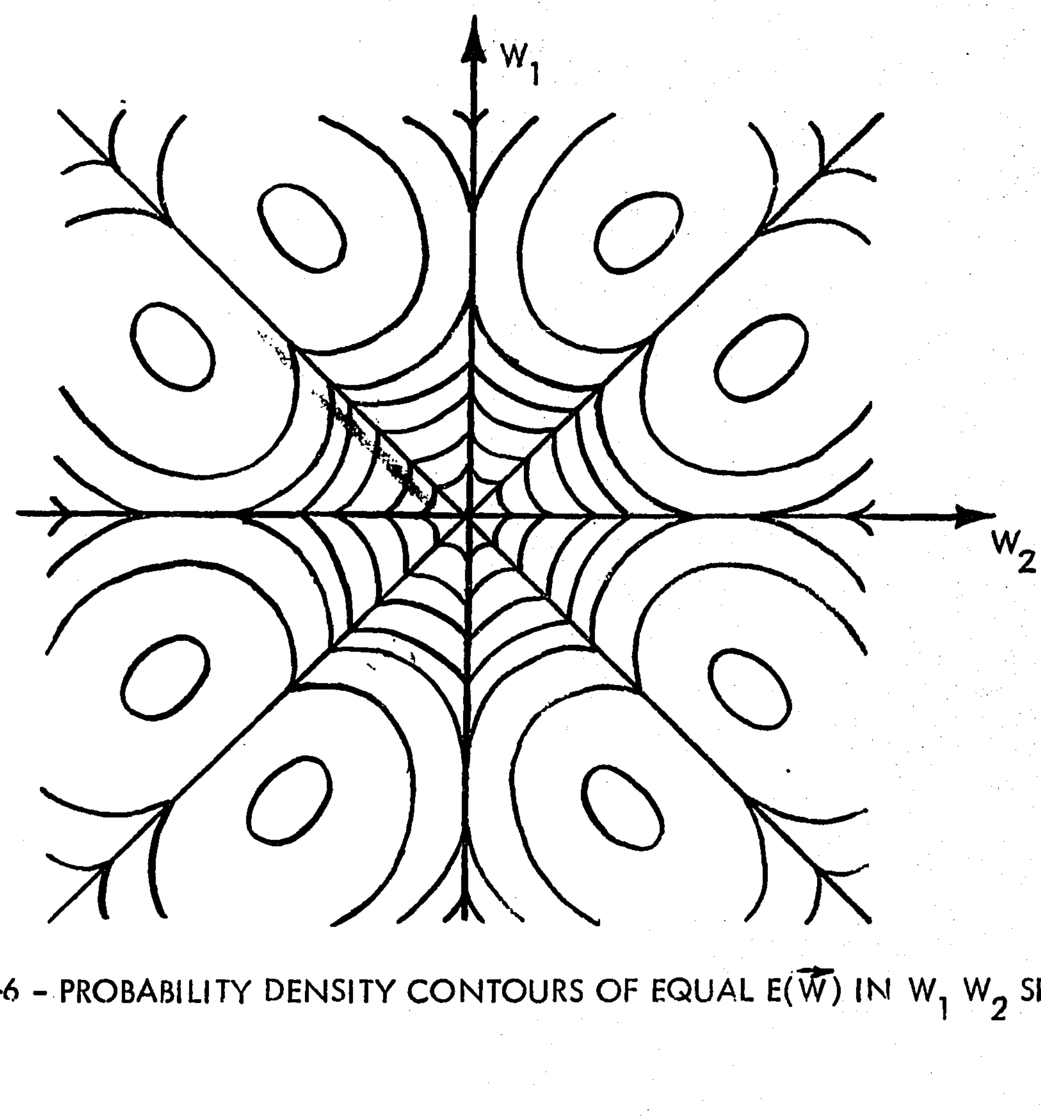


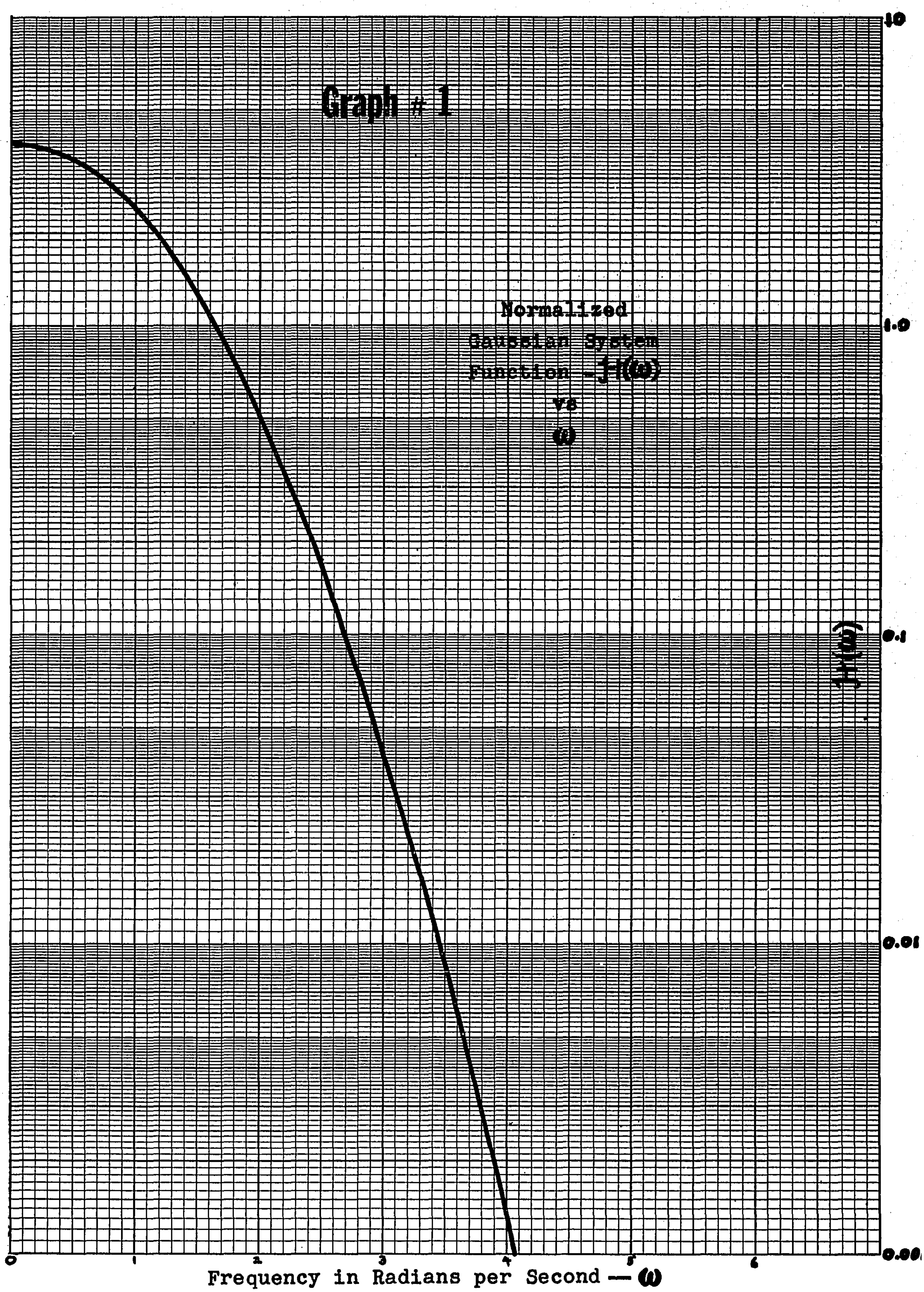
FIGURE 4-6 - PROBABILITY DENSITY CONTOURS OF EQUAL $E(\vec{W})$ IN W_1 W_2 SPACE

CHAPTER V

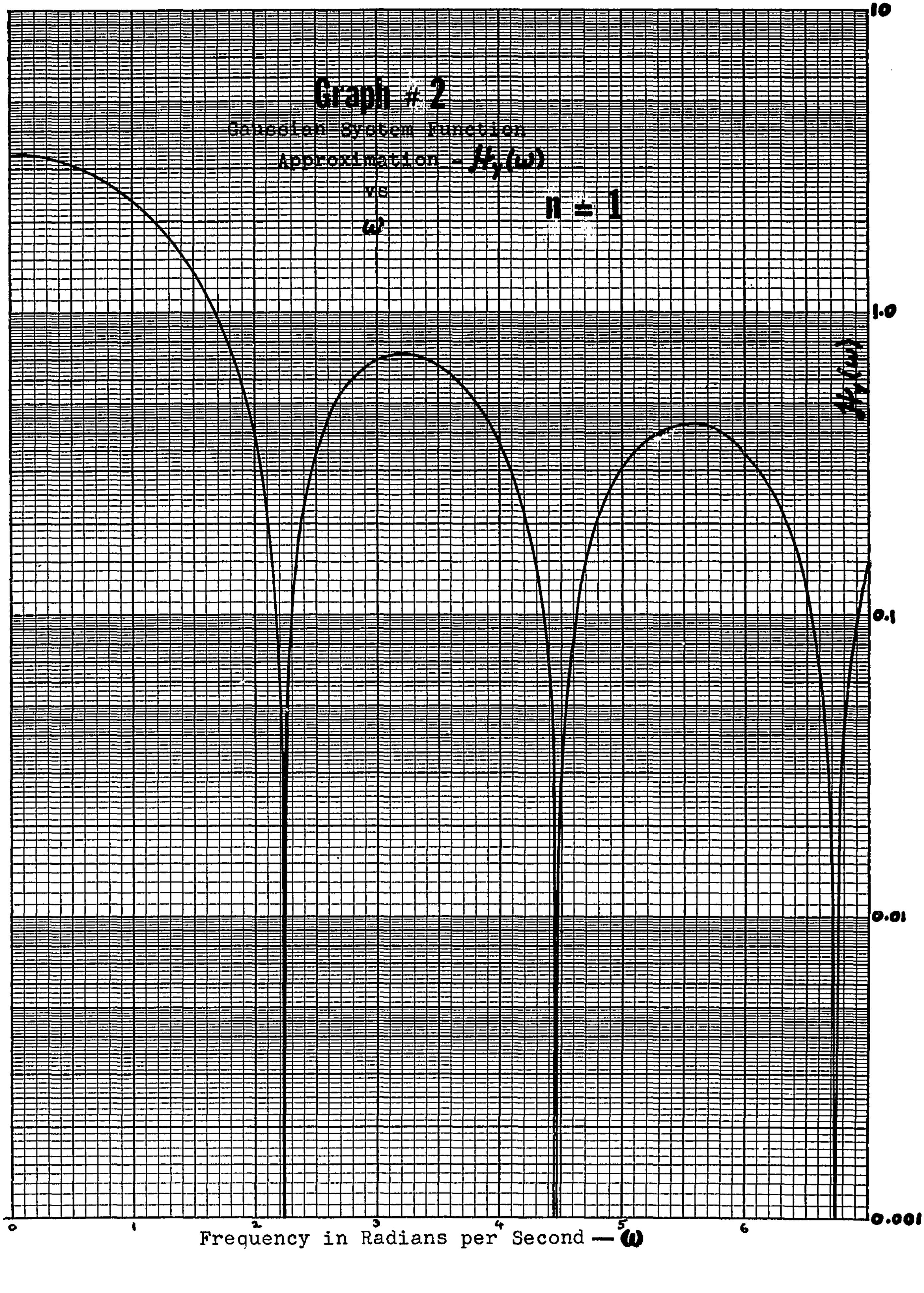
CONCLUSIONS

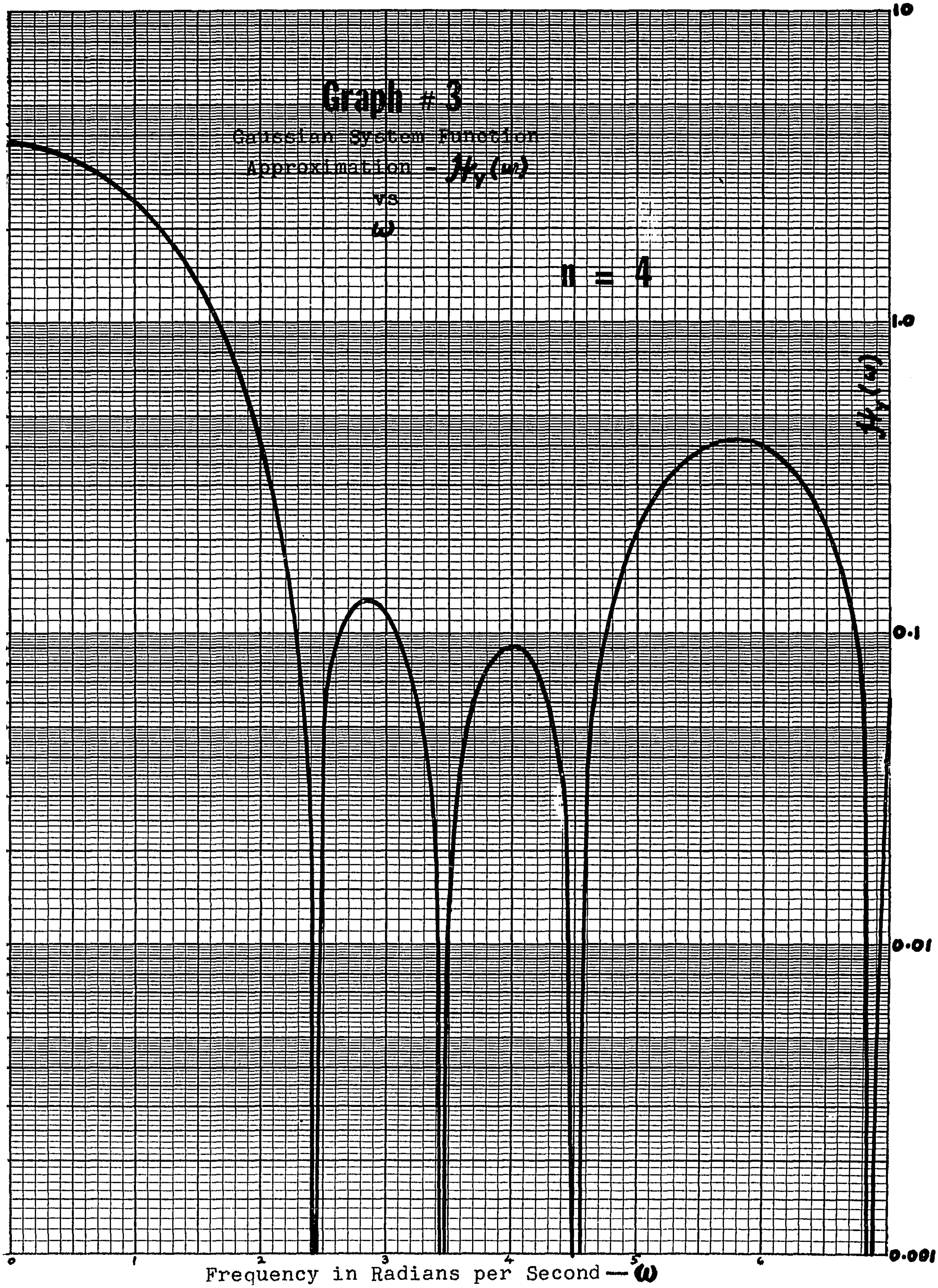
Chapter II contains the solutions to the problem of the autocorrelation function and power spectral density of an arbitrarily weighted shift register with a random binary process as the input. These results are extended for the case of maximum length pseudo-random shift register sequences. It is also shown that the autocorrelation function and power spectral density of a signal obtained in this manner is equivalent to that obtained by passing white noise through a non-recursive digital filter with a sample and hold. Using this property, we are able to simplify the problem of approximating a prescribed power spectral density in Chapter III. The Gaussian and brickwall power spectral density are considered in greater detail, and the optimum weights for both of these particular cases are listed in Appendix A and B.

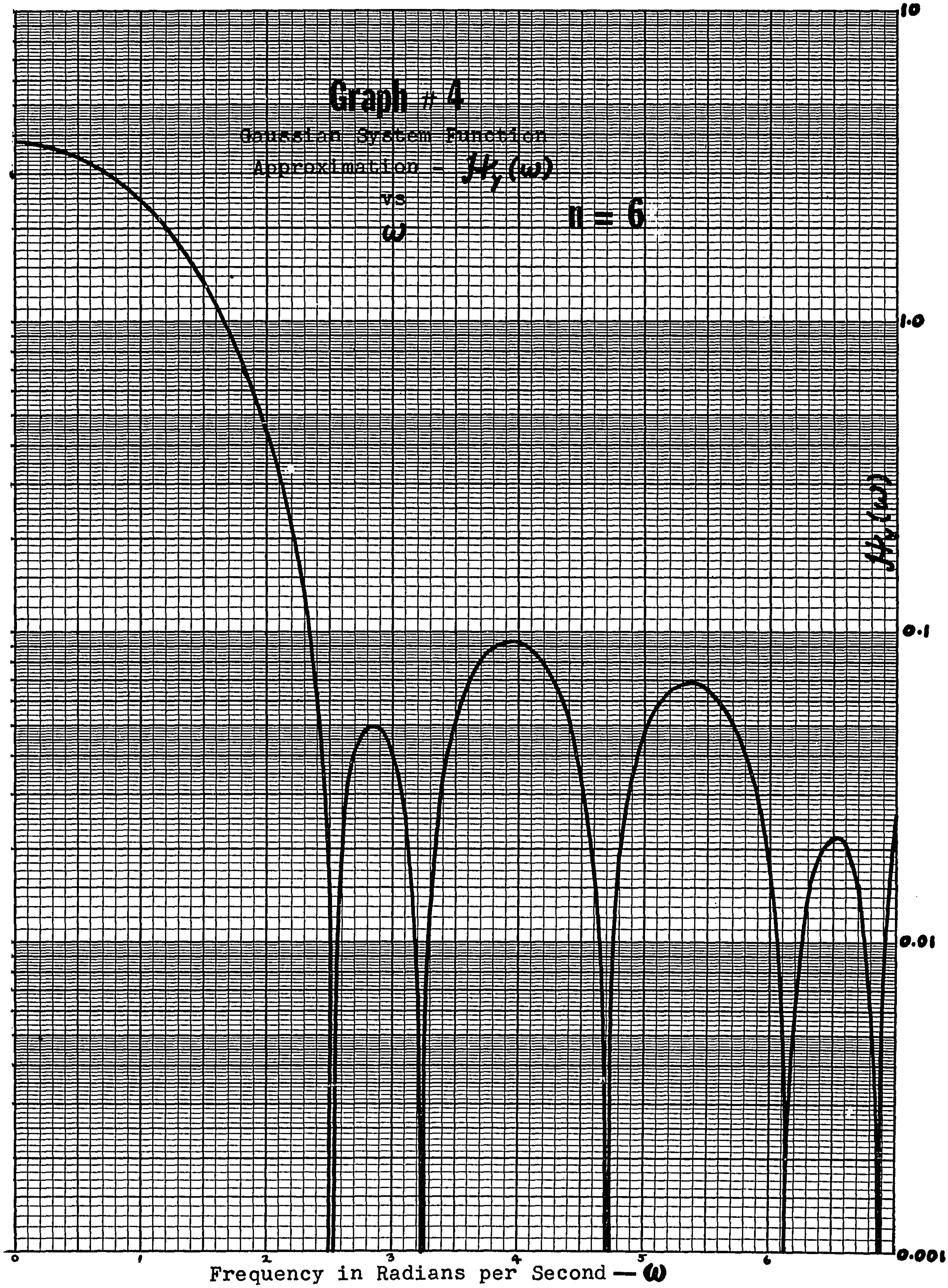
In Chapter IV, we consider the probability density function of arbitrarily weighted registers. An approximation criterion is introduced between the probability density of the weighted register and that of a Gaussian process. Solutions for the weights which minimize this error are obtained by using direct optimization methods. The errors corresponding to these optimum weights are compared to the errors of equal, binomial and the Gaussian spectrum weights discussed in Chapter III. The results as plotted in Graph 9, indicate that a great improvement is obtained over the equally weighted case, discussed by Kramer[8] and Davies[9], by using the optimum weights listed in Appendix G.



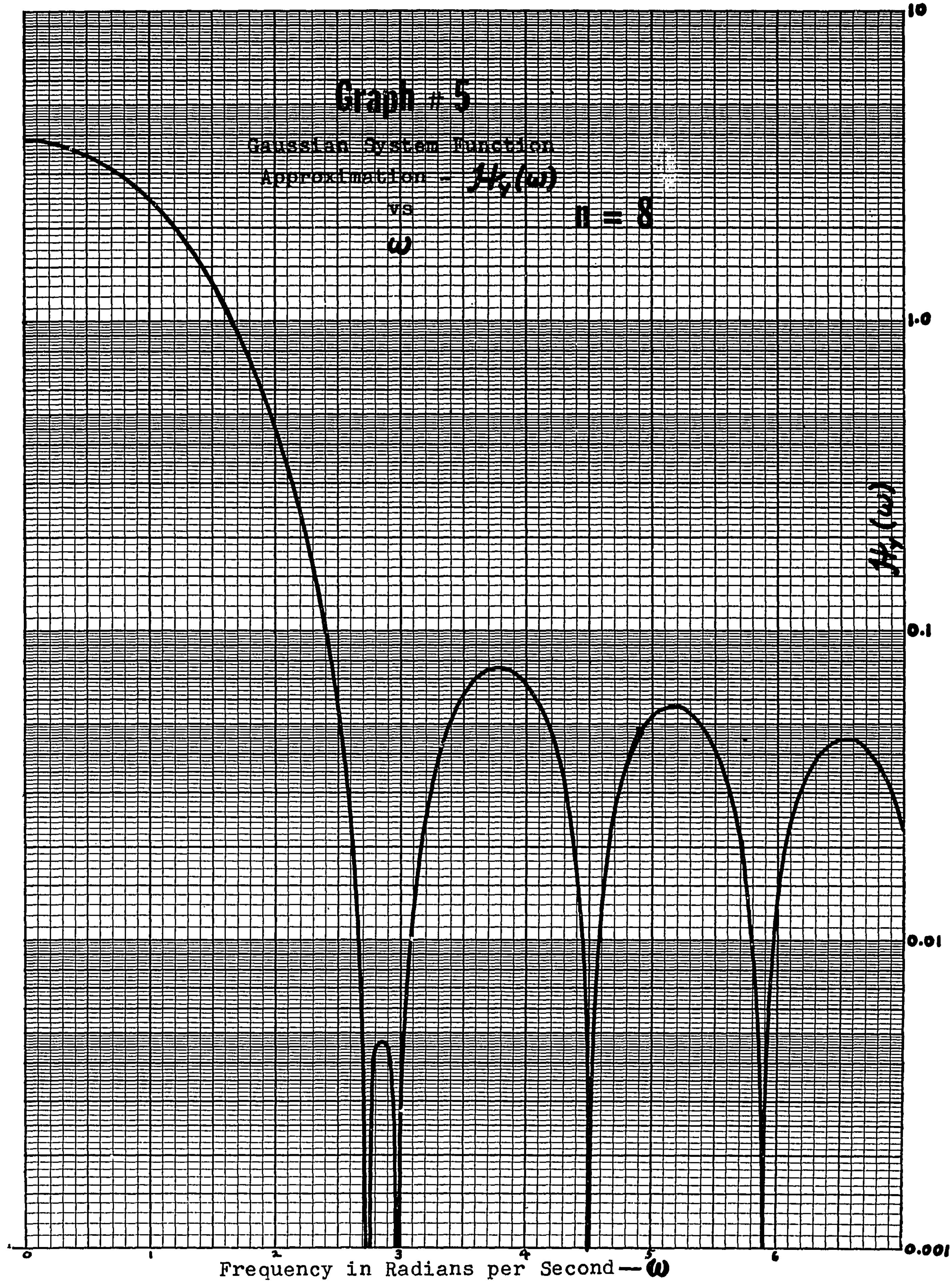
Graph # 1
Normalized
Gaussian System
Function — H(ω)
vs
ω
10
1.0
0.1
0.01
0.001
H(ω)
0
1
2
3
4
5
6
Frequency in Radians per Second — ω


Graph # 2
Gaussian System Function
Approximation - $H_y(\omega)$
vs
ω
n = 1
10
1.0
0.1
0.01
0.001
$H_y(\omega)$
0
1
2
3
4
5
6
Frequency in Radians per Second — ω

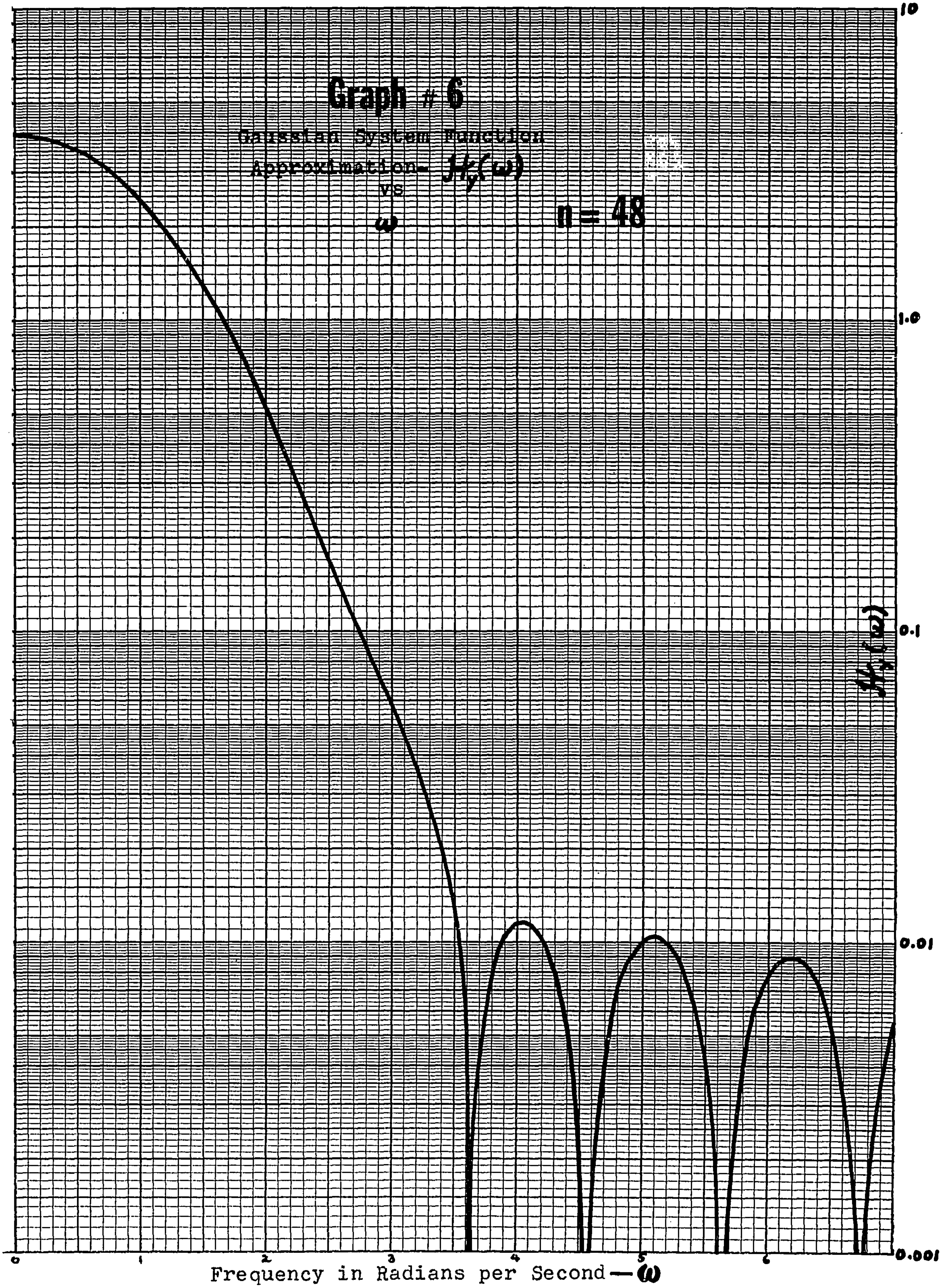

Graph # 3
Gaussian System Function
Approximation - $\mathcal{H}_y(\omega)$
vs
ω
n = 4
10
1.0
0.1
0.01
0.001
$\mathcal{H}_y(\omega)$
0
1
2
3
4
5
6
Frequency in Radians per Second — ω

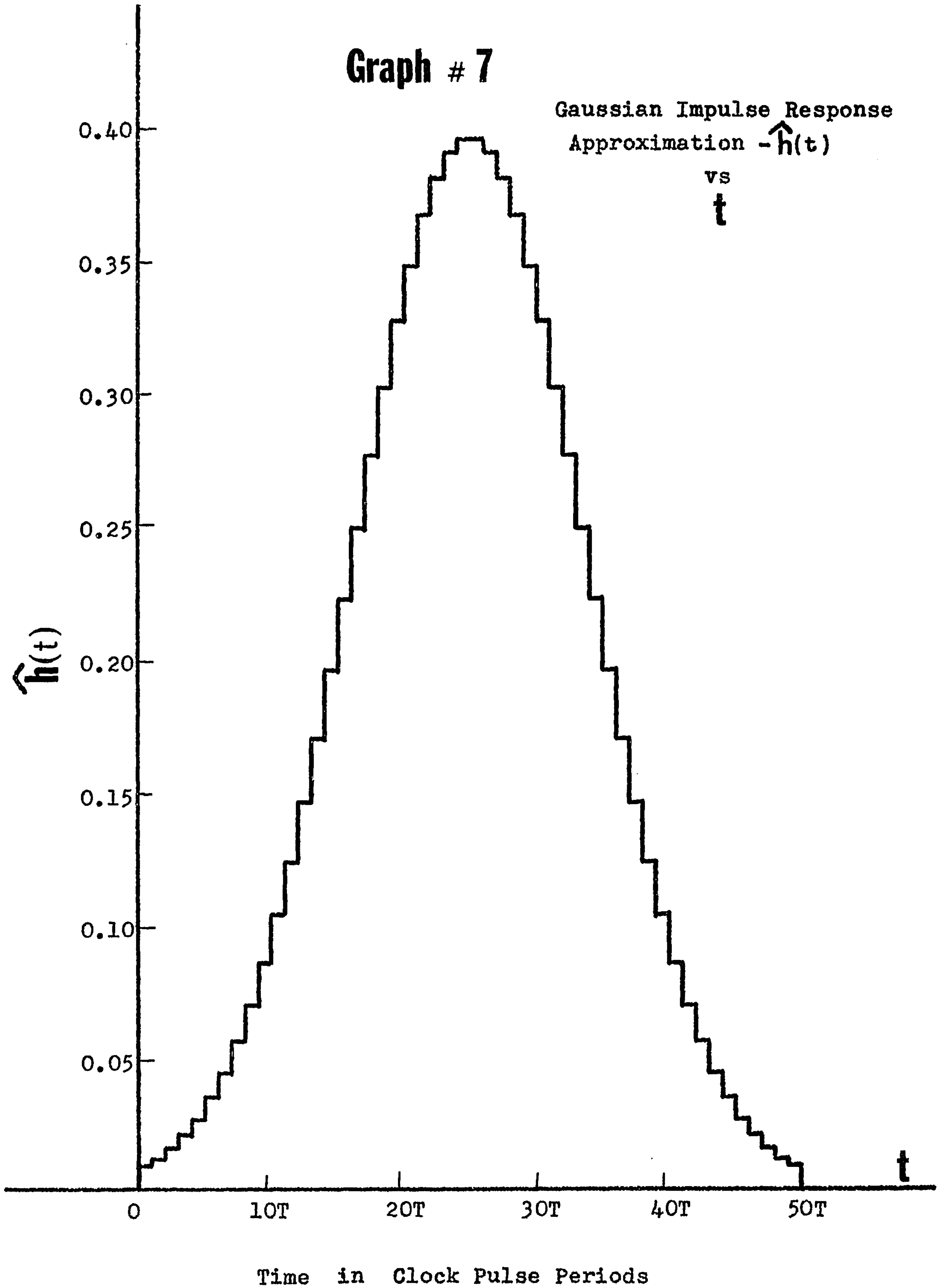


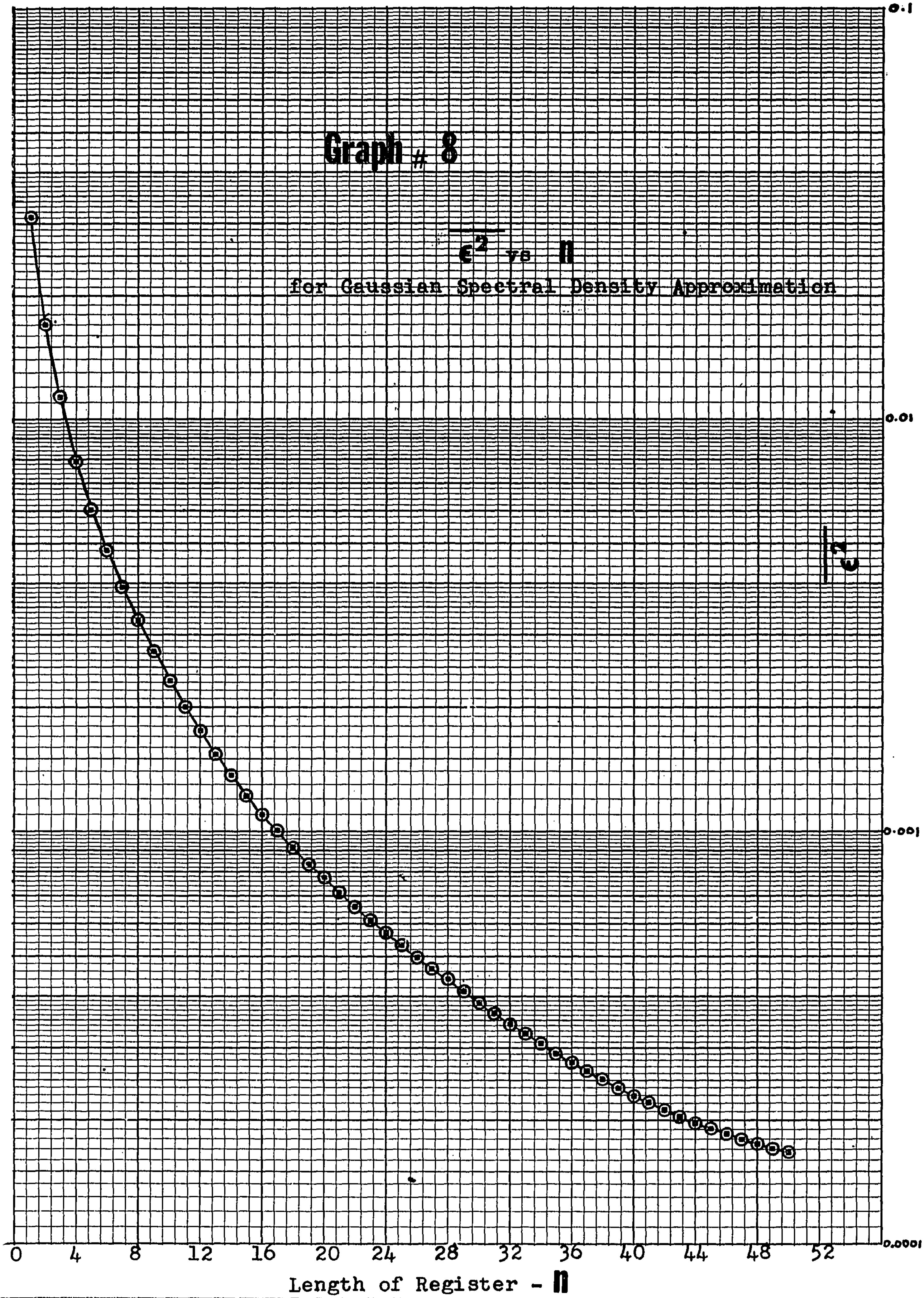
Graph # 4
Gaussian System Function
Approximation - $H_\gamma(\omega)$
vs
ω
n = 6
10
1.0
0.1
0.01
0.001
$H_\gamma(\omega)$
0
1
2
3
4
5
6
Frequency in Radians per Second — ω



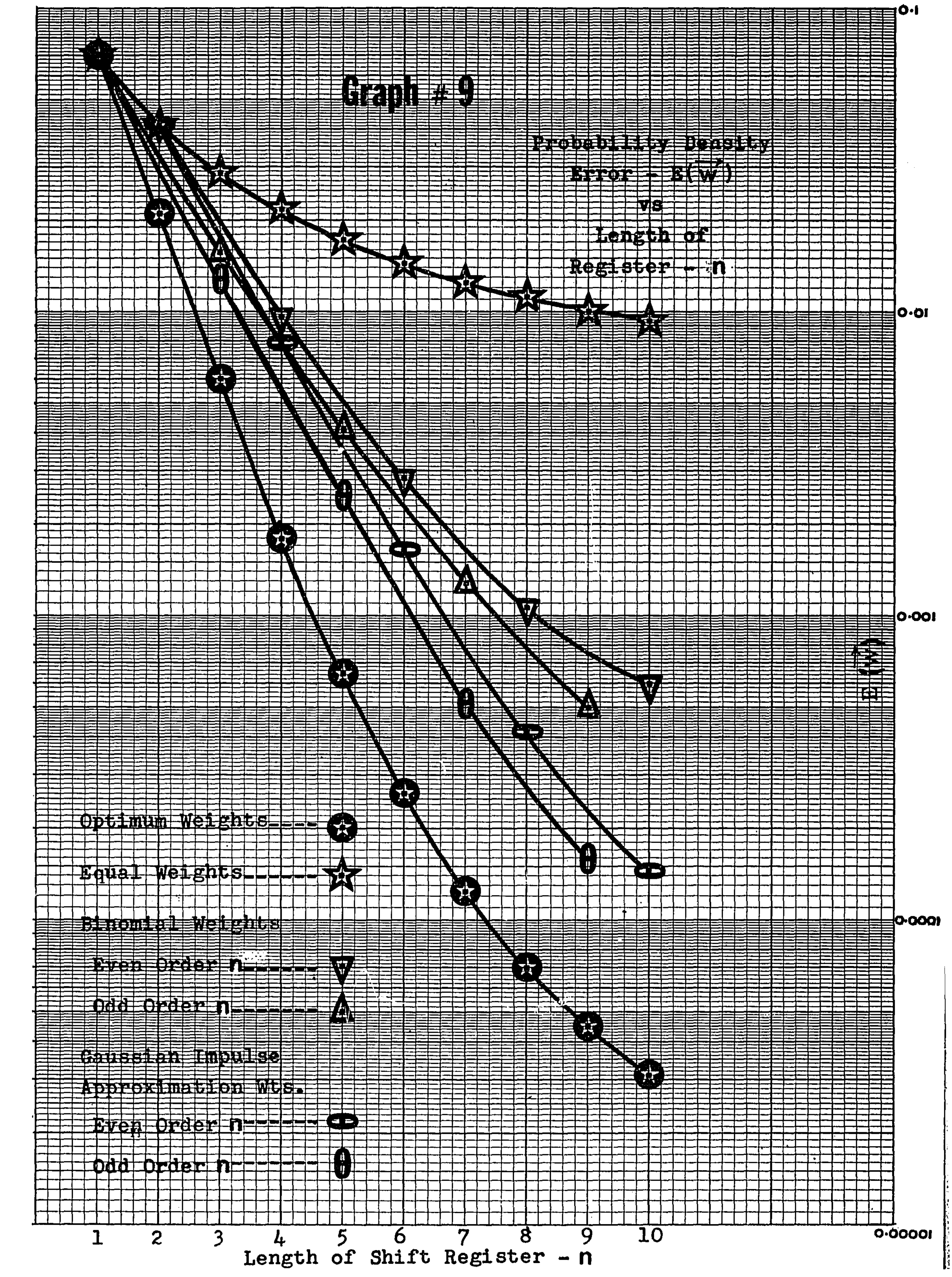
Graph # 5
Gaussian System Function
Approximation - $\mathcal{H}_y(\omega)$
vs
ω
n = 8
10
1.0
0.1
0.01
0.001
$\mathcal{H}_y(\omega)$
0
1
2
3
4
5
6
Frequency in Radians per Second — ω


Graph # 6
Gaussian System Function
Approximation— $H_y(\omega)$
vs
ω
n = 48
10
1.0
0.1
0.01
0.001
$H_y(\omega)$
0
1
2
3
4
5
6
Frequency in Radians per Second — ω


Graph # 7
Gaussian Impulse Response
Approximation - ĥ(t)
vs
t
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
ĥ(t)
t
0
10T
20T
30T
40T
50T
Time in Clock Pulse Periods



Graph # 8
$\overline{\epsilon^2}$ vs n
for Gaussian Spectral Density Approximation
0.1
0.01
0.001
0.0001
$\overline{\epsilon^2}$
0
4
8
12
16
20
24
28
32
36
40
44
48
52
Length of Register - n



Graph # 9
Probability Density
Error - E(W)
vs
Length of
Register - n
Optimum Weights
Equal Weights
Binomial Weights
Even Order n
Odd Order n
Gaussian Impulse
Approximation Wts.
Even Order n
Odd Order n
0.1
0.01
0.001
0.0001
0.00001
E(W)
1
2
3
4
5
6
7
8
9
10
Length of Shift Register - n

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APPENDIX A

OPTIMUM WEIGHTS FOR BEST APPROXIMATION TO A GAUSSIAN POWER SPECTRAL DENSITY

**   *
****
*   **

THE SPACING PARAMETER FOR ORDER 1 IS 2.79999E 00

AND THE WEIGHT IS --------
2.99461E-01

THE MEAN-SQUARE-ERROR IS 3.10023E-02 .

THE SPACING PARAMETER FOR ORDER 2 IS 1.39999E 00

AND THE WEIGHT IS --------
2.99461E-01

THE MEAN-SQUARE-ERROR IS 3.10023E-02 .

THE SPACING PARAMETER FOR ORDER 3 IS 1.12632E 00

AND THE WEIGHTS ARE ------
3.78821E-01 2.14059E-01

THE MEAN-SQUARE-ERROR IS 1.72427E-02 .

THE SPACING PARAMETER FOR ORDER 4 IS 9.09550E-01

AND THE WEIGHTS ARE ------
3.50140E-01 1.61708E-01

THE MEAN-SQUARE-ERROR IS 1.15079E-02 .

THE SPACING PARAMETER FOR ORDER 5 IS 7.70189E-01

AND THE WEIGHTS ARE ------
3.89297E-01 2.93560E-01 1.25816E-01

THE MEAN-SQUARE-ERROR IS 8.24191E-03 .

THE SPACING PARAMETER FOR ORDER 6 IS 6.69750E-01

AND THE WEIGHTS ARE ------
3.71021E-01 2.40843E-01 1.01454E-01

THE MEAN-SQUARE-ERROR IS 6.21846E-03 .

THE SPACING PARAMETER FOR ORDER 7 IS 5.93761E-01

AND THE WEIGHTS ARE ------
3.93156E-01 3.31309E-01 1.98244E-01 8.42157E-02

THE MEAN-SQUARE-ERROR IS 4.87397E-03 .

THE SPACING PARAMETER FOR ORDER 8 IS 5.34112E-01

AND THE WEIGHTS ARE ------
3.80758E-01 2.88184E-01 1.65076E-01 7.15567E-02

THE MEAN-SQUARE-ERROR IS 3.93222E-03 .

THE SPACING PARAMETER FOR ORDER 9 IS 4.85951E-01

AND THE WEIGHTS ARE ------
3.95050E-01 3.51864E-01 2.48620E-01 1.39353E-01
6.19577E-02

THE MEAN-SQUARE-ERROR IS 3.24594E-03 .

THE SPACING PARAMETER FOR ORDER 10 IS 4.46203E-01

AND THE WEIGHTS ARE ------
3.86090E-01 3.17429E-01 2.14563E-01 1.19235E-01
5.44727E-02

THE MEAN-SQUARE-ERROR IS 2.72874E-03 .

THE SPACING PARAMETER FOR ORDER 11 IS 4.12790E-01

AND THE WEIGHTS ARE ------
3.96127E-01 3.64214E-01 2.83086E-01 1.86003E-01
1.03311E-01 4.85058E-02

THE MEAN-SQUARE-ERROR IS 2.32980E-03 .

THE SPACING PARAMETER FOR ORDER 12 IS 3.84290E-01

AND THE WEIGHTS ARE ------
3.89335E-01 3.36494E-01 2.51345E-01 1.62261E-01
9.05302E-02 4.36517E-02

THE MEAN-SQUARE-ERROR IS 2.01429E-03 .

THE SPACING PARAMETER FOR ORDER 13 IS 3.59677E-01

AND THE WEIGHTS ARE ------
3.96798E-01 3.72209E-01 3.07194E-01 2.23078E-01
1.42535E-01 8.01299E-02 3.96344E-02

THE MEAN-SQUARE-ERROR IS 1.76049E-03 .

THE SPACING PARAMETER FOR ORDER 14 IS 3.38185E-01

AND THE WEIGHTS ARE ------
3.91466E-01 3.49540E-01 2.78675E-01 1.98381E-01
1.26096E-01 7.15643E-02 3.62645E-02

THE MEAN-SQUARE-ERROR IS 1.55315E-03 .

THE SPACING PARAMETER FOR ORDER 15 IS 3.19248E-01

AND THE WEIGHTS ARE ------

3.97249E-01	3.77684E-01	3.24557E-01	2.52098E-01
1.76995E-01	1.12321E-01	6.44259E-02	3.34018E-02

THE MEAN-SQUARE-ERROR IS 1.38157E-03 .

THE SPACING PARAMETER FOR ORDER 16 IS 3.02429E-01

AND THE WEIGHTS ARE ------

3.92942E-01	3.58847E-01	2.99275E-01	2.27932E-01
1.58534E-01	1.00697E-01	5.84100E-02	3.09405E-02

THE MEAN-SQUARE-ERROR IS 1.23755E-03 .

THE SPACING PARAMETER FOR ORDER 17 IS 2.87384E-01

AND THE WEIGHTS ARE ------

3.97566E-01	3.81600E-01	3.37424E-01	2.74864E-01
2.06272E-01	1.42606E-01	9.08268E-02	5.32919E-02
2.88059E-02			

THE MEAN-SQUARE-ERROR IS 1.11578E-03 .

THE SPACING PARAMETER FOR ORDER 18 IS 2.73841E-01

AND THE WEIGHTS ARE ------

3.94010E-01	3.65716E-01	3.15079E-01	2.51954E-01
1.87010E-01	1.28838E-01	8.23857E-02	4.88983E-02
2.69382E-02			

THE MEAN-SQUARE-ERROR IS 1.01143E-03 .

THE SPACING PARAMETER FOR ORDER 19 IS 2.61586E-01

AND THE WEIGHTS ARE ------

3.97799E-01	3.84504E-01	3.47199E-01	2.92891E-01
2.30824E-01	1.69948E-01	1.16895E-01	7.51149E-02
4.50930E-02	2.52891E-02		

THE MEAN-SQUARE-ERROR IS 9.21612E-04 .

THE SPACING PARAMETER FOR ORDER 20 IS 2.50437E-01

AND THE WEIGHTS ARE ------

3.94807E-01	3.70933E-01	3.27417E-01	2.71524E-01
2.11556E-01	1.54861E-01	1.06503E-01	6.88152E-02
4.17739E-02	2.38247E-02		

THE MEAN-SQUARE-ERROR IS 8.43485E-04 .

THE SPACING PARAMETER FOR ORDER 21 IS 2.40240E-01

AND THE WEIGHTS ARE ------

3.97981E-01	3.86719E-01	3.54793E-01	3.07338E-01
2.51365E-01	1.94113E-01	1.41532E-01	9.74345E-02
6.33315E-02	3.88671E-02	2.25215E-02	

THE MEAN-SQUARE-ERROR IS 7.75201E-04 .

THE SPACING PARAMETER FOR ORDER 22 IS 2.30899E-01

AND THE WEIGHTS ARE ------

3.95421E-01	3.74986E-01	3.37219E-01	2.87579E-01
2.32568E-01	1.78358E-01	1.29713E-01	8.94591E-02
5.85074E-02	3.62862E-02	2.13414E-02	

THE MEAN-SQUARE-ERROR IS 7.15174E-04 .

THE SPACING PARAMETER FOR ORDER 23 IS 2.22274E-01

AND THE WEIGHTS ARE ------

3.98118E-01	3.88448E-01	3.60812E-01	3.19051E-01
2.68577E-01	2.15235E-01	1.64204E-01	1.19259E-01
8.24569E-02	5.42743E-02	3.40087E-02	2.02868E-02

THE MEAN-SQUARE-ERROR IS 6.61849E-04 .

THE SPACING PARAMETER FOR ORDER 24 IS 2.14308E-01

AND THE WEIGHTS ARE ------

3.95909E-01	3.78204E-01	3.45133E-01	3.00868E-01
2.50550E-01	1.99318E-01	1.51470E-01	1.09960E-01
7.62562E-02	5.05175E-02	3.19698E-02	1.93270E-02

THE MEAN-SQUARE-ERROR IS 6.14377E-04 .

THE SPACING PARAMETER FOR ORDER 25 IS 2.06930E-01

AND THE WEIGHTS ARE ------

3.98232E-01	3.89825E-01	3.65659E-01	3.28662E-01
2.83070E-01	2.33617E-01	1.84755E-01	1.40005E-01
1.01665E-01	7.07390E-02	4.71656E-02	3.01333E-02
1.84480E-02			

THE MEAN-SQUARE-ERROR IS 5.72333E-04 .

THE SPACING PARAMETER FOR ORDER 26 IS 2.00066E-01

AND THE WEIGHTS ARE ------

3.96291E-01	3.80800E-01	3.51598E-01	3.11939E-01
2.65929E-01	2.17841E-01	1.71470E-01	1.29691E-01
9.42549E-02	6.58225E-02	4.41691E-02	2.84798E-02
1.76452E-02			

THE MEAN-SQUARE-ERROR IS 5.34647E-04 .

THE SPACING PARAMETER FOR ORDER 27 IS 1.93659E-01

AND THE WEIGHTS ARE ------

3.98317E-01	3.90939E-01	3.69626E-01	3.36639E-01
2.95352E-01	2.49615E-01	2.03221E-01	1.59379E-01
1.20407E-01	8.76279E-02	6.14311E-02	4.14859E-02
2.69882E-02	1.69126E-02		

THE MEAN-SQUARE-ERROR IS 5.00491E-04 .

THE SPACING PARAMETER FOR ORDER 28 IS 1.87675E-01

AND THE WEIGHTS ARE ------

3.96612E-01	3.82925E-01	3.56954E-01	3.21259E-01
2.79156E-01	2.34200E-01	1.89706E-01	1.48359E-01
1.12021E-01	8.16644E-02	5.74793E-02	3.90610E-02
2.56277E-02	1.62345E-02		

THE MEAN-SQUARE-ERROR IS 4.69468E-04 .

THE SPACING PARAMETER FOR ORDER 29 IS 1.82069E-01

AND THE WEIGHTS ARE ------

3.98382E-01	3.91864E-01	3.72902E-01	3.43328E-01
3.05812E-01	2.63542E-01	2.19730E-01	1.77244E-01
1.38323E-01	1.04439E-01	7.62908E-02	5.39168E-02
3.68652E-02	2.43867E-02	1.56075E-02	

THE MEAN-SQUARE-ERROR IS 4.41540E-04 .

THE SPACING PARAMETER FOR ORDER 30 IS 1.76804E-01

AND THE WEIGHTS ARE ------

3.96869E-01	3.84693E-01	3.61437E-01	3.29162E-01
2.90564E-01	2.48624E-01	2.06205E-01	1.65773E-01
1.29178E-01	9.75721E-02	7.14361E-02	5.06954E-02
3.48722E-02	2.32510E-02	1.50272E-02	

THE MEAN-SQUARE-ERROR IS 4.15908E-04 .

THE SPACING PARAMETER FOR ORDER 31 IS 1.71848E-01

AND THE WEIGHTS ARE ------

3.98446E-01	3.92626E-01	3.75655E-01	3.48983E-01
3.14793E-01	2.75707E-01	2.34468E-01	1.93609E-01
1.55228E-01	1.20843E-01	9.13436E-02	6.70409E-02
4.77758E-02	3.30585E-02	2.22106E-02	1.44894E-02

THE MEAN-SQUARE-ERROR IS 3.92630E-04 .

THE SPACING PARAMETER FOR ORDER 32 IS 1.67176E-01

AND THE WEIGHTS ARE ------

3.97092E-01	3.86172E-01	3.65227E-01	3.35920E-01
3.00467E-01	2.61366E-01	2.21102E-01	1.81900E-01
1.45532E-01	1.13232E-01	8.56801E-02	6.30485E-02
4.51193E-02	3.14007E-02	2.12522E-02	1.39881E-02

THE MEAN-SQUARE-ERROR IS 3.71139E-04 .

THE SPACING PARAMETER FOR ORDER 33 IS 1.62762E-01

AND THE WEIGHTS ARE ------

3.98485E-01	3.93272E-01	3.77984E-01	3.53813E-01
3.22548E-01	2.86375E-01	2.47625E-01	2.08534E-01
1.71033E-01	1.36616E-01	1.06279E-01	8.05205E-02
5.94142E-02	4.26961E-02	2.98825E-02	2.03681E-02
1.35211E-02			

THE MEAN-SQUARE-ERROR IS 3.52025E-04 .

THE SPACING PARAMETER FOR ORDER 34 IS 1.58592E-01

AND THE WEIGHTS ARE ------

3.97277E-01	3.87428E-01	3.68462E-01	3.41736E-01
3.09094E-01	2.72639E-01	2.34522E-01	1.96737E-01
1.60948E-01	1.28405E-01	9.99033E-02	7.58017E-02
5.60883E-02	4.04731E-02	2.84816E-02	1.95461E-02
1.30810E-02			

THE MEAN-SQUARE-ERROR IS 3.33480E-04 .

THE SPACING PARAMETER FOR ORDER 35 IS 1.54631E-01

AND THE WEIGHTS ARE ------

3.98538E-01	3.93819E-01	3.79972E-01	3.57967E-01
3.29283E-01	2.95755E-01	2.59375E-01	2.22111E-01
1.85713E-01	1.51619E-01	1.20864E-01	9.40771E-02
7.14986E-02	5.30580E-02	3.84450E-02	2.71999E-02
1.87902E-02	1.26744E-02		

THE MEAN-SQUARE-ERROR IS 3.17009E-04 .

THE SPACING PARAMETER FOR ORDER 36 IS 1.50882E-01

AND THE WEIGHTS ARE ------

3.97432E-01	3.88502E-01	3.71250E-01	3.46780E-01
3.16652E-01	2.82648E-01	2.46621E-01	2.10356E-01
1.75393E-01	1.42956E-01	1.13899E-01	8.87094E-02
6.75378E-02	5.02651E-02	3.65682E-02	2.60072E-02
1.80795E-02	1.22866E-02		

THE MEAN-SQUARE-ERROR IS 3.01468E-04 .

THE SPACING PARAMETER FOR ORDER 37 IS 1.47321E-01

AND THE WEIGHTS ARE ------

3.98569E-01	3.94293E-01	3.81677E-01	3.61558E-01
3.35160E-01	3.04024E-01	2.69876E-01	2.34424E-01
1.99271E-01	1.65754E-01	1.34922E-01	1.07472E-01
8.37712E-02	6.38973E-02	4.76943E-02	3.48368E-02
2.49001E-02	1.74164E-02	1.19208E-02	

THE MEAN-SQUARE-ERROR IS 2.87436E-04 .

THE SPACING PARAMETER FOR ORDER 38 IS 1.43924E-01

AND THE WEIGHTS ARE ------

3.97566E-01	3.89429E-01	3.73661E-01	3.51178E-01
3.23300E-01	2.91547E-01	2.57524E-01	2.22823E-01
1.88848E-01	1.56778E-01	1.27493E-01	1.01554E-01
7.92367E-02	6.05599E-02	4.53370E-02	3.32462E-02
2.38809E-02	1.68028E-02	1.15806E-02	

THE MEAN-SQUARE-ERROR IS 2.74118E-04 .

THE SPACING PARAMETER FOR ORDER 39 IS 1.40689E-01

AND THE WEIGHTS ARE ------

3.98608E-01	3.94688E-01	3.83171E-01	3.64700E-01
3.40324E-01	3.11372E-01	2.79304E-01	2.45633E-01
2.11800E-01	1.79053E-01	1.48405E-01	1.20598E-01
9.60833E-02	7.50528E-02	5.74794E-02	4.31592E-02
3.17716E-02	2.29323E-02	1.62275E-02	1.12584E-02

THE MEAN-SQUARE-ERROR IS 2.61907E-04 .

THE SPACING PARAMETER FOR ORDER 40 IS 1.37613E-01

AND THE WEIGHTS ARE ------

3.97685E-01	3.90235E-01	3.75756E-01	3.55042E-01
3.29175E-01	2.99481E-01	2.67365E-01	2.34218E-01
2.01340E-01	1.69835E-01	1.40579E-01	1.14180E-01
9.10026E-02	7.11724E-02	5.46190E-02	4.11316E-02
3.03943E-02	2.20391E-02	1.56816E-02	1.09487E-02

THE MEAN-SQUARE-ERROR IS 2.50329E-04 .

THE SPACING PARAMETER FOR ORDER 41 IS 1.34666E-01

AND THE WEIGHTS ARE ------

3.98625E-01	3.95053E-01	3.84459E-01	3.67451E-01
3.44880E-01	3.17897E-01	2.87759E-01	2.55808E-01
2.23324E-01	1.91467E-01	1.61206E-01	1.33295E-01
1.08237E-01	8.63142E-02	6.75943E-02	5.19861E-02
3.92637E-02	2.91230E-02	2.12135E-02	1.51750E-02
1.06608E-02			

THE MEAN-SQUARE-ERROR IS 2.39591E-04 .

THE SPACING PARAMETER FOR ORDER 42 IS 1.31851E-01

AND THE WEIGHTS ARE ------

3.97783E-01	3.90947E-01	3.77605E-01	3.58444E-01
3.34394E-01	3.06598E-01	2.76270E-01	2.44654E-01
2.12934E-01	1.82137E-01	1.53110E-01	1.26495E-01
1.02708E-01	8.19592E-02	6.42759E-02	4.95405E-02
3.75256E-02	2.79361E-02	2.04395E-02	1.46965E-02
1.03857E-02			

THE MEAN-SQUARE-ERROR IS 2.29479E-04 .

THE SPACING PARAMETER FOR ORDER 43 IS 1.29155E-01

AND THE WEIGHTS ARE ------

3.98657E-01	3.95360E-01	3.85603E-01	3.69876E-01
3.48922E-01	3.23728E-01	2.95387E-01	2.65074E-01
2.33942E-01	2.03057E-01	1.73338E-01	1.45522E-01
1.20153E-01	9.75686E-02	7.79189E-02	6.11989E-02
4.72727E-02	3.59124E-02	2.68311E-02	1.97152E-02
1.42474E-02	1.01257E-02		

THE MEAN-SQUARE-ERROR IS 2.20148E-04 .

THE SPACING PARAMETER FOR CRDER 44 IS 1.26567E-01

AND THE WEIGHTS ARE ------

3.97876E-01	3.91561E-01	3.79240E-01	3.61462E-01
3.39062E-01	3.12992E-01	2.84348E-01	2.54220E-01
2.23683E-01	1.93687E-01	1.65053E-01	1.38419E-01
1.14241E-01	9.27915E-02	7.41704E-02	5.83466E-02
4.51695E-02	3.44143E-02	2.58028E-02	1.90398E-02
1.38266E-02	9.88062E-03		

THE MEAN-SQUARE-ERROR IS 2.11299E-04 .

THE SPACING PARAMETER FOR CRDER 45 IS 1.24105E-01

AND THE WEIGHTS ARE ------

3.98683E-01	3.95627E-01	3.86612E-01	3.72020E-01
3.52528E-01	3.28944E-01	3.02260E-01	2.73503E-01
2.43698E-01	2.13833E-01	1.84759E-01	1.57205E-01
1.31715E-01	1.08676E-01	8.82965E-02	7.06450E-02
5.56576E-02	4.31820E-02	3.29905E-02	2.48201E-02
1.83878E-02	1.34146E-02	9.63719E-03	

THE MEAN-SQUARE-ERROR IS 2.03192E-04 .

THE SPACING PARAMETER FOR CRDER 46 IS 1.21724E-01

AND THE WEIGHTS ARE ------

3.97950E-01	3.92115E-01	3.80679E-01	3.64150E-01
3.43220E-01	3.18740E-01	2.91659E-01	2.62959E-01
2.33598E-01	2.04472E-01	1.76345E-01	1.49854E-01
1.25472E-01	1.03514E-01	8.41438E-02	6.73935E-02
5.31848E-02	4.13553E-02	3.16842E-02	2.39185E-02
1.77913E-02	1.30385E-02	9.41541E-03	

THE MEAN-SQUARE-ERROR IS 1.95328E-04 .

THE SPACING PARAMETER FOR ORDER 47 IS 1.19447E-01

AND THE WEIGHTS ARE ------

3.98695E-01	3.95875E-01	3.87503E-01	3.73942E-01
3.55747E-01	3.33650E-01	3.08493E-01	2.81206E-01
2.52700E-01	2.23872E-01	1.95526E-01	1.68353E-01
1.42906E-01	1.19588E-01	9.86595E-02	8.02411E-02
6.43378E-02	5.08567E-02	3.96316E-02	3.04473E-02
2.30601E-02	1.72177E-02	1.26742E-02	9.19767E-03

THE MEAN-SQUARE-ERROR IS 1.87952E-04 .

THE SPACING PARAMETER FOR ORDER 48 IS 1.17247E-01

AND THE WEIGHTS ARE ------

3.98023E-01	3.92607E-01	3.81961E-01	3.66557E-01
3.46965E-01	3.23938E-01	2.98324E-01	2.70984E-01
2.42790E-01	2.14569E-01	1.87040E-01	1.60816E-01
1.36388E-01	1.14090E-01	9.41390E-02	7.66150E-02
6.15028E-02	4.86991E-02	3.80345E-02	2.93003E-02
2.22645E-02	1.66867E-02	1.23360E-02	8.99556E-03

THE MEAN-SQUARE-ERROR IS 1.81008E-04 .

THE SPACING PARAMETER FOR ORDER 49 IS 1.15142E-01

AND THE WEIGHTS ARE ------

3.98707E-01	3.96089E-01	3.88308E-01	3.75654E-01
3.58639E-01	3.37892E-01	3.14161E-01	2.88244E-01
2.60991E-01	2.33205E-01	2.05638E-01	1.78945E-01
1.53664E-01	1.30220E-01	1.08903E-01	8.98772E-02
7.31976E-02	5.88304E-02	4.66611E-02	3.65212E-02
2.82096E-02	2.15027E-02	1.61749E-02	1.20072E-02
8.79562E-03			

THE MEAN-SQUARE-ERROR IS 1.74562E-04 .

THE SPACING PARAMETER FOR ORDER 50 IS 1.13100E-01

AND THE WEIGHTS ARE ------

3.98090E-01	3.93036E-01	3.83129E-01	3.68710E-01
3.50340E-01	3.28662E-01	3.04400E-01	2.78356E-01
2.51309E-01	2.24006E-01	1.97136E-01	1.71288E-01
1.46937E-01	1.24448E-01	1.04064E-01	8.59123E-02
7.00267E-02	5.63540E-02	4.47751E-02	3.51230E-02
2.72026E-02	2.08000E-02	1.57027E-02	1.17043E-02
8.61289E-03			

THE MEAN-SQUARE-ERROR IS 1.68183E-04 .

APPENDIX B

OPTIMUM WEIGHTS FOR BEST APPROXIMATION TO A BRICKWALL POWER SPECTRAL DENSITY (EVEN ORDER APPROXIMATION)

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THE SPACING PARAMETER FOR ORDER 2 IS 2.15343E 00

AND THE WEIGHT IS --------
7.75526E-01

THE MEAN-SQUARE-ERROR IS 5.51279E-01 .

THE SPACING PARAMETER FOR ORDER 4 IS 1.26311E 00

AND THE WEIGHTS ARE ------
9.15492E-01 4.97409E-01

THE MEAN-SQUARE-ERROR IS 3.99265E-01 .

THE SPACING PARAMETER FOR ORDER 6 IS 8.98097E-01

AND THE WEIGHTS ARE ------
9.56259E-01 7.18120E-01 3.49124E-01

THE MEAN-SQUARE-ERROR IS 3.53873E-01 .

THE SPACING PARAMETER FOR ORDER 8 IS 1.32688E 00

AND THE WEIGHTS ARE ------
9.07203E-01 4.57256E-01 -3.66434E-02 -2.00249E-01

THE MEAN-SQUARE-ERROR IS 2.92677E-01 .

THE SPACING PARAMETER FOR ORDER 10 IS 1.09664E 00

AND THE WEIGHTS ARE ------
9.35549E-01 6.01097E-01 1.49008E-01 -1.54843E-01
-1.89524E-01

THE MEAN-SQUARE-ERROR IS 2.49393E-01 .

THE SPACING PARAMETER FOR ORDER 12 IS 9.33952E-01

AND THE WEIGHTS ARE ------
9.52790E-01 6.97795E-01 3.11312E-01 -3.09909E-02
-1.99064E-01 -1.72607E-01

THE MEAN-SQUARE-ERROR IS 2.23887E-01 .

THE SPACING PARAMETER FOR ORDER 14 IS 8.13155E-01

AND THE WEIGHTS ARE ------
9.63985E-01 7.64437E-01 4.39999E-01 1.06695E-01
-1.28516E-01 -2.11176E-01 -1.55976E-01

THE MEAN-SQUARE-ERROR IS 2.07641E-01 .

THE SPACING PARAMETER FOR ORDER 16 IS 1.05155E 00

AND THE WEIGHTS ARE ------
9.40568E-01 6.28515E-01 1.92040E-01 -1.28220E-01
-2.02569E-01 -8.16538E-02 7.17212E-02 1.21191E-01

THE MEAN-SQUARE-ERROR IS 1.96089E-01 .

THE SPACING PARAMETER FOR ORDER 18 IS 9.47882E-01

AND THE WEIGHTS ARE ------
9.51409E-01 6.89787E-01 2.96764E-01 -4.45315E-02
-2.02655E-01 -1.63752E-01 -2.09704E-02 9.87437E-02
1.17425E-01

THE MEAN-SQUARE-ERROR IS 1.78701E-01 .

THE SPACING PARAMETER FOR ORDER 20 IS 8.62204E-01

AND THE WEIGHTS ARE ------

9.59610E-01	7.38022E-01	3.87378E-01	4.66648E-02
-1.65817E-01	-2.04946E-01	-1.10322E-01	2.59345E-02
1.13992E-01	1.12086E-01		

THE MEAN-SQUARE-ERROR IS 1.65842E-01 .

THE SPACING PARAMETER FOR ORDER 22 IS 7.90456E-01

AND THE WEIGHTS ARE ------

9.65931E-01	7.76331E-01	4.64379E-01	1.36150E-01
-1.07251E-01	-2.08919E-01	-1.73828E-01	-5.85351E-02
6.02278E-02	1.21902E-01	1.06301E-01	

THE MEAN-SQUARE-ERROR IS 1.56078E-01 .

THE SPACING PARAMETER FOR ORDER 24 IS 7.29591E-01

AND THE WEIGHTS ARE ------

9.70895E-01	8.07102E-01	5.29420E-01	2.19537E-01
-3.81500E-02	-1.85208E-01	-2.06587E-01	-1.30800E-01
-1.40138E-02	8.45075E-02	1.25233E-01	1.00574E-01

THE MEAN-SQUARE-ERROR IS 1.48497E-01 .

THE SPACING PARAMETER FOR ORDER 26 IS 6.77355E-01

AND THE WEIGHTS ARE ------

9.74858E-01	8.32110E-01	5.84344E-01	2.95146E-01
3.42495E-02	-1.43269E-01	-2.12035E-01	-1.80981E-01
-8.65825E-02	2.21539E-02	1.01263E-01	1.25713E-01
9.51231E-02			

THE MEAN-SQUARE-ERROR IS 1.42496E-01 .

THE SPACING PARAMETER FOR ORDER 28 IS 8.30753E-01

AND THE WEIGHTS ARE ------

9.62442E-01	7.55069E-01	4.21095E-01	8.45636E-02
-1.43266E-01	-2.10624E-01	-1.40542E-01	-9.65782E-03
9.58813E-02	1.23151E-01	7.25283E-02	-1.24623E-02
-7.63579E-02	-8.46784E-02		

THE MEAN-SQUARE-ERROR IS 1.36916E-01 .

THE SPACING PARAMETER FOR ORDER 30 IS 7.80007E-01

AND THE WEIGHTS ARE ------

9.66809E-01	7.81731E-01	4.75592E-01	1.50029E-01
-9.66234E-02	-2.06711E-01	-1.81139E-01	-7.13696E-02
4.89707E-02	1.18522E-01	1.12696E-01	4.83178E-02
-3.14734E-02	-8.25618E-02	-8.21155E-02	

THE MEAN-SQUARE-ERROR IS 1.29954E-01 .

THE SPACING PARAMETER FOR ORDER 32 IS 7.34924E-01

AND THE WEIGHTS ARE ------

9.70475E-01	8.04475E-01	5.23755E-01	2.12005E-01
-4.48844E-02	-1.88353E-01	-2.04752E-01	-1.24864E-01
-6.83083E-03	8.94405E-02	1.25491E-01	9.60352E-02
2.56390E-02	-4.65805E-02	-8.64196E-02	-7.93487E-02

THE MEAN-SQUARE-ERROR IS 1.24170E-01 .

THE SPACING PARAMETER FOR ORDER 34 IS 6.94660E-01

AND THE WEIGHTS ARE ------

9.73576E-01	8.23976E-01	5.66277E-01	2.69768E-01
9.01423E-03	-1.59407E-01	-2.12872E-01	-1.66239E-01
-6.24750E-02	4.53361E-02	1.13521E-01	1.21725E-01
7.66594E-02	5.35005E-03	-5.84209E-02	-8.86074E-02
-7.65331E-02			

THE MEAN-SQUARE-ERROR IS 1.19320E-01 .

THE SPACING PARAMETER FOR ORDER 36 IS 6.58512E-01

AND THE WEIGHTS ARE ------

9.76220E-01	8.40789E-01	6.03842E-01	3.23077E-01
6.30485E-02	-1.23176E-01	-2.07930E-01	-1.94375E-01
-1.12125E-01	-5.20637E-03	8.32701E-02	1.24491E-01
1.11159E-01	5.67227E-02	-1.22998E-02	-6.75915E-02
-8.96046E-02	-7.37494E-02		

THE MEAN-SQUARE-ERROR IS 1.15212E-01 .

THE SPACING PARAMETER FOR ORDER 38 IS 6.25895E-01

AND THE WEIGHTS ARE ------

9.78490E-01	8.55371E-01	6.37087E-01	3.71982E-01
1.15897E-01	-8.22904E-02	-1.92566E-01	-2.09813E-01
-1.52632E-01	-5.56500E-02	4.19322E-02	1.07977E-01
1.25732E-01	9.66786E-02	3.74814E-02	-2.73702E-02
-7.45987E-02	-8.97473E-02	-7.10531E-02	

THE MEAN-SQUARE-ERROR IS 1.11702E-01 .

THE SPACING PARAMETER FOR ORDER 40 IS 5.96330E-01

AND THE WEIGHTS ARE ------

9.80452E-01	8.68082E-01	6.66573E-01	4.16672E-01
1.66726E-01	-3.87848E-02	-1.69244E-01	-2.13946E-01
-1.82766E-01	-1.01614E-01	-4.17082E-03	7.76079E-02
1.21608E-01	1.20172E-01	8.02834E-02	1.96035E-02
-4.00754E-02	-7.98788E-02	-8.92776E-02	-6.84634E-02

THE MEAN-SQUARE-ERROR IS 1.08683E-01 .

THE SPACING PARAMETER FOR ORDER 42 IS 7.05426E-01

AND THE WEIGHTS ARE ------

9.72763E-01	8.18838E-01	5.54968E-01	2.54134E-01
-6.07414E-03	-1.68327E-01	-2.12037E-01	-1.56042E-01
-4.73940E-02	5.85919E-02	1.19001E-01	1.16950E-01
6.38170E-02	-9.66617E-03	-6.87014E-02	-8.94127E-02
-6.74928E-02	-1.76898E-02	3.47171E-02	6.60179E-02
6.42740E-02			

THE MEAN-SQUARE-ERROR IS 1.05469E-01 .

THE SPACING PARAMETER FOR ORDER 44 IS 6.75480E-01

AND THE WEIGHTS ARE ------

9.74996E-01	8.32980E-01	5.86293E-01	2.97912E-01
3.70553E-02	-1.41385E-01	-2.11780E-01	-1.82445E-01
-8.91680E-02	1.95188E-02	9.96863E-02	1.25858E-01
9.69275E-02	3.28353E-02	-3.58411E-02	-8.09115E-02
-8.71124E-02	-5.65106E-02	-5.73210E-03	4.22241E-02
6.78533E-02	6.26718E-02		

THE MEAN-SQUARE-ERROR IS 1.01922E-01 .

THE SPACING PARAMETER FOR ORDER 46 IS 6.47906E-01

AND THE WEIGHTS ARE ------

9.76970E-01	8.45594E-01	6.14726E-01	3.38912E-01
7.98302E-02	-1.10720E-01	-2.04104E-01	-2.00540E-01
-1.25928E-01	-2.13165E-02	7.11502E-02	1.21139E-01
1.17897E-01	7.08514E-02	3.54589E-03	-5.64133E-02
-8.76035E-02	-8.17320E-02	-4.51456E-02	5.05462E-03
4.84030E-02	6.90056E-02	6.10425E-02	

THE MEAN-SQUARE-ERROR IS 9.88045E-02 .

THE SPACING PARAMETER FOR ORDER 48 IS 6.22453E-01

AND THE WEIGHTS ARE ------

9.78723E-01	8.56875E-01	6.40553E-01	3.77172E-01
1.21680E-01	-7.75346E-02	-1.90316E-01	-2.10805E-01
-1.56516E-01	-6.10628E-02	3.69226E-02	1.05171E-01
1.26146E-01	1.00159E-01	4.27524E-02	-2.22510E-02
-7.15210E-02	-8.98498E-02	-7.42972E-02	-3.38890E-02
1.46685E-02	5.34604E-02	6.96334E-02	5.94142E-02

THE MEAN-SQUARE-ERROR IS 9.60500E-02 .

THE SPACING PARAMETER FOR ORDER 50 IS 5.98893E-01

AND THE WEIGHTS ARE ------

9.80286E-01	8.67000E-01	6.64043E-01	4.12790E-01
1.62219E-01	-4.27908E-02	-1.71625E-01	-2.13994E-01
-1.80469E-01	-9.77141E-02	6.52882E-05	8.07806E-02
1.22665E-01	1.18764E-01	7.68808E-02	1.53586E-02
-4.37272E-02	-8.16946E-02	-8.86422E-02	-6.55907E-02
-2.30658E-02	2.31614E-02	5.75698E-02	6.98568E-02
5.78103E-02			

THE MEAN-SQUARE-ERROR IS 9.36034E-02 .

APPENDIX C

EVALUATION OF AN INDEFINITE INTEGRAL

An integral which appears when analysing the probability density of weighted shift registers is

$$I(x) = \int \left[A - P(x) \right]^2 dx \qquad (C.1)$$

where A is a constant,

$$P(x) = \int_{-\infty}^{x} Z(t)\, dt \qquad (C.2)$$

and Z(t) is the standardized Gaussian curve

$$Z(t) = \frac{\exp(-t^2/2)}{\sqrt{2\pi}} \qquad (C.3)$$

Ignoring the constant of integration, we expand (C.1) to obtain

$$I(x) = \int A^2\, dx - 2A \int P(x)\, dx + \int P^2(x)\, dx \qquad (C.4)$$

We assume that the second integral in the right hand side of (C.4) may be written as

$$\int P(x)\, dx = (A + Bx)\, P(x) + (C + Dx)\, Z(x) \qquad (C.5)$$

Differentiating both sides of (C.5) with respect to x, and making use of

$$\frac{d\, P(x)}{dx} = Z(x) \quad \text{and} \quad \frac{d\, Z(x)}{dx} = -x\, Z(x)$$

we obtain

$$P(x) = B\, P(x) + \left[A + D + (B - C)\, x - Dx^2 \right] Z(x) \qquad (C.6)$$

In order that (C.6) hold true, we must have

$$B = C = 1, \quad A = D = 0. \tag{C.7}$$

Substituting (C.7) into (C.5), we obtain

$$\int P(x)\,dx = x\,P(x) + Z(x) \quad . \tag{C.8}$$

Similarly, let us assume* that the third integral in the right hand side of (C.4) may be written as

$$\int P^2(x)\,dx = (A + Bx)\,P^2(x) + (C + Dx)\,P(x)\,Z(x) + E\,P(\sqrt{2}\,x) \tag{C.9}$$

Differentiating both sides of (C.9), we obtain

$$P^2(x) = B\,P^2(x) + \left[2A + D + (2B - C)\,x + D(1 - x^2)\right] P(x)\;Z(x)$$
$$+ (C + Dx)\;Z^2(x) + EF\;Z(Fx).$$

Making use of

$$Z(Fx) = \sqrt{2\pi}\;Z^2\left[\frac{Fx}{\sqrt{2}}\right] \tag{C.10}$$

---

* There is no logical reason for making these assumptions other than the fact that they lead to the solution for the integrals involved.

we may write

$$P(x)^2 = BP^2(x) + \left[2A + D + (2B-C)x + D(1-x^2)\right] P(x)\ Z(x)$$
$$+ (C + 2\sqrt{\pi}\ E + Dx)\ Z^2(x) \qquad (C.11)$$

In order that (C.11) hold true, we must have

$$\begin{aligned} B &= 1 \\ 2A + D &= 0 \\ 2B - C &= 0 \\ D &= 0 \\ C + 2\sqrt{\pi}E &= 0 \end{aligned}$$

from which we obtain

$$B = 1, \quad C = 2, \quad E = -\frac{1}{\sqrt{\pi}}, \quad A = D = 0. \qquad (C.12)$$

Substituting (C.12) into (C.9), we obtain

$$\int P(x)^2\, dx = x P^2(x) + 2P(x)\ Z(x) - \frac{1}{\sqrt{\pi}}\ P(\sqrt{2}\, x) \qquad (C.13)$$

Since

$$\int A^2\, dx = A^2 x \qquad (C.14)$$

we find, by substituting (C.8), (C.13) and (C.14) into (C.4), that

$$I(x) = \int (A - P(x))^2\, dx$$

$$= A^2 x + x P^2(x) + 2P(x)\ Z(x) - \frac{P(\sqrt{2}\, x)}{\sqrt{\pi}} - 2Ax\ P(x) - 2A\ Z(x)$$

$$(C.15)$$

which is the solution of the required integral.

APPENDIX D

PROBABILITY DENSITY SOLUTION - ORDER 1

Although evaluation of the optimum weights is trivial for a one stage register (n = 1), it is useful for obtaining insight as to the behaviour of the error criterion functional.

Assuming the weight W is positive, it is easy to see from (4.4-4) that

$$\eta_1 = -W \; ; \; \eta_2 = W$$

From (4.4-11), we see that for n = 1

$$E(W) = 2\left\{-W\left[-\frac{1}{4} + P(-W)\right] + Z(-W)\right\} - \frac{1}{\sqrt{\pi}} \qquad (D.1)$$

Making use of

$$P(-x) = 1 - P(x)$$

and

$$Z(-x) = Z(x)$$

(D.1) reduces to

$$E(W) = 2\,W\,P(W) + 2\,Z(W) - \frac{3\,W}{2} - \frac{1}{\sqrt{\pi}} \qquad (D.2)$$

Differentiating with respect to W and equating to zero, we obtain

$$\frac{d\,E(W)}{dW} = 2\,P(W) - \frac{3}{2} \equiv 0 \qquad (D.3)$$

Thus we see that

$$P(W) = \frac{3}{4} \tag{D.4}$$

from which we may obtain, using mathematical tables,[24]

$$W = 0.67449 \tag{D.5}$$

which satisfies the condition $W > 0$.

In order to see whether (D.5) gives rise to a maximum or a minimum of E(W), we differentiate (D.3) with respect to W. Thus

$$\frac{d^2 E(W)}{d W^2} = 2 Z(W) \tag{D.6}$$

which is greater than zero for all W in the domain of interest; and therefore the value (D.5) gives rise to a minimum as desired.

Similarly, for $W < 0$*,

$$\eta_1 = W, \quad \eta_2 = -W$$

and from (4.4-11)

$$E(W) = 2 \left\{ W \left[-\frac{1}{4} + P(W) \right] + Z(W) \right\} - \frac{1}{\sqrt{\pi}}$$

which reduces to

$$E(W) = 2 W P(W) + 2 Z(W) - \frac{W}{2} - \frac{1}{\sqrt{\pi}} .$$

---

* As explained in Chapter III, a negative W can be realized by weighting the complementary output of the shift register stage with a positive weight.

After differentiation, we obtain

$$\frac{d\,E(W)}{dW} = 2\,P(W) - \frac{1}{2} \equiv 0$$

from which $P(W) = \frac{1}{4}$ or $W = -0.67449$.

Thus, for n = 1, the optimum weight is

$$\pm\,0.67449\ .$$

APPENDIX E

PROBABILITY DENSITY SOLUTION - ORDER 2

For n = 2, we make the following assumption for the weights :

$$W_2 > W_1 > 0 \tag{E.1}$$

Thus, from (4.4-4) we find

$$\eta_1 = -W_2 - W_1, \; \eta_2 = -W_2 + W_1, \; \eta_3 = W_2 - W_1, \text{ and}$$
$$\eta_4 = W_2 + W_1 \tag{E.2}$$

From (4.4-11), we obtain

$$E = \eta_1 P(\eta_1) + \eta_2 P(\eta_2) + Z(\eta_1) + Z(\eta_2) - \frac{\eta_1}{8} - \frac{3\eta_2}{8} - \frac{1}{\sqrt{\pi}} \tag{E.3}$$

Taking the partial derivatives of (E.3) with respect to W_1 and W_2, we obtain

$$P(\eta_2) - P(\eta_1) = \frac{1}{4}$$
$$P(\eta_2) + P(\eta_1) = \frac{1}{2} \tag{E.4}$$

where we have made use of

$$\frac{\partial \eta_1}{\partial W_1} = \frac{\partial \eta_1}{\partial W_2} = \frac{\partial \eta_2}{\partial W_2} = -1; \; \frac{\partial \eta_2}{\partial W_1} = +1.$$

From (E.4) we get

$$P(\eta_1) = \frac{1}{8}, \; P(\eta_2) = \frac{3}{8} \tag{E.5}$$

Thus we find[24]

$$\eta_1 = -1.15035$$

$$\eta_2 = -0.31864$$

and using (E.2) we obtain

$$W_2 = 0.73445$$
$$W_1 = 0.41590 \qquad \text{(E.6)}$$

In the same way as in Appendix D, the following assumption will yield other pairs of solutions.

For example,

$W_1 > W_2 > 0$ $\quad$ $W_1 = 0.73445$

$W_2 = 0.41590$

$W_1 < W_2 < 0$ $\quad$ $W_1 = -0.73445$

$W_2 = -0.41590$

$W_2 < W_1 < 0$ $\quad$ $W_2 = -0.73445$

$W_1 = -0.41590$

etc.

All told, there are 8 solutions symmetrical about the W_1 W_2 axes as is shown in Figure 4.6.

APPENDIX F

DIRECT OPTIMIZATION TECHNIQUE - METHOD OF FLOOD AND LEON

This method is fully described in two papers written jointly by M.M. Flood and A. Leon[30]. It may be described briefly as follows[31] :

"1. Initialization. The optimization process is initiated by picking up, as the starting point, an arbitrary point inside the operating space."

"2. Order of Analysis. Once the function has been evaluated at the starting point, the independent variables to be changed are changed in an order selected initially by the experimenter." For the purposes of evaluating the optimum value of (4.4-11), the order of analysis is unimportant due to the symmetry of the function with respect to the weight parameters. Thus, for convenience, the weight parameters are analysed in their natural sequence.

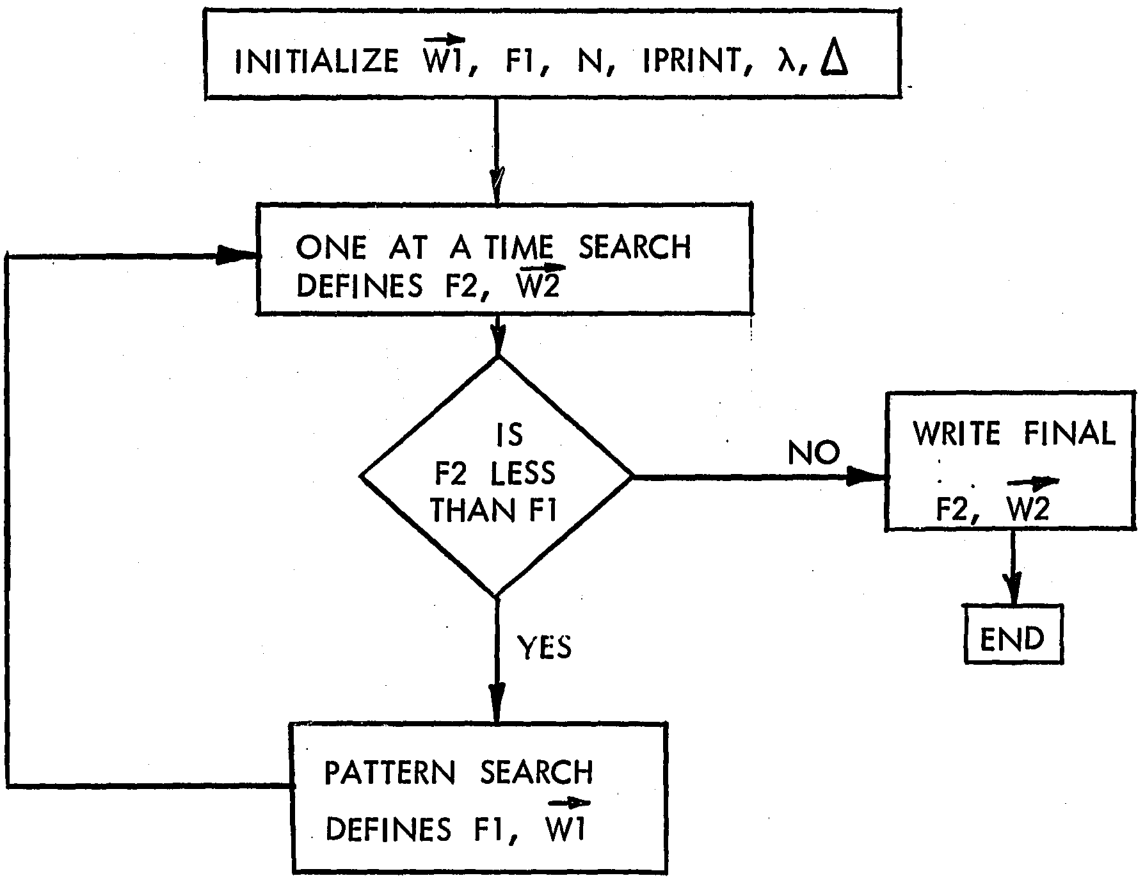
"3. One-at-a-time search. After deciding upon the order in which to search, the one-at-a-time search is initiated. Let W1 (1) be the first variable under study; this variable is incremented by an amount Δ, holding the other variables at their initial values. If the functional value at this point is better than the one at the preceding point, there is some reason for trying further in the same direction. A larger step size is now used, taken equal to $\lambda\Delta$ (where $\lambda > 1$), and if a better functional value (comparing against the immediately previous one) is obtained, a step of length $\lambda^2\Delta$ is used next. This is continued in the same direction of powers of λ until no further improvement is obtained. Assume that step $\lambda^{n+1}\Delta$ was the first unsuccessful one; in this case the preceding base point is kept, namely, the one obtained by step $\lambda^n\Delta$ and a new sequence is started from this point with initial step size equal to Δ following the same scheme as before. If a step of Δ in the positive direction does not bring a better point, then a step of length Δ in the negative direction is tried; if this happens to be a successful step, then

$\lambda \Delta$ is tried in the same negative direction continuing in the same fashion as was done in the positive direction. Finally, a point is reached where no improvement is obtained by moving the variable W1 (1) either Δ or $-\Delta$; this point is considered to be the best temporarily for variable W1 (1). After the best point in the W1(1) direction is found the second variable W1 (2) is ready to be analysed. The process is repeated until all of the n variables to be analyzed have been studied and a point $\overrightarrow{W2}$ presenting the best functional value of the round is reached."

"4. Pattern Move. If the functional value F2 at the end of step 3 is better than the initial one F1 then the pattern move is tried. The coordinates $\overrightarrow{W2}$ of the F2 point are incremented by an amount proportional to the change experienced for the coordinates in going from F1 to F2. This rate of change will be greater than one. If F3, after the initial pattern move, happens to be better than F2, a new step of length $\lambda \vec{D}$ is taken in the same direction. The process here follows the same scheme explained in phase 3. As before, when a point is reached where no improvement is obtained by moving the vector either $\vec{D}$ or $-\vec{D}$, this point is considered the best of this series of pattern moves.

If the point obtained after a series of pattern moves is better than the point at the beginning of the series (i.e., at the end of the one-at-a-time round), a new round of the one-variable-at-a-time phase, as it was previously described, is attempted, and the process is kept going until no better points are found. If the pattern move phase happens to be a failure, a one-at-a-time round will be tried, resulting either in the final point, i.e., the optimum searched (as far as the technique can tell), or in the continuation of the optimization calculation".

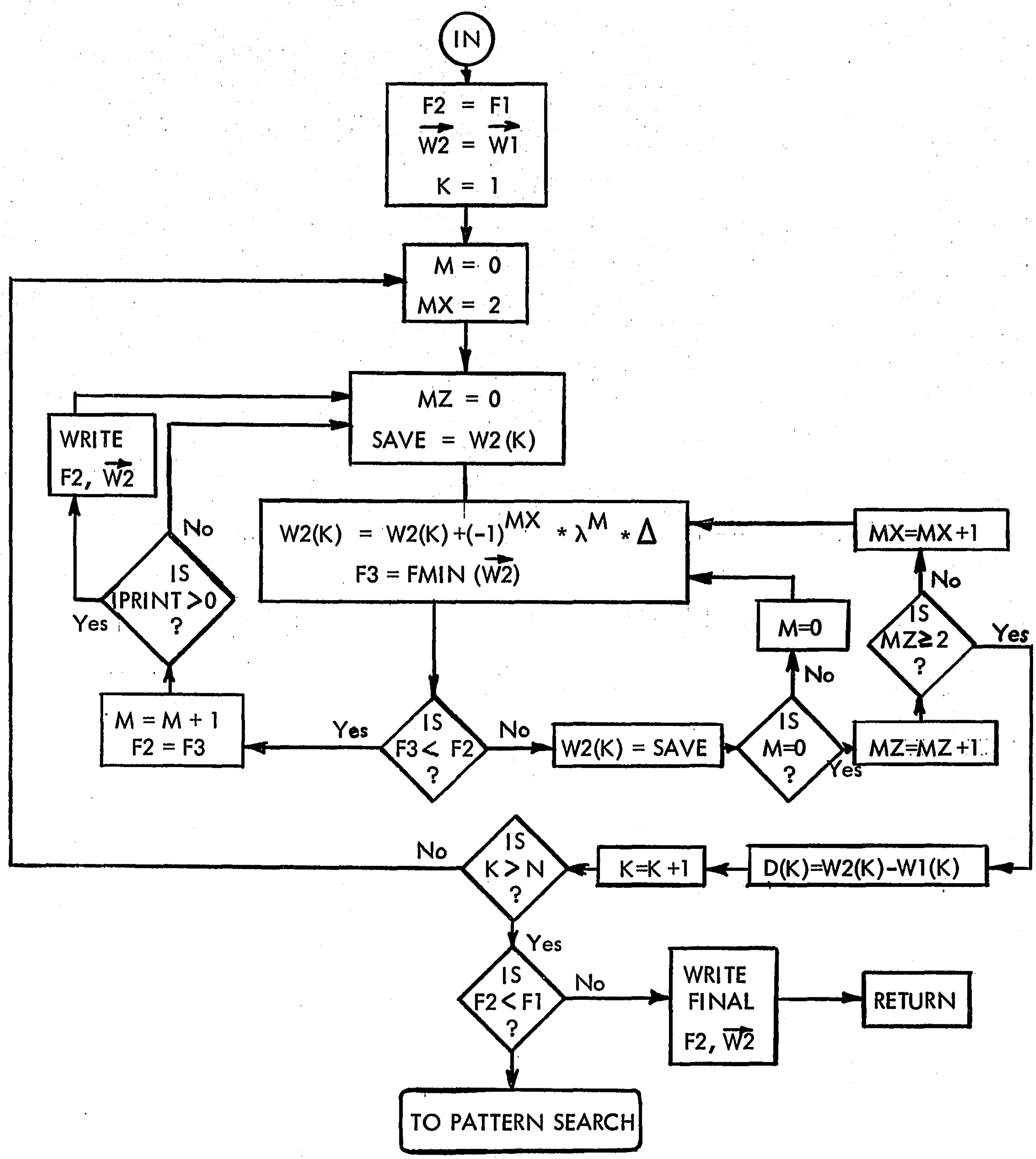
A simplified flow chart of this subroutine is illustrated in Figure F-1(a) and detailed in F-1(b) and F-1(c). In Figure F-2, the flow chart for FMIN (i.e., $E(\vec{W})$ of (4.4-11) is given.



(a)

FIGURE F-1 - OPTIMIZATION SUBROUTINE METHOD OF FLOOD AND LEON

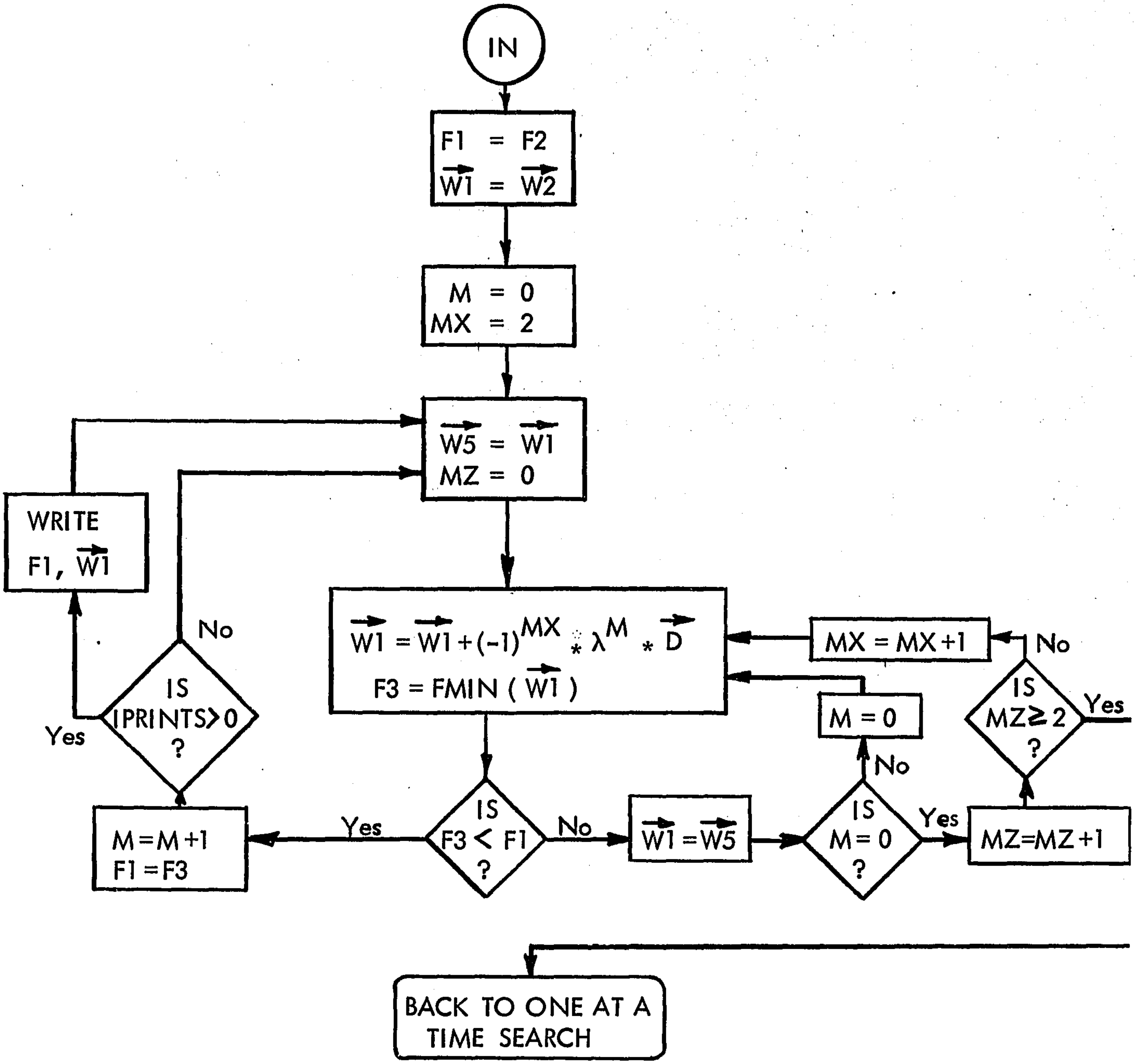
(a) Simplified Flow Chart.



(b)

FIGURE F-1 - OPTIMIZATION SUBROUTINE METHOD OF FLOOD AND LEON.

(b) Flow Chart For One At A Time Search.



(c)

FIGURE F-1 - OPTIMIZATION SUBROUTINE METHOD OF FLOOD AND LEON

(c) Flow Chart for Pattern Search. IPRINT is a parameter which is set to an integer greater than 0 if all intermediate values of FMIN and the weight vector are to be printed. If IPRINT is set less or equal to 0, only the final values are printed.

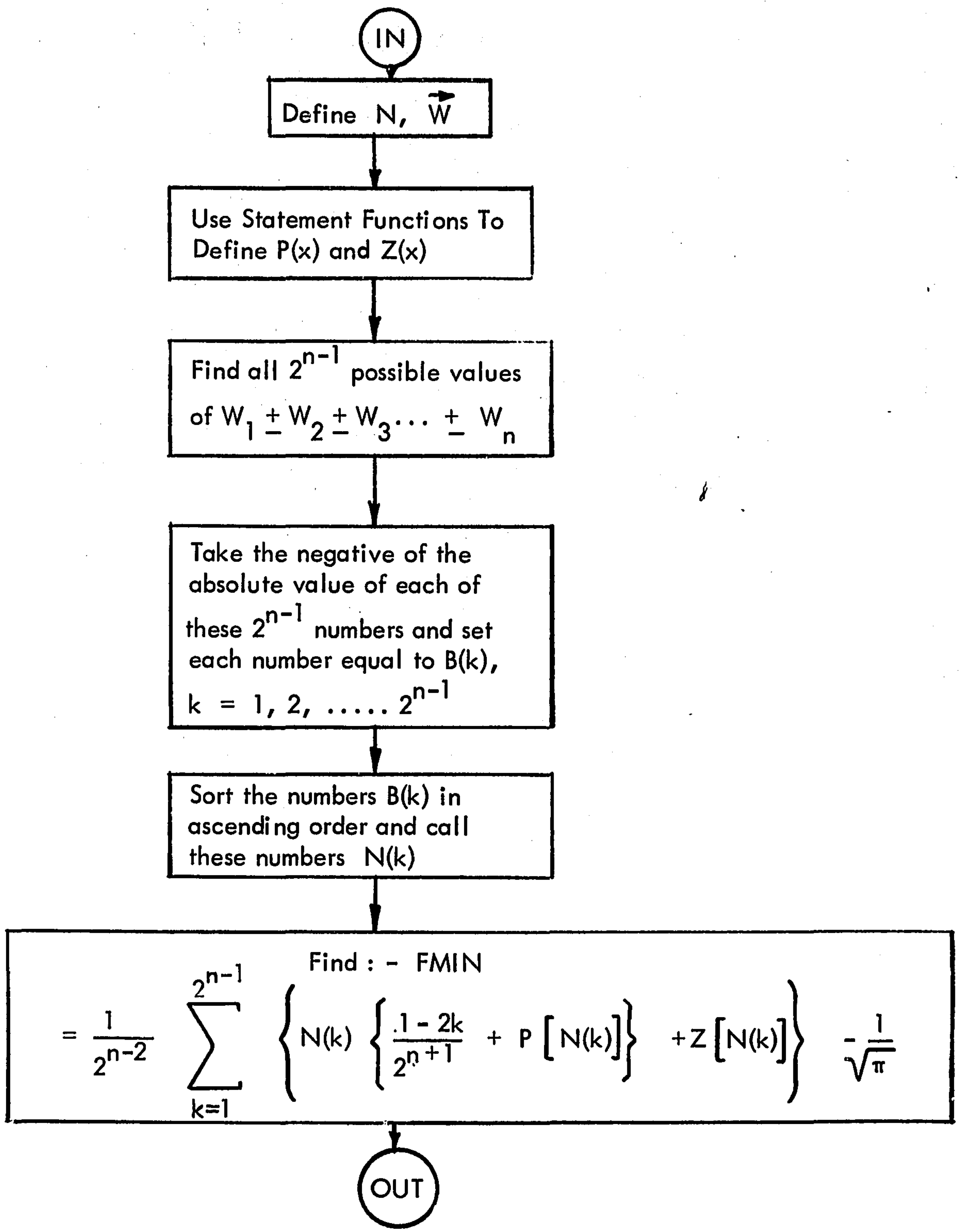


FIGURE F-2 - FLOW DIAGRAM OF FMIN, (PROBABILITY DENSITY ERROR FUNCTION TO BE MINIMIZED).

APPENDIX G

OPTIMUM WEIGHTS FOR BEST APPROXIMATION TO A GAUSSIAN AMPLITUDE PROBABILITY DENSITY FUNCTION

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THE OPTIMUM WEIGHT FOR ORDER 1 IS-------

0.6745

AND THE MEAN-SQUARE-ERROR IS 7.1364E-02 .

THE OPTIMUM WEIGHTS FOR ORDER 2 ARE-----

0.7344 0.4156

AND THE MEAN-SQUARE-ERROR IS 2.0859E-02 .

THE OPTIMUM WEIGHTS FOR ORDER 3 ARE-----

0.6883 0.5227 0.3232

AND THE MEAN-SQUARE-ERROR IS 5.9124E-03 .

THE OPTIMUM WEIGHTS FOR ORDER 4 ARE-----

0.6129 0.5247 0.4276 0.2730

AND THE MEAN-SQUARE-ERROR IS 1.7972E-03 .

THE OPTIMUM WEIGHTS FOR ORDER 5 ARE-----

0.6033 0.4896 0.4033 0.3131 0.2466

AND THE MEAN-SQUARE-ERROR IS 6.6593E-04 .

THE OPTIMUM WEIGHTS FOR ORDER 6 ARE-----

0.5454 0.5046 0.4033 0.3425 0.2701 0.1860

AND THE MEAN-SQUARE-ERROR IS 2.6423E-04 .

THE OPTIMUM WEIGHTS FOR ORDER 7 ARE-----

0.5312 0.4445 0.3894 0.3649 0.3052 0.2376 0.1907

AND THE MEAN-SQUARE-ERROR IS 1.2664E-04 .

THE OPTIMUM WEIGHTS FOR ORDER 8 ARE-----

0.4949 0.4292 0.3784 0.3546 0.3320 0.2628 0.2263 0.1727

AND THE MEAN-SQUARE-ERROR IS 6.9365E-05 .

THE OPTIMUM WEIGHTS FOR ORDER 9 ARE-----

0.4607 0.4079 0.3639 0.3541 0.3383 0.3161 0.2515 0.1744 0.1378

AND THE MEAN-SQUARE-ERROR IS 4.5374E-05 .

THE OPTIMUM WEIGHTS FOR ORDER 10 ARE----

0.4529 0.3812 0.3383 0.3219 0.3176 0.3039 0.2700 0.2308 0.2194 0.1732

AND THE MEAN-SQUARE-ERROR IS 3.1747E-05 .