Heat full statistics and regularity of perturbations in quantum statistical mechanics

by

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À la mémoire de

Ludger Turbide

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Résumé On présente une étude du rôle de la régularité des perturbations dans le contrôle des fluctuations de la chaleur qui entrent en jeu dans le premier principe de la thermodynamique en mécanique quantique. Ces fluctuations sont décrites par les queues de la mesure dite *statistiques complètes* de la chaleur, motivée par un protocole de mesure à deux temps de l'énergie aux temps 0 et t. On introduit les notions appropriées de régularité de la perturbation qui assurent une décroissance polynomiale ou exponentielle de ladite mesure à l'infini. On obtient aussi un contrôle en temps et la nécessité de ces conditions dans des modèles concrets de théorie quantique des champs.

Abstract We present a study of the role of regularity of perturbations in the control of quantum fluctuations appearing in the first law of thermodynamics in quantum mechanics. These fluctuations are encoded in the so-called heat full statistics measure, motivated by considering differences in energy measurements at times 0 and t. We introduce the suitable notions of regularity of the perturbation that ensure polynomial or exponential decay of this measure at infinity. We also obtain control in t, and necessity of these conditions for decay in concrete models of quantum field theory.

Préface

This thesis is centred around the paper [BPR17] written in collaboration with Tristan Benoist (IMT Toulouse) and Annalisa Panati (CPT Toulon), from whom I have gained valuable knowledge and insight over the past year and a half. *Ce fut un véritable plaisir de travailler avec vous.* Although not an author of the paper, my supervisor and mentor Vojkan Jakšić played a crucial role in the early stage of this project by suggesting key ideas and tools, and remained available for guidance and insightful discussion throughout the process. More generally, he deserves credit for a considerable part of my overall formation as a young researcher in mathematical physics, with Alain Joye (Insitut Fourier, Grenoble) and Yan Pautrat (LMO, Paris-Saclay) also deserving substantial credit.

With some of Vojkan's conjectures in mind, I arrived in Toulouse in May 2016 to work out the proofs and optimise the hypotheses with Tristan. Having settled many technical questions (essentially the contents of Appendix A and Section 4.1 here), I left Toulouse to work on another project in Grenoble, but met back with Tristan (and virtually with Annalisa) in Autrans, working out more results (most of the results of Section 4.2 here) during afternoon breaks of the *Stochastic Methods in Quantum Mechanics* summer school in July — thanks to the organizers. During the following four months, more results were slowly gathered between classes, seminars and other research projects, allowing me to write a first pseudo-official draft of the paper.

I then took some time in Montréal to expand several proofs and remarks and write Chapters 1 and 3 in order to achieve a certain self-containedness, turning it into this thesis as the last pieces of the paper were falling into place, the last pieces being the details of the proofs for Chapter 2, the self-adjointness results of Appendix B (mostly thought through by Annalisa), and Tristan's introduction (the introduction here is in different both in style and content).

On a personal level, I would like to sincerely thank my family, especially my parents Josée and Mario, for inculcating me with curiosity, for always supporting me, and more importantly for their care and unconditional love. *Je vous aime et ne pourrai jamais assez vous remercier*. I also wish to thank my friends and roommates, Nicolas, Erick and Joey (also my unofficial typography consultant), for both entertaining me and peer-pressuring me into working harder and later. I am also grateful to a handful of quality friends for directly or indirectly supporting me through this project, and more importantly for filling the past few years with moments I will almost surely always remember with fondness.

Lastly, I truly hope the reader will appreciate this work; I have tried to make it as readable and as enjoyable as my imperfect knowledge of this language allows.

Renaud Raquépas,

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Contents

Introduction

The problem of interest concerns the contribution of heat in the first law of thermodynamics — the appropriate form of energy conservation for extended physical systems — when a system undergoes a pertubation. Results on the first law of thermodynamics in quantum mechanics used to be typically obtained by considering work as an observable on the quantum system. However, since the work of Kurchan and Tasaki [Kur00, Tas00], the use of *full statistics* (FS)[1] in a redefinition of work has revealed many new research avenues concerning the nature of heat and work, energy conservation, and fluctuation relations. Although the FS approach yields expectations that agree with those obtained when heat and work are considered as an observable, it represents a significantly different viewpoint and reveals a very rich theory that finds no counterpart in the earlier literature.

The basic physical picture behind the mathematics we are doing is the following. Consider initially non-interacting subsystems at time 0. Make a measurement of the energy of the whole system, and call the result ϵ_i. Now, connect the subsystems by turning on an internal interaction (potential, perturbation) and let the system evolve jointly for a time $t > 0$. After that, measure the energy of the system again (the same way you did initially, negelecting the energy in the interaction) and label the result ϵ_j. You have two numbers at hand, and the first law concerns their difference $\Delta Q = \epsilon_j - \epsilon_i$. We are interested in the probability distribution governing ΔQ: the heat full statistics measure $\mathbf{P}_t$.

For large t, if the first law of thermodynamics is robust to perturbations, this measure should, in one way or another, concentrate around values for which $\frac{1}{t}|\Delta Q|$ is small. This is the case in a very strong sense in classical physiscs (see e.g. the discussion in [BPR17]), but the picture in

[1] The term *full counting statistics* (FCS) can also be found in the literature for historical reasons: the method was first introduced in the context of charge transport, where the counting makes sense.

quantum physics is not as clear: at finite time t, how do the tails (probability of a large $|\Delta Q|$) behave? It turns out — and this is the whole point of this thesis — that the key property to look at to understand these tails is a suitable notion of *regularity* of the interaction. This link between the tails of the energy FS measure and the regularity of interactions is a purely quantum phenomenon that has no counterpart in the classical theory.

For confined quantum systems, the probability measure $\mathbf{P}_t$ arising from this two-time energy measurement protocol can be naively computed in terms of traces of matrices. However, since phenomena of thermodynamics truly concern extended systems (usually obtained through a so-called *thermodynamic limit*), the case of confined systems is not satisfactory. Many notions used in the naive definition of $\mathbf{P}_t$ (like what it means to measure the energy) will not survive the thermodynamical limit. We need to reformulate the definition of the heat FS measure in terms of more abstract objects that still hold a meaning in the realm of extended quantum systems. The appropriate language for doing this is found in the algebraic framework of quantum statistical mechanics, developed since the seminal work of John von Neumann in the 1930s. It turns out that, in this setup, the separation into subsystems is somehow artificial — as long as the whole system is in an invariant state for the initial dynamics — and hence does not explicitly appear in the general mathematical formulation.

Chapter 1 starts with an introduction to the *à la Heisenberg* quantum mechanics of confined quantum systems, sufficient to provide a first definition of the heat full statistics measure $\mathbf{P}_t$. Then, we turn to a brief exposition of the algebraic framework of quantum statistical mechanics: C^*-algebras, quantum dynamical systems, and perturbation theory.

With this formalism at hand, we introduce the desired general definition of the heat full statistics measure $\mathbf{P}_t$ and discuss general results for bounded perturbations of C^*-dynamical systems in Chapter 2.

In Chapter 3, we present the basic theory of many-body quantum mechanics which will allow us to construct examples of physical models to later investigate the optimality of the conditions of our general theorems. Doing so, we introduce some elementary notions of quantum field theory: Fock spaces, fields, and quasi-free Bose and Fermi gases.

In Chapter 4, we study three physical models, the fermionic Wigner–Weisskopf atom model, the bosonic Wigner–Weisskopf model, and the van Hove Hamiltonian system. These models, which are described using the language of Chapter 3, allow us to investigate the optimality of the hypothses of the general results of Chapter 2 and see how these results extend to some classes of unbounded perturbations of W^*-dynamical systems.

Chapter 1

Overview of the mathematical structure

In this chapter, we gather the elements of mathematical formalism that will be necessary for the treatment of our problem. Starting from the matrix description of the quantum mechanics of confined systems, we then move to the more general algebraic description of non-relativistic quantum theory.

We assume that the reader is familiar with elementary measure theory and the basic theory of operators on Hilbert spaces. To be completely honest, previous exposure to notions of quantum mechanics will be of great help to make sense of the motivation and nomenclature. More details can be found in the classic references [RS72, RS75] and [BR87, BR97].

The reader interested in the historical development might be interested in consulting the seminal original works [Seg47], [GN43], and [vN55].

1.1 Quantum mechanics of confined system

Confined quantum mechanical systems are described by a finite dimensional Hilbert space (isomorphic to $\mathbf{C}^n$ with the standard euclidian inner product[1]). States of such a system correspond to n-by-n **density matrices**, i.e. non-negative hermitian matrices with trace equal to one. Experiments are described in terms of hermitian matrices, referred to as observables. Recall that by the spectral theorem, a hermitian n-by-n matrix A has a spectral decomposition $A = \sum_{i=1}^{r} a_i P_i$

[1]Our inner products follow the "physicist convention": they are linear in the second variable, and antilinear in the first variable.

where $a_i \in \mathbf{R}$, $1 \leq i \leq r$, are distinct eigenvalues of A and P_i, $1 \leq i \leq r$, are orthogonal projectors onto the corresponding eigenspaces. When the system is in a state corresponding to the density matrix ρ, a measurement of the observable A will yield the result $a_i \in \mathrm{sp}(A)$ with probability

$$\mathrm{Prob}(a_i) = \mathrm{tr}(P_i \rho).$$

For such a measurement, the expected value is given by

$$\sum_{i=1}^{r} a_i \,\mathrm{Prob}(a_i) = \sum_{i=1}^{r} a_i \,\mathrm{tr}(P_i \rho) = \mathrm{tr}(A\rho),$$

which is commonly denoted by $\rho(A)$, identifying the density matrix with a linear functional on the space of n-by-n matrices.

A profound particularity of quantum mechanics is that, after a measurement the observable A yielding result a_i, the state of the system "collapses" to the state represented by the density matrix

$$\frac{P_i \rho P_i}{\mathrm{tr}(P_i \rho)}.$$

Only then can one perform a measurement of another observable, and the non-commutativity of observables (or incompatibility of their spectral decompositions) comes into play, preventing us from having a natural notion of a joint distribution. In some sense, the quantum mechanics of confined quantum systems is a non-commutative analogue of probability theory on finite sample spaces.

The physics of a time-independent quantum system is encoded in a distinguished n-by-n matrix H_0 called the **Hamiltonian**, which corresponds to the energy[2] observable of the quantum mechanical system and implements its dynamics. If the system is in a state ρ at time 0, then at time t it is in the state

$$\rho_t = \mathrm{e}^{-\mathrm{i}tH_0} \rho \mathrm{e}^{\mathrm{i}tH_0}.$$

A particular kind of state, for which the density matrix is of the form $\psi \langle \psi, \cdot \rangle$ for some unit vector $\psi \in \mathbf{C}^n$, are often of special interest and are called **pure states**. The vector ψ is sometimes referred to as the **wave function**. If the system is in a pure state corresponding to the unit vector ψ at time 0, it remains a pure state under this evolution, and the corresponding unit vector at time t

[2] We work in units where $\hbar \equiv 1$.

is $\psi_t = \mathrm{e}^{-\mathrm{i}tH_0}\psi$, i.e. the vector solving the renowned Schrödinger equation

$$\begin{cases} H_0\psi_t = \mathrm{i}\partial_t\psi_t \\ \psi_0 = \psi \end{cases}.$$

Equivalently, one can instead consider the dual evolution on the observables, i.e. the unique group $(\tau^t)_{t\in\mathbf{R}}$ of automorphisms such that $\mathrm{tr}(\rho\tau^t(A)) = \mathrm{tr}(\rho_t A)$ for all A. It is explicitly given by the conjugation

$$\tau^t(A) = \mathrm{e}^{\mathrm{i}tH_0}A\mathrm{e}^{-\mathrm{i}tH_0}.$$

Because we wish to discuss the physics of extended systems, there are several notions of linear algebra that we need to transfer to a more general setup. Extracting the defining essential properties of states, observables, and dynamics will lead us to the abstract framework of C^*-dynamical systems.

1.2 The algebraic framework of quantum statistical mechanics

1.2.1 Basic definitions

Now we wish to motivate the algebraic structure that is used in the following chapters. However, we do not aim at a thorough exposition of the subject: all facts are provided without proof. The classical reference [BR87, BR97] provides a very complete treatment of what is glimpsed at in this section, and much more, but is far from being an easy read. More accessible discussions can be found in the excellent master's thesis of Jane Panangaden [Pan16] and in the lecture notes [Pil06] and [JOPP11]. We assume that the reader is familiar with the basics of functional analysis and linear operators on Hilbert spaces (operator topologies, self-adjointess, spectral measures, functional calculus *etc.*), which can be found for example in the classical reference [RS72], also see [Joy06].

For confined quantum systems, we recall that observables correspond to self-adjoint elements of the space of n-by-n matrices over $\mathbf{C}$, which we denote by $\mathrm{Mat}_{n\times n}(\mathbf{C})$. This space is naturally endowed with the structure of a unital algebra (with the usual matrix addition and multiplication), a norm ($\|\cdot\|$, the operator norm), and an involution ($*$, the hermitian conjugation).

The first important property is that the normed vector space $(\mathrm{Mat}_{n\times n}(\mathbf{C}), \|\cdot\|)$ is a Banach space, i.e. it is such that

(B1) All Cauchy sequences converge.

Moreover, the norm and matrix multiplication satisfy the compatibility condition that for all $A, B \in \mathrm{Mat}_{n\times n}(\mathbf{C})$,

(B2) $\|AB\| \leq \|A\|\|B\|$.

The matrix addition and multiplication and the hermitian conjugation also satisfy the important properties

($*$1) $(AB)^* = B^*A^*$

($*$2) $(A + \lambda B)^* = A^* + \overline{\lambda}B^*$

for all $A, B \in \mathrm{Mat}_{n\times n}(\mathbf{C})$ and $\lambda \in \mathbf{C}$. Finally, it is an elementary fact of linear algebra that

(B$*$) $\|A^*A\| = \|A\|^2$

for all $A \in \mathrm{Mat}_{n\times n}(\mathbf{C})$. These abstract properties, at the center of the following definition, encapsulate the essential structure that is needed to discuss observables on non-relativistic quantum systems.

Definition 1.1. An algebra $\mathfrak{A}$ together with a norm $\|\cdot\|$ is said to form a **Banach algebra** if it satisfies (B1) and (B2) for all $A, B \in \mathfrak{A}$. An algebra $\mathfrak{A}$ together with an involution $*$ is said to form a **involutive algebra** if it satisfies ($*$1) and ($*$2) for all $A, B \in \mathfrak{A}$ and $\lambda \in \mathbf{C}$. An algebra $\mathfrak{A}$ that has both a Banach algebra structure and involutive algebra structure is said to be a **B^*-algebra** if it moreover satisfies (B$*$) for all $A \in \mathfrak{A}$.

Remark 1.2. These conditions imply that the involution $*$ is an isometry, which in turn implies

(C$*$) $\|A^*A\| = \|A\|\|A^*\|$

for all $A \in \mathfrak{A}$, known as the C^*-identity. In fact, it turns out that with (B1) and (B2), and ($*$1) and ($*$2), the conditions (B$*$) and (C$*$) are equivalent.

In what follows, we always assume a B^*-algebra $\mathfrak{A}$ has an identity, which we denote by $\mathbf{1}$. The fact that $\mathrm{Mat}_{n\times n}(\mathbf{C})$ is a B^*-algebra has a straightforward generalization: any norm-closed self-adjoint algebra of bounded operators on a Hilbert space $\mathfrak{H}$, denoted $\mathfrak{A} \subseteq \mathfrak{L}(\mathfrak{H})$, is a B^*-algebra. The perhaps more surprising result is that this statement has a converse, which we state without proof.

Theorem 1.3. *Any B^*-algebra is isomorphic to a norm-closed self-adjoint algebra of bounded operators on a Hilbert space.*

As a consequence of this structure theorem, the term ***C**-algebra**, originally used to describe norm-closed self-adjoint algebras of bounded operators on a Hilbert space, is nowadays used for both. Sometimes, the term *concrete* C^*-algebra is used to emphasize the concrete underlying Hilbert space structure.

From now on, we interpret **observables** as self-adjoint elements of a C^*-algebra. In this language, the map $\mathrm{Mat}_{n\times n}(\mathbf{C}) \ni A \mapsto \mathrm{tr}(\rho A)$, where ρ is a density matrix, is merely a positive bounded linear functional on the C^*-algebra $\mathrm{Mat}_{n\times n}(\mathbf{C})$ that moreover satisfies the normalisation condition $\mathrm{tr}(\rho\mathbf{1}) = 1$. This motivates the following definition.

Definition 1.4. A bounded linear functional ω is said to be **non-negative** if $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$. A **state** over the C^*-algebra $\mathfrak{A}$ is a non-negative bounded linear functional ω on $\mathfrak{A}$ satisfying the normalisation condition

$$\omega(\mathbf{1}) = 1.$$

Remark 1.5. Linearity and the non-negativity condition $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$ actually guarantee boundedness with $\|\omega\| = \omega(\mathbf{1})$.

Example 1.6. Consider the commutative C^*-algebra $C(X)$, the algebra of complex valued continuous functions on a compact Hausdorff space X, equipped with the supremum norm. By the Riesz–Markov representation theorem, states correspond to Borel probability measures on X.

Along the same vein as Theorem 1.3, there exists a canonical way to make a C^*-algebra $\mathfrak{A}$ concrete that is particularly useful to compute expectations with respect to a fixed state ω. To state this theorem we need a few definitions.

Definition 1.7. A **representation** of a C^*-algebra $\mathfrak{A}$ is a pair $(\mathfrak{H}, \pi)$, where $\mathfrak{H}$ is a Hilbert space and $\pi : \mathfrak{A} \to \mathfrak{L}(\mathfrak{H})$ is a $*$-morphism. A representation is said to be **faithful** if it has trivial kernel. If a unit vector $\Omega \in \mathfrak{H}$ is cyclic for $\pi(\mathfrak{A})$, then the triple $(\mathfrak{H}, \pi, \Omega)$ is called a **cyclic representation** of $\mathfrak{A}$.

The existence of a particularly convenient cyclic representation, that will be used crucially in our abstract definition of the heat full statistics measure, is guaranteed by the following celebrated theorem attributed to Gelfand, Naĭmark and Segal.

Theorem 1.8. *Let ω be a state over the C^*-algebra $\mathfrak{A}$. Then, there exists a cyclic representation $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ satisfying*

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle_{\mathfrak{H}_\omega} \text{ for all } A \in \mathfrak{A}.$$

Moreover, this representation is unique up to unitary equivalence.

This representation is known as a **GNS representation** of $\mathfrak{A}$ with respect to the state ω. Because of uniqueness up to unitary equivalence, we will sometimes write *the* GNS representation.

Example 1.9. Consider the C^*-algebra $\mathrm{Mat}_{n\times n}(\mathbf{C})$ together with the state $\omega_\rho : A \mapsto \mathrm{tr}(\rho A)$ for some strictly positive density matrix ρ. Considering $\mathrm{Mat}_{n\times n}(\mathbf{C})$ as a Hilbert space $\mathfrak{H}$ with Hilbert–Schmidt inner product $\langle A, B\rangle_{\mathfrak{H}} = \mathrm{tr}(A^* B)$, the map $l : \mathrm{Mat}_{n\times n}(\mathbf{C}) \to \mathfrak{L}(\mathfrak{H})$ sending A to left multiplication by A, denoted $l(A) \in \mathfrak{L}(\mathfrak{H})$, is a cyclic representation for the vector $\Omega_\rho = \rho^{1/2} \in \mathfrak{H}$ and

$$\begin{aligned} \omega_\rho(A) &= \mathrm{tr}(A\rho) \\ &= \mathrm{tr}((\rho^{1/2})^* A \rho^{1/2}) \\ &= \langle \Omega_\rho, l(A)\Omega_\rho \rangle_{\mathfrak{H}} . \end{aligned}$$

The triple $(\mathfrak{H}, l, \Omega_\rho)$ is thus a GNS representation for $\mathrm{Mat}_{n\times n}(\mathbf{C})$ with respect to the state ω_ρ. We refer the reader to [JOPP11, §4.3.11][3] for more details.

Having treated observables and states, we now turn to providing a C^*-algebraic definition of the notion of time evolution. We recall that for confined quantum systems, a matrix A at time $t = 0$ is time-evolved to a matrix $\tau^t(A) = \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH}$ at time t. This evolution satisfies the key properties:

(Gr) $\tau^{t_1+t_2} = \tau^{t_1} \circ \tau^{t_2}$,

(Au) $\tau^t(AB + \lambda C^*) = \tau^t(A)\tau^t(B) + \lambda\tau^t(C)^*$,

(SC) $\lim_{t\to 0} \|\tau^t(A) - A\| = 0$,

[3]Section 4.3.11. *The standard representations of $\mathcal{O}$* of the published version corresponds to Section 2.11 of the version available on the *arXiv*.

for all $t, t_1, t_2 \in \mathbf{R}$, $\lambda \in \mathbf{C}$ and $A, B, C \in \mathrm{Mat}_{n\times n}(\mathbf{C})$.

Definition 1.10. A family $(\tau^t)_{t\in\mathbf{R}}$ of maps on a C^*-algebra is called a **one-parameter group of $*$-automorphisms** if it satisfies (Gr) and (Au). It is said to be **strongly continous** if it moreover satisfies (SC).

For fixed $t \in \mathbf{R}$, τ^t is a $*$-automorphism (condition (Au)). In particular, $\tau^t(A - \lambda\mathbf{1}) = \tau^t(A) - \lambda\mathbf{1}$ and τ^t preserves the spectrum of A. Because the norm of self-adjoint element of $\mathfrak{A}$ equals its spectral radius, the condition (B$*$) ensures that τ^t is then an isometry of $\mathfrak{A}$.

With this definition at hand, we are ready to encapsulate the physics of many quantum systems in a single abstract structure, that of a C^*-dynamical system.

Definition 1.11. We call a **C^*-dynamical system** a triple $(\mathfrak{A}, \tau, \omega)$, where $\mathfrak{A}$ is a C^*-algebra, $(\tau^t)_{t\in\mathbf{R}}$ is a strongly continuous one-parameter group of $*$-automorphisms of $\mathfrak{A}$, and ω is a state on $\mathfrak{A}$.

However, for some quantum systems (e.g. Bose gases), this is not the suitable structure; we need to consider a different type of dynamics, on a specific class of C^*-algebras.

Definition 1.12. A concrete C^*-algebra that is closed with respect to the weak operator topology is called a **W^*-algebra** or **von Neumann algebra**.

We call a **W^*-dynamical system** a triple $(\mathfrak{M}, \tau, \omega)$, where $\mathfrak{M}$ is a W^*-algebra, $(\tau^t)_{t\in\mathbf{R}}$ is a σ-weakly continuous one-parameter group of $*$-automorphisms of $\mathfrak{M}$, and ω is a state on $\mathfrak{M}$.

Remark 1.13. It turns out that the closure of a concrete C^*-algebra under several important operator topologies (e.g. weak, strong, strong*, σ-weak, σ-strong, and σ-strong*) coincide. There is also an equivalent definition of W^*-algebras as C^*-algebras who are their own bicommutant (commutant of the commutant). The proof of this equivalence, known as the bicommutant theorem, is attributed to von Neumann.

In both the W^* and C^* case, a state ω is said to be **invariant** for $(\tau^t)_{t\in\mathbf{R}}$ if $\omega = \omega \circ \tau^t$ for all $t \in \mathbf{R}$. To define the heat full statistics measure, we will need a last mathematical object in the GNS representation: the ω-Liouvillean, which suitably implements the dynamics in the GNS Hilbert space. Precisely, we have the following result.

Theorem 1.14. *Suppose ω is invariant under the C^*-dynamics $(\tau^t)_{t\in\mathbf{R}}$ on $\mathfrak{A}$* [respectively W^*-dynamics $(\tau^t)_{t\in\mathbf{R}}$ on $\mathfrak{M}$] *and let $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$ be a GNS representation of $\mathfrak{A}$* [resp. $\mathfrak{M}$] *with respect to ω. Then, there exists a unique self-adjoint operator L_ω on $\mathfrak{H}_\omega$ such that*

$$\pi_\omega(\tau^t(A)) = \mathrm{e}^{\mathrm{i}tL_\omega}\pi_\omega(A)\mathrm{e}^{-\mathrm{i}tL_\omega}, \tag{1.1}$$

$$L_\omega\Omega_\omega = 0 \tag{1.2}$$

for all $t \in \mathbf{R}$ and $A \in \mathfrak{A}$ [resp. $\mathfrak{M}$] .

Definition 1.15. We call L_ω above the **ω-Liouvillean** of the dynamical system (for the GNS representation $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$).

1.2.2 Infinitesimal generators and perturbation theory

Consider a C^*-dynamical system $(\mathfrak{A}, \tau, \omega)$ with ω an invariant state. The heuristics of the introduction to this thesis suggest that we might want to "perturb" this dynamics in one way or another. To properly introduce the notion of perturbation, we will need an object known as the (infinitesimal) generator, which characterizes one-parameter groups.

Definition 1.16. Let $(\tau^t)_{t\in\mathbf{R}}$ be a strongly continuous one-parameter group of $*$-automorphisms of $\mathfrak{A}$. Then, the subspace $\operatorname{Dom}\delta \subseteq \mathfrak{A}$ on which the limit

$$\lim_{t\to 0} \frac{\tau^t(A) - A}{t}$$

exists in $\mathfrak{A}$ in the norm sense is a dense subspace $\mathfrak{A}$. We call the **generator** of $(\tau^t)_{t\in\mathbf{R}}$ the closed linear operator $\delta : \operatorname{Dom}\delta \to \mathfrak{A}$ defined by this limit.

The group can then be expressed as $\tau^t = \mathrm{e}^{t\delta}$ in accordance with the functional calculus. For a proof that $\operatorname{Dom}\delta$ is indeed dense, we refer the reader to [BR87, §3.1.2] or [RS75, §X.8]. Moreover, generators enjoy the following important properties. We also mention in passing the celebrated Hille–Yosida theorem. This theorem deals with a considerably more general situation and will not be directly applied in this work, but is important enough anyway. We refer the reader to [RS75, §X.8] for a discussion and generalization (the Hille–Yosida–Phillips theorem) of this theorem.

Theorem 1.17 (Hille–Yosida theorem). *A closed linear operator A on a Banach space X generates a strongly conitnuous contraction semi-group if and only if the following conditions are satisfied*

1. *The half-line $(0, \infty)$ is contained in the resolvent of A;*
2. *For all $\lambda \in (0, \infty)$, $\|(\lambda \mathbf{1} + A)^{-1}\| \leq \lambda^{-1}$.*

Moreover, if these conditions are satisfied, then the whole half-plane $\{z : \operatorname{Re} z > 0\}$ is actually contained in the resolvent of A.

Proposition 1.18. *The generator δ of a strongly continuous one-parameter group $(\tau^t)_{t\in\mathbf{R}}$ of $*$-automorphisms always satisfies*

1. *the domain $\operatorname{Dom}\delta$ is a $*$-sub-algebra of $\mathfrak{A}$;*
2. *the product rule $\delta(AB) = \delta(A)B + A\delta(B)$ holds for all $A, B \in \operatorname{Dom}\delta$;*
3. *the involution is preserved: $\delta(A^*) = \delta(A)^*$ for all $A, B \in \operatorname{Dom}\delta$.*

In other words, δ is a ***derivation****.*

Lemma 1.19. *Let δ be a derivation. Repeated application of the product rule reads*

$$\delta^n(AB) = \sum_{k=0}^{n} \binom{n}{k} \delta^{n-k}(A)\delta^k(B)$$

for all $A, B \in \operatorname{Dom}\delta^n$.

It is through this generator that we choose to define what we mean by a perturbation of a C^*-dynamical system $(\mathfrak{A}, \tau, \omega)$.

Definition 1.20. Let $(\mathfrak{A}, \tau, \omega)$ be a C^*-dynamical system and let V be a self-adjoint element of $\mathfrak{A}$. The corresponding **perturbed dynamics** $(\tau_V^t)_{t\in\mathbf{R}}$ is defined as the unique strongly continuous one-parameter group of $*$-automorphisms of $\mathfrak{A}$ with generator $\delta + \mathrm{i}[V, \cdot]$.

Example 1.21. In the case of quantum confined systems discussed above, the perturbed dynamics associated to a self-adjoint matrix V is precisely the dynamics corresponding to the perturbed Hamiltonian $H_0 + V$.

Proposition 1.22. *The perturbed dynamics $(\tau_V^t)_{t\in\mathbf{R}}$ has the equivalent definition in terms of the norm-convergent* ***Dyson series***

$$\tau_V^t(A) = \sum_{n=0}^{N} \mathrm{i}^n \int_0^t \cdots \int_0^{t_{n-1}} [\tau^{t_n}(V), [\ldots, [\tau^{t_1}(V), \tau^t(A)] \ldots]] \, \mathrm{d}t_n \cdots \mathrm{d}t_1. \tag{1.3}$$

for all $t \in \mathbf{R}$ and all $A \in \mathfrak{A}$. Moreover, in the GNS representation, the self-adjoint operator $L_\omega + \pi_\omega(V)$ implements the dynamics $(\tau_V^t)_{t\in\mathbf{R}}$ in the sense that

$$\pi(\tau_V^t(A)) = \mathrm{e}^{\mathrm{i}t(L_\omega+\pi_\omega(V))}\pi_\omega(A)\mathrm{e}^{-\mathrm{i}t(L_\omega+\pi_\omega(V))}$$

for all $t \in \mathbf{R}$ and $A \in \mathfrak{A}$.

Remark 1.23. It is typically *not* true that the operator $L_\omega + \pi_\omega(V)$ is the ω-Liouvillean for the perturbed dynamics $(\tau_V^t)_{t\in\mathbf{R}}$: there is no reason for $\pi_\omega(V)$ to vanish on Ω_ω.

The definitions above related to generators adapt to W^*-dynamical systems $(\mathfrak{M}, \tau, \omega)$ by requiring the defining limit to exist in the suitable sense, and the perturbed dynamics is defined analogously for self-adjoint elements $V \in \mathfrak{M}$. However, in some physical situations such as Bose gases, perturbations of interest might be unbounded, requiring more technical machinery. We will be very brief on this subject and refer the reader to [DJP03] for more details.

Proposition 1.24 (Theorem 3.3 in [DJP03])**.** *Let $(\mathfrak{M}, \tau, \omega)$ be a W^*-dynamical system, where $\mathfrak{M} \subseteq \mathfrak{L}(\mathfrak{H})$ and τ is implemented by the self-adjoint operator L with domain* $\operatorname{Dom} L \subseteq \mathfrak{H}$*. Let V be a self-adjoint operator with domain* $\operatorname{Dom} V \subseteq \mathfrak{H}$ *that satisfies.*

1. [4] $V(\mathbf{1} + V^*V)^{-1/2} \in \mathfrak{M}$;

2. *$L + V$ is essentially self-adjoint on* $\operatorname{Dom} L \cap \operatorname{Dom} V$.

Then,

$$\tau_V^t(A) := \mathrm{e}^{\mathrm{i}t(L+V)} A \mathrm{e}^{-\mathrm{i}t(L+V)}$$

defines a W^*-dynamics on $\mathfrak{M}$.

[4] For V is closed, the condition $V(\mathbf{1} + V^*V)^{-1/2} \in \mathfrak{M}$ is a condition for (concrete) **affilitation** to $\mathfrak{M}$. Since V is self-adjoint, it is equivalent to requiring that all Borel bounded functions of V (via the functional calculus) are in $\mathfrak{M}$. See [DJP03, §2.3].

Chapter 2

The heat full statistics: general considerations

In this chapter, we discuss the central object of this thesis: the heat full statistics measure $\mathbf{P}_t$. It is introduced as arising from a two-time measurement of energy and is given an abstract Definition 2.1. We then discuss the probability it gives to large heat fluctuations and the link of this decay to suitable notions of regularity of the perturbation. The main results of this chapter are Theorems 2.4 and 2.10.

2.1 Defining the heat FS measure

2.1.1 Two-time measurement of energy in confined quantum systems

Consider a confined quantum system with unperturbed Hamiltonian $H_0 \in \mathrm{Mat}_{n\times n}(\mathbf{C})$ and initial state ω_ρ described by a strictly positive density matrix $\rho \in \mathrm{Mat}_{n\times n}(\mathbf{C})$ that commutes with H_0. Denote the spectral decomposition of H_0 by $\sum_i \epsilon_i P_i$. Also consider a self-adjoint perturbation $V \in \mathrm{Mat}_{n\times n}(\mathbf{C})$.

A measurement of H_0 (i.e. of the energy) at time $t = 0$ yields a result ϵ_i with probability

$$\mathrm{tr}(P_i \rho)$$

and the state of the system collapses to

$$\frac{P_i \rho P_i}{\operatorname{tr}(P_i \rho)}.$$

Turning on the pertubation V and letting time run until $t > 0$, the state of the system evolves to

$$\frac{\mathrm{e}^{-\mathrm{i}t(H_0+V)} P_i \rho P_i \mathrm{e}^{\mathrm{i}t(H_0+V)}}{\operatorname{tr}(P_i \rho)}.$$

Given the first result ϵ_i, a second measurement of the energy observable H_0 (*not* of $H_0 + V$) yields a result ϵ_j with probability

$$\frac{\operatorname{tr}(P_j \mathrm{e}^{-\mathrm{i}t(H_0+V)} P_i \rho P_i \mathrm{e}^{\mathrm{i}t(H_0+V)})}{\operatorname{tr}(P_i \rho)}.$$

Hence, the probability of measuring a difference ΔQ in this two-time measurement protocol — interpreted as the heat variation — is

$$\begin{aligned}
\mathbf{P}_t(\Delta Q) &= \sum_{\substack{i,j \\ \epsilon_j - \epsilon_i = \Delta Q}} \frac{\operatorname{tr}(P_j \mathrm{e}^{-\mathrm{i}t(H_0+V)} P_i \rho P_i \mathrm{e}^{\mathrm{i}t(H_0+V)})}{\operatorname{tr}(P_i \rho)} \operatorname{tr}(P_i \rho) \\
&= \sum_{\substack{i,j \\ \epsilon_j - \epsilon_i = \Delta Q}} \operatorname{tr}(P_j \mathrm{e}^{-\mathrm{i}t(H_0+V)} P_i \rho P_i \mathrm{e}^{\mathrm{i}t(H_0+V)}).
\end{aligned}$$

Note that this measure $\mathbf{P}_t$ is supported on finitely many values (the elements of $\operatorname{sp} H_0 - \operatorname{sp} H_0$).

We now wish to describe this measure $\mathbf{P}_t$ in terms of objects that survive the thermodynamic limit, objects that make sense in the realm of the algebraic structure of quantum statistical mechanics presented in Chapter 1. To do this, we look at the characteristic function $\mathcal{E}_t$ of $\mathbf{P}_t$. Manipulating the definition of the characteristic function, we have

$$\begin{aligned}
\mathcal{E}_t(\alpha) &= \sum_{\Delta Q \in \operatorname{sp} H_0 - \operatorname{sp} H_0} \mathrm{e}^{\mathrm{i}\alpha \Delta Q} \sum_{i,j : \epsilon_j - \epsilon_i = \Delta Q} \operatorname{tr}(P_j \mathrm{e}^{-\mathrm{i}t(H_0+V)} P_i \rho \mathrm{e}^{\mathrm{i}t(H_0+V)}) \\
&= \sum_{i,j} \operatorname{tr}(\mathrm{e}^{\mathrm{i}\alpha\epsilon_j} P_j \mathrm{e}^{-\mathrm{i}t(H_0+V)} \mathrm{e}^{-\mathrm{i}\alpha\epsilon_i} P_i \rho \mathrm{e}^{\mathrm{i}t(H_0+V)}) \\
&= \operatorname{tr}(\mathrm{e}^{\mathrm{i}\alpha H_0} \mathrm{e}^{-\mathrm{i}t(H_0+V)} \mathrm{e}^{-\mathrm{i}\alpha H_0} \rho \mathrm{e}^{\mathrm{i}t(H_0+V)}).
\end{aligned}$$

Recall that for such confined systems, the standard GNS representation is given by left multiplication in $\operatorname{Mat}_{n\times n}(\mathbf{C})$ equipped with the Hilbert–Schmidt inner product, where the vector representative of ω_ρ is the matrix $\Omega_\rho = \rho^{1/2}$. We also saw the the corresponding Liouvillean is $L = [H_0, \cdot]$.

In these terms we rewrite

$$\begin{aligned}
\mathcal{E}_t(\alpha) &= \operatorname{tr}(\mathrm{e}^{\mathrm{i}t(H_0+V)}\mathrm{e}^{\mathrm{i}\alpha(H_0+V-V)}\mathrm{e}^{-\mathrm{i}t(H_0+V)}\mathrm{e}^{-\mathrm{i}\alpha H_0}\rho) \\
&= \operatorname{tr}(\mathrm{e}^{\mathrm{i}\alpha(H_0+V-\mathrm{e}^{\mathrm{i}t(H_0+V)}V\mathrm{e}^{-\mathrm{i}t(H_0+V)})}\mathrm{e}^{-\mathrm{i}\alpha H_0}\rho) \\
&= \operatorname{tr}(\mathrm{e}^{\mathrm{i}\alpha(H_0+V-\tau_V^t(V))}\mathrm{e}^{-\mathrm{i}\alpha H_0}\rho) \\
&= \operatorname{tr}(\rho^{1/2}\mathrm{e}^{\mathrm{i}\alpha([H_0,\cdot]+V-\tau_V^t(V))}\rho^{1/2}) \\
&= \langle \Omega_\rho, \mathrm{e}^{\mathrm{i}\alpha(L+\pi(V)-\pi(\tau_V^t(V)))}\Omega_\rho \rangle \,.
\end{aligned}$$

2.1.2 Abstract definition in C^*-dynamical systems

Based on the discussion of the last section, we make the following definition of the heat full statistics measure in the context of C^*-dynamical systems. Examples of W^*-dynamical systems are presented in the following chapters.

Definition 2.1. We define the **heat full statistics measure $\mathbf{P}_t$** associated to $(\mathfrak{A}, \tau, \omega)$, where ω is faithful and invariant for τ, and the self-adjoint perturbation $V \in \mathfrak{A}$ to be the unique probability measure with characteristic function

$$\mathcal{E}_t(\alpha) = \langle \Omega_\omega, \mathrm{e}^{\mathrm{i}\alpha(L_\omega+\pi_\omega(V)-\pi_\omega(\tau_V^t(V)))}\Omega_\omega \rangle \,, \tag{2.1}$$

or equivalently the spectral measure for the operator

$$L_\omega + \pi_\omega(V) - \pi_\omega(\tau_V^t(V)), \tag{2.2}$$

with respect to the vector representative Ω_ω of ω in the GNS representation $(\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega)$. We denote expectation with respect to $\mathbf{P}_t$ by $\mathbf{E}_t$.

Remark 2.2. Definition 2.1 extends naturally to the case of W^*-dynamical system $(\mathfrak{M}, \tau, \omega)$, where $\mathfrak{M}$ is a concrete W^*-algebra, $(\tau^s)_{s\in\mathbf{R}}$ is a σ-weakly continuous one-parameter group of $*$-automorphisms of $\mathfrak{M}$ and ω is a τ-invariant state on $\mathfrak{M}$, together with a bounded self-adjoint perturbation $V \in \mathfrak{M}$. Theorems 2.4 and 2.10 then also generalise to this case.

When considering V merely affiliated to the W^*-algebra $\mathfrak{M}$, one may in some cases mimic the construction Definition 2.1 and obtain results analogous to the ones for C^*-algebras. For example, we have seen in Proposition 1.24 that, under suitable assumptions for V, the construction of the perturbed group $(\tau_V^s)_{s\in\mathbf{R}}$ is indeed a W^*-dynamics (also see [DJP03]). We do not discuss the general theory of heat FS measures for such unbounded perturbations, but treat some examples below.

Remark 2.3. In the framework of Tomita–Takesaki theory, with ω a (τ, β)-KMS state, the Liouvillean is $L_\omega = \beta^{-1} \log \Delta_\omega$, where Δ_ω denotes the modular operator for the state ω. In this context, one may equivalently define $\mathbf{P}_t$ as spectral measure for $\beta^{-1} \log \Delta_{\omega\circ\tau_V^{-t}|\omega}$, where $\Delta_{\omega\circ\tau_V^{-t}|\omega}$ denotes the relative modular operator between $\omega \circ \tau_V^{-t}$ and ω, with respect to the vector representative Ω of ω, see [JPPP15] and [Pan16].

2.2 Tails of the heat FS measure and regularity of perturbations

We are interested in the extent to which $\mathbf{P}_t$ concentrates around small values of $\frac{1}{t}|\Delta Q|$ as t gets large. The key property to consider in the study of these tails is the regularity of the map

$$\begin{aligned} \tau^{\cdot}(V) : \mathbf{R} &\to \mathfrak{A} \\ s &\mapsto \tau^s(V). \end{aligned}$$

Differentiability of this map is understood in the norm sense. As a Banach space valued function (recall that a C^*-algebra $\mathfrak{A}$ is by definition a Banach space), we say that it extends analytically to an open neighbourhood $U \subseteq \mathbf{C}$ of the real line if there exists a function $U \ni z \mapsto G(z) \in \mathfrak{A}$ with $G(s) = \tau^s(V)$ for all $s \in \mathbf{R}$ and such that the limit

$$\lim_{w\to z} \frac{G(w) - G(z)}{w - z}$$

exists in the norm sense for all $z \in U$. The existence of this limit is equivalent to requiring that the function $\eta \circ G : U \to \mathbf{C}$ is analytic for all bounded linear functionals η, see [RS72, §VI.3].

To see how such regularity conditions could come into play, note that it is an immediate consequence of the definitions that the $(2n+2)$th moment of $\mathbf{P}_t$ is given by

$$\begin{aligned}\mathbf{E}_t[|\Delta Q|^{2n+2}] &= \langle \Omega_\omega, (L_\omega + \pi(V) - \pi_\omega(\tau_V^t(V)))^{2n+2}\Omega_\omega\rangle \\ &= \|(L_\omega + \pi_\omega(V) - \pi_\omega(\tau_V^t(V)))^n(L_\omega + \pi_\omega(V) - \pi_\omega(\tau_V^t(V)))\Omega_\omega\|^2,\end{aligned}$$

where the the norm is possibly formally infinite (as an improper integral). Since L_ω vanishes on Ω_ω by (1.2) of the definition of the Liouvillean,

$$\mathbf{E}_t[|\Delta Q|^{2n+2}] = \|(L_\omega + \pi_\omega(V) - \pi_\omega(\tau_V^t(V)))^n(\pi_\omega(V) - \pi_\omega(\tau_V^t(V)))\Omega_\omega\|^2. \tag{2.3}$$

This hints at the fact that the analysis of products involving L_ω^n and $\pi_\omega(V) - \pi_\omega(\tau_V^t(V))$ will be determinant in our analysis of the $(2n+2)$th moment of $\mathbf{P}_t$. By Chebyshëv's inequality, a bound on such a moment of the measure yield polynomial decay of the tails:

$$\mathbf{P}_t(|\Delta Q| \geq M) \leq \frac{\mathbf{E}_t[|\Delta Q|^{2n+2}]}{M^{2n+2}}. \tag{2.4}$$

On the other hand, the nth derivative of the map $s \mapsto \tau^s(V)$ is also expressed, in the GNS representation, in terms of products involving L_ω^n and $\pi_\omega(V)$ (see the formula (1.1)).

This idea that the tails of the heat FS are controlled by the regularity rather than the strength of the perturbation is the *key point* to be taken from this thesis. The work [BJP+15] already provides an important result in this direction for quantum dynamical systems arising as the limit of a sequence of confined systems.

The appearance of a regularity condition for the potential in the control of energy fluctuations should not be completely surprising the reader. Indeed, the fairly general Fermi golden rule yields that, to first order in the size of the perturbation, that the probability of transition between states that differ greatly in energy are linked to the tails of an energy profile associated to the perturbation, which is itself obtained from the momentum representation of the perturbation by a change of variable (dispersion relation). By the general Fourier theory, decay of the momentum representation of the potential at infinity corresponds to regularity of its Fourier transform, the position representation of the potential.

A manifestation of this type of link between a notion of regularity of the perturbation and control of energy fluctuations can be seen by considering a free particle on $[0, 2\pi]$ with periodic boundary condition, originally in the ground state for the Laplacian ∇^2, $\psi_0 = (2\pi)^{-1/2}$. Then, an initial measurement of the energy (associated to the Laplacian) surely yields $\epsilon_i = 0$. With a perturbation mult(u) that is multiplication by a small continuous bounded function u of the position x, Fermi's golden rule gives a first order approximation to the probability of measuring an energy $\epsilon_j = N^2$ (and hence $\Delta Q = N^2$) after time $t > 0$:

$$\mathbf{P}_t(N^2) \approx 4|\langle\psi_0, V\psi_N\rangle|^2 \frac{\sin^2 \frac{N^2 t}{2}}{N^2} + 4|\langle\psi_0, V\psi_{-N}\rangle|^2 \frac{\sin^2 \frac{N^2 t}{2}}{N^2}.$$

Hence,

$$\begin{aligned}
\mathbf{E}_t[|\Delta Q|^p] &= \sum_{N \in \mathbf{N}} N^{2p} \mathbf{P}_t(N^2) \\
&\sim \sum_{N \in \mathbf{N}} N^{2p} 4|\langle\psi_0, V\psi_N\rangle|^2 \frac{\sin^2 \frac{N^2 t}{2}}{N^2} + N^{2p} 4|\langle\psi_0, V\psi_{-N}\rangle|^2 \frac{\sin^2 \frac{N^2 t}{2}}{N^2} \\
&= 4 \sum_{N \in \mathbf{N}} N^{2p-2} \Big(\Big| \frac{1}{2\pi} \int_0^{2\pi} u(x) \mathrm{e}^{\mathrm{i}Nx} \,\mathrm{d}x \Big|^2 + \Big| \frac{1}{2\pi} \int_0^{2\pi} u(x) \mathrm{e}^{-\mathrm{i}Nx} \,\mathrm{d}x \Big|^2 \Big) \sin^2 \frac{N^2 t}{2} \\
&= 4 \sum_{n \in \mathbf{Z}} n^{2p-2} |\hat{u}_n|^2 \sin^2 \frac{n^2 t}{2}.
\end{aligned}$$

The use of "$\sim$" here is loose, given that there really is a series in N and a perturbative series in the size of u that we are manipulating recklessly. If the last series converges and is integrable in t on an interval, then the series

$$\sum_{n \in \mathbf{Z}} (1 + |n|^{p-1})^2 |\hat{u}_n|^2$$

converges and a classical Fourier series result gives that u is in the Sobolev space $W^{p-1,2}([0, 2\pi])$. This suggests that boundedness of the pth moment is related to u having $p - 1$ weak derivatives in L^2.

Here, this notion of regularity is not naturally expressed in terms of the differentiability of a map $s \mapsto \tau^s(\mathrm{mult}(u))$ (note for example that the operator $\psi \mapsto [\nabla^2, \mathrm{mult}(u)]\psi$ is a first order differential operator that is only bounded in trivial cases); this system and the interaction considered are not of the type we are interested in. We emphasize that we are interested in sytems of *many* particles. Nevertheless, it illustrates the fact that notions regularity, which should be suitably formulated for the context, naturally play a role in the control of energy fluctuations.

2.3 General locally perturbed C^*-dynamical systems: two results

We now describe how differentiability of the map $s \mapsto \tau^s(V)$ introduced earlier influences the tails of the heat full statistics measure $\mathbf{P}_t$. One easily verifies that the first and second moment of $\mathbf{P}_t$ are always finite. For higher order even moments, we have the following control from regularity.

Theorem 2.4. *Let $(\mathfrak{A}, \tau, \omega)$ be a C^*-dynamical system, with ω a τ-invariant state, and $\mathbf{P}_t$ be the heat FS measure associated to a self-adjoint perturbation $V \in \mathfrak{A}$. Suppose that the map $\mathbf{R} \ni s \mapsto \tau^s(V) \in \mathfrak{A}$ is n times norm-differentiable. Then,*

$$\sup_{t \in \mathbf{R}} \mathbf{E}_t[|\Delta Q|^{2n+2}] < \infty.$$

Before we proceed with the proof, we note that we have the following corollary as an immediate consequence of Chebyshëv's inequality.

Corollary 2.5. *Under the conditions of the previous theorem,*

$$\mathbf{P}_t(\tfrac{1}{t}|\Delta Q| \geq R) \leq \frac{1}{(Rt)^{2n+2}} \sup_{t \in \mathbf{R}} \mathbf{E}_t[|\Delta Q|^{2n+2}]. \tag{2.5}$$

We note that because δ is the generator of a strongly continuous one-paramater group of automorphisms, $V \in \operatorname{Dom} \delta^n$ implies $V \in \operatorname{Dom} \delta^k$ for $k = 1, \dots, n-1, n$. We also introduce several lemmas that will be used in the proof of Theorem 2.4.

Lemma 2.6. *If $V \in \operatorname{Dom} \delta^n$, then $V - \tau_V^t(V) \in \operatorname{Dom} \delta^n$ and*

$$c_n := \sup_{t \in \mathbf{R}} \|\delta^n(V - \tau_V^t(V))\| < \infty.$$

Proof. Because $\|\delta^n(V)\| < \infty$ is independent of t, it suffices to show uniform boundedness of $\|\delta^n(\tau_V^t(V))\|$. We proceed by induction on n, noting that because δ is a generator, $V \in \operatorname{Dom} \delta^n$ implies $V \in \operatorname{Dom} \delta^k$ for $k = 1, \dots, n-1, n$.

For $n = 1$, by definition of τ_V^t from $\delta_V = \delta + \mathrm{i}[V, \cdot]$,

$$\begin{aligned} \delta^1(\tau_V^t(V)) &= \delta_V^1(\tau_V^t(V)) - \mathrm{i}[V, \tau_V^t(V)] \\ &= \tau_V^t(\delta_V(V)) - \mathrm{i}[V, \tau_V^t(V)] \\ &= \tau_V^t(\delta(V)) - \mathrm{i}[V, \tau_V^t(V)] \end{aligned}$$

and thus

$$\begin{aligned}\|\delta^1(\tau_V^t(V))\| &\le \|\tau_V^t(\delta(V))\| + \|[V, \tau_V^t(V)]\| \\ &\le \|\delta(V)\| + 2\|V\|^2 \\ &=: c_1.\end{aligned}$$

Now suppose that the statement holds for n and that $V \in \operatorname{Dom}\delta^{n+1}$. Then,

$$\begin{aligned}\delta^{n+1}(\tau_V^t(V)) &= \delta_V(\delta^n(\tau_V^t(V))) - \mathrm{i}[V, \delta^n(\tau_V^t(V))] \\ &= \delta_V^{n+1}(\tau_V^t(V)) - \mathrm{i}\sum_{k=0}^{n} \delta_V^k[V, \delta^{n-k}(\tau_V^t(V))] \\ &= \tau_V^t(\delta_V^{n+1}(V)) - \mathrm{i}\sum_{k=0}^{n} \delta_V^k[V, \delta^{n-k}(\tau_V^t(V))]\end{aligned}$$

and thus, using Lemma 1.19,

$$\begin{aligned}\|\delta^{n+1}(\tau_V^t(V))\| &\le \|\tau_V^t(\delta_V^{n+1}(V))\| + \sum_{k=0}^{n} \|\delta_V^k[V, \delta^{n-k}(\tau_V^t(V))]\| \\ &\le \|\delta_V^{n+1}(V)\| + \sum_{k=0}^{n}\sum_{j=0}^{k} \binom{k}{j} 2\|\delta_V^j(V)\|\|\delta_V^{k-j}\delta^{n-k}(\tau_V^t(V))\|.\end{aligned}$$

By repeated application of the product rule of Lemma 1.19, this may in turn be bounded uniformly in t by a combination of powers of $c_1, \dots, c_n$, $\|V\|$, $\|\delta(V)\|, \dots, \|\delta^n(V)\|$ and $\|\delta^{n+1}(V)\|$, which are all finite by induction hypothesis. □

Lemma 2.7. *Let $n \in \mathbf{N}$. If $X_t \in \operatorname{Dom}\delta^k$ with $\|\delta^k(X_t)\| \le d_k < \infty$ for all $t \in \mathbf{R}$ and for all $1 \le k \le n$, then $X_t^p \in \operatorname{Dom}\delta^n$ for all $p \in \mathbf{N}$ and $\sup_{t\in\mathbf{R}} \|\delta^n(X_t^p)\| < \infty$.*

Proof. Note that by the product rule of Lemma 1.19,

$$\delta^n(X_t^{p+1}) = \sum_{k=0}^{n} \binom{n}{k} \delta^{n-k}(X_t^p)\delta^k(X_t)$$

has uniformly bounded norm if $\delta^k(X_t^p)$ does for all $k \le n$. Hence, the result then follows by induction on $p \in \mathbf{N}$. □

Lemma 2.8. *Let* ad_{L_ω} *be defined (on a dense subspace) by* $\mathrm{ad}_{L_\omega}(W) = [L_\omega, W]$ *and let* $n \in \mathbf{N}$. *If* $\mathrm{ad}_{L_\omega}^k(W)$ *extends to a bounded linear operator for all* $k = 1, \ldots, n-1, n$*, then*

$$L_\omega^n W = \sum_{k=0}^{n} \binom{n}{k} \mathrm{ad}_{L_\omega}^k(W) L_\omega^{n-k}.$$

Proof. The result follows by induction on $n \in \mathbf{N}$ and a standard combinatorial argument. □

Proof of Theorem 2.4. Let $X_t := \pi_\omega(V) - \pi_\omega(\tau_V^t(V))$. It is immediate from the definition of the heat FS and $L_\omega \Omega_\omega = 0$ that

$$\begin{aligned}\mathbf{E}_t[|\Delta Q|^{2n+2}] &= \|(L_\omega + X_t)^n X_t \Omega_\omega\|^2 \\ &= \left\| \sum_{A_i = L_\omega, X_t} A_1 \cdots A_n X_t \Omega_\omega \right\|^2\end{aligned}$$

and each term in the sum is of the form $L_\omega^{\alpha_1} X_t^{\beta_1} \cdots L_\omega^{\alpha_m} X_t^{\beta_m} X_t$ for a set of nonnegative integer exponents satisfying $\sum_{i=1}^m \alpha_i + \beta_i = n$.

By a repeated application of Lemma 2.8, we obtain an combination of terms of the form $\mathrm{ad}_{L_\omega}^k(X_t^p)$ for $k \leq n$ and $p \leq n+1$. Since $\mathrm{ad}_{L_\omega}^k(A) = \pi_\omega(\delta^k(A))$ for all $A \in \mathrm{Dom}\,\delta^k$, Lemmas 2.6 and 2.7 yield the desired result. □

We now turn to describing analytic extendibility of the map $s \mapsto \tau^s(V)$ influences the tails of the heat full statistics measure $\mathbf{P}_t$. This is essentially the result of [BJP+15] reformulated for general perturbed C^*-dynamical systems. The proof there uses matrix inequalities that are not directly applicable to extended systems. We provide a proof that is adapted to our framework.

Theorem 2.9 ([BJP+15], original form)**.** *Consider a sequence of confined quantum systems indexed by* N*, with unperturbed Hamiltonian* $H_0^{(N)}$*, initial invariant state* $\omega^{(N)}$ *and perturbation* $V^{(N)}$*. Suppose that the corresponding sequence* $(\mathbf{P}_t^{(N)})_N$ *of heat FS measures is such that their characteristic functions converge pointwise to a function that is continuous in* $\alpha = 0$*. If there exists* $\gamma_0 > 0$ *such that*

$$r(\gamma_0) := \sup_N \sup_{x \in [-\frac{1}{2}, \frac{1}{2}]} \|\mathrm{e}^{x\gamma H_0^{(N)}} V^{(N)} \mathrm{e}^{-x\gamma H_0^{(N)}}\| < \infty$$

then

$$\sup_{t\in\mathbf{R}} \mathbf{E}_t[\mathrm{e}^{\gamma_0|\Delta Q|}] \leq 2\mathrm{e}^{2\gamma_0 r(\gamma_0)},$$

where $\mathbf{E}_t$ *denotes expectation with respect to* $\mathbf{P}_t = \text{w-lim}_N \mathbf{P}_t^{(N)}$.

Theorem 2.10 ([BJP+15], adapted)**.** *Let* $(\mathfrak{A}, \tau, \omega)$ *be a* C^**-dynamical system, with* ω *a* τ*-invariant state, and* $\mathbf{P}_t$ *be the heat FS measure associated to a self-adjoint perturbation* $V \in \mathfrak{A}$*. Suppose that the map* $\mathbf{R} \ni s \mapsto \tau^s(V) \in \mathcal{O}$ *has an norm-analytic extension to the strip* $\{z \in \mathbf{C} : |\operatorname{Im} z| < \frac{1}{2}\gamma_0\}$ *with* $v_0 = \sup_{|\operatorname{Im} z|<\frac{1}{2}\gamma_0} \|\tau^z(V)\| < \infty$*. Then,*

$$\sup_{t\in\mathbf{R}} \mathbf{E}_t[\mathrm{e}^{\gamma_0|\Delta Q|}] \leq 2\mathrm{e}^{2\gamma_0 v_0}.$$

Before we provide the proof, we mention that this result also has an interesting corollary, whose proof is a direct application of Chebyshëv's inequality.

Corollary 2.11 ([BJP+15])**.** *Under the assumptions of the previous theorem,*

$$\mathbf{P}_t(\tfrac{1}{t}|\Delta Q| \geq R) \leq 2\mathrm{e}^{2\gamma_0 v_0}\mathrm{e}^{-R\gamma_0 t}. \tag{2.6}$$

Proof of Theorem 2.10. By hypothesis, the function of $s \in \mathbf{R}$ defined by the truncated Araki–Dyson series

$$\mathbf{1} + \sum_{n=1}^{N} \mathrm{i}^n \int_0^s \cdots \int_0^{s_{n-1}} \pi_\omega(\tau^{s_n}(V)) \cdots \pi_\omega(\tau^{s_1}(V))\, \mathrm{d}s_n \cdots \mathrm{d}s_1, \tag{2.7}$$

has an analytic extension to the strip $\{z \in \mathbf{C} : |\operatorname{Im} z| < \frac{1}{2}\gamma_0\}$ for each $N \in \mathbf{N}$. By the Vitali–Porter convergence theorem[1] and equivalence of the weak and strong analyticity, we obtain a limiting norm-analytic $\mathfrak{L}(\mathfrak{H}_\omega)$-valued function $z \mapsto E_V(z)$ satisfying

$$\|E_V(z)\| \leq 1 + \sum_{n\in\mathbf{N}} \frac{|z|^n}{n!} \sup_{|\operatorname{Im} z'|<\frac{1}{2}\gamma_0} \|\tau^{z'}(V)\|^n \leq \mathrm{e}^{|z|v_0}$$

for $z \in \{z \in \mathbf{C} : |\operatorname{Im} z| < \frac{1}{2}\gamma_0\}$. On the other hand, the truncated Araki–Dyson series (2.7) converges to $\mathrm{e}^{\mathrm{i}s(L_\omega+\pi_\omega(V))}\mathrm{e}^{-\mathrm{i}sL_\omega}$ for all $s \in \mathbf{R}$ [Pil06]. Then, temporarily omitting the representation

[1] This theorem, attributed to Vitali and Porter independently (see [Vit04] and [Por05]), is somehow not standard in basic complex analysis textbooks, but is nevertheless quite useful in applications.

maps π_ω and the subscripts ω for notational simplicity, we have

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}s(L+V-\tau_V^t(V))}\Omega &= \mathrm{e}^{\mathrm{i}t(L+V)}\mathrm{e}^{-\mathrm{i}s(L+V-V)}\mathrm{e}^{-\mathrm{i}t(L+V)}\Omega \\
&= \mathrm{e}^{\mathrm{i}t(L+V)}\mathrm{e}^{-\mathrm{i}sL}\mathrm{e}^{\mathrm{i}s(L+V)}\mathrm{e}^{-\mathrm{i}t(L+V)}\mathrm{e}^{-\mathrm{i}s(L+V)}\Omega \\
&= \mathrm{e}^{\mathrm{i}t(L+V)}E_V(-s)^*\mathrm{e}^{-\mathrm{i}t(L+V)}E_V(-s)\Omega,
\end{aligned}$$

with the operators on the right-hand side admitting analytic extensions as functions of s to the strip $\{z \in \mathbf{C} : |\mathrm{Im}\, z| < \frac{1}{2}\gamma_0\}$. Therefore, for all $\gamma \in (-\gamma_0, \gamma_0)$,

$$\begin{aligned}
\mathbf{E}_t[\mathrm{e}^{\gamma|\Delta Q|}] &\leq \mathbf{E}_t[\mathrm{e}^{-\gamma\Delta Q}] + \mathbf{E}_t[\mathrm{e}^{\gamma\Delta Q}] \\
&= \|\mathrm{e}^{-\frac{1}{2}\gamma(L_\omega+\pi_\omega(V)-\pi_\omega(\tau_V^t(V)))}\Omega_\omega\|^2 + \|\mathrm{e}^{\frac{1}{2}\gamma(L_\omega+\pi_\omega(V)-\pi_\omega(\tau_V^t(V)))}\Omega_\omega\|^2 \\
&\leq \|\tau_V^t(E_V(-\tfrac{\mathrm{i}}{2}\gamma)^*)\|^2\|E_V(-\tfrac{\mathrm{i}}{2}\gamma)\|^2\|\Omega_\omega\|^2 + \|\tau_V^t(E_V(\tfrac{\mathrm{i}}{2}\gamma)^*)\|^2\|E_V(\tfrac{\mathrm{i}}{2}\gamma)\|^2\|\Omega_\omega\|^2 \\
&\leq \mathrm{e}^{2\gamma_0 v_0}. \qquad \square
\end{aligned}$$

Chapter 3

Identical particles and ideal quantum gases

The goal of this section is to introduce the mathematical tools that are required to discuss the physics of many identical particles, non-interacting quantum gases. The book [DG13] provides a very thorough treatment of the mathematical formalism presented here, and much more. However, the shorter expositions in [BR97, §5.2] or [RS75, §X.7] are sufficient for most purposes. A very pedagocial approach through finite dimensional system is found in [JOPP11].

3.1 From tensor products to Fock spaces

As a first step, we introduce the tensor product of two Hilbert spaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$ over $\mathbf{C}$ in order to mathematically describe two quantum systems that are combined. Consider the set $\operatorname{span}(\mathfrak{h}_1 \times \mathfrak{h}_2)$ of finite linear combinations over $\mathbf{C}$ of ordered pairs $(\psi, \phi) \in \mathfrak{h}_1 \times \mathfrak{h}_2$. Let $\mathfrak{I}$ be the subspace of $\operatorname{span}(\mathfrak{h}_1 \times \mathfrak{h}_2)$ spanned by elements of the form

$$
\begin{aligned}
&(\psi, \phi + \phi') - (\psi, \phi) - (\psi, \phi'), && (\psi, \lambda\phi) - \lambda(\psi, \phi), \\
&(\psi + \psi', \phi) - (\psi, \phi) - (\psi', \phi), && (\lambda\psi, \phi) - \lambda(\psi, \phi),
\end{aligned}
$$

where $\psi, \psi' \in \mathfrak{h}_1$, $\phi, \phi' \in \mathfrak{h}_2$ and $\lambda \in \mathbf{C}$. We define the **algebraic tensor product**

$$
\mathfrak{h}_1 \otimes_{\text{alg}} \mathfrak{h}_2 := \operatorname{span}(\mathfrak{h}_1 \times \mathfrak{h}_2)/\mathfrak{I},
$$

and denote equivalence classes by

$$
\psi \otimes \phi := (\psi, \phi) + \mathfrak{I}
$$

for $(\psi, \phi) \in \mathfrak{h}_1 \times \mathfrak{h}_2$.

The **Hiblert space tensor product** $\mathfrak{h}_1 \otimes \mathfrak{h}_2$ of the Hilbert spaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$ over $\mathbf{C}$ is defined as the completion of the algebraic tensor product $\mathfrak{h}_1 \otimes_{\text{alg}} \mathfrak{h}_2$ with respect to the metric induced by the inner product

$$\langle \psi \otimes \phi, \psi' \otimes \phi' \rangle := \langle \psi, \psi' \rangle \langle \phi, \phi' \rangle$$

extended by linearity. The Hilbert space $\mathfrak{h}_1 \otimes \mathfrak{h}_2$ is then the suitable space for describing the system composed of two distinguishable parts described by $\mathfrak{h}_1$ and $\mathfrak{h}_2$ respectively.

From now on, we will consider only Hilbert spaces over $\mathbf{C}$. We also introduce a shorthand for the repeated tensor product of the same Hilbert space: $\mathfrak{h}^{\otimes 0} := \mathbf{C}$ and

$$\mathfrak{h}^{\otimes n} := \underbrace{\mathfrak{h} \otimes \cdots \otimes \mathfrak{h}}_{n \text{ times}}$$

for $n \in \mathbf{N}^*$. We endow the direct sum $\bigoplus_{n \leq N} \mathfrak{h}^{\otimes n}$ with the inner product

$$\langle (\Psi^{(0)}, \Psi^{(1)}, \dots, \Psi^{(N)}), (\Phi^{(0)}, \Phi^{(1)}, \dots, \Phi^{(N)}) \rangle := \langle \Psi^{(0)}, \Phi^{(0)} \rangle + \langle \Psi^{(1)}, \Phi^{(1)} \rangle + \cdots + \langle \Psi^{(N)}, \Phi^{(N)} \rangle$$

for $\Psi^{(n)}, \Phi^{(n)} \in \mathfrak{h}^{\otimes n}$, $n = 0, \dots, N$. There is a natural injection of $\bigoplus_{n \leq N} \mathfrak{h}^{\otimes n}$ into $\bigoplus_{n \leq N+1} \mathfrak{h}^{\otimes n}$ and we let

$$\Gamma_{\text{fin}}(\mathfrak{h}) := \bigcup_{N \in \mathbf{N}} \bigoplus_{n \leq N} \mathfrak{h}^{\otimes n}.$$

A priori the physics of a quantum system composed of an unknown (possibly fluctuating, unbounded) number of indistinguishable particles, each described on $\mathfrak{h}$, could be described on the Hilbert space

$$\Gamma(\mathfrak{h}) := \overline{\Gamma_{\text{fin}}(\mathfrak{h})},$$

called the full Fock space over $\mathfrak{h}$. A vector $\Psi \in \Gamma(\mathfrak{h})$ is a sequence $(\Psi^{(n)})_{n \in \mathbf{N}}$ where $\Psi^{(n)} \in \mathfrak{h}^{\otimes n}$ corresponds (up to normalization) to a wave function of n particles.

However, it turns out that "nature" imposes an extra condition for a physically accurate description of quantum systems. This condition is stated in terms of exchanges of particles. For a permutation $\pi \in S_n$, let $\Pi_\pi^{(n)}$ denote the operator on $\mathfrak{h}^{\otimes n}$ defined by

$$\Pi_\pi^{(n)}(\psi_1 \otimes \cdots \otimes \psi_n) := \psi_{\pi(1)} \otimes \cdots \otimes \psi_{\pi(n)}$$

and extended by linearity. This operator is interpreted as permuting, according to π, the particles in the n-particle subspace of $\Gamma(\mathfrak{h})$. It is used to define the antisymmetrization and symmetrization operators

$$P_-^{(n)} := \frac{1}{n!}\sum_{\pi\in S_n} \operatorname{sign}(\pi)\Pi_\pi^{(n)},$$
$$P_+^{(n)} := \frac{1}{n!}\sum_{\pi\in S_n} \Pi_\pi^{(n)}.$$

Note that these operators are projections. It is a central fact of quantum physics that systems of identical particles can be put into to categories: those whose for which the n-particle wave functions are invariant under $P_-^{(n)}$, and those whose for which the n-particle wave functions are invariant under $P_+^{(n)}$. Those of the first kind are called **fermionic**, or are said to satisfy the Fermi statistics; those of the second the second kind, **bosonic**, satisfying the Bose statistics. It turns out that this dichotomy is related to the notion of spin in quantum mechanics, which we won't discuss here. Systems of fermions and bosons are thus suitably described on the antisymmetric (fermionic) **Fock space** $\Gamma^-(\mathfrak{h})$ and symmetric (bosonic) Fock space $\Gamma^+(\mathfrak{h})$ respsectively, which are defined as the Hilbert space closures of

$$\Gamma_{\text{fin}}^-(\mathfrak{h}) := \bigcup_{N\in\mathbf{N}} \bigoplus_{n\le N} P_-^{(n)}\mathfrak{h}^{\otimes n},$$

and

$$\Gamma_{\text{fin}}^+(\mathfrak{h}) := \bigcup_{N\in\mathbf{N}} \bigoplus_{n\le N} P_+^{(n)}\mathfrak{h}^{\otimes n}.$$

We now turn to a discussion of ways of passing from operators on $\mathfrak{h}$, referred to as one-particle or one-body operators, to operators on all of $\Gamma(\mathfrak{h})$ and $\Gamma^\pm(\mathfrak{h})$.

Consider a bounded operator U on $\mathfrak{h}$. We define, $U^{\otimes 0} = \mathbf{1}$ on $\mathbf{C}$, $U^{\otimes n}$ on $\mathfrak{h}^{\otimes n}$ by

$$U^{\otimes n}(\psi_1 \otimes \cdots \otimes \psi_n) = U\psi_1 \otimes \cdots \otimes U\psi_n$$

(extended by linearity), and $\Gamma(U)$ on $\Gamma(\mathfrak{h})$ by

$$\Gamma(U) := \bigoplus_{n\in\mathbf{N}} U^{\otimes n}.$$

Consider a self-adjoint operator A with domain $\mathrm{Dom}\, A \subset \mathfrak{h}$. We define $\mathrm{d}\Gamma^{(0)}(A) = 0$ on $\mathfrak{h}^{\otimes 0}$ and, for $n \in \mathbf{N}^*$, $\mathrm{d}\Gamma^{(n)}(A)$ by

$$\mathrm{d}\Gamma^{(n)}(A) := \sum_{j=1}^{n} \mathbf{1}^{\otimes(j-1)} \otimes A \otimes \mathbf{1}^{\otimes(n-j)}.$$

on the dense subspace $(\mathrm{Dom}\, A)^{\otimes_{\mathrm{alg}} n}$ of $\mathfrak{h}^{\otimes n}$. The direct sum $\bigoplus_{n\in\mathbf{N}} \mathrm{d}\Gamma^{(n)}$ is then essentially self-adjoint and we denote its opeartor closure by $\mathrm{d}\Gamma(A)$. With these definitions,

$$\Gamma(\mathrm{e}^{\mathrm{i}tA}) = \mathrm{e}^{\mathrm{i}t\,\mathrm{d}\Gamma(A)}.$$

When working on $\Gamma^{\pm}(\mathfrak{h})$, we restrict $\Gamma(U)$ and $\mathrm{d}\Gamma(A)$ to the appropriate subspace. The process of passing from objects on $\mathfrak{h}$ to objects in $\Gamma^{\pm}(\mathfrak{h})$ is known as second quantisation.[1]

Example 3.1. We give a particular importance to the operator

$$N := \mathrm{d}\Gamma(\mathbf{1})$$

defined through this process, called the **number operator**. Explicitly, $N\Psi^{(n)} = n\Psi^{(n)}$ for all $\Psi^{(n)} \in \mathfrak{h}^{\otimes n}$ and the domain of N is taken to be

$$\mathrm{Dom}\, N = \{\Psi = (\Psi^{(n)})_{n\in\mathbf{N}} \in \Gamma(\mathfrak{h}) : \sum_{n\in\mathbf{N}} n^2 \|\Psi^{(n)}\|^2 < \infty\}.$$

3.2 Elements of quantum field theory

We call the vector $(1, 0, 0, \dots) \in \Gamma(\mathfrak{h})$ the **vacuum vector** and denote it by the letter Ω. For $\psi \in \mathfrak{h}$, we define $a^*(\psi)$ on $\Gamma_{\mathrm{fin}}(\mathfrak{h})$ by

$$a^*(\psi)\Omega = \psi,$$
$$a^*(\psi)(\psi_1 \otimes \cdots \otimes \psi_n) = \sqrt{n+1}\, \psi \otimes \psi_1 \otimes \cdots \otimes \psi_n$$

for $\psi, \psi_1, \dots, \psi_n \in \mathfrak{h}$. This operator extends by linearity to $\mathrm{Dom}\, N^{1/2}$. Note that $a^*(\psi)$ takes elements of $\mathfrak{h}^{\otimes n}$ (n-particle wave functions) to elements of $\mathfrak{h}^{\otimes n+1}$ ($(n+1)$-particle wave functions).

[1] First quantisation refers to the procedure by which one goes from a classical system to a corresponding quantum system.

It is called the **creation operator** associated to ψ. On the other hand, we also define **annihilation operators** by

$$a(\psi)\Omega = 0,$$
$$a(\psi)\psi_1 = \langle \psi, \psi_1 \rangle \, \Omega,$$
$$a(\psi)(\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n) = \sqrt{n} \, \langle \psi, \psi_1 \rangle \, \psi_2 \otimes \cdots \otimes \psi_n$$

extended by linearity to $\mathrm{Dom}\, N^{1/2}$.

In fermionic systems, we consider

$$a_-^*(\psi) := P_- a^*(\psi) P_- = P_- a^*(\psi),$$
$$a_-(\psi) := P_- a(\psi) P_- = a(\psi) P_-,$$

called the **fermionic creation and annihilation operators**. In bosonic systems, we consider

$$a_+^*(\psi) := P_+ a^*(\psi) P_+ = P_+ a^*(\psi),$$
$$a_+(\psi) := P_+ a(\psi) P_+ = a(\psi) P_+,$$

called the **bosonic creation and annihilation operators**.

Proposition 3.2. *Let A be a self-adjoint operator with domain $\mathrm{Dom}\, A \subseteq \mathfrak{h}$ and U be a bounded operator on $\mathfrak{h}$. For all $\psi \in \mathrm{Dom}\, A$ and $\Psi \in \bigcup_{M \in \mathbf{N}} \bigoplus_{n \leq M} (\mathrm{Dom}\, A)^{\otimes_{\mathrm{alg}} n}$*

$$[\mathrm{d}\Gamma(A), a^*(\psi)]\Psi = a^*(A\psi)\Psi \tag{3.1}$$

and for all $\psi \in \mathfrak{h}$ and $\Psi \in \mathrm{Dom}\, N^{1/2}$

$$\Gamma(U) a^*(\psi)\Psi = a^*(U\psi)\Gamma(U)\Psi. \tag{3.2}$$

Proposition 3.3. *The maps $\psi \mapsto a_\pm^*(\psi)$ and $\psi \mapsto a_\pm(\psi)$ are respectively linear and anti-linear. The operators $a_\pm^*(\psi)$ and $a_\pm(\psi)$ have closed extensions (which we denote by the same symbol) and then, as suggested by the notation, $a_\pm^*(\psi)$ is the adjoint of $a_\pm(\psi)$.*

Example 3.4. Let $g_1, g_2 \in \mathfrak{h}$. Then, for any $\psi_1, \ldots, \psi_n \in \mathfrak{h}$,

$$\begin{aligned} \mathrm{d}\Gamma(g_2 \langle g_1, \cdot \rangle)(\psi_1 \otimes \cdots \otimes \psi_n) &= P_\pm \sum_{j=1}^{n} \langle g_1, \psi_j \rangle \, \psi_1 \otimes \cdots \otimes \psi_{j-1} \otimes g_2 \otimes \psi_{j+1} \otimes \cdots \otimes \psi_m \\ &= a_\pm^*(g_2) a_\pm(g_1)(\psi_1 \otimes \cdots \otimes \psi_n). \end{aligned}$$

The operator

$$\mathrm{d}\Gamma(g_2 \langle g_1, \cdot\rangle + g_1 \langle g_2, \cdot\rangle) = a_\pm^*(g_2)a_\pm(g_1) + a_\pm^*(g_1)a_\pm(g_2)$$

is a hopping interaction between g_1 and g_2 and will be of particular importance later on.

It is also immediate from these definitions that

$$\|a_\pm^\sharp(\psi)\Psi\| \leq \|\psi\| \|(N+\mathbf{1})^{1/2}\Psi\|$$

for all $\Psi \in \mathrm{Dom}\, N^{1/2}$, where $a^\sharp$ stands for a^* or a. A straight forward computation then shows that for any $\psi_1, \psi_2 \in \mathfrak{h}$ and $\Psi^{(n)} \in \mathfrak{h}^{\otimes n}$

$$\begin{aligned}
\|a^{\sharp_1}(\psi_1)a^{\sharp_2}(\psi_2)(\mathbf{1}+N)^{-1}\Psi^{(n)}\| &\leq \|a^{\sharp_1}(\psi_1)a^{\sharp_2}(\psi_2)\tfrac{1}{n+1}\Psi^{(n)}\| \\
&\leq \tfrac{1}{n+1}\|\psi_1\| \|(N+\mathbf{1})^{1/2}a^{\sharp_2}(\psi_2)\Psi^{(n)}\| \\
&\leq \tfrac{1}{n+1}\|\psi_1\|(n+2)^{1/2}\|a^{\sharp_2}(\psi_2)\Psi^{(n)}\| \\
&\leq \tfrac{(n+2)^{1/2}}{n+1}\|\psi_1\| \|\psi_2\| \|(N+\mathbf{1})^{1/2}\Psi^{(n)}\| \\
&\leq 2\|\psi_1\| \|\psi_2\| \|\Psi^{(n)}\|,
\end{aligned}$$

independently of $n \in \mathbf{N}$. Hence, $a^{\sharp_1}(\psi_1)a^{\sharp_2}(\psi_2)(\mathbf{1}+N)^{-1}$ extends to a bounded operator on $\Gamma(\mathfrak{h})$ and its norm satisfies

$$\|a^{\sharp_1}(\psi_1)a^{\sharp_2}(\psi_2)(\mathbf{1}+N)^{-1}\| \leq 2\|\psi_1\| \|\psi_2\|$$

The seemingly innocent restrictions in the definition of the fermionic and bosonic creation and annihilation operators have deep consequences. First, note that the fermionic creation operator is such that

$$a_-^*(\psi)a_-^*(\psi) = 0$$

for all $\psi \in \mathfrak{h}$. This is the manifestation of the **Pauli exclusion principle**: two identical fermions cannot be created in the same quantum state. There is no such exclusion principle for bosons. Further consequences of the restrictions are presentend in the following theorem.

Theorem 3.5. *The fermionic and bosonic creation and annihilation operators enjoy the following properties*

1. *we have the anticommutation relations*

$$\{a_-(\psi), a_-(\varphi)\} = 0, \qquad \{a_-(\psi), a_-^*(\varphi)\} = \langle\psi|\varphi\rangle\, \mathbf{1},$$
$$\{a_-^*(\psi), a_-^*(\varphi)\} = 0,$$

known as the ***Canonical anticommutation relations*** *(CAR), and the commutation relations*

$$[a_+(\psi), a_+(\varphi)] = 0, \qquad [a_+(\psi), a_+^*(\varphi)] = \langle\psi|\varphi\rangle\, \mathbf{1},$$
$$[a_+^*(\psi), a_+^*(\varphi)] = 0,$$

on Dom N *known as the* ***Canonical commutation relations*** *(CCR);*

2. *The fermionic creation and annihilation* $a_-^*(\psi)$ *and* $a_-(\psi)$ *extend to bounded operators on* $\Gamma^-(\mathfrak{h})$ *with* $\|a_-^*(\psi)\| = \|a_-(\psi)\| = \|\psi\|$*, whereas the bosonic creation and annihilation operators* $a_+(\psi)$ *and* $a_+^*(\psi)$ *are unbounded.*

3.3 The quasi-free Fermi gas

We build the quasi-free Fermi gas from a one-particle Hilbert space $\mathfrak{h}$ and a one-particle unperturbed Hamiltonian, a self-adjoint operator h_0 on $\mathfrak{h}$. We consider the C^*-algebra generated by

$$\{a_-^*(\phi) : \phi \in \mathfrak{h}\} \subset \mathfrak{L}(\Gamma^-(\mathfrak{h}))$$

(with the addition, multiplication, involution and norm inherited from $\mathfrak{L}(\Gamma^-(\mathfrak{h}))$). It is known as the Fock space representation of the **canonical anticommutation relations algebra** of $\mathfrak{h}$ and is denoted by CAR($\mathfrak{h}$). In this context, we drop the "–" subscript on a^* and a, and use $a^\sharp$ to denote generically either a^* or a.

We consider the C^*-dynamical system consisting of the C^*-algebra $\mathfrak{A} = \mathrm{CAR}(\mathfrak{h})$, and the dynamics given by

$$\tau^t(a^\sharp(\phi)) = a^\sharp(\mathrm{e}^{\mathrm{i}th_0}\phi)$$

for $\phi \in \mathfrak{h}$ and extended to CAR($\mathfrak{h}$) by linearity. The one-parameter group $(\tau^t)_{t\in\mathbf{R}}$ is strongly continuous on $\mathfrak{A}$.

We let ω be the quasi-free state on CAR($\mathfrak{h}$) generated by a density $0 \leq T \leq \mathbf{1}$ in $\mathfrak{L}(\mathfrak{h})$, i.e. the unique state satisfying

$$\omega(a^*(\psi_n)\cdots a^*(\psi_1)a(\phi_1)\cdots a(\phi_m)) = \delta_{n,m}\det\{\langle\phi_i, T\psi_j\rangle\}_{i,j}$$

for all $\psi_1, \ldots \psi_n, \phi_1, \ldots, \phi_m \in \mathfrak{h}$. At times, we will rewrite T as $\gamma(\mathbf{1}+\gamma)^{-1}$ where γ is a positive operator on $\mathfrak{h}$. We make the assumption that T commutes with $\mathrm{e}^{\mathrm{i}th_0}$ for all $t \in \mathbf{R}$ so that ω is τ-invariant. A typical example of density is the Fermi–Dirac density $T = (\mathbf{1} + \mathrm{e}^{\beta h_0})^{-1}$, or equivalently $\gamma = \mathrm{e}^{-\beta h_0}$, for some inverse temperature $\beta > 0$, in which case ω is a β-KMS state for τ.[2]

When considering this C^*-dynamical system, the GNS representation we will work in is known as the (left glued) **Araki–Wyss representation**, which was introduced in its earliest form in [AW64]. The Hilbert space for this representation is the anti-symmetric Fock space $\mathfrak{H} = \Gamma^-(\mathfrak{h}\oplus\mathfrak{h})$ and the representation is determined by

$$\pi(a^*(\phi)) = a^*(\sqrt{\mathbf{1}-T}\phi \oplus 0) + a(0 \oplus \sqrt{T}\,\overline{\phi}), \tag{3.3}$$

$$\pi(a(\phi)) = a(\sqrt{\mathbf{1}-T}\phi \oplus 0) + a^*(0 \oplus \sqrt{T}\,\overline{\phi}), \tag{3.4}$$

for all $\phi \in \mathfrak{h}$. Finally the vector representative Ω of ω is the vacuum in $\Gamma^-(\mathfrak{h} \oplus \mathfrak{h})$ and the Liouvillean is

$$L = \mathrm{d}\Gamma(h_0 \oplus -h_0).$$

For proofs of these facts, we refer the reader to [DG13, §17] and [Der06, §10].

3.4 The quasi-free Bose gas

We want to construct a bosonic analogue of the construction of the last section. The fact that the bosonic creation and annihilation operators a_+^* and a_+ are unbounded (Theorem 3.5) represents an obstacle to this construction. Hence, we take a slightly different route than in the fermionic case: we construct the algebra from the so-called Weyl operators instead of the creation and annihilation operators. We omit the "+" subscripts on a^* and a through the rest of this section.

[2]Given a dynamics τ and an inverse temperature β, the Kubo–Martin–Schwinger (KMS) condition reads $\omega(A\tau^t(B)) = \omega(BA)$ for all A and B in a norm dense set of $\mathfrak{A}$, see [BR97, §5.3].

On $\Gamma^+(\mathfrak{k})$ (for any Hilbert space $\mathfrak{k}$), the **Segal field operators**

$$\varphi(y) := \tfrac{1}{\sqrt{2}}(a^*(y) + a(y))$$

have self-adjoint extensions for any $y \in \mathfrak{k}$ (which we denote by the same symbol). The **Weyl operators**

$$W(y) := \exp(\mathrm{i}\varphi(y))$$

obtained from them are then unitary operators and hence elements of $\mathfrak{L}(\Gamma^+(\mathfrak{k}))$. Then, the CCR of Theorem 3.5 become

$$W(y)W(x) = \mathrm{e}^{\frac{1}{2}\operatorname{Im}\langle y,x\rangle} W(y+x), \tag{3.5}$$

known as the **Weyl form of the CCR**.

For a one-particle Hilbert space $\mathfrak{h}$ and a one-particle unperturbed Hamiltonian h_0 on $\mathfrak{h}$, we consider the W^*-algebra generated by

$$\{W(\sqrt{\mathbf{1}+\rho}\,\phi \oplus \sqrt{\rho}\,\overline{\phi}) : \phi \in \operatorname{Dom}\rho^{1/2}\} \subset \mathfrak{L}(\Gamma^+(\mathfrak{h}\oplus\mathfrak{h})).$$

for $\rho = \gamma(\mathbf{1}-\gamma)^{-1}$ for some $\gamma \in \mathfrak{L}(\mathfrak{h})$ with $0 \leq \gamma \leq \mathbf{1}$. We make the assumption that γ commutes with $\mathrm{e}^{\mathrm{i}th_0}$ for all $t \in \mathbf{R}$. This algebra, which we denote by $\mathfrak{M}^{\mathrm{AW}}$ is known as the (left glued) **Araki–Woods representation** of the CCR over $\mathfrak{h}$ with respect to ρ, which goes back to [AW63]. A typical choice of density is the Bose–Einstein density $\rho = (\mathrm{e}^{\beta h_0} - \mathbf{1})^{-1}$, or equivalently $\gamma = \mathrm{e}^{-\beta h_0}$. The operators $W(\sqrt{\mathbf{1}+\rho}\,\phi \oplus \sqrt{\rho}\,\overline{\phi})$, which we denote by $W^{\mathrm{AW}}(\phi)$, satisfy the Weyl form of the CCR

$$W^{\mathrm{AW}}(\phi)W^{\mathrm{AW}}(\psi) = \mathrm{e}^{\frac{1}{2}\operatorname{Im}\langle\phi,\psi\rangle} W^{\mathrm{AW}}(\phi+\psi) \tag{3.6}$$

for all $\phi, \psi \in \operatorname{Dom}\rho^{1/2}$.

The maps

$$\tau^t(W^{\mathrm{AW}}(\phi)) = W^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}th_0}\phi) \tag{3.7}$$

for $\phi \in \operatorname{Dom}\rho^{1/2}$ extend to a W^*-dynamics $(\tau^t)_{t\in\mathbf{R}}$ on $\mathfrak{M}^{\mathrm{AW}}$.

The state $\omega : A \mapsto \langle\Omega, A\Omega\rangle$ on $\mathfrak{M}^{\mathrm{AW}}$, where Ω is the vacuum in $\Gamma^+(\mathfrak{h}\oplus\mathfrak{h})$, is a quasi-free state with density ρ: it satisfies the evaluation identity

$$\langle\Omega, W^{\mathrm{AW}}(\phi)\Omega\rangle = \mathrm{e}^{-\frac{1}{4}\langle\phi,\phi\rangle - \frac{1}{2}\langle\phi,\rho\phi\rangle} \tag{3.8}$$

for all $\phi \in \operatorname{Dom} \rho^{1/2}$. The state ω is τ-invariant. It is a β-KMS state for τ in the case of the Bose–Einstein density. The triple $(\Gamma^+(\mathfrak{h} \oplus \mathfrak{h}), \mathbf{1}, \Omega)$ provides a GNS representation of the W^*-dynamical system $(\mathfrak{M}^{\mathrm{AW}}, \tau, \omega)$.

The corresponding Liouvillean is

$$L = \mathrm{d}\Gamma(h_0 \oplus -h_0).$$

For proofs of these facts, we refer the reader to [DG13, §17] and [Der06, §9].

When need be, we will denote the generator of $s \mapsto W^{\mathrm{AW}}(s\phi)$ by $\varphi^{\mathrm{AW}}(\phi)$. It is explicitly given by

$$\varphi^{\mathrm{AW}}(\phi) = \varphi(\sqrt{1+\rho}\,\phi \oplus \sqrt{\rho}\,\overline{\phi}), \tag{3.9}$$

where φ denotes the Segal field operators on $\Gamma^+(\mathfrak{h} \oplus \mathfrak{h})$. We will also use the affiliated creation and annihilation operators, $a^{\mathrm{AW}*}(\phi)$ and $a^{\mathrm{AW}}(\phi)$, related to $\varphi^{\mathrm{AW}}(\phi)$ by the formula $\varphi^{\mathrm{AW}}(\phi) = \frac{1}{\sqrt{2}}(a^{\mathrm{AW}*}(\phi) + a^{\mathrm{AW}}(\phi))$. They are explicitly given by the formulae

$$a^{\mathrm{AW}*}(\phi) = a^*(\sqrt{1+\rho}\,\phi \oplus 0) + a(0 \oplus \sqrt{\rho}\,\overline{\phi}), \tag{3.10}$$

$$a^{\mathrm{AW}}(\phi) = a(\sqrt{1+\rho}\,\phi \oplus 0) + a^*(0 \oplus \sqrt{\rho}\,\overline{\phi}). \tag{3.11}$$

Chapter 4

Further results in specific models

4.1 Optimality of the regularity conditions

The level of generality of the first two chapters is too important to allow for an in-depth study of the optimality of the condition in Theorem 2.4. However, this can be done in typical physical models using the formalism of the previous chapter.

Example 4.1. Consider a Fermi gas specified by the CAR on the Hilbert space $\mathfrak{h} = \mathbf{C} \oplus L^2(\mathbf{R}_+, \mathrm{d}e)$ and the unperturbed hamiltonian $h_0 = \epsilon_0 \oplus \hat{e}$, where ϵ_0 denotes multiplication by the constant $\epsilon_0 > 0$ and $\hat{e}$ denotes the multiplication by the energy variable e in $L^2(\mathbf{R}_+, \mathrm{d}e)$: $(\hat{e} f)(e) = ef(e)$. Let the system be initially in the quasi-free state specified by the Fermi–Dirac density $T = (\mathbf{1} + \mathrm{e}^{\beta h_0})^{-1}$ and consider perturbations of the form

$$V = a^*(\psi_f)a(\psi_{\mathrm{imp}}) + a^*(\psi_{\mathrm{imp}})a(\psi_f)$$

where $\psi_f = 0 \oplus f \in \mathbf{C} \oplus L^2(\mathbf{R}_+, \mathrm{d}e)$ and $\psi_{\mathrm{imp}} = 1 \oplus 0 \in \mathbf{C} \oplus L^2(\mathbf{R}_+, \mathrm{d}e)$. Recall that $V = \mathrm{d}\Gamma(v)$ for the one-particle rank-two pertubation

$$v = \psi_{\mathrm{imp}} \langle \psi_f, \cdot \rangle + \psi_f \langle \psi_{\mathrm{imp}}, \cdot \rangle .$$

We will refer to this model as the **fermionic Wigner–Weisskopf atom model**. A thorough discussion of this model can be found in [JKP06].

Remark 4.2. The model is described in so-called **energy representation**, where the Hamiltonian acts by mutliplication by an energy variable. For example, starting with a spherically symmetric system in position space with Hilbert space $L^2(\mathbf{R}^d, \mathrm{d}x)$ and Hamiltonian given by the Laplacian on $\mathbf{R}^d$, one first goes to momentum space by mapping the function

$$x \mapsto g(x)$$

in $L^2(\mathbf{R}^d, \mathrm{d}x)$ to its Fourier transform

$$p \mapsto \hat{g}(p)$$

in $L^2(\mathbf{R}^d, \mathrm{d}p)$, where the Laplacian acts as multiplication by $|p|^2$. With polar coordinates $r := |p|$, $L^2(\mathbf{R}^d, \mathrm{d}p)$ is identified with $L^2(\mathbf{R}_+, r^{d-1}\,\mathrm{d}r) \otimes L^2(\mathbf{S}^{d-1}, \mathrm{d}\sigma)$ and the energy representation is then obtained by mapping the function $r \mapsto \hat{g}(r)$ in $L^2(\mathbf{R}_+, r^{d-1}\,\mathrm{d}r)$ to the function

$$e \mapsto 2^{-1/2} e^{\frac{d-2}{4}} \hat{g}(\sqrt{e})$$

in $L^2(\mathbf{R}_+, \mathrm{d}e)$, where the Laplacian now acts as multiplication by the energy variable e. See [JKP06] for more details.

For this class of models, Theorem 2.4 can be strengthened to an equivalence relation. The proofs are given with reference to one-body operator estimates from Appendix A.

Proposition 4.3. *For the Wigner–Weisskopf atom model of Example 4.1, the following are equivalent*

1. *the* $(2n+2)$*th moment of* $\mathbf{P}_t$ *is uniformly bounded in time,*
$$\sup_{t\in\mathbf{R}} \mathbf{E}_t[|\Delta Q|^{2n+2}] < \infty;$$
2. *there exist a non-trivial time interval* $[t_1, t_2]$ *over which the* $(2n+2)$*th moment of* $\mathbf{P}_t$ *is integrable,*
$$\int_{t_1}^{t_2} \mathbf{E}_t[|\Delta Q|^{2n+2}] < \infty;$$
3. *the map* $\mathbf{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}s\hat{e}} f \in L^2(\mathbf{R}_+, \mathrm{d}e)$ *is n times norm differentiable.*

Proof. The fact that the first statement implies the second one is trivial. The fact that the third one implies the first one is immediate from Theorem 2.4.

To prove that the the second statement implies the third one, we proceed by induction on n. The result holds for $n = 0$. Assume it holds for some $n-1 \in \mathbf{N}$ and suppose that there exists a non-trivial interval $[t_1, t_2]$ on which $t \mapsto \mathbf{E}_t(|\Delta Q|^{2n+2})$ is integrable. Then, $t \mapsto \mathbf{E}_t(|\Delta Q|^{2n})$ is also integrable by Jensen's inequality and $f \in \operatorname{Dom} \hat{e}^{n-1}$ by the induction hypothesis.

Consider the set

$$G_p(t) := \{(\mathbf{1}-T)^{1/2} h_0^p \mathrm{e}^{\mathrm{i}sh} \psi_f \oplus 0, (\mathbf{1}-T)^{1/2} h_0^p \mathrm{e}^{\mathrm{i}sh} \psi_{\mathrm{imp}} \oplus 0,$$
$$0 \oplus T^{1/2}(-h_0)^p \mathrm{e}^{-\mathrm{i}sh} \psi_f, 0 \oplus T^{1/2}(-h_0)^p \mathrm{e}^{-\mathrm{i}sh} \psi_{\mathrm{imp}}\}_{s=0,t}.$$

By Lemma A.1, these sets are bounded uniformly in $t \in [t_1, t_2]$ for each $p = 0, 1, \ldots, n-1$. We denote their bound by M_p. We also set $X_t := \pi(V) - \pi(\tau_V^t(V))$ and omit the representation maps π.

We first note by expanding $(L + X_t)^n$, that the moment $\mathbf{E}_t(\Delta Q^{2n+2}) = \|(L + X_t)^n X_t \Omega\|^2$ can be written as $\|L^n X_t \Omega + \sum_{p=0}^{n-1} Q_{p,n} \Omega\|^2$, where $Q_{p,n}$ is a sum of $\binom{n}{p}$ products containing p (not necessarily adjacent) copies of L and $n+1-p$ (not necessarily adjacent) copies of X_t. Then,

$$\int_{t_1}^{t_2} \|L^n X_t \Omega\|^2 \,\mathrm{d}t \leq 2 \int_{t_1}^{t_2} \mathbf{E}_t(|\Delta Q|^{2n+2}) \,\mathrm{d}t + 2 \sum_{p=0}^{n-1} \int_{t_1}^{t_2} \|Q_{p,n} \Omega\|^2 \,\mathrm{d}t. \tag{4.1}$$

Recall that X_t is a sum of eight products of two creation or annihilation operators corresponding to functions in $G_0(t)$. Using the commutation relation (3.1) between L and $a^\sharp$, and the defining property $L\Omega = 0$ of L, each $Q_{p,n}$ applied to the vacuum Ω can be rewritten as at most $\binom{n}{p} 8^{n+1-p}(2n+2-2p)^p$ products of $2n+2-2p$ creation or annihilation operators corresponding to functions in $\bigcup_{k=0}^p G_k(t)$.

But since $\|a^\sharp(\phi)\| = \|\phi\| \leq M_k$ for all $\phi \in G_k(t)$, such polynomials $Q_{p,n}$ must satisfy

$$\|Q_{p,n} \Omega\| \leq \binom{n}{p} 8^{n+1-p} (2n+2-2p)^p \Big(\max_{0 \leq k \leq p} M_k \Big)^{2n+2-2p}. \tag{4.2}$$

We conclude that $\|L^n X_t \Omega\|^2$ is integrable on $[t_1, t_2]$.

Using the explicit expressions in Fock space, the non-trivial part of $L^n X_t \Omega$ lives in the two-particle subspace of $\Gamma^-(\mathfrak{h} \oplus \mathfrak{h})$ only, and the wave function there (although not a priori of finite norm) is

$$\begin{aligned}
&(\sqrt{\mathbf{1}-T}h_0^n\psi_{\mathrm{imp}}\oplus 0)\wedge(0\oplus\sqrt{T}\,\overline{\psi_f})+(\sqrt{\mathbf{1}-T}\psi_{\mathrm{imp}}\oplus 0)\wedge(0\oplus\sqrt{T}\,h_0^n\overline{\psi_f})\\
&\quad+(\sqrt{\mathbf{1}-T}h_0^n\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}\psi_{\mathrm{imp}})+(\sqrt{\mathbf{1}-T}\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}h_0^n\psi_{\mathrm{imp}})\\
&\quad-(\sqrt{\mathbf{1}-T}h_0^n\mathrm{e}^{\mathrm{i}th}\psi_{\mathrm{imp}}\oplus 0)\wedge(0\oplus\sqrt{T}\mathrm{e}^{-\mathrm{i}th}\overline{\psi_f})-(\sqrt{\mathbf{1}-T}\mathrm{e}^{\mathrm{i}th}\psi_{\mathrm{imp}}\oplus 0)\wedge(0\oplus\sqrt{T}h_0^n\mathrm{e}^{-\mathrm{i}th}\overline{\psi_f})\\
&\quad-(\sqrt{\mathbf{1}-T}h_0^n\mathrm{e}^{\mathrm{i}th}\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}\mathrm{e}^{-\mathrm{i}th}\psi_{\mathrm{imp}})-(\sqrt{\mathbf{1}-T}\mathrm{e}^{\mathrm{i}th}\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}h_0^n\mathrm{e}^{-\mathrm{i}th}\psi_{\mathrm{imp}}).
\end{aligned}$$

All terms except for $(\sqrt{\mathbf{1}-T}h_0^n\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}\psi_{\mathrm{imp}})$ and $-(\sqrt{\mathbf{1}-T}h_0^n\mathrm{e}^{\mathrm{i}th}\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}\mathrm{e}^{-\mathrm{i}th}\psi_{\mathrm{imp}})$ are guaranteed to be of square integrable norm by Lemmas A.1 and A.2. But because $\|L^nX_t\Omega\|^2$ is integrable on $[t_1,t_2]$, so is the term

$$\|(\sqrt{\mathbf{1}-T}h_0^n\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}\psi_{\mathrm{imp}})-(\sqrt{\mathbf{1}-T}h_0^n\mathrm{e}^{\mathrm{i}th}\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}\mathrm{e}^{-\mathrm{i}th}\psi_{\mathrm{imp}})\|^2.$$

Using Lemma A.2 again, we obtain that

$$\int_{t_1}^{t_2}\|(\sqrt{\mathbf{1}-T}h_0^n\psi_f\oplus0)\wedge(0\oplus\sqrt{T}\psi_{\mathrm{imp}})-(\sqrt{\mathbf{1}-T}h_0^n\mathrm{e}^{\mathrm{i}th_0}\psi_f\oplus0)\wedge(0\oplus\sqrt{T}\mathrm{e}^{-\mathrm{i}th}\psi_{\mathrm{imp}})\|^2\,\mathrm{d}t<\infty$$

which yields

$$\int_{t_1}^{t_2}\int_{\mathbf{R}_+}\left|\frac{\mathrm{e}^{\beta e}}{\mathrm{e}^{\beta e}+1}\right|e^{2n}|f(e)|^2\|\sqrt{T}\psi_{\mathrm{imp}}-\mathrm{e}^{\mathrm{i}te}\sqrt{T}\mathrm{e}^{-\mathrm{i}th}\psi_{\mathrm{imp}}\|^2\,\mathrm{d}e\,\mathrm{d}t<\infty. \tag{4.3}$$

By Fubini's theorem,

$$\int_1^{\infty}\left|\frac{\mathrm{e}^{\beta e}}{\mathrm{e}^{\beta e}+1}\right|e^{2n}|f(e)|^2\int_{t_1}^{t_2}\|\sqrt{T}\psi_{\mathrm{imp}}-\mathrm{e}^{\mathrm{i}te}\sqrt{T}\mathrm{e}^{-\mathrm{i}th}\psi_{\mathrm{imp}}\|^2\,\mathrm{d}t\,\mathrm{d}e<\infty. \tag{4.4}$$

Then, by Lemma A.6,

$$\int_1^{\infty}e^{2n}|f(e)|^2\,\mathrm{d}e<\infty. \tag{4.5}$$

But $\int_0^1 e^{2n}|f(e)|^2\,\mathrm{d}e<\infty$ because $e\mapsto e^{2n}$ is bounded on $[0,1]$ and $f\in L^2(\mathbf{R}_+,\mathrm{d}e)$. The implication holds true for n. □

4.2 Extension and optimality in examples of W^*-dynamical systems

As we mentioned earlier, the definition of the energy FS measure extends to some classes of affiliated unbounded perturbations of W^*-dynamical systems. We present two examples in which we do not only get extensions of Theorems 2.4 and 2.10, but stronger equivalences.

Example 4.4. Consider a Bose gas specified by the left glued Araki–Woods representation of the CCR on the Hilbert space $\mathfrak{h} = \mathbf{C} \oplus L^2(\mathbf{R}_+, \mathrm{d}e)$, the unperturbed Hamiltonian $h_0 = \epsilon_0 \oplus \hat{e}$, and the Bose–Einstein density $\rho = (\mathrm{e}^{\beta h_0} - \mathbf{1})^{-1}$. Consider perturbations V affiliated to $\mathfrak{M}^{\mathrm{AW}}$ of the form

$$V = a^{\mathrm{AW}*}(\psi_{\mathrm{imp}}) a^{\mathrm{AW}}(\psi_f) + a^{\mathrm{AW}*}(\psi_f) a^{\mathrm{AW}}(\psi_{\mathrm{imp}}),$$

where $\psi_f = 0 \oplus f \in \operatorname{Dom} \rho^{1/2} \subset \mathfrak{h}$ and $\psi_{\mathrm{imp}} = 1 \oplus 0 \in \mathfrak{h}$. If moreover $f \in \operatorname{Dom} \hat{e}$, by Lemma B.4 the operator $L + V$ is essentially self-adjoint on $\operatorname{Dom} L \cap \operatorname{Dom} V$ and then by Proposition 1.24

$$\tau_V^t(A) := \mathrm{e}^{\mathrm{i}t(L+V)} A \mathrm{e}^{-\mathrm{i}t(L+V)}$$

defines a W^*-dynamics on $\mathfrak{M}^{\mathrm{AW}}$. For this model, which we will refer to as the **bosonic Wigner–Weisskopf model**, we define the heat FS measure $\mathbf{P}_t$ to be the spectral measure of the operator

$$L + V - \mathrm{e}^{\mathrm{i}t(L+V)} V \mathrm{e}^{-\mathrm{i}t(L+V)} = \mathrm{e}^{\mathrm{i}t(L+V)} L \mathrm{e}^{-\mathrm{i}t(L+V)}$$

with respect to the vacuum vector Ω.

Example 4.5. Consider a Bose gas specified by the left glued Araki–Woods representation of the CCR on the Hilbert space $\mathfrak{h} = L^2(\mathbf{R}_+, \mathrm{d}e)$, the unperturbed hamiltonian $h_0 = \hat{e}$, and the Bose–Einstein density $\rho = (\mathrm{e}^{\beta h_0} - \mathbf{1})^{-1}$. Consider a perturbation of the form $\varphi^{\mathrm{AW}}(f)$ for $f \in \operatorname{Dom} \hat{e}^{1/2}$. If moreover $f \in \operatorname{Dom} \hat{e}$, then by Lemma B.5 the operator $L + \varphi^{\mathrm{AW}}(f)$ is essentially self-adjoint on $\operatorname{Dom} L \cap \operatorname{Dom} \varphi^{\mathrm{AW}}(f)$ and by Proposition 1.24 the maps defined by

$$\tau_V^t(W^{\mathrm{AW}}(\phi)) = \mathrm{e}^{\mathrm{i}t(L+\varphi^{\mathrm{AW}}(f))} W^{\mathrm{AW}}(\phi) \mathrm{e}^{-\mathrm{i}t(L+\varphi^{\mathrm{AW}}(f))}$$

for $\phi \in \operatorname{Dom} \rho^{1/2}$ extend to a W^*-dynamics on $\mathfrak{M}^{\mathrm{AW}}$.

For this model, which we refer to as the **van Hove model**, we set the heat FS measure $\mathbf{P}_t$ to be the spectral measure for the operator

$$L + \varphi^{\mathrm{AW}}(f) - \mathrm{e}^{\mathrm{i}t(L+\varphi^{\mathrm{AW}}(f))} \varphi^{\mathrm{AW}}(f) \mathrm{e}^{-\mathrm{i}t(L+\varphi^{\mathrm{AW}}(f))}$$

with respect to the vacuum vector Ω.

Remark 4.6. Our definitions of $\mathbf{P}_t$ in these examples of perturbed W^*-dynamical systems are made by analogy with our abstract definition in the context of C^*-dynamical systems. One may prefer to obtain these measures through a thermodynamic limit of confined systems for which the two-time measurement protocol of Section 2.1.1 is well-defined. We provide a short discussion of how these two approaches are related in Appendix C.

Lemma 4.7. *For the van Hove Hamiltonian system of Example 4.5, the characteristic function $\mathcal{E}_t(\alpha)$ of $\mathbf{P}_t$ has explicit expression*

$$\mathcal{E}_t(\alpha) = \exp\left(-2\int_{\mathbf{R}_+} (\mathrm{i}\sin(e\alpha) + 2\frac{1+\mathrm{e}^{-\beta e}}{1-\mathrm{e}^{-\beta e}}\sin^2(\tfrac{e\alpha}{2}))\sin^2(\tfrac{et}{2})e^{-2}|f(e)|^2\,\mathrm{d}e\right). \tag{4.6}$$

Proof. Using the defining properties of the Liouvillean L, the characteristic function of the energy full statistics measure can be rewritten as

$$\begin{aligned}\mathcal{E}_t(\alpha) &= \langle \mathrm{e}^{-\mathrm{i}t(L+\varphi^{\mathrm{AW}}(f))}\Omega, \mathrm{e}^{\mathrm{i}\alpha L}\mathrm{e}^{-\mathrm{i}t(L+\varphi^{\mathrm{AW}}(f))}\mathrm{e}^{-\mathrm{i}\alpha L}\Omega\rangle \\ &= \langle \mathrm{e}^{-\mathrm{i}t(L+\varphi^{\mathrm{AW}}(f))}\Omega, \mathrm{e}^{-\mathrm{i}t(L+\varphi^{\mathrm{AW}}(\mathrm{e}^{+\mathrm{i}\alpha h_0}f))}\Omega\rangle\,.\end{aligned}$$

Using the Lie–Trotter product formula (see e.g. Theorem VIII.31 in [RS72], also c.f. Lemma B.5) and equation (3.6) we get

$$\begin{aligned}\mathcal{E}_t(\alpha) &= \langle \mathrm{e}^{-\mathrm{i}t(L+\varphi^{\mathrm{AW}}(f))}\Omega, \mathrm{e}^{-\mathrm{i}t(L+\varphi^{\mathrm{AW}}(\mathrm{e}^{+\mathrm{i}\alpha h_0}f))}\Omega\rangle \\ &= \lim_{n\to\infty}\langle (\mathrm{e}^{-\mathrm{i}\frac{t}{n}L}\mathrm{e}^{-\mathrm{i}\frac{t}{n}\varphi^{\mathrm{AW}}(f)})^n\Omega, (\mathrm{e}^{-\mathrm{i}\frac{t}{n}L}\mathrm{e}^{-\mathrm{i}\frac{t}{n}\varphi^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}\alpha h_0}f)})^n\Omega\rangle \\ &= \lim_{n\to\infty}\langle \prod_{k=1}^n \mathrm{e}^{-\mathrm{i}\frac{t}{n}\varphi^{\mathrm{AW}}(\mathrm{e}^{-\mathrm{i}\frac{kt}{n}h_0}f)}\Omega, \prod_{k=1}^n \mathrm{e}^{-\mathrm{i}\frac{t}{n}\varphi^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}\alpha h_0}\mathrm{e}^{-\mathrm{i}\frac{kt}{n}h_0}f)}\Omega\rangle \\ &= \lim_{n\to\infty}\langle \mathrm{e}^{-\mathrm{i}\varphi^{\mathrm{AW}}(\sum_{k=1}^n \frac{t}{n}\mathrm{e}^{-\mathrm{i}\frac{kt}{n}h_0}f)}\Omega, \mathrm{e}^{-\mathrm{i}\varphi^{\mathrm{AW}}(\sum_{k=1}^n \frac{t}{n}\mathrm{e}^{\mathrm{i}\alpha h_0}\mathrm{e}^{-\mathrm{i}\frac{kt}{n}h_0}f)}\Omega\rangle \\ &= \langle \mathrm{e}^{-\mathrm{i}\varphi^{\mathrm{AW}}(\int_0^t \mathrm{e}^{-\mathrm{i}sh_0}f\,\mathrm{d}s)}\Omega, \mathrm{e}^{-\mathrm{i}\varphi^{\mathrm{AW}}(\int_0^t \mathrm{e}^{\mathrm{i}\alpha h_0}\mathrm{e}^{-\mathrm{i}sh_0}f\,\mathrm{d}s)}\Omega\rangle \\ &= \mathrm{e}^{-\frac{\mathrm{i}}{2}\operatorname{Im}\int_{\mathbf{R}_+}(\int_0^t \mathrm{e}^{\mathrm{i}sh_0}\overline{f}\,\mathrm{d}s)(e)(\int_0^t \mathrm{e}^{\mathrm{i}\alpha h_0}\mathrm{e}^{-\mathrm{i}sh_0}f\,\mathrm{d}s)(e)\,\mathrm{d}e}\langle\Omega, \mathrm{e}^{\mathrm{i}\varphi^{\mathrm{AW}}(\int_0^t(\mathbf{1}-\mathrm{e}^{\mathrm{i}\alpha h_0})\mathrm{e}^{-\mathrm{i}sh_0}f\,\mathrm{d}s)}\Omega\rangle \\ &= \mathrm{e}^{-\frac{\mathrm{i}}{2}\int_{\mathbf{R}_+}4\sin^2(\frac{et}{2})\sin(e\alpha)e^{-2}|f(e)|^2\,\mathrm{d}e}\mathrm{e}^{-\frac{1}{4}\int_{\mathbf{R}_+}4\sin^2(\frac{e\alpha}{2})\frac{1+\mathrm{e}^{-\beta e}}{1-\mathrm{e}^{-\beta e}}4\sin^2(\frac{et}{2})e^{-2}|f(e)|^2\,\mathrm{d}e}.\end{aligned}$$

Finally, the evaluation identity (3.8) and a simple rearrangement yield (4.6). □

Proposition 4.8. *For the bosnic Wigner–Weisskopf models of Example 4.4 and the van Hove model of Example 4.5, the following are equivalent*

1. *the* $(2n+2)$*th moment of* $\mathbf{P}_t$ *is uniformly bounded in time,*

$$\sup_{t\in\mathbf{R}} \mathbf{E}_t[|\Delta Q|^{2n+2}] < \infty;$$

2. *there exist a non-trivial time interval* $[t_1, t_2]$ *over which the* $(2n+2)$*th moment of* $\mathbf{P}_t$ *is integrable,*

$$\int_{t_1}^{t_2} \mathbf{E}_t[|\Delta Q|^{2n+2}]\,\mathrm{d}t < \infty;$$

3. *the map* $\mathbf{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}s\hat{e}} f \in L^2(\mathbf{R}_+, \mathrm{d}e)$ *is* n *times differentiable in the* L^2*-norm sense.*

Proof for the bosonic Wigner–Weisskopf model. The fact that the first statement implies the second one is trivial. The fact that the third one implies the first one is immediate from Theorem 2.4.

To prove that the the second statement implies the third one, we proceed by induction on n. The result holds for $n = 0$. Assume it holds for some $n-1 \in \mathbf{N}$ and suppose that there exists a non-trivial interval $[t_1, t_2]$ on which $t \mapsto \mathbf{E}_t[|\Delta Q|^{2n+2}]$ is integrable. Then, $t \mapsto \mathbf{E}_t[|\Delta Q|^{2n}]$ is also integrable by Jensen's inequality and $f \in \operatorname{Dom} \hat{e}^{n-1}$ by the induction hypothesis.

Consider the set

$$G_p(t) := \{(\mathbf{1}-T)^{1/2} h_0^p \mathrm{e}^{\mathrm{i}sh} \psi_f \oplus 0, (\mathbf{1}-T)^{1/2} h_0^p \mathrm{e}^{\mathrm{i}sh} \psi_{\mathrm{imp}} \oplus 0,$$
$$0 \oplus T^{1/2}(-h_0)^p \mathrm{e}^{-\mathrm{i}sh}\psi_f, 0 \oplus T^{1/2}(-h_0)^p \mathrm{e}^{-\mathrm{i}sh}\psi_{\mathrm{imp}}\}_{s=0,t}.$$

By Lemma A.1, these sets are bounded uniformly in $t \in [t_1, t_2]$ for each $p = 0, 1, \ldots, n-1$. We denote their bound by M_p. We also set $X_t := \pi(V) - \pi(\tau_V^t(V))$ and omit the representation maps π.

We first note by expanding $(L + X_t)^n$, that the moment $\mathbf{E}_t[|\Delta Q|^{2n+2}] = \|(L + X_t)^n X_t \Omega\|^2$ can be written as $\|L^n X_t \Omega + \sum_{p=0}^{n-1} Q_{p,n}\Omega\|^2$, where $Q_{p,n}$ is a sum of $\binom{n}{p}$ products containing p (not necessarily adjascent) copies of L and $n+1-p$ (not necessarily adjacent) copies of X_t. Then,

$$\int_{t_1}^{t_2} \|L^n X_t \Omega\|^2\,\mathrm{d}t \le 2\int_{t_1}^{t_2} \mathbf{E}_t[|\Delta Q|^{2n+2}]\,\mathrm{d}t + 2\sum_{p=0}^{n-1}\int_{t_1}^{t_2} \|Q_{p,n}\Omega\|^2\,\mathrm{d}t. \tag{4.7}$$

Recall that X_t is a sum of eight products of two creation or annihilation operators corresponding to functions in $G_0(t)$. Using the commutation relation (3.1) between L and $a^\sharp$, and the defining property $L\Omega = 0$ of L, each $Q_{p,n}$ applied to the vacuum Ω can be rewritten as at most

$\binom{n}{p} 8^{n+1-p}(2n+2-2p)^p$ products of $2n+2-2p$ creation or annihilation operators corresponding to functions in $\bigcup_{k=0}^{p} G_k(t)$.

But since $\|a^\sharp(\phi)\| = \|\phi\| \leq M_k$ for all $\phi \in G_k(t)$, such polynomials $Q_{p,n}$ must satisfy

$$\|Q_{p,n}\Omega\| \leq \binom{n}{p} 8^{n+1-p}(2n+2-2p)^p \Big(\max_{0\leq k\leq p} M_k\Big)^{2n+2-2p}. \tag{4.8}$$

We conclude that $\|L^n X_t \Omega\|^2$ is integrable on $[t_1, t_2]$.

Using the explicit expressions in Fock space, the non-trivial part of $L^n X_t \Omega$ lives in the two-particle subspace of $\Gamma^-(\mathfrak{h}\oplus\mathfrak{h})$ only, and the wave function there (although not a priori of finite norm) is

$$\begin{aligned}
&(\sqrt{\mathbf{1}-T}h_0^n\psi_{\mathrm{imp}}\oplus 0)\wedge(0\oplus\sqrt{T}\,\overline{\psi_f}) + (\sqrt{\mathbf{1}-T}\psi_{\mathrm{imp}}\oplus 0)\wedge(0\oplus\sqrt{T}\,h_0^n\overline{\psi_f})\\
&\quad + (\sqrt{\mathbf{1}-T}h_0^n\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}\psi_{\mathrm{imp}}) + (\sqrt{\mathbf{1}-T}\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}h_0^n\psi_{\mathrm{imp}})\\
&\quad - (\sqrt{\mathbf{1}-T}h_0^n\mathrm{e}^{\mathrm{i}th}\psi_{\mathrm{imp}}\oplus 0)\wedge(0\oplus\sqrt{T}\mathrm{e}^{-\mathrm{i}th}\overline{\psi_f}) - (\sqrt{\mathbf{1}-T}\mathrm{e}^{\mathrm{i}th}\psi_{\mathrm{imp}}\oplus 0)\wedge(0\oplus\sqrt{T}h_0^n\mathrm{e}^{-\mathrm{i}th}\overline{\psi_f})\\
&\quad - (\sqrt{\mathbf{1}-T}h_0^n\mathrm{e}^{\mathrm{i}th}\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}\mathrm{e}^{-\mathrm{i}th}\psi_{\mathrm{imp}}) - (\sqrt{\mathbf{1}-T}\mathrm{e}^{\mathrm{i}th}\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}h_0^n\mathrm{e}^{-\mathrm{i}th}\psi_{\mathrm{imp}}).
\end{aligned}$$

All terms except for $(\sqrt{\mathbf{1}-T}h_0^n\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}\psi_{\mathrm{imp}})$ and $-(\sqrt{\mathbf{1}-T}h_0^n\mathrm{e}^{\mathrm{i}th}\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}\mathrm{e}^{-\mathrm{i}th}\psi_{\mathrm{imp}})$ are guaranteed to be of square integrable norm by Lemmas A.1 and A.2. But because $\|L^n X_t\Omega\|^2$ is integrable on $[t_1, t_2]$, so is the term

$$\|(\sqrt{\mathbf{1}-T}h_0^n\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}\psi_{\mathrm{imp}}) - (\sqrt{\mathbf{1}-T}h_0^n\mathrm{e}^{\mathrm{i}th}\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}\mathrm{e}^{-\mathrm{i}th}\psi_{\mathrm{imp}})\|^2.$$

Using Lemma A.2 again, we obtain that

$$\int_{t_1}^{t_2}\|(\sqrt{\mathbf{1}-T}h_0^n\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}\psi_{\mathrm{imp}}) - (\sqrt{\mathbf{1}-T}h_0^n\mathrm{e}^{\mathrm{i}th_0}\psi_f\oplus 0)\wedge(0\oplus\sqrt{T}\mathrm{e}^{-\mathrm{i}th}\psi_{\mathrm{imp}})\|^2\,\mathrm{d}t < \infty$$

which yields

$$\int_{t_1}^{t_2}\int_{\mathbf{R}_+}\left|\frac{\mathrm{e}^{\beta e}}{\mathrm{e}^{\beta e}+1}\right| e^{2n}|f(e)|^2\|\sqrt{T}\psi_{\mathrm{imp}} - \mathrm{e}^{\mathrm{i}te}\sqrt{T}\mathrm{e}^{-\mathrm{i}th}\psi_{\mathrm{imp}}\|^2\,\mathrm{d}e\,\mathrm{d}t < \infty. \tag{4.9}$$

By Fubini's theorem,

$$\int_1^\infty\left|\frac{\mathrm{e}^{\beta e}}{\mathrm{e}^{\beta e}+1}\right| e^{2n}|f(e)|^2\int_{t_1}^{t_2}\|\sqrt{T}\psi_{\mathrm{imp}} - \mathrm{e}^{\mathrm{i}te}\sqrt{T}\mathrm{e}^{-\mathrm{i}th}\psi_{\mathrm{imp}}\|^2\,\mathrm{d}t\,\mathrm{d}e < \infty. \tag{4.10}$$

By Lemma A.6,

$$\int_1^\infty e^{2n}|f(e)|^2\,\mathrm{d}e < \infty. \tag{4.11}$$

But $\int_0^1 e^{2n}|f(e)|^2\,\mathrm{d}e < \infty$ because $e \mapsto e^{2n}$ is bounded on $[0,1]$ and $f \in L^2(\mathbf{R}_+, \mathrm{d}e)$. The implication holds true for n. □

Proof for the van Hove model. The fact that the first statement implies the second one is trivial.

To prove that the the second statement implies the third one, we proceed by induction on n. The implication is obvious for $n = 0$. Assume the result holds for $n-1 \in \mathbf{N}$, and suppose that there exists $t_1, t_2 \in \mathbf{R}$ with $t_1 < t_2$ such that $t \mapsto \mathbf{E}_t(\Delta Q^{2n+2})$ is integrable on $]t_1, t_2[$. Then, $\mathbf{E}_t(\Delta Q^{2n})$ is finite, and $\mathcal{E}_t(\alpha)$ is $2n$ times differentiable in $\alpha = 0$, for Lebesgue-almost every $t \in]t_1, t_2[$. Then, so is the argument of the exponential in (4.6), which we denote by $2F_t(\alpha)$. By the Leibniz formula for differentiation under the integral sign, for $\alpha \in \mathbf{R}$ small enough,

$$F_t^{(2n)}(\alpha) = (-1)^{n+1}\int_{\mathbf{R}_+}\left(-\mathrm{i}\sin(e\alpha) + \frac{1+\mathrm{e}^{-\beta e}}{1-\mathrm{e}^{-\beta e}}(\cos^2(\tfrac{e\alpha}{2}) - \sin^2(\tfrac{e\alpha}{2}))\right)\sin^2(\tfrac{et}{2})e^{2n-2}|f(e)|^2\,\mathrm{d}e.$$

so that

$$F_t^{(2n+2)}(0) = (-1)^n \lim_{h\to 0}\int_{\mathbf{R}_+}\frac{1+\mathrm{e}^{-\beta e}}{1-\mathrm{e}^{-\beta e}}\left(\frac{2-2\cos^2(\frac{eh}{2})}{e^2h^2} + \frac{2\sin^2(\frac{eh}{2})}{e^2h^2}\right)\sin^2(\tfrac{et}{2})e^{2n}|f(e)|^2\,\mathrm{d}e.$$

By Fatou's lemma, for n odd,

$$\begin{aligned}-F_t^{(2n+2)}(0) &\geq \int_{\mathbf{R}_+}\frac{1+\mathrm{e}^{-\beta e}}{1-\mathrm{e}^{-\beta e}}\liminf_{h\to 0}\left(\frac{2-2\cos^2(\frac{eh}{2})}{e^2h^2} + \frac{2\sin^2(\frac{eh}{2})}{e^2h^2}\right)\sin^2(\tfrac{et}{2})e^{2n}|f(e)|^2\,\mathrm{d}e \\ &= \int_{\mathbf{R}_+}\frac{1+\mathrm{e}^{-\beta e}}{1-\mathrm{e}^{-\beta e}}\sin^2(\tfrac{et}{2})e^{2n}|f(e)|^2\,\mathrm{d}e,\end{aligned}$$

and for n even, similarly,

$$F_t^{(2n+2)}(0) \geq \int_{\mathbf{R}_+}\frac{1+\mathrm{e}^{-\beta e}}{1-\mathrm{e}^{-\beta e}}\sin^2(\tfrac{et}{2})e^{2n}|f(e)|^2\,\mathrm{d}e.$$

In any case,

$$\int_{t_1}^{t_2}\int_1^{\infty}\frac{1+\mathrm{e}^{-\beta e}}{1-\mathrm{e}^{-\beta e}}\sin^2(\tfrac{et}{2})e^{2n}|f(e)|^2\,\mathrm{d}e\,\mathrm{d}t \leq \int_{t_1}^{t_2}|F_t^{(2n+2)}(0)|\,\mathrm{d}t < \infty.$$

Finally, by Fubini's theorem

$$\int_1^{\infty} e^{2n}|f(e)|^2\,\mathrm{d}e < \infty.$$

But $\int_0^1 e^{2n}|f(e)|^2\,\mathrm{d}e < \infty$ because $e \mapsto e^{2n}$ is bounded on $[0,1]$ and $f \in L^2(\mathbf{R}_+, \mathrm{d}e)$. The implication is true for n.

Under the assumption that $f \in \operatorname{Dom}\hat{e}^n$, differentiation under the integral sign in (4.6) yields that $\mathcal{E}_t(\alpha)$ is $2n+2$ times differentiable at $\alpha = 0$, with a derivative that is uniformly bounded in t. This in turn implies that the $(2n+2)$th moment of $\mathbf{P}_t$ is uniformly bounded in t. This shows that the third statement implies the first one. □

Remark 4.9. The induction argument and use of Fatou's lemma above is an adaptation of a common proof of the fact that the $2n$th moment of of a random variable $X : \Omega \to \mathbf{R}$ exists and is finite if and only if its characteristic function $\alpha \mapsto \mathbf{E}(\mathrm{e}^{\mathrm{i}\alpha X})$ is $2n$ times differentiable in $\alpha = 0$; see for example [Shi96, II§12].

Proposition 4.10. *For the bosonic Wigner–Wisskopf model of Example 4.4, if there exists $\gamma_0 > 0$ such that $\psi_f \in \operatorname{Dom} \mathrm{e}^{\frac{1}{2}\gamma_0 h_0}$, then there exists $\gamma_0' > 0$ such that*

$$\mathbf{E}_t[\mathrm{e}^{\gamma_0'|\Delta Q|}] < \infty. \tag{4.12}$$

Lemma 4.11. *If $f \in \hat{e}^n$ for all $n \in \mathbf{N}$, then for each $t \in \mathbf{R}$, there exists a real open interval on which*

$$\mathrm{e}^{\mathrm{i}s(L+X_t)}\mathrm{e}^{-\mathrm{i}sL}\Omega = \Omega + \sum_{n=1}^{N} \mathrm{i}^n \int_0^s \cdots \int_0^{s_{n-1}} \tau^{s_n}(X_t) \cdots \tau^{s_1}(X_t)\Omega \,\mathrm{d}s s_n \cdots \mathrm{d}s_1 + R_N \tag{4.13}$$

where $R_N(s) \to 0$ in norm as $N \to \infty$.

Proof. First, note that for Ψ made of finitely many particles with wave functions in $\operatorname{Dom}(h_0 \oplus -h_0)^n$ for all $n \in \mathbf{N}$, we have

$$\partial_s \mathrm{e}^{\mathrm{i}s(L+X_t)}\mathrm{e}^{-\mathrm{i}sL}\Psi = \mathrm{i}\mathrm{e}^{\mathrm{i}s(L+X_t)}\mathrm{e}^{-\mathrm{i}sL}\tau^s(X_t)\Psi.$$

If $f \in \operatorname{Dom} \hat{e}^n$ for all $n \in \mathbf{N}$, then Lemma A.1 ensures that $\tau^{s_n}(X_t) \cdots \tau^{s_1}(X_t)\Omega$ is still a linear combination of symmetrized tensor products of finitely many one-particle vectors in $\bigcap_{n \in \mathbf{N}} \operatorname{Dom}(h_0 \oplus -h_0)^n$. Then, repeated application of the fundamental theorem of calculus yields that for any $N \in \mathbf{N}$,

$$\langle \Phi, \mathrm{e}^{\mathrm{i}s(L+X_t)}\mathrm{e}^{-\mathrm{i}sL}\Omega \rangle = 1 + \sum_{n=1}^{N} \mathrm{i}^n \int_0^s \cdots \int_0^{s_{n-1}} \langle \Phi, \tau^{s_n}(X_t) \cdots \tau^{s_1}(X_t)\Omega \rangle \,\mathrm{d}s_n \cdots \mathrm{d}s_1 + r_N^\Phi \tag{4.14}$$

for any unit vector $\Phi \in \Gamma^+(\mathfrak{h} \oplus \mathfrak{h})$ where

$$r_N^\Phi = \mathrm{i}^{N+1} \int_0^s \cdots \int_0^{s_N} \langle \Phi, \mathrm{e}^{\mathrm{i}s_{N+1}(L+X_t)}\mathrm{e}^{-\mathrm{i}s_{N+1}L}\tau^{s_{N+1}}(X_t) \cdots \tau^{s_1}(X_t)\Omega \rangle \,\mathrm{d}s_{N+1} \cdots \mathrm{d}s_1.$$

Then,

$$|r_N^\Phi| \leq \frac{|s|^{N+1}}{(N+1)!} \sup_{s_1,\dots,s_{N+1} \in [0,s]} |\langle \Phi, \mathrm{e}^{\mathrm{i}s_{N+1}(L+X_t)}\mathrm{e}^{-\mathrm{i}s_{N+1}L}\tau^{s_{N+1}}(X_t) \cdots \tau^{s_1}(X_t)\Omega \rangle|,$$

but since $\tau^s(X_t)$ is a polynomial with 16 monomials (or 8...) of degree 2 in creation an annihilation of unitary evolutions of functions in $G_0(t)$,

$$|r_N^\Phi| \leq \frac{|s|^{N+1}}{(N+1)!}\sqrt{(2N+3)!}16^{N+1} \sup_{\phi\in G_0(t)} \|\phi\|^{2N+2}$$

Using Lemma A.1, this bound converges to 0 as $N \to \infty$ for real s small enough, uniformly in $\Phi \in \{\Psi \in \Gamma^+(\mathfrak{h}\oplus\mathfrak{h}) : \|\Psi\| = 1\}$. □

Proof of Proposition 4.10. Let

$$F_{t,N} : \{\alpha \in \mathbf{C} : |\operatorname{Im}\alpha| < \tfrac{1}{2}\gamma_0\} \to \Gamma^+(\mathfrak{h}\oplus\mathfrak{h})$$

be defined by the natural analytic extension of the truncated series in (4.13):

$$F_{t,N}(\alpha) := \mathbf{1} + \sum_{n=1}^{N} \mathrm{i}^n \int_0^{\alpha} \cdots \int_0^{\alpha_{n-1}} \langle \Omega, \tau^{\alpha_n}(X_t)\cdots\tau^{\alpha_1}(X_t)\Omega\rangle \, \mathrm{d}\alpha_n \cdots \mathrm{d}\alpha_1,$$

where the integral are taken along straight lines in the complex planes. Note that

$$\begin{aligned}\tau^{\alpha_j}(X_t) = {}& a^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}\alpha_j h_0}\psi_f)a^{\mathrm{AW}*}(\mathrm{e}^{\mathrm{i}\alpha_j h_0}\psi_{\mathrm{imp}}) + a^{\mathrm{AW}*}(\mathrm{e}^{\mathrm{i}\alpha_j h_0}\psi_f)a^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}\alpha_j h_0}\psi_{\mathrm{imp}}) \\ & - a^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}\alpha_j h_0}\mathrm{e}^{\mathrm{i}t(h_0+v)}\psi_f)a^{\mathrm{AW}*}(\mathrm{e}^{\mathrm{i}\alpha_j h_0}\mathrm{e}^{\mathrm{i}t(h_0+v)}\psi_{\mathrm{imp}}) \\ & - a^{\mathrm{AW}*}(\mathrm{e}^{\mathrm{i}\alpha_j h_0}\mathrm{e}^{\mathrm{i}t(h_0+v)}\psi_f)a^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}\alpha_j h_0}\mathrm{e}^{\mathrm{i}t(h_0+v)}\psi_{\mathrm{imp}}).\end{aligned}$$

By hypothesis and Lemma A.8, the functions in the creation and annihilation operators above are indeed in $\operatorname{Dom}\rho^{1/2}$ as long as $|\operatorname{Im}\alpha_j| < \frac{1}{2}\gamma_0$. Because Ω defines a quasi-free state, each $F_{t,N}$ is analytic on $\{\alpha \in \mathbf{C} : |\operatorname{Im}\alpha| < \frac{1}{2}\gamma_0\}$. Lemma 4.11 yields convergence of the sequence $(F_{t,N})_{N\in\mathbf{N}}$ on a subset of $\{\alpha \in \mathbf{C} : |\operatorname{Im}\alpha| < \frac{1}{2}\gamma_0\}$ that has an accumulation point. Also, because

$$\begin{aligned}\left\| \int_0^{\alpha} \cdots \int_0^{\alpha_{n-1}} \langle \Omega, \tau^{\alpha_n}(X_t)\cdots\tau^{\alpha_1}(X_t)\Omega\rangle \, \mathrm{d}\alpha_n \cdots \mathrm{d}\alpha_1 \right\| &\leq \frac{|\alpha|^n}{n!}\|\tau^{\alpha_n}(X_t)\cdots\tau^{\alpha_1}(X_t)\Omega\| \\ &\leq \frac{|\alpha|^n}{n!}\sqrt{(2n+1)!}16^n B_t(\alpha)^{2n},\end{aligned}$$

where by Lemma A.8,

$$B_t(\alpha) := \max_{g\in G_0(t)} \|\mathrm{e}^{\pm \operatorname{Im}\alpha(h_0\oplus -h_0)}g\| < \infty,$$

the sequence $(F_{t,N})_{N\in\mathbf{N}}$ is uniformly bounded on $\{\alpha \in \mathbf{C} : |\operatorname{Im}\alpha| < \frac{1}{2}\gamma_0, |\alpha| < \frac{1}{32}B_t(\frac{\mathrm{i}}{2}\gamma_0)^{-2}\}$. We conclude by the Vitali–Porter convergence theorem that $(F_{t,N})_{N\in\mathbf{N}}$ converges there to an analytic function. Therefore, the arguments used to prove Theorem 2.10 yield the result for $\gamma_0' > 0$ small enough. □

Proposition 4.12. *For the van Hove model of Example 4.5, for $\gamma_0 > 0$, the following are equivalent*

1. *for all $0 < \gamma < \gamma_0$,*
$$\sup_{t\in\mathbf{R}} \mathbf{E}_t[\mathrm{e}^{\gamma|\Delta Q|}] < \infty;$$
2. *for all $0 < \gamma < \gamma_0$, there exist a non-trivial time interval $[t_1(\gamma), t_2(\gamma)]$ such that*
$$\int_{t_1(\gamma)}^{t_2(\gamma)} \mathbf{E}_t[\mathrm{e}^{\gamma|\Delta Q|}]\,\mathrm{d}t < \infty;$$
3. *the map $\mathbf{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}s\hat{e}} f \in L^2(\mathbf{R}_+, \mathrm{d}e)$ extends analytically in the L^2-norm sense to the strip $\{z \in \mathbf{C} : |\mathrm{Im}\, z| < \frac{1}{2}\gamma_0\}$.*

Proof. The fact that the first statement implies the second is trivial.

Assume that the second statement holds and let $0 < \gamma < \gamma_0$. It ensures that the characteristic function $\mathcal{E}_t$ can be extended analytically to $\{\alpha \in \mathbf{C} : |\mathrm{Im}\,\alpha| < \frac{1}{2}(\gamma + \gamma')\}$ for Lebesgue-almost all t in an interval $[t_1, t_2]$, this extension is in particular valid at $\mathrm{i}\gamma$) and by our explicit formule

$$\int_{t_1}^{t_2} \mathrm{e}^{2\int_{\mathbf{R}_+}\left(\sinh(e\gamma)+2\frac{1+\mathrm{e}^{-\beta e}}{1-\mathrm{e}^{-\beta e}}\sinh^2(\frac{e\gamma}{2})\right)\sin^2(\frac{et}{2})e^{-2}|f(e)|^2\,\mathrm{d}e}\,\mathrm{d}t < \infty.$$

Because $0 < x < \mathrm{e}^{2x}$ for all $x > 0$, this and the positivity of the integrand in the above exponential in turn imply that

$$\int_{t_1}^{t_2}\int_1^{\infty}\left(\sinh(e\gamma) + 2\frac{1+\mathrm{e}^{-\beta e}}{1-\mathrm{e}^{-\beta e}}\sinh^2(\tfrac{e\gamma}{2})\right)\sin^2(\tfrac{et}{2})e^{-2}|f(e)|^2\,\mathrm{d}e\,\mathrm{d}t < \infty.$$

By Fubini's theorem, $\int_1^\infty \sinh(e\gamma)e^{-2}|f(e)|^2\,\mathrm{d}e < \infty$. Since $\gamma \in [0, \gamma_0[$ was arbitrary, we conclude that $f \in \mathrm{Dom}\,\mathrm{e}^{\frac{1}{2}\gamma\hat{e}}$ for all $\gamma \in [0, \gamma_0[$. The map $\mathbf{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}s\hat{e}} f \in L^2(\mathbf{R}_+, \mathrm{d}e)$ extends analytically to the strip $\{z \in \mathbf{C} : |\mathrm{Im}\, z| < \frac{1}{2}\gamma_0\}$.

On the other hand, suppose that the map $\mathbf{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}s\hat{e}} f \in L^2(\mathbf{R}_+, \mathrm{d}e)$ extends analytically to the strip $\{z \in \mathbf{C} : |\mathrm{Im}\, z| < \frac{1}{2}\gamma_0\}$. Then for each $0 < \gamma < \gamma_0$

$$\int_{\mathbf{R}_+} \mathrm{e}^{\gamma e}|f(e)|^2\,\mathrm{d}e < \infty. \tag{4.15}$$

Hence, by (4.6), $\mathcal{E}_t$ extends analytically to the strip $\{\alpha \in \mathbf{C} : |\mathrm{Im}\,\alpha| < \gamma_0\}$ with

$$|\mathcal{E}_t(\mathrm{i}\gamma)| \le \mathrm{e}^{2\int_{\mathbf{R}_+}\left(\sinh(e|\gamma|)+2\frac{1+\mathrm{e}^{-\beta e}}{1-\mathrm{e}^{-\beta e}}\sinh^2(\frac{e\gamma}{2})\right)\sin^2(\frac{et}{2})e^{-2}|f(e)|^2\,\mathrm{d}e} \tag{4.16}$$

there. Now, using $x^{-2}\sinh^2 x \le \mathrm{e}^{2|x|}$,

$$\int_0^1 \frac{1+\mathrm{e}^{-\beta e}}{1-\mathrm{e}^{-\beta e}} \sinh^2(\tfrac{e\gamma}{2})\sin^2(\tfrac{et}{2}) e^{-2}|f(e)|^2\,\mathrm{d}e \le \frac{\gamma^2}{4}\mathrm{e}^{\gamma}\int_0^1 \frac{1+\mathrm{e}^{-\beta e}}{1-\mathrm{e}^{-\beta e}}|f(e)|^2\,\mathrm{d}e,$$

which is finite since $f \in \operatorname{Dom}\hat{e}^{-1/2}$ by construction, and

$$\int_1^\infty \frac{1+\mathrm{e}^{-\beta e}}{1-\mathrm{e}^{-\beta e}} \sinh^2(\tfrac{e\gamma}{2})\sin^2(\tfrac{et}{2}) e^{-2}|f(e)|^2\,\mathrm{d}e \le \frac{1+\mathrm{e}^{-\beta}}{1-\mathrm{e}^{-\beta}}\int_1^\infty \mathrm{e}^{\gamma e}|f(e)|^2\,\mathrm{d}e,$$

which is finite by (4.15), and independent of t. The other term in the integral in (4.16) is treated similarly to obtain a bound on $\mathcal{E}_t(\mathrm{i}\gamma)$ that is independent of t. But then

$$\sup_{t\in\mathbf{R}}\int_{\mathbf{R}} \mathrm{e}^{\gamma|\Delta Q|}\,\mathrm{d}\mathbf{P}_t \le \sup_{t\in\mathbf{R}}|\mathcal{E}_t(\mathrm{i}\gamma)| + \sup_{t\in\mathbf{R}}|\mathcal{E}_t(-\mathrm{i}\gamma)|$$

and the first statement follows.

□

Conclusion

The quantum theory of statistical mechanics and thermodynamics exhibits phenomena that have no counterpart in the classical theory and that are still not completely understood. In this thesis and in [BPR17], we show how the regularity of interactions is determining in the control of fluctuations about a form of the first law of thermodynamics, contributing to a better understanding such quantum phenomena. This regularity condition is fundamentally different from the perhaps more intuitive "weak interaction assumption" that is sometimes found in the theoretical physics literature. This criterion plays no role in classical analogues of the systems we have studied: a section of the paper [BPR17] is devoted to this point.

In more technical terms, we have considered perturbations V of a dynamics τ and the associated heat full statistics measure $\mathbf{P}_t$ motivated by a two-time measurement protocol of the energy, and showed how n-differentiability of the map $t \mapsto \tau^t(V)$ yields control on the moments $\mathbf{E}_t[|\Delta Q|^{2n+2}]$ of $\mathbf{P}_t$ and how analytic extendibility controls $\mathbf{E}_t[\mathrm{e}^{\gamma|\Delta Q|}]$. This was done in a general setup for a C^*-dynamical system with bounded perturbation. We have treated an example of such dynamical systems and two examples of W^*-dynamical systems with affiliated unbounded perturbation, in which we can obtain converses to the general results. In the van Hove model, which describes the interaction of a free field of bosons with classical (infinitely heavy) electric charges, the regularity of the map $t \mapsto \tau^t(V)$ depends crucially on the chosen regularisation of the charge distribution. These results can therefore be of interest to physicists for their choice of charge regularisation in modelling. More generally, the results provide a general rule of thumb for the extent to which the interaction can be neglected in energy fluctuations.

One can hope to obtain statements in the same spirit for more general classes of W^*-dynamical systems with affiliated perturbation using in a more refined way the tools of perturbation theory found in [DJP03]. Also, the study of this problem for the case of a time-dependent perturbation could reveal additional structure, but would require the use of significantly different techniques.

Appendix A

One-body operator estimates

Some of the results of Section 4 require technical lemmas regarding the domains of powers of one-particle Hamiltonians h_0 and $h = h_0 + v$. In this section, we provide norm estimates for their repeated application to some sets of vectors in $\mathfrak{h}$. Throughout this section, $h_0 = \epsilon_0 \oplus \hat{e}$ on $\mathfrak{h} = \mathbf{C} \oplus L^2(\mathbf{R}_+, \mathrm{d}e)$, and the one-particle perturbation v is of the form $\psi_{\mathrm{imp}} \langle \psi_f, \cdot \rangle + \psi_f \langle \psi_{\mathrm{imp}}, \cdot \rangle$ for $\psi_f = 0 \oplus f \in \mathbf{C} \oplus L^2(\mathbf{R}_+, \mathrm{d}e)$ and $\psi_{\mathrm{imp}} = 1 \oplus 0 \in \mathbf{C} \oplus L^2(\mathbf{R}_+, \mathrm{d}e)$.

First note that $\operatorname{Dom} h_0 = \operatorname{Dom} h$ and, if $f \in \operatorname{Dom} \hat{e}^{n-1}$, $\operatorname{Dom} h^n = \operatorname{Dom} h_0^n$. Moreover, we have the following lemmas.

Lemma A.1. *Suppose $f \in \operatorname{Dom} \hat{e}^r$ for some $r \in \mathbf{R}$. If $\phi \in \operatorname{Dom} h_0^r$, then $\mathrm{e}^{\mathrm{i}th}\phi \in \operatorname{Dom} h_0^r$ and there exists an affine function $t \mapsto C_r(t)$ such that*

$$\|h_0^r \mathrm{e}^{\mathrm{i}th} \phi\| \leq C_r^{\phi}(t). \tag{A.1}$$

Proof. Let $\psi \in \operatorname{Dom} h_0^r$. Using Duhamel's formula and the triangle inequality

$$|\langle h_0^r \psi, \mathrm{e}^{\mathrm{i}th} \phi \rangle| \leq \|\psi\| \|h_0^r \phi\| + \int_0^t |\langle \psi, \mathrm{e}^{\mathrm{i}(t-s)h_0} h_0^r v \mathrm{e}^{\mathrm{i}sh} \psi \rangle| \,\mathrm{d}s.$$

Clearly, $f \in \operatorname{Dom} \hat{e}^r$ implies $\psi_f \in \operatorname{Dom} h_0^r$ and $\|h_0^r v\| < \infty$. By the Cauchy–Schwarz inequality,

$$|\langle h_0^r \psi, \mathrm{e}^{\mathrm{i}th} \phi \rangle| \leq \|\psi\| (\|h_0^r \phi\| + t \|h_0^r v\| \|\phi\|).$$

Take $C_r^{\phi}(t) := \|h_0^r \phi\| + t \|h_0^r v\| \|\phi\|$. □

Lemma A.2. *Let $n \in \mathbf{N}^*$ and assume $f \in \operatorname{Dom} \hat{e}^{n-1}$. If $\phi \in \operatorname{Dom} h_0$ then $(\mathrm{e}^{\mathrm{i}th_0} - \mathrm{e}^{\mathrm{i}th})\phi \in \operatorname{Dom} h_0^n$ and there exists an affine function $t \mapsto L_n^\phi(t)$ such that*

$$\|h_0^n(\mathrm{e}^{\mathrm{i}th_0} - \mathrm{e}^{\mathrm{i}th})\phi\| \leq L_n^\phi(t). \tag{A.2}$$

Proof. Let $\chi \in \operatorname{Dom}(h_0^n)$. Using Duhamel's formula,

$$\begin{aligned}\langle h_0^n\chi, (\mathrm{e}^{\mathrm{i}th_0} - \mathrm{e}^{\mathrm{i}th})\phi\rangle =& \mathrm{i}\int_0^t \langle h_0^n\chi, \mathrm{e}^{\mathrm{i}(t-s)h_0} v \mathrm{e}^{\mathrm{i}sh}\phi\rangle\, \mathrm{d}s,\\ =& \mathrm{i}\int_0^t \langle \chi, (\partial_s e^{\mathrm{i}(t-s)h_0}) h_0^{n-1} v \mathrm{e}^{\mathrm{i}sh}\phi\rangle\, \mathrm{d}s,\end{aligned}$$

Since $\phi \in \operatorname{Dom}(h)$, $s \mapsto \langle \psi_{\mathrm{imp}}, \mathrm{e}^{\mathrm{i}sh}\phi\rangle$ and $s \mapsto \langle \psi_f, \mathrm{e}^{\mathrm{i}sh}\phi\rangle$ are in $C^1(\mathbf{R})$. Similarly, $s \mapsto \langle \chi, \mathrm{e}^{\mathrm{i}(t-s)h_0} h_0^{n-1}\psi_{\mathrm{imp}}\rangle$ and $s \mapsto \langle \chi, \mathrm{e}^{\mathrm{i}(t-s)h_0} h_0^{n-1}\psi_f\rangle$ are in $C^1(\mathbf{R})$ using $\chi \in \operatorname{Dom}(h_0^n)$ for the second function. Integrating by parts,

$$\int_0^t \langle \chi, (\partial_s \mathrm{e}^{\mathrm{i}(t-s)h_0}) h_0^{n-1} v \mathrm{e}^{\mathrm{i}sh}\phi\rangle\, \mathrm{d}s = [\langle \chi, e^{\mathrm{i}(t-s)h_0} h_0^{n-1} v \mathrm{e}^{\mathrm{i}sh}\phi\rangle]_0^t - \mathrm{i}\int_0^t \langle \chi, e^{\mathrm{i}(t-s)h_0} h_0^{n-1} v h \mathrm{e}^{\mathrm{i}sh}\phi\rangle\, \mathrm{d}s.$$

Since $\psi_{\mathrm{imp}} \in \operatorname{Dom}(h_0^{n-1})$ and $\psi_f \in \operatorname{Dom}(h_0^{n-1})$,

$$\|h_0^{n-1} v\| < \infty \quad \text{and} \quad \|h_0^{n-1} v h\| < \infty.$$

The inequality,

$$|\langle h_0^n \chi, (\mathrm{e}^{\mathrm{i}th_0} - \mathrm{e}^{\mathrm{i}th})\phi\rangle| \leq \|\chi\|\ \|\phi\| \big(2\|h_0^{n-1} v\| + t\|h_0^{n-1} v h\|\big).$$

yields the Lemma with the affine bound

$$L_n^\phi(t) = \|\phi\| \big(2\|h_0^{n-1} v\| + t\|h_0^{n-1} v h\|\big).$$

□

Lemma A.3. *Let $n \in \mathbf{N}^*$ and suppose $f \in \operatorname{Dom} \hat{e}^{n-1}$. If $\phi \in \operatorname{Dom} h_0^n$, then for all $p \in \{1, \ldots, n\}$,*

$$\sup_t \|h_0^p \mathrm{e}^{\mathrm{i}th}\phi\| < \infty \tag{A.3}$$

and

$$\sup_t \|h_0^{p-1/2} \mathrm{e}^{\mathrm{i}th}\phi\| < \infty. \tag{A.4}$$

In the notation of Lemma A.1, the affine functions $C_p^\phi(t)$ and $C_{p-1/2}^\phi(t)$ can then be made constant.

Proof. By a concavity argument, it suffices to show (A.3) for $p = n$. Since $f \in \operatorname{Dom} \hat{e}^{n-1}$ implies $\operatorname{Dom} h^n = \operatorname{Dom} h_0^n$, the commutation $[\mathrm{e}^{\mathrm{i}th}, h^n] = 0$ yields directly that $\mathrm{e}^{\mathrm{i}th}\phi \in \operatorname{Dom} h_0^n$.

Now, by the triangle inequality

$$\|h_0^n \mathrm{e}^{\mathrm{i}th}\phi\| \leq \|(h_0^n - h^n)\mathrm{e}^{\mathrm{i}th}\phi\| + \|h^n\phi\|. \tag{A.5}$$

Developing $h_0^n - h^n$, it is a polynomial in operator variables $h_0^k v h_0^l$ with $k, l \in \{0, \ldots, n-1\}$. Since $f \in \operatorname{Dom}(\hat{e}^{n-1})$, $h_0^k v h_0^l$ is bounded for any $k, l \in \{0, \ldots, n-1\}$. Hence, $h_0^n - h^n$ is bounded and $\sup_t \|h_0^n \mathrm{e}^{\mathrm{i}th}\phi\| < \infty$ follows from $\phi \in \operatorname{Dom}(h^n)$. □

Lemma A.4. *Suppose $\|\hat{e}^{-1/2} f\| \neq \epsilon_0^{1/2}$ and $\phi \in \operatorname{Dom} h_0^{-1/2}$, then*

$$\sup_{t\in\mathbf{R}} \|h_0^{-1/2}\mathrm{e}^{\mathrm{i}th}\phi\| < \infty.$$

In the notation of Lemma A.1, the affine function, $C_{-1/2}^{\phi}(t)$ can then be made constant.

Proof. Since $\ker h_0 = \{0\}$, $h_0^{\pm 1/2} h_0^{\mp 1/2} = \mathbf{1}$ and

$$h_0^{-1/2} h h_0^{-1/2} = \mathbf{1} + (\epsilon_0^{-1/2}\psi_{\mathrm{imp}})\,\langle h_0^{-1/2}\psi_f, \cdot\rangle + (h_0^{-1/2}\psi_f)\,\langle \epsilon_0^{-1/2}\psi_{\mathrm{imp}}, \cdot\rangle\,.$$

Since $\psi_f \in \operatorname{Dom} h_0^{-1/2}$, $h_0^{-1/2} h h_0^{-1/2}$ is bounded and so is $h_0^{-1/2} h^{1/2}$.

The operator $h_0^{-1/2} h h_0^{-1/2}$ has spectrum

$$\operatorname{sp}(h_0^{-1/2} h h_0^{-1/2}) = \{1, 1 \pm \epsilon_0^{-1/2}\|\hat{e}^{-1/2} f\|\}.$$

Let $\psi \in \ker h$, $\psi \neq 0$. Then $\psi \in \operatorname{Dom}(h) = \operatorname{Dom}(h_0)$ and therefore, $\psi \in \operatorname{Dom}(h_0^{1/2})$. Then $\psi' = h_0^{1/2}\psi$ is an eigenvector of $h_0^{-1/2} h h_0^{-1/2}$ of eigenvalue 0. Since, $\|\hat{e}^{-1/2} f\| \neq \epsilon_0^{1/2}$ implies 0 is not an eigenvalue of $h_0^{-1/2} h h_0^{-1/2}$. It follows that $\|\hat{e}^{-1/2} f\| \neq \epsilon_0^{1/2}$ implies $\ker h = \{0\}$ and $h^{\pm 1} h^{\mp 1} = \mathbf{1}$.

Hence, $\|\hat{e}^{-1/2} f\| \neq \epsilon_0^{1/2}$ implies that $(h_0^{-1/2} h h_0^{-1/2})^{-1} = h_0^{1/2} h^{-1} h_0^{1/2}$ is bounded, and so is $h_0^{1/2} h^{-1/2}$.

The inequality

$$|\langle \psi, h^{-1/2}\phi\rangle| \leq \|h_0^{1/2} h^{-1/2}\|\|\psi\|\|h_0^{-1/2}\phi\|.$$

for all $\psi \in \operatorname{Dom} h_0^{-1/2}$ shows that $\phi \in \operatorname{Dom} h^{-1/2}$. Therefore,

$$\|h_0^{-1/2}\mathrm{e}^{\mathrm{i}th}\phi\| = \|h_0^{-1/2} h^{1/2}\mathrm{e}^{\mathrm{i}th} h^{-1/2}\phi\| \leq \|h_0^{-1/2} h^{1/2}\|\|h^{-1/2}\phi\|$$

which is independent of t. $\square$

Remark A.5. As shown in the proof, assumption $\|\hat{e}^{-1/2} f\| \neq \epsilon_0^{1/2}$ implies $\ker h = \{0\}$ which might be a more recognizable assumption to the reader. Assuming $f \in \operatorname{Dom}(\hat{e}^{-1})$, the implication is an equivalence. If $f \notin \operatorname{Dom}(\hat{e}^{-1})$ and $\|\hat{e}^{-1/2} f\| = \epsilon_0^{1/2}$, 0 may be in the singular continuous spectrum of h.

Let us introduce a piece of notation: for $e_0 \in \mathbf{R}_+^*$, and $x \oplus g \in \mathbf{C} \oplus L^2(\mathbf{R}_+, \mathrm{d}e)$ let

$$\mathbf{1}_{\hat{e} \geq e_0}(x \oplus g) := x \oplus g_{\geq e_0},$$

where

$$g_{\geq e_0}(e) = \begin{cases} 0 & \text{if } e < e_0, \\ g(e) & \text{else,} \end{cases}.$$

Obviously, $\mathbf{1}_{\hat{e} \geq e_0}^2 = \mathbf{1}_{\hat{e} \geq e_0}$ and $\|\mathbf{1}_{\hat{e} \geq e_0}\| = 1$.

Lemma A.6. *Let A be a continuous function of h_0 with* $\ker A = \{0\}$ *and* $\operatorname{Dom} A = \operatorname{Dom} \rho^r$ *for some $r \in \mathbf{R}$. Assume furthermore that $\psi \in \operatorname{Dom} A \setminus \{0\}$ is not an eigenvector of h. Then, for any $t_1 < t_2$,*

$$\inf_{e \in [1, \infty)} \int_{t_1}^{t_2} \|A(1 - \mathrm{e}^{\mathrm{i}t(\bar{h} - e)} \psi)\| \, \mathrm{d}t > 0.$$

Remark A.7. This lemma applies with the square root of the Fermi–Dirac and Bose–Einstein densities, $T^{1/2}$ and $\rho^{1/2}$, playing the role of A (with $r = 0$ and $r = -1/2$ respectively).

Proof. Let $y : \mathbf{R} \to \mathbf{R}_+$ be the function defined by

$$y(x) = \int_{t_1}^{t_2} 1 - \mathrm{e}^{\mathrm{i}tx} \, \mathrm{d}t.$$

The function y is continuous and bounded by $2(t_2 - t_1)$ on $\mathbf{R}$ and $y(x) = 0 \Leftrightarrow x = 0$. Since $(1 - \mathrm{e}^{\mathrm{i}t(\bar{h} - e)})\psi \in \operatorname{Dom} A$ by Lemma A.1, Fubini's Theorem implies,

$$F(e) := \left\| \int_{t_1}^{t_2} A(1 - \mathrm{e}^{\mathrm{i}t(\bar{h} - e)})\psi \, \mathrm{d}t \right\| = \|Ay(\bar{h} - e)\psi\|.$$

Since $\ker A = \{0\}$, it follows from $y(x) = 0 \Leftrightarrow x = 0$ that $F(e) = 0 \Leftrightarrow \bar{h}\psi = e\psi$ which contradicts the assumptions of the Lemma. Hence on any compact $E \subset \mathbf{R}$,

$$\inf_{e \in E} F(e) > 0.$$

Hence $\inf_{e\in\mathbf{R}} F(e) = 0$ only if $\liminf_{e\to\infty} F(e) = 0$.

Since $\mathbf{R} \ni x \mapsto y(x-e)$ converges simply to $t_2 - t_1$ when $e \to \infty$ and y is bounded by Lebesgue's dominated convergence Theorem,

$$\lim_{e\to\infty} y(\bar{h} - e)\psi = (t_2 - t_1)\psi$$

in norm. Let $e_0 > 0$ be small enough such that

$$\|\mathbf{1}_{\hat{e}\geq e_0}\psi\| > 0.$$

Considering,

$$\|Ay(\bar{h} - e)\psi\| \geq \|\mathbf{1}_{\hat{e}\geq e_0} Af(\bar{h} - e)\psi\|$$

and using $\|\mathbf{1}_{\hat{e}\geq e_0} A\| < \infty$,

$$\liminf_{e\to\infty} F(e) \geq \lim_{e\to\infty} \|\mathbf{1}_{\hat{e}\geq e_0} Ay(\bar{h} - e)\psi\| = (t_2 - t_1)\|A\mathbf{1}_{\hat{e}\geq e_0}\psi\|.$$

The triviality of the kernel of A and the triangle inequality yield the result. $\square$

Lemma A.8. *If there exists $\gamma_0 > 0$ such that $f \in \operatorname{Dom} \mathrm{e}^{\frac{1}{2}\gamma_0\hat{e}}$, then for any $x \in [0, \frac{1}{2}[$,*

$$\max_{g\in G_0(t)} \|\mathrm{e}^{\pm x\gamma_0(h_0\oplus -h_0)} g\| < \infty.$$

Proof. We want to bound

$$\|\mathrm{e}^{\pm x\gamma_0(h_0\oplus -h_0)} g\| \leq \|g\| + \sum_{n\geq 1} \frac{x^n\gamma_0^n}{n!} \|(h_0 \oplus -h_0)^n g\| \tag{A.6}$$

for $g \in G_0(t)$. We consider two cases which exhaust the elements of $G_0(t)$.

Case 1: elements of the form $\rho^{1/2}\mathrm{e}^{\mathrm{i}sh}\psi$ with $\psi = \psi_f, \psi_{\mathrm{imp}}$ and $s = 0, t$. For $n \geq 1$, note that

$$\|h_0^n \rho^{1/2}\mathrm{e}^{\mathrm{i}sh}\psi\| \leq \|\rho^{1/2} h_0^{1/2}\| \|h_0^{n-1/2}\mathrm{e}^{\mathrm{i}sh}\psi\|$$

using a Duhamel expansion

$$\leq \|\rho^{1/2} h_0^{1/2}\| \|h_0^{n-1/2}\psi\| + s\|\rho^{1/2} h_0^{1/2}\| \|h_0^{n-1/2} v\| \|\psi\|$$

using Jensen (and the spectral theorem) for the function $y \mapsto y^{\frac{n-1/2}{n}}$

$$\leq \|\rho^{1/2} h_0^{1/2}\| \|h_0^n \psi\|^{\frac{n-1/2}{n}} \|\psi\| + s\|\rho^{1/2} h_0^{1/2}\| \|h_0^n v\|^{\frac{n-1/2}{n}} \|\psi\|^2.$$

To check that the series (A.6) converges, it is no loss of generality to assume $\|h_0^n \psi\|$ and $\|h_0^n v\|$ are at least 1. Then,

$$\begin{aligned}
&\|e^{\pm x\gamma_0 h_0} \rho^{1/2} e^{ish} \psi\| \\
&\quad \leq \|\rho^{1/2} e^{ish} \psi_f\| + \|\rho^{1/2} h_0^{1/2}\| \|\psi\| \Big(\sum_{n\geq 1} \frac{x^n \gamma_0^n}{n!} \|h_0^n \psi_f\| + s\|\psi\| \sum_{n\geq 1} \frac{x^n \gamma_0^n}{n!} \|h_0^n v\| \Big).
\end{aligned}$$

Using

$$\|h_0^n \psi\| \leq \|e^{\frac{1}{2}\gamma_0 h_0} \psi\| \frac{n! 2^n}{\gamma_0^n},$$

(note that $\|e^{\frac{1}{2}\gamma_0 h_0} \psi_{\mathrm{imp}}\| < \infty$ and $\|e^{\frac{1}{2}\gamma_0 h_0} \psi_f\| < \infty$ by hypothesis), we get

$$\begin{aligned}
&\|e^{\pm x\gamma_0 h_0} \rho^{1/2} e^{ish} \psi\| \\
&\quad \leq \|\rho^{1/2} e^{ish} \psi\| + \|\rho^{1/2} h_0^{1/2}\| \|\psi\| \Big(\sum_{n\geq 1} 2^n x^n \|e^{\frac{1}{2}\gamma_0 h_0} \psi\| + s\|\psi\| \sum_{n\geq 1} 2^n x^n \|e^{\frac{1}{2}\gamma_0 h_0} v\| \Big) \\
&\quad = \|\rho^{1/2} e^{ish} \psi\| + \|\rho^{1/2} h_0^{1/2}\| \|\psi\| \frac{2x}{1-2x} (\|e^{\frac{1}{2}\gamma_0 h_0} \psi\| + s\|\psi\| \|e^{\frac{1}{2}\gamma_0 h_0} v\|)
\end{aligned}$$

and the bound on (A.6) follows.

Case 2: elements of the form $(1+\rho)^{1/2} e^{ish} \psi$ with $\psi = \psi_f, \psi_{\mathrm{imp}}$ and $s = 0, t$. First split the norm of $h_0^n (1+\rho)^{1/2} \psi$ according to $\mathbf{1}_{\hat{e}<1}$:

$$\begin{aligned}
\|h_0^n (1+\rho)^{1/2} e^{ish} \psi\| &\leq \|\mathbf{1}_{\hat{e}<1} h_0^n (1+\rho)^{1/2} e^{ish} \psi\| + \|\mathbf{1}_{\hat{e}\geq 1} h_0^n (1+\rho)^{1/2} e^{ish} \psi\| \\
&\leq \|\mathbf{1}_{\hat{e}<1} (1+\rho)^{1/2} h_0^{1/2}\| \|h_0^{n-1/2} e^{ish} \psi\| + \|\mathbf{1}_{\hat{e}\geq 1} (1+\rho)^{1/2}\| \|h_0^n e^{ish} \psi\|
\end{aligned}$$

and repeat the argument of Case 1.

$\square$

Appendix B

Self-adjointess

In this appendix, we take care of the technicalities regarding self-adjointness of operators that appear in the bosonic Wigner–Weisskopf and van Hove models. This analysis is centred around the celebrated Nelson commutator theorem [Nel72], which is discussed for example in [RS75, §X.5].

We recall a few basic definitions. An operator A with dense domain $\operatorname{Dom} A$ is called **essentially self-adjoint** if its operator closure is self-adjoint. A dense subspace D in the domain of a core operator G is called a **core** for G if the operator closure of the restriction of G to D equals G. In pratice there are several ways to check that D is a core for G. For example, if G is self-adjoint, by Theorem VIII.11 of [RS72], it suffices to check that $D \subseteq \operatorname{Dom} G$ is dense and that D is invariant under $\mathrm{e}^{\mathrm{i}sG}$ for all $s \in \mathbf{R}$.

Theorem B.1 (Nelson's commutator theorem)**.** *Let $G \geq \mathbf{1}$ be a self-adjoint operator and let A be a symmetric operator on D which is a core for G. If the conditions*

1. *there exists a constant $c > 0$ such that $\|A\Psi\| \leq c\|G\Psi\|$ for all $\Psi \in D$;*
2. *there exists a constant $d > 0$ such that $|\langle A\Psi, G\Psi\rangle - \langle G\Psi, A\Psi\rangle| \leq d\|G^{1/2}\Psi\|^2$ for all $\Psi \in D$;*

are satisfied, then A is essentially self-adjoint on D and its closure is essentially self-adjoint on any other core for G.

We will apply this theorem with $G := \mathbf{1} + \mathrm{d}\Gamma(\varepsilon + \mathbf{1})$, where $\varepsilon := h_0 \oplus h_0$ and

$$D := \bigcup_{N \in \mathbf{N}} \bigoplus_{n \leq N} (\operatorname{Dom} \varepsilon)^{\otimes_{\mathrm{alg}} n} \subset \Gamma^+(\mathfrak{h} \oplus \mathfrak{h}),$$

the space of vectors with a bounded number of particles in the domain of ε. We first check core the conditions.

Lemma B.2. *The subspace D is a core for G.*

Proof. The subspace D is dense. Because $\mathrm{e}^{\mathrm{i}sG} = \mathrm{e}^{\mathrm{i}s}\Gamma(\mathrm{e}^{\mathrm{i}s(1+\varepsilon)})$ does not change the number of particles and because $\mathrm{e}^{\mathrm{i}s(1+\varepsilon)}$ preserves the domain of ε, $\mathrm{e}^{\mathrm{i}sG}$ preserves D. We conclude that D is a core for G. □

Lemma B.3. *In the bosonic Wigner–Weisskopf model, the subspace* $\operatorname{Dom} L \cap \operatorname{Dom} V$ *is a core for* G.

Proof. First note that $\operatorname{Dom} L \cap \operatorname{Dom} V$ is dense. Morevoer, for $\Psi \in \operatorname{Dom} L$

$$\begin{aligned} \|L\mathrm{e}^{\mathrm{i}sG}\Psi\| &= \|\,\mathrm{d}\Gamma(h_0 \oplus -h_0)\Gamma(\mathrm{e}^{\mathrm{i}s(1+(h_0\oplus h_0))})\Psi\| \\ &= \|\Gamma(\mathrm{e}^{\mathrm{i}s(1+(h_0\oplus h_0))})\,\mathrm{d}\Gamma(h_0 \oplus -h_0)\Psi\| \\ &= \|L\Psi\| \end{aligned}$$

and for $\Psi \in \operatorname{Dom} V$, by the commutation relation (3.2),

$$\begin{aligned} \|a^*(g_1)a(g_2)\mathrm{e}^{\mathrm{i}sG}\Psi\| &= \|a^*(g_1)a(g_2)\Gamma(\mathrm{e}^{\mathrm{i}s(1+\varepsilon)})\Psi\| \\ &= \|a^*(g_1)\Gamma(\mathrm{e}^{\mathrm{i}s(1+\varepsilon)})a(\mathrm{e}^{\mathrm{i}s(1+\varepsilon)}g_2)\Psi\| \\ &= \|a^*(\mathrm{e}^{-\mathrm{i}s(1+\varepsilon)}g_1)a(\mathrm{e}^{\mathrm{i}s(1+\varepsilon)}g_2)\Psi\| \end{aligned}$$

(and similarly for terms of the form $a(g_1)a^*(g_2)$). We conclude that $\mathrm{e}^{\mathrm{i}sG}$ preserves $\operatorname{Dom} L \cap \operatorname{Dom} V$ and hence that $\operatorname{Dom} L \cap \operatorname{Dom} V$ is a core for G. □

Lemma B.4. *If $f \in$* $\operatorname{Dom} \hat{e} \cap \operatorname{Dom} \hat{e}^{-1/2}$, *then $L + V$ in the bosonic Wigner–Weisskopf model is essentially self-adjoint on* $\operatorname{Dom} L \cap \operatorname{Dom} V$.

Proof. First, recall that

$$\|a^{\sharp_1}(g_1)a^{\sharp_2}(g_2)(\mathbf{1}+N)^{-1}\| \leq 2\|g_1\|\|g_2\|$$

for all $g_1, g_2 \in \mathfrak{h}$. With g_1 and g_2 chosen amongst $\rho^{1/2}\psi_f$, $\rho^{1/2}\psi_{\mathrm{imp}}$, $(\mathbf{1}+\rho)^{1/2}\overline{\psi_f}$, $(\mathbf{1}+\rho)^{1/2}\psi_{\mathrm{imp}}$, and $\Psi \in D \subset \operatorname{Dom} N \subset \operatorname{Dom} G$, this yields

$$\begin{aligned}\|V\Psi\| \leq& \|V(\mathbf{1}+N)^{-1}\|\|(\mathbf{1}+N)\Psi\| \\ &\leq 16\max\{\|(\mathbf{1}+\rho)^{1/2}\psi_f\|, \|(\mathbf{1}+\rho)^{1/2}\psi_{\mathrm{imp}}\|\}^2\|(\mathbf{1}+N)\Psi\| \\ &\leq c'\|G\Psi\|.\end{aligned}$$

But because L is bounded with respect to G, we get Condition 1 of Nelson's commutator theorem.

From the commutation relations (3.1), for vectors $\Psi \in D \subset \operatorname{Dom} G$ and $g_1, g_2 \in \operatorname{Dom} \varepsilon$ (this is the case for $\rho^{1/2}\psi_f$, $\rho^{1/2}\psi_{\mathrm{imp}}$, $(\mathbf{1}+\rho)^{1/2}\overline{\psi_f}$ and $(\mathbf{1}+\rho)^{1/2}\psi_{\mathrm{imp}}$ because $f \in \operatorname{Dom} \hat{e} \cap \operatorname{Dom} \hat{e}^{-1/2}$),

$$\begin{aligned}&|\langle a^{\sharp_1}(g_1)a^{\sharp_2}(g_2)\Psi, G\Psi\rangle - \langle G\Psi, a^{\sharp_1}(g_1)a^{\sharp_2}(g_2)\Psi\rangle| \\ &\qquad \leq |\langle \Psi, a^{\sharp_1}((\varepsilon+\mathbf{1})g_1)a^{\sharp_2}(g_2)\Psi\rangle| + |\langle \Psi, a^{\sharp_1}(g_1)a^{\sharp_2}((\varepsilon+\mathbf{1})g_2)\Psi\rangle|.\end{aligned}$$

But then

$$\begin{aligned}|\langle \Psi, a^{\sharp_1}((\varepsilon+\mathbf{1})g_1)a^{\sharp_2}(g_2)\Psi\rangle| &\leq \|(\varepsilon+\mathbf{1})g_1\|\|g_2\|\langle \Psi, (\mathbf{1}+N)\Psi\rangle \\ &\leq \|(\varepsilon+\mathbf{1})g_1\|\|g_2\|\|G^{1/2}\Psi\|^2\end{aligned}$$

and

$$|\langle \Psi, a^{\sharp_1}(g_1)a^{\sharp_2}((\varepsilon+\mathbf{1})g_2)\Psi\rangle| \leq \|g_1\|\|(\varepsilon+\mathbf{1})g_2\|\|G^{1/2}\Psi\|^2.$$

Since

$$\langle L\Psi, G\Psi\rangle - \langle G\Psi, L\Psi\rangle = 0$$

for all $\Psi \in D$, we also have Condition 2. Therefore, by Nelson's commutator theorem and our two last lemmas on D,

$$L + V = L + a^{\mathrm{AW}^*}(\psi_{\mathrm{imp}})a^{\mathrm{AW}}(\psi_f) + a^{\mathrm{AW}^*}(\psi_f)a^{\mathrm{AW}}(\psi_{\mathrm{imp}})$$

is essentially self-adjoint on $\operatorname{Dom} L \cap \operatorname{Dom} V$. $\square$

Lemma B.5. *If $f \in \operatorname{Dom} \hat{e} \cap \operatorname{Dom} \hat{e}^{-1/2}$, then the operator $L + \varphi^{AW}(f)$ in the van Hove model is essentially self-adjoint on* $\operatorname{Dom} L \cap \varphi^{AW}(f)$.

Proof. The result follows from an application of Nelson's commutator theorem, as in the proof of Lemma B.4 □

Appendix C

A note on the thermodynamic limit

In this appendix, we justify furthermore the definition of the heat full statistics measure by obtaining the measure from a thermodynamic limit for the van Hove Hamiltonian system.

For $\delta > 0$, consider a family $(P_\delta^{(L)})_{L\in\mathbf{N}}$ of finite-rank orthogonal projections on $L^2(\mathbf{R}_+, \mathrm{d}e)$ that converge strongly to $\mathbf{1}_\delta \equiv \mathbf{1}_{L^2([\delta,\infty),\mathrm{d}e)}$ and such that $P_\delta^{(L)} h_0 P_\delta^{(L)}$ converges to $h_0 \mathbf{1}_\delta$ in the strong resolvent sense as $L \to \infty$. Such a sequence is typically obtained in position space by restriction to a box of size of order L, imposing convenient boundary condition and frequency cutoffs. The choice of boundary condition involves a certain amount of arbitrariness, whose effects however disappear in the thermodynamic limit $L \to \infty$.

Let $\mathbf{P}_{t,\delta}^{(L)}$ the unique probability measure with characteristic function

$$\mathcal{E}_{t,\delta}^{(L)}(\alpha) := \omega_\delta^{(L)}(\mathrm{e}^{\mathrm{i}t(H_{0,\delta}^{(L)}+\varphi(P_\delta^{(L)}f))}\mathrm{e}^{\mathrm{i}\alpha H_{0,\delta}^{(L)}}\mathrm{e}^{-\mathrm{i}t(H_{0,\delta}^{(L)}+\varphi(P_\delta^{(L)}f))}\mathrm{e}^{-\mathrm{i}\alpha H_{0,\delta}^{(L)}}).$$

This measure corresponds to a two-time measurement of the energy $H_{0,\delta}^{(L)} = \mathrm{d}\Gamma(h_{0,\delta}^{(L)})$ for the initial state $\omega_\delta^{(L)} : A \mapsto \frac{1}{Z}\operatorname{tr}(\mathrm{e}^{-\beta H_{0,\delta}^{(L)}} A)$, and the perturbation $\varphi(P_\delta^{(L)} f)$.

The CCR over $P_\delta^{(L)} L^2(\mathbf{R}_+, \mathrm{d}e)$ associated to the Bose–Einstein density $\rho_\delta^{(L)}$ obtained from the Hamiltonian $h_{0,\delta}^{(L)} := P_\delta^{(L)} h_0 P_\delta^{(L)}$ can all be realised in the common space $\Gamma^+(L^2(\mathbf{R}_+, \mathrm{d}e) \oplus L^2(\mathbf{R}_+, \mathrm{d}e))$, just like the Araki–Woods representation of the CCR over $L^2(\mathbf{R}_+, \mathrm{d}e)$ associated to the Bose–Einstein density ρ obtained from h_0.

Lemma C.1. *As $L \to \infty$ and then $\delta \downarrow 0$, the measures $\mathbf{P}_{t,\delta}^{(L)}$ defined from the above two-time measurement converge weakly to the measure $\mathbf{P}_t$ defined through a perturbation of the Liouvillean for the extended system as in Definition 2.1.*

Proof. Using the Araki–Woods representation, which is unitarily equivalent to the standard representation,

$$\begin{aligned}
\mathcal{E}_{t,\delta}^{(L)}(\alpha) &= \langle \Omega, \mathrm{e}^{\mathrm{i}t(\mathrm{d}\Gamma(h_{0,\delta}^{(L)}\oplus - h_{0,\delta}^{(L)})+\varphi^{\mathrm{AW}}(P_\delta^{(L)}f))}\mathrm{e}^{\mathrm{i}\alpha\,\mathrm{d}\Gamma(h_{0,\delta}^{(L)}\oplus - h_{0,\delta}^{(L)})}\mathrm{e}^{-\mathrm{i}t(\mathrm{d}\Gamma(h_{0,\delta}^{(L)}\oplus - h_{0,\delta}^{(L)})+\varphi^{\mathrm{AW}}(P_\delta^{(L)}f))}\Omega\rangle \\
&= \langle \mathrm{e}^{-\mathrm{i}t(\mathrm{d}\Gamma(h_{0,\delta}^{(L)}\oplus - h_{0,\delta}^{(L)})+\varphi^{\mathrm{AW}}(P_\delta^{(L)}f))}\Omega, \\
&\qquad \mathrm{e}^{-\mathrm{i}t(\mathrm{d}\Gamma(h_{0,\delta}^{(L)}\oplus - h_{0,\delta}^{(L)})+\mathrm{e}^{\mathrm{i}\alpha\,\mathrm{d}\Gamma(h_{0,\delta}^{(L)}\oplus - h_{0,\delta}^{(L)})}\varphi^{\mathrm{AW}}(P_\delta^{(L)}f)\mathrm{e}^{-\mathrm{i}\alpha\,\mathrm{d}\Gamma(h_{0,\delta}^{(L)}\oplus - h_{0,\delta}^{(L)})}}\Omega\rangle.
\end{aligned}$$

Expanding the vector on the left of this inner product in a Dyson series,

$$\begin{aligned}
&\mathrm{e}^{-\mathrm{i}t(\mathrm{d}\Gamma(h_{0,\delta}^{(L)}\oplus - h_{0,\delta}^{(L)})+\varphi^{\mathrm{AW}}(P_\delta^{(L)}f))}\Omega \\
&\quad = -\sum_{k\in\mathbf{N}} \mathrm{i}^k \int\cdots\int_{0\le s_k\le\cdots\le s_1\le t} \varphi^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}s_k h_{0,\delta}^{(L)}}P_\delta^{(L)}f)\cdots\varphi^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}s_1 h_{0,\delta}^{(L)}}P_\delta^{(L)}f)\Omega\,\mathrm{d}s_k\cdots\mathrm{d}s_1 \\
&\quad = \mathrm{e}^{-\mathrm{i}t(\mathrm{d}\Gamma(h_0\mathbf{1}_\delta\oplus -h_0\mathbf{1}_\delta)+\varphi^{\mathrm{AW}}(\mathbf{1}_\delta f))}\Omega \\
&\qquad -\sum_{k\in\mathbf{N}}\sum_{1\le j\le k} \mathrm{i}^k \int\cdots\int_{0\le s_k\le\cdots\le s_1\le t} \varphi^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}s_k h_0}\mathbf{1}_\delta f)\cdots(\varphi^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}s_j h_{0,\delta}^{(L)}}P_\delta^{(L)}f)-\varphi^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}s_j h_0}\mathbf{1}_\delta f)) \\
&\qquad\qquad \cdots\varphi^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}s_1 h_{0,\delta}^{(L)}}P_\delta^{(L)}f)\Omega\,\mathrm{d}s_k\cdots\mathrm{d}s_1
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathrm{e}^{-\mathrm{i}t(\mathrm{d}\Gamma(h_{0,\delta}^{(L)}\oplus - h_{0,\delta}^{(L)})+\varphi^{\mathrm{AW}}(P_\delta^{(L)}f))}\Omega - \mathrm{e}^{-\mathrm{i}t(\mathrm{d}\Gamma(h_0\mathbf{1}_\delta\oplus -h_0\mathbf{1}_\delta)+\varphi^{\mathrm{AW}}(\mathbf{1}_\delta f))}\Omega\| \\
&\quad \le \sum_{k\in\mathbf{N}}\sum_{1\le j\le k}\frac{t^k}{k!}\|\varphi^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}s_k h_0}\mathbf{1}_\delta f)\cdots\varphi^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}s_j h_{0,\delta}^{(L)}}P_\delta^{(L)}f-\mathrm{e}^{\mathrm{i}s_j h_0}\mathbf{1}_\delta f)\cdots\varphi^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}s_1 h_{0,\delta}^{(L)}}P_\delta^{(L)}f)\Omega\| \\
&\quad \le \sum_{k\in\mathbf{N}} k\frac{t^k}{k!}\sqrt{(k+1)!}\,4^k(\|\sqrt{1+\rho}\,\mathbf{1}_\delta f\|+\|\sqrt{1+\rho_\delta^{(L)}}\,P_\delta^{(L)}f\|)^{k-1} \\
&\qquad\qquad \sup_{1\le s\le t}\|\mathrm{e}^{\mathrm{i}sh_{0,\delta}^{(L)}}\sqrt{1+\rho_\delta^{(L)}}P_\delta^{(L)}f-\mathrm{e}^{\mathrm{i}sh_0}\sqrt{1+\rho}\,\mathbf{1}_\delta f\|.
\end{aligned}$$

The other vector in the inner product is estimated similarly. Because the Borel function $e\mapsto \mathrm{e}^{\mathrm{i}se}(1+\frac{\mathrm{e}^{-\beta e}}{1-\mathrm{e}^{-\beta e}})^{1/2}\chi_{[\delta,\infty)}(e)$ is bounded and because s-r-$\lim_{L\to\infty} P_\delta^{(L)}h_0P_\delta^{(L)} = h_0\mathbf{1}_\delta$ by construction, each summand converges to 0 as $L\to\infty$. Therefore, by Lebesgue dominated convergence, the characteristic functions $\mathcal{E}_{t,\delta}^{(L)}$ obtained from the two-time measurement of $H_{0,\delta}^{(L)}$ converge pointwise to $\mathcal{E}_{t,\delta}$ from the abstract Definition 2.1 as $L\to\infty$.

Similarly,

$$\|\mathrm{e}^{-\mathrm{i}t(\mathrm{d}\Gamma(h_0\mathbf{1}_\delta\oplus -h_0\mathbf{1}_\delta)+\varphi^{\mathrm{AW}}(\mathbf{1}_\delta f))}\Omega - \mathrm{e}^{-\mathrm{i}t(\mathrm{d}\Gamma(h_0\oplus -h_0)+\varphi^{\mathrm{AW}}(f))}\Omega\|$$

$$\leq \sum_{k\in\mathbf{N}} k\frac{t^k}{k!}\sqrt{(k+1)!}\,4^k 2^{k-1}(\|\sqrt{1+\rho}\,f\|)^{k-1}\sup_{1\leq s\leq t}\|\mathrm{e}^{\mathrm{i}sh_0}\sqrt{1+\rho}\,\mathbf{1}_\delta f - \mathrm{e}^{\mathrm{i}sh_0}\sqrt{1+\rho}\,f\|.$$

Because $\hat{e}^{-1/2}f \in L^2(\mathbf{R}_+, \mathrm{d}e)$, each summand converges to 0 as $\delta \downarrow 0$ and the characteristic functions $\mathcal{E}_{t,\delta}$ converge pointwise to $\mathcal{E}_t$ as $\delta \downarrow 0$ by Lebesgue dominated convergence. □

Remark C.2. For the Wigner–Weisskopf model, the same procedure may be applied, but the degree of the polynomials in creation and annihilation operators is doubled, yielding weaker bounds of the form

$$\|\mathrm{e}^{-\mathrm{i}t(\mathrm{d}\Gamma(h_{0,\delta}^{(L)}\oplus -h_{0,\delta}^{(L)})+\varphi^{\mathrm{AW}}(P_\delta^{(L)}f))}\Omega - \mathrm{e}^{-\mathrm{i}t(\mathrm{d}\Gamma(h_{0,\delta}\oplus -h_{0,\delta})+V_\delta^{\mathrm{AW}})}\Omega\|$$

$$\leq C\sum_{k\in\mathbf{N}} k\frac{t^k}{k!}\sqrt{(2k+3)!}\,16^k\|\sqrt{1+\rho}\,f\|^{k-1}$$

$$\sup_{1\leq s\leq t}\|\mathrm{e}^{\mathrm{i}sh_{0,\delta}^{(L)}}\sqrt{1+\rho_\delta^{(L)}}P_\delta^{(L)}f - \mathrm{e}^{\mathrm{i}sh_0}\sqrt{1+\rho}\,\mathbf{1}_\delta f\|.$$

The series is summable only for t small enough. Using the same techniques as in the linear case the sequence of characteristic only shows that the functions $\mathcal{E}_{t,\delta}^{(L)}$ converge pointwise to $\mathcal{E}_t$ defined via the perturbation of the Liouvillean.

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