# Singularities of Schubert Varieties

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**Abstract**

Let $G$ be a semi-simple linear algebraic group defined over an algebraically closed field and $B$ a Borel subgroup. A Schubert variety is the closure of an orbit of the group $B$ in the flag variety $G/B$. The present thesis studies the algebraic curves with a torus action in general and in the case of Schubert varities. It also presents two proofs of Peterson's Theorem and describes the singular locus of Schubert varieties in terms of the Peterson map.

**Résumé**

Soient $G$ un groupe algébrique linéaire semi-simple défini sur un corps algébriquement fermé et $B$ un sous-groupe de Borel de $G$. Nous appelons variété de Schubert l'adhérence d'une orbite de $B$ dans la variété de drapeaux $G/B$. Dans ce travail nous étudions les courbes algébriques avec une action d'un tore, d'abord dans un contexte général et ensuite dans le cas des variétés de Schubert. Nous présentons deux preuves du Théorème de Peterson et décrivons les parties singulières des variétés de Schubert selon l'application de Peterson.

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# Contents

# Introduction

The aim of this thesis is to give a review of the main results on Schubert varieties and a detailed exposition of the new results about the singularities and their nature, mainly due to Carrell, Kuttler and Peterson (see [CK99]). The approach uses the so called Peterson translate, which is the degeneracy of the tangent space to the Schubert variety along some curve. This leads to an algorithm for finding all the singularities of a given Schubert variety.

The results presented here are all over an algebraically closed field of characteristic $p \geq 0$, where in [CK99] they are over the complex field. Let $G$ be a semisimple connected algebraic group and $B$ a Borel subgroup. The homogeneous space $G/B$ is a projective variety with a left action of the group $G$. It is known that the orbits of the Borel subgroup $B$ on $G/B$ are parametrized by the elements of the Weyl group $W = N_G(T)/T$, where $T$ is a maximal torus in $B$. We denote them by $X_\omega$ for $\omega \in W$. Define the Schubert variety $S_\omega$ to be the Zariski closure of a $B$-orbit $X_\omega$ corresponding to $\omega \in W$. Being the closure of a $B$-orbit, the variety $S_\omega$ is a disjoint union of $B$-orbits. The question is: for which $x \in W$ does the corresponding $B$-orbit $X_x$ lie in the singular locus of $S_\omega$?

Let $x \in W$ be such that $X_x \subseteq S_\omega$. Assume that for all $y \in W$ such that $X_x \subsetneq S_y \subseteq S_\omega$, the $B$-orbit $X_y$ lies in the smooth locus of $S_\omega$. Let $X$ be a closed connected $T$-stable curve lying in $G/B$ such that $X_x \cap C \neq \emptyset$ and $C \cap X_y \neq \emptyset$ for some $y \in W$ such that $X_x \subsetneq S_y \subseteq S_\omega$. We will show that $C \cap X_x$ is a single point. We define the Peterson translate along $C$ to be the degeneracy of the tangent space of $S_\omega$ along $C \cap X_y$ to the unique point of $C$ lying in $X_x$. Peterson's Theorem says that the Peterson translate for all such curves are the same if and only if the $B$-orbit $X_x$ lies in the smooth locus of $S_\omega$.

The thesis is divided as follows. Chapter I contains some basic results from the theory of algebraic groups and Lie algebras. Chapter II concerns the well-known concepts involving combinatorics in the Weyl group such as the Bruhat decomposition, Bruhat-Chevalley order, reduced decomposition, the Bott-Samelson variety, etc. A study of varieties and curves with an action of a torus is contained in Chapter III. Finally we present the main results, some examples and applications in Chapter IV.

There exists many other results in this field. Two very important papers of Khazdan and Lusztig [KL79a, KL79b] study the smoothness of Schubert varieties in terms of the Hecke algebra. Kumar in [Kum96] gives a smoothness condition in terms of the nil Hecke algebras.

# Chapter I

# Preliminaries

We start by a general overview of the basic concepts from the theory of linear algebraic groups and Lie algebras. For the first subject, we are basing ourself on classical works in this field, namely the books of Borel [Bor91], Humphreys [Hum81] and less frequently Springer [Spr81]. For the theory of Lie algebras and root systems we follow the other book of Humphreys [Hum97] and Bourbaki [Bou68]. The material and the notation are fairly standard, therefore the reader with a basic knowledge of this subject can jump directly to the next chapter.

## 1 Semisimple and Unipotent Elements

Let $G$ be a linear algebraic group defined over an algebraically closed field $K$. We can assume that $G$ is a subgroup of some $GL(V)$ (see [Hum81, 8.6], [Bor91, 5.6]). We say that $g \in G$ is **semisimple** (resp. **unipotent**) if $g$ is diagonalizable in $GL(V)$ (resp. $g$ has eigenvalues equal to 1). Similarly, if $\mathfrak{g}$ is a Lie algebra, then we say that $x \in \mathfrak{g}$ is **semisimple** (resp. **nilpotent**) if $\operatorname{ad} x$ is a diagonalizable endomorphism of $\mathfrak{g}$ (resp. $\operatorname{ad} x$ has eigenvalues equal to 0).

A group is called a **torus** if it is isomorphic to the diagonal subgroup $D(n, K)$

of $GL(n, K)$ for some $n$, or equivalently, if it is a connected algebraic group consisting of semisimple elements. Any representation of a torus splits in a direct sum of one dimensional ones (see [Bor91, 8.5], [Hum81, 16.2], [Spr81, 2.5 and 6.11]). A torus $T$ in $G$ of maximal dimension (or equivalently a torus not properly included in any other) is called a **maximal torus**. Maximal tori in a connected algebraic group are conjugate.

A closed subgroup consisting of unipotent elements is called **unipotent subgroup** (see [Bor91, 11.10], [Hum81, 22.2], [Spr81, 3.3]). When $G$ is connected, each semisimple element of $G$ lies in some maximal torus, and each unipotent element of $G$ lies in some maximal connected unipotent subgroup (see [Bor91, 11.10], [Hum81, 22.2]). Similarly in $\mathfrak{g}$, an element $x \in \mathfrak{g}$ is semisimple if and only if it lies in a Lie algebra of a maximal torus of $G$ (or in **toral subalgebra** of $\mathfrak{g}$, that is a subalgebra consisting of semisimple elements), while $x$ is nilpotent if and only if it lies in a Lie algebra of a closed unipotent subgroup of $G$ ([Bor91, 14.26]).

An algebraic group $G$ has a unique maximal closed connected normal solvable subgroup $\mathcal{R}G$ called the **radical**, and a unique maximal closed connected normal unipotent subgroup $\mathcal{R}_uG$ called the **unipotent radical**. We say that $G$ is **semisimple** (resp. **reductive**) if $\mathcal{R}G$ (resp. $\mathcal{R}_uG$) is trivial. Note that a semisimple group is reductive.

## 2 Borel Subgroups

Perhaps the most important objects in the study of the structure of algebraic groups are Borel subgroups. They generalize the concept of the subgroup of triangular matrices in $GL(n, K)$. In this section we state some important results about Borel subgroups (see [Bor91, 11], [Hum81, Ch. VIII], [Spr81, Ch. 7]). Assume that $G$ is a connected reductive algebraic group.

**Definition A.** A Borel subgroup $B$ of $G$ is a maximal connected solvable subgroup of $G$.

Let $B$ be a Borel subgroup of $G$, and consider the homogeneous space $G/B$. It is isomorphic to a closed $G$-orbit in a flag variety of some vector space, and since the flag varieties are projective, $G/B$ must be projective. The **flag variety** $\mathcal{B}$ of $G$ is the set of all Borel subgroups of $G$. The Borel subgroups are conjugate in $G$ and if $B \in \mathcal{B}$ then the normalizer $N_G(B)$ is exactly $B$. Using these facts we can identify the flag variety $\mathcal{B}$ with the homogeneous space $G/B$.

Maximal tori of $G$ and maximal tori of the Borel subgroups correspond. If $T$ is a maximal torus of a Borel subgroup $B$, then $B = T \ltimes B_u$ where $B_u$ is the nilpotent subgroup of $B$ consisting of all unipotent elements and $B_u$ is equal to the commutator group $(B, B)$. The set of all Borel subgroups containing a maximal torus $T$ is denoted by $\mathcal{B}^T$. The group $G$ is generated by all $B \in \mathcal{B}^T$ (see [Bor91, 13.7]). For a fixed maximal torus $T$ in $B$, there exits a unique Borel subgroup $B^-$ such that $B^- \cap B = T$ called the **Borel subgroup opposite** $B$ (see [Hum81, 26.2C]).

# 3 Parabolic Subgroups

A subgroup $P$ of $G$ is called a **parabolic subgroup** if the homogeneous space $G/P$ is complete (if and only if it is a projective variety). If $P$ contains a Borel subgroup $B$ then $G/B \to G/P$ is a surjective morphism from a complete variety, so $G/P$ is complete. Conversely, by the Fixed Point Theorem ([Hum81, 21.2], [Bor91, 10.4]), a Borel subgroup $B$ has a fixed point in the complete variety $G/P$, so some conjugate of $B$ lies in $P$. This shows that a subgroup of $G$ is parabolic if and only if it contains a Borel subgroup.

# 4 Root Space Decomposition

Let $G$ be a linear algebraic group defined over an algebraically closed field $K$. Denote by $K^*$ the multiplicative group in $K$. Let $T$ be a maximal torus in $G$. A homomorphism $T \to K^*$ is called a **character** of $T$. The set of all characters of $T$, denoted by $X(T)$, is called the **character group** of $T$. The adjoint action of $T$ on the Lie algebra $\mathfrak{g}$, $\mathrm{Ad} : T \to \mathrm{Aut}(\mathfrak{g})$, is a morphism of algebraic groups. Therefore $\mathrm{Ad}(T)$ is a torus in $\mathrm{Aut}(\mathfrak{g})$ and we can decompose $\mathfrak{g}$ as a direct sum of weight spaces of $\mathrm{Ad}(T)$.

$$\mathfrak{g} = \bigoplus_{\alpha \in X(T)} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{Ad}(t)(x) = \alpha(t)x. \forall t \in T\}$.

*Remark A.* It is usual to denote the operation of $X(T)$ in an additive way. This is due to an identification of the character group $X(T)$ with a lattice in the dual space $\mathfrak{t}^*$ given by the differential. More precisely, the algebra $\mathfrak{t}$ acts on $\mathfrak{g}$ by the ad-action (the morphism $\mathrm{ad} : \mathfrak{g} \to \mathfrak{g}$ is the differential of $Ad : \mathfrak{g} \to \mathfrak{g}$ and defines an action of $\mathfrak{g}$ on itself). Moreover this ad-action is diagonalizable and we have a weight decomposition of $\mathfrak{g}$ similar to the weight decomposition with respect to the Ad-action of $T$. Note that the differential of a character $\alpha \in X(T)$ is an element of the dual space $\mathfrak{t}^*$. It is clear that the weight space for the Ad-action of $T$ corresponding to the character $\alpha \in X(T)$ is the same as the weight space for the ad-action of $\mathfrak{t}$ corresponding to $d\alpha \in \mathfrak{t}^*$. It follows also that for $\alpha \in X(T)$, $h \in \mathfrak{t}$ and $x \in \mathfrak{g}_\alpha$, we have $[h, x] = (d\alpha)(h)x$.

The non-zero characters $\alpha \in X(T)$ for which $\mathfrak{g}_\alpha \neq 0$ are called **roots** and the set of them, denoted by $\Phi$, is called the **root system** (since we are using the additive notation for $X(T)$, by non-zero characters we mean characters which are not uniformly 1). We call the space $\mathfrak{g}_\alpha$ the **root space corresponding to** $\alpha \in \Phi$. Note that the weight space $\mathfrak{g}_0$ is the **infinitesimal centralizer** of $T$:

$\mathfrak{g}_0 = \mathfrak{c}_\mathfrak{g}(T) = \{x \in \mathfrak{g} \mid \mathrm{Ad}(t)(x) = x, \forall t \in T\}$. When $G$ is a reductive group, $\mathfrak{c}_\mathfrak{g}(T)$ turns out to be equal to $\mathfrak{t}$, the Lie algebra of $T$. Moreover $\mathfrak{t}$ becomes a Cartan subalgebra of $\mathfrak{g}$ (i.e. a nilpotent algebra equal to its normalizer in $\mathfrak{g}$) and the decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

becomes a Cartan decompostion of $\mathfrak{g}$ where the root spaces are one dimensional (see [Hum97, 8.1]). Denote by $[.,.]$ the lie bracket on $\mathfrak{g}$. For $\alpha$, $\beta \in \Phi$ we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. If moreover $\alpha + \beta \in \Phi$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. Now if $x \in \mathfrak{g}_\alpha$, then for any $y \in \mathfrak{g}$, and any integer $n$, $\mathrm{ad}^n(x)(y) \in \bigoplus_{\gamma \in \Phi} \mathfrak{g}_{n\alpha+\gamma}$ which is 0 for $n$ big enough. This shows that elements of the root spaces are all nilpotent (by definition $x \in \mathfrak{g}$ is **nilpotent** if $\mathrm{ad}\, x$ is a nilpotent endomorphism of $\mathfrak{g}$, that is there is an integer $n$ such that $\mathrm{ad}^n x = 0$).

In fact the root system $\Phi$ is an abstract root system in $X(T) \otimes_\mathbb{Z} \mathbb{R}$, in the sense of the next section (see [Hum81, 14.8], [Bor91, 27]).

# 5 Abstract Root System

Let $V$ be a finite dimensional vector space over $\mathbb{R}$. Suppose $(\ ,\ )$ is a nondegenerate symmetric bilinear form on $V$. Let $\alpha \in V$ and define the **reflexion relative to** $\alpha$ to be a linear map $\sigma_\alpha : V \to V$ given by:

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

for $\beta \in V$. Let write $\langle \beta, \alpha \rangle$ for $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. Let $V$ be a finite dimensional vector space over $\mathbb{R}$, and $\Phi$ a subset of $V$. We say that $\Phi$ is a (reduced) **root system in** $V$ if

1. $\Phi$ is finite, don't contain 0 and generates $V$,

2. $\Phi$ is closed under all reflections $\sigma_\alpha$ with $\alpha \in \Phi$,

3. if $\alpha \in \Phi$, then the only multiple of $\alpha$ in $\Phi$ are $\alpha$ and $-\alpha$,

4. if $\alpha$, $\beta \in \Phi$, then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

The elements of $\Phi$ are called **roots** and the dimension of $V$ is called the **rank** of the root system $\Phi$. Let $\alpha$, $\beta \in \Phi$. Then there is a simple expression relating the angle between $\alpha$ and $\beta$ and the integers $\langle\alpha,\beta\rangle$ and $\langle\beta,\alpha\rangle$, namely:

$$\begin{aligned}\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle &= \frac{4(\beta,\alpha)^2}{(\alpha,\alpha)(\beta,\beta)}\\ &= 4\cos^2\theta\end{aligned}$$

where $\theta$ is the angle between $\alpha$ and $\beta$. Using the hypothesis that $\langle\alpha,\beta\rangle$ and $\langle\beta,\alpha\rangle$ are integers and the fact that cos takes only values between $-1$ and 1, we deduce the following properties:

1. $4\cos^2\theta \in \{0,1,2,3,4\}$,

2. $4\cos^2\theta = 4 \Leftrightarrow \alpha = \pm\beta$,

3. $\langle\alpha,\beta\rangle$ and $\langle\beta,\alpha\rangle$ have the same sign.

Let $\alpha$ and $\beta$ be two roots such that $\alpha \neq \pm\beta$ and $(\alpha,\alpha) \leq (\beta,\beta)$. Then the only possibilities for the angles between $\alpha$ and $\beta$ and for the values of $\langle\beta,\alpha\rangle$ and $\langle\alpha,\beta\rangle$ are:

| $\langle\alpha,\beta\rangle$ | $\langle\beta,\alpha\rangle$ | $\theta$ | $\frac{(\beta,\beta)}{(\alpha,\alpha)}$ |
|---|---|---|---|
| 0 | 0 | $\pi/2$ | undetermined |
| 1 | 1 | $\pi/3$ | 1 |
| $-1$ | $-1$ | $2\pi/3$ | 1 |
| 1 | 2 | $\pi/4$ | 2 |
| $-1$ | $-2$ | $3\pi/4$ | 2 |
| 1 | 3 | $\pi/6$ | 3 |
| $-1$ | $-3$ | $5\pi/6$ | 3 |

(I.1)

**Proposition A.** *Let $\alpha$, $\beta \in \Phi$.*

1. *If $\langle \alpha, \beta \rangle > 0$ then $\alpha - \beta$ is a root unless $\alpha = \beta$.*

2. *If $\langle \alpha, \beta \rangle < 0$ then $\alpha + \beta$ is a root unless $\alpha = -\beta$.*

*Proof.* We use the table I.1. If $\langle \alpha, \beta \rangle > 0$, then either $\langle \alpha, \beta \rangle = 1$ or $\langle \beta, \alpha \rangle = 1$. In the first case, $\sigma_\beta(\alpha) = \alpha - \beta \in \Phi$. In the second case, $\sigma_\alpha(\beta) = \beta - \alpha \in \Phi$. But $\Phi = -\Phi$. and both of the cases implies that $\alpha - \beta \in \Phi$. The second statement is proved similarly. □

For more details see [Hum97, Ch. III] or [Bou68, Ch. VI].

# 6 Root Strings

As an application of Proposition 5 A, consider two nonproportional roots $\alpha$ and $\beta \in \Phi$. Look at the set of all roots of the form $\alpha + n\beta$ where $n$ is an integer. Let $p$ and $q$ be the bigest nonnegative integers such that $\alpha + p\beta \in \Phi$ and $\alpha - q\beta \in \Phi$. If $\alpha + i\beta$ is not a root for some $i \in \mathbb{Z}$, $-q < i < p$, then there exist $-q < s < t < p$ such that $\alpha + s\beta$ and $\alpha + t\beta$ are roots but $\alpha + (s+1)\beta$ and $\alpha + (t-1)\beta$ are not roots. Proposition 5 A implies that $\langle \alpha + s\beta, \beta \rangle \geq 0$ and $\langle \alpha + t\beta, \beta \rangle \leq 0$, and substracting the two we get $\langle (t-s)\beta, \beta \rangle \leq 0$ which is a contradiction. This shows that $\alpha + j\beta$ are roots for $j \in [-q, p]$. The set of all roots of this form is called the **$\beta$-string through $\alpha$**.

We can obtain an upper limit for the **length of a $\beta$-string** through $\alpha$ (which is by definition its number of roots). It is easy to see that the reflection $\sigma_\beta$ reverses the $\beta$-string, in particular $\sigma_\beta(\alpha + p\beta) = \alpha - q\beta$. But the left side is $\alpha - \langle \alpha, \beta \rangle \beta - p\beta$, which yields $q - p = \langle \alpha, \beta \rangle$. If the integer $p$ is greater than zero, we can replace $\alpha$ by $\alpha + p\beta$. Now $p = 0$ and $q = \langle \alpha, \beta \rangle \leq 3$. Hence the length of any string is at most 4.

# 7 The Weyl Group

Let $T$ be a maximal torus of a connected reductive group $G$. Let $W = N_G(T)/T$ and call this group the **Weyl group**. The very first property of the Weyl group is its simply transitive action on the set of $T$-stable Borel subgroups $\mathcal{B}^T$. Let $B_1$, $B_2 \in \mathcal{B}^T$. By the conjugacy theorem of Borel subgroups (see Section 2), there exists $x \in G$ such that $xB_2x^{-1} = B_1$. Then $T$ and $xTx^{-1}$ are both maximal tori of $B_1$, hence they are conjugate: there exists $y \in B_1$ such that $yxTx^{-1}y^{-1} = T$. This implies that $yx \in N_G(T)$ and since $N_G(B_1) = B_1$, $yxB_2x^{-1}y^{-1} = yB_1y^{-1} = B_1$ and the coset of $yx$ in $W$ sends $B_2$ to $B_1$, which shows that the action is transitive.

Now suppose for some $x \in N_G(T)$ and $B \in \mathcal{B}^T$ we have $xBx^{-1} = B$. Then $x \in B$ since $N_G(B) = B$. If we show that $N_G(T) \cap B = T$ we will have that $x \in T$ and the coset of $x$ in $W$ is the identity. To show this, consider the canonical map $\pi : B \to B/B_u$ where $B_u$ is the subgroup of unipotent elements of $B$ (see Section 2), and restrict this projection to the maximal torus $T$. This restriction is injective since $T$ consists of semisimple elements. Note also that $B/B_u$ is a torus and hence it is commutative. Therefore if $x \in N_G(T) \cap B$ and $y \in T$, then $xyx^{-1} \in T$ and $\pi(xyx^{-1}) = \pi(x)\pi(y)\pi(x^{-1}) = \pi(y)$. The restriction of $\pi$ to $T$ being injective, we have $xyx^{-1} = y$ that is $x \in C_G(T) = T$, and $N_G(T) \cap B = T$.

The Weyl group also acts on the root system. If $\sigma \in W$ and $\alpha \in \Phi$, then we define $\sigma(\alpha)$ as follows: for $t \in T$,

$$\sigma(\alpha)(t) = \alpha(\dot{\sigma}^{-1} t \dot{\sigma})$$

where $\dot{\sigma} \in N_G(T)$ is a representative of the coset $\sigma$. It has to be verified that $\sigma(\alpha)$ is a root in $\Phi$. But first we note that the Weyl group permutes the eigenspaces of $T$ in $\mathfrak{g}$ as follows: $\mathrm{Ad}(\dot{\sigma})(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma(\alpha)}$. This shows that the eigenspace $\mathfrak{g}_{\sigma(\alpha)}$ is non-zero and thus $\sigma(\alpha)$ is a root. To summarize:

**Proposition A.** *The Weyl group $W = N_G(T)/T$ acts simply transitively on the* *set of $T$-stable Borel subgroups $\mathcal{B}^T$. It acts on the root system and permutes correspondingly the eigenspaces of $T$ in $\mathfrak{g}$.*

# 8 Root Subgroups

Let $B$ be a Borel subgroup of $G$ and $U = B_u$ be the group of unipotent elements in $B$. Let $T$ be a maximal torus in $B$ and $\Phi$ be the root system determined by $T$. To each root $\alpha$ corresponds a unique closed connected 1-dimensional subgroup $U_\alpha$ of $G$ normalized by $T$ and having $\mathfrak{g}_\alpha$ as the Lie algebra (see [Bor91, 13.18], [Hum81, 26.3], [Spr81, 9.2.6]). The group $U_\alpha$ is called the **root subgroup** corresponding to $\alpha$. The following Proposition shows how to construct root subgroups.

**Proposition A.** *Let $\alpha \in \Phi$. Define $Z_\alpha$ to be the centralizer $Z_G(T_\alpha)$ where $T_\alpha = \ker(\alpha)^\circ \subseteq T$. Then $Z_\alpha$ is a reductive group whose root system has rank 1, $B_\alpha = B \cap Z_\alpha$ (resp. $B_{-\alpha} = B^- \cap Z_\alpha$) is a Borel subgroup of $Z_\alpha$ and $(B_\alpha)_u$ (resp. $(B_{-\alpha})_u$) is the root subgroup $U_\alpha$ (resp. $U_{-\alpha}$) of $G$.*

The following result allows us to decompose unipotent groups in a cartesian product of root subgroups (see [Bor91, 14.2], [Hum81, 21.1]).

**Proposition B.** *Let $H$ be a closed $T$-stable subgroup of $U = B_u$. Then $H$ is connected and the product morphism $U_{\alpha_1} \times \cdots \times U_{\alpha_l} \to H$ is an isomorphism of varieties, where $\alpha_i \in \Phi$ are all roots such that $U_{\alpha_i} \subseteq H$, taken in any order.*

# 9 Base for a Root System

Let $\Phi^+$ (resp. $\Phi^-$) be the set of all roots $\alpha \in \Phi$ such that $U_\alpha \subset U = B_u$ (resp. $U_\alpha \subseteq U^- = B_u^-$, where $B^-$ is the Borel subgroup opposite $B$, Section 2). The elements of $\Phi^+$ (resp. $\Phi^-$) are called **positive roots** (resp. **negative roots**).

We write $\alpha > 0$ (resp. $\alpha < 0$) if $\alpha$ is a positive (resp. negative) root. We will give now three equivalent definitions of a base for a root system.

**Definition A.** A subset $\Delta$ of $\Phi^+$ is called the **base of $\Phi$ corresponding to $B$** if one of the following equivalent conditions is satisfied:

1. The set $\Delta$ contains all the roots $\alpha \in \Phi^+$ such that $\sigma_\alpha$ permutes the set $\Phi^+ \setminus \{\alpha\}$.

2. Each root $\beta \in \Phi$ can be written as a linear combination of elements in $\Delta$, say $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$, with integral coefficients $k_\alpha$ which are all nonnegative or all nonpositive.

3. The set $\Delta$ contains all the roots $\alpha \in \Phi^+$ such that the set $B \cup B\sigma_\alpha B$ is a group.

The elements of the base $\Delta$ are called **simple roots.** A **simple reflection** is a reflection $\sigma_\alpha \in W$ corresponding to a simple root $\alpha$. The simple relections generate the Weyl group $W$.

# Chapter II

# Combinatorics in Semisimple Groups

In this chapter we present some results involving the combinatorics in the Weyl group which follows from the inclusion of Schubert varieties. As the Bruhat decomposition is closely linked to this subject, we present it in this chapter even though we don't give a proof. We define the reduced decomposition and the length of an element in the Weyl group and relate them to some geometric properties of the Bruhat cells in Section 11. Section 12 concerns the subproducts of a reduced decomposition. We define the Bott-Samelson variety in Section 13. The Bruhat-Chevalley order is defined and related to the inclusion of Schubert varieties in Section 14. Finally, we present some results relating the reduced decomposition to some special subsets of the root system in the last section.

Let $G$ be a connected reductive group defined over an algebraically closed field $K$, $B$ a Borel subgroup containing a maximal torus $T$, $W$ be the Weyl group $N_G(T)/T$. Let $\Phi = \Phi(T)$ be the root system corresponding to $T$ and $\Delta$ the base associated with $B$. Let $S$ be the set of all simple reflections $\sigma_\alpha$ $(\alpha \in \Delta)$.

# 10 Bruhat Decomposition

The Bruhat decomposition is a normal form for elements of the reductive group $G$ parametrized by a Borel subgroup $B$ and the Weyl group $W$. In the $GL(n, K)$ case, this result is a familiar one: multiplying a matrix on the left and right by upper triangular matrices (which correspond to elementary row and column operations) we obtain a permutation matrix.

**Theorem A.** *Let $G$ be a reductive group. Then*

1. *(Bruhat decomposition). $G$ is the disjoint union of the double cosets $B\omega B$ with $\omega \in W$. If $\omega \in W$ and $\dot{\omega} \in N_G(T)$ is any representative of $\omega$ then the morphism $U \cap \omega(U^-) \times B \to B\omega B$ given by $(x, y) \mapsto x\dot{\omega}y$ is an isomorphism of varieties.*

2. *(Cellular decomposition of $G/B$). $G/B$ is the disjoint union of the $B$-orbits $B\omega B/B$ with $\omega \in W$. If $\omega \in W$ then the morphism $U \cap \omega(U^-) \to B\omega B/B$ given by $u \mapsto u\omega B/B$ is an isomorphism of varieties.*

We need another result closely related to this Theorem.

**Proposition B.** *If $\sigma \in W$ is a simple reflection and $\omega \in W$, then $\sigma B\omega \subseteq B\omega B \cup B\sigma\omega B$.*

For further informations and proofs, we refer the reader to [Hum81, 28.3], [Bor91, Ch IV, 14.11], [Spr81, 10.2.7], [Bou68, Ch IV, §2] and [CG98, 3.1].

# 11 Reduced Decomposition

Since simple reflections generate the Weyl group (see Section 9), every element $\omega$ of the Weyl group $W$ distinct from the unit element can be written as a product

$\sigma_1 \cdots \sigma_l$ of simple reflections $\sigma_i \in S$, $1 \leq i \leq l$. If the number of simple reflections involved in this product is minimal, we say that $(\sigma_1, \ldots, \sigma_l)$ is a **reduced decomposition** of $\omega$, and the number of simple reflections $l$ is called the **length** of $\omega$ and is denoted by $\mathrm{l}(\omega)$. By convention, $\mathrm{l}(\omega) = 0$ if and only if $\omega$ is the unit element of $W$. The following is a well-known result, the proof follows [Hum81, 29.3]. For another treatment look in [Bou68, Ch. IV, §2].

**Lemma A.** *Let $\sigma \in S$ and $\omega \in W$. Then*

1. $\mathrm{l}(\sigma\omega) \geq \mathrm{l}(\omega)$ *implies* $\sigma B\omega \subseteq B\sigma\omega B$.
2. $\mathrm{l}(\sigma\omega) \leq \mathrm{l}(\omega)$ *implies* $\sigma B\omega \cap B\omega B \neq \emptyset$.
3. $\mathrm{l}(\sigma\omega) = \mathrm{l}(\omega) \pm 1$.

*Proof.* We prove the first statement by induction on $\mathrm{l}(\omega)$, the result being trivial for $\mathrm{l}(\omega) = 0$. Suppose $\mathrm{l}(\omega) > 0$, and write $\omega = x\rho$ where $\rho \in S$ and $\mathrm{l}(x) = \mathrm{l}(\omega) - 1$. Suppose on the contrary that $\sigma B\omega \not\subseteq B\sigma\omega B$, that is, in view of Proposition 10 B, $\sigma B\omega \cap B\omega B \neq \emptyset$. By multiplying on the right by $\rho$, it follows that $\sigma Bx \cap B\omega B\rho \neq \emptyset$. Since $\mathrm{l}(x) = \mathrm{l}(\omega) - 1$, the induction hypothesis implies that $\sigma Bx \subseteq B\sigma xB$ and it follows that $B\sigma xB \cap B\omega B\rho \neq \emptyset$. Again by Proposition 10 B, $B\omega B\rho \subseteq B\omega B \cup B\omega\rho B = B\omega B \cup BxB$ and Theorem 10 A implies that $B\sigma xB$ is equal to either $B\omega B$ or $BxB$, that is $\sigma x$ is either $\omega$ or $x$ by Theorem 10 A. The second case is clearly impossible as it would imply that $\sigma = e$ which is absurd. If $\sigma x = \omega$, then $x = \sigma\omega$ and the hypothesis $\mathrm{l}(\sigma\omega) \geq \mathrm{l}(\omega) > \mathrm{l}(x)$ leads also to a contradiction. It follows that $\sigma B\omega \subseteq B\sigma\omega B$.

We prove now the second statement. We know from Proposition 10 B that $\sigma B\sigma \subseteq B\sigma B \cup B$. But $\sigma B\sigma \neq B$ as the Weyl group acts simply transitively on $\mathcal{B}^T$ (see Section 7). Thus $\sigma B\sigma \cap B\sigma B \neq \emptyset$. Multiplying on the right by $\sigma\omega$, we get $\sigma B\omega \cap B\sigma B\sigma\omega \neq \emptyset$. The first part then implies that $B\sigma B\sigma\omega \subseteq B\omega B$, that is $\sigma B\omega \cap B\omega B \neq \emptyset$.

In view of Proposition 10 B, the first and second statements are mutually exclusive, thus $l(\sigma\omega) \neq l(\omega)$. But $l(\sigma\omega)$ can't differ from $l(\omega)$ by more than 1, which yields $l(\sigma\omega) = l(\omega) \pm 1$. □

A direct application of Lemma A yields the following result.

**Proposition B.** *Let $(\sigma_1, \dots, \sigma_l)$ be a reduced decomposition of $\omega \in W$. Then*

$$(B\sigma_1 B) \cdots (B\sigma_l B) = B\omega B.$$

# 12 Subproducts

Suppose $(\sigma_1, \dots, \sigma_l)$ is a sequence of simple reflections. Let $1 \le i_1 < \cdots < i_j \le l$ be a subsequence of $1, \dots, l$. Then $\sigma_{i_1} \cdots \sigma_{i_j}$ is called a **subproduct** of $(\sigma_1, \dots, \sigma_l)$. We will show in Section 14 that the set of subproducts of a reduced decomposition of $\omega$ depends only on $\omega$, and not on the reduced decomposition. Let $\sigma \in S$ and define $P_\sigma = B\sigma B \cup B$. By Definition 9 A, $P_\sigma$ is a subgroup of $G$ (it is a parabolic subgroup as it contains $B$).

**Lemma A.** *An element $x$ of the Weyl group is a subproduct of $(\sigma_1, \dots, \sigma_l)$ if and only if $BxB \subseteq P_{\sigma_1} \cdots P_{\sigma_l}$.*

*Proof.* Let $1 \le i_1 < \cdots < i_j \le l$ be a subsequence of $1, \dots, l$. Note that

$$\begin{aligned} B\sigma_{i_1} \cdots \sigma_{i_l} B &\subseteq (B\sigma_{i_1} B) \cdots (B\sigma_{i_j} B) \\ &\subseteq (B\sigma_1 B \cup B) \cdots (B\sigma_l B \cup B) \\ &= P_{\sigma_1} \cdots P_{\sigma_l}, \end{aligned}$$

which shows that $BxB \subseteq P_{\sigma_1} \cdots P_{\sigma_l}$.

Conversely, note that

$$P_{\sigma_1} \cdots P_{\sigma_l} = \bigcup_{1 \le i_1 < \cdots < i_j \le l} (B\sigma_{i_1} B) \cdots (B\sigma_{i_j} B).$$

Therefore for each sequence $1 \leq i_1 < \cdots < i_j \leq l$, by applying successively Proposition 10 B, $(B\sigma_{i_1}B)\cdots(B\sigma_{i_j}B)$ is contained in the union of the Bruhat cells corresponding to subproducts of $(\sigma_{i_1},\ldots,\sigma_{i_j})$. This shows that $P_{\sigma_1}\cdots P_{\sigma_l}$ is contained in the union of all $BxB$ where $x$ runs over the subproducts of the sequence of simple reflections $(\sigma_1,\ldots,\sigma_l)$. □

# 13 Bott-Samelson Variety

Let $X$ be a $G$-variety and let $B$ be a (any) subgroup of $G$. Define an action of $B$ on $G \times X$ by: $b \cdot (g,x) = (gb^{-1}, b\cdot x)$. Then the quotient, $G \times_B X$ of $G \times X$ by the action of $B$, is the set of $B$-orbits with the structure of variety induced by the bijection $\phi : G \times_B X \to G/B \times X$ given by $(g,x) \mapsto (gB, g\cdot x)$.

Recall from Section 9 that $P_\sigma = B\sigma B \cup B$ is a parabolic subgroup for $\sigma \in S$. Let $\omega \in W$ have a reduced decomposition $(\sigma_1,\ldots,\sigma_l)$ $(\sigma_i \in S,\ 1 \leq i \leq l)$. For simplicity, denote by $\Gamma$ the reduced decomposition $(\sigma_1,\ldots,\sigma_l)$. Let $X_l = G/B$ and define by induction $X_i = G \times_B X_{i+1}$ for $1 \leq i < l$. Denote by $G^l/B^l$ the space $X_1$. Note that there is a natural projection $G^l \to G^l/B^l$ where $G^l$ denotes the cartesian product of $l$ copies of $G$. Let $Z_\Gamma$ be the image of $P_{\sigma_1} \times P_{\sigma_2} \times \cdots \times P_{\sigma_l}$ in $G^l/B^l$. An usual formula for $Z_\Gamma$ is $P_{\sigma_1} \times_B P_{\sigma_2} \times_B \cdots \times_B P_{\sigma_l}/B$. The space $Z_\Gamma$ is projective since $G^l/B^l \cong (G/B)^l$ is projective. Moreover $Z_\Gamma$ is of dimension $l = \mathrm{l}(\omega)$ as it is an iterated bundle over $\mathbb{P}^1$ with one dimensional fibers.

The group product in $G$ induces a morphism $\psi : Z_\Gamma \to G/B$.

**Proposition A.** *The morphism $\psi : Z_\Gamma \to \overline{B\omega B/B}$ induced by the product morphism in $G$ is proper and surjective.*

*Proof.* The variety $Z_\Gamma$ being projective, the morphism $\psi$ is proper. Hence its image under $\psi$ is a closed subvariety of $G/B$. Since $B\sigma_1 B \times \cdots \times B\sigma_l B \subseteq P_{\sigma_1} \times \cdots \times P_{\sigma_l}$. $Z_\Gamma$ contains the projection of the image of $B\sigma_1 B \times \cdots \times B\sigma_l B$ in

$G^l/B^l$, and therefore $\psi$ contains $(B\sigma_1 B)\cdots(B\sigma_l B)/B$ which is equal to $B\omega B/B$ by Proposition 11 B. This shows that $\overline{B\omega B/B}$ is contained in the image of $\psi$. Since $B\sigma_1 B\times\cdots\times B\sigma_l B$ is dense in $P_{\sigma_1}\times\cdots\times P_{\sigma_l}$, the image of $B\sigma_1 B\times\cdots\times B\sigma_l B$ in $G^l/B^l$ is dense in $Z_\Gamma$ which shows that the image of $\psi$ is exactly $\overline{B\omega B/B}$. □

We will prove later (Section 15) that $\psi$ is birational. This is due to Bott and Samelson [BS58]. Therefore the variety $Z_\Gamma$ is called the **Bott-Samelson variety corresponding to the reduced decomposition** $\Gamma$ and the map $\psi : Z_\Gamma \to \overline{B\omega B/B}$ is called the **Bott-Samelson map of** $Z_\Gamma$. The Bott-Samelson variety $Z_\Gamma$ is smooth and irreducible but is not a resolution of singularities of $\overline{B\omega B/B}$ as the fiber of $\psi$ above a smooth point of $\overline{B\omega B/B}$ is not necessarily a single point. A standard reference for this section is the paper of Demazure [Dem74].

# 14 Bruhat-Chevalley order

A **Schubert variety** is the Zariski closure of a Bruhat cell. If $\omega \in W$, we will denote by $S_\omega$ the Schubert variety $\overline{B\omega B/B}$. Since the Schubert variety $S_\omega$ is the closure of a $B$-orbit, it is a union of $B$-orbits in $G/B$, that is of Bruhat cells. We will now answer the question: which are the Bruhat cells that lie in the Schubert variety $S_\omega$?

**Proposition A.** *Let $\omega$, $x \in W$. Then $BxB/B \subseteq S_\omega$ if and only if $x$ is a subproduct of some reduced decomposition of $\omega$.*

*Proof.* By Lemma 12 A and Proposition 13 A, $x$ is a subproduct of $\Gamma$ if and only if $BxB \subseteq P_{\sigma_1}\cdots P_{\sigma_l}$ if and only if $BxB/B \subseteq S_\omega$. □

From this it is clear that the set of subproducts of a reduced decomposition of $\omega$ depends only on $\omega$. If $x$ is a subproduct of some reduced decomposition of $\omega$, we can say that $x$ is a **subproduct** of $\omega$, and write $x \leq \omega$. This defines a partial order on the Weyl group, called the **Bruhat-Chevalley order**. Therefore, in view of Proposition A, we can write $S_\omega = \cup_{x \leq \omega} BxB/B$. Note also that the order $\leq$ on $W$ corresponds to the inclusion of Schubert varieties: $x \leq \omega$ if and only if $S_x \subseteq S_\omega$. A classic reference for this section is the paper of Chevalley [Che94].

# 15 Special Sets of Roots

Fix $\omega \in W$. In view of Section 14 we can ask the following two questions: which are the reflections $\sigma_\beta$ such that $\sigma_\beta\omega \leq \omega$, and which are the roots $\beta \in \Phi^+$ such that $\omega^{-1}(\beta) < 0$? Proposition B answers the second question. The full answer for the first question will be given only in Section 23 using algebraic geometry, here we can only give a partial result.

Let $\omega \in W$ have a reduced decomposition $\sigma_{\alpha_1} \cdots \sigma_{\alpha_l}$, where $\sigma_{\alpha_i}$ is the simple reflection corresponding to the simple root $\alpha_i \in \Delta$. Define $\omega_0 = e$ (the unity in $W$), $\omega_i = \sigma_{\alpha_1}\sigma_{\alpha_2}\ldots\sigma_{\alpha_i}$ for $1 \leq i \leq l$, and $\beta_i = \omega_{i-1}(\alpha_i)$ for $1 \leq i \leq l$. The following Lemma gives a sufficient condition for $\sigma_\beta\omega$ and for $\omega\sigma_\beta$ to be subproducts of $\omega$.

**Lemma A.** *Let $\beta \in \Phi^+$. If $\omega^{-1}(\beta) < 0$ then $\sigma_\beta\omega \leq \omega$. If $\omega(\beta) < 0$ then $\omega\sigma_\beta \leq \omega$.*

*Proof.* Let $\beta \in \Phi^+$ be such that $\omega^{-1}(\beta) < 0$. Let $s$ be the smallest integer, $1 \leq s \leq l$, such that $\omega_s^{-1}(\beta) < 0$. The choice of $s$ implies that $\omega_{s-1}^{-1}(\beta) > 0$. Since $\omega_s^{-1}(\beta) = \sigma_{\alpha_s}\omega_{s-1}^{-1}(\beta)$ we must have $\omega_{s-1}^{-1}(\beta) = \alpha_s$ because the only positive root which is send to a negative root by $\sigma_{\alpha_s}$ is $\alpha_s$ (see Section 9). It follows that

$$\begin{aligned}\sigma_\beta\omega &= \sigma_{\omega_{s-1}(\alpha_s)}\omega \\ &= \omega_{s-1}\sigma_{\alpha_s}\omega_{s-1}^{-1}\omega \\ &= \sigma_{\alpha_1}\cdots\sigma_{\alpha_{s-1}}\sigma_{\alpha_{s+1}}\cdots\sigma_{\alpha_l},\end{aligned}$$

which gives $\sigma_\beta\omega \leq \omega$. To get the second statement, replace $\omega$ by $\omega^{-1}$ in the first. This yields $\sigma_\beta\omega^{-1} \leq \omega^{-1}$, which is equivalent to $\omega\sigma_\beta \leq \omega$. □

The next result relates the set of roots $\beta \in \Phi^+$ such that $\omega^{-1}(\beta) < 0$ to a reduced decomposition of $\omega$.

**Proposition B.** *Let $\omega \in W$ and suppose $\omega = \sigma_{\alpha_1}\cdots\sigma_{\alpha_l}$ is a reduced decomposition of $\omega$. Define $\omega_0 = e$ (the unity in $W$), $\omega_i = \sigma_{\alpha_1}\sigma_{\alpha_2}\ldots\sigma_{\alpha_i}$ for $1 \leq i \leq l$, and $\beta_i = \omega_{i-1}(\alpha_i)$ for $1 \leq i \leq l$. Then*

$$\Phi^+ \cap \omega(\Phi^-) = \{\beta_1, \ldots, \beta_l\} \subseteq \{\beta \in \Phi^+ \mid \sigma_\beta\omega \leq \omega\}$$

*Proof.* The inclusion $\Phi^+ \cap \omega(\Phi^-) \subseteq \{\beta \in \Phi^+ \mid \sigma_\beta\omega \leq \omega\}$ is just Lemma A. Let $1 \leq j \leq l$. The second statement of Lemma A implies that $\beta_j = \omega_{j-1}(\alpha_j) > 0$, which yields $\beta_j \in \Phi^+$. Note that $\omega^{-1}(\beta_j) = \sigma_{\alpha_l}\cdots\sigma_{\alpha_j}(\alpha_j)$ where $\sigma_{\alpha_j}(\alpha_j) = -\alpha_j$. Again by Lemma A, $\sigma_{\alpha_l}\cdots\sigma_{\alpha_{j+1}}(\alpha_j) > 0$, thus $\omega^{-1}(\beta_j) < 0$ and $\beta_j \in \omega(\Phi^-)$.

Conversely, let $\gamma \in \Phi^+ \cap \omega(\Phi^-)$. Then $\sigma_{\alpha_l}\cdots\sigma_{\alpha_1}(\gamma) < 0$. Suppose there is no $j$, $1 \leq j \leq l$, such that $\gamma = \beta_j$. Then $\gamma \neq \alpha_1 = \beta_1$ which implies that $\sigma_{\alpha_1}(\gamma) > 0$. But $\sigma_{\alpha_1}(\gamma) \neq \alpha_2$, which implies that $\sigma_{\alpha_2}\sigma_{\alpha_1}(\gamma) > 0$ (since the simple reflection $\sigma_{\alpha_1}$ permutes $\Phi^+ \setminus \{\alpha_1\}$, see Section 9). Continuing this way, we get that $\omega^{-1}(\gamma) = \sigma_{\alpha_l}\cdots\sigma_{\alpha_1}(\gamma) > 0$ which is absurd. This shows that $\gamma$ is an element of $\{\beta_1, \ldots, \beta_l\}$. □

We can now show that the Bott-Samelson morphism is a birational map, i.e. an isomorphism over a dense open set.

**Corollary C.** *Let* $\Gamma = (\sigma_{\alpha_1}, \dots, \sigma_{\alpha_l})$ *be a reduced decomposition of* $\omega \in W$. *The Bott-Samelson morphism* $\psi : Z_\Gamma \to S_\omega$ *restricts to an isomorphism* $\psi : \psi^{-1}(B\omega B/B) \to B\omega B/B$.

*Proof.* Suppose $X$ is a $G$-variety. Write $g * x$ for the image of $(g, x) \in G \times X$ in $G \times_B X$. If we express $x \in \psi^{-1}(B\omega B/B)$ as $x_1 * x_2 * \cdots * x_l$ where $x_i \in P_{\sigma_{\alpha_i}}$, then each $x_i$ must be in $B\sigma_{\alpha_i}B$, otherwise $\psi(x)$ wouldn't lie in $B\omega B/B = (B\sigma_{\alpha_1}B)\cdots(B\sigma_{\alpha_l}B)/B$. Now it is clear that an element $x$ of $\psi^{-1}(B\omega B/B)$ is of the form $b_1\dot{\sigma}_{\alpha_1} * b_2\dot{\sigma}_{\alpha_2} * \cdots * b_l\dot{\sigma}_{\alpha_l}B/B$ where $b_i \in B$, and $\dot{\sigma}_{\alpha_i} \in N_G(T)$ are representatives of $\sigma_{\alpha_i} \in W$. Since $B = TU = UT$, and $T$ is normalized by the elements of the Weyl group, the element $x$ can be written as $u_1\dot{\sigma}_{\alpha_1} * u_2\dot{\sigma}_{\alpha_1} * \cdots * u_l\dot{\sigma}_{\alpha_l}B/B$, where $u_i \in U$. The group $U$ is isomorphic as a variety to the cartesian product of the one dimensional root subgroups $U_\alpha$ $(\alpha \in \Phi^+)$ in any order (see Section 8), and applying recursively the fact that the only positive root sent to a negative root by $\sigma_{\alpha_i}$ is $\alpha_i$ (see Section 9), we get that $x = u_1\dot{\sigma}_{\alpha_1} * u_2\dot{\sigma}_{\alpha_1} * \cdots * u_l\dot{\sigma}_{\alpha_l}B/B$ where $u_i \in U_{\alpha_i}$. Applying the first part of Lemma A, we get that $\omega_{j-1}(U_{\alpha_j}) \subseteq B$, which shows that $x$ is of the form $u\dot{\sigma}_{\alpha_1} * \dot{\sigma}_{\alpha_1} * \cdots * \dot{\sigma}_{\alpha_l}B/B$ where $u \in U_{\alpha_1}U_{\omega_1(\alpha_2)}\cdots U_{\omega_{l-1}(\alpha_l)}$. But Proposition B yields $U_{\alpha_1}U_{\omega_1(\alpha_2)}\cdots U_{\omega_{l-1}(\alpha_l)} = U \cap \omega(U^-)$, and since $U \cap \omega(U^-) \to B\omega B/B$, given by $u \mapsto u\omega B$ is a bijection (see Theorem 10 A), we get that $\psi : \psi^{-1}(B\omega B/B) \to B\omega B/B$ is a bijection, and hence an isomorphism. □

It follows that the dimension of the Schubert variety $\overline{B\omega B/B}$, which is equal to the dimension of the dense Bruhat cell $B\omega B/B$, is equal to the length $l(\omega)$ of $\omega$ by Section 13.

# Chapter III

# Algebraic Torus Actions and $T$-Stable Algebraic Curves

The varieties with a torus action and more particulary the algebraic curves with a torus action will be the central objects in the study of singularities of Schubert varieties. In this chapter, we will first review some concepts from algebraic geometry. Then in Section 17, we will give well-known results about torus actions. We describe a very important class of algebraic actions with attractive points in Section 18. In Section 19, we show that the number of $T$-stable curves in an algebraic variety with a torus action is bounded below by its dimension. In Section 20 and Section 21, we describe the basic properties of the $T$-stable curves. Finally, the sections 22 and 23 concern the $T$-stable curves in projective spaces $\mathbb{P}^n$ and in flag varieties.

## 16 Finite and étale morphisms

A morphism of affine varieties $p : X \to Y$ is said to be **finite** if the ring $K[X]$ is finitely generated $p^*K[Y]$-module. A finite morphism is **quasi-finite**, that is a morphism with finite fibers. We will need the following two facts about finite morphisms (see [Spr81, 4.2]):

**Proposition A.**
- *A quasi-finite surjective morphism of irreducible affine varieties is finite.*
- *Let $p : X \to Y$ be a dominant morphism of irreducible affine varieties. Let $x \in X$ be such that $p^{-1}(p(x))$ is finite. There is an open affine neighbourhood $U$ of $p(x)$ in $Y$ such that $p^{-1}(U)$ is an open affine neighbourhood of $x$ and the restriction morphism $p : p^{-1}(U) \to U$ is finite (or equivalently quasi-finite and surjective).*

By definition, a point $x \in X$ is a smooth point if and only if the dimension of the tangent space $T_xX$ is equal to the dimension of the variety $X$ if and only if the tangent cone $C_xX$ of $X$ at $x$ is equal to the tangent space $T_xX$ (see [Dan94]). In general, the tangent cone lies in the tangent space. Let $f : X \to Y$ be a morphism. The morphism $d_xf : T_xX \to T_{f(x)}Y$ of tangent spaces restricts to a morphism of tangent cones $d_xf : C_xX \to C_{f(x)}Y$. The morphism $f$ is said to be **étale** at $x \in X$ if $d_xf : C_xX \to C_{f(x)}Y$ is an isomorphism of the tangent cones considered as schemes and it is said to be **unramified** at $x \in X$ if $d_xf : T_xX \to T_{f(x)}Y$ is injective. We say that $f$ is étale (resp. unramified) if it is étale (resp. unramified) at all points of $X$.

A finite étale morphism is called an **étale covering**. Over the field of complex numbers, with the classical topology, such morphisms are locally trivial bundles with finite fibres. In particular the number of points in the fibre of such a morphism is the same above any point in the same connected component. This is also true for an arbitrary algebraically closed field (see [Dan94, Ch. 2, §5.4]).

**Theorem B (Conservation of Number).** *Suppose $f : X \to Y$ is an étale covering, and $Y$ is connected. Then the number of points in a fiber $f^{-1}(y)$ is independent of $y \in Y$.*

# 17 Torus Actions

We begin with a well-known result of Sumihiro ([Sum74]):

**Theorem A (Sumihiro).** *Let $T$ be a torus acting on a normal algebraic variety $X$. Suppose $x \in X^T$. Then there exists a $T$-stable open affine neighbourhood $U$ of $x$ in $X$.*

The proof of the following corollary was suggested by A. Broer.

**Corollary B.** *Let $T$ be a torus acting on an algebraic variety $X$. Suppose $x \in X^T$ is a smooth point of $X$. Then there exist a $T$-stable open affine neighbourhood $U$ of $x$ in $X$, and a $T$-equivariant étale morphism $\beta : U \to T_xX$ sending $x$ to 0. Moreover $\beta$ can be chosen such that $d_x\beta : T_xU \to T_xX$ is the identity.*

*Proof.* Since the smooth locus of $X$ is open and $T$-stable, we may as well assume that $X$ is smooth. By Theorem A, there exists an open affine $T$-stable neighbourhood $U$ of $x$. Let $\mathfrak{m}_x \subseteq K[U]$ be the maximal ideal corresponding to $x \in U$. Let $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathfrak{m}_x$ be any $T$-equivariant split of the projection $\mathfrak{m}_x \to \mathfrak{m}_x/\mathfrak{m}_x^2$. The inclusion $\mathfrak{m}_x \to K[U]$ induces $\mathfrak{m}_x/\mathfrak{m}_x^2 \to K[U]$. This map extends uniquely to a $T$-equivariant map $S(\mathfrak{m}_x/\mathfrak{m}_x^2) \to K[U]$, where $S(\mathfrak{m}_x/\mathfrak{m}_x^2)$ is the symmetric algebra on $\mathfrak{m}_x/\mathfrak{m}_x^2$. This yields a $T$-equivariant map $\beta : U \to (\mathfrak{m}_x/\mathfrak{m}_x^2)^* = T_xX$, which is étale at $x$ by construction. Note that the set where this map is not étale is closed and $T$-stable. Again by Theorem A, there exists a $T$-stable open affine neighbourhood $U' \subseteq U$ of $x$ such that the restriction $\beta : U' \to T_xX$ is étale. Then $U'$ is the desired neighbourhood. Let $\beta' = (d_x\beta)^{-1} \circ \beta|_{U'}$. Then $\beta'$ is étale and $d_x\beta'$ is the identity. □

The following is due to Bialynicki-Birula (Theorem 2.1 in [BB73]) but the proof presented here is much simpler and more intuitive as it follows directly from Corollary B. Moreover Bialynicki-Birula assumes that $x$ is smooth on $X$ which is not necessary.

**Theorem C ([BB73]).** *Let $T$ be a torus acting on an algebraic variety $M$. Let $X$ be an irreducible $T$-stable subvariety of $M$. Suppose $x \in X^T$ is a smooth point of $M$ and let $V$ be a $T$-stable subspace of the tangent space $T_xM$ containing $T_xX$. Then there exists $Y$ an irreducible $T$-stable subvariety of $M$ such that $X \subseteq Y$, $x$ is smooth on $Y$, and $T_xY = V$.*

*Proof.* Let $U \subseteq M$ be an affine $T$-stable neighbourhood of $x$ and $\beta : U \to T_xM$ be the étale morphism given by Corollary B. Take $Y' = \beta^{-1}(V)$. Then $Y'$ is smooth since $\beta$ is étale and $V$ is smooth in $T_xM$, moreover $T_xY' = d_x\beta(T_xY') = V$ and $X \subseteq Y'$. Then the required $Y$ is the closure of $Y'$ in $M$. □

# 18 Attractive Points

The ideas contained in this section are well-known, unfortunately we couldn't find any good reference. Recall that every finite dimensional representation of a torus splits in a direct sum of one dimensional representations. Suppose a torus $T$ acts linearly on $V$. Then $V$ is a direct sum of one dimensional irreducible submodules. Write $V_\alpha$ for the direct sum of the one dimensional submodules of $V$ of weight $\alpha \in X(T)$. Then $V = V_{\alpha_1} \oplus \cdots \oplus V_{\alpha_l}$, where $V_{\alpha_i} \neq 0$ is the weight space of weight $\alpha_i \in X(T)$. Denote by $\mathcal{W}(V)$ the set of all weights $\alpha_i$, $1 \leq i \leq l$.

Suppose $\mathbb{G}_m$ acts on an algebraic variety $X$. Embed $\mathbb{G}_m$ in $\mathbb{P}^1$ such that $\mathbb{P}^1 = \mathbb{G}_m \cup \{0, \infty\}$. Let $y \in X$ and suppose the morphism $\phi_y : \mathbb{G}_m \to X$, $s \mapsto s \cdot y$ extends to a morphism $\tilde{\phi_y} : \mathbb{G}_m \cup \{0\} \to X$ (resp. $\tilde{\phi_y} : \mathbb{G}_m \cup \{\infty\} \to X$), then denote by $\lim_{s\to 0} s \cdot y$ (resp. by $\lim_{s\to\infty} s \cdot y$) the point $\tilde{\phi}_y(0)$ (resp. $\tilde{\phi}_y(\infty)$).

**Definition A.** A point $x \in X$ is called **attractive** if $x \in X^T$ and there exists a one-parameter group $\lambda \in Y(T)$ such that for all $\alpha \in \mathcal{W}(T_xX)$, $\langle \alpha, \lambda \rangle > 0$. Let $x \in X$ be an attractive point and fix a $\lambda \in Y(T)$ as before. Denote by $X_x$ the set of $y \in X$ such that $\lim_{s\to 0} \lambda(s) \cdot y = x$ and call it the **$x$-attracted set.**

*Remark A.* Let $G$ be a semi-simple linear algebraic group defined over an algebraically closed field, $T$ be a maximal torus in a Borel subgroup $B$ of $G$, $W$ be the Weyl group $N_G(T)/T$, $\Phi$ be the root system corresponding to $T$ and $\Phi^+$ be set of positive roots corresponding to $B$. For any $\omega \in W$, the set $\mathcal{W}(T_{\omega B}G/B)$ is just $\omega(\Phi^+)$. It is well-known that there exists a one-parameter group $\lambda \in Y(T)$ such that $\langle \alpha, \lambda \rangle > 0$ for all $\alpha \in \omega(\Phi^+)$ (see [Hum97, 10.1], [Hum81, 25.4]). This shows that the $T$-fixed points $\omega B$, $\omega \in W$, are attractive in $G/B$.

Note that when $V$ is a $T$-module and $0 \in V$ is attractive, then for all $y \in V$ we have $\lim_{s \to 0} \lambda(s) \cdot y = 0$, which implies that the 0-attracted set $V_0$ is the whole $V$.

**Lemma B.** *Suppose $V$ is a $T$-module and $0 \in V$ is attractive. Then the only $T$-stable open neighbourhood of $0 \in V$ is $V$ and $V_0 = V$.*

*Proof.* Let $U$ be a $T$-stable neighbourhood of 0. Let $\lambda \in Y(T)$ be such that $\langle \alpha, \lambda \rangle > 0$ for all $\alpha \in \mathcal{W}(T_0 V)$. Let $y \in V$, and let $C$ be the curve $\{\lambda(s) \cdot y \mid s \in \mathbb{G}_m\}$. Since $0 \in \overline{C}$, $U \cap C$ is non-empty $T$-stable open in $C$, and hence it is the whole $C$. This shows that $y \in U$. □

In the case that $x \in X$ is attractive, Corollary 17 B can be precised as follows:

**Theorem C ([BB73]).** *Let $T$ be a torus acting on an algebraic variety $X$. Suppose $x \in X^T$ is a smooth attractive point of $X$. Then there exist a $T$-stable open affine neighbourhood $U$ of $x$ in $X$, and a $T$-equivariant isomorphism $\beta : U \to T_x X$.*

*Proof.* Let $\beta : U \to T_x X$ be the $T$-equivariant étale morphism given by Corollary 17 B. Since 0 is the unique $T$-fixed point of $T_x X$, $\beta(U^T) = 0$, and since $\beta$ is étale, $U^T$ is finite. Applying again Corollary 17 B on $X = U \setminus X^T \cup \{x\}$, we find $U$ so that $U^T = \{x\}$.

Since $\beta$ is étale, $\beta(U)$ is an open $T$-stable neighbourhood of $\beta(x) = 0$. But by Lemma B, the only $T$-stable open neighbourhood of 0 in $T_xX$ is $T_xX$ itself. This shows that $\beta : U \to T_xX$ is a quasi-finite surjective morphism of affine varieties, hence finite by Proposition 16 A. Thus $\beta : U \to T_xX$ is an étale covering, and $|\beta^{-1}(y)|$ doesn't depend on $y \in T_xX$ by Theorem 16 B. But $\beta^{-1}(0) = x$ as the only $T$-fixed point in $U$ is $x$. Therefore $\beta$ is a bijective étale covering, hence an isomorphism. □

An easy application of the theorem and the preceding lemma gives:

**Corollary D.** *If $x \in X$ is an attractive smooth point of $X$, then $X_x$ is an open affine $T$-stable neighbourhood of $x$ in $X$, and any other open $T$-stable neighbourhood of $x$ in $X$ contains $X_x$.*

# 19 Existence of Enough $T$-Stable Curves

An **algebraic curve** is an irreducible algebraic variety of dimension 1. We need two properties of algebraic curves:

- A smooth algebraic curve $C$ is an open subset of a unique complete smooth curve $\bar{C}$.
- A morphism $\phi : C \to X$ from a smooth algebraic curve $C$ to a complete variety $X$ extends uniquely to a morphism $\bar{\phi} : \bar{C} \to X$.

**Definition A.** An algebraic curve with an action of a torus $T$ is called a **$T$-stable curve**.

Let $T$ be a torus acting on a variety $X$. Denote by $X^T$ the set of the $T$-fixed points of $X$. In this section we will show that the number of $T$-stable curves through a $T$-fixed point of $X$ has to be at least the dimension of $X$. First we show that $T$-stable divisors exist.

**Lemma B.** *Suppose $M$ is a $T$-variety with* $\dim M > 0$. *Let $x \in M^T$ be a smooth point of $M$. Then there exists $M'$ a $T$-stable irreducible subvariety of codimension* 1 *in $M$ containing $x$ such that $x$ is smooth on $M'$.*

*Proof.* There exists a $T$-stable hyperplane $V$ in $T_xM$. The result then follows from Theorem 17 C. □

**Proposition C.** *Suppose $X$ is a $T$-stable irreducible subvariety of a $T$-variety $M$ with* $\dim X > 0$. *Let $x \in X^T$ and suppose $x$ is smooth on $M$. Then there exists $Z$ a $T$-stable irreducible subvariety of codimension* 1 *in $X$ containing $x$.*

*Proof.* By Lemma B, there exists $M'$ a $T$-stable irreducible subvariety of codimension 1 in $M$ containing $x$ such that $x$ is smooth on $M'$. Let $Z$ be an irreducible component of $M' \cap X$ containing $x$. Then

$$\dim X \geq \dim Z \geq \dim X + \dim M' - \dim M = \dim X - 1.$$

If $\dim X = \dim Z$ replace $M$ by $M'$, $X$ by $Z$ and choose again $M'$ and $Z$. Since $\dim M' = \dim M - 1$, eventually we will get $Z$ such that $\dim Z = \dim X - 1$. □

Suppose $T$ acts on a variety $X$. Denote the set of all closed $T$-stable curves in $X$ by $E(X)$ and let $E(X, x)$ be the set of closed $T$-stable curves in $X$ containing the point $x$. Denote by $TE(X, x)$ the subspace of $T_xX$ spanned by the $T_xC$ for all $C \in E(X, x)$. The next theorem is due to Carrell [Car94], but the proof presented here does not make use of a local equivariant embedding in a projective space.

**Theorem D.** *Let $M$ be a smooth $T$-variety and $X$ an irreducible $T$-stable subvariety. Then for every $x \in X^T$*

1. $|E(X, x)| \geq \dim X$,

2. $\dim TE(X, x) \geq \dim X$.

*Proof.* The result being trivial for $\dim X = 1$ and $\dim X = 0$, we can assume that $\dim X > 1$, and we use induction on $\dim X$. Choose $Y$ to be a $T$-stable irreducible subvariety of codimension 1 in $X$ containing $x$. By induction hypothesis $|E(Y,x)| \geq \dim Y = \dim X - 1 \geq 1$. Hence we can choose $C \in E(Y,x)$. Let $L$ be a $T$-stable line in the tangent space $T_xC$ and let $V$ be a $T$-stable complement to $L$ in $T_xM$. By Theorem 17 C, there exists $M'$ a $T$-stable irreducible subvariety of $M$ containing $x$ such that $x$ is smooth on $M'$ and $T_xM' = V$. Let $Z'$ be an irreducible component of $X \cap M'$ containing $x$. Then

$$\dim X \geq \dim Z' \geq \dim X + \dim M' - \dim M = \dim X - 1.$$

If $\dim X = \dim Z'$ apply Proposition C to $Z'$ and $M'$. In any case we get $Z$ a $T$-stable irreducible subvariety of codimension 1 in $X$ containing $x$ and contained in $M'$. Note that $L$ doesn't lie in $T_xZ \subset T_xM' = V$, which implies that $T_xC$ is not contained in $T_xZ$ and $C$ is not contained in $Z$. By induction hypothesis $|E(Z,x)| \geq \dim Z = \dim X - 1$ (resp. $\dim TE(Z,x) \geq \dim X - 1$). But $X$ contains all the curves of $Z$ and $C$ (resp. $TE(X,x)$ contains $TE(Z,x)$ and $T_xC$), which shows that $|E(X,x)| \geq \dim X$ (resp. $\dim TE(X,x) \geq \dim X$). □

# 20 A Smoothness Criterion for $T$-Stable Curves

**Lemma A.** *Suppose any two distinct weights in $\mathcal{W}(V)$ are linearly independent characters of $T$. Then every closed $T$-stable curve in $V$ is a line through* 0.

*Proof.* Let $C \in E(V)$. Let $z \in C \setminus \{0\}$. Write $z$ as $v_1 + \cdots + v_l$ where $v_i \in V_{\alpha_i}$, $1 \leq i \leq l$. Then $t \cdot z = \alpha_1(t)v_1 + \cdots + \alpha_l(t)v_l$. By hypothesis any two $\alpha_i$ $(1 \leq i \leq l)$ are linearly independent characters, which implies that all $v_i$ are 0 but one: otherwise $\dim T \cdot C \geq 2$ for $\dim T > 1$ and if $\dim T = 1$ we must have $l = 1$ as any two weights of $T$ are linearly dependent. This shows that $C$ is the line spanned by $v_i$. □

Let $x \in X^T$ and suppose that any two distinct weights in $\mathcal{W}(T_xX)$ are linearly independent characters, then $x$ is isolated in $X^T$: if $x$ is not isolated in $X^T$, then $T$ acts trivially on $T_x(X^T) \neq 0$, and $\mathcal{W}(T_xX)$ contains the trivial character which is impossible.

**Proposition B.** *Suppose $x \in X$ is a smooth $T$-fixed point and any two distinct weights in $\mathcal{W}(T_xX)$ are linearly independent. Then every $T$-stable curve $C \in E(X, x)$ is smooth. If moreover there exist an open $T$-stable neighbourhood $U$ of $x$ and a $T$-equivariant isomorphism $\beta : U \to T_xX$, then any two distinct $T$-stable curves have distinct tangent spaces.*

*Proof.* By Corollary 17 B, there exists $U \subseteq X$ a $T$-stable open neighbourhood of $x$, an étale morphism $\beta : U \to T_xX$ such that $x$ is sent to $0 \in T_xX$. By Lemma A, $\beta(C)$ is a line. Hence $\dim T_0\beta(C) = 1$. But since $\beta$ is étale, $d_x\beta : T_xC \to T_0\beta(C)$ is injective, which implies that $\dim T_xC = 1$, and the first statement follows. The second statement follows directly from Lemma A. □

# 21 Fixed Points of a $T$-Stable Curve

It is well-known that a torus acting linearly on a $n$-dimensional projective variety, has at least $n+1$ $T$-fixed points (see [Bor91, Ch. IV, 13.5]). In particular, any $T$-stable complete curve has at least two fixed points. The following result precises this statement in a particular situation.

**Proposition A.** *Suppose $X$ is a complete $T$-variety. Suppose $C \in E(X)$ is such that if $x \in C^T$ then $x$ is smooth in $X$ and isolated in $X^T$. Then $C^T$ contains exactly two $T$-fixed points.*

*Proof.* Since $X$ is complete, $C$ is complete which implies, by the Fixed Point Theorem, that $C$ has at least one fixed point $x \in C^T$. Let $U$ be an open affine

$T$-stable neighbourhood of $x$ (see Theorem 17 A). Since $x$ is isolated in $X^T$, $T$ acts non-trivially on $C$. A one parameter group $\lambda$ of $T$ induces a $\mathbb{G}_m$-action on $X$ by: $\mathbb{G}_m \times X \to X$, $(s, x) \mapsto \lambda(s) \cdot x$. Let $\lambda$ be a one parameter group of $T$ such that the induced $\mathbb{G}_m$-action on $C$ is non-trivial. Let $z \in C \cap U$ be a point which is not fixed by the action of $\mathbb{G}_m$ and define $\phi_z : \mathbb{G}_m \to C$ by $s \mapsto \lambda(s) \cdot z$. Since $C$ is a complete variety and $\phi_z$ is dominant, $\phi_z$ extends to a surjective morphism $\bar{\phi}_z : \mathbb{P}^1 \to C$. Note that since $\phi_z(\mathbb{G}_m)$ is a $\mathbb{G}_m$-orbit, the only $T$-fixed points of $C$ are $\bar{\phi}_z(0)$ and $\bar{\phi}_z(\infty)$. Thus either $\bar{\phi}_z(0) = x$ or $\bar{\phi}_z(\infty) = x$. Suppose that $\bar{\phi}_z(0) = \bar{\phi}_z(\infty)$, then $\bar{\phi}_z$ is a morphism from a projective variety $\mathbb{P}^1$ into an affine variety $U$, and hence it is constant. This is impossible, and $C$ has exactly two $T$-fixed points. □

## 22 $T$-Stable Curves in $\mathbb{P}(V)$

Suppose $T$ acts naturally on $\mathbb{P}(V)$, that is this action is induced by a linear action on the vector space $V$. Let $X \subseteq \mathbb{P}(V)$ be a $T$-stable closed irreducible subvariety. Let $C \in E(X)$, and suppose that $T$ doesn't act trivially on $C$. Let $z \in C \setminus C^T$. Express $z$ in homogeneous coordinates $z = (v_1 + \cdots + v_l)/\mathbb{G}_m$ such that each $v_i$, $1 \leq i \leq l$, lies in the weight space $V_{\alpha_i}$, and assume $v_i \neq 0$ for all $i$, $1 \leq i \leq l$. Note that $l > 1$ as $T$ acts non-trivially on $C$. If $t \in T$, then

$$t \cdot z = (\alpha_1(t)v_1 + \cdots + \alpha_l(t)v_l)/\mathbb{G}_m,$$

where $\alpha_i$ is the weight of $v_i$. Let $x \in C^T$, and let $\alpha_x$ be its weight. Then $\alpha_x$ must appear among $\alpha_1, \ldots, \alpha_l$, since $C$ lies in the projection of the space $V_{\alpha_1} \oplus \cdots \oplus V_{\alpha_l}$, and $x$ is a point of $C$. Suppose that $\alpha_1 = \alpha_x$. We can use affine coordinates in a neighbourhood of $x$ to express $z$: $z = \frac{v_2}{v_1} + \cdots + \frac{v_l}{v_1}$. If $t \in T$ then

$$t \cdot z = (\alpha_2 - \alpha_1)(t)\frac{v_2}{v_1} + \cdots + (\alpha_l - \alpha_1)(t)\frac{v_l}{v_1}.$$

Since the affine neighbourhood of $x$ is isomorphic to the tangent space of $\mathbb{P}(V)$ at $x$, and since the curve $C$ lies in $X$, the weights $\alpha_2 - \alpha_1, \dots, \alpha_l - \alpha_1$ appear in $\mathcal{W}(T_x\mathbb{P}(V))$. Among $\alpha_2 - \alpha_1, \dots, \alpha_l - \alpha_1$ there can't be two weights which are linearly independent, otherwise the $T$-orbit $T \cdot z$ would be at least 2-dimensional, which is a contradiction.

The next Proposition allows to make a rough classification of $T$-stable curves in $T$-stable subvariety $X$ of $\mathbb{P}(V)$, when for each $T$-fixed point $x \in X^T$ the weights in $\mathcal{W}(T_xX)$ are linearly independent.

**Proposition A ([Car94]).** *Suppose $T$ acts naturally on $\mathbb{P}(V)$. Let $X \subseteq \mathbb{P}(V)$ be a $T$-stable closed irreducible subvariety. Assume that for every $x \in X^T$, any two different weights in $\mathcal{W}(T_xX)$ are linearly independent.*

1. *Let $C \in E(X)$, $C^T = \{x, y\}$. Then $C$ is the unique closed $T$-stable curve containing its $T$-fixed points $C^T = \{x, y\}$. In particular, it is the projection on $\mathbb{P}(V)$ of the direct sum $x \oplus y$, and it is smooth.*

2. *There exists a character $\alpha$ of $T$ such that $T_xC$ has weight $\alpha$ and $T_yC$ has weight $-\alpha$. Moreover $C$ is the unique $T$-stable curve in $E(X, x)$ such that $T_xC$ has weight $\alpha$.*

3. *$E(X)$ is finite.*

4. *If $x \in X^T$, then two distinct closed $T$-stable curves in $E(X, x)$ have distinct tangent spaces at $x$.*

5. *If $x \in X^T$ is a smooth point of $X$, then $x$ lies on exactly* $\dim X$ *distinct complete $T$-stable curves.*

*Proof.* 1. By Theorem 17 C there exists a $T$-stable subvariety $M \subseteq \mathbb{P}(V)$ such that $x$ is smooth in $M$, $X \subseteq M$ and $T_xM = T_xX$. By the property of projective spaces, there is an affine $T$-stable neighbourhhod $U$ of $x$ in

$\mathbb{P}(V)$, and a $T$-equivariant isomorphism $\beta : U \to T_x\mathbb{P}(V)$. Since any two weights in $\mathcal{W}(T_xX)$ are linearly independent, the curve $C$ is smooth and the tangent space $T_xD$ to any other curve $D \in E(X,x)$ is different from $T_xC$ (see Proposition 20 B). It is easy to see that for any two distinct lines $L_1$ and $L_2$ in $T_x\mathbb{P}(V)$ the closure of $\beta^{-1}(L_1)$ and the closure of $\beta^{-1}(L_2)$ intersects only in $x$. Applying this to $T_xC$ and $T_xD$ we see that $C$ and $D$ intersects only in $x$. Therefore $C$ is the unique $T$-stable curve containing $C^T = \{x, y\}$.

2. Since any two different weights in $\mathcal{W}(T_xX)$ are linearly independent, the 0 weight is not in $\mathcal{W}(T_xX)$ and therefore the torus $T$ acts non-trivially on any $T$-stable curve. If $\alpha_1$ is the weight of $x$ and $\alpha_2$ is the weight of $y$, then by the preceding discussion, $T_xC$ has weight $\alpha = \alpha_2 - \alpha_1$ and $T_yC$ has weight $-\alpha = \alpha_1 - \alpha_2$. If $D \in E(X,x)$ is such that $T_xD$ has weight $\alpha$, and if $z \in C^T \setminus \{x\}$ is the other $T$-fixed point, then $z$ has weight $\alpha + \alpha_1 = \alpha_2$, which forces $z = y$ and $D = C$.

3. Follows from the preceding part and the fact that $X^T$ is finite.

4. This follows from the explicit computation of the weight of $T_xX \subseteq T_x\mathbb{P}(V)$ in the preceding parts.

5. If $x \in X^T$ is a smooth point of $X$, then $\dim T_xX = \dim X$ which implies that $T_xX$ decomposes in at most $\dim X$ distinct weight spaces. But since no two smooth $T$-stable curve in $E(X,x)$ have the same tangent space at $x$, there is are most $\dim X$ $T$-stable curves in $E(X,x)$. The result follows from Theorem 19 D.

□

*Remark A.* Suppose $X$ is the flag variety $G/P$, where $P$ is a parabolic subgroup of $G$ and $T$ is a maximal torus in $G$. There exists a representation $G \to GL(V)$

and a line $L \subseteq V$ such that $B = \mathrm{Stab}_G(L)$ (see [Hum81, 11.2]). Passing to the projective space, the $G$-orbit $G \cdot L$ in $\mathbb{P}(V)$ is isomorphic to $G/P$. Since $G/P$ is complete, $G \cdot L$ is closed, and we have identified $G/P$ with a closed $T$-stable subspace of $\mathbb{P}(V)$. Note that $(G/P)^T$ is finite, and the $T$-weights of $T_eG/P$ are all distinct elements of $\Phi^-$, thus any two of them are linearly independent. For any other point $x \in (G/P)^T$, $T_xG/P$ is the translate of $T_eG/P$ by $x$, thus its weights are also distinct. We can then apply Proposition A to $G/P \subseteq \mathbb{P}(V)$.

Suppose that a torus $T$ acts on a vector space $V$. Then it induces an action on the Grassmannian $\mathbb{G}_d(V)$ for any positive integer $d$. But $\mathbb{G}_d(V) \cong SL(V)/P$ for a maximal parabolic subgroup $P$. Therefore the preceding discussion applies to $T$-actions on Grassmannians, assuming that $\mathbb{G}_d(V)^T$ is finite.

# 23 $T$-Stable Curves in $G/B$

We would like to classify the $T$-stable curves in $G/B$. In view of Section 22, the natural question is: under which condition two $T$-fixed points are joined by a $T$-stable curve. We know that the map $W \to (G/B)^T$, $\omega \mapsto \omega B$ is a bijective correspondence between the Weyl group and the $T$-fixed points of $G/B$ (see Section 7). In particular $(G/B)^T$ is finite. Consider $G/B$ as a subset of some $\mathbb{P}(V)$ as in Remark 22 A. Let $\omega$, $x \in W$. We know (see Proposition 22 A) that the projection of $\omega B \oplus xB$ defines a $T$-stable curve in $\mathbb{P}(V)$. In this section we will show that this curve lies in $G/B$ if and only if $\omega x^{-1}$ is a reflection. We follow the ideas of Springer [Spr98] and Carrell [Car94].

Let $\beta \in \Phi$. Let $Z_\beta$ be the centralizer $Z_G(T_\beta)$ where $T_\beta = \ker(\beta)^\circ$. We know that $Z_\beta$ is a reductive group containing $U_\beta$, $U_{-\beta}$ and the maximal torus $T$ and whose root system has rank 1 (see Proposition 8 A). Let $\omega \in W$ and define $C_{\omega,\beta} = Z_\beta \omega B/B \subseteq G/B$. The stabilizer of $\omega B$ in $Z_\beta$ is the intersection of the Borel subgroup $\omega B \omega^{-1}$ with $Z_\beta$. By Proposition 8 A, this intersection is a Borel subgroup of $Z_\beta$ containing $T$: $B_\beta$ or $B_{-\beta}$, and since $Z_\beta = Z_{-\beta}$, we can assume that it is $B_\beta$. The map $Z_\beta/B_\beta \to Z_\beta\omega B/B$ is an equivariant isomorphism of varieties, which identifies $C_{\omega,\beta}$ with the flag variety of $Z_\beta$. This shows that $C_{\omega,\beta}$ is a smooth closed $T$-stable curve in $G/B$.

**Proposition A.** *Let $\omega \in W$ and $\beta \in \Phi^+$. Then the $T$-stable curve $C_{\omega,\beta}$ is equal to $\overline{U_\beta\omega B/B}$ if $\omega^{-1}(\beta) \leq 0$, otherwise it is equal to $\overline{U_{-\beta}\omega B/B}$. In particular $\mathcal{W}(T_\omega C_{\omega,\beta}) = \pm\beta$.*

*Proof.* The sets $U_\beta B_{-\beta}$ and $U_{-\beta}B_\beta$ are open in $Z_\beta$ (they are the big cells). If $\omega^{-1}(\beta) < 0$ then $U_\beta B_{-\beta}\omega B/B = U_\beta\omega B/B$ is dense in $C_{\omega,\beta}$, otherwise $\omega^{-1}(\beta) > 0$ and $U_{-\beta}B_\beta\omega B/B = U_{-\beta}\omega B/B$ is dense in $C_{\omega,\beta}$. Since the weight of $T_eU_\beta = \mathfrak{g}_\beta$ (resp. $T_eU_{-\beta} = \mathfrak{g}_{-\beta}$) is $\beta$ (resp. $-\beta$), the second statement follows. □

This proposition allows the following definition:

**Definition B.** Let $\omega$, $x \in W$ and suppose that $\omega B$ and $xB$ are two distinct $T$-fixed points of some $T$-stable curve of the form $C = C_{\omega,\beta}$. We define $\beta(\omega, x)$ to be the unique root such that $C = \overline{U_{\beta(\omega,x)}\omega B/B}$ (or equivalently let $\beta(\omega, x)$ be the weight of $T_\omega C$). Note that $\beta(\omega, x) = \pm\beta$.

Let $S_\omega$ denotes the Schubert variety $\overline{B\omega B/B}$. There is no confusion if we write $\omega$ for the $T$-fixed point $\omega B \in (G/B)^T$. The following lemma classifies the closed $T$-stable curves in Schubert varieties.

**Lemma C.** *Let $\omega \in W$ and $\beta \in \Phi$.*

1. *The variety $C_{\omega,\beta}$ is a smooth closed $T$-stable curve with fixed points $\omega$ and $\sigma_\beta\omega$.*

2. *The weight of the tangent space $T_\omega C_{\omega,\beta}$ (resp. $T_{\sigma_\beta\omega}C_{\omega,\beta}$) is $\beta(\omega, \sigma_\beta\omega)$ (resp. $-\beta(\omega, \sigma_\beta\omega)$).*

3. *Any closed $T$-stable curve in $G/B$ is of the form $C_{\omega,\beta}$.*

4. *A closed $T$-stable curve $C$ lies in the Schubert variety $S_\omega$ if and only if its fixed points lie in $S_\omega$.*

*Proof.* The curve $C_{\omega,\beta}$ is a closed smooth $T$-stable curve by the preceding discussion. Since $Z_\beta = B_\beta \cup B_\beta\sigma_\beta B_\beta$ the only two $T$-fixed points of $G/B$ contained in $C_{\omega,\beta} = Z_\beta\omega B/B$ are $\omega$ and $\sigma_\beta\omega$. This shows the first statement.

Suppose $\beta = \beta(\omega, \sigma_\beta\omega)$. Then $T_\omega C_{\omega,\beta}$ is isomorphic (as a $T$-module) to $T_\omega(U_\beta\omega B/B)$, which in turn is isomorphic to $\mathrm{Lie}(U_\beta) = \mathfrak{g}_\beta$. But the weight of $\mathfrak{g}_\beta$ is $\beta$ which shows the second part.

Let $\gamma$ be the weight of $T_\omega C$. We want to show that $C = C_{\omega,\gamma}$. If we show that $T_\omega C_{\omega,\gamma} = T_\omega C$, then the result follows from Proposition 22 A. The weight of the tangent space $T_\omega C_{\omega,\gamma} = T_\omega Z_\gamma\omega B/B$ is $\beta(\omega, \sigma_\gamma\omega) = \pm\gamma$ (see Proposition A). Suppose it is $-\gamma$. Then $C_{\omega,\gamma} = \overline{U_\gamma\omega B/B}$ and $U_{-\gamma}$ doesn't fix $\omega$. Hence $U_\gamma$ fixes $\omega$ and $U_\gamma \subseteq \mathrm{Stab}_G(\omega B) = \omega B\omega^{-1}$, which implies that $\gamma$ is not a weight of $T_\omega G/B$. But this contradicts that $T_\omega C \subseteq T_\omega G/B$ has weight $\gamma$. Therefore $\beta(\omega, \sigma_\gamma\omega) = \gamma$, and this shows $T_\omega C_{\omega,\gamma} = T_\omega C$.

For the last statement, suppose that $C \in E(G/B)$ and $C^T = \{x, y\} \subseteq S_\omega$. Note that by the second part, either $\beta(x,y) > 0$ or $\beta(y,x) > 0$, so we may as well assume that $\beta(x,y) > 0$. Therefore $U_{\beta(x,y)} \subseteq B$, and $U_{\beta(x,y)}xB/B \subseteq S_\omega$ because $S_\omega$ is closed under the action of $B$. This implies that $C = \overline{U_{\beta(x,y)}xB/B} \subseteq S_\omega$, as $S_\omega$ is closed. □

Let $x \in S_\omega$ be a smooth point. The dimension of $T_x S_\omega$ is equal to the dimension of $S_\omega$ that is $l(\omega)$ (see Section 15). In view of this fact, of Theorem 19 D and of Lemma C, it is clear that the number of curves in $E(S_\omega, x)$ is exactly $l(\omega)$. Lemma C shows also that $E(S_\omega, x)$ is in bijective relation with the set

$\{\beta \in \Phi^+ \mid \sigma_\beta x \leq \omega\}$. It follows that

$$\left|\{\beta \in \Phi^+ \mid \sigma_\beta x \leq \omega\}\right| = \mathrm{l}(\omega).$$

In particular for $x = \omega$, using Proposition 15 B,

$$\Phi^+ \cap \omega(\Phi^-) = \{\beta \in \Phi^+ \mid \sigma_\beta \omega \leq \omega\}.$$

An important combinatorial result is the Deodhar inequality. It has been conjectured by Deodhar, and proved by Lakshmibai-Seshadri([LS84]), Carrell-Peterson([Car94]) and Polo([Pol94]) in the context of Schubert varieties and by Dyer([Dye93]) in a more general setup of Coxeter groups. We follow the proof of [Car94].

**Proposition D (Deodhar).** *Let $x \leq \omega$ be in $W$. Then the number of reflections $\sigma \in W$ such that $x < \sigma x \leq \omega$ is at least $\mathrm{l}(\omega) - \mathrm{l}(x)$, that is*

$$\left|\{\beta \in \Phi^+ \mid x < \sigma_\beta x \leq \omega\}\right| \geq \mathrm{l}(\omega) - \mathrm{l}(x).$$

*Proof.* By Lemma C, the number of reflections $\sigma \in W$ such that $\sigma x \in S_\omega$ is equal to $|E(S_\omega, x)|$ which is at least $\dim S_\omega = \mathrm{l}(\omega)$ (see Theorem 19 D and Section 15). By the preceding discussion, the number of reflections $\sigma \in W$ such that $\sigma x \in S_x \subseteq S_\omega$ is exactly $\mathrm{l}(x)$. Hence the number of reflections $\sigma$ such that $\sigma x \in S_\omega$ and $\sigma x \notin S_x$ is at least $\mathrm{l}(\omega) - \mathrm{l}(x)$. The result follows from the fact that the Bruhat-Chevalley order on $W$ corresponds to the inclusion of Schubert varieties. □

We will use this result in proving Peterson's Theorem.

# Chapter IV

# Smoothness Criterion for Schubert Varieties

This is the main chapter of this thesis. We develop here criteria for smoothness of varieties with torus action and more particulary of Schubert varieties. The first two sections concern the topology of the tangent and cotangent bundles. In Section 26 we state Peterson's Theorem which gives an easily computable criterion for smoothness of $T$-fixed points in a Schubert variety. In sections 27 and 29 we develop a more general result for $T$-varieties and use it to prove Peterson's Theorem. In Section 30, we give another criterion involving Cohen-Macaulay varieties, and give another proof of Peterson's Theorem. We develop tools to compute the Peterson translate in Section 31 and apply it to examples in Section 32. Sections 33 and 34 are just restatement of preceding results in a nicer form.

## 24 Closure of the Tangent Bundle

Let $X$ be a smooth variety, and denote by $TX$ (resp. $T^*X$) the tangent (resp. cotangent) bundle of $X$. Let $Y$ be a smooth subvariety of $X$. Denote by $T_Y^*X$ the conormal bundle of $Y$ in $T^*X$, i.e. the set of all covectors over $Y$ which annihilate the subbundle $TY$ of $TX$. First a general result.

**Proposition A.** *If $Y$ is a closed smooth subvariety of $X$, then the tangent bundle $TY$ (resp. conormal bundle $T^*_YX$) is closed in the tangent bundle $TX$ (resp. in the cotangent bundle $T^*X$).*

*Proof.* Since $TY$ is a subvector bundle of $TX|_Y$, it is closed in $TX|_Y$. Moreover, if $\pi : TX \to X$ is the projection, $TX|_Y = \pi^{-1}(Y)$ so $TX|_Y$ is closed in $TX$. This shows that $TY$ is closed in $TX$.

The conormal bundle $T^*_YX$ satisfies the following exact sequence:

$$0 \to T^*_YX \to T^*X|_Y \to T^*Y \to 0$$

Therefore $T^*_YX$ is closed in $T^*X|_Y$. But $T^*X|_Y$ is closed in $T^*X$ because it is equal to $\pi^{*-1}(Y)$ (where $\pi^* : T^*X \to X$ is the projection) and $Y$ is closed in $X$. This shows that $T^*_YX$ is closed in $T^*X$. □

**Corollary B.** *Let $X$ be a smooth algebraic variety and $Y$ a locally closed smooth subset. Then the closure of the tangent bundle (resp. conormal bundle) of $Y$ restricted to the smooth locus of the closure of $Y$ is equal to the tangent bundle (resp. conormal bundle) of the smooth locus of the closure of $Y$. In notation:*

$$\overline{TY}|_{\overline{Y}^{reg}} = T\overline{Y}^{reg}$$

$$\overline{T^*_YX}|_{\overline{Y}^{reg}} = T^*_{\overline{Y}^{reg}}X$$

*Proof.* Let $\tilde{X}$ be the complement in $X$ of the singular locus of $\overline{Y}$: $\tilde{X} = X \setminus \overline{Y}^s$ ($\overline{Y}^s$ denotes the singular locus of $\overline{Y}$). Note that the closure of $TY$ (resp. $T^*_YX$) restricted to $\overline{Y}^{reg}$ (the smooth locus of $\overline{Y}$) is equal to the closure of $TY$ (resp. $T^*_YX$) in $T\tilde{X}$ (resp. $T^*\tilde{X}$). But since $\overline{Y}^{reg}$ is a smooth and closed subvariety of $\tilde{X}$, it is a closed submanifold of $\tilde{X}$. By Proposition A, $T\overline{Y}^{reg}$ (resp. $T^*_{\overline{Y}^{reg}}\tilde{X}$) is also closed in $T\tilde{X}$ (resp. $T^*\tilde{X}$). Therefore

$$\overline{TY}|_{Y^{reg}} \subseteq T\overline{Y}^{reg}$$
$$\overline{T_Y^*X}|_{Y^{reg}} \subseteq T^*_{\overline{Y}^{reg}}\tilde{X}(= T^*_{\overline{Y}^{reg}}X).$$

The dimension of the fibers of $\overline{TY}$ (resp. $\overline{T_Y^*X}$) must be at least the dimension of the fiber above $Y$. But since $T\overline{Y}^{reg}$ (resp. $T^*_{\overline{Y}^{reg}}X$) has fibers of this dimension, we must get the equality: $\overline{TY}|_{Y^{reg}} = T\overline{Y}^{reg}$ (resp. $\overline{T_Y^*X}|_{Y^{reg}} = T^*_{\overline{Y}^{reg}}X$). □

It follows that if we consider the tangent (resp. conormal) bundle $TX_\omega$ (resp. $T^*_{X_\omega}\mathcal{B}$) of a Bruhat cell $X_\omega = B\omega B/B$, then its closure $\overline{TX_\omega}$ (resp. $\overline{T^*_{X_\omega}\mathcal{B}}$) above a smooth point of the Schubert variety $S_\omega = \overline{X_\omega}$ is just the tangent space (resp. conormal space) at this point. In fact, in next sections, this will lead us to a smoothness criterion for Schubert varieties.

# 25 Extensions of Vector Bundles

Let $\pi : \mathcal{E} \to X$ be a map. Denote by $\mathcal{E}_x$ the fiber $\pi^{-1}(x)$ above a point $x \in X$, by $\mathcal{E}|_Y$ the inverse image $\pi^{-1}(Y)$ for a subset $Y \subseteq X$. If $\pi : \mathcal{E} \to X$ is a vector bundle, we define $\mathcal{G}_d(\mathcal{E})$ to be a fiber bundle over $X$ with fibers being Grassmannians of $d$-dimensional subspaces in the fibers of $\mathcal{E}$.

**Proposition A.** *If $X$ is complete then $\mathcal{G}_d(\mathcal{E})$ is complete.*

*Proof.* If $X$ is complete, then $\mathcal{G}_d(\mathcal{E}) \to X$ is a fiber bundle with complete fibers, thus $\mathcal{G}_d(\mathcal{E})$ is complete. □

Suppose $\pi_{\mathcal{E}} : \mathcal{E} \to X$ is a vector bundle. Let $Y$ be a (smooth) algebraic subset of $X$. Let $\pi_{\mathcal{V}} : \mathcal{V} \to Y$ be a subbundle of $\mathcal{E}|_Y \to Y$ of rank $d$ (the dimension of the fibers). The bundle $\mathcal{V}$ defines a section $[\mathcal{V}] : Y \to \mathcal{G}_d(\mathcal{E}|_Y)$ as follows: $[\mathcal{V}](y) = \mathcal{V}_y \in \mathcal{G}_d(\mathcal{E}_y)$ for any $y \in Y$. Denote by $[\mathcal{V}]$ the image of this section in

$\mathcal{G}_d(\mathcal{E})$. Note that the Zariski closure of $\mathcal{V}$ (resp. of $[\mathcal{V}]$) in $\mathcal{E}$ (resp. in $\mathcal{G}_d(\mathcal{E})$) is not necessarily a subbundle of $\mathcal{E}$ (resp. of $\mathcal{G}_d(\mathcal{E})$). The following lemma shows that the closure of $[\mathcal{V}]$ in $\mathcal{G}_d(\mathcal{E})$ and the closure of $\mathcal{V}$ in $\mathcal{E}$ behave in the same way.

**Lemma B.** *Let $\mathcal{T}$ be the tautological bundle of $\mathcal{G}_d(\mathcal{E})$, that is the following closed subvariety $\mathcal{T} = \{(v, V) \in \mathcal{E} \times \mathcal{G}_d(\mathcal{E}) \mid v \in V\} \subseteq \mathcal{E} \times \mathcal{G}_d(\mathcal{E})$ with the projection $\tau : \mathcal{T} \to \mathcal{G}_d(\mathcal{E})$. Let $\mu$ be the projection $\mathcal{T} \to \mathcal{E}$.*

1. *If $\mathcal{D}$ is a subset of $\mathcal{G}_d(\mathcal{E})$, then $\tau^{-1}(\overline{\mathcal{D}}) = \overline{\tau^{-1}(\mathcal{D})}$.*
2. *If $\mathcal{F}$ is a subset of $\mathcal{T}$, then $\mu(\overline{\mathcal{F}}) = \overline{\mu(\mathcal{F})}$.*
3. *Suppose that $\mathcal{V} \to Y$ is a vector subbundle of $\mathcal{E}|_Y \to Y$ of rank d, then*

$$\overline{\mathcal{V}} = \mu(\tau^{-1}(\overline{[\mathcal{V}]})).$$

*Proof.* The bundle morphism $\tau : \mathcal{T} \to \mathcal{G}_d(\mathcal{E})$ is flat and the first statement follows from this fact. For the second statement, note that $\mu : \mathcal{T} \to \mathcal{E}$ is proper as its fibers are all complete. It follows that $\mu(\overline{\mathcal{F}}) = \overline{\mu(\mathcal{F})}$. For the third part, we have $\mathcal{V} = \mu(\tau^{-1}([\mathcal{V}]))$, and the two preceding results imply $\overline{\mathcal{V}} = \mu(\tau^{-1}(\overline{[\mathcal{V}]}))$.

□

Next we will show that sections of $\mathcal{G}_d(\mathcal{E}) \to X$ give rise to vector subbundles of $\mathcal{E} \to X$.

**Lemma C.** *Let $Y$ be an algebraic subset of $X$, and suppose $s : Y \to \mathcal{G}_d(\mathcal{E}|_Y)$ is a section of $\mathcal{G}_d(\mathcal{E}|_Y)$. Then $\mathcal{V} = \{v \in \mathcal{E} \mid v \in s(x), x \in Y\}$ is a vector bundle over $Y$.*

*Proof.* Note that $\mu|_{\tau^{-1}(s(Y))}$ is an isomorphism $\tau^{-1}(s(Y)) \to \mathcal{V}$. But $\tau^{-1}(s(Y))$, as a space over $Y$, is a vector bundle, hence so is $\mathcal{V}$. □

# 26 Peterson's Theorem

Let $G$ be a semisimple algebraic group over an algebraicallly closed field $K$, $B$ a Borel subgroup, $U \subseteq B$ its unipotent part, $T \subseteq B$ a maximal torus, $\Phi$ the root system given by $T$, $\Delta \subset \Phi$ the basis determined by $B$, $\Phi^+$ the set of postive roots, $W = N_G(T)/T$ the Weyl group. Denote by $\sigma_\beta \in W$ the reflection corresponding to the root $\beta \in \Phi$.

For $\alpha \in \Phi$ let $T_\alpha = (\ker \alpha)^\circ$, $Z_\alpha$ the centraliser of $T_\alpha$, $B_\alpha = Z_\alpha \cap B$ and $B_{-\alpha} = Z_\alpha \cap B^-$ the two Borel subgroups of $Z_\alpha$ containing $T$, $U_\alpha$ and $U_{-\alpha}$ their unipotent parts. Let $\mathfrak{n}$ (resp. $\mathfrak{g}_\alpha$) be the Lie algebra of $U$ (resp. of $U_\alpha$).

Denote by $\mathcal{B}$ the flag variety $G/B$. There is a natural action of the group $G$ on $T\mathcal{B}$, the tangent bundle of the flag variety, given by $(g, v) \mapsto d_x l_g(v) \in T_{g\cdot x}\mathcal{B}$, where $l_g : \mathcal{B} \to \mathcal{B}$ is the left translation by $g$. This induces a $G$-action on $\mathcal{G}_d(T\mathcal{B})$ for any positive integer $d$ (see Section 25). Denote by $\pi : \mathcal{G}_d(T\mathcal{B}) \to \mathcal{B}$ the projection. Let $\mathfrak{q} \in \mathcal{G}_d(TX)$ be a $T$-fixed point, $\omega B = \pi(\mathfrak{q})$ and suppose $C \in E(\mathcal{B}, \omega B)$. By Proposition 23 A and Lemma 23 C, there exists $\beta \in \Phi$ such that $C = \overline{U_\beta \omega B/B}$. We define $\tau(\mathfrak{q}, C) \in \mathbb{G}_d(T\mathcal{B})$ to be the fiber of $\overline{U_\beta \cdot \mathfrak{q}} \to C$ above the $T$-fixed point $\sigma_\beta\omega \in C^T$, that is $\tau(\mathfrak{q}, C) = \pi^{-1}(\sigma_\beta\omega) \cap \overline{U_\beta \cdot \mathfrak{q}}$. The space $\tau(\mathfrak{q}, C)$ is called the **Peterson translate** of $\mathfrak{q}$ along $C$. Let $\mathcal{P} = \{(\mathfrak{q}, C) \in \mathcal{G}_d(T\mathcal{B})^T \times E(\mathcal{B}) \mid \pi(\mathfrak{q}) \in C^T\}$, then $\tau$ can be viewed as a map

$$\tau : \mathcal{P} \to \mathcal{G}_d(T\mathcal{B})^T,$$

and it is called the **Peterson map**.

Let $\omega, x \in W$. Recall that $E(S_\omega, x)$ denotes the set of all closed $T$-stable curves in $S_\omega$ containing $x$ (as before, if $\omega \in W$ we denote by $\omega$ the point $\omega B \in \mathcal{B}$). If $C \in E(S_\omega, x)$, let $y_C \in W$ be the $T$-fixed point of $C$ distinct from $x$.

**Theorem A (Peterson).** *Let $x, \omega \in W$ such that $x < \omega$. Suppose that $S_\omega$ is smooth at every point $y \in W$ such that $x < y \leq \omega$. If for any $C$, $D \in$*

*$E(S_\omega, x) \setminus E(S_x, x)$ we have $\tau(T_{y_C} S_\omega, C) = \tau(T_{y_D} S_\omega, D)$, then $S_\omega$ is smooth at $x$.*

In the same way, we can define the "cotangent" version of the Peterson translate. Denote by $\pi^* : T^*\mathcal{B} \to \mathcal{B}$ the projection of the cotangent bundle. The $G$ action on $T^*\mathcal{B}$ is naturally defined as follows: for a cotangent vector $\mu \in T_z^*\mathcal{B}$ and $g \in G$, let $g \cdot \mu$ be the element of $T_{gz}^*\mathcal{B}$ given by $(g \cdot \mu)(v) = \mu(g \cdot v)$ for $v \in T_z\mathcal{B}$. If $\mathfrak{q} \in \mathcal{G}_d(T^*\mathcal{B})$ and $C \in E(X)$ such that $\pi^*(\mathfrak{q}) \in C^T$, then write $\tau(\mathfrak{q}, C)$ for the fiber of $\overline{U_\beta \cdot \mathfrak{q}}$ above $\sigma_\beta x$. Theorem A can be restated in terms of this new map:

**Theorem B.** *Let $x, \omega \in W$ such that $x < \omega$. Suppose that $S_\omega$ is smooth at every point $y \in W$ such that $x < y \leq \omega$. If for any $C$, $D \in E(S_\omega, x) \setminus E(S_x, x)$ we have $\tau(T_{y_C}^\perp S_\omega, C) = \tau(T_{y_D}^\perp S_\omega, D)$, then $S_\omega$ is smooth at $x$.*

Where $T_y^\perp Y$ denotes the annihilator bundle of $T_y Y$ in $T_y^* X$, that is the space that fits in the short exact sequence

$$0 \to T_y^\perp Y \to T_y^* X \to T_y^* Y \to 0,$$

where $T_y^\perp Y \to T_y^* X$ is the inclusion and $T_y^* X \to T_y^* Y$ is the restriction to $T_y Y$.

This theorem was first proved by Peterson (unpublished) and then, in a more general context, by Carrell and Kuttler ([CK99]).

# 27 Peterson Translate

In this section we will proof a more general version of Peterson's Theorem, due to Carrell and Kuttler ([CK99]), which gives a nice criterion for smoothness of $T$-varieties. We will first redefine the Peterson map in more generality.

Recall that if $M$ is a smooth $T$-variety, there is a natural action of $T$ on the tangent bundle $TM$. If $d$ is any integer, there is also an action of $T$ on $\mathcal{G}_d(TM)$, the fiber bundle of $d$-dimensional Grassmannians in $TM$.

**Lemma A.** *Let $M$ be a smooth $T$-variety. Suppose $C \in E(M)$ is a curve with a non-trivial action of $T$. Suppose that $\mathfrak{q} \in \mathcal{G}_d(TM)|_{C \setminus C^T}$ is such that* $\mathrm{Stab}_T(\mathfrak{q}) = \mathrm{Stab}_T(\pi(\mathfrak{q}))$. *Then $T \cdot \mathfrak{q}$ is a section of $\mathcal{G}_d(TM|_{C \setminus C^T})$.*

*Proof.* Since $\mathrm{Stab}_T(\pi(\mathfrak{q}))$ has codimension 1 in $T$, $T \cdot \mathfrak{q} \cong T/\,\mathrm{Stab}_T(\mathfrak{q})$. Note that $T/\,\mathrm{Stab}_T(\pi(\mathfrak{q})) \cong C \setminus C^T$, therefore the map $t \cdot \pi(\mathfrak{q}) \mapsto t \cdot \mathfrak{q}$ is the required section. $\square$

We will now define the Peterson translate for two cases. Let $C \in E(M)$ be a curve with a non-trivial action of $T$.

1. Let $x \in C^T$ and suppose that $\mathfrak{q} \in \mathcal{G}_d(TM)|_{C \setminus C^T}$ is such that $\mathrm{Stab}_T(\mathfrak{q}) = \mathrm{Stab}_T(\pi(\mathfrak{q}))$. Then define $\tau(\mathfrak{q}, x)$ to be the fiber of the Zariski closure $\overline{T \cdot \mathfrak{g}} \subseteq \mathcal{G}_d(TM)$ above $x$.

2. Suppose that $\mathcal{V} \to C \setminus C^T$ is a $T$-stable vector subbundle of $TM|_{C \setminus C^T}$. Let $x \in C^T$ and define $\tau(\mathcal{V}, x)$ to be the fiber of the Zariski closure $\overline{[\mathcal{V}]} \subseteq \mathcal{G}_d(TM)$ above $x$ (where $d$ is the rank of $\mathcal{V}$).

We give some properties of the Peterson translate in the following three results.

**Proposition B.** *Let $X$ be an irreducible $T$-stable subvariety of a smooth $T$-variety $M$ and suppose $x \in X$ is a $T$-fixed point of $X$. Let $C \in E(X, x)$ and suppose $C \setminus C^T$ lies in the smooth locus of $X$. Moreover suppose there exists $p : X \to Y$ an $T$-equivariant morphism of $T$-varieties such that $d_x p$ restricted to $\tau(TX|_{C \setminus \{x\}}, x)$ is injective. Then $p$ is unramified at the points of $C \setminus C^T$.*

*Proof.* Let $z \in C \setminus C^T$ and suppose $p$ is ramified at $z$. Since $z$ is a smooth point of $X$, we have that $L = \ker d_z p \neq 0$. Let $d = \dim L$. As an element of $\mathcal{G}_d(TM)$, $L$ is an element of the fiber above $z$. Note that $L$ is fixed under the action of $\mathrm{Stab}_T(z)$ on $T_z C$, which implies that $\tau(L, x)$, the Peterson translate of $L$ in $\mathcal{G}_d(TM_x)$, is well defined (see Lemma A). We claim that $\tau(L, x)$ lies in $\ker d_x p$.

First note that $\ker dp$ is a closed in $TX$. Next the vector bundle $\mu(\tau^{-1}(T \cdot L))$ (we are using the notation from Lemma 25 B) defined by $T \cdot L \subseteq \mathcal{G}_d(TM|_{C\setminus C^T})$ is contained in $\ker dp$. Since $\tau(L, x)$ is an element of $\overline{T \cdot L}$, and $\mu(\tau^{-1}(\overline{T \cdot L})) = \overline{\mu(\tau^{-1}(T \cdot L))}$ by Lemma 25 B, $\tau(L, x)$ lies in $\ker d_x p$, which shows the claim. On the other hand $\tau(L, x)$ lies in $\tau(TX|_{C\setminus\{x\}}, x)$ because $L \subseteq T_z X$. The hypothesis says that $\ker d_x p \cap \tau(TX|_{C\setminus\{x\}}, x) = 0$ which implies that $\tau(L, x) = 0$ which is a contradiction. Therefore $p$ is unramified at $z$. □

**Proposition C.** *Let $X \subseteq M$ be as before, $x \in X^T$ and let $C$ be a smooth $T$-stable curve in $E(X, x)$ such that $C \setminus C^T$ lies in the smooth locus of $X$. Let $Y \subseteq X$ be a smooth $T$-stable algebraic subset of $X$ with $x \in Y$ such that $Y \cap C = \{x\}$. Suppose there exists $Z$ a $T$-stable smooth algebraic subset of $X$ such that $Y \subseteq Z$ and $C \cap X^{reg} \subseteq Z \cap X^{reg}$. Then $T_x Y \subseteq \tau(TX|_{C\cap X^{reg}}, x)$.*

*Proof.* Since $Z$ is smooth, Corollary 24 B implies that $\overline{TZ|_{Z\cap X^{reg}}}|_Z = TZ$ (where $X^{reg}$ denotes the smooth locus of $X$). In particular it follows that

$$\overline{TZ|_{C\cap X^{reg}}} = TZ|_{C\cap Z}.$$

Hence $T_x Y \subseteq T_x Z = \tau(TZ|_{C\cap X^{reg}}, x)$ in view of Lemma 25 B.

From $TZ|_{C\cap X^{reg}} \subseteq TX|_{C\cap X^{reg}}$, it follows that $\overline{TZ|_{C\cap X^{reg}}} \subseteq \overline{TX|_{C\cap X^{reg}}}$, which yields $\tau(TZ|_{C\cap X^{reg}}, x) \subseteq \tau(TX|_{C\cap X^{reg}}, x)$ by using Lemma 25 B. The result follows from this and the preceding paragraph. □

**Corollary D.** *Let $\omega$ and $C \in E(S_\omega, \omega)$. Let $x \in W$ be the other fixed point of $C$. Then $T_x S_x \subseteq \tau(T(B\omega B/B)|_{C\cap B\omega B/B})$.*

*Proof.* To show this we use Proposition C on $X = S_\omega$, $Y = BxB/B$ and $Z = (U \cap x(U^-)) \cdot U_{-\beta} xB/B$. It is clear that $Y = BxB/B = U \cap x(U^-)xB/B \subseteq Z$. By Proposition 23 A, we see that $C \setminus C^T \subseteq U_\beta xB/B$, thus $C \cap X^{reg} = C \setminus C^T \subseteq Z$. Moreover $Z$ is smooth as the product morphism $U \cap x(U^-) \times U_{-\beta} \times \{x\} \to G$ followed by the projection $G \to G/B$ is a smooth morphism. □

# 28 A Finiteness Criterion

**Proposition A.** *Let $M$ be an irreducible variety with an action of the torus $T$. Let $X$ be a $T$-stable irreducible subvariety of $M$ and suppose $x \in X$ is an attractive point of $X$, smooth in $M$. Let $V$ be a vector space with a linear action of $T$ such that* $\dim X = \dim V$ *and let $p : X_x \to V$ be a $T$-equivariant morphism such that* $d_xp(T_xC) \neq 0$ *for all $C \in E(X,x)$. Then $p$ is finite.*

*Proof.* Suppose $C \in E(p^{-1}(0))$. Since $x$ is attractive, $C$ contains $x$, that is $C \in E(X_x, x)$. The hypothesis $d_xp(T_xC) \neq 0$ implies that $p(C) \neq 0$, which is a contradiction. Therefore $|E(p^{-1}(0))| = 0$, and by Theorem 19 D, $\dim p^{-1}(0) = 0$. In particular this implies that $p$ is dominant.

Let us show that $X_x$ is affine. By Theorem 17 C, there exists $Y$ an irreducible $T$-stable subvariety of $M$ such that $X \subseteq Y$, $x$ is smooth on $Y$, and $T_xY = T_xX$. Note that $x$ is an attractive smooth point in $Y$. Hence, by Corollary 18 D, there exists $Y_x$ an affine $x$-attracted neighbourhood of $x$ in $Y$. Then $X_x = X \cap Y_x$ is an affine $x$-attracted neighbourhood of $x$ in $X$.

By Proposition 16 A, there exists an open neighbourhood $U$ of $p(x) = 0$ such that $p^{-1}(U)$ is an open neighbourhood of $x$ and $p : p^{-1}(U) \to U$ is surjective and quasi-finite (even finite). Note that $U$ can be chosen to be $T$-stable. Therefore by Lemma 18 B, $U = V$, and $p : X_x \to V$ is a quasi-finite surjective morphism of affine varieties. It follows that $p$ is finite as any quasi-finite surjective morphism of irreducible affine varieties is finite (see Proposition 16 A). □

# 29 Peterson-Carrell-Kuttler Theorem

Recall (Section 16) that a morphism $f : X \to Y$ is said to be **étale** at $x \in X$ if the differential $d_xf : C_xX \to C_{f(x)}Y$ is an isomorphism of the tangent cones considered as schemes, and is said to be unramified at $x$ if the differential $d_xf$ :

$T_xX \to T_yY$ is injective. The set of all points where $f$ is not étale is called the **branch locus** of $f$. We will need a theorem of Zariski-Nagata on the purity of the branch locus of a finite map.

**Theorem A (Zariski-Nagata,[Dan94, ch.3, §1.3]).** *Suppose $f : X \to Y$ is a finite dominant morphism, where $Y$ is smooth and $X$ is normal. Then the branch locus of $f$ has pure codimension 1 in $X$.*

**Corollary B.** *Let $T$ be a torus. Let $p : X \to Y$ be a finite dominant equivariant morphism of $T$-varieties with $Y$ smooth and irreducible. Suppose that there exists an attractive point $x \in X^T$ such that $X = X_x$. Then the branch locus of $p$ has codimension 1 in $X$ or is empty.*

*Proof.* Suppose that the branch locus of $p$ has codimension greater than or equal to 2. Let $\pi : \tilde{X} \to X$ be a normalization of $X$, and let $\tilde{p} = p \circ \pi$. Note that the branch locus of $\tilde{p}$ is in codimension 2 or more since the normalization $\pi$ is an isomorphism in codimension 1. Therefore $\tilde{p}$ is étale by Theorem A.

As $p$ is finite and dominant it is surjective. By hypothesis $X = X_x$, hence $Y = p(X) = p(X_x) = Y_{p(x)}$. It follows that $p(x)$ is an attractive point of $Y$ as $Y$ is smooth.

Let $\tilde{x} \in \pi^{-1}(x)$. Then since $\tilde{p}$ is étale, $d_{\tilde{x}}\tilde{p} : T_{\tilde{x}}\tilde{X} \to T_{p(x)}Y$ is an isomorphism and $\tilde{x}$ is an attractive point of $\tilde{X}$. Therefore we can assume that $\tilde{X} = \tilde{X}_{\tilde{x}}$. But then $\tilde{p}$ is an étale covering with $\tilde{p}^{-1}(p(x)) = \tilde{x}$, hence an isomorphism by Theorem 16 B. Thus $p$ is birational. It follows that $p$ is an isomorphism as any finite birational morphism to a smooth (even normal) variety is an isomorphism. □

We will now state and prove the central result in this thesis.

**Theorem C (Peterson-Carrell-Kuttler Theorem).** *Let $M$ be an irreducible variety with an action of the torus $T$. Let $X$ be a $T$-stable irreducible subvariety* *and suppose $x \in X^T$ is an attractive smooth point of $M$. Suppose moreover that there is a subset $E \subseteq E(X, x)$ such that*

1. *every curve in $E$ is not contained in the singular locus of $X$,*
2. $|E(X, x) \setminus E| \leq \dim X - 2$,
3. *for all $C$, $D \in E$, $\tau(TX|_{C \cap X^{reg}}, x) = \tau(TX|_{D \cap X^{reg}}, x)$ where $X^{reg}$ denotes the smooth locus of $X$,*
4. *if $\tau(E)$ denotes the common value of $\tau(TX|_{C \cap X^{reg}}, x)$ for $C \in E$, then $T_xC \cap \tau(E) \neq 0$ for all curves $C \in E(X, x)$.*

*Then $x$ is a smooth point of $X$.*

*Proof.* Since $x$ is attractive in $M$, $M_x$ is a $T$-stable affine neighbourhood of $x$ $T$-equivariantly isomorphic to $T_xM$ by Theorem 18 C and Corollary 18 D. Let $i : M_x \to T_xM$ be this $T$-equivariant isomorphism. By Corollary 17 B, we can choose $i$ so that $d_xi : T_xM_x \to T_xM$ is the identity. In particular there is a closed $T$-equivariant embedding $i : X_x \to T_xM$ of $X_x = X \cap M_x$ in $T_xM$.

Let $\pi : T_xM \to \tau(E)$ be any $T$-equivariant projection, and let $p = \pi \circ i$. We have the following commutative diagram.

$$\begin{array}{ccc} X_x & \xrightarrow{\ i\ } & T_xM \\ & \searrow^{p} & \big\downarrow \pi \\ & & \tau(E) \end{array}$$

Suppose $C \in E(X_x, x)$. By hypothesis $T_xC \cap \tau(E) \neq 0$, which implies that $d_xp(T_xC) \neq 0$ as $d_xp$ is a projection on $\tau(E)$. Then Proposition 28 A implies that $p$ is finite.

Let $Z$ be the branch locus of $p$. We will show that $\dim Z \leq \dim X_x - 2$, which will imply, by Corollary B, that $Z$ is empty and therefore $X_x$ and $X$ are smooth at $x$. It is clear that $Z$ is a $T$-stable closed subvariety of $X_x$ (see [Dan94, II,2.5.4]). Suppose $C \in E(Z,x) \cap E$. Since $C \in E$ is not contained in the singular locus of $X_x$, the $T$-orbit $C^\circ = C \setminus \{x\}$ lies in the smooth locus of $X_x$. At smooth points of $X_x$, $p : X_x \to \tau(E)$ is étale if and only if it is unramified because $\tau(E)$ is smooth and $\dim \tau(E) = \dim X_x$. Hence $p$ is ramified at the points of $C \setminus \{x\}$.

Note that $\ker d_x p \cap \tau(E) = \emptyset$ because $d_x p$ is the projection on $\tau(E) = \tau(TX|_{C\setminus\{x\}}, x)$, and applying Proposition 27 B, we get that $p$ is unramified at the points of $C \setminus \{x\}$. This is a contradiction with the last paragraph. Hence $E(Z,x) \cap E = \emptyset$ and by Theorem 19 D and the second hypothesis, $\dim Z \leq |E(Z,x)| \leq |E(X_x,x) \setminus E| \leq \dim X_x - 2$.

□

We give now the proof of Theorem 26 A

*Proof of Theorem 26 A.* In the theorem let $M = G/B$ and $X = S_\omega$. We know from Remark 18 A, that the points $\omega B$ and $xB$ are attractive in $M$ and hence in $X$. We set $E$ to be $E(S_\omega, x) \setminus E(S_x, x)$. For $C \in E$, the fixed point $y_C$ is a smooth point by hypothesis. Hence $C$ is not contained in the singular locus of $S_\omega$, which yields the first condition of the theorem. The third condition follows directly from the hypothesis of Peterson's Theorem.

Corollary 27 D implies that $T_x D \subseteq \tau(E)$ for all $D \in E(S_x, x)$. Now for $D \in E = E(S_\omega, x) \setminus E(S_x, x)$, we have $TD|_{D\cap X^{reg}} \subseteq TX|_{D\cap X^{reg}}$ hence $T_x D = \tau(TD|_{D\setminus D^T}, x) \subseteq \tau(E)$. This yields the fourth condition.

Since any two distinct weights in $\mathcal{W}(T_x S_\omega)$ are linearly independent, the curves $C \in E(S_\omega, x)$ are all smooth, and the tangent spaces at $x$ of two distinct curves are distinct (see Proposition 20 B). Moreover, since the dimension of a weight space of $T_x S_\omega$ is 1, $|E(S_\omega, x)| = \dim TE(S_\omega, x)$. By the preceding paragraph $TE(S_\omega, x) \subseteq \tau(E)$, thus $|E(S_\omega, x)| \leq \dim \tau(E) = \dim S_\omega$. Note that we can assume that $\mathrm{l}(\omega) - \mathrm{l}(x) \geq 2$ (if $\mathrm{l}(\omega) - \mathrm{l}(x)$ is 0 or 1 the result is trivial). Hence by the Deodhars inequality in Proposition 23 D, we have that $|E| \geq 2$, thus $|E(S_\omega, x) \setminus E| \leq \dim S_\omega - 2$ which gives the second condition of the theorem, and the result follows. □

# 30 Another Smoothness Criterion

We will now give another smoothness argument and we will use it to give another proof of Peterson's Theorem. The argument is based on the Cohen-Macaulay property of Schubert varieties. By definition $X$ has the **Cohen-Macaulay property** if and only if every finite dominant morphism $f : X \to Y$, with $Y$ smooth, is locally free (see [Dan94, Ch 2,§6.6]). We will use the following application of the Principle of Conservation of Number (see [Dan94, Ch. 2,§5.7]):

**Lemma A.** *Let $p : X \to Y$ be a finite locally free morphism, and suppose $X$ is connected. If there exists a point $x \in X$ such that $p$ is unramified at $x$ and $|p^{-1}(p(x))| = 1$, then $p$ is unramified.*

*Proof.* The degree of $p$ at $x$ is 1 and by the Principle of Conservation of Numbers, the degree of $p$ is 1 because $|p^{-1}(p(x))| = 1$. This shows that $p$ is unramified. □

**Theorem B.** *Suppose $X$ is a Cohen-Macaulay irreducible closed $T$-stable subvariety of a smooth $T$-variety $M$. Suppose $x \in X$ is an attractive point of $X$. Suppose moreover that there exists a $T$-stable curve $C \in E(X, x)$ which doesn't lie in the singular locus of $X$ and such that $\tau(TX|_{C \cap X^{reg}}, x) = TE(X, x)$. Then $x$ is smooth in $X$.*

*Proof.* We can assume that $X = X_x$. Let $U$ be an open $T$-stable neighbourhood of $x$ in $M$ isomorphic to $T_xM$. This gives us a closed embedding $i : X_x \to T_xX \subseteq T_xM$. Let $\pi : T_xX \to TE(X, x)$ be any $T$-equivariant split of the inclusion $TE(X, x) \subsetneq T_xX$. Let $p = \pi \circ i$. Note that $\dim TE(X, x) =$

$\dim \tau(TX|_{C\cap X^{reg}}, x) = \dim X_x$ and $d_x p(T_x C) \neq 0$. Hence by Proposition 28 A, $p$ is finite.

Now $p : X_x \to T_x X$ is a finite morphism, $X_x$ is Cohen-Macaulay and $T_x X$ is smooth, hence $p$ is locally free (or flat). Therefore if we show that $p$ is unramified, then $p$ is an étale covering and the smoothness of $x$ follows.

That $p$ is unramified at the points of $C \setminus \{x\}$ follows from Proposition 27 B. As the restriction $i|_C : C \to T_x C$ is a closed embedding and $\pi|_{T_x C} : T_x C \to TE(X, x)$ is the identity, we have that $p|_C : C \to T_x C$ is a bijection. Moreover the only $T$-stable curve in $p^{-1}(T_x C)$ is $C$, as any other would have tangent space at $x$ different from $T_x C$ (see Proposition 20 B). This shows that each fiber of $p$ above a point in $T_x C$ contains only one element, moreover this fiber lies in $C$. As $p$ is unramified at the points of $C \setminus \{x\}$, Lemma A shows that $p$ is unramified. □

We can now give another proof of Peterson's Theorem.

*Proof of Theorem 26 A*. Let $M = G/B$ and $X = S_\omega$. We use again Remark 18 A. The hypothesis of Peterson's Theorem implies that for every curve $C \in E(S_\omega, x)$ which doesn't lie in $S_x$ we have $T_x C \subseteq \tau(TX|_{C\cap S_\omega^{reg}}) = \tau(E)$. Moreover $T_x S_x \subseteq \tau(E)$ by Corollary 27 D. It follows that $TE(S_\omega, x) \subseteq \tau(E)$. But by Theorem 19 D, $\dim TE(S_\omega, x) \geq \dim S_\omega$, hence $TE(S_\omega, x) = \tau(E)$. Peterson's Theorem then follows from the fact that Schubert varieties are Cohen-Macaulay (see [Ram85]). □

# 31 Root String Translation

Let $S$ be a subset of $\Phi$. We can write $S$ as disjoint union of $\beta$-strings contained in $S$:

$$S = \cup_{\gamma \in I} S_\gamma^\beta$$

where $S^\beta_\gamma$ is the $\beta$-string through $\gamma$ in $S$ and $I$ is the set of $\gamma \in S$ such that $\gamma - r\beta \notin S$ for all $r > 0$ (i.e. $\gamma$ is the minimal element in the $\beta$-string $S^\beta_\gamma$). For each $\beta$-string $S^\beta_\gamma$, we define $t_\beta(S^\beta_\gamma)$ to be the $\beta$-string:

$$t_\beta(S^\beta_\gamma) = \{\gamma + (m-i)\beta \,|\, 0 \leq i < |S^\beta_\gamma|\}$$

where $m$ is the greatest integer such that $\gamma + m\beta \in \Phi$. Therefore $t_\beta(S^\beta_\gamma)$ is the connected $\beta$-string with $|S^\beta_\gamma|$ elements containing the maximal element of the $\beta$-string trough $\gamma$ in $\Phi$. We can define $t_\beta(S)$ as the union of $t_\beta(S^\beta_\gamma)$ with $\gamma \in I$:

$$t_\beta(S) = \bigcup_{\gamma \in I} t_\beta(S^\beta_\gamma).$$

Note that the operator $t_\beta$ preserves the cardinality. Suppose now that $\mathfrak{q}$ is some $T$-stable linear subspace of $\mathfrak{n}$. Then $\mathfrak{q}$ is the direct sum of its root spaces. If we set $S$ to be all the roots $\gamma$ such that $\mathfrak{g}_\gamma \subseteq \mathfrak{q}$, then we can define $t_\beta(\mathfrak{q})$ to be the linear subspace of $\mathfrak{n}$ with root spaces corresponding to the roots of $t_\beta(S)$:

$$t_\beta(\mathfrak{q}) = \bigoplus_{\gamma \in t_\beta(S)} \mathfrak{g}_\gamma.$$

Here we note that the operator $t_\beta$ preserves the dimension.

**Proposition A.** *Let $\mathfrak{q}$ be a $T$-stable linear subspace of $\mathfrak{n}$ (the Lie algebra of $U$) of dimension $d$. Let $\beta \in \Phi^+$ and suppose that $U_\beta$ doesn't fix $\mathfrak{q}$. Then the $T$-fixed points of the $T$-stable curve $\overline{U_\beta \cdot \mathfrak{q}} \subseteq \mathcal{G}_d(\mathfrak{n})$ are $\mathfrak{q}$ and $t_\beta(\mathfrak{q})$.*

*Proof.* Let $S$ be the set of roots whose root groups lie in $\mathfrak{q}$. We decompose $S$ as a disjoint union of $\beta$-strings: $S = \cup_{\gamma \in I} S^\beta_\gamma$ (see Section 6). For each $\gamma \in I$, let $\mathfrak{q}_\gamma = \bigoplus_{\lambda \in S^\beta_\gamma} \mathfrak{g}_\lambda$. The space $\mathfrak{q}$ is a direct sum of the $\mathfrak{q}_\gamma$ and the Ad-action of $U_\beta$ distributes over each summand:

$$\mathrm{Ad}(u)(\mathfrak{q}) = \bigoplus_{\gamma \in I} \mathrm{Ad}(u)(\mathfrak{q}_\gamma).$$

Therefore we can assume that $S$ is a single $\beta$-string. The curve $\overline{U_\beta \mathfrak{q}}$ has two fixed points by Proposition 21 A. We know that $\mathfrak{q}$ is one of those and we denote by $\mathfrak{q}'$ the other fixed point of $\overline{U_\beta \cdot \mathfrak{q}}$. There exists $\gamma \in S$ such that $\gamma - i\beta \notin S$ for all $i \geq 1$. Since $S$ is a $\beta$-string, $\mathfrak{q} \subseteq \mathfrak{g}_\gamma \oplus \mathfrak{g}_{\gamma+\beta} \oplus \cdots \oplus \mathfrak{g}_{\gamma+m\beta}$ where $m$ is the biggest integer such that $\mathfrak{g}_{\gamma+m\beta}$ is a root space. It follows that

$$U_\beta \mathfrak{q} \subseteq \mathfrak{g}_\gamma \oplus \mathfrak{g}_{\gamma+\beta} \oplus \cdots \oplus \mathfrak{g}_{\gamma+m\beta}.$$

This yields that $\mathfrak{q}' \subseteq \mathfrak{g}_\gamma \oplus \mathfrak{g}_{\gamma+\beta} \oplus \cdots \oplus \mathfrak{g}_{\gamma+m\beta}$. The space $\mathfrak{q}'$ being $T$-stable, $U_\beta$-stable, and having dimension $d$, we must have

$$\mathfrak{q}' = \mathfrak{g}_{\gamma+(m-d+1)\beta} \oplus \cdots \oplus \mathfrak{g}_{\gamma+m\beta}.$$

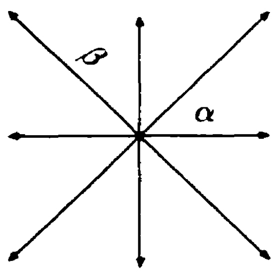
This shows that $\mathfrak{q}' = t_\beta(\mathfrak{q})$. □

# 32 Examples

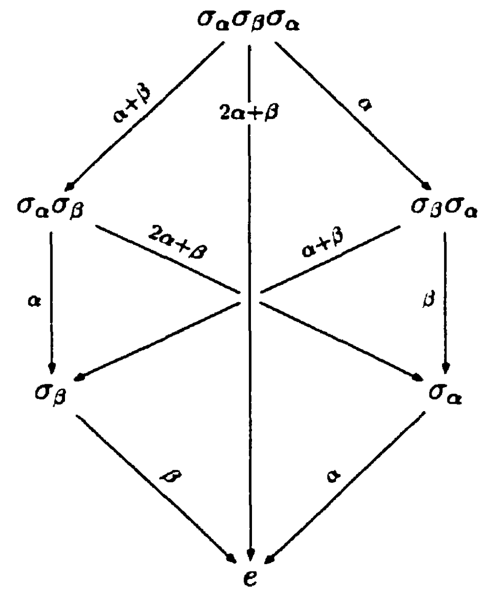
A nice property of the cotangent bundle of the flag variety is that it is naturally isomorphic (as a $G$-variety) to the doubles variety $\{(n, gB) \in \mathcal{N} \times G/B \mid n \in Lie(gBg^{-1})\}$. If $X_\omega = B\omega B/B$ is a Bruhat cell in $\mathcal{B} = G/B$ then the conormal bundle to $X_\omega$, $T^*_{X_\omega}\mathcal{B}$, is identified with the set $\{(n, b\omega B) \mid n \in \mathfrak{n} \cap Ad(b\omega)(\mathfrak{n}), b \in B\}$ (see [BB85] or [CG98]). Therefore the fibers of $T^*_{X_\omega}\mathcal{B}$ all lie in $\mathfrak{n}$, in particular, the fiber above $\omega B$ is $T^\perp_{\omega B} X_\omega = \mathfrak{n} \cap Ad(\omega)(\mathfrak{n}) = \mathfrak{n}^\omega$.

## Root System of Type $C_2$

The Cartan diagram of the root system of type $C_2$ is:



Let $\omega = \sigma_\alpha\sigma_\beta\sigma_\alpha$. The $T$-stable curves in $S_\omega$ are represented in the following diagram:




Write $\mathcal{W}(\mathfrak{q})$ for the set of weights of the space $\mathfrak{q}$. The weights of the space $T_\omega^\perp S_\omega$ are $\Phi^{+\omega} = \{\beta\}$. Let's compute the weights of the Peterson translates of $T_\omega^\perp S_\omega$.

1. $\mathcal{W}(T^\perp_{\sigma_\beta\sigma_\alpha} S_\omega) = t_\alpha(\beta) = 2\alpha + \beta$.

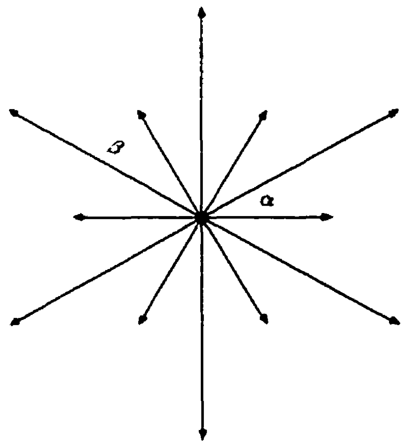
2. $\mathcal{W}(T^\perp_{\sigma_\alpha\sigma_\beta} S_\omega) = t_{\alpha+\beta}(\beta) = \beta$.

3. Since $t_\alpha(\beta) = 2\alpha + \beta = t_{\alpha+\beta}(2\alpha+\beta)$, $S_\omega$ is smooth at $\sigma_\beta$ and $\mathcal{W}(T^\perp_{\sigma_\beta} S_\omega) = 2\alpha + \beta$.

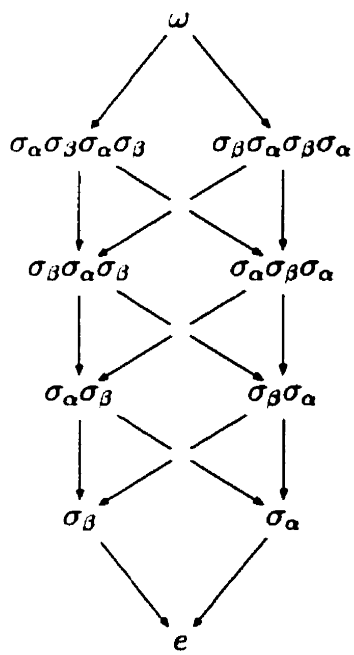
4. Since $t_\beta(2\alpha + \beta) = 2\alpha + \beta$ and $t_{2\alpha+\beta}(\beta) = \beta$ are not equal, $S_\omega$ is singular at $\sigma_\alpha$.

## Root System of Type $G_2$

The Cartan diagram of the root system of type $G_2$ is:




Let $\omega = \sigma_\beta\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta$. The Bruhat-Chevalley order in $S_\omega$ is described in the following diagram:



The following table gives the weights of the Peterson translates:

| $x$ | $y_C$(for $C \in E^u(S_\omega, x)$) | $\beta(y_C, x)$ | $\mathcal{W}(\tau(T^{\perp}_{y_C} S_\omega, x))$ |
|---|---|---|---|
| $\omega$ | | | $\alpha$ |
| $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta$ | $\omega$ | $\beta$ | $\alpha+\beta$ |
| $\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_\alpha$ | $\omega$ | $3\alpha+\beta$ | $\alpha$ |
| $\sigma_\alpha\sigma_\beta\sigma_\alpha$ | $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta$ | $3\alpha+2\beta$ | $\alpha+\beta$ |
| | $\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_\alpha$ | $\beta$ | $\alpha+\beta$ |
| $\sigma_\beta\sigma_\alpha\sigma_\beta$ | $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta$ | $\alpha$ | $3\alpha+\beta$ |
| | $\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_\alpha$ | $2\alpha+\beta$ | $3\alpha+\beta$ |
| $\sigma_\alpha\sigma_\beta$ | $\omega$ | $\alpha+\beta$ | $3\alpha+2\beta$ |
| | $\sigma_\alpha\sigma_\beta\sigma_\alpha$ | $2\alpha+\beta$ | $3\alpha+2\beta$ |
| | $\sigma_\beta\sigma_\alpha\sigma_\beta$ | $\beta$ | $3\alpha+2\beta$ |
| $\sigma_\beta\sigma_\alpha$ | $\omega$ | $2\alpha+\beta$ | $3\alpha+\beta$ |
| | $\sigma_\alpha\sigma_\beta\sigma_\alpha$ | $\alpha$ | $3\alpha+\beta$ |
| | $\sigma_\beta\sigma_\alpha\sigma_\beta$ | $3\alpha+2\beta$ | $3\alpha+\beta$ |
| $\sigma_\alpha$ | $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta$ | $2\alpha+\beta$ | $3\alpha+2\beta$ |
| | $\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_\alpha$ | $\alpha+\beta$ | $3\alpha+2\beta$ |
| | $\sigma_\alpha\sigma_\beta$ | $3\alpha+\beta$ | $3\alpha+2\beta$ |
| | $\sigma_\beta\sigma_\alpha$ | $\beta$ | $3\alpha+2\beta$ |
| $\sigma_\beta$ | $\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta$ | $3\alpha+\beta$ | $\alpha+\beta$ |
| | $\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_\alpha$ | $3\alpha+2\beta$ | $\alpha$ |
| | $\sigma_\alpha\sigma_\beta$ | $\alpha$ | $3\alpha+2\beta$ |
| | $\sigma_\beta\sigma_\alpha$ | $\alpha+\beta$ | $3\alpha+\beta$ |

Only at $x = \sigma_\beta$ the values of $\tau(T^{\perp}_{y_C} S_\omega, x)$ are different for distinct curves $C \in E^u(S_\omega, x)$, and Peterson's Theorem implies that the singular locus of $S_\omega$ is $S_{\sigma_\beta}$. Note that $x = \sigma_\beta$ is a "rationally smooth" point of $S_\omega$ as the number of curves in $E^u(S_\omega, x)$ is equal to $l(\omega) - l(x) = 4$ (see [Car94]).

# 33 $T$-Stable Paths in Schubert Varieties

If two $T$-stable curves have a common $T$-fixed point, we can take the union of them to obtain a $T$-stable closed connected one dimensional subvariety of $\mathcal{B} = G/B$. In general we can take a union of an arbitrary finite collection of $T$-stable curves with overlaping $T$-fixed points.

**Definition A.** Let $(x_1, \ldots, x_n)$ be an order $n$-tuple of elements in $W$ with $n \geq 2$. We say that $(x_1, \ldots, x_n)$ is a **$T$-stable path** in $G/B$ if for each $1 \leq i \leq n-1$, $x_i$ and $x_{i+1}$ are two distinct $T$-fixed points of the same $T$-stable curve in $G/B$. Call $x_iB \in G/B$, $1 \leq i \leq n$, the **vertices** of $(x_1, \ldots, x_n)$. We say that $(x_1, \ldots, x_n)$ lies in the Schubert variety $\overline{B\omega B/B}$ if each of its vertices lies in $\overline{B\omega B/B}$.

**Lemma B.** *$(x_1, \ldots, x_n)$ is a $T$-stable path if and only if $x_{i+1}x_i^{-1}$ are reflections for all $1 \leq i \leq n-1$.*

*Proof.* If $(x_1, \ldots, x_n)$ is a $T$-stable path, then by Lemma 23 C, $x_i$ and $x_{i+1}$ lie in a unique $T$-stable curve which is of the form $C_{x_i,\beta}$ with some $\beta \in \Phi$. Therefore by the same lemma, $x_{i+1}B$ must be the $T$-fixed point $\sigma_\beta x_i B$,i.e. $x_{i+1}x_i^{-1} = \sigma_\beta$.

Conversely, if $x_{i+1}x_i^{-1} = \sigma_\beta$ for some $\beta \in \Phi$, then $x_i$ and $x_{i+1}$ are fixed points of the $T$-stable curve $C_{x_i,\beta}$. □

We will say that the $T$-stable path $(x_1, \ldots, x_n)$ is **decreasing** (resp. **increasing**) if $x_i \geq x_{i+1}$ (resp. $x_i \leq x_{i+1}$) for $1 \leq i \leq n-1$. Recall that $\beta(x_i, x_{i+1})$ denotes the unique root such that the unique $T$-stable curve containing $x_i$ and $x_{i+1}$ is $\overline{U_{\beta(x_i,x_{i+1})}x_iB}$. A simple lemma:

**Lemma C.** *A $T$-stable path $(x_1, \ldots, x_n)$ is decreasing (resp. increasing) if and only if $\beta(x_i, x_{i+1})$ are positive (resp. negative) roots for all $1 \leq i \leq n-1$.*

*Proof.* This follows from Proposition 23 A. □

# 34 Fibres

In this section, we are going to study the closure of the tangent and conormal bundle to the Bruhat cell $X_\omega = B\omega B/B$ in respectively $T\mathcal{B}$ and $T^*\mathcal{B}$. We are going to describe the fibers of those two spaces above the Bruhat cells contained in the smooth locus of the Schubert variety $S_\omega = \overline{X_\omega}$ and we will give a criterion for smoothness of Schubert varieties. Let $\pi : T\mathcal{B} \to \mathcal{B}$ and $\pi^* : T^*\mathcal{B} \to \mathcal{B}$ be the projections. For each $\omega \in W$ define $\pi_\omega : \overline{TX_\omega} \to S_\omega$ and $\pi^*_\omega : \overline{T^*_{X_\omega}\mathcal{B}} \to S_\omega$ to be their respective restrictions. For each $p \in X_\omega$, Denote by $T^\perp_p X_\omega$ the fiber of $T^*_{X_\omega}\mathcal{B}$ at $p$. i.e.. the annihilator in $T^*_p\mathcal{B}$ of $T_pX_\omega$ the tangent space of $X_\omega$ at $p$.

First recall that the cotangent bundle of $G/B$ can be identified with the set $\{(n, gB) \in \mathcal{N} \times G/B \,|\, n \in Lie(gBg^{-1})\}$. The conormal bundle to $X_\omega$, $T^*_{X_\omega}\mathcal{B}$, is identified with the set $\{(n, b\omega B) \,|\, n \in \mathfrak{n} \cap Ad(b\omega)(\mathfrak{n}), b \in B\}$. Therefore the fibers of $T^*_{X_\omega}\mathcal{B}$ all lie in $\mathfrak{n}$, in particular, the fiber above $\omega$ is $T^\perp_\omega X_\omega = \mathfrak{n} \cap Ad(\omega)(\mathfrak{n}) = \mathfrak{n}^\omega$.

Let $P = (\omega_1, \ldots, \omega_n)$ be a decreasing $T$-stable path in $S_\omega$ from $\omega$. Let $\beta_i = \beta(\omega_i, \omega_{i+1})$, $1 \leq i \leq n-1$, be the unique root such that $C_{\omega_i,\beta_i} = \overline{U_{\beta_i}\omega_i B/B}$ (see Section 23). If $\mathfrak{q} \subseteq T^\perp_{\omega_1}X_\omega$, then denote by $\tau_P(\mathfrak{q})$ the following sequence of Peterson translates of $\mathfrak{q}$:

$$\tau_{C_{\omega_{n-1},\beta_{n-1}}} \circ \cdots \circ \tau_{C_{\omega_1,\beta_1}}(\mathfrak{q})$$

**Theorem A.** *Let $x < \omega$ be elements of $W$. Let $P$ be a decreasing $T$-stable path in the Schubert variety $\overline{X_\omega}$ from $\omega$ to $x$. The fiber in $\overline{T^*_{X_\omega}\mathcal{B}}$ over the point $xB$ contains the $T$-stable vector space $\tau_P(T^\perp_{\omega B}X_\omega)$:*

$$\tau_P(T^\perp_{\omega B}X_\omega) \subseteq \pi^{*-1}_\omega(xB).$$

*The equality holds if the point $x$ is smooth in $S_\omega$.*

*Proof.* Denote by $[T^*_{X_\omega}\mathcal{B}]$ the subbundle of $\mathcal{G}_d(T^*\mathcal{B})|_{X_\omega}$ defined by $T^*_{X_\omega}\mathcal{B}$. $[T^*_{X_\omega}\mathcal{B}]$ is closed under the action of $B$, so is its closure $\overline{[T^*_{X_\omega}\mathcal{B}]}$ . Therefore $T$-stable curve $U_{\beta_1} \cdot T_\omega^\perp X_\omega$ is contained in $\overline{[T^*_{X_\omega} \mathcal{B}]}$, and its closure too. This shows that $\tau_{C_{\omega_1,\beta_1}}(T_\omega^\perp X_\omega)$ is contained in $\pi_\omega^{*-1}(x)$. The same applies to the other iterations of the Peterson translates.

If $S_\omega$ is smooth at $x$, then the conormal space to $S_\omega$ at $x$ has the same dimension as the conormal space at $\omega$. The space $\tau_P(T_{\omega B}^\perp X_\omega)$ lies in $\pi_\omega^{*-1}(x)$ which is equal to $T_x^\perp S_\omega$ by Corollary 24 B. But the Peterson translate preserves the dimension, which implies $\tau_P(T_{\omega B}^\perp X_\omega) = T_x^\perp S_\omega$. □

Note that if $\omega_n B$ is a smooth point then the theorem gives a construction of the conormal space to this point and hence of the tangent plane by taking the orthogonal complement. Clearly it doesn't depend on the choice of the $T$-stable path. The converse is also true, and we get a smoothness criterion for points in Schubert varieties. The proof uses a generalization of the Peterson's Theorem.

**Theorem B.** *Let $x < \omega$ be elements of $W$, and suppose $l(\omega) - l(x) \geq 2$. Then $x$ is a smooth point in the Schubert variety $\overline{X_\omega}$ if and only if for any two decreasing $T$-stable paths $P_1$ and $P_2$ from $\omega$ to $x$ in $\overline{X_\omega}$ we have $\tau_{P_1}(T_\omega^\perp X_\omega) = \tau_{P_2}(T_\omega^\perp X_\omega)$ and $S_\omega$ is smooth at every $y \in W$ such that $x < y \leq \omega$.*

*Proof.* For the case when $x$ is a maximal singular point, that is every $y \in W$, $x < y \leq \omega$, $y$ is a smooth point in $S_\omega$, the result is clear from Theorem 26 A. □

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