Efficiency Evaluations of North American University Libraries by Data Envelopment Analysis

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Abstract

A method for ranking efficient decision making units in Data Envelopment Analysis is suggested. The method uses parametric programming and calculates "rigidity" of efficient units relative to perturbations of input and output data.

Theoretical results are applied to 108 North American university libraries. The efficiency of the libraries is determined by the Charnes-Cooper-Rhodes tests. The radius of rigidity approach is applied to efficient libraries. In particular, we focus on the McGill University library and compare its efficiency to other libraries using different data sets.

Une méthode pour classifier l'efficacité des unités décisionnelles dans l'-analyse par enveloppement des données (DEA) est suggérée. Cette méthode utilise la programmation paramétrique et calcule la "rigidité" des unités efficaces relative aux perturbations des données de base et des résultats.

Les résultats théoriques sont mis en pratique pour 108 bibliotèques universitaires Nord Americaines. L'efficacité des bibliotèques est déterminée par le test de Charnes-Cooper-Rhodes. Des bibliotèques efficaces sont classifiées par la méthode de rayon de rigidité. En particulier, on prend le cas de l'Université McGill et on compare son efficacité aux autres bibliotèques en utilisant différents ensembles de données.

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Contents

Chapter 1

Introduction

Data Envelopment Analysis (DEA) was originally proposed in the late seventies as a means of determining and improving the relative efficiencies of not-for-profit organizations, such as hospitals, schools and government agencies, whose goals, organization and structure, did not fit the standard methods of analysis. DEA allows comparison of a number of decision making units (DMU's) based on a finite number of generally incomparable inputs and outputs which are common to all units.

The theoretical objective of this thesis is to study stability of the basic efficiency evaluation models in DEA. One of the major difficulties in DEA is a large number of efficient DMU's, which are not comparable. This typically occurs when there is a large number of different input and output data in comparison to the number of units being studied. We suggest two approaches for ranking the efficient DMU's. One of these is briefly described as follows: Take an arbitrary efficient DMU_k. Then find the largest perturbation of each variable in every other DMU_i, $i \neq k$, which preserves efficiency of that DMU_k. For each variable, the smallest of these numbers is termed the "radius of rigidity" of DMU_k. In an ideal situation, all efficient DMU's are then ordered by their radii of rigidity. Academic models for calculating the radii of rigidity can be highly stable (with unbounded radii of rigidity) or highly unstable (with zero radii). It is shown here that these extreme cases also occur in practical situations.

The "radius of rigidity" approach is applied in this thesis to selected libraries from the 108 North American university libraries which are members of the Association of Research Libraries. Our DEA analysis of these libraries confirms the known fact that the number of efficient DMU's increases with the number of data. (In a particular DEA model of 13 Canadian libraries, 12 were declared efficient.) This warrants a study of how to order efficient DMU's.

A particular feature of this thesis is that it relates the performance of McGill University library to other university libraries. McGill's performance, when a small number of different types of data is considered, was, until recently, not very impressive. It consistently ranked below 87^{th} place until 1994-95 when its efficiency ranking jumped to 11^{th} place. However, McGill's performance appears to be improved when more different data is compared. (Its library was actually declared efficient in 1993-94 and 1994-95 in a group of 30 libraries with five input and seven output data.) When McGill library is declared inefficiently run, then DEA identifies "close" efficiently run libraries that McGill should "emulate" to achieve efficiency. DEA also shows how this can be done. It is interesting that the University of Illinois at Urbana-

Champaign has been consistently declared as one of the libraries McGill should emulate to improve efficiency.

The thesis is organized as follows: In Chapters 2 and 3, we recall the general tools and various DEA models from the literature, with a focus on the Charnes-Cooper-Rhodes model (see, e.g., [14, 59, 16, 17, 50, 52]). Also, we recall some notions from the study of stable parametric optimization (see, e.g., [9, 34, 60]). Chapter 4 surveys the methods which have been used for sensitivity and post-optimality analysis in DEA models (see, e.g., [3, 10, 11, 18, 19, 20, 21, 22, 23, 24, 25, 39, 45, 46, 55, 56, 57]. Each method is illustrated by a simple example. In Chapter 5, we introduce two radius of rigidity models and two simplified marginal value formulae. The marginal value formula given in Theorem 5.6 is proved under new assumptions. These formulae can be used in input optimization (see, e.g., [9, 60, 62]) to calculate the radii of rigidity. We also adjust characterizations of locally and globally optimal inputs from input optimization to fit our radius of rigidity models. Chapters 6 through 9 deal with applications. The first two of these chapters present the results of having applied DEA to North American university libraries for a period of five years. A full model consists of 108 libraries. Also, a subset of 30 libraries with various sets of data is studied. Chapter 8 looks at the application of the radius of rigidity approaches in detail, including algorithms of the technique, simple examples and an application to library data. All empirical data used in this thesis are taken from [47, 30, 31, 40, 41].

The reader interested in the results for McGill, but not necessarily in the details of the theory, may proceed directly to Chapter 9 which looks at how the McGill library fared in the analysis.

Chapter 2

Terminology

2.1 DEA Models

In this section, we survey the most popular models used in Data Envelopment Analysis (DEA). The basic DEA model takes the form of a non-linear, non-convex fractional programming problem. We will assume that we have N decision making units (DMU's), each with a set of strictly positive inputs represented by $X^j \in \Re^m$, $j = 1, ..., N$ and a set of strictly positive outputs represented by $Y^j \in \Re^s$, $j = 1, ..., N$. It should be pointed out that the number of DMU's must be sufficiently large as compared to the number of inputs and the number of outputs for any confidence in the statistical reliability of the input and output evaluations. DEA is units invariant in the sense that the variables do not necessarily have to be comparable amongst themselves. Thus, we can include monetary variables such as expenditures, as well as unitary variables such as the number of employees, in the same model without any difficulties. For each, say the k^{th}, DMU, the model seeks to maximize a weighted ratio of 'virtual' outputs to 'virtual' inputs with respect to a common pool of weighted ratios determined by all N DMU's:

$$\begin{aligned} Max_{(u,v)} \quad & (u, Y^k)/(v, X^k) & & \\ s.t. \quad & (u, Y^j)/(v, X^j) & \leq 1, \qquad & j = 1, ..., N \\ & u, v & \geq 0. & \end{aligned}$$

These programs must be solved for each of the $k = 1, ..., N$ DMU's. In each case, the DMU under consideration selects its own best possible set of weights. Each program determines the coordinates of DMU_k in the production possibility set: the set of all possible production levels as determined by the current production levels of the DMU's being analyzed. Those DMU that lie on the boundary of the production possibility set are said to lie on the envelopment surface or the efficiency frontier, and are deemed to be efficient: under given technology and feasible production levels, it is not possible to produce higher levels of output without increasing levels of input. The reader may recognize that this is directly related to the Pareto-Koopmans concept of efficiency in economic theory. The work described above was an extension of that of Farrell who considered the single output case which proved to be somewhat restrictive in application. For more on these topics, the reader is referred to [33, 13].

Due to an observation made by Charnes and Cooper in fractional programming, it is possible to convert the above model into a linear program, a number of versions of which exist in the literature. The idea is to maximize a ratio of a sum of weighted outputs to a sum of weighted inputs (an output oriented model) or to minimize the reciprocal ratio (an input oriented model). The two primary models are the ratio model, developed by Charnes, Cooper and Rhodes (CCR), and the additive model.

The CCR model for DMU_k, as presented in the original paper by Charnes, Cooper and Rhodes [14], is given by

$$\begin{aligned} Max \quad & \theta \\ s.t. \quad & Y\lambda \geq Y_k\theta \\ & X\lambda \leq X_k \\ & \lambda \geq 0. \end{aligned}$$

An alternate version which employs the non-Archimedean construct ϵ is given by

$$\begin{aligned} Min \quad & \theta - \epsilon(e^T s^+ + e^T s^-) \\ s.t. \quad & Y\lambda - s^+ = Y_k \\ & \theta X_k - X\lambda - s^- = 0 \\ & \lambda \geq 0 \\ & s^+ \geq 0 \\ & s^- \geq 0. \end{aligned}$$

The existence of the solution to the CCR model is assured, see e.g., [20]. In [13], the authors claim that the non-Archimedian extension is required in the CCR model for rigorous theory and usage because it is necessary for an algebraically closed system of the linear programming type. This guarantees optimal solutions at finite non-zero extremal points. Efficiency is achieved, for the output oriented model, if the optimal value is one and all slack variables are zero. The presence of non-zero slacks in the optimal tableau indicates a source of inefficiency.

The so-called additive model is given by the program

$$\begin{aligned} Min \quad & -e^T s^+ - e^T s^- \\ s.t. \quad & Y\lambda - s^+ && = Y_k \\ & -X\lambda - s^- && = -X_k \\ & \sum \lambda_j && = 1, \end{aligned}$$

which classifies DMU_k as efficient if the optimal solution is zero.

In [36], these two models are classified according to the type of envelopment surface that is constructed, either a variable or a constant returns-to-scale function. A variable returns-to-scale envelopment surface consists of a reference set which is the convex hull of all the DMU vectors augmented by the non-negative input-output possibilities that are dominated by this convex hull. Such a set is provided by the additive model. A constant returns-to-scale envelopment surface consists of a reference set which is the conical hull of all the DMU vectors similarly augmented. Such a set is provided by the ratio model.

Two other models commonly appear in the literature: the Banker, Charnes and Cooper (BCC) model and the multiplicative model. Equivalent formulations to the CCR models exist for the BCC models, the only difference being the inclusion of a convexity constraint on the weights, namely, $\sum_{j=1}^{N} \lambda_j = 1$. Conditions for efficiency are the same as above. The multiplicative model uses a logarithmic transformation from fractional to linear form. Thus, the multiplicative model would take the form

$$\begin{array}{lll} Max & \prod_{r=1}^{s} Y_{rk}^{\mu_r} / \prod_{i=1}^{m} X_{ik}^{v_i} & \\ s.t. & \prod_{r=1}^{s} Y_{rj}^{\mu_r} / \prod_{i=1}^{m} X_{ij}^{v_i} \leq 1, & j = 1, \ldots, N \\ & \mu_r, v_i \geq 1 & \end{array}$$

and is converted to the following linear program

$$\begin{array}{lll} Max & \sum_{r=1}^{s} \mu_r \widehat{Y}_{rk} - \sum_{i=1}^{m} v_i \widehat{X}_{ik} & \\ s.t. & \sum_{r=1}^{s} \mu_r \widehat{Y}_{rj} - \sum_{i=1}^{m} v_i \widehat{X}_{ij} \leq 0, & j = 1, \ldots, N \\ & -\mu_r \leq -1 \quad r = 1, \ldots, s & \\ & -v_i \leq -1 \quad i = 1, \ldots, m. & \end{array}$$

Here the carat indicates the log of the original data point. For this formulation, the efficiency frontier is a piecewise log-linear function. Efficiency is achieved if the optimal value and all the slacks are zero. The N linear programs differ only in the objective function.

A measure of efficiency which has recently been introduced into the DEA literature by Thompson, Dharmapala and Thrall (see, e.g., [29]) is given as a solution to the problem

$$Max_{(u,v)} \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} / \frac{\sum_{r=1}^{s} u_r y_{rk}}{\sum_{i=1}^{m} v_i x_{ik}}$$

where, for any choice of (u, v),

$$\frac{\sum_{r=1}^{s} u_r y_{rk}}{\sum_{i=1}^{m} v_i x_{ik}} = max \left\{ \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \mid j = 1, \ldots, n \right\}.$$

This is a measure of relative efficiency of DMU_o. It may be considered as an extreme value statistic and treated by appropriately adapted versions of the statistical theory of extreme values or it may be solved as a mathematical programming problem.

The selection of the model and, consequently, the envelopment surface used for analysis can frequently be determined by various assumptions, economic and otherwise, regarding the data set. The CCR ratio model gives an overall efficiency evaluation. The BCC model provides information about returns-to-scale and gives a technical efficiency evaluation. The multiplicative model provides a log-linear envelopment of the production process. And the additive model relates efficiency results to the economic concept of Pareto optimality. In addition, a number of alterations can be made to the selected model to more realistically fit the data. For instance, a use of exact weights may be replaced by upper and lower bounds while allowing DEA to determine a best set of values from past performances. Or, if certain input or output levels are not within the control of the DMU, for example, they may be specified by government regulations or by population, these particular inputs or outputs may be included in the determination of the reference set, but excluded from the objective function and thereby do not affect the efficiency rating. One can also include ordinal relationships among virtual multipliers to reflect the relative worth of certain inputs or outputs, or one may include multiple time frames in the same model. For these topics, the reader is referred to, for example, [24, 4, 49].

2.2 The CCR Model

This section provides theory and terminology for the CCR model and closely follows the development in [59].

For each DMU_k, $k = 1, ..., N$, the problem of determining its efficiency rating is given by the fractional program

$$\begin{aligned} Max \quad & (u, Y^k)/(v, X^k) & & \\ s.t. \quad & (u, Y^j)/(v, X^j) & \leq 1, \qquad & j = 1, ..., N \\ & u & \geq 0, & \\ & v & \geq 0, & \end{aligned}$$

where the constraints represent the common pool of all DMU's. Thus, higher efficiency ratings imply producing more outputs with less inputs.

We will assume that all inputs and outputs are strictly positive and that only those optimal solutions (u^*, v^*) for which $v^* \neq 0$ are acceptable. The above fractional form can then be transformed into a linear programming problem.

Using the substitution

$$\tau = \frac{1}{(v, X^k)}, \quad x = \tau v, \quad y = \tau u$$

we find that

$$\frac{(u, Y^k)}{(v, X^k)} = (y, Y^k)$$

with

$$(2.1) \qquad (x, X^k) = \tau(v, X^k) = 1$$

Hence

$$(u, Y^j) \leq (v, X^j), \qquad j = 1, ..., N,$$

becomes

(2.2) $$(y, Y^j) \leq (x, X^j), \quad j = 1, ..., N.$$

Thus, the CCR model is

$$\begin{array}{llll} (CCR, k) & Max_{(x,y)} & (y, Y^k) & \\ & s.t. & (y, Y^j) & \leq (x, X^j), \qquad j = 1, ..., N \\ & & (x, X^k) & = 1 \\ & & x & \geq 0 \\ & & y & \geq 0. \end{array}$$

Alternately, it can be shown that the constraint $(X^k, x) = 1$ can be replaced by $(X^k, x) \leq 1$. The two resulting programs are equivalent.

Theorem 2.1 *The constraint* $(X^k, x) = 1$ *can be replaced by* $(X^k, x) \leq 1$ *in (CCR, k). The resulting optimal solutions and values are the same.*

Proof: If $(\tilde{x}, \tilde{y})$ is an optimal solution of the latter problem, it suffices to show that $(X^k, \tilde{x}) = 1$. If this were not the case, then we can define $\hat{x} = \delta^{-1}\tilde{x}$ and $\hat{y} = \delta^{-1}\tilde{y}$ where $\delta = (X^k, \tilde{x}) \in (0, 1)$. The point $(\hat{x}, \hat{y})$ is feasible for the problem with the constraint $(X^k, x) \leq 1$ since $\delta > 0$ and $(X^k, \hat{x}) = 1$. But $(Y^k, \hat{y}) > (Y^k, \tilde{y})$ since the optimal values are positive and $\delta < 1$. This contradicts the optimality of $(\tilde{x}, \tilde{y})$.

Notice that one must solve such a problem for each of the N DMU's. Each DMU chooses its optimal weighting from the common feasible pool. Efficiency is defined as follows:

Definition 2.2 *A DMU is* **efficient** *if the optimal value q^* of (CCR, k) is 1. It is inefficient if the optimal value is such that* $0 < q^* < 1$.

One may recognize that the CCR test presented here bears a different form from the one presented in the original paper on DEA by Charnes, Cooper, and Rhodes [14]. To avoid any non-Archimedean constructs, we have employed the dual of the program used in that paper. The two are equivalent by linear programming theory.

Not only does DEA provide a means of ranking a number of similar DMU's according to their efficiency, but it also allows identification of sources of inefficiency and an estimate of the overall amount of inefficiency. Thus, we can determine how an inefficient DMU can be made efficient. Let us show how this is done.

Each inefficient DMU can be associated with a set of efficient DMU's, which determine the boundaries of a facet of the efficiency frontier closest to that inefficient DMU. It is onto that facet that we project the inefficient DMU. The associated efficient DMU's form the set $\mathcal{R}_k = \{j : p_j^* > 0\}$ where p^* is the optimal solution of the dual variables associated with the common feasibility constraints given by 2.2. One notes that q^* is the dual variable associated wtih the constraint given in 2.1.

Theorem 2.3 *The decision making units in the reference set $\mathcal{R}_k$ of an inefficient DMU_k are efficient.*

Proof: Let x^* and y^* be optimal primal solutions and p^* and q^* be optimal dual solutions for an inefficient DMU_k. The corresponding complementarity conditions from linear programming are

$$p_j^*[(y^*, Y^j) - (x^*, X^j)] = 0, \quad j = 1, ..., N.$$

For $j \in \mathcal{R}_k$, we have $p_j^* > 0$. So

$$(2.3) \qquad (y^*, Y^j) = (x^*, X^j), \quad j \in \mathcal{R}_k.$$

Now, for any $\bar{j} \in \mathcal{R}_k$, define

$$y^{\bar{j}} = \alpha y^*, \quad x^{\bar{j}} = \alpha x^* \qquad for\ some\ \alpha > 0.$$

We claim that $(x^{\bar{j}}, y^{\bar{j}})$ is a feasible solution of the primal efficiency test for $\text{DMU}_{\bar{j}}$. First,

$$(y^{\bar{j}}, Y^j) - (x^{\bar{j}}, X^j) = \alpha[(y^*, Y^j) - (x^*, X^j)] \leq 0, \quad j = 1, ..., N,$$

by feasibility of (x^*, y^*) for DMU_k. Also,

$$\begin{aligned} (x^{\bar{j}}, X^{\bar{j}}) &= \alpha(x^*, X^{\bar{j}}) \\ &= \alpha(y^*, Y^{\bar{j}}), \quad by \quad (2.3) \\ &= 1 \end{aligned}$$

for the choice

$$(2.4) \qquad \alpha = \frac{1}{(y^*, Y^{\bar{j}})} > 0.$$

It is impossible that $(y^*, Y^{\bar{j}}) = 0$, because of the positivity assumption on the data. That would mean $y^* = 0$ and therefore $(y^*, Y^k) = 0$ which implies $q^* = 0$ since the optimal values are equal. But $q^* = 0$ is impossible because it would imply $p^* = 0$ which would imply $Y^k \leq 0$ contradicting the positivity assumption on the data. So feasiblity is proven.

Finally,

$$(y^{\bar{j}}, Y^{\bar{j}}) = \alpha(y^*, Y^{\bar{j}}) = 1$$

by the choice of (2.4) for α. Hence $\text{DMU}_{\bar{j}}$ is efficient.

Definition 2.4 *For an inefficient DMU_k, we define its* **ideal DMU** *to have the ideal input and output vectors:*

$$X^\circ = \sum_{j \in R_k} p_j^* X^j \qquad Y^\circ = \sum_{j \in R^k} p_j^* Y^j.$$

The ideal DMU, given by X° and Y°, uses strictly less input to produce at least as much output. This is guaranteed since

$$Y^\circ = \sum_{j \in \Re_k} p_j^* Y^j \geq Y^k$$

by feasibility of p^* in the dual problem, and

$$X^\circ = \sum_{j \in \Re_k} p_j^* X^j \leq q^* X^k < X^k$$

by feasibility of q^* and p^* in the dual problem and because $0 < q^* < 1$.

Also, the ideal DMU is efficient with respect to the efficient DMU's from the reference set:

Theorem 2.5 *An ideal decision making unit is efficient.*

Proof: Let (x^*, y^*) and (p^*, q^*) be the optimal solutions of the primal and dual problems of the efficiency test for DMU_k, assumed inefficient. The complementarity condition

$$(x^*, [-\sum_{j=1}^{N} p_j^* X^j + q^* X^k]) = 0,$$

can be expressed as

$$q^*(x^*, X^k) = \sum_{j=1}^{N} p_j^*(x^*, X^k). \tag{2.5}$$

Define

$$\bar{x} = \frac{1}{q^*}x^* \quad and \quad \bar{y} = \frac{1}{q^*}y^*. \tag{2.6}$$

We will show that $(\bar{x}, \bar{y})$ is feasible and that the ideal DMU reaches efficiency at that point. First, since (x^*, y^*) is feasible,

$$(y^*, Y^j) \leq (x^*, X^j), \quad j = 1, ..., N,$$

so we can make the substitution (2.6) and we have

$$(\bar{y}, Y^j) \leq (\bar{x}, X^j), \quad j = 1, ..., N. \tag{2.7}$$

Since

$$\frac{(y^*, Y^\circ)}{(x^*, X^\circ)} = \frac{\sum_{j \in \mathcal{R}_k} p_j^*(y^*, Y^j)}{\sum_{j \in \mathcal{R}_k} p_j^*(x^*, X^j)} = 1$$

by (2.3), it therefore follows that

$$(\bar{x}, X^\circ) = (\bar{y}, Y^\circ).$$

It remains to show that $(\bar{x}, X^\circ) = 1$. We have:

$$\begin{aligned}
(\bar{x}, X^\circ) &= \frac{1}{q^*}(x^*, X^\circ), \quad by \quad (2.6) \\
&= \frac{1}{q^*} \sum_{j \in \mathcal{R}_k} p_j^*(x^*, X^j), \quad by \quad definition \quad of \quad X^\circ \\
&= \frac{1}{q^*} \sum_{j=1}^{N} p_j^*(x^*, X^j), \quad since \quad p_j^* = 0 \quad if \quad j \notin \mathcal{R}_k \\
&= (x^*, X^k), \quad by \quad (2.5) \\
&= 1, \quad by \quad feasibility \quad of \quad x^*.
\end{aligned}$$

Hence, the ideal DMU (X°, Y°) is efficient.

So X°, Y° provide the required input and output levels to ensure an efficient ranking. If all of the components of the optimal solution x^* are positive, then the efficiency ratio q^* indicates the necessary proportionate decrease in current input levels, while maintaining the same level of output, in order to obtain efficiency. Thus,

$$X^\circ = q^* X^k,$$

or, the input should be proportionately decreased by $100(1 - q^*)\%$.

2.3 Scaling in CCR Models

One property of DEA is that the models are units invariant in the sense that the variables representing the inputs and outputs do not all have to be measured by the same unit. However, this may mean a wide range of data values which could cause numerical instability and inconsistencies. Ali [2] recommends that DEA problems should be scaled in order to avoid ill-conditioned problems. Let us investigate exactly what happens if we scale the data in the CCR tests.

Theorem 2.6 *Consider (CCR, k). If the inputs $X_i^j, j = 1, ..., N$ are scaled by a factor $\alpha_i > 0, i = 1, ..., m$, and the outputs Y_l^j are scaled by a factor $\beta_l > 0, l = 1, ..., s$, the efficiency rating for every DMU_k, $k = 1, ..., N$ remains the same. Moreover, the ideal DMU of an inefficient DMU does not change. The optimal solutions of the scaled DMU are multiples of the optimal solutions of the unscaled DMU by the corresponding reciprocal scaling factors.*

Proof: Take $\bar{x}_i = x_i^*/\alpha_i$ and $\bar{y}_i = y_i^*/\beta_i$, where (x^*, y^*) is the optimal solution of the unscaled problem. Let the scaled data be referred to by $\hat{X}^j = (\alpha_i X_i^j) \in \Re^m$ and $\hat{Y}^j = (\beta_i Y_i^j) \in \Re^s$, $j = 1, ..., N$.

Claim: $(\bar{x}, \bar{y}) \in F$:

$$(\bar{x}, \hat{X}^k) = \sum_{i=1}^{m} \frac{x_i^*}{\alpha_i}(\alpha_i X_i^k) = \sum_{i=1}^{s} x_i^* X_i^k = 1,$$

by feasibility of x^*. Also,

$$(\bar{y}, \hat{Y}^j) - (\bar{x}, \hat{X}^j) = \sum_{i=1}^{s} \frac{y_i^*}{\beta_i}(\beta_i Y^j) - \sum_{i=1}^{m} \frac{x_i^*}{\alpha_i}(\alpha_i X^j) \leq 0$$

by feasibility of (x^*, y^*).

Claim: $(\bar{x}, \bar{y})$ is optimal. First,

$$(\bar{y}, \hat{Y}^k) = \sum_{i=1}^{s} \frac{y_i^*}{\beta_i}(\beta_i Y_i^k) = (y^*, Y^k) = q^*,$$

where q^* is the optimal value of the unscaled problem.

If $(\bar{x}, \bar{y})$ was not optimal, then there would exist $(x', y') \neq (\bar{x}, \bar{y})$ such that $(y', \beta \cdot Y^k) = q' > q^*$, where

$$q' = \sum_{i=1}^{s} y_i' \beta_i Y^k \quad \textit{and} \quad q^* = \sum_{i=1}^{s} y_i^* Y^k.$$

But then we could take $y_i^* = y_i'\beta_i$ and $x_i^* = x_i'\alpha_i$, contradicting the optimality of (x^*, y^*) in the original unscaled problem.

Claim: $p_j^* = \bar{p}_j$, $j = 1, ..., N$ where p^* is an optimal solution of the unscaled dual problem and $\bar{p}$ is an optimal solution of the scaled dual problem.

The complementarity condition from linear programming theory gives

$$\bar{p}_j[\sum_{i=1}^{s} \frac{y_i^*}{\beta_i}(\beta_i Y_i^j) - \sum_{i=1}^{m} \frac{x_i^*}{\alpha_i}(\alpha_i X_i^j)] = 0, \quad j = 1, ..., N.$$

Thus, $p_j^* = \bar{p}_j = 0$ for all $j \notin \mathcal{R}_k$, where $\mathcal{R}_k$ is the reference set for DMU_k as defined in the previous section. When $\bar{p}_j \neq 0$, we know that $(\bar{y}, \beta \cdot Y^j) = (\bar{x}, \alpha \cdot X^j)$ or, equivalently, $(y^*, Y^j) = (x^*, X^j)$, where DMU_j is efficient $(j \in \mathcal{R}_k)$. Since $q^* = \bar{q}$, we have from the complementarity condition

$$\sum_{j \in \mathcal{R}_k} p_j^*(x^*, X^j) = \sum_{j \in \mathcal{R}_k} \bar{p}_j(x^*, X^j). \tag{2.8}$$

Thus,

$$\sum_{j \in \mathcal{R}_k} (x^*, p_j^* X^j - \bar{p}_j X^j) = 0$$

and

$$(x^*, \sum_{j \in \mathcal{R}_k} (p_j^* X^j - \bar{p}_j X^j)) = 0.$$

Now $(X^k, x^*) = 1$ and $x^* \geq 0$ by feasibility. Since $X^k > 0$, this implies that $x^* > 0$. Otherwise, if $x^* = 0$, the first condition would not be satisfied. So

$$\sum_{j \in \mathcal{R}_k} p_j^* X^j = \sum_{j \in \mathcal{R}_k} \bar{p}_j X^j,$$

and hence the ideal input vector is the same for the scaled and the unscaled problem, based on calculations with unscaled data. Because $(y^*, Y^j) = (x^*, X^j)$, $j \in \mathcal{R}_k$, we also have from (2.8) that

$$\sum_{j \in \mathcal{R}_k} p_j^*(y^*, Y^j) = \sum_{j \in \mathcal{R}_k} \bar{p}_j(y^*, Y^j).$$

We showed in the proof of Theorem 2.3 that $y^* > 0$. So in a similar fashion, we can derive that the ideal output vector is the same for the scaled and unscaled problem. Again, this is based on calculations with unscaled data.

Chapter 3

Point-to-Set Mappings

This chapter introduces basic theory that is necessary to understand perturbed DEA models. In particular, we recall (see, e.g., [60]) basic notions of parametric optimization and stability and state the Karush-Kuhn-Tucker conditions for optimality.

3.1 Point-to-Set Mappings in Parametric Optimization

Parametric optimization deals with models of the form

$$\begin{aligned} (P,\theta) \qquad & Min_x \quad f(x,\theta) \\ & s.t. \quad f^i(x,\theta) \le 0, \qquad i \in \mathcal{P}. \end{aligned}$$

It involves both the decision variable $x \in \Re^n$ and a parameter, or input, $\theta \in \Re^p$. Each choice of θ determines a feasible set, $\mathrm{F}(\theta)$, a set of optimal solutions $\tilde{F}(\theta) = \{\tilde{x}(\theta)\}$ and an optimal value $\tilde{f}(\theta) = f(\tilde{x}(\theta), \theta)$. The first two objects are expressed by point-to-set mappings

$$F\colon \theta \rightarrow F(\theta) = \{x \in \Re^n : f^i(x,\theta) \leq 0, i \in \mathcal{P}\}$$

and

$$\tilde{F}\colon \theta \rightarrow \tilde{F}(\theta) = \{\tilde{x}(\theta)\}.$$

(Here, $\tilde{x}(\theta)$ represents an optimal solution for the given θ.) For each choice of input θ, one seeks $x \in F(\theta)$ which provides the best optimal value. The set of all θ for which there exist feasible points is denoted by $\mathcal{F} = \{\theta\colon F(\theta) \neq \emptyset\}$. Thus, in the parametric optimization problem, we seek to

$$\begin{aligned} Min_x \quad & f(x,\theta) = \tilde{f}(\theta) \\ s.t. \quad & f^i(x,\theta) \leq 0, \; i \in \mathcal{P}, \end{aligned}$$

in order to subsequently

$$\begin{aligned} Min \quad & \tilde{f}(\theta) \\ s.t. \quad & \theta \in \mathcal{F}. \end{aligned}$$

We will assume that $f(x,\theta)$ and $f^i(x,\theta)$, $i \in \mathcal{P}$ are continuous functions.

We will also define some point-to-set mappings that will be used in Chapter 5:

$$\begin{aligned} \mathcal{P}^{=}(\theta) &= \{i \in \mathcal{P} : f^i(x,\theta) = 0\}; \\ \mathcal{P}^{<}(\theta) &= \{i \in \mathcal{P} : f^i(x,\theta) < 0, \; for \; some \; x \in F(\theta)\}; \\ F^{=}(\theta) &= \{x \in \Re^n : f^i(x,\theta) \leq 0, i \in \mathcal{P}^{=}(\theta)\}; \\ F^{=}_{*}(\theta) &= \{x \in \Re^n : f^i(x,\theta) \leq 0, i \in \mathcal{P}^{=}(\theta^*)\}. \end{aligned}$$

Note that the latter is given relative to a fixed $\theta^* \in \mathcal{F}$. The first two represent the active and inactive constraints, respectively, for a given θ. $\mathcal{P}^=(\theta)$ is called the minimal index set of active constraints. These two mappings give an indication of the relative freedom of the constraints within the problem.

At this point, we require some basic topology for point-to-set mappings in order to define their continuity. In particular, we need the notions of open and closed mappings.

Definition 3.1 *A point-to-set mapping* $\Gamma: \Re^p \to \Re^n$ *is* **closed** *at* $\theta^* \in \Re^p$ *if, given any sequence* $\theta^k \to \theta^*$ *and a sequence* $x^k \in \Gamma(\theta^k)$ *such that* $x^k \to x^*$*, it follows that* $x^* \in \Gamma(\theta^*)$.

Definition 3.2 *A point-to-set mapping* $\Gamma: \Re^p \to \Re^n$ *is* **open** *at* $\theta^* \in \Re^p$ *if, given any sequence* $\theta^k \to \theta^*$ *and any* $x^* \in \Gamma(\theta^*)$*, there exists a sequence* $x^k \in \Gamma(\theta^k)$ *such that* $x^k \to x^*$.

We can now define continuity of point-to-set mappings.

Definition 3.3 *A point-to-set mapping* $\Gamma: \Re^p \to \Re^n$ *is* **continuous** *at* $\theta^* \in \Re^p$ *if it is both open and closed at* θ^*.

A closely related concept is that of lower-semicontinuity of the point-to-set mapping. In fact, it can be shown that openness and lower-semicontinuity of mappings are equivalent notions. Also, continuity of the constraint functions guarantees that the feasible set mapping F is closed, so continuity of F requires that openness, or equivalently lower-semicontinuity, be satisfied. Let us formalize these statements.

Definition 3.4 *A point-to-set mapping* $\Gamma: \Re^p \to \Re^n$ *is* **lower-semicontinuous** *at* $\theta^* \in \Re^p$ *if, for each open set* $\mathcal{A} \subset \Re^n$ *satisfying*

$$\mathcal{A} \cap \Gamma(\theta^*) \neq \emptyset,$$

there exists a neighbourhood $N(\theta^*)$ *of* θ^* *such that*

$$\mathcal{A} \cap \Gamma(\theta) \neq \emptyset \text{ for each } \theta \in N(\theta^*).$$

Theorem 3.5 *A point-to-set mapping* $\Gamma: \Re^p \to \Re^n$ *is open at* θ^* *if, and only if, it is lower-semicontinuous at* θ^*.

Theorem 3.6 *Consider the model (P,* θ*) where the functions* $f^i : \Re^{n+p} \to \Re$, $i \in \mathcal{P}$, *are continuous. The point-to-set mapping* F *is closed at every* $\theta^* \in \Re^p$.

3.2 Stability

We will now define stability of a model. For our needs, it suffices to consider only convex models. These are models for which the functions $f(\cdot, \theta), f^i(\cdot, \theta) : \Re^n \to \Re$ are convex functions for every $\theta \in \Re^p$, $i \in \mathcal{P}$. Such models will be studied around a fixed, but arbitrary, θ^*. For simplification, we will use the following notion:

Definition 3.7 *Consider the convex model (P,* θ*) around some* θ^*. *The objective function* f *is said to be realistic at* θ^* *if*

$$\widetilde{F}(\theta^*) \neq \emptyset \text{ and bounded.}$$

We will also use uniformly bounded sequences of sets:

Definition 3.8 *A sequence of sets $\Gamma(\theta)$ as $\theta \to \theta^*$ is said to be* **uniformly bounded** *at θ^* if $\Gamma(\theta) \subset K$ for every $\theta \in N(\theta^*)$ where K is a ball of finite radius and $N(\theta^*)$ is some neighbourhood of θ^*.*

The following theorem, borrowed from [60], will be used repeatedly. Recall that $F\colon \theta \to F(\theta)$ is continuous at some θ^* if, and only if, it is open, or, equivalently, lower-semicontinuous, at θ^*.

Theorem 3.9 (Characterization of Continuity of $\mathbf{F}(\theta)$) *Consider the convex model (P, θ) around some θ^*. The following statements are equivalent:*

(i) The point-to-set mapping F is continuous at θ^*.

(ii) For every realistic objective function f there exists a neighbourhood $N(\theta^*)$ of θ^* such that

$$\tilde{F}(\theta) \neq \emptyset \text{ and uniformly bounded}$$

for every $\theta \in N(\theta^*)$. Moreover, all the limit points of $\tilde{x}(\theta) \in \tilde{F}(\theta)$, as $\theta \to \theta^*, \theta \in N(\theta^*)$, are contained in $\tilde{F}(\theta^*)$.

(iii) For every realistic objective function f there exists a neighbourhood $N(\theta^*)$ of θ^* such that

$$\tilde{F}(\theta) \neq \emptyset \text{ for every } \theta \in N(\theta^*) \text{ and } \theta \to \theta^* \text{ implies } \tilde{f}(\theta) \to \tilde{f}(\theta^*).$$

An accompanying note is that for any model (P, θ) with a continuous feasible set mapping F, the optimal solution mapping $\tilde{F}$ is closed. The above characterization is crucial because it is integral to our definition of stability.

Definition 3.10 *A convex model (P, θ) is said to be* **stable** *at a given* $\theta^* \in \mathcal{F}$ *if the objective function is realistic and the feasible set mapping F is continuous at* θ^*.

A convex model (P, θ) will not necessarily react continuously to continuous perturbations of an input θ. Those perturbations that do preserve the lower-semicontinuity of the feasible set mapping F form a region of stability.

Definition 3.11 *Consider the convex model (P, θ) around some* θ^* *with a realistic objective function. A set* $S \subset \Re^p$, *containing* θ^*, *is called a* **region of stability** *at* θ^* *if, for every open set* $\mathcal{A} \subset \Re^n$ *satisfying*

$$\mathcal{A} \cap F(\theta^*) \neq \emptyset$$

there is a neighbourhood $N(\theta^*)$ *of* θ^* *such that*

$$\mathcal{A} \cap F(\theta) \neq \emptyset \text{ for each } \theta \in N(\theta^*) \cap S.$$

3.3 KKT Conditions

To finish this chapter, we state the two versions of the Karush-Kuhn-Tucker (KKT) conditions which serve to characterize optimality at a point x^* and we provide a saddle point condition which is necessary and sufficient for the optimality of the parameter θ^* for programs with linear constraints. The conditions are stated for a convex problem of the form:

$$\begin{array}{lll} (CP) & Min & f(x) \\ & s.t. & f^i(x) \leq 0, \qquad i \in \mathcal{P}. \end{array}$$

The first version makes use of the Lagrangian over a restricted set. The KKT conditions are said to hold, if the system

$$\nabla f(x^*) + \sum_{i \in \mathcal{P}(x^*)} u_i \nabla f^i(x^*) = 0,$$
$$u_i \geq 0, i \in \mathcal{P}(x^*),$$

is consistent, where $\mathcal{P}(x^*) = \{i : f^i(x^*) = 0\}$ denotes the set of active constraints at x^*. The latter version circumvents the restriction on the summation by incorporating the complementarity conditions:

$$\nabla f(x^*) + \sum_{i \in \mathcal{P}} u_i \nabla f^i(x^*) = 0,$$
$$u_i f^i(x^*) = 0, i \in \mathcal{P},$$
$$u_i \geq 0, i \in \mathcal{P}.$$

Note that the two systems are simultaneously either consistent or inconsistent.

Corollary 3.12 (Sufficiency of the KKT conditions) *Consider the convex program (CP), where all functions are assumed differentiable, and a feasible point x^*. If the KKT conditions are satisfied at x^*, then x^* is an optimal solution.*

The above tools extend to the parametric case for **fixed** θ. Thus, the KKT conditions, for the convex model (P, θ) with a fixed θ and a feasible point $x^* \in F(\theta)$, can be expressed as

$$\nabla f(x^*, \theta) + \sum_{i \in \mathcal{P}} u_i(\theta) \nabla f^i(x^*, \theta) = 0,$$
$$u_i(\theta) f^i(x^*, \theta) = 0, i \in \mathcal{P},$$
$$u_i(\theta) \geq 0, i \in \mathcal{P}.$$

As θ varies, so do the Lagrange multipliers (i.e., they become functions of the parameter θ). Thus, the KKT conditions also provide a characterization of optimality for the parametric problem with fixed θ.

Finally, we include a saddle-point condition for optimality of some θ^* which is based on the classic Lagrangian

$$L(x, u; \theta) = f(x, \theta) + \sum_{i \in \mathcal{P}} u_i f^i(x, \theta).$$

It is stated for linear models (i.e., convex models where $f(\cdot, \theta), f^i(\cdot, \theta): \Re^m \to \Re$, $i \in \mathcal{P}$ are linear functions. The results of this section can be extended to the so-called LFS functions (see, e.g., [60, 44]).

Theorem 3.13 (Optimal Inputs for Linear Models) *Consider the linear model (P, θ) around some θ^*. Assume that the optimal value function $\tilde{f}: \Re^p \to \Re$ exists on $\mathcal{F}$. Let x^* be an optimal solution of the program (P, θ^*). Then θ^* minimizes $\tilde{f}$ on $\mathcal{F}$ if, and only if, there exists a non-negative vector function $u^*: \mathcal{F} \to u^*(\theta) \in \Re^p_+$ such that*

$$L(x^*, u; \theta^*) \le L(x^*, u^*(\theta^*); \theta^*) \le L(x, u^*(\theta); \theta)$$

for every $u \in \Re^p_+$, every $x \in \Re^n$ and every $\theta \in \mathcal{F}$.

Chapter 4

Survey of Approaches to Post-Optimality Analysis

4.1 Radius of Stability

In [39], Kuntz and Scholtes present means of determining what they call condition numbers for an efficient DMU. These quantities, called radii of stability, provide information about the data perturbations which preserve the current efficient status of a DMU. Thus, the method is only applicable to efficient DMU's. The authors consider perturbations of the input-output vector for a particular DMU, of a particular commodity and of all sampled production vectors. The condition numbers, or radii of stability, associated with each of these perturbations are referred to as the individual condition number, the condition number with respect to a commodity i and the total condition number, respectively.

This is a cone-based CCR efficiency measure approach in which the technology cone (i.e., the set of all production possibilities given current technology as represented by the sample production vectors), is explicitly constructed. The data is structured in a production matrix A, where each column represents both inputs and outputs for a DMU. A is in turn embedded into a parametric family $A(\lambda)$ (i.e., $A = A(\lambda_\circ)$ for some parameter $\lambda_\circ$). The sensitivity analysis of an efficient DMU, say A_1, seeks to determine the maximal number ϵ such that $A_1(\lambda)$ is an efficient element of the technology cone $T[A(\lambda)] = \{w \in \Re^{m+s} | \exists z \geq 0 \colon A(\lambda)z \geq w\}$ with $\|\lambda - \lambda_\circ\| \leq \epsilon$.

The individual condition number allows for perturbations of a particular efficient DMU (i.e., $A(\lambda) = (\lambda, A_2, ..., A_N)$). It coincides with the optimal value of the program

$$\begin{aligned} Min \quad & \|x - A_1\| \\ s.t. \quad & x \in T[A_2, ..., A_N]. \end{aligned}$$

For the l_1 and l_∞ norms, this can be converted to a linear program by standard transformations.

The condition number with respect to a commodity allows for perturbations of a single, say the first, commodity. So

$$A(\lambda) = \begin{pmatrix} \lambda_1, & ..., & \lambda_N \\ a_1, & ..., & a_N \end{pmatrix},$$

where a_i are the remaining $m + s - 1$ commodities for each DMU_i. If there exists a vector $v \geq 0$ such that $a_1 v \geq 1$ and $a_i v \leq 0$ for every $i = 2, ..., N$, then the condition number of A_1 with respect to commodity 1 is infinity.

Otherwise, the condition number is bounded below by the optimal value of the program

$$\begin{aligned} Min \quad & \|\lambda - \lambda_o\| \\ s.t. \quad & -a_1 v - \lambda_1 \leq 0 \\ & a_i v + \lambda_i \leq 0, \; i = 2, ..., N \\ & v \geq 0. \end{aligned}$$

Here, λ_o represents the original unperturbed data and once again, for the l_1 and l_∞ norms, the above can be transformed into a linear program.

Finally, the total condition number allows for unrestricted perturbations. Thus, $A(\lambda) = [\lambda_1, ..., \lambda_N]$, $\lambda_i \in \Re^{m+s}$, $i = 1, ..., N$. This condition number is given by the optimal value of the program

$$\begin{aligned} Min \quad & \|[A_1, ..., A_N] - [\tilde{A}_1, ..., \tilde{A}_N]\| \\ s.t. \quad & \tilde{A}_1 v \geq 1 \\ & \tilde{A}_i v \leq 0, \; i = 2, ..., N \\ & v \geq 0. \end{aligned}$$

In all cases, the condition number is dependent upon the choice of norm. The theory is based on the result that, for a given production matrix A, if A_j, representing a sample production vector (i.e. a single vector of inputs and outputs which represents a DMU) does not belong to the technology cone spanned by the remaining sample production vectors, then A_j is an efficient element of the technology cone spanned by all sample production vectors. In terms of our terminology, if a DMU does not belong to the cone generated by the other DMU's, then it is efficient with respect to the cone generated by all DMU's. Let us illustrate the above ideas on a concrete example, borrowed from [60].

Example 4.1 Consider two DMU's, each with one input and two outputs. The data is presented in the table below.

	DMU_1	DMU_2
X	1	1
Y_1	1	1
Y_2	1	2

The CCR test gives the following models to be solved for the respective DMU's.

$$\begin{array}{llll} (CCR,1) & Max & y_1 + y_2 & \\ & s.t. & y_1 + y_2 & \leq x_1 \\ & & y_1 + 2y_2 & \leq x_1 \\ & & x_1 & = 1 \\ & & x_1,\ y_1,\ y_2 & \geq 0; \end{array}$$

$$\begin{array}{llll} (CCR,2) & Max & y_1 + 2y_2 & \\ & s.t. & y_1 + y_2 & \leq x_1 \\ & & y_1 + 2y_2 & \leq x_1 \\ & & x_1 & = 1 \\ & & x_1,\ y_1,\ y_2 & \geq 0. \end{array}$$

Both DMU's are efficient since the objective is 1 in the optimal solution. Optimal solutions are $[y_1, y_2, x_1]^T = [1, 0, 1]^T$ in both cases and dual solutions are $[p_1, p_2]^T = [1, 0]^T$ and $[0, 1]^T$ respectively. We will specifically consider DMU_2.

For our example, the individual condition number is given by the optimal value of

$$\begin{array}{lll} Min & \|x - A_2\| & \\ s.t. & -y_1 & \geq x_1 \\ & y_1 & \geq x_2 \\ & y_1 & \geq x_3 \\ & y_1 & \geq 0, \end{array}$$

where $A_2 = (-1, 1, 2)^T$ is the production vector for DMU_2. The authors in [39] use negative entries to represent inputs and positive entries for outputs. For the l_1 norm, the condition number is 4 and for the l_∞ norm, it is 2. Thus, within a cell of size 4 centered at $(-1, 1, 2)$, DMU_2 will retain its efficiency.

A lower bound for the condition number with respect to commodity 1 (the input) for DMU_2 is given by

$$\begin{array}{lll} Min & \|[\lambda_1 + 1, \lambda_2 + 1]\| & \\ s.t. & -v_1 - 2v_2 - \lambda_1 & \leq 0 \\ & v_1 + v_2 + \lambda_2 & \leq 0 \\ & v_1, v_2 & \geq 0. \end{array}$$

This bound is 2 for the l_1 norm and 1 for the l_∞ norm. So, if we perturb the input for both DMU's up to a value of 1, DMU_2 will remain efficient. Also for DMU_2, the lower bound for the condition number with respect to either output for both of the previously employed norms was zero. A cursory consideration of the example will make it clear that this bound is not very tight. This condition number indicates that a pertubation of 0 of an output will cause DMU_2 to remain efficient. This is not particularly informative and it is clear from an inspection of the data, that we could decrease output 2 by 1, retaining positivity of output 2 for DMU_1 and clearly retaining the efficiency of DMU_2.

4.2 Radius of Classification Preservation

In a series of papers [25, 48, 19] by Rousseau and Semple et al., the concept of a radius of classification preservation (RCP) is defined and discussed. Program formulations are also presented. The RCP is the largest radius of a ball centered at the current input-output position of any DMU, either efficient or inefficient, such that all the points in the interior of this ball preserve the current efficiency classification of that DMU. Clearly, the RCP is dependent on the chosen norm. Formulations, based on the ratio model, are given for both the l_1 and l_∞ norms. Different formulations are required depending on whether the DMU is classified as efficient or inefficient. For an efficient DMU, the idea is to determine by how much it can be worsened before it lies within the convex hull of the production vectors of the other DMU's, thereby making it inefficient. In the case of an inefficient DMU, one seeks to measure the minimum distance between the DMU and an unstable point. (A point is unstable if, and only if, for $\epsilon > 0$, the open ball, centered at that point, of radius ϵ, contains both efficient and inefficient points.)

There is also a combined formulation for a DMU_k which suffices for either efficient or inefficient DMU's and which gives the classification as well as the RCP:

$$\begin{aligned} Max \quad & \alpha^+ - \alpha^- \\ s.t. \quad & Y^{(k)}\lambda - s^+ + \alpha^+ e_s - \alpha^- e_s = y_k \\ & X^{(k)}\lambda + s^- - \alpha^+ e_m + \alpha^- e_m = x_k \\ & \lambda, s^+, s^-, \alpha^-, \alpha^+ \geq 0. \end{aligned}$$

Here e_s and e_m are vectors of 1's of the appropriate dimensions and $Y^{(k)}$ and $X^{(k)}$ represent matrices of outputs and inputs respectively with the column for DMU_k omitted. Based on the uniform norm, the sign of the optimal value indicates the classification, a negative identifying an inefficient DMU, a positive identifying an efficient DMU, and the optimal value gives the RCP.

Example 4.2 Consider again the two DMU's from Example 4.1. The RCP of DMU_2 for the l_∞ norm under the combined formulation is given by the optimal value of the program

$$\begin{aligned} Min \quad & \alpha^+ - \alpha^- & \\ s.t. \quad & \lambda_1 - s_1^+ + \alpha^+ - \alpha^- & = 1 \\ & 2\lambda_1 - s_2^+ + \alpha^+ - \alpha^- & = 2 \\ & \lambda_1 + s^- - \alpha^+ + \alpha^- & = 1 \\ & \lambda, s^+, s^-, \alpha^+, \alpha^- & \geq 0. \end{aligned}$$

The optimal solution is 0.5 suggesting that within a ball of radius 0.5 centered at (1,1,2), DMU_2 will remain efficient. For the l_1 norm, the RCP, given by the optimal value of

$$\begin{aligned} Min \quad & w_1^+ + w_2^+ + w^- \\ s.t. \quad & \lambda_1 - s_1^+ + w_1^+ = 1 \\ & 2\lambda_1 - s_2^+ + w_2^+ = 1 \\ & \lambda_1 + s^- + w^- = 1 \\ & \lambda_1, s_1^+, s_2^+, s^-, w_1^+, w_2^+, w^- \geq 0, \end{aligned}$$

is 1. Thus, within a cell of size 1, DMU_2 will remain efficient. The RCP of 1 is the greatest perturbation of all variables combined that will preserve this classification.

4.3 Sensitivity Analysis

In [20], Charnes, Cooper et al., introduce a sensitivity analysis method for the additive model which provides sufficiency conditions for the continued efficiency of a currently efficient DMU. Their method can be applied to inefficient DMU's, but is much simplified by restricting its application only to efficient ones. The authors restrict their attention to output reductions, one at a time, of a particular DMU, that will maintain the efficiency of that DMU.

Neralić [21] has further developed this work. He considers efficient DMU's where allowable perturbations are defined by an increase of inputs and a decrease of outputs and by a decrease of inputs and an increase of outputs for inefficient DMU's. In this manner, he is improving the status of inefficient DMU's and worsening that of efficient ones. The fundamental idea is to determine by how much inputs or outputs can be perturbed before there is a change in the optimum tableau of the simplex method. He finds sufficient conditions for the optimal basis to remain unchanged with the optimal value still equal to 1, thus preserving efficiency.

Sufficiency conditions for preserving efficiency are developed for perturbations of a single input, a single output, for the simultaneous change of all inputs and all outputs, both separately and combined, of a particular DMU. More recently, in [22, 23, 45, 46], Neralić has introduced sufficiency conditions for the simultaneous change of all data, for the decrease of all outputs of efficient DMU's with fixed inputs, for the decrease of a single output of efficient DMU's with an increase of the same single output of inefficient DMU's and, similarly, for the increase of a single input of efficient DMU's and a decrease of the same input for inefficient DMU's, and for changes in inputs and outputs of different proportionalities. Let us illustrate Neralić's ideas on the DMU's from Example 4.1.

Example 4.3 Applying Neralić's methods of sensitivity analysis to our two DMU example, we first chose to decrease output 2 of DMU_2 by α, making the output vector $[1, 2-\alpha]$. The sufficiency conditions reduce to $0 \leq \alpha \leq 1$. Reduction of output 2 for DMU_2 by 1 results in two identical and necessarily efficient DMU's. Reducing output 2 by more than 1 results in DMU_2 being in a worse situation that DMU_1, in the sense that it produces less output for the same amount of input, thereby making it inefficient. Similarly, for output 1, with the output vector $[1-\alpha, 2]$, sufficiency conditions for continued efficiency of DMU_2 require α to be zero. If we increase the input by β and decrease output 2 by α, the conditions imply that $\beta = 0$ and $0 \leq \alpha \leq 1$. So, for DMU_2, with input $1 + \beta$ and output $[1, 2 - \alpha]$, we must not increase the input and may decrease the output 2 by at most 1 in order to retain the efficiency of DMU_2 in comparison to an unchanged DMU_1. These results are easily seen by an inspection of the original data. It should be noted that the determination of these sufficiency conditions were lengthy to compute, even for such a simple example.

4.4 Cone-Ratio Model

Charnes, Cooper, Huang, Sun and Wei [10, 11, 18] have developed theory for a cone-ratio model, specifically for polyhedral cones, and its relation to multi-objective programming. This cone-ratio CCR model is based on the original CCR model, but it allows for infinitely many DMU's and arbitrary closed convex cones for the virtual multipliers. This method can also be adapted to other models such as the additive model. The cone-ratio model allows for the inclusion of additional relevant information in the construction of a more adequate DEA model either by direct modification of the production possibility set or by restricting the ranges on the marginal rates of substitution of inputs or outputs.

The cone-ratio CCR model in the case of a finite number of DMU's is given by

$$\begin{aligned} Max \quad & \mu^T Y_k \\ s.t. \quad & \omega^T X + \mu^T Y \leq 0 \\ & \omega^T X_k = 1 \\ & \omega \in V, \ \mu \in U. \end{aligned}$$

X and Y represent matrices with input and output vectors respectively as columns for all DMU's. X_k and Y_k are the input and output vectors of the particular DMU under consideration. A DMU is efficient if $\mu^T Y_k = 1$, $\omega^* \in Int\ V$ and $\mu^* \in Int\ U$, where '$*$' indicates an optimal solution of the above problem.

In particular, for polyhedral cones, (i.e., cones which can be represented as an intersection of a finite number of half-spaces) one can write

$$\begin{aligned} V &= \{v \in \Re^m : v = A^T y \ for\ some\ y \geq 0\} \\ U &= \{u \in \Re^s : u = B^T z \ for\ some\ z \geq 0\}, \end{aligned}$$

for some m x l matrix A and some s x k matrix B. Thus, the problem can be reformulated as

$$\begin{aligned} Max \quad & z^T(BY_k) \\ s.t. \quad & y^T(AX) + z^T(BY) \leq 0 \\ & y^T(AX_k) = 1 \\ & y \geq 0, \ z \geq 0 \\ & y \in E^l, \ z \in E^k. \end{aligned}$$

The authors show that certain fundamentals from linear programming theory hold for the cone-ratio model, such as weak and strong duality theorems. They establish the existence of at least one efficient DMU in the cone-ratio model and the projection of inefficient DMU's onto the efficiency surface, as well as the equivalence of an efficient DMU and a non-dominated solution from multi-objective programming. It is proven that a DMU ranked as efficient under the CCR test will also be efficient in the cone-ratio model if the optimal dual solution from the CCR test is contained in the constraint cone used in the cone-ratio model. Otherwise, it will be ranked as inefficient.

The additional constraint cones may be selected so as to emphasize particular inputs or outputs or so as to favour individual DMU's. A constraint cone tilted towards any objective, whether an input or an output, will emphasize that objective. To favour certain DMU's, one can place bounds on the virtual multipliers. Those DMU's with ratios of multipliers falling within the ranges specified by the bounds will remain efficient. In the case of polyhedral constraint cones, one can use the cone spanned by the optimal CCR test dual vectors of those DMU's deemed most efficient by expert opinion.

Example 4.4 Let us use this method for the two DMU's in Example 4.1. We leave the input multipliers unaffected and constrain the ouput multipliers by defining the cone $U = \{A^T\alpha \colon \alpha \geq 0\}$ with

$$A^T = \begin{bmatrix} 1 & a^2 \\ a^1 & 1 \end{bmatrix}.$$

This is equivalent to perturbing the data in the following manner:

	DMU_1	DMU_2
X	1	1
Y_1	$1 + a^1$	$1 + 2a^1$
Y_2	$1 + a^2$	$2 + a^2$

If a^1 is sufficiently small and a^2 is sufficiently large, this approximates to

	DMU_1	DMU_2
X	1	1
Y_1	1	1
Y_2	a^2	a^2

Thus, a^1 is dominated by the observed value of y_1, and a^2 dominates the observed value of y_2. The DMU's originally producing the most y_2 will still be efficient. Here, the efficiency of both DMU's will be preserved under the additional restrictions. If a^2 is sufficiently small and a^1 is sufficiently large, the situation is reversed. In that case, only DMU_2 would maintain its efficiency. Similar cones can be constructed for the inputs. Only those DMU producing the least amount of the emphasized input would retain their efficiency ranking. In order for us to initiate an emphasis towards a DMU by using appropriate optimal virtual multipliers to construct the cone, we would require a larger sample size.

4.5 Assurance Regions

Thompson [55] has done a great deal of work on a concept that he refers to as Assurance Regions (AR). The method was first used in determining the optimal location for a Superconducting Super Collider physics laboratory in the U.S. when standard DEA techniques returned all six possible locations as efficient. The technique is applicable to any model; it seeks to include 'price-cost' inequality bounds for the mathematical multipliers in the DEA problem, thereby reducing the number of efficient DMU's. One can use survey data and expert opinion to specify or estimate boundary conditions for the virtual multipliers.

For DEA problems with a finite number of DMU's and a well-defined data domain, Thompson defines a Strict AR to be a subset of virtual multipliers W such that the vectors w, excluded from the AR, are not reasonable virtual multipliers. A Flexible AR is defined such that it permits a low probability for the exclusion of some reasonable virtual multipliers. Here, the set of all virtual multipliers W is equal to the union of the AR and the set of excluded virtual multipliers.

Initially, Thompson considers a sequence of Assurance Regions which gradually refine the original feasible set. Under this AR-I principle, the regions can be specified as separable cones by using homogeneous sets of linear inequalities. Specifically, he looks at the use of cone-ratios, but this restriction is not necessary. In particular, he employs the use of bounds on the ratios of virtual multipliers. Thus, an AR-I model can be described by:

$$
\begin{aligned}
Max \quad & y^T Y^k & \\
s.t. \quad & x^T X^k & = 1 \\
& y^T Y - x^T X & \leq 0 \\
& y_j \alpha_{ij}^k & \leq y_i, \; j = 1, ..., s, \; i = 1, ..., s \\
& x_i \beta_{ij}^k & \leq x_j, \; j = 1, ..., m, \; i = 1, ..., m \\
& x, y & \geq 0,
\end{aligned}
$$

where α and β represent bounds on the changes in consumption or production levels. This reflects a more realistic input-output behaviour dynamic. Practically, one can construct an input cone and an output cone. The AR-I consists of all vectors $w = [v, \omega]^T$ where v belongs to the output cone and ω belongs to the input cone.

AR-I does not specify any interrelationships between the input and output prices. So Thompson theoretically considers an additional tightening AR-II which specifies the input and outputs cones and defines linkage constraints $\alpha_{ij} \leq y_j/x_i \leq \beta_{ij}$ between the input and output virtual multipliers. These non-separable cones allow the inclusion of price ratios relative to marginal rates of substitution in the model.

4.6 Dynamic Efficiency

As complement to Thompson's development of Assurance Regions, Van Rooyen [56] uses parametric analysis as a means of interpretating the CCR efficiency test. He also introduces a modified test which measures the dynamic efficiency of a DMU.

For the CCR test, he shows that

$$y_i^{(k)} = \frac{\partial E^{(k)}}{\partial Y_i^{(k)}} \; and \; x_j^{(k)} = -\left(\frac{\partial E^{(k)}}{\partial X_j^{(k)}}\right)\frac{1}{E^{(k)}},$$

where $E^{(k)}$ is the efficiency evaluation for an inefficient DMU_k and $x_i^{(k)}$ and $y_j^{(k)}$ are the largest individual components of the optimal solution (x^*, y^*). So $y_i^{(k)}$ is the rate of increase of efficiency of DMU_k for an increase in output i while $x_j^{(k)}$ is the rate of decrease of efficiency for an increase in input j. In addition, if the efficiency remains constant under perturbations of the data, then $x_i^{(k)}/y_j^{(k)}$, when well defined, is the shadow price of outputting y_j relative to inputting x_i.

The modified test is

$$\begin{aligned} Max \quad & (Y^k, y) \\ s.t. \quad & (Y^k, y) - (X^k, x) \leq \epsilon e_k \\ & (X^k, x) = 1 \\ & x \geq 0 \\ & y \geq 0. \end{aligned}$$

This program is dependent upon a fixed ϵ, $|\epsilon| < 1$, chosen to reflect the marginal efficiency over the usual efficiency. It measures how much better an efficient DMU can become in relation to its efficient peers. Similarly, one can determine by how much such a DMU can be worsened by minimizing the above model.

Example 4.5 In an application to the two DMU's from Example 4.1, we have imposed linkage constraints $\alpha x \leq y_1 \leq \beta x$ on the feasible set, inducing an increasingly restrictive cone. This is equivalent to placing bounds on the marginal rate of substitution of output 1 with respect to the input. Thompson refers to this as AR-II. It is also suggested in Van Rooyen and in the cone-ratio methods described earlier. The results obtained were:

α	β	DMU_1	DMU_2
2	4	*infeasible*	*infeasible*
1	3	1	1
0.5	1	1	1
0.5	0.75	0.875	1
0	0.5	0.75	1

Van Rooyen's dynamic model for determining how much better an efficient DMU can become than its efficient peers gives the following results:

ϵ	DMU_1	DMU_2
0.5	1	1.5
0.1	1	1.1
0.01	1	1.01
0.001	1	1.001
0.0001	1	1.0001

Clearly, DMU_2 is more resilient and its efficient ranking is more secure than DMU_1. An attempt to determine how much worse a DMU could become resulted in unbounded solutions for both DMU's for the the values of ϵ given above.

4.7 Structural Efficiency

In [60], Zlobec suggests a form of parametric post-optimality analysis, the goal of which is to reduce the number of DMU's classified as efficient. The feasible set of an efficient DMU_k is systematically decreased in an economically meaningful way using the model:

$$\begin{aligned}
(PO, \theta) \qquad & Max \quad (y, Y^k) \\
& s.t. \quad (y, Y^k) \leq (x, X^k), \qquad j = 1, ..., N \\
& \qquad (x, X^k) = 1 \\
& \qquad x, y \geq 0 \\
& \qquad x \in C_1(\theta), \ \ y \in C_2(\theta).
\end{aligned}$$

Here $C_1(\theta)$ and $C_2(\theta)$ are cones of the form

$$\begin{aligned}
C_1(\theta) &= \{x : x = A(\theta)u, \ u \geq 0\} \\
C_2(\theta) &= \{y : y = B(\theta)v, \ v \geq 0\},
\end{aligned}$$

and $A(\theta)$ and $B(\theta)$ are matrices whose elements are continuous functions of the vector parameter θ. Thus, a structurally efficient DMU, relative to some prescribed θ°, is defined to be an efficient DMU that remains efficient for every sufficiently small stable perturbation from θ° in the model (PO, θ).

Zlobec notes that the classification of a DMU as structurally efficient is dependent on the initial θ° and on the allowable perturbations. Also, every efficient DMU that remains efficient after applying (PO, θ), without restricting perturbations to stable paths, is structurally efficient but the reverse is not necessarily true. The intent of the method is related to Thompson's Assurance Regions or to the Cone-Ratio methods with the additional requirement of stable perturbations.

Example 4.6 Recall the two DMU's from Example 4.1. Notice that the CCR model for DMU_2 in our example reduces to a program in two variables:

$$\begin{aligned} Max \quad & y_1 + 2y_2 \\ s.t. \quad & y_1 + y_2 \quad \leq 1 \\ & y_1 + 2y_2 \quad \leq 1 \\ & y_1,\ y_2 \quad \geq 0. \end{aligned}$$

We will take

$$C_1(\theta) = \{x \in \Re | x = u,\ \ u \geq 0\}$$

and

$$C_2(\theta) = \{y \in \Re^2 | y = \begin{pmatrix} 1 & -\theta \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},\ v \geq 0\},$$

where θ is allowed to vary in the interval $[0, 2]$. We are imposing no change on the inputs from the usual CCR test and we are invoking a decrease in each of the outputs by the parameter θ. For $1 \leq \theta \leq 2$, the feasible set consists of the single point $[y_1, y_2]^T = [0, 0]^T$ and so the model has an objective value equal to 0. For $0 < \theta < 1$, the feasible set is a triangular region determined by the constraints

$$\begin{aligned} v_1 + v_2 \leq \frac{1}{(1-\theta)} \\ (1-2\theta)v_1 + (2-\theta)v_2 \leq 1 \\ v_1,\ v_2 \geq 0 \end{aligned}$$

where the original variables y_1 and y_2 are transformed by

$$\begin{aligned} y_1 &= v_1 - \theta v_2 \\ y_2 &= -\theta v_1 + v_2. \end{aligned}$$

The optimal value is 1 so DMU_2 remains efficient for these perturbations. For $\theta = 0$, we recover the original problem.

An increase in θ starting from 0 does not affect the efficiency ranking. A local increase from $\theta = 1$ causes DMU_2 to lose its efficiency, but such perturbations are not stable because the feasible set drops to a point and, therefore, DMU_2 is considered structurally efficient. Note that it would not remain efficient under equivalent cone restrictions in the cone-ratio model.

4.8 Summary

In this chapter, we have studied some of the popular post-optimality analysis approaches in DEA. Every approach described in this chapter has different goals and is structured to permit particular, and differing, perturbations of the data. The methods can be divided into one of two classes: those that seek to gradually restrict the feasible set in a meaningful way in order to eliminate some of the DMU's classified as efficient, and those that attempt to determine those perturbations which preserve the efficiency classification of a particular DMU while the others remain unperturbed. Those that fall into the former category include the cone-ratio model, assurance regions and structural efficiency. A gradual elimination process is effected by adding and then tightening additional constraints based on economic interpretations.

The idea of a structurally efficient DMU is advantageous because there are no limitations as to the allowable perturbations of the data. Also, interest is restricted to stable regions so that the feasible set and the optimal solutions change in a continuous (economically meaningful) fashion.

The latter methods are dependent on the choice of norm and are only readily solvable for the l_1 and l_∞ norms. In these cases, the programs can be reduced to linear ones, using familiar substitutions. Perturbations are, in general, restricted to very particular cases in the data of a single DMU. In addition, results may be misleading if the method is dependent on maintaining the original optimal basis. In this case, a global maximum may not be found since DEA problems are degenerative in nature, and thus multiple optimal bases are likely to exist.

Chapter 5

Radius of Rigidity Approach

In this chapter, we study a new approach in which we classify efficient DMU's by their "radius of rigidity". The radius of rigidity problem seeks to determine the maximal radius that guarantees the continued efficiency of a currently efficient DMU. Much of the work that has been done in this field involves perturbing the data only for the DMU currently under consideration, whether it be efficient or inefficient. Here, we shall restrict our interest to efficient DMU's, but we will allow perturbations of the data for all DMU's except the DMU currently under consideration.

5.1 Theory for Radius of Rigidity Approach

In the theory that follows, the maximal radius of rigidity problem is formulated as the parametric optimization problem

$$
\begin{aligned}
(L,\theta,k) \qquad & Max \quad \| \theta \| \\
& s.t. \quad (Y^j(\theta), y) \leq (X^j(\theta), x) \qquad j = 1, ..., N,\ j \neq k \\
& \qquad (X^k, x) = 1 \\
& \qquad (Y^k, y) = 1 \\
& \qquad y \geq 0 \\
& \qquad x \geq 0.
\end{aligned}
$$

We have referred to our problem as (L, θ, k). The θ indicates a parametrized problem; the L stands for linear. Since the objective function is a norm, it is convex in θ and, for fixed θ, all constraint functions are linear in x and y. Notice that we are maximizing a convex function. We will thus restrict our attention to norms which can be transformed into linear functions, for example, l_1 or l_∞. More work needs to be done for the l_∞ norm since the usual transformation for minimizing this norm does not hold for a maximization problem. For the moment, we will specifically consider the l_1 norm.

We initially consider an efficient DMU_k at $\theta = 0$, an unperturbed problem. The rounded brackets indicate the Euclidean inner product and θ is a vector determined by the perturbations which one wishes to consider. The constraints are identical to those in the original CCR test except for the addition of the constraint $(Y^k, y) = 1$ and the deletion of one of the reference set constraints, $(Y^k, y) \leq (X^k, x)$. The additional constraint forces DMU_k to remain efficient for perturbations of θ not equal to zero. The constraint that was removed was necessarily active but was redundant because the problem already specifies that $(Y^k, y) = 1$ and $(X^k, x) = 1$. As in the CCR test, we assume that all components of X^j and Y^j, $j = 1, ..., N$, are strictly greater than zero. We can now formally define the radius of rigidity.

Definition 5.1 *The* **radius of rigidity** *of an efficient* DMU_k *is the optimal value of* (L, θ, k).

The radius of rigidity for an efficient DMU_k is a uniform measure of the greatest allowable specified perturbation of the data which ensures the continued efficiency of DMU_k. The goal of the radius of rigidity problem is to provide a means of ranking the efficient DMU's. The interpretation of the radius of rigidity is as follows: it provides an indication of how secure DMU_k's efficient ranking is with respect to the assumed attempts at improvement by other DMU's. In other words, it indicates how long an efficient DMU_k can remain static and still be efficient while the other DMU's continue to improve. In order to make comparison meaningful, we scale all input and output data to be between zero and one. (This can be done by dividing each input (output) by the largest one of the same type. Recall Section 2.3 for a justification.)

If we assume that inefficient DMU's will attempt to improve their efficiency, then we should define the perturbations for these DMU's by

$$X_i(\theta) = [X - \theta]_i, \; i = 1, ..., m \tag{5.1}$$

$$Y_j(\theta) = [Y + \theta]_j, \; j = 1, ..., s \tag{5.2}$$

$$\theta \geq 0. \tag{5.3}$$

Note that for $X(0)$, $Y(0)$, i.e., when $\theta = 0$, DMU_j is unperturbed. For an efficient DMU, there are two possible scenarios. Either we assume that these efficient DMU's are attempting to increase their competitive edge, in which case perturbations for these DMU's would be specified as above. Or, we may assume that they choose to cut back on production while remaining efficient. In that case, the perturbations of their data should be defined by

$$X_i(\theta) = [X + \theta]_i, \; i = 1, ..., m \tag{5.4}$$

$$Y_i(\theta) = [Y - \theta]_i, \; i = 1, ..., s \tag{5.5}$$

$$\theta \geq 0. \tag{5.6}$$

One may tailor the perturbations according to the type of situation regarding which one seeks information.

Let us point out that although we consider the radius of rigidity for an efficient DMU, the problem could be formulated for an inefficient DMU if, for instance, one wished to determine how resilient an inefficient DMU was to perturbations of the data. This could be done simply by replacing the constraint $(Y^k, y) = 1$ by $(Y^k, y) \leq 1$ or by forcing (Y^k, Y) to be greater than the actual efficiency evaluation.

An alternate approach to the radius of rigidity problem is given by

$$
\begin{array}{lll}
(L', \theta, k) & Max & \| \theta \| + (X^k, x) \\
 & s.t. & (Y^j(\theta), y) \leq (X^j(\theta), x) \qquad j = 1, ..., N, \; j \neq k \\
 & & (X^k, x) \leq 1 \\
 & & (Y^k, y) = 1 \\
 & & y \geq 0 \\
 & & x \geq 0,
\end{array}
$$

provided the optimal value of $(X^k, x) = 1$. However, it remains to be shown that the optimal value of the expression (X^k, x) is always 1, and that infeasibility indicates that DMU_k can no longer remain efficient under the perturbation θ. This problem is equivalent to a multi-objective program with equal weights on the two objectives.

It is well known that the KKT conditions are necessary and sufficient conditions for optimality of programs with linear constraints. (For the KKT conditions, see Section 3.3.) However, the above model is not a linear program (jointly in the (x, y, θ) variable). So to check optimality, we need different optimality conditions. First, we observe the equivalence between the KKT optimality conditions and the saddle point characterization of optimality for a fixed θ. Consider the linear model:

$$\begin{array}{lll} (L,\theta) & Min_x & f(x,\theta) \\ & s.t. & f^i(x,\theta) \leq 0, \ i \in \mathcal{P}, \end{array}$$

where $f(\cdot,\theta)$ and $f^i(\cdot,\theta)$, $i \in \mathcal{P}$ are assumed to be linear for every θ. Let us construct the corresponding classic Lagrangian

$$L(x, u; \theta) = f(x, \theta) + \sum_{i \in \mathcal{P}} u_i f^i(x, \theta)$$

and define m to have the cardinality of $\mathcal{P}$.

Theorem 5.2 *Consider (L, θ) where all functions are differentiable in x around some θ. Then, for every fixed $\theta \in \Re^p$, at an arbitrary $x^* \in F(\theta)$, the KKT conditions*

$$(5.7) \qquad \nabla_x f(x^*, \theta) + \sum_{i \in \mathcal{P}} \tilde{u}_i(\theta) \nabla_x f^i(x^*, \theta) \ = \ 0$$

(5.8) $$\tilde{u}_i(\theta) \geq 0$$

(5.9) $$\tilde{u}_i(\theta) f^i(x^*, \theta) = 0, \ i \in \mathcal{P}$$

are satisfied if, and only if, for the same vector function $\tilde{u}(\theta)\colon \Re^p \to \Re^m_+$, *we have*

(5.10) $$L(x^*, u; \theta) \leq L(x^*, \tilde{u}(\theta); \theta) \leq L(x, \tilde{u}(\theta); \theta)$$

for every $x \in \Re^n$ *and for every* $u \in \Re^m_+$.

This is a well-known fact. (See, e.g., [62].)

In order to solve the problem (L, θ, k), we will use a marginal value formula. First, we need conditions for the boundedness and continuity of the Lagrange multiplier function. (We do not know how to prove the marginal value formula if these functions are discontinuous.) It is well known that Lagrange multipliers are unbounded if the constraints do not satisfy Slater's condition.

We will establish a form of the marginal value formula which uses a restricted Lagrangian. For a general convex model

$$\begin{aligned} (P, \theta) \qquad & Min_x \quad f(x, \theta) \\ & s.t. \quad f^i(x, \theta) \leq 0, \qquad i \in \mathcal{P}, \end{aligned}$$

around a fixed $\theta^* \in \mathcal{F}$, define

(5.11) $$L^<_*(x, u, \theta) = f(x, \theta) + \sum_{i \in \mathcal{P}^<(\theta^*)} u_i(\theta) f^i(x, \theta),$$

where $\mathcal{P}^<(\theta^*)$ is as defined in Chapter 3. Let c be the cardinality of $\mathcal{P}^<(\theta^*)$.

Theorem 5.3 *Consider the convex model (P, θ) around some θ^* and a region of stability S at θ^*. Then there exists a neighbourhood $N(\theta^*)$ of θ^* such that, for every fixed $\theta \in N(\theta^*) \cap S$, $\tilde{x}(\theta) \in F_*^=(\theta)$ is an optimal solution of the program (P, θ) if, and only if, there exists a vector function $\tilde{u}: N(\theta^*) \cap S \to \Re_+^c$ such that*

$$L_*^<(\tilde{x}(\theta), u; \theta) \leq L_*^<(\tilde{x}(\theta), \tilde{u}(\theta); \theta) \leq L_*^<(x, \tilde{u}(\theta); \theta) \tag{5.12}$$

for every $u \in \Re_+^c$ and every $x \in F_*^=(\theta)$.

The proof of this theorem is given in [60]. The proof given there also establishes the fact that

$$\sum_{i \in \mathcal{P}^<(\theta^*)} \tilde{u}_i(\theta) f^i(\tilde{x}(\theta), \theta) = 0. \tag{5.13}$$

Theorem 5.4 *Consider the convex model (P, θ) around some θ^* with a realistic objective funciton. Let S be a region of stability at θ^* and let $F_*^=$ be lower-semicontinuous at θ^* relative to S. Consider a sequence of sets of Lagrange multipliers $\{\tilde{u}(\theta)\}$ from the saddle-point condition (5.12) as $\theta \to \theta^*$, $\theta \in S$. Then every sequence $u(\theta) \in \{\tilde{u}(\theta)\}$ is uniformly bounded for all θ sufficiently close to θ^* and all its accumulation points are in $\{\tilde{u}(\theta^*)\}$.*

The assumptions for the MVF require lower-semicontinuity of $F^=(\theta)$ and uniqueness of the optimal solution and of the saddle point. We will also require certain assumptions about θ. Let us consider perturbations in (a particular region of stability):

$$S = \{\theta: \mathcal{P}^=(\theta) = \mathcal{P}^=(\theta^*)\} \cap \{\theta: \tilde{F}(\theta^*) \subset F^=(\theta), \tilde{F}(\theta) \subset F^=(\theta^*)\}.$$

Theorem 5.5 (Marginal Value Formula) *Consider the convex model (P,θ) with a realistic objective function at some θ^*. Let us assume that the mapping $F^=: \theta \to F^=(\theta)$ is lower-semicontinuous at θ^* relative to the set S, and that the saddle point $\{\tilde{x}(\theta^*), \tilde{u}(\theta^*)\}$ is unique. Also suppose that the gradients $\nabla_\theta f(x,\theta)$, $\nabla_\theta f^i(x,\theta)$, $i \in \mathcal{P}^<(\theta^*)$ are continuous at $(\tilde{x}(\theta^*), \theta^*)$. Then, for every sequence $\theta \in S$, $\theta \to \theta^*$ for which the limit*

$$l = \lim_{\theta \in S; \theta \to \theta^*} \frac{\theta - \theta^*}{||\theta - \theta^*||}$$

exists, we have

$$(MVF) \qquad \lim_{\theta \in S; \theta \to \theta^*} \frac{\tilde{f}(\theta) - \tilde{f}(\theta^*)}{||\theta - \theta^*||} = \nabla_\theta L_*^<(\tilde{x}(\theta^*), \tilde{u}(\theta^*); \theta^*) \cdot l.$$

Proof: First we note that, for perturbations in S,

$$\begin{aligned} F_*^=(\theta) &= \{x: f^i(x,\theta) \le 0, \ i \in \mathcal{P}^=(\theta^*)\} \\ &= \{x: f^i(x,\theta) \le 0, \ i \in \mathcal{P}^=(\theta)\} \\ &= F^=(\theta). \end{aligned}$$

Therefore, the requirement on lower-semicontinuity of $F_*^=$ in Theorem 5.4 and the usual MVF (see [60], Theorem 10.1) can be replaced by requiring lower-semicontinuity of $F^=$. But the latter implies lower-semicontinuity of the mapping F. (See, e.g., [53], Lemma 1.4.) Hence, S is a region of stability.

We also note that for perturbations in S, the two inclusions can be rewritten as

$$\tilde{F}(\theta^*) \subset F_*^=(\theta), \ \tilde{F}(\theta) \subset F_*^=(\theta^*).$$

Using the saddle-point condition from Theorem 5.3 for $\theta \in N(\theta^*) \cap S$, we have

$$L_*^<(\tilde{x}(\theta), u; \theta) \le L_*^<(\tilde{x}(\theta), \tilde{u}(\theta); \theta) \le L_*^<(x, \tilde{u}(\theta); \theta)$$

for every $x \in F_*^{=}(\theta)$, and for every $u \in \Re_+^c$, and, in particular, at $\theta = \theta^*$,

$$L_*^<(\tilde{x}(\theta^*), v; \theta^*) \leq L_*^<(\tilde{x}(\theta^*), \tilde{u}(\theta^*); \theta^*) \leq L_*^<(z, \tilde{u}(\theta^*); \theta^*)$$

for every $z \in F_*^{=}(\theta^*)$, and for every $v \in \Re_+^c$.

By (5.13),

$$\sum_{i \in \mathcal{P}^<(\theta)} \tilde{u}_i(\theta) f^i(\tilde{x}(\theta), \theta) = 0$$

and

$$\sum_{i \in \mathcal{P}^<(\theta^*)} \tilde{u}_i(\theta^*) f^i(\tilde{x}(\theta^*), \theta^*) = 0.$$

Thus

$$L_*^<(\tilde{x}(\theta), \tilde{u}(\theta); \theta) = \tilde{f}(\theta)$$

and

$$L_*^<(\tilde{x}(\theta^*), \tilde{u}(\theta^*); \theta^*) = \tilde{f}(\theta^*).$$

After rearranging the above saddle-point conditions and adding and subtracting the term $L_*^<(\tilde{x}(\theta^*), \tilde{u}(\theta); \theta)$, we obtain

$$\begin{aligned} \tilde{f}(\theta) - \tilde{f}(\theta^*) \quad &\leq \quad L_*^<(x, \tilde{u}(\theta), \theta) - L_*^<(\tilde{x}(\theta^*), \tilde{u}(\theta); \theta) \\ &\qquad + L_*^<(\tilde{x}(\theta^*), \tilde{u}(\theta); \theta) - L_*^<(\tilde{x}(\theta^*), v; \theta^*) \end{aligned}$$

for every $x \in F_*^{=}(\theta)$, and for every $v \in \Re_+^c$.

By choosing $v = \tilde{u}(\theta^*) \in \Re_+^c$ and $x = \tilde{x}(\theta^*) \in F_*^{=}(\theta)$ by the assumptions, we have

$$\begin{aligned} \tilde{f}(\theta) - \tilde{f}(\theta^*) \quad &\leq \quad L_*^<(\tilde{x}(\theta^*), \tilde{u}(\theta), \theta) - L_*^<(\tilde{x}(\theta^*), \tilde{u}(\theta); \theta) \\ &\qquad + L_*^<(\tilde{x}(\theta^*), \tilde{u}(\theta); \theta) - L_*^<(\tilde{x}(\theta^*), \tilde{u}(\theta^*); \theta^*) \\ &\leq \quad \sum_{i \in \mathcal{P}^<(\theta^*)} [\tilde{u}_i(\theta) - \tilde{u}_i(\theta^*)] f^i(\tilde{x}(\theta^*), \theta) + \nabla_\theta L_*^<(\tilde{x}(\theta^*), \tilde{u}(\theta^*); w)(\theta - \theta^*) \end{aligned}$$

for some point w between θ and θ^*, by the Mean Value Theorem.

The index set $\mathcal{P}^<(\theta^*)$ can be split into two parts: the active constraints

$$\mathcal{P}_a = \{i \in \mathcal{P}^<(\theta^*) \colon f^i(\tilde{x}(\theta^*), \theta^* = 0\}$$

and the nonactive constraints

$$\mathcal{P}_n = \{i \in \mathcal{P}^<(\theta^*) \colon f^i(\tilde{x}(\theta^*), \theta^* < 0\}.$$

Note that

$$\sum_{i \in \mathcal{P}_n} [\tilde{u}_i(\theta) - \tilde{u}_i(\theta^*)] f^i(\tilde{x}(\theta^*), \theta) \leq 0$$

for θ's close to θ^*. (Recall that $\tilde{u}_i(\theta^*) = 0$, $i \in \mathcal{P}_n$, by the KKT condition). Also note that

$$\begin{aligned} & \sum_{i \in \mathcal{P}_a} [\tilde{u}_i(\theta) - \tilde{u}_i(\theta^*)] f^i(\tilde{x}(\theta^*), \theta) \\ = & \sum_{i \in \mathcal{P}_a} [\tilde{u}(\theta) - \tilde{u}_i(\theta^*)][f^i(\tilde{x}(\theta^*), \theta) - f^i(\tilde{x}(\theta^*), \theta^*)] \end{aligned}$$

and hence

$$\lim_{\theta \in S; \theta \to \theta^*} \sum_{i \in \mathcal{P}_a} [\tilde{u}(\theta) - \tilde{u}_i(\theta^*)] \frac{f^i(\tilde{x}(\theta^*), \theta) - f^i(\tilde{x}(\theta^*), \theta^*)}{\|\theta - \theta^*\|} = 0$$

whenever $\tilde{u}_i(\theta) \to \tilde{u}_i(\theta^*)$, $i \in \mathcal{P}^<(\theta^*)$, by Theorem 5.4, and because the functions $f^i(\tilde{x}(\theta^*), \cdot)$, $i \in \mathcal{P}$ are assumed differentiable.

On the other hand,

$$\tilde{f}(\theta) - \tilde{f}(\theta^*) \geq L_*^<(\tilde{x}(\theta), u; \theta) - L_*^<(z, \tilde{u}(\theta^*); \theta^*)$$

for every $x \in F_*^=(\theta^*)$ and $u \in \Re_+^c$.

After adding and subtracting the same term, and specifying $u = \tilde{u}(\theta^*)$ and $z = \tilde{x}(\theta) \in F_*^=(\theta^*)$ by the assumtion, the first and third terms cancel. This yields

$$\tilde{f}(\theta) - \tilde{f}(\theta^*) \geq \nabla_\theta L_*^<(\tilde{x}(\theta), \tilde{u}(\theta^*); w)(\theta - \theta^*)$$

for some w between θ and θ^* by the Mean Value Theorem. Bounding $\tilde{f}(\theta) - \tilde{f}(\theta^*)$ from both sides with the above inequalities, dividing by $||\theta - \theta^*||$ and letting $\theta \to \theta^*$ completes the proof.

The above proof was essentially given in [62]. We have reproduced it here for the sake of completeness. The same formula holds under different assumptions, as will be shown in Theorem 5.6. But first, let us explain in the next section why such a formula is important.

5.2 Input Optimization

Input optimization employs the marginal value formula in an iterative manner to improve the current optimal value of the objective function. In our particular case, we wish to find the largest possible radius of rigidity and, therefore, are maximizing the objective function. In application, we will consider the problem as the minimization of the negative objective function. Thus, we wish to find a path along which $\tilde{f}(\theta)$ is smaller than $\tilde{f}(\theta^*)$ or, in other words, where the left-hand side of the marginal value formula is strictly negative. The path should preserve continuity of the feasible point-to-set mapping F.

The input optimization process is two-fold. At each step, one determines a stable path of improvement, and then determines the maximal step that can be taken along that path so that the path remains feasible and there is local improvement of the objective. Once we have determined a better input, we repeat the process until a locally optimal input has been determined. Note that different initial choices for θ generally result in different locally optimal inputs. However, in our case, we will always begin with $\theta^0 = 0$ (i.e., unperturbed data).

While any choice of path is acceptable, we simplify our calculations by considering only linear paths. Thus, we are interested in changes to some known θ^n along the path

$$\theta^{n+1} = \theta^n + \alpha d^n, \qquad \alpha \geq 0, \tag{5.14}$$

where d^n is the direction of improvement. A direction d^n which guarantees that the left hand side of the MVF be negative and, therefore, that the optimal value of the objective function is locally decreasing, can be determined by

$$d^n = -\nabla_\theta L_*^<(\tilde{x}(\theta^n), \tilde{u}(\theta^n); \theta^n).$$

Step two concerns the step size problem: how to determine an optimal α_n. By substituting $\theta^{n+1} = \theta^n + \alpha_n d^n$ into (L, θ, k), one is left with the problem

$$
\begin{aligned}
Min \quad & -||\theta^n + \alpha_n d^n|| \\
s.t. \quad & (Y^j(\alpha_n), y) \leq (X^j(\alpha_n), x), \; j = 1, ..., N, \; j \neq k \\
& (X^k, x) = 1 \\
& (Y^k, y) = 1 \\
& x \geq 0 \\
& y \geq 0 \\
& \alpha_n \geq 0.
\end{aligned}
$$

As $\tilde{f}(\alpha_k)$ is generally not known explicitly, we would use approximation techniques such as the Golden Section Method. This is a numerical approximation technique which allows us to localize an optimum of a function by gradually reducing the interval in which the optimum lies. The process works for unimodal functions in one variable. (These are functions which have a local minimum x^* and for which the following properties hold:

$$if \; x_1 < x_2 \leq x^*, \; then \; f(x_1) > f(x_2)$$

and

$$if \; x^* \leq x_1 < x_2, \; then \; f(x_2) > f(x_1).)$$

The first step is to localize the interval $[a_0, b_0]$ which ensures feasibility and contains the local minimum by evaluating the function at two points. Then we symmetrically reduce the original interval by calculating the value of the function at only one point. (For details, see, e.g., [60]). We will return to this method in Sections 8.2 and 8.3.

5.3 The Marginal Value Formula Revisited

In this section, we derive the marginal value formula under different assumptions than those in Theorem 5.5. These new assumptions appear to be more appropriate for the radius of rigidity approach defined by (L′, θ, k). Our problem is of the form

$$\begin{array}{lll} (L',\theta) & Min_x & f(x,\theta) \\ & s.t. & f^i(x,\theta) \leq 0,\ i \in Q \\ & & A(\theta)x = b. \end{array}$$

Define the restricted Lagrangian involving only the objective and the inequality constraints as

$$L^{\subseteq}(x,u;\theta) = f(x,\theta) + \sum_{i\in Q} u_i(\theta) f^i(x,\theta)$$

and the saddle-point condition

$$(5.15) \qquad L^{\subseteq}(\tilde{x}(\theta),u;\theta) \leq L^{\subseteq}(\tilde{x}(\theta),\tilde{u}(\theta);\theta) \leq L^{\subseteq}(x,\tilde{u}(\theta);\theta)$$

for $x \in F_*^{=}(\theta)$ and $u \in \Re_+^{c'}$, where c' is the cardinality of Q.

The new assumptions require that there exists a generalized Slater's point at θ^*. We say that the constraints of (L′, θ^*) satisfy the generalized Slater's condition if there exists a point x' such that

$$\begin{array}{rcll} f^i(x',\theta^*) & < & 0, & i \in Q \\ A(\theta^*)x' & = & b. & \end{array}$$

Theorem 5.6 (MVF under Generalized Slater's Condition) *Consider the model (L', θ) with a realistic objective function at some θ^* and a region of stability S at θ^* intersected with the set $\{\theta\colon \tilde{F}(\theta^*) \subseteq F^=(\theta),\ \tilde{F}(\theta) \subseteq F^=(\theta^*)\}$. Assume that the saddle point $\{\tilde{x}(\theta^*), \tilde{u}(\theta^*)\}$ is unique and that the gradients $\nabla_\theta f(x,\theta)$, $\nabla_\theta f^i(x,\theta), i \in \mathcal{Q}$ are continuous at $\{\tilde{x}(\theta^*), \theta^*\}$. Also assume that there exists a generalized Slater's point at θ^*. Then for every sequence $\theta \in S, \theta \to \theta^*$ for which the limit*

$$l = \lim_{\theta \in S; \theta \to \theta^*} \frac{\theta - \theta^*}{\| \theta - \theta^* \|}$$

exists, we have:

$$\text{(5.16)} \quad (MVF') \quad \lim_{\theta \in S; \theta \to \theta^*} \frac{\tilde{f}(\theta) - \tilde{f}(\theta^*)}{\| \theta - \theta^* \|} = \nabla_\theta L^{\subseteq}(\tilde{x}(\theta^*), \tilde{u}(\theta^*); \theta^*) \cdot l.$$

Proof: First, we note that when there exists a generalized Slater's point at θ^*, then the set $\mathcal{Q}$ from (L', θ) is the usual $\mathcal{P}^<(\theta^*)$ from (P, θ) and $\mathcal{P}^=(\theta^*)$ is the set of equality constraints. On the region of stability S, there exists a neighbourhood $N(\theta^*)$ of θ^* such that $\mathcal{P}^=(\theta) \subseteq \mathcal{P}^=(\theta^*)$. (See, e.g., [60], Theorem 7.13.) But $\mathcal{P}^=(\theta^*)$ is already just the set of equality constraints, so the active constraints for all $\theta \in N(\theta^*) \cap S$ are also the equality constraints. In other words, in terms of our original notation, $\mathcal{P}^=(\theta^*) = \mathcal{P}^=(\theta)$. This means that $L^{\subseteq} = L^<_*$ with $c' = card\{\mathcal{Q}\}$. Also, lower-semicontinuity of $F(\theta)$ implies lower-semicontinuity of $F^=_*(\theta) = F^=(\theta)$ under the generalized Slater's condition.

Using the saddle-point condition from Theorem 5.3 for $\theta \in N(\theta^*) \cap S \cap \{\theta\colon \tilde{F}(\theta^*) \subseteq F^=(\theta),\ \tilde{F}(\theta) \subseteq F^=(\theta^*)\}$, we have

$$L^{\subseteq}(\tilde{x}(\theta), u; \theta) \leq L^{\subseteq}(\tilde{x}(\theta), \tilde{u}(\theta); \theta) \leq L^{\subseteq}(x, \tilde{u}(\theta); \theta)$$

for every $x \in F_*^=(\theta)$, and for every $u \in \Re_+^{c'}$, and, in particular, at $\theta = \theta^*$,

$$L^{\subseteq}(\tilde{x}(\theta^*), v; \theta^*) \leq L^{\subseteq}(\tilde{x}(\theta^*), \tilde{u}(\theta^*); \theta^*) \leq L^{\subseteq}(z, \tilde{u}(\theta^*); \theta^*)$$

for every $z \in F_*^=(\theta^*)$, and for every $v \in \Re_+^{c'}$.

By (5.13),

$$\sum_{i \in Q} \tilde{u}_i(\theta) f^i(\tilde{x}(\theta), \theta) = 0$$

and

$$\sum_{i \in Q} \tilde{u}_i(\theta^*) f^i(\tilde{x}(\theta^*), \theta^*) = 0.$$

Thus

$$L^{\subseteq}(\tilde{x}(\theta), \tilde{u}(\theta); \theta) = \tilde{f}(\theta)$$

and

$$L^{\subseteq}(\tilde{x}(\theta^*), \tilde{u}(\theta^*); \theta^*) = \tilde{f}(\theta^*).$$

After rearranging the above saddle-point conditions and adding and subtracting the term $L^{\subseteq}(\tilde{x}(\theta^*), \tilde{u}(\theta); \theta)$, we obtain

$$\begin{aligned} \tilde{f}(\theta) - \tilde{f}(\theta^*) \;\; &\leq \;\; L^{\subseteq}(x, \tilde{u}(\theta), \theta) - L^{\subseteq}(\tilde{x}(\theta^*), \tilde{u}(\theta); \theta) \\ &\quad + L^{\subseteq}(\tilde{x}(\theta^*), \tilde{u}(\theta); \theta) - L^{\subseteq}(\tilde{x}(\theta^*), v; \theta^*) \end{aligned}$$

for every $x \in F_*^=(\theta)$, and for every $v \in \Re_+^{c'}$.

By choosing $v = \tilde{u}(\theta) \in \Re_+^c$ and $x = \tilde{x}(\theta^*) \in \tilde{F}(\theta^*) \subseteq F^=(\theta) = F_*^=(\theta)$, since $\mathcal{P}^=(\theta) = \mathcal{P}^=(\theta^*)$, and after cancellation,

$$\begin{aligned} \tilde{f}(\theta) - \tilde{f}(\theta^*) \;\; &\leq \;\; L^{\subseteq}(\tilde{x}(\theta^*), \tilde{u}(\theta); \theta) - L^{\subseteq}(\tilde{x}(\theta^*), \tilde{u}(\theta); \theta^*) \\ &= \;\; \nabla_\theta L^{\subseteq}(\tilde{x}(\theta^*), \tilde{u}(\theta); w) \cdot (\theta - \theta^*) \end{aligned}$$

where w is some point between θ and θ^* by the Mean Value Theorem.

Similarly,

$$\begin{aligned}\tilde{f}(\theta) - \tilde{f}(\theta^*) &\leq L^{\subseteq}(\tilde{x}(\theta), u; \theta) - L^{\subseteq}(\tilde{x}(\theta), \tilde{u}(\theta^*); \theta) \\ &\quad + L^{\subseteq}(\tilde{x}(\theta), \tilde{u}(\theta^*); \theta) - L^{\subseteq}(z, \tilde{u}(\theta^*); \theta^*)\end{aligned}$$

for every $z \in F_*^=(\theta^*)$, and for every $u \in \Re_+^{c'}$.

Take $u = \tilde{u}(\theta^*) \in \Re_+^{c'}$ and $z = \tilde{x}(\theta) \in \tilde{F}(\theta) \subseteq F^=(\theta^*) = F_*^=(\theta^*)$ since $\mathcal{P}^=(\theta) = \mathcal{P}^=(\theta^*)$. Thus

$$\begin{aligned}\tilde{f}(\theta) - \tilde{f}(\theta^*) &\geq L^{\subseteq}(\tilde{x}(\theta), \tilde{u}(\theta^*); \theta) - L^{\subseteq}(\tilde{x}(\theta), \tilde{u}(\theta^*); \theta^*) \\ &= \nabla_\theta L^{\subseteq}(\tilde{x}(\theta), \tilde{u}(\theta^*); w') \cdot (\theta - \theta^*)\end{aligned}$$

where w' is some point between θ and θ^* by the Mean Value Theorem.

Dividing by $\| \theta - \theta^* \|$, we have

$$\frac{\nabla_\theta L^{\subseteq}(\tilde{x}(\theta), \tilde{u}(\theta^*); w') \cdot (\theta - \theta^*)}{\| \theta - \theta^* \|} \leq \frac{\tilde{f}(\theta) - \tilde{f}(\theta^*)}{\| \theta - \theta^* \|} \leq \frac{\nabla_\theta L^{\subseteq}(\tilde{x}(\theta^*), \tilde{u}(\theta); w) \cdot (\theta - \theta^*)}{\| \theta - \theta^* \|}.$$

As $\theta \to \theta^*$, w and w' approach θ^*, $\tilde{x}(\theta) \to \tilde{x}(\theta^*)$ and $\tilde{u}(\theta) \to \tilde{u}(\theta^*)$ by Theorem 3.9 and by Theorem 5.4, for $\theta \in S$, respectively, and by the uniqueness assumption on the saddle point.

So taking the limit, we have

$$\lim_{\theta \to \theta^*; \theta \in S} \frac{\tilde{f}(\theta) - \tilde{f}(\theta^*}{\| \theta - \theta^* \|} = \nabla_\theta L^{\subseteq}(\tilde{x}(\theta^*), \tilde{U}(\theta^*); \theta^*) \cdot l.$$

All of the above results require that we work in a region of stability. A linear model with a realistic objective function that satisfies lower-semicontinuity of $F^=(\theta)$ and a generalized Slater's condition at θ^* is stable at any θ in a feasible neighbourhood of θ^*. (See [60], Theorem 14.1.) Hence the model (L', θ, k) is stable at θ^*. We give a direct proof.

Theorem 5.7 *The radius of rigidity tests (L', θ, k), where k represents an efficient DMU, are stable at every θ^* for which the generalized Slater's condition holds.*

Proof: Consider X^j and Y^j, $j = 1, .., N$ as the parameter evaluated at some θ^*. Note that the mapping $F^=: \theta \to F^=(\theta) = \{y: (Y^k, y) = 1\}$, where $Y^k = Y^k(\theta)$, is lower-semicontinuous because $Y^k > 0$.

Clearly, the objective function is realistic. Since $X^k > 0$ and $Y^k > 0$, the set $F(\theta)$ is bounded. Thus, $\tilde{F}(\theta)$ is bounded and there exists an optimal solution for an initial θ^*.

Next we will construct a Slater's point for the inequality constraints. Take $y^* = (y_i^*) \in \Re^s$ such that $y_i^* = (sY_i^k)^{-1}, i = 1, ..., s$. Clearly, $y^* > 0$ and $(Y^k, y^*) = 1$. One can now construct $x^* \in \Re^m$ such that $x^* > 0$ and $(Y^j, y^*) < (X^j, x^*)$, $j = 1, .., N$ and $(X^k, x^*) < 1$, since the kth constraint has been removed from the reference set. Thus (L$'$, θ, k) is stable at θ^*.

5.4 A Modification for the Radius of Rigidity Approach

If a MVF does not hold, it is still possible to find a better θ, in the sense of a larger optimal value function. For each variable, we solve a single parameter model where $\theta \in \Re_+$ affects the variable under consideration for each DMU except an efficient DMU_k. The model is solved by increasing θ until the problem is no longer feasible. We recall that since the CCR model requires positivity of all data and we are subtracting θ from the inputs, bounds must be established to ensure this. As long as the data has been normalized, we can compare radii of rigidity for different variables. One may then compare radii for different DMU's over different parameters. In particular, for each DMU, one can take the minimal radius of rigidity across all variables which does not represent an imposed bound and then rank DMU's by taking the maximum. We will designate this the max min ranking.

5.5 Characterizations of Locally and Globally Optimal Inputs

We have seen that for each choice of the parameter θ, we obtain a model which we minimize over all feasible decision variables $x \in \Re^n$. We have also shown that we can improve the optimal objective value in x and θ via methods such as input optimization. Consequently, it is important in our study of these models to be able to identify those inputs that are locally or globally optimal. After calculating the radius of rigidity, we can verify its optimality by an optimality condition. This is done in this section for (L, θ, k), but the theory also holds for (L$'$,θ,k).

Definition 5.8 *Consider the radius of rigidity model (L, θ, k) around some θ^*. Let S be a region of stability at θ^*. If*

$$\tilde{f}(\theta^*) \leq \tilde{f}(\theta)$$

for every $\theta \in N(\theta^*) \cap S$, *where* $N(\theta^*)$ *is a neighbourhood of* θ^*, *then* θ^* *is a* **locally optimal input** *relative to* S.

Definition 5.9 *Consider the radius of rigidity model* (L, θ, k) *around some* θ^*. *If*

$$\tilde{f}(\theta^*) \leq \tilde{f}(\theta)$$

for all $\theta \in \mathcal{F}$, *then* θ^* *is a* **globally optimal input**.

Theorem 5.10 (Characterization of Locally Optimal Inputs) *Consider the radius of rigidity model* (L, θ, k) *at some* θ^*. *Let* $(\tilde{x}(\theta^*), \tilde{y}(\theta^*))$ *be a corresponding optimal solution and let* S *be a region of stability at* θ^*. *Then* θ^* *is a locally optimal input with respect to* S *if, and only if, there exists a non-negative vector function* $\tilde{u}$: $N(\theta^*) \cap S \to \Re^c_+$ *such that, whenever* $\theta \in N(\theta^*) \cap S$

$$L((\tilde{x}(\theta^*), \tilde{y}(\theta^*)), u; \theta^*) \leq L((\tilde{x}(\theta^*), \tilde{y}(\theta^*)), \tilde{u}(\theta^*); \theta^*) \leq L((x, y), \tilde{u}(\theta^*); \theta^*)$$

for every $u \in \Re^c_+$, *and for every* $(x, y) \in \Re^{m+s}$.

Proof: (*Necessity*)

We know that $\tilde{f}(\theta)$ exists and that $\tilde{f}(\theta^*) \leq \tilde{f}(\theta)$ for all $\theta \in N(\theta^*) \cap S$. But, from the proof of the saddle point characterization of optimality (Theorem 5.2), this is equivalent to

$$\begin{aligned} L((\tilde{x}(\theta^*), \tilde{y}(\theta^*)), \tilde{u}(\theta^*); \theta^*) &\leq L((\tilde{x}(\theta^*), \tilde{y}(\theta^*)), \tilde{u}(\theta^*), \theta^*) \\ &\leq L((x, y), \tilde{u}(\theta); \theta) \end{aligned}$$

for all $(x, y) \in \Re^{m+s}$, by the same characterization. This proves the right-hand side of the inequality.

The left-hand side of the inequality follows immediately from feasibility of $(\tilde{x}(\theta^*), \tilde{y}(\theta^*))$ and by positivity of u.

(*Sufficiency*)

The left-hand side of the saddle-point condition gives

$$\begin{aligned} & f(\tilde{x}(\theta^*), \tilde{y}(\theta^*), \theta^*) && + \sum_{i \in \mathcal{P}} u_i f^i(\tilde{x}(\theta^*), \tilde{y}(\theta^*), \theta^*) \\ \leq \; & f(\tilde{x}(\theta^*), \tilde{y}(\theta^*), \theta^*) && + \sum_{i \in \mathcal{P}} \tilde{u}_i(\theta^*) f^i(\tilde{x}(\theta^*), \tilde{y}(\theta^*), \theta^*), \end{aligned}$$

for all $u \in \Re^c_+$. (We are using f^i to represent the constraint functions and the set $\mathcal{P}$ to represent all constraints. Recall that any equality constraints can be converted to two inequality constraints.) Setting $u = 0$ gives

$$\begin{aligned} f(\tilde{x}(\theta^*), \tilde{y}(\theta^*), \theta^*) & \leq f(\tilde{x}(\theta^*), \tilde{y}(\theta^*), \theta^*) + \sum_{i \in \mathcal{P}} \tilde{u}_i(\theta^*) f^i(\tilde{x}(\theta^*), \tilde{y}(\theta^*), \theta^*) \\ & \leq f(\tilde{x}(\theta^*), \tilde{y}(\theta^*), \theta^*) \end{aligned}$$

by feasibility of $(\tilde{x}(\theta^*), \tilde{y}(\theta^*))$. But this implies that

$$\sum_{i \in \mathcal{P}} \tilde{u}_i(\theta^*) f^i(\tilde{x}(\theta^*), \tilde{y}(\theta^*), \theta^*) = 0.$$

Hence, by the right-hand side inequality

$$\tilde{f}(\theta^*) \leq f(x, y, \theta) + \sum_{i \in \mathcal{P}} u_i(\theta) f^i(x, y, \theta)$$

for all $(x, y) \in \Re^{m+s}$. In particular, for $(x, y) = (\tilde{x}(\theta), \tilde{y}(\theta))$, we have

$$\tilde{f}(\theta^*) \leq \tilde{f}(\theta) + \sum_{i \in \mathcal{P}} u_i(\theta) f^i(\tilde{x}(\theta), \tilde{y}(\theta), \theta) \leq \tilde{f}(\theta)$$

for all $\theta \in N(\theta^*) \cap S$ by non-negativity of $u(\theta)$ and by feasibility of $(\tilde{x}(\theta), \tilde{y}(\theta))$. Thus, θ^* is a locally optimal input.

Theorem 5.11 (Characterization of Globally Optimal Inputs) *Consider the radius of rigidity model (L, θ, k) at some θ^*. Let $(\tilde{x}(\theta^*), \tilde{y}(\theta^*))$ be a corresponding optimal solution. Assume that $\tilde{f}: \mathcal{F} \to \Re$ exists. Then θ^* is a globally optimal input if, and only if, there exists a non-negative vector function $\tilde{u}: \mathcal{F} \to \Re^c_+$ such that*

$$L((\tilde{x}(\theta^*), \tilde{y}(\theta^*)), u; \theta^*) \leq L((\tilde{x}(\theta^*)\tilde{y}(\theta^*)), \tilde{u}(\theta^*); \theta^*) \leq L((x, y), \tilde{u}(\theta^*); \theta^*)$$

for every $u \in \Re^c_+$, and for every $(x, y) \in \Re^{m+s}$.

The proof proceeds exactly as in the case for locally optimal inputs except that now $\theta \in \mathcal{F}$.

Chapter 6

Application of CCR Tests to North American University Libraries

In this chapter, we provide the results of the CCR tests (CCR, k) applied to 108 North American university libraries which constitute the university affiliated members of the Association of Research Libraries (ARL). CCR tests were applied to all 108 libraries for the last five years of available data, with the exception of 1990-1991. The University of Auburn was not a member of the ARL in 1990-1991, so the CCR tests for that year are based on the remaining 107 libraries. The GAMS (General Algebraic Modelling System) optimization software package was employed for this purpose under a student licensed copy on a 486 IBM compatible to double precision. All data are obtained from [47, 30, 31, 40, 41]. They are scaled so that values fall below a thousand (see Theorem 2.6).

Initially, DEA was based on the use of five variables: total volumes held, volumes added (gross) over the course of the academic year, number of current serials, total expenditures and the sum of professional and non-professional staff (not including student assistants). These are the five variables employed in the determination of the yearly ARL Membership Index. Table 6.1 ranks the 108 members in decreasing order of efficiency based on these five variables for 1994-95 using DEA (more precisely CCR) efficiency tests. The letter "E" indicates an efficient library. Note that McGill library ranked 11^{th} in that group in the 1994 academic year. (This appears to be a result of an additional 36 000 volumes added over the previous year and an approximately $400 000 decrease in expenditures.) Prior to that year, during the period 1990-1994, McGill ranked consistently between 87^{th} and 96^{th}. (See Table 6.2 for details.)

Subsequently, in an attempt to determine more realistic efficiency evaluations, the CCR test was used with one additional variable: the number of microform units. The ARL publications [47, 30, 31, 40, 41] provide a complete set of data for all 108 libraries and it was included in the model as an output. It is interesting to note that in all instances, the inclusion of this additional output resulted in either no change or an increase in the efficiency rating. There were certain libraries which consistantly exhibited a large increase in their efficiency rating with the additional data included in the model. These are Florida State University, University of Kentucky, University of Missouri, University of New Mexico, and both Virginia Polytechnic Institute and Virginia State University counted as one university library (VPI & SU). Each

Table 6.1: DEA ranking of efficiency for 1994-1995:
Inputs: Staff, Total Expenditures
Outputs: Total Volumes, Volumes Added (Gross), Current Serials

E.	Chicago	1.00000	29.	Iowa	0.77280
E.	Georgia Tech	1.00000	30.	Minnesota	0.76850
E.	Illinois, Urbana	1.00000	31.	Laval	0.76716
E.	Oklahoma State	1.00000	32.	North Carolina State	0.75805
			33.	Maryland	0.75241
5.	Auburn	0.96319	34.	California, Berkley	0.75213
6.	Yale	0.92591	35.	South Carolina	0.74212
7.	Rice	0.92330	36.	Virginia	0.73623
8.	Dartmouth	0.91871	37.	Notre Dame	0.73605
9.	California, Davis	0.91641	38.	Arizona	0.72408
10.	California, L.A.	0.91271	39.	Kansas	0.71774
			40.	B.C.	0.71083
11.	McGill	0.89881	41.	Stanford	0.70956
			42.	Indiana	0.70774
12.	Duke	0.87861	43.	Pennsylvania	0.70765
13.	Massachusetts	0.87322	44.	SUNY, Buffalo	0.70136
14.	Alberta	0.86487	45.	Harvard	0.69408
15.	Michigan State	0.86093	46.	Columbia	0.69404
16.	California, Santa Barbara	0.85986	47.	Ohio State	0.69238
17.	Hawaii	0.85953	48.	Syracuse	0.69114
18.	Queen's	0.83981	49.	Michigan	0.68485
19.	Colorado State	0.83452	50.	Oregon	0.68314
20.	Guelph	0.83085	51.	Washington	0.67926
21.	Texas	0.82815	52.	Missouri	0.67664
22.	Washington State	0.82769	53.	Wayne State	0.67623
23.	Louisiana State	0.82286	54.	Purdue	0.67158
24.	Rochester	0.81837	55.	Kent State	0.66719
25.	Oklahoma	0.80956	56.	North Carolina	0.65928
26.	Georgia	0.80571	57.	Colorado	0.65918
27.	Howard	0.79375	58.	Boston	0.65721
28.	Princeton	0.78971	59.	Nebraska	0.65707

Table 6.1 cont'd.

60.	Toronto	0.65338	85.	VPI & SU	0.56321
61.	Brown	0.64926	86.	McMaster	0.55745
62.	Southern Illinois	0.63939	87.	Southern California	0.55485
63.	Florida State	0.63560	88.	Iowa State	0.55436
64.	SUNY, Albany	0.63248	89.	Western Ontario	0.55389
65.	Washington, St. Louis	0.63146	90.	Georgetown	0.54826
66.	M.I.T.	0.63015	91.	York	0.54638
67.	Cornell	0.62479	92.	California, San Diego	0.52737
68.	Wisconsin	0.62319	93.	Vanderbilt	0.52665
69.	Tulane	0.62251	94.	Florida	0.51459
70.	Alabama	0.62237	95.	Emory	0.51296
71.	California, Riverside	0.62200	96.	Houston	0.50538
72.	Conneticut	0.62168	97.	Brigham Young	0.48383
73.	Case Western Reserve	0.62102	98.	California, Irvine	0.48116
74.	SUNY, Stony Brook	0.60872	99.	Kentucky	0.48098
75.	Johns Hopkins	0.60861	100.	Pennsylvania State	0.48028
76.	Delaware	0.60169	101.	Manitoba	0.47750
77.	Miami	0.59254	102.	Cincinnati	0.47132
78.	Arizona State	0.58596	103.	Tennessee	0.47086
79.	Temple	0.58366	104.	Rutgers	0.45759
80.	Waterloo	0.57726	105.	Texas A&M	0.44871
81.	Northwestern	0.57607	106.	New York	0.44485
82.	Saskatchewan	0.57387	107.	Illinois, Chicago	0.40728
83.	Pittsburgh	0.57287	108.	New Mexico	0.39749
84.	Utah	0.57075			

of these libraries showed an average increase in efficiency rating of at least 0.3, suggesting a very large microform collection in comparison to the rest of their collection and in comparison to other libraries.

The ARL Membership Index provides a very different ranking of the member libraries as compared to the DEA technique. We compare the two rankings in Table 6.2. There the first row for each library gives the ARL Index rank; the second row gives the DEA rank. Once again, "E" indicates an efficient library. (These indices, as well as the raw data used in the DEA models, are available in the yearly ARL Statistical Publications [47, 30, 31, 40, 41].)

The ARL index is determined by the use of factor analysis, a set of statistical techniques commonly used in the social sciences. The technique attempts to find and characterize underlying dimensions or factors in a large set of data. The specific method employed is principal component analysis, in which the original set of variables is represented in terms of a number of common factors. These factors are determined in sequence so that at each successive stage a maximum of the total variance of the original variables is accounted for. Thus, the first principal component is a linear combination of the original variables contributing a maximum to their total variance; the second, uncorrelated with the first, contributes a maximum of the residual variance, etc. In practice, only a few of the components need to be retained, particularly if they account for a large percentage of the total variance.

In the case of the ARL data, using factor analysis, it was determined that there exist four strong patterns of library relations, the first of which clearly represents library size and resource deployment. By analyzing the correlation matrix, one determines a subset of five variables, as indicated above, associated with the hypothetical component representing library size. From the correlation coefficients between these five variables and the hypothetical factor, one derives the factor loadings which are essentially weights associated with each variable. The index score is then found by multiplying these weights by the data and summing over the five variables for each individual library. It was found that the ARL data is lognormal rather than normal, so logarithms of the original data are used. Also, since the technique is dependent on the total variance of the original variables, it is sensitive to linear transformations such as change of units. Consequently, all values are expressed in standard normal form. The resulting set of scores approximates a normal curve with Harvard consistently ranking number one overall. Based on its expenditures, its collections and the number of staff, Harvard is larger than any of the other ARL members. ARL membership criteria requires a score of over -1 and membership may be withdrawn if a library consistently earns a score of less than -1.75. Any score greater than 0 indicates that a library is above the median size. All scores are relative to the group of libraries and so changes in rank from year to year are also relative. Since the hypothetical component represents library size, the ranking gives no indication of quality of collections or services or of efficiency of operation. For more information on factor analysis or its specific application to the ARL data, one may refer to [37, 38] and [54].

As already indicated, there are discrepancies between the ranking of libraries as determined by the use of factor analysis and by DEA. The former seeks to rank the libraries according to relative library size, whereas the latter attempts to determine the relative efficiencies. The first thing that becomes apparent upon an inspection of the ARL and DEA rankings provided in Table 6.2 is the greater consistency of a library's position in the ARL Index as opposed to the DEA rankings. This is readily explained by the differing intents of the two methods: the ARL Index seeks to measure the relative size of its members which is unlikely to change drastically over a short period of time. Comparatively, DEA seeks to rank the relative operating efficiencies. While it is unlikely that the operating policies governing the levels of these variables will change radically from one year to another, it is possible that factors or events outside library control may result in sharp cuts in expenditures or staff or an unexpectedly large number of new volumes added to the library, thus altering the efficiency levels. (An illustration is the McGill library jump to 11^{th} position in 1994-95 from 95^{th} position a year earlier. This was a result of an increase in volumes added and a decrease in total expenditures.) What can be seen from Table 6.2 is that, while there are greater yearly fluctuations in a library's position under DEA, the libraries tend to remain within a certain range; thus, the top ten remain within the top ten, the bottom ten stay at the bottom, and similar patterns occur in groupings of libraries ranked in the middle.

The second noticeable difference between the two rankings is the discrepancies between a library's rank in the ARL Index and under the DEA evaluation. It is clear that there is very little direct correlation between size and efficiency. In fact, in many instances it would seem that the smaller libraries (i.e., those that rank low in the ARL Index) tend to be more efficient and the larger libraries less so. Thus, "big" does not necessarily mean "better" in the sense of more efficient. It is likely that structure and organization of individual libraries plays a much larger role in determining their efficiency ranking than does their size. (Harvard is ranked 1^{st} in 1994-95 by the ARL Index and only 45^{th} when its efficiency was questioned by DEA.) An additional consideration is the choice of variables. DEA was based on the same variables as were used in the factor analysis in order to allow for a direct comparison. This may not have been appropriate given the distinctly different objectives of the two methods. DEA is specifically concerned with input to output ratios. The ARL data, up until 1994, had not provided any variables which would serve to give an indication of library use, such as circulation transactions. Nor are there any indicators of the quality of the services provided or of the collections themselves. Also, given the number of DMU's used, it would likely have been more appropriate to use more variables, but in many instances, the ARL data is incomplete.

The top ten ARL members which have been consistently the largest for the last five years according to the ARL Index are Harvard, the University of California at Berkeley, Yale, the University of California at L.A., the University of Illinois at Urbana-Champaign, the University of Michigan, Columbia, Stanford, the University of Toronto and Cornell, in approximately decreasing order. Of these ten, only two, the University of Illinois at Urbana-Champaign and the University of California at L.A., ranked as efficient under DEA during the same five year period.

The University of Illinois at Urbana-Champaign was the only library to maintain efficient status during the entire five year period.

The University of California at L.A., as well as Georgia Tech, ranked as efficient two years out of the five. However, the University of California at L.A. ranked higher during the other three years than did Georgia Tech. Other libraries to rank as efficient during the five year period were the University of Chicago, Oklahoma State, Michigan State, the University of Nebraska, the University of Pennsylvania, the University of Kentucky, Colorado State and Wayne State. In general, there were three or four libraries out of 108 that were ranked as efficient each year.

Table 6.2: Comparison of ARL and DEA rankings.

Inputs: Staff, Total Expenditures.
Outputs: Total Volumes, Volumes Added (Gross), Current Serials.

	'90	**'91**	**'92**	**'93**	**'94**	
Alabama	95	101	99	91	95	**ARL Index**
	38	39	40	46	70	**DEA Rank**
Alberta	32	33	26	34	25	
	82	83	27	64	14	
Arizona	23	28	27	27	29	
	33	37	28	29	38	
Arizona State	26	26	29	28	31	
	73	91	74	80	78	
Auburn	-	68	75	71	75	
	-	31	7	5	5	
Boston	55	66	63	59	61	
	69	54	59	56	58	
Brigham Young	71	70	72	68	74	
	17	15	18	51	97	
BC	25	27	25	25	27	
	96	101	85	71	40	
Brown	69	73	69	50	72	
	56	57	49	17	61	
California, Berkley	2	2	3	2	2	
	13	19	6	23	34	
California, Davis	29	23	31	35	34	
	18	10	23	10	9	
California, Irvine	74	74	84	84	84	
	93	93	100	90	98	
California, L.A.	4	4	4	5	3	
	12	5	**E**	**E**	10	
California, Riverside	102	105	107	107	107	
	22	52	54	50	71	

		'90	'91	'92	'93	'94	
California, San Diego		34	31	38	40	41	**ARL Index**
		91	86	103	93	92	**DEA Rank**
California, Santa Barbara		51	63	62	75	63	
		66	82	87	88	16	
Case Western Reserve		104	102	105	104	98	
		59	72	83	72	73	
Chicago		16	17	17	17	16	
		10	9	10	8	**E**	
Cincinnati		62	67	71	78	65	
		103	102	99	103	102	
Colorado		54	46	46	47	49	
		32	28	32	38	57	
Colorado State		98	96	100	99	96	
		E	4	8	9	19	
Columbia		10	8	7	7	7	
		11	67	63	54	46	
Conneticut		46	54	53	55	48	
		34	18	17	47	72	
Cornell		11	9	10	11	11	
		63	46	43	63	67	
Dartmouth		81	79	68	73	68	
		62	56	33	32	8	
Delaware		67	71	78	80	87	
		53	49	64	69	76	
Duke		19	24	24	23	20	
		16	21	35	44	12	
Emory		50	48	48	45	50	
		85	85	86	97	95	
Florida		28	41	39	37	35	
		74	73	97	101	94	
Florida State		58	80	82	79	81	
		24	66	68	55	63	
Georgetown		52	51	49	52	45	
		95	92	89	79	90	

		'90	'91	'92	'93	'94	
Georgia		30	32	32	30	30	**ARL Index**
		5	8	24	21	26	**DEA Rank**
Georgia Tech		105	108	108	108	105	
		E	6	14	30	**E**	
Guelph		99	100	104	102	108	
		20	32	42	12	20	
Harvard		1	1	1	1	1	
		46	59	70	74	45	
Hawaii		44	42	40	48	47	
		7	3	4	6	17	
Houston		106	104	101	94	92	
		68	69	58	78	96	
Howard		64	72	79	90	103	
		81	75	75	73	27	
Illinois, Chicago		66	59	64	74	70	
		107	105	108	108	107	
Illinois, Urbana		6	6	5	4	5	
		E	**E**	**E**	**E**	**E**	
Indiana		15	14	13	14	14	
		15	11	9	28	42	
Iowa		43	38	34	31	33	
		26	23	19	15	29	
Iowa State		79	77	80	81	73	
		70	58	65	70	88	
Johns Hopkins		35	36	36	39	39	
		88	97	90	96	75	
Kansas		39	39	35	33	38	
		52	55	44	26	39	
Kent State		87	93	102	105	91	
		28	60	60	45	55	
Kentucky		59	34	52	58	51	
		54	**E**	57	60	99	
Laval		56	52	51	54	55	
		104	106	98	86	31	

		'90	'91	'92	'93	'94	
Louisiana State		65	64	81	76	77	**ARL Index**
		19	29	15	11	23	**DEA Rank**
McGill		41	40	42	44	43	
		94	87	96	95	11	
McMaster		86	87	97	97	99	
		102	99	105	87	86	
Manitoba		94	99	98	101	100	
		105	108	107	104	101	
Maryland		42	50	50	41	42	
		97	84	92	81	33	
Massachusetts		93	90	77	77	79	
		14	12	22	22	13	
M.I.T.		61	58	56	61	66	
		80	81	88	89	66	
Miami		73	60	54	57	62	
		77	96	76	98	77	
Michigan		7	11	11	6	8	
		43	27	30	35	49	
Michigan State		40	43	41	38	37	
		23	22	13	**E**	15	
Minnesota		14	15	15	15	15	
		39	14	21	25	30	
Missouri		72	62	59	65	60	
		44	43	51	43	52	
Nebraska		76	76	74	56	76	
		64	62	56	**E**	59	
New Mexico		60	53	61	53	52	
		106	107	106	105	108	
New York		27	29	28	26	23	
		92	94	102	106	106	
North Carolina		22	19	20	19	19	
		48	64	66	82	56	

Table 6.2 cont'd.

	'90	'91	'92	'93	'94	
North Carolina State	101	98	90	64	58	**ARL Index**
	99	98	95	67	32	**DEA Rank**
Northwestern	31	30	30	29	32	
	36	36	45	53	81	
Notre Dame	57	55	66	62	59	
	25	26	46	40	37	
Ohio State	21	21	21	22	26	
	42	38	34	27	47	
Oklahoma	97	89	91	86	88	
	9	7	11	13	25	
Oklahoma State	100	103	93	95	90	
	30	45	5	7	**E**	
Oregon	88	88	88	87	80	
	45	42	50	58	50	
Pennsylvania	24	25	16	21	22	
	51	48	E	57	43	
Pennsylvania State	18	18	19	16	18	
	89	79	81	100	100	
Pittsburgh	38	37	33	32	28	
	65	68	79	92	83	
Princeton	17	16	18	18	17	
	35	33	38	52	28	
Purdue	68	65	70	72	69	
	87	78	73	59	54	
Queen's	83	82	83	82	85	
	86	88	72	39	18	
Rice	107	107	103	103	104	
	41	44	29	48	7	
Rochester	77	84	87	92	93	
	21	24	20	19	24	
Rutgers	20	20	22	24	24	
	101	104	104	107	104	

	'90	'91	'92	'93	'94	
Saskatchewan	92	91	96	98	101	**ARL Index**
	83	89	61	36	82	**DEA Rank**
South Carolina	75	69	65	67	54	
	27	35	52	76	35	
Southern California	36	35	37	36	36	
	6	40	84	18	87	
Southern Illinois	84	81	86	85	86	
	29	13	16	24	62	
Stanford	9	7	9	8	9	
	47	51	55	65	41	
SUNY, Albany	103	106	106	100	102	
	78	76	69	66	64	
SUNY, Buffalo	45	44	44	43	44	
	49	41	36	42	44	
SUNY, Stony Brook	78	83	94	96	97	
	76	71	80	84	74	
Syracuse	80	97	73	69	64	
	67	47	62	37	48	
Temple	82	75	76	83	83	
	61	70	82	85	79	
Tennessee	70	78	67	70	67	
	75	74	77	91	103	
Texas	8	10	8	10	10	
	8	17	26	14	21	
Texas A&M	49	49	45	49	56	
	79	90	31	94	105	
Toronto	5	5	6	9	6	
	72	77	78	61	60	
Tulane	90	95	85	88	89	
	58	61	53	41	69	
Utah	96	92	60	51	53	
	50	63	67	68	84	

	'90	'91	'92	'93	'94	
Vanderbilt	47	57	58	60	57	**ARL Index**
	84	100	94	102	93	**DEA Rank**
Virginia	33	22	23	20	21	
	57	25	37	20	36	
VPI & SU	85	86	89	89	82	
	71	65	47	62	85	
Washington	12	12	12	12	12	
	40	30	41	34	51	
Washington State	91	94	95	93	94	
	60	53	48	31	22	
Washington, St. Louis	48	47	47	42	40	
	55	50	71	75	65	
Waterloo	89	85	92	106	106	
	100	80	93	99	80	
Wayne State	37	45	43	46	46	
	E	20	25	33	53	
Western Ontario	53	56	55	63	71	
	98	103	101	77	89	
Wisconsin	13	13	14	13	13	
	37	34	39	49	68	
Yale	3	3	2	3	4	
	31	16	12	16	6	
York	63	61	57	66	78	
	90	95	91	83	91	

Chapter 7

CCR Tests with More Data

In this chapter, we select particular libraries for which more data is available. The libraries were selected according to the following criteria: all Canadian university affiliated libraries were included, also U.S. libraries whose size and organization are comparable to that of McGill and finally a few libraries that were largest and most efficient overall as determined by the ARL Membership Index and by DEA respectively. The sets of five and six variables are identical to those used in the previous section: total volumes, volumes added (gross) in the given year, current serials, number of professional and non-professional staff (not including student assistants), total expenditures, and the sixth variable, number of microform units. Efficiency evaluations listed in decreasing order for 1994-1995 are given in Tables 7.1 and 7.2. For these thirty libraries based on a five variable analysis, only two are found to be efficient (the University of California at L.A. and the University of Illinois at Urbana-Champaign).

Table 7.1: Selected libraries ranked by DEA in 1994-1995:
Inputs: Staff, Total Expenditures;
Outputs: Total Volumes, Volumes Added (Gross), Current Serials.

E.	California, L.A.	1.00000
E.	Illinois, Urbana	1.00000
3.	Duke	0.97697
4.	Hawaii	0.94214
5.	McGill	0.89881
6.	Rochester	0.85951
7.	Yale	0.85774
8.	Queen's	0.83428
9.	South Carolina	0.82973
10.	Louisiana State	0.82286
11.	SUNY-Buffalo	0.81889
12.	Conneticut	0.81871
13.	Guelph	0.78828
14.	Laval	0.76298
15.	Alberta	0.74814
16.	Brown	0.70872
17.	BC	0.69775
18.	Syracuse	0.67581
19.	Southern California	0.61935
20.	Washington, St. Louis	0.61276
21.	Pittsburgh	0.60238
22.	Toronto	0.59722
23.	California, San Diego	0.58210
24.	York	0.56209
25.	Saskatchewan	0.55626
26.	Johns Hopkins	0.55304
27.	Waterloo	0.53008
28.	McMaster	0.51468
29.	Western Ontario	0.51239
30.	Manitoba	0.47750

Table 7.2: Selected libraries ranked by DEA in 1994-1995:
Inputs: Staff, Total Expenditures;
Outputs: Total Volumes, Volumes Added (Gross), Current Serials, Microforms.

E.	California, L.A.	1.00000
E.	Hawaii	1.00000
E.	Illinois, Urbana	1.00000
E.	Louisiana State	1.00000
5.	Rochester	0.98947
6.	Duke	0.98038
7.	Queen's	0.97531
8.	SUNY-Buffalo	0.90185
9.	South Carolina	0.90049
10.	McGill	0.89881
11.	Conneticut	0.88122
12.	Yale	0.86042
13.	Guelph	0.81327
14.	Alberta	0.77629
15.	Laval	0.76298
16.	Syracuse	0.76174
17.	BC	0.76078
18.	Southern California	0.73555
19.	Saskatchewan	0.72101
20.	Brown	0.71331
21.	York	0.67830
22.	Washington, St. Louis	0.64958
23.	Pittsburgh	0.62898
24.	Western Ontario	0.61312
25.	Johns Hopkins	0.60540
26.	California, San Diego	0.60311
27.	Toronto	0.59722
28.	Waterloo	0.55222
29.	McMaster	0.55085
30.	Manitoba	0.49729

The set of twelve variables consists of the following inputs: number of professional and non-professional staff combined, number of student assistants, total expenditures, expenditures on serials, expenditures on monographs, total salaries and wages and total volumes borrowed in inter-library loans; and the outputs: total volumes in library, volumes added (gross) in the given year, total number of current serials, total microform units in the library and total volumes loaned in inter-library loans. Results can be found in Table 7.3.

Originally, a variable indicating university size which summed the number of staff and the number of full-time and part-time undergraduate and graduate students was included, but this was dropped based on the fact that this is not a factor over which the library has direct control. An additional model was run on 1994-1995 data which included as an output variable total circulation transactions. This variable was included as an indication of library use. Efficiency evaluations are provided in Table 7.4. The selection of libraries and variables was, to a large extent, determined by the data that was available.

As in the analysis of all 108 libraries, the efficiency evaluations showed either no change or an increase with the inclusion of additional data in the model. It is clear that too many variables in a model lead to a large proportion of relatively efficient DMU's. This warrants additional analysis such as the radius of rigidity approach. (Only two DMU's were classified as efficient in 1994-1995 based on a model using five variables. This number increased to four in the six variable model and jumped to twenty-two and twenty-three respectively in the twelve and thirteen variable models.) Those

Table 7.3: Selected libraries ranked by DEA in 1994-1995:
Inputs: Staff, Student Assistants, Expenditures, Expenditures: on Serials, on Monographs, Salaries & Wages, Volumes Borrowed in Inter-library Loans;
Outputs: Total Volumes, Volumes Added (Gross), Current Serials, Microforms, Volumes Loaned in Inter-library Loans.

E.	Alberta	1.00000
E.	BC	1.00000
E.	California, L.A.	1.00000
E.	Conneticut	1.00000
E.	Duke	1.00000
E.	Guelph	1.00000
E.	Hawaii	1.00000
E.	Illinois, Urbana	1.00000
E.	Laval	1.00000
E.	Louisiana State	1.00000
E.	McGill	1.00000
E.	Queen's	1.00000
E.	Rochester	1.00000
E.	Saskatchewan	1.00000
E.	South Carolina	1.00000
E.	SUNY, Buffalo	1.00000
E.	Syracuse	1.00000
E.	Toronto	1.00000
E.	Washington, St. Louis	1.00000
E.	Western Ontario	1.00000
E.	Yale	1.00000
E.	York	1.00000
23.	Pittsburgh	0.94496
24.	Manitoba	0.91004
25.	Waterloo	0.90182
26.	Johns Hopkins	0.82289
27.	McMaster	0.81592
28.	Brown	0.76617
29.	Southern California	0.74137
30.	California, San Diego	0.65855

Table 7.4: Selected libraries ranked by DEA in 1994-1995:
Inputs: Staff, Student Assistants, Expenditures, Expenditures: on Serials, on Monographs, Salaries & Wages, Volumes Borrowed in Inter-library Loans;
Outputs: Total Volumes, Volumes Added (Gross), Current Serials, Microforms, Volumes Loaned in Inter-library Loans, Total Circulation.

E.	Alberta	1.00000
E.	BC	1.00000
E.	California, L.A.	1.00000
E.	Conneticut	1.00000
E.	Duke	1.00000
E.	Guelph	1.00000
E.	Hawaii	1.00000
E.	Illinois, Urbana	1.00000
E.	Laval	1.00000
E.	Louisiana State	1.00000
E.	McGill	1.00000
E.	Manitoba	1.00000
E.	Queen's	1.00000
E.	Rochester	1.00000
E.	Saskatchewan	1.00000
E.	South Carolina	1.00000
E.	SUNY, Buffalo	1.00000
E.	Syracuse	1.00000
E.	Toronto	1.00000
E.	Washington, St. Louis	1.00000
E.	Western Ontario	1.00000
E.	Yale	1.00000
E.	York	1.00000
24.	Pittsburgh	0.96432
25.	Waterloo	0.95046
26.	Southern California	0.85340
27.	Johns Hopkins	0.83255
28.	McMaster	0.81592
29.	Brown	0.79742
30.	California, San Diego	0.67334

libraries that had poor efficiency ratings tended to remain inefficient when more data was added, thus adhering to the consistent grouping of libraries that was witnessed in the 108 DMU models. On the other hand, any library that obtained efficiency in some model would retain its efficient status when more variables were added.

Table 7.5 gives the rank of each of the thirty libraries for each of five years for models with five, six and twelve variables. As can be seen, the overall rankings remained consistent, despite the enormous increases in the number of efficient DMU's. While there are no clear guidelines as to the optimal selection of variables, it would appear that five variables were too few and twelve were too many. Since more variables in the model seem to imply a higher relative efficiency, one may question the choice of inputs and outputs that produce too many efficient DMU's.

	DEA 1990			DEA 1991			DEA 1992			DEA 1993			DEA 1994		
Alberta	20	18	E	24	23	E	E	E	E	22	19	E	15	14	E
BC	22	20	E	23	22	19	12	17	25	15	16	E	17	17	E
Brown	18	23	25	21	24	26	16	19	28	E	E	E	16	20	28
California, L.A.	6	10	E	6	9	E	E	E	E	E	E	E	E	E	E
California, San Diego	12	15	21	E	E	E	26	26	29	21	23	30	23	26	30
Conneticut	7	8	E	E	E	E	E	E	E	9	11	E	12	11	E
Duke	E	E	E	5	7	E	10	14	E	14	15	E	3	6	E
Guelph	5	7	E	10	14	E	13	16	E	6	7	E	13	13	E
Hawaii	3	E	E	E	E	E	E	E	E	5	E	E	4	E	E
Illinois, Urbana	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E
Johns Hopkins	24	25	30	29	29	29	28	27	30	27	27	29	26	25	26
Laval	27	26	E	22	26	24	23	25	E	18	21	E	14	15	E
Louisiana State	8	E	E	13	10	E	8	6	E	E	E	E	10	E	E
McGill	26	27	23	17	20	22	27	28	22	28	29	E	5	10	E
McMaster	29	28	29	27	28	21	29	29	23	25	25	23	28	29	27
Manitoba	30	30	24	30	30	30	30	30	24	30	30	25	30	30	24
Pittsburgh	14	14	26	18	19	27	18	21	27	26	26	28	21	23	23
Queen's	25	21	E	26	17	E	14	12	E	8	9	E	8	7	E
Rochester	9	6	E	11	5	E	9	7	E	13	10	E	6	5	E
Saskatchewan	15	11	E	12	13	E	11	9	E	7	6	E	25	19	E
Southern California	4	5	E	7	12	E	15	15	26	11	8	24	19	18	29
South Carolina	10	9	E	15	8	E	19	11	E	24	22	27	9	9	E
SUNY-Buffalo	13	13	E	8	11	20	6	8	E	12	12	E	11	8	E
Syracuse	19	12	20	20	6	E	21	13	E	16	14	E	18	16	E
Toronto	17	19	E	19	21	E	17	20	E	17	18	E	22	27	E
Washington, St. Louis	16	17	27	14	16	25	24	23	E	23	24	26	20	20	E
Waterloo	28	29	E	16	18	28	22	24	E	29	28	E	27	28	25
Western Ontario	23	22	22	28	27	E	25	22	E	19	17	E	29	24	E
Yale	11	16	E	9	15	E	7	10	E	10	13	E	7	12	E
York	21	24	28	25	25	23	20	18	21	20	20	E	24	21	E

Table 7.5: DEA ranking based on 5, 6 and 12 variables for each of 5 years.

Chapter 8

Numerical Results for the Radius of Rigidity Approach

This chapter demonstrates the numerical implementation of the radius of rigidity approach as given by (L, θ, k) with the objective function defined by the l_1 norm and some $\theta \in \Re^p$. We begin with the familiar academic example seen repeatedly in Chapter 4 which illustrates two extreme stability situations. Then, we provide the results of the radius of rigidity approach for a subset of North American university libraries.

8.1 Pathologies in an Academic Example

We return once again to the example which was used in Chapter 4 to demonstrate the various methods currently employed in post-optimality DEA. Recall that both DMU's in the example are efficient with inputs and outputs as given below.

	DMU_1	DMU_2
X	1	1
Y^1	1	1
Y^2	1	2

We will show that a radius of rigidity model can be highly unstable (resulting in a zero radius of rigidity). Moreover, within the same set of DMU's, the radius of rigidity model can be highly stable (resulting in an unbounded radius of rigidity). Let us first consider the radius of rigidity for DMU_2. We want DMU_2 to remain efficient but unperturbed. DMU_1 decreases its input and increases its outputs, thereby improving its efficiency. Initially, we will consider a single positive θ imposing a perturbation on all variables. Thus, the radius of rigidity problem is given by

$$\begin{array}{lcl} Max & \theta & \\ s.t. & x & = 1 \\ & y_1 + 2y_2 & = 1 \\ & (1+\theta)y_1 + (1+\theta)y_2 & \leq (1-\theta)x \\ & x, y_1, y_2, \theta & \geq 0. \end{array}$$

The second constraint specifies that $y_1 = 1 - 2y_2$. Since y_1 and y_2 are positive, this implies that $y_2 \leq 1/2$. Substituting the first two constraints into the third gives $(1+\theta)(1-y_2) \leq (1-\theta)$. After isolating y_2, we have

$$y_2 \geq \frac{2\theta}{1+\theta}.$$

So feasibility is guaranteed if

$$\frac{1}{2} \geq \frac{2\theta}{1+\theta}$$

or equivalently, for the interval $0 \leq \theta \leq 1/3$. Thus, the radius of rigidity is 1/3. Note that it is possible to prove global optimality of this θ using the saddle-point condition for linear functions (see Theorem 3.13).

Let us now consider the radius of rigidity for DMU_1. The model is given by

$$\begin{array}{lll} Max & \theta & \\ s.t. & x & = 1 \\ & y_1 + y_2 & = 1 \\ & (1+\theta)y_1 + (2+\theta)y_2 & \leq (1-\theta)x \\ & x, y_1, y_2, \theta & \geq 0. \end{array}$$

The constraints imply that $y_2 \leq -2\theta$, which is inconsistent for $\theta > 0$. So here, the radius of rigidity is 0. Since the radius of rigidity for DMU_2 is greater than that for DMU_1, we conclude that DMU_2 is better than DMU_1 since it can maintain its efficiency status under larger perturbations of the other DMU. Notice that a generalized Slater's point exists in the radius of rigidity model for DMU_2 for all θ in $\mathcal{F} = \{\theta : 0 \leq \theta \leq 1/3\}$ except for the optimal point $\theta = 1/3$ where F(1/3) reduces to a single point. Thus $\mathcal{P}^{=}(\theta) = \mathcal{P}^{=}(\theta^{*})$ and the two inclusions required by the MVF hold for all feasible θ except $\theta = 1/3$. In the model for DMU_1, stability breaks down as $\mathcal{F} = \{0\}$ with $F(0) = \{(x, y_1, y_2) = (1, 1, 0)\}$. Thus, the MVF is not applicable here.

Next, we take a two parameter model in which outputs are perturbed equally by θ_1 and the input by θ_2. For DMU_2, the radius of rigidity model is

$$\begin{array}{lll} Max & \theta_1 + \theta_2 & \\ s.t. & x & = 1 \\ & y_1 + 2y_2 & = 1 \\ & (1+\theta_1)y_1 + (1+\theta_1)y_2 & \leq (1-\theta_2)x \\ & x, y_1, y_2, \theta_1, \theta_2 & \geq 0. \end{array}$$

Once again, feasibility dictates that $y_2 \leq 1/2$. Combining the constraints, we have

$$1/2 \leq 1 - y_2 = y_1 + y_2 \leq \frac{1-\theta_2}{1+\theta_1} \leq 1$$

and therefore,

$$\theta_1 + 2\theta_2 \leq 1 \quad and \quad \theta_1 + \theta_2 \geq 0.$$

The latter constraint is necessarily satisfied by positivity of θ_i. Since we wish to maximize $||\theta||_1$, the first constraint implies that optimality occurs for $\theta^* = (\theta_1, \theta_2) = (1, 0)$.

For DMU_1, the two parameter radius of rigidity model is

$$\begin{array}{lll} Max & \theta_1 + \theta_2 & \\ s.t. & x & = 1 \\ & y_1 + y_2 & = 1 \\ & (1+\theta_1)y_1 + (2+\theta_1)y_2 & \leq (1-\theta_2)x \\ & x, y_1, y_2, \theta_1, \theta_2 & \geq 0. \end{array}$$

Feasibility requires that $(y_1, y_2, x) = (1, 0, 1)$, but this implies that $0 \leq -\theta_1 - \theta_2$ which is only possible if $(\theta_1, \theta_2) = (0, 0)$. We conclude that DMU_2 will remain efficient while DMU_1's combined input and output effort is less than or equal to 1. However, DMU_1 will not remain efficient if DMU_2 makes any improvements to its current operating levels. Once again, DMU_2 ranks better than DMU_1.

Finally, we will consider a three parameter radius of rigidity model for DMU_2 in which each variable is affected by a different perturbation. The model is

$$\begin{array}{lcl} Max & \theta_1 + \theta_2 + \theta_3 & \\ s.t. & x & = 1 \\ & y_1 + 2y_2 & = 1 \\ & (1+\theta_1)y_1 + (1+\theta_2)y_2 & \leq (1-\theta_3)x \\ & x, y_1, y_2, \theta_1, \theta_2, \theta_3 & \geq 0. \end{array}$$

We claim that the point $(y_1, y_2, x) = (0, 1/2, 1)$ is feasible and results in an unbounded objective. Using this point, we have

$$\frac{1}{2} \leq \frac{1-\theta_3}{1+\theta_2}$$

which gives $\theta_2 + 2\theta_3 \leq 1$. As long as we take θ_2 and θ_3 satisfying this constraint, we can send θ_1 to infinity and feasibility is preserved. This results in an infinite radius of rigidity. This seeming anomaly can be seen in Figure 8.1, which also depicts the efficiency frontier after the perturbations to DMU_1 determined by the one and two parameter models. It is interesting that these extreme situations also appear when real library data are used.

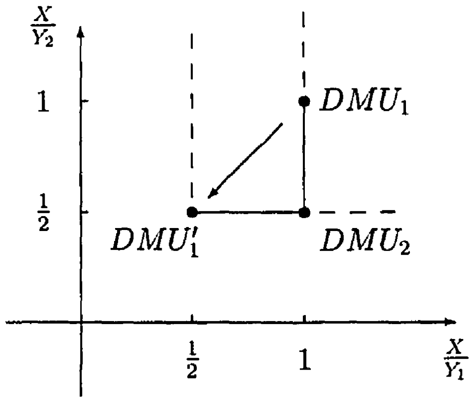

Figure 8.1:

8.2 Implementation and Interpretation

In this section we consider the specifics of solving the radius of rigidity problem (L, θ, k) with the objective function defined by the l_1 norm.

First, we will discuss the input optimization method for some general perturbation $\theta \in \Re^p$. The previous example showed that the MVF is not always applicable to solving the radius of rigidity problem. Thus, we must verify the assumptions that guarantee the validity of the MVF before we use it. For each iteration, at some initial θ^*, these are as follows:

1. The parameter θ must remain in the set $S = \{\theta\colon \mathcal{P}^=(\theta) = \mathcal{P}^=(\theta^*)\} \cap \{\theta\colon \tilde{F}(\theta^*) \subset F^=(\theta),\ \tilde{F}(\theta) \subset F^=(\theta^*)\}$;

2. The point-to-set mappping $F^=(\theta)$ must be lower-semicontinuous at θ^* relative to S;

3. The saddle point $\{\tilde{x}(\theta^*), \tilde{u}(\theta^*)\}$ must be unique;

4. The gradients $\nabla_\theta f(x,\theta)$, $\nabla_\theta f^i(x,\theta)$, $i \in \mathcal{P}^<(\theta)$ must be continuous;

5. The limit

$$l = \lim_{\theta \in S; \theta \to \theta^*} \frac{\theta - \theta^*}{\|\theta - \theta^*\|}$$

must exist.

Let us consider each of these.

First, as long as there exists a generalized Slater's point, we have $\mathcal{P}^=(\theta) = \mathcal{P}^=(\theta^*)$ for every θ close enough to θ^*. In addition, since the equality constraints are independent of θ, the two inclusions will hold as well. The mapping $F^=(\theta)$ is lower-semicontinuous because Y^k and X^k are assumed strictly positive. Uniqueness of $\tilde{x}(\theta)$ can be guaranteed by Tikhonov's regularization. (The term $-\xi(\|x\|_2^2 + \|y\|_2^2)$ is added to the objective function.) The optimal solution in (x, y) is then unique and by sending ξ to 0, we obtain, in the limit, the solution to the original problem. (See, e.g., [58].) Uniqueness of $\tilde{u}(\theta)$ is not necessarily guaranteed. However, the set $\{\tilde{u}(\theta)\}$ is closed since the conditions of Theorem 5.4 are satisfied. (Indeed, $F_*^=(\theta)$ is lower-semicontinuous, since $\mathcal{P}^=(\theta) = \mathcal{P}^=(\theta^*)$ implies that $F_*^=(\theta) = F^=(\theta)$. Also, the objective function is realistic since positivity of the data ensures that $F(\theta)$ is bounded. This set, also closed, ensures that there exists an optimal solution for every feasible θ. Finally, it was shown in the proof of Theorem 5.5 that S is a region of stability.) So, if one is not concerned with the locally fastest improvement of $f(\theta)$, then uniqueness of $\tilde{u}(\theta^*)$ can be relaxed. Continuity of the gradients is assured in (L, θ, k). Finally, for linear search directions where $\theta^{k+1} = \theta^k + \alpha d^k$, $\alpha \geq 0$, $l = d/\|d\|$ exists as long as $d \neq 0$. Furthermore, if all the assumptions of the MVF hold, then, since every θ in the numerical method is located on a region of stability and since the objective function is only a function of θ, then we are guaranteed continuity of the optimal value function. One may choose to normalize the data to ensure greater numerical consistency.

The algorithm is as follows:

(i) Set ξ equal to say 0.1; $n = 0$;

(ii) Set $\theta^0 = 0$;

(iii) Solve (L, θ, k) with $\theta = \theta^n$ using the Tikhonov regularization condition with the current ξ to obtain a unique solution. Notice that (L, θ, k) also depends on ξ. Since the objective is now quadratic in x and y, we may use a gradient method (see, e.g., [61], Section 5.3);

(iv) Calculate $d^n = -\nabla_\theta L(\tilde{x}(\theta^n), \tilde{u}(\theta^n); \theta^n)$ with the problem expressed as a minimization problem. This guarantees a direction of local improvement since

$$\lim_{\theta \to \theta^*} \frac{\tilde{f}(\theta) - \tilde{f}(\theta^*)}{||\theta - \theta^*||} = -d^T d / ||d||$$

by the MVF;

(v) Determine the best step-size α_n, where $\theta^{n+1} = \theta^n + \alpha_n d^n$. Since the objective is a function of θ only, as long as all components of d^n are positive, this reduces to finding the largest α_n which guarantees feasibility of the linear program. (One can use the Golden Section Method to determine the best α_n. This entails maximizing $f(\theta)$ over an interval of α that guarantees feasibility of $\theta^n + \alpha_n d^n$ (see, e.g., [62] for details));

(vi) Set $\theta^{n+1} = \theta^n + \alpha_n d^n$ $(n = n+1)$. Go to (iii) until $\alpha_n \leq \epsilon_1$ or $d^n \leq \epsilon_2$ or $\|\tilde{f}(\theta) - \tilde{f}(\theta^*)\| \leq \epsilon_3$, where ϵ_i, $i = 1,2,3$, is chosen to the desired accuracy;

(vii) Replace ξ by $\xi/10$. If $\xi \leq \epsilon_4$, for some sufficiently small ϵ_4, then stop. Else, go to (ii).

At each new $\theta \in \Re^p$, it is important to ensure that a generalized Slater's point holds. Otherwise, the MVF may not guarantee a local direction of improvement. The interpretation is straight-forward: the higher the radius of rigidity, the greater the resilience to improved efficiency by the other DMU's and, therefore, the better the ranking.

8.3 Numerical Example

We will apply two techniques, the first described in the previous section and the second described in Section 5.4, to the following example borrowed from Neralić [46]:

	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5
X^1	4	12	8	6	2
X^2	6	8	2	6	8
Y	2	4	2	3	2

After solving the CCR model for each DMU, it is discovered that DMU_1 and DMU_2 are inefficient with efficiency evaluations of 0.857 while DMU's 3,4 and 5 are efficient.

First, we apply input optimization using the MVF to a two parameter model for DMU_3. The model is

$$\begin{array}{lll} Max & \theta_1 + \theta_2 & \\ s.t. & 2y & = 1 \\ & 8x_1 + 2x_2 & = 1 \\ & (2+\theta_1)y - (4-\theta_2)x_1 - (6-\theta_2)x_2 & \leq 0 \\ & (4+\theta_1)y - (12-\theta_2)x_1 - (8-\theta_2)x_2 & \leq 0 \\ & (3+\theta_1)y - (6-\theta_2)x_1 - (6-\theta_2)x_2 & \leq 0 \\ & (2+\theta_1)y - (2-\theta_2)x_1 - (8-\theta_2)x_2 & \leq 0 \\ & x_1, x_2, y & \geq 0. \end{array}$$

To ensure positivity of the data, we impose the restriction $\theta_2 \leq 2$. After some algebraic manipulation of the data, we determine that the set of feasible θ's is given by $\mathcal{F} = \{\theta \in \Re^2 \colon \theta_1 + \theta_2 \leq 3; \theta_2 \leq 2; \theta_i \geq 0,\ i = 1,2\}$. The results can be found in Table 8.1.

The optimal solution at $\theta^* = (1.5, 1.5)$ is $(\tilde{y}(\theta^*), \tilde{x_1}(\theta^*), \tilde{x_2}(\theta^*)) = (0.5, 0, 0.5)$ and $\tilde{u}(\theta^*) = (u_1, u_2, u_3, u_4, 0, 0, 0, 0, 0, 0, 0)$ with $u_1 = u_2$ and $u_3 = u_4$. (Recall that each of the equality constraints produce two inequality constraints.) We can prove global optimality of this point using the saddle point condition from Theorem 3.13. The Lagrange multipliers which satisfy the saddle point are not unique and are identical to $\tilde{u}(\theta^*)$. Notice that θ^* is not a unique global optimum. Two iterations were required for each value of ξ because a maximal α which preserved feasibility was used, and the direction of improvement for all iterations was approximately $(1, 1)$. Thus the first iteration

ξ	θ_1	θ_2	*Iterations*
0.1	1.49763	1.50181	2
0.01	1.49966	1.50008	2
0.001	1.49991	1.49996	2
0.0001	1.49999	1.5	2
0.00001	1.5	1.5	2
0.000001	1.5	1.5	2

Table 8.1: Radii of Rigidity for DMU_3

served to find the best θ and the second to confirm this. For values of ξ up to and including $\xi = 0.0001$, a generalized Slater's point exists so the assumptions for the MVF hold. For $\theta^* = (1.5, 1.5)$, the feasible point reduces to a single point and two additional constraints become active. Thus MVF does not necessarily guarantee a direction of improvement from θ^*. When this method is applied to DMU_4, a radius of rigidity of 2/3 is obtained and, for DMU_5, the radius of rigidity is 2. We can thus rank these efficient DMU's by their radii of rigidity. In decreasing order of rigidity, they are: DMU_3, DMU_5, and DMU_4.

While this method typically produces a better parameter θ, it may not converge to a globally or even locally optimal solution. From experience, the Lagrange multipliers consistently appear to take on values close or equal to zero. Thus, the direction vector causes approximately equal increases in all parameters.

Next, we apply the modified radius of rigidity approach in which we solve the single parameter model for each variable. (This approach was described in Section 5.4.) For instance, the radius of rigidity model for DMU_3 for the output is given by

$$\begin{array}{lll} Max & \theta & \\ s.t. & 8x_1 + 2x_2 & = 1 \\ & 2y & = 1 \\ & (2+\theta)y - 4x_1 - 6x_2 & \leq 0 \\ & (4+\theta)y - 12x_1 - 8x_2 & \leq 0 \\ & (3+\theta)y - 6x_1 - 6x_2 & \leq 0 \\ & (2+\theta)y - 2x_1 - 8x_2 & \leq 0 \\ & x_1, x_2, y & \geq 0. \end{array}$$

Note that when perturbing the inputs, the parameter θ is subtracted from the original data value. Since all data must be strictly positive, we set the following limits on θ for each variable: for y, the limit is some arbitrarily large number, for x_1, the limit is 2 and for x_2, it is 6. These limits are dependent on the variable and the DMU begin considered. The results are given in Table 8.2.

Since $r_{3i} \geq r_{ji}$ for every $j = 4, 5$ and every $i = 1, 2, 3$ (here j represents the DMU and i represents the variable), we conclude that the "best" DMU is DMU_3, followed by DMU_5, and then DMU_4. This is the same ordering as was obtained with the input optimization method.

	DMU_3		DMU_4		DMU_5	
	Radius	Limit	Radius	Limit	Radius	Limit
X^1	2	2	2	2	2	4
X^2	3	6	2	2	2	2
Y	3	100	0.54	100	2	100

Table 8.2: Radii of Rigidity

We remark that the method which employs the MVF gives greater flexibility in terms of the allowable perturbations. Thus, depending on the situation and the desired analysis and interpretation, it may be preferable to attempt to use that method. The latter method, however, requires no assumptions be satisfied and thus is always applicable.

8.4 Application to North American University Libraries

We have applied the modified radius of rigidity approach to various subsets of ARL members. All data was normalized. The code found in [9] was adapted for our purposes. First, all Canadian university libraries were selected for a model with twelve different data sets, as seen in Chapter 7. In that group, only one library (McMaster) was inefficient. In solving the radius of rigidity problem, bounds were set to ensure positivity of the data. In almost all cases, the radius of rigidity attained the prescribed bound. Thus, for outputs, a tendency towards an "infinite" radius of rigidity was noticed (An upper bound of 100 was set which, given that the data was normalized, was deemed "sufficiently large".) On the other hand, for inputs, the radius of rigidity typically reached the lower limit at which positivity of the data would be lost. The larger the number of inputs and outputs, in comparison to the number of DMU's, and the higher the number of efficient DMU's, the more typical this behaviour.

Then, the model with all Canadian libraries was scaled down to six data sets (the same six as were used in Chapter 7). Here, the efficient libraries were the University of Alberta, Guelph, McGill, Queen's, Saskatchewan and Waterloo. The same sort of behaviour recurred. It appears that the max-min interpretation described in Section 8.2 is appropriate here. Using this interpretation, we have the following ordering of efficient libraries: Alberta, Queen's, Saskatechewan, Waterloo, Guelph and McGill. (Alberta is the most rigid efficiently run, McGill is the least rigid efficiently run in this group.)

Finally, a subset of fifteen libraries was taken from the group of thirty seen in Chapter 7. The analysis was based on six data sets from Chapter 7. The results can be found in Table 8.3. For those libraries that are efficient, the bounds are the same in all cases: $0 \leq \theta_i \leq 100$, $i = 1, ..., 4$, $0 \leq \theta_5 \leq 0.330166$, $0 \leq \theta_6 \leq 0.191436$. As can be seen from the table, the university libraries of British Columbia, California at L.A., Hawaii, Illinois at Urbana-Champaign and Louisiana State are efficient. After determining the radius of rigidity for each variable in turn, the following ranking of these efficient libraries can be established: Illinois and British Columbia are comparably the best since the radius of rigidity reaches the imposed bounds for all variables.

Hawaii would rank below these two since the bounds were reached in all cases except one and, in that case, the radius of rigidity was better than in all other instances. Louisiana State is next, followed by California since two radii of rigidity of 0 would suggest a precarious efficiency evaluation. Note that the McGill library is ranked inefficient in this group.

	Outputs				Inputs		Rank
	Y_1	Y_2	Y_3	Y_4	X_1	X_2	
Alberta				**Inefficient**			
B.C.	∞	∞	∞	∞	0.33	0.191	1-2
Brown				**Inefficient**			
California, L.A.	∞	∞	0	0.1482	0	0.191	5
California, San Diego				**Inefficient**			
Conneticut				**Inefficient**			
Duke				**Inefficient**			
Guelph				**Inefficient**			
Hawaii	∞	∞	∞	1.2521	0.33	0.191	3
Illinois, Urbana	∞	∞	∞	∞	0.33	0.191	1-2
John Hopkins				**Inefficient**			
Laval				**Inefficient**			
Louisiana State	∞	∞	∞	0.5601	0.33	0.0705	4
McGill				**Inefficient**			
McMaster				**Inefficient**			

Table 8.3: Radii of Rigidity for 15 Libraries in 1994-1995.

Inputs: Staff and Expenditures;

Outputs: Volumes, Volumes Added, Serials and Microforms.

Chapter 9

Efficiency Evaluations for McGill

In this chapter, we will focus our attention on the University of McGill library (all libraries are considered as a single unit); its efficiency evaluations in the various models and its ideal operating levels. We will interpret the results in non-technical terms. (For mathematical qualifications and terminology, see, e.g., Chapters 2 and 8.)

When all 108 libraries and five data sets (staff, total expenditures, total volumes, volumes added gross, and current serials) are considered, McGill ranks as about the 40^{th} largest university library with respect to the ARL Index. However, it has shown poor performance in regard to the efficiency analysis using DEA. Under DEA, McGill had been consistently placed approximately 95^{th} overall with the same five data, with efficiency evaluations ranging from 0.47 to 0.53. Surprisingly, during the 1994 academic year, it exhibited a jump to 11^{th} position with an efficiency evaluation of 0.90. Based on a single improved observation, it is impossible to determine whether or not this is indicative of a trend of increased relative efficiency. This improvement may well be based on a sharp decrease in that year's total expenditures (by approximately $400 000) and an increase in gross volumes added (by approximately 36 000 volumes) for McGill libraries.

Table 9.1 gives actual and ideal operating levels for McGill over a five year period. This is based on results from the DEA model with five variables and all 108 university libraries. The ideal values are calculated from the optimal dual variables associated with the efficient libraries in McGill's reference set. The reference set consists of those efficiently run libraries which are "closest" to McGill and the associated dual variable gives a numerical representation of the importance of that library in providing an ideal profile for McGill. (For more specifics, see Section 2.2.) It is interesting that the University of Illinois at Urbana-Champaign has turned out in all efficiency evaluations as the library McGill should emulate to improve its efficiency. For example, in 1994-1995, McGill's reference set consisted only of the University of Illinois at Urbana-Champaign (with the associated dual equal to 0.535). Thus, McGill's ideal input and output vectors are given by the associated dual times the input and output vectors for Illinois:

$$X^{\circ} = 0.535\ [520,\ 22\ 388\ 297]^T = [278,\ 11\ 977\ 739]^T$$

and

$$Y^{\circ} = 0.535\ [8\ 665\ 814,\ 191\ 077,\ 90\ 969]^T = [4\ 636\ 211,\ 102\ 226,\ 48\ 668]^T$$

(If there had been more than one library in the reference set, the ideal vectors would have been given by the sum of the dual variable times the vectors for the particular library over all libraries in the reference set.) The ideal profile means the following: during that year, McGill libraries employed 315 people, spent \$13 299 751, contained 2 878 716 volumes, added 102 164 volumes (gross) and had 17 424 current serials. During that year, the McGill library was run inefficiently according to DEA. In order to run efficiently, McGill should have reduced its library staff to 278 people, decreased its expenditures to \$11 977 739 and increased its volumes to 4 636 211, its new volumes (gross) to 102 226 and its current serials to 48 668. If McGill had been operating at these levels, it would have been ranked as efficient in comparison to the other libraries. DEA seems to be suggesting that for a library of McGill's size it should be running on a lower budget with fewer employees in order to be ranked as efficient in relation to other university libraries. However, DEA also suggests that an increased budget which is used to increase volumes and serials (i.e., outputs) will also improve its relative efficiency. One must keep in mind that these DEA results are based on a particular set of variables.

Table 9.2 gives McGill's reference sets for each of the five years (i.e., libraries McGill should look up to to improve its efficiency). We also give the optimal value of the dual variable associated with each library in the set. (These values can be interpreted as the "intensities" of the influences on McGill in order to make McGill efficient.) Note that the reference sets are dominated by the presence of the University of Illinois at Urbana-Champaign. It is again this library which is the "closest" efficiently run library to McGill.

For the models based on a subset of thirty libraries, McGill is efficient

		Staff	Expenditures	Volumes	Volumes Added	Serials
1990	Actual	280	14 684 399	2 570 377	70 614	17 812
	Ideal	131	6 897 508	2 571 936	70 726	29 052
1991	Actual	279	15 155 419	2 621 044	95 887	17 541
	Ideal	140	7 473 280	2 624 337	96 012	29 440
1992	Actual	279	13 989 375	2 766 775	68 580	18 524
	Ideal	132	6 948 980	2 766 775	68 580	29 928
1993	Actual	276	13 704 719	2 824 083	66 040	17 739
	Ideal	132	7 252 265	2 825 277	66 040	30 320
1994	Actual	315	13 299 751	2 878 716	102 164	17 424
	Ideal	278	11 977 739	4 636 211	102 226	48 668

Table 9.1: DEA Analysis of McGill in a group of 108 libraries.

Inputs: Staff, Total Expenditures.

Outputs: Volumes, Volumes Added (Gross), Current Serials.

	Reference Set	Associated Dual
1990	Illinois, Urbana	0.292
	Wayne State	0.098
	Colorado State	0.005
1991	Illinois, Urbana	0.264
	Kentucky	0.198
1992	Illinois, Urbana	0.315
	Pennsylvania	0.038
1993	Illinois, Urbana	0.328
	Nebraska	0.020
1994	Illinois, Urbana	0.535

Table 9.2: DEA Reference Sets for McGill in a group of 108 libraries.

only in the twelve variable model and in the two most recent years. However, in that model, in one year, twenty-one and, in the other, twenty-two other libraries from the set of thirty were efficient. Its position dropped in the six data model, even though its efficiency evaluation did not change. (Some inefficient libraries improved their ranking but McGill did not.) With an efficiency evaluation of 0.9, McGill ranked 5^{th} when five variables were considered and 10^{th} when six variables were considered. In the four years 1990-1994, McGill had consistently placed low in approximately the 12^{th} percentile, just as it did in the full 108 library model. Efficiency evaluations were very close for the five and six data models, around 0.52. Again, any change in McGill's relative placement was due to the displacement of other libraries as opposed to McGill itself. For each of the four years respectively, the efficiency evaluations increased in the twelve data model with 30 libraries, taking on the values 0.7, 0.97, 0.95 and 1.

Tables 9.3, 9.4 and 9.5 provide information pertaining to an ideal profile of McGill. In a group of 30 libraries, McGill was run inefficiently, and these tables identify the "closest" efficiently run libraries and their desirable intensities (associated dual variable) towards McGill to make it efficient. Table 9.3 gives the ideal input and output vectors, based on a six data model for each of the five years considered. The actual values refer to McGill's observed operating levels. The "ideal" data are the operating levels that would have been necessary for McGill to have been ranked as efficient. Table 9.4 gives both the libraries in the reference sets and their associated optimal dual variables for both the five and six data models for each of the five years. One can see how comparable these two models are and, once again, the importance of the University of Illinois at Urbana-Champaign as an efficient model for McGill. Table 9.5 gives the libraries belonging to McGill's reference set (i.e., the libraries McGill should look up to) and their associated optimal dual variable (i.e., intensities towards McGill) for the twelve data model. By a comparison of Tables 9.4 and 9.5, one remarks that the inclusion of additional data to the model clearly alters the efficiency frontier (i.e., efficiency structure of the libraries). This is evidenced by the fact that the reference sets for the twelve data model are essentially different from those for the five and six data model. This suggests that the inclusion of more data results in a geometric alteration of the model, rather than a progression of inefficient libraries towards a static efficiency frontier.

Finally, we recall that, in Chapter 8, we applied the radius of rigidity approach to various DEA models. The purpose of this was to be able to rank efficiently run libraries by their "rigidity" or resilience to improvements in operating levels of the other libraries. Based on a model with twelve variables which included all thirteen Canadian ARL members, McGill was efficient along with 11 other libraries. (Only the University of McMaster library was inefficient in that model.) For all twelve variables, McGill's radii of rigidity attained the bounds imposed to ensure positivity of all data. A radius of "infinity" was found for all five outputs. Thus, the other 12 libraries can increase their outputs as much as they choose without affecting McGill's efficient evaluation. The radii of rigidity for the inputs were 0.263, 0.094, 0.22, 0.3, 0.15, 0.2 and 0.134. Once all of the 12 other libraries have increased the relevant input by the values given above, McGill is still efficient and these values cannot be increased any more or the model will no longer be viable

		Staff	Expenditures	Volumes	Volumes Added	Serials	Microform
1990	Actual	280	14 684 399	2 570 377	70 614	17 812	1 161 647
	Ideal	142	7 273 347	2 570 377	70 614	27 554	1 161 647
1991	Actual	279	15 155 419	2 621 044	95 887	17 541	1 204 911
	Ideal	193	10 500 410	2 621 629	95 887	28 986	1 754 081
1992	Actual	279	13 989 375	2 766 775	68 580	18 524	1 245 360
	Ideal	141	7 254 464	2 766 775	68 580	29 084	1 489 305
1993	Actual	276	13 704 719	2 824 083	66 040	17 739	1 298 576
	Ideal	133	7 321 777	2 881 411	66 040	31 048	1 420 411
1994	Actual	315	13 299 751	2 878 716	102 164	17 424	1 347 565
	Ideal	278	11 977 739	4 636 211	102 226	48 668	2 347 427

Table 9.3: DEA Analysis of McGill in a group of 30 libraries.
Inputs: Staff, Total Expenditures.
Outputs: Volumes, Volumes Added (Gross), Current Serials, Microform Units.

	5 Variable Model		6 Variable Model	
	Reference Set	Associated Dual	Reference Set	Associated Dual
1990	Illinois, Urbana	0.229	Illinois, Urbana	0.228
	Duke	0.188	Duke	0.187
			Hawaii	0.004
1991	California, San Diego	0.342	California, San Diego	0.342
	Illinois, Urbana	0.217	Illinois, Urbana	0.217
	Conneticut	0.049	Connecticut	0.049
1992	Illinois, Urbana	0.299	Illinois, Urbana	0.299
	Alberta	0.085	Alberta	0.085
1993	Illinois, Urbana	0.340	Illinois, Urbana	0.340
1994	Illinois, Urbana	0.535	Illinois, Urbana	0.535

Table 9.4: DEA Reference Sets for McGill in a group of 30 libraries.

	12 Variable Model	
	Reference Set	Associated Dual
1990	Alberta	0.280
	Duke	0.176
	California, L.A.	0.069
	Yale	0.061
1991	Alberta	0.228
	Toronto	0.205
	California, San Diego	0.137
	Duke	0.113
	California, L.A.	0.003
1992	California, L.A.	0.168
	Yale	0.125
	Waterloo	0.110
	Alberta	0.100

Table 9.5: DEA Reference Sets for McGill in a group of 30 libraries.

(since some of the data will become negative). Since most of the other Canadian libraries in this subset also attained their bounds, comparison of the results proved to be almost impossible. For the same set of 13 Canadian ARL members for which six variables were considered, McGill was again efficient. Here, the radii of rigidity for the outputs were "infinity" with the exception of the radius for the number of volumes added (gross) which was low at 0.0436. The radii of rigidity for the inputs attained their bounds of 0.263 and 0.229. These results placed McGill 6^{th} among six efficient libraries in the set of thirteen. In a final analysis of a subset of fifteen libraries, McGill was inefficient and so was not considered in the radius of rigidity analysis. (For the full set of 15 libraries, see Table 8.4.)

Overall, McGill library did not place well in the efficiency evaluations with the exception of the 1994 academic year in which it placed 11^{th} out of 108 libraries. This sudden jump in its ranking appears to be the result of a decrease in expenditures and an increase in the number of gross volumes added. DEA appears to suggest that for a library of McGill's size to be run efficiently, it should reduce its expenditures and cut back on staff. One must keep in mind that this was determined using a few select variables (originally chosen by the ARL to indicate library size) which give no indication of the quality of the collections or of the service provided. While McGill had much better efficiency evaluations in smaller models with more data, so did the majority of the other libraries included in those models. The University of Illinois at Urbana-Champaign consistently appeared as one of the libraries which McGill should take as an example of an efficiently run library.

Chapter 10

Conclusion

We have studied stability properties of several models used for efficiency testing and ranking of efficient DMU's in DEA. In particular, we have suggested a parametric programming approach to ranking of the efficient units by calculating their radii of rigidity.

The results have been applied to a group of 108 ARL university affiliated libraries. Our numerical results confirm some of the claims made in the literature, e.g., that an increase in the number of variables used in efficiency tests generally increases the efficiency of the DMU's. This fact confirms the importance of ranking the efficient DMU's and warrants a follow-up study of the radius of rigidity approaches initiated in this thesis.

We have found it interesting that some of the pathological behaviour relative to stability found in our academic models extends to the models with real-life data. Our DEA comparison of the efficiency of the libraries has produced essentially different results than those based on factor analysis with the same data. Particular attention was paid to the University of McGill libraries, considered as one unit. We have obtained McGill's ranking in different sets of libraries and with different data over a period of several years. When its library was declared inefficient, we used DEA to identify the libraries that McGill should look up to in order to improve its efficiency. The results of the thesis could also be used to study how efficiently the libraries are run within McGill itself. This could form a possible topic for future research.

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