POST'S PROBLEM - PRIORITY METHOD SOLUTION

by

Leonda S. Adler, B.A.

Department of Mathematics

McGill University

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ABSTRACT

In 1944 E.L. Post posed a question which was to remain open for twelve years and evoke two simultaneous[2,5], independent solutions depending upon very intricate constructions both of which use a technique which has been called "the priority method"[8]. This thesis is an intuitive exposition of this method and is divided into five parts.

The first is a statement of notation and definitions. The second discusses Post's classification of recursively enumerable sets and leads to the question : "Can there be two recursively enumerable but non-recursive sets such that the first is recursive relative to the second, but not vice-versa?". The third is an expansion of the work by G.E. Sacks which abstracts the mechanism of priority from the proofs of Friedberg and Mučnik and uses it to exhibit the two recursively enumerable sets about which Post conjectures. Sections four and five explain the actual constructions of Friedberg and Mučnik.

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CHAPTER I

INTRODUCTION

In 1944 Emil Post published his paper "Recursively Enumerable Sets of Positive Integers and Their Decision Problems"[6]. This paper initiated the classification of recursively enumerable sets, and raised the question : Can there be two recursively enumerable but non-recursive sets such that the first is recursive relative to the second but not vice-versa? This question, which came to be known as Post's problem, was answered in the affirmative by Richard Friedberg[2] and A.A. Mučnik[5] in 1956.

As a preliminary step to the discussion of this problem and its solutions, it is necessary to state certain definitions.

> A function, ϕ, is a partial recursive function in the sense of Herbrand-Gödel-Kleene iff ϕ is defined inductively by a finite set of equations, E, where f is the principal function letter of E and for each n-tuple $(x_1, \ldots, x_n)$ in some set called the domain of definition of ϕ there is exactly one numeral, $\overline{x}$, such that from E it can be shown that $f(\overline{x}_1, \ldots, \overline{x}_n) = \overline{x}$, and the value of $\phi(x_1, \ldots, x_n)$ for the natural numbers $x_1, \ldots, x_n$ as arguments is the natural number x, for which $\overline{x}$ is the numeral.
>
> A function, ϕ, is a general recursive function iff it is a partial recursive function and its domain of definition is the set of all n-tuples.
>
> A function general recursive in another function, h, is essentially the same as above except that the system E can involve in addition to the initial functions, the function, h.

A recursive set is a set whose characteristic function is recursive. A set recursive in a function, h, is a set whose characteristic function is recursive in h.

A non-empty set, R, is recursively enumerable iff it is of the form $\{\, t \mid [E y\, \phi(y) = t]$ and ϕ is a partial recursive function $\}$.

A set is recursively enumerable in a function, h, iff it is as above, but ϕ is general recursive in h.

The decision problem for a set, S, is a matter of effectively determining whether an arbitrary positive integer is or is not a member of S.

The degree (of unsolvability) of a function, f, is zero iff the function is recursive. Its degree is greater than the degree of a function, g, iff g is recursive in f, but not vice-versa. The degree of f is the set of all functions, h, such that f is recursive in h and h is recursive in f. The degrees of f and k are incomparable if k is not recursive in f and vice-versa.

The degree (of unsolvability or undecidability) of a set is the degree of its characteristic function.

In particular A and B are any two sets whose degrees of unsolvability are incomparable iff the decision problem for A cannot be reduced to that of B, and vice-versa, or in other words, iff if it is impossible to tell whether any arbitrary element is in A on the basis of it (or some elements calculable from it) being in B, or vice-versa.

There are two problems here. The first, that of the existence of a recursively enumerable, non-recursive set is quite straight forward and will be shown here ; the second, the existence of a pair of recursively enumerable sets whose degrees are incomparable will be proven in Chapter III, Corollary 1.

Let T and U be the primitive recursive predicates defined in Kleene[3] (IM) :

> $T(e,x,y)$: y is the Gödel number of a calculation starting from a set of equations with Gödel number, e, which define a partial recursive function $f(x)$.
>
> $U(y)$: If y is the least value making $T(e,x,y)$ true, then $U(y)$ is the value of $f(x)$ given by this calculation.

Post[6] proves that a set of positive integers is recursive when and only when both it and its complement with respect to the natural numbers are recursively enumerable.

Using the above we can now prove the existence of a set which is recursively enumerable, but non-recursive.

Proof : Consider the set

$$R = \{ t \mid E y [t = U(y) \text{ and } E x T(x,x,y)]\} .$$

R is the range of U, a primitive recursive function defined for all y such that $E x T(z,x,y)$ and certainly for those values of y for which $E x(T(x,x,y))$, so R is recursively enumerable. We must now show that the complement of R, $N - R$, is not recursively enumerable.

Suppose it were. Then

$$N - R = \{ t \mid \neg [E y(t = U(y) \text{ and } E x T(x,x,y))]\}$$

would be the range of a partial recursive function. That means that the function

$[\,A\,y\,t \neq U(y)$ or $A\,y\,\neg E\,x\,T(x,x,y)]$ which can be rewritten $[A\,y(t \neq U(y))$ or $A\,x\,\neg E\,y(T(x,x,y))]$ is partial recursive. As U is defined for all y such that $E\,y\,T(x,x,y)$, it must follow that $\neg E\,y\,T(x,x,y)$ is partial recursive. Since $\neg E\,y\,T(x,x,y)$ is partial recursive, by Kleene's Theorem XIX, an extension of the Enumeration theorem[3], there is a number, z_o, such that for all x

$$E\,y\,T(z_o,x,y) \text{ iff } \neg E\,y(T(x,x,y)).$$

Since this holds for all x, it must hold when $x = z_o$, giving

$$E\,y\,T(z_o,z_o,y) \text{ iff } \neg E\,y\,T(z_o,z_o,y)$$

which is absurd.

So N – R is not recursively enumerable and R is recursively enumerable but non-recursive.

A slightly more involved version of this T predicate also due to Kleene will be used in the construction of the recursively enumerable sets of different degrees of unsolvability:

$$\tilde{f}^i_s(y) = \prod_{j<y} p_j^{f^i_s(y)} = \begin{cases} 1 & \text{if } i=0 \text{ and } y \in A, \text{ or} \\ & \quad i=1 \text{ and } y \in B. \\ \prod_{j<y} p_j & \text{otherwise.} \end{cases}$$

is the Course-of-Values function for f^i_s.

$T_1^1(\tilde{f}_s^i(y), e, x, y)$ as the primitive recursive predicate : e is the Gödel number of a formal procedure for calculating one function, g, given another, and y is the Gödel number of a formal application of this procedure starting with a set of equations having f_s^i as its principal function letter and substituting x for the variable of f_s^i.

U(y) as the particular primitive recursive function which takes on the value, g(x), for the least y which makes $T_1^1(\tilde{f}_s^i(y), e, x, y)$ true.

The concept degree of recursive undecidability is based on that of reducibility of decision problems. There have been three precise formulations of this idea : Turing reducibility, Kleene's general recursive reducibility, and Post's canonical reducibility. These concepts have all been proven to be equivalent. Using the notation of Kleene, where f is a function and $\underline{f}$ is its degree : there is a Gödel number associated with the system of equations defining any function, f, recursive in another function, g. Thus there can be only countably many degrees $\underline{f}$ such that $\underline{f} \leqslant \underline{g}$ for any given $\underline{g}$; each degree has at most countably many predecessors and consists of at most countably many functions. Since there is a continuum of functions, there must be a continuum of degrees.

Spector[9] showed that the degrees are not dense in themselves. That is, given degrees a, c with $a < c$, there does not always exist a degree b with $a<b<c$. In fact for any degree a, there is a degree $c > a$ with no degree between. Thus among all degrees $> a$, there is a minimal one. In particular there is a minimal degree of recursive unsolvability.

It will be shown in Chapter III, Corollary 2 that there is a sequence of simultaneously recursively enumerable sets whose degrees are independent. Therefore, there must be at least countably many different degrees of recursively enumerable sets. Kleene and Post[(4)] abstract from the degrees of unsolvability of sets to degrees as such and prove that the degrees less than or equal to 0' form an upper semi-lattice, where 0' is the degree of the completion of a recursive function.

Remark 1 : The set of degrees form an upper semi-lattice.

Let A_1, A_2 be two sets ; and let degree (A_1) be a_1. Let the set $A_1 \vee A_2 = \{2^n\ 3^y \mid n \in A_1 \,\&\, y \in A_2\}$. If degree $(A_1 \vee A_2)$ is defined to be $(a_1 \cup a_2)$, and $(a_1 \cup a_2)$ is called the least upper bound of a_1 and a_2, then $a_1 < (a_1 \cup a_2)$ and $a_2 < (a_1 \cup a_2)$. Therefore given any two degrees of sets, their least upper bound is also a degree of a set. The degrees of sets form an upper semi-lattice.

Remark 2 : It will be useful to note that every countable, partially ordered set, P, is isomorphic to a subset of a set N_R, or in other words imbeddable in N_R, where N_R is defined as follows : N_R is the range of the relation, R, N is the natural numbers, the partial ordering $\leqslant_R$ is any recursive, reflexive, anti-symmetric and transitive relation and $R = \{(x, y) \mid x, y, \in N \text{ and } x \leqslant_R y\}$. We say that $\leqslant_R$ is recursive if the predicate $(m \leqslant_R n)$ is recursive.

CHAPTER II

POST'S CLASSIFICATION

Post's paper[6] presents a portion of the theory of recursive functions of positive integers in an intuitive and informal style. He notes that if a problem, P_1, has been reduced to a problem, P_2, a solution for P_2 yields a solution to P_1, while if P_1 is proved to be unsolvable, P_2 must also be unsolvable. As in the definition for sets : two unsolvable problems are of the same degree of unsolvability if each is reducible to the other ; one of lower degree of unsolvability than another if it is reducible to the other, but not vice-versa ; and of incomparable degrees of unsolvability if neither is reducible to the other. Post states that the problem of determining the degrees of unsolvability of unsolvable decision problems is a primary problem in the theory of recursively enumerable sets. He shows at an early stage that there is a highest degree of recursive unsolvability, that of a set he calls the complete set, K, which is any set such that the decision problem for every recursively enumerable set is reducible to the decision problem for it.

One of the simplest ways in which the decision problem of a set of natural numbers, S_1, would be reducible to a set of natural numbers, S_2, would arise if there was an effective method which would determine for each positive integer, n, a positive integer, m, such that n is, or is not, in S_1 according as m is, or is not, in S_2. If somehow we determined whether m is, or is not, in S_2, we would determine n to be, or not be in S_1. We say S_1 is <u>many-one reducible</u> to S_2 if $m = f(n)$ where $f(n)$ is a recursive function ; and <u>one-one reducible</u> if in addition different n's lead to different m's.

Post[6] defines four non-empty classes of recursively enumerable, non-recursive sets :

i) The class of sets called complete sets is the family of sets, K, with the property that the decision problem for all recursively enumerable, non-recursive sets can be reduced in a one-one manner to that of a complete set.

ii) The class of creative sets is the family of sets, R, such that there is a recursive function, f, defined on some set of natural numbers A in the complement of R and for all $n \in A$ $f(n)$ is in the complement of R, but not in A. Post shows that a complete set K is one-one reducible to a creative set.

iii) The class of simple sets is the family of sets, S, whose complements are infinite, but contain no infinite recursively enumerable set. Post shows that a creative set is many-one reducible to a simple set.

iv) The class of hypersimple sets is the family of sets, H, whose complements are infinite and have the property that there is no infinite recursively enumerable set of mutually exclusive finite sequences of positive integers such that each sequence has at least one member in the complement. Post shows that every hypersimple set has a lower degree of recursive unsolvability than every complete set relative to many-one and one-one reducibility.

Post leaves open the question : "Do there exist sets of degree of unsolvability less than K?".

If this question is to be answered in the affirmative, he offers the class of hypersimple sets as a possible class of candidates from which to construct such a set. This conjecture was disproved by Dekker[1] who showed that to any recursively enumerable non-recursive set G there corresponds a hypersimple set, H such that G and H are reducible to each other.

CHAPTER III

SACKS : THE PRIORITY METHOD

Sacks designates as the "priority method" any method of proof which owes its inspiration to the solution of Post's problem given by Friedberg or Mučnik. He sets up some new apparatus to reveal the details of a recursive construction in which one attempts to set up a certain status quo at an early level and make decisions in the future which will be unlikely to disturb this arrangement. In effect one gives a priority to the existing situation even if it means definitely not creating the advantageous situation at a later level.

Sacks defines the following machinery :

Requirement : $R = \{(F^i, H^i) \mid i \in I \text{ (i is a positive integer)}\}$,

$F^i \cap H^i = 0$, and $F^i = \{n_o, \ldots, n_{F_i}\}$, $H^i = \{m_o, \ldots, m_{H_i}\}$.

A requirement R is a sequence of ordered pairs of finite disjoint sets of positive integers.

Meet : A set T meets a requirement, R, iff for some $i \in I$ $F^i \subseteq T$ and $H^i \cap T = 0$.

$j : L \rightarrow I$: where L is a finite subset of I. If $L = \{h_o, \ldots, h_m\}$ then $j(L) = 2^{h_o} + \ldots + 2^{h_m}$. As this function is the core of all constructions, it is shown to be a 1 - 1, onto function :

<u>Function</u> : Let L be the finite set $\{h_o, \ldots, h_m\}$.

$$j(L) = 2^{h_o} + 2^{h_1} + \ldots + 2^{h_m}.$$

Let L' be another finite set equal to L.

If $L = L'$, then L' has the same members as L, and

$$j(L') = 2^{h_o} + 2^{h_1} + \ldots + 2^{h_m} = j(L), \text{ so } j \text{ is a function.}$$

<u>Onto</u> : Let n be any natural number : if n is even, $n = 2^{\alpha_o} b_o$, $(\alpha_o > 0)$, and if n is odd, $n = 2^{\alpha_o} m$ $(\alpha_o = 0)$, and $m = (1 + b_1)$ where b_1 is even and b_o, b_1 are less than n, so this decomposition can be done at most a finite number of times.

$$\begin{aligned} n &= 2^{\alpha_o}(1 + 2^{\alpha_1}(1 + \ldots + 2^{\alpha_m}(1)) \ldots) \\ &= 2^{\alpha_o} + 2^{\alpha_o + \alpha_1} + \ldots + 2^{\alpha_o + \alpha_1 + \ldots + \alpha_m} \\ &= 2^{h_o} + \ldots + 2^{h_m}. \end{aligned}$$

<u>One-one</u> : If $j(L) = j(L') = n$. It is easy to show $L = L'$ by complete induction on n.

Suppose for all $k < n$: $k = j(L_o) = j(L'_o)$ implies $L_o = L'_o$ where L_o, L'_o are finite sets. There are two cases : <u>n even</u>: Then $n = 2b$ where b is unique and b is less than n, so $b = j(L_o) = j(L'_o)$ implies $L_o = L'_o = \{h_o, \ldots, h_m\}$. Set $L = h_o + 1, h_1 + 1, \ldots, h_m + 1$. Since multiplication is well-defined, this is the unique set L with $j(L) = 2b$ and so $L = L'$. <u>n odd</u> :

Then $n = 1 + b$ where b is unique, less than n and even.

$b = j(L_o) = j(L'_o)$ implies $L_o = L'_o$. $(L_o \cup \{0\})$ is the unique set, L such that $j(L) = n$, since addition is a well-defined operation. Therefore $L = L'$.

Enumerate requirements : Let t be a function whose domain and range are included in the natural numbers. Restrict domain t to those s which make $(j^{-1}((t(s))_o) \cap j^{-1}((t(s))_1) = 0)$ a true statement, and calculate $(t(s))_2$.

The symbol $(t(s))_i$ is defined to be : the power to which the i^{th} prime is raised in the unique prime number decomposition of the natural number $t(s)$. If

$j^{-1}((t(s))_o)$ is called F^s,

$j^{-1}((t(s))_1)$ is called H^s, and $g(t(s))_2$ is called $g(s)$, then t enumerates requirements by assigning the number $k = g(s)$ to the set R ; $R_k = \{(F^s, H^s) \mid g(s) = k\}$.

Priority Set : T, of the function t : $T = \bigcup_{s=0}^{\infty} T_s$, where T_s is defined inductively :

$s = 0$ $T_o = 0$

$s > 0$ There are two cases :

i) if (a), (b), or (c) is true, then $T_s = T_{s-1}$;

(a) There is an $r \leq s$ such that $g(r) < g(s)$,

$r > 0$, $F^r \nsubseteq T_{r-1}$, $F^r \subseteq T_r$, $H^r \cap T_{s-1} = 0$, and

$H^r \cap F^s \neq 0$.

(b) There is an $r < s$ such that $g(r) = g(s)$,

$r > 0$, $F^r \nsubseteq T_{r-1}$, $F^r \subseteq T_r$, and $H^r \cap T_{s-1} = 0$.

(c) $H^s \cap T_{s-1} \neq 0$.

ii) if none of these conditions are true : $T_s = T_{s-1} \cup F^s$.

It will be useful to note that the set T_s is dependent only on the values of $t(i)$ for $i \leq s$, and the sets, T_i for $i < s$.

Met at stage s : R_k is met at stage s if $s > 0$, $g(s) = k$; $F^s \nsubseteq T_{s-1}$, and $F^s \subseteq T_s$.

Injured at stage s : R_k is injured at stage s if there is an $r < s$, R_k was met at stage r, $H^r \cap T_{s-1} = 0$ and $H^r \cap T_s \neq 0$.

The following observations on the definitions will be needed later in many of the proofs.

If R is met at stage s, then

i) Clauses (a), (b), and (c) must be false at stage s ;

ii) $H^s \cap T_s = 0$; and

iii) if R is not injured at any stage after s, T meets R_k.

<u>Proof</u> :

i) If not, then $T_s = T_{s-1}$. If R_k is met at stage s, $F^s \nsubseteq T_{s-1}$, but $F^s \subseteq T_s$. This contradicts $T_s = T_{s-1}$, so (a), (b), and (c) must be false at stage s.

ii) Since (c) is false at stage s, then $H^s \cap T_{s-1} = 0$. Since R_k is a requirement, $F^s \cap H^s = 0$, and $T_s = T_{s-1} \cup F^s$ implies $H^s \cap T_s = 0$.

iii) As R_k is met at stage s and not injured at any stage $s' \geq s$, $F^s \subseteq T_s$ and $H^s \cap T_{s'} = 0$ for all $s' \geq s$, but then $F^s \subseteq T$ and $H^s \cap T = 0$, and T meets R_k.

Looking at this in another way, we have defined T in a manner which will enable us to make T meet g(s) at stage s whenever we set $T_s = T_{s-1} \cup F^s$ and manage to keep $T_u \cap H^s = 0$ for all $u \geq s$. If we don't want or have been unable to make T meet $R_{g(s)}$ at stage s we set $T_s = T_{s-1}$.

i) If (c) is true, $H^s \cap T_{s-1} \neq 0$, so T cannot meet $R_{g(s)}$.

ii) If (b) is true, then for some $r < s$:

1) $R_{g(r)}$ was met at stage r.

<u>Proof</u> : $F^r \nsubseteq T_{r-1}$ and $F^r \subseteq T_r$.

2) $R_{g(r)}$ is not injured at any stage u, $r < u \leq s$.

<u>Proof</u> : If $R_{g(r)}$ was injured at stage u, then $H^r \cap T_u \neq 0$. Since $T_u \subseteq T_{s-1}$, $H^r \cap T_{s-1} \neq 0$ contradicting the statement $H^r \cap T_{s-1} = 0$.

3) As we do not wish to injure $R_{g(r)}$ at any future stage, we set $T_s = T_{s-1}$.

iii) If (a) is true, then for some $r < s$:

1) $R_{g(r)}$ was met at stage r.

Proof : same as ii) 1).

2) $R_{g(r)}$ is not injured at any stage u, $r < u < s$.

Proof : same as ii) 2).

3) $R_{g(r)}$ would be injured at stage s, if $T_s = T_{s-1} \cup F^s$.

Proof : Since $H^r \cap F^s \neq 0$, $H^r \cap T_s \neq 0$. Therefore $T_s = T_{s-1}$, because we do not wish to injure $R_{g(r)}$ at stage s for the sake of meeting $R_{g(s)}$ at stage s. Thus a higher priority has been assigned to $R_{g(r)}$ than to $R_{g(s)}$.

Lemma I : If $r < s$ and R_k is met at stage r and at stage s, then there is a u such that $r < u < s$ and R_k is injured at stage u.

Proof : If R_k is met at stage r and s, $k = g(r) = g(s)$, and $F^r \not\subseteq T_{r-1}$, $F^r \subseteq T_r$ and $H^r \cap T^r = 0$; $F^s \not\subseteq T_{s-1}$, $F^s \subseteq T_s$ and $H^s \cap T_s = 0$. Here $T_s = T_{s-1} \cup F^s$, so (b) is not true, (for if (b) were true, then $T_s = T_{s-1}$). The only clause of (b) it is still possible to contravene is $H^r \cap T_{s-1} = 0$: R_k is met at stage r so $H^r \cap T_{r-1} = 0$ and $H^r \cap T_r = 0$ must follow, since $H^r \cap F^r = 0$. If $H^r \cap T_{s-1} \neq 0$, there is a u $r < u < s$ such that $T_u \cap H^r \neq 0$, and since there are only a finite number of T_i's between T_r and T_s, there is a first u, and R_k

is injured at this u.

It will be convenient to define m_i to be the number of times a requirement R_i can be met. R_i is then injured at least $m_i - 1$ times, since between every two stages at which it is met, it must be injured at least once.

<u>Lemma II</u> : For each k, the set $\{s \mid R_k \text{ is injured at stage } s\}$ has cardinality less than 2^k, and the set $\{s \mid R_k \text{ is met at stage } s\}$ has cardinality at most 2^k.

<u>Proof</u> : By complete induction on k :

Assume for $i < k$: the cardinality of $\{s \mid \text{some } R_i \text{ is injured at stage } s\}$ is less than 2^i.

When $i = 0, 1, \ldots, k-1$, how many injuries can be done to any of the R_i's? For any i, there can be at most $2^i - 1$,

$$\sum_{i=0}^{k-1} (2^i - 1) = \sum_{i=0}^{k-1} 2^i - \sum_{i=0}^{k-1} 1 = 2^k - 1 - k < 2^k - k.$$

<u>Set</u> : $R = \{s \mid \text{there is } i < k \text{ such that } R_i \text{ is injured at stage } s\}$, so $\overline{\overline{R}} < 2^k - k$;

$$S = \{s \mid \text{there is } i \ (i < k) \text{ and } R_i \text{ is met at stage } s\}.$$

Since each R can be met at most once more than it is injured, $\overline{\overline{S}}$ can be at most

$$\sum_{i=0}^{k-1} m_i = \sum_{i=0}^{k-1} (m_i - 1) + k < 2^k - k + k < 2^k.$$

In order to prove that an R_k is injured at any stage s less than 2^k times, it is sufficient to show :

If R_k is injured at stage s, there is an $i < k$ such that R_i is met at stage s.

Proof : Suppose R_k is injured at stage s :

By definition there is an $r < s$ such that R_k was met at stage r ; $H^r \cap T_{s-1} = 0$ and $H^r \cap T_s \neq 0$ which implies $H^r \cap F^s \neq 0$. Since R_k is met at stage r, $F^r \nsubseteq T_{r-1}$, $F^r \subseteq T_r$, and $H^r \cap T_r = 0$. From above $H^r \cap T_{s-1} = 0$, and $H^r \cap T_s \neq 0$ implies $H^r \cap F^s \neq 0$, and as $H^r \cap T_{s-1} \neq H^r \cap T_s$, $T_{s-1} \neq T_s$. Therefore $T_s = T_{s-1} \cup F^s$ and (a), (b) and (c) are all false. It is true that $F^r \nsubseteq T_{r-1}$, $F^r \subseteq T_r$, $H^r \cap T_{s-1} = 0$ and $H^r \cap F^s \neq 0$, so (a) and (b) can only be a false if it is not true that $g(r) \leqslant g(s)$, or $g(s) < g(r) = k$.

Now we have to show that $i = g(s) < g(r) = k$ and $R_i = R_{g(s)} = \{(F^s, H^s) \mid g(s) = i\}$ is met at stage s. We have : $F^s \nsubseteq T_{s-1}$ and $F^s \subseteq T_s$; from above

$H^s \cap T_{s-1} = 0$ or (c) would be true, contradicting $T_s \neq T_{s-1}$;

$H^s \cap F^s = 0$ as R_i is a requirement, so

$H^s \cap T_s = 0$.

Therefore R_i is met at stage s for some $i < k$.

Remark 3. For any requirement $R_i (i \leqslant k)$, there is a u, such that for any $s \geqslant u$, R is neither injured nor met at stage s.

Proof : We know that R_k is injured only a finite number of times, and that between any two meets there is an injury. Therefore, there are only a finite number of stages at which R_k is met, and there is a finite u at or after which R_k is neither met nor injured.

A requirement is defined to be t-dense iff for each finite set L, there is an $s > 0$ such that $g(s) = k$, $F^s \not\subseteq T_{s-1}$, $H^s \cap T_{s-1} = 0$ and $L \cap F^s = 0$.

Theorem I : If t enumerates requirements, then

i) T, the priority set of t, is recursively enumerable in t ; and

ii) T meets every t-dense requirement.

Proof : It is necessary to show that if T_i is recursively enumerable in t for all i, and T_{i+1} is obtained from T_i by a recursive function, then T is recursively enumerable in t. From the definition of T, $T = \bigcup_{i=0}^{\infty} T_i$ where $T_i \supseteq T_{i-1}$, $T_{i-1} = T_i$ if (a), (b) or (c) is true, and $T_i = T_{i-1} \cup F^i$ if (a) and (b) and (c) are false.

By complete induction on i : Assume $r > 0$, $T_k (k < r)$ is recursively enumerable in t. It must be shown that T_r is recursively enumerable in t. In particular, $r - 1 < r$, T_{r-1} is recursively enumerable in t. This means $T_{r-1} = \{ y_o \mid h(x) = y_o$ and h is recursively enumerable in t $\}$.

Case 1 : If $T_r = T_{r-1}$, obviously T_r is recursively enumerable in t.

<u>Case 2</u> : If $T_r = T_{r-1} \cup F^r$ where $F^r = j^{-1}((t(s))_o)$: j^{-1} is a one-one function, therefore $j^{-1}(t(s))_o$ is completely defined whenever $(t(s))_o$ is, and $j^{-1}((t(s))_o)$ is recursively enumerable in t ; so F^r is recursively enumerable in t.

<u>Enumerate T_r</u> : Since T_{r-1} and F^r are both finite sets, T_r can be enumerated by calculating a term of T_{r-1} and then a term of F^r and so on. The union of two sets recursively enumerable in t is recursively enumerable in t.

ii) If T is to meet a requirement, R_k ; R_k must be met at some stage and not injured thereafter. There is some stage u at or after which no requirement, R_i, $(i \leq k)$ is met or injured. Choose this u, and define $L = \bigcup_{w \leq u} (H^w \cup F^w)$. We determine s, by noting that since R_k is t-dense, there is an $s > 0$ such that :

i) $g(s) = k$,

ii) $F^s \nsubseteq T_{s-1}$,

iii) $H^s \cap T_{s-1} = 0$, and

iv) $F^s \cap L = 0$.

Choose the least s which satisfies these conditions. We have set $T_s = T_{s-1}$ at this stage s, since s is greater than u. To determine r : We recall that if condition (b) is true at stage $s > 0$, there is an $r < s$ such that R_k is met at r, not injured up to stage s, and $g(r) = g(s) = k$. It will be shown in a few lines that (b) must be true when u, s, and r are chosen thusly.

We now note that

i) $s > u$, since if $s \leq u$, $F^s \subseteq L$, contradicting $F^s \cap L = 0$; and

ii) $r < u$, since R_k is met at stage r, and u was chosen so that at or after u, no requirement $R_i (i \leq k)$ is met or injured.

If condition (b) is true at stage s, then R_k is met at stage $r < s$, and neither injured up to or at stage s, nor at or after stage $u <$ stage s. Therefore R_k is met at stage r and not injured at any stage after r. If condition (b) is true at stage s, T meets R_k. To show that condition (b) must be true at stage s:

1. If $T_s \neq T_{s-1} \cup F^s$, then (a), or (b), or (c) is true at stage s. Recall we did set $T_s = T_{s-1}$. By the definition of R_k is t-dense, $F^s \nsubseteq T_{s-1}$, and so $T_s = T_{s-1} \neq T_{s-1} \cup F^s$.

2. If both conditions (c) and (a) are false, then (b) is true.

   i) If (c) is true, $H^s \cap T_{s-1} \neq 0$, contradicting R_k is t-dense. So (c) is false.

   ii) If (a) is true: There was an $r < s$ such that $g(r) < g(s) = k$, $H^r \cap F^s \neq 0$, and $R_{k=g(r)}$ was met at stage r. Since r is less than u, $H^r \subset L$, but $H^r \cap F^s \neq 0$ implies $L \cap F^s \neq 0$ contradicting the t-denseness of R_k. So (a) is false.

To generate particular sets A and B which are non-recursive, recursively enumerable and have incomparable degrees of unsolvability. These sets will be generated inductively by stages, s :

$$s = 0, \text{ let } T_o = 0,\ F^o = H^o = 0, \text{ and } g(s) = 0\ ;$$

$$s > 0, \text{ let } A_s = \{ n \mid 2n \in T_{s-1} \}\ ;\ B_s = \{ n \mid 2n+1 \in T_{s-1} \}\ .$$

Let f_s^o and f_s^1 be the characteristic functions of A_s and B_s ;

$$i = 1 - (s)_o = \begin{cases} 0 & \text{if } s \text{ is even}\ ; \quad e = (s)_o \\ 1 & \text{if } s \text{ is odd.} \end{cases}$$

<u>Case 1</u> : $(Em)\ 0 \leq m \leq s\ (Ey)\ y \leq s\ [T_1^1(\tilde{f}_s^i(y), e, p_e^m, y) \text{ and } U(y) = 1]$ is a true statement. $\tilde{f}_s^i(y)$, $T_1^1(\tilde{f}_s^i(y), e, p_e^m, y)$, and $U(y)$ are defined in the introduction as in IM, Theorem IX * and relevant surrounding material, and p_e^m is the e^{th} prime raised to the m^{th} power.

If r is the greatest value for $m \leq s$ which makes this predicate true, then set :

$$F^s = \{2p_e^r + 1 - i\}$$
$$H^s = \{2n + i \mid f_s^i(n) = 1 \text{ and } n \leq s\}$$
$$g(s) = 2e + i + 1.$$

<u>Case 2</u> : Otherwise, set $F^s = H^s = 0$ and $g(s) = 0$. Now $T_s = T_{s-1}$ if conditions (a), (b), or (c) on page 13 are true, and $T_s = T_{s-1} \cup F^s$ otherwise. We let $T = \bigcup_{i=1}^{\infty} T_i$.

Remark 4. From these definitions $T_{s-1} \cap H^s = 0$ for all s. If $i = 0$, $H^s = \{2n \mid f_s^o(n) = 1 \text{ and } n \leq s\}$, so if $2n \in H^s$, n is not in A_s and $2n \notin T_{s-1}$. If $i = 1$, $H^s = \{2n + 1 \mid f_s^1(n) = 1 \text{ and } n \leq s\}$, so if $2n + 1$ is in H^s, n is not in B, and $2n + 1$ is not in T_{s-1}.

Let $t(s) = 2^{j(F^s)} \cdot 3^{j(H^s)} \cdot 5^{g(s)}$; then t is defined for all s. If the one element of F^s is even, then $i = 1$, and all the elements of H^s are odd, and vice versa. Therefore, t enumerates requirements, and T is the priority set for t. Hence T is recursively enumerable in t.

Let $A = \{n \mid 2n \in T\}$ and $B = \{n \mid 2n + 1 \in T\}$; then A, B are both recursively enumerable, and $A = \bigcup_{s>o} A_s$; $B = \bigcup_{s>o} B_s$ where $\{A_s\}_{s>o}$ and $\{B_s\}_{s>o}$ are nested sequences of sets. If f^o and f^1 are the characteristic functions for A and B then $i = 0, 1$: $f_s^i(n) = f^i(n)$ for a large enough s.

Corollary 1 : There are two recursively enumerable sets whose degrees of unsolvability are incomparable.

Proof : It must be shown that A and B are two recursively enumerable sets whose degrees of unsolvability are incomparable. By the Normal Form Theorem[IM] Theorem IX* : For every function, h, which is recursive in a function, g, an e can be found such that $h(n) = U(\mu y((j)j < y \ \ g(j) \text{ is defined and } T_1^1(\tilde{g}(y), e, n, y)))$ where U and T_1^1 are the particular primitive recursive functions.

If B were recursive in A, there would be a finite set of equations, $E = E_1$, $E_2, \ldots, E_n$, such that each $E_j (1 \leq j \leq n)$ is one of the primitive recursive schemata, or the characteristic function for A, or is derivable from the characteristic function for A by a finite number of applications of the primitive recursive schemata, and E_n is the characteristic function for B. That is, f^1 would be recursive in f^o, and f^1 would be expressible as :

$$f^1(n) = U(\mu y((j)\, j < y\; f^o(j) \text{ is defined } \& \; T_1^1(\tilde{f}^o(y), e, n, y))) \text{ for some e and all n.}$$

To show that B is not recursive in A: The set B is not recursive in A if the set of equations defined by any Gödel number, e, is not a set fitting the description of E, or the set of equations defined by e does not forecast whether or not some element is in B. Let us define the function

$$\{e\}^{f^o}(n) = U(\mu y((j)\, j < y\; f^o(j) \text{ is defined } \& \; T_1^1(\tilde{f}^o(y), e, n, y))).$$

There is no loss in generality in assuming that $\{e\}^{f^o}$ is the characteristic function for some set of natural numbers, since if it is not then it is not equal to f^1, the characteristic function for B. If for all e, there is at least one n such that $\{e\}^{f^o}(n) \neq f^1(n)$ then f^1 is not recursive in f^o.

If R_{2e+1} is a requirement, either it is or it is not t-dense.

If R_{2e+1} is t-dense, then by Theorem I, T meets R_{2e+1}, and there is an s with the following properties :

i) $g(s) = 2e + 1$. Hence $i = 1 - (s)_o = 0$, but $g(s) \neq 0$ implies that Case 2 is not true, since if Case 2 were true, $g(s) = 0$. Case 1 is true, and there must exist an r and a y such that

$$y \leqslant s \ \& \ T_1^1(\tilde{f}_s^o(y),\ e,\ p_e^r,\ y) \text{ and } U(y) = 1 : \qquad \text{EQ.1.}$$

$$F^s = \{2p_e^r + 1\} \ ; \text{ and } H^s = \{2n \mid f_s^o(n) = 1 \text{ and } n \leqslant s\} .$$

ii) $F^s \subseteq T$. Thus $p_e^r \in B = \{n \mid 2n + 1 \in T\}$, that is, $f^1(p_e^r) = 0$.

iii) $H^s \cap T = 0$. Consequently, if any element, $2n$ is in H^s, it is not in T, and n is not in A ; $f^o(n) = 1$. From the definition of H^s, $f_s^o(n) = 1$ for all $n \leqslant s$, so $f_s^o(n) = f^o(n)$ for all $n \leqslant s$. Since $y \leqslant s$, it follows from EQ.1 that

$$\{e\}^{f^o}(p_e^r) = U(\mu y((j)\ j < y\ f^o(j) \text{ is defined and } T_1^1(\tilde{f}^o(y),\ e,\ p_e^r,\ y))) = 1.$$

So for all e, there is an r whereby $1 = \{e\}^{f^o}(p_e^r) \neq f^1(p_e^r) = 0$.

If R_{2e+1} is not t-dense, there is a finite set, L, such that for all $s > 0$

i) $H^s \cap T_{s-1} = 0$,

ii) $L \cap F^s = 0$, and

iii) $g(s) = 2e + 1$;

imply iv) $F^s \subseteq T_{s-1}$.

We know (i) is true by Remark 4.

Since L is finite, there is an $m > 0$ assuring $2p_e^m + 1$ is greater than all members of L.

Suppose p_e^m is in B : Then $2p_e^m + 1$ is in T, and for some $s > 0$, $2p_e^m + 1$ is in $T_s - T_{s-1}$. We can be sure that this $T_s = T_{s-1} \cup F^s \neq T_{s-1}$, so ; F^s is non-empty in fact F^s is the set $\{2p_e^m + 1\}$. As m was chosen to ensure $(2p_e^m + 1)$ would be greater than all members of L, $F^s \cap L = \emptyset$, so (ii) holds.

Case 2 is not true, since if Case 2 were true, F^s would be empty. Case 1 is true, whereby $g(s) = 2e + 1$; (iii) is verified for this s.

It follows from $T_{s-1} \cup F^s \neq T_{s-1}$ that $F^s \not\subseteq T_{s-1}$. When p_e^m is in B there is an s for which (i), (ii), (iii) hold and (iv) is false, hence p_e^m is not in B, and $f^1(p_e^m) = 1$.

It is now necessary to show that p_e^m is in A by showing that $\{e\}^{f^o}(p_e^m) = 0$. Suppose the contrary :

$$\{e\}^{f^o}(p_e^m) = 1 = U(\mu y((j)\ j < y\ f^o(j) \text{ is defined } \& \ T_1^1(\tilde{f}^o(y),\ e,\ p_e^m,\ y))).$$

In particular there is a y such that $f^o(j)$ is defined for all $j < y$. Since $f_s^o(j) = f^o(j)$ for all sufficiently large s, choose an s with $i = 1 - (s)_o$, $e = (s)_1$, $m \leq s$, and $y \leq s$. Case 1 is true for this s.

This means $F^s = 2p_e^r + 1$ where $r \geq m$, so $2p_e^r + 1$ is greater than or equal to

$2p_e^m + 1$ which is greater than all members of L, so $F^s \cap L = 0$. (ii) is true.

At this stage s, as Case 1 is true, $g(s) = 2e + 1$, so (iii) holds.

From our earlier argument, we know p_e^m is not in B for all $m > 0$ such that $2p_e^m + 1$ is greater than all members of L. Certainly then for $r \geqslant m$, p_e^r is not in B ; $2p_e^r + 1$ is not in T ; F^s is not a subset of T, and, a fortiori, $F^s \not\subseteq T_{s-1}$ contradicting the hypothesis that R_{2e+1} is not t-dense.

Therefore $\{e\}^{f^o}(p_e^m) = 0$ and $f^1(p_e^m) = 1$.

B is not recursive in A.

To show that A is not recursive in B :

If A were recursive in B, f^o would be recursive in f^1 and f^o would be expressible as :

$f^o(n) = U(\mu y((j)\ j < y\ f^1(j)$ is defined and $T_1^1\ (\tilde{f}^1(y), e, n, y)))$ for some e and all n.

Let us define the function

$$\{e\}^{f^1}(n) = U(\mu y((j)\ j < y\ f^1(y) \text{ is defined and } T_1^1\ (\tilde{f}^1(y), e, n, y))),$$

a characteristic function for some set of natural numbers. If for all e, there is at least one n such that $\{e\}^{f^1}(n) \neq f^o(n)$ then f^o is not recursive in f^1, and A is not recursive in B. This argument is identical to the one showing B is not recursive in A with the exception of the notational changes from $\{e\}^{f^o}$ to $\{e\}^{f^1}$.

To expand this idea we can show that there is a sequence of non-recursive, recursively enumerable sets which have incomparable degrees of unsolvability. The construction of this sequence $\{A_i\}$ $i \geqslant 0$ is very similar to the construction of sets A and B in the previous corollary. We generate the sets F^s, H^s, T_s, $A^s_m = \{n \mid p_m^{1+n} \in T_{s-1}\}$ by induction on stage s, and define $A_m = \{n \mid p_m^{1+n} \in T\}$.

$s = 0$ $\qquad F^o = H^o = T_o = 0$ and $g(0) = 0$.

$s > 0$ $\qquad$ For each $m > 0$, let $\lambda n\, A^s(m,n)$ be the representing function of A^s_m, $u = (s)_o$, $e = (s)_1$. The lambda notation is due to A. Church(10). If A(n) is a term and A contains n as a free variable then $\lambda n A$ is the function defined by the condition that $(\lambda n A(n))(\bar{n}) = A(\bar{n})$.

Let f^u_s denote the function $\lambda m n \mid A\,(m + sg((m+1) \dot{-} u), n)$ where the subscript

$$m + sg((m+1) \dot{-} u) = \begin{cases} m+1 & \text{if } u < m \\ m & \text{if } u \geqslant m, \end{cases}$$

is a device to ensure that the first component of the set classification does not equal u. For reading ease, name this quantity v(m). Again there are two cases :

<u>Case 1</u> : $\qquad (Em)\, 0 < b \leqslant s\ \ (Ey)\, y \leqslant s\ (T^2_1(\breve{f}^u_s(y,y),\ e,\ p^b_e,\ y)\ \&\ U(y) = 1)$.

If r is the greatest value of b which makes the predicate true, set :

$$F^s = \{p_u^{1+p_e^r}\}$$

$$H^s = \{p_{v(m)}^{1+n} \mid f^u_s(m,n) = 1\ m \leqslant s\ \&\ n \leqslant s\}$$

$$g(s) = 2^u \cdot 3^e.$$

<u>Case 2</u> : $\qquad$ Otherwise, set $F^s = H^s = 0$, and $g(s) = 0$. In either case $T_s = T_{s-1}$ if (a), (b) or (c) are true, $T_s = T_{s-1} \cup F^s$ otherwise, and $T = \bigcup_{i=1}^{\infty} T_i$.

For every s : F^s is either empty or a power of the u^{th} prime and H^s is either empty or the set of powers of some primes, but never the u^{th} prime. In other words for every s, $F^s \cap H^s = 0$. Defining $t(s) = 2^{i(F^s)} \cdot 3^{i(H^s)} \cdot 5^{g(s)}$ t enumerates requirements, T is the priority set for t, and T is recursively enumerable in t.

Remark 5. In this sequence it is also true that $H^s \cap T_{s-1} = 0$ for all s. $H^s = \{ p_{v(m)}^{1+n} \mid f_s^u(m,n) = 1 \mid m \leq s \ \& \ n \leq s \}$ or, more simply, if $p_{v(m)}^{1+n}$ is in H^s, then n is not in A_m^s for $m \neq u$, which by definition of A_m^s, ensures that $p_{v(m)}^{1+n}$ is not in T_{s-1}.

Let $A_u = \{ n \mid p_u^{1+n} \in T \}$ for all $u \geq 0$. and $\lambda n\, A(u,n)$ be the representing function for A_u.

Since T is recursively enumerable, A_o, A_1, ... are all recursively enumerable, and $A_u = \bigcup_{s > o} A_u^s$ where for each u, $\{A_u^s\}$ s > o is a nested sequence of sets. If f^u denotes the function $\lambda m n \mid A(v(m), n)$, we know

$$f^u(m,n) = \begin{cases} 0 & n \in A \\ 1 & n \notin A, \end{cases}$$

where m can be any number not equal to u. For a big enough s, $f^u(m,n) = f_s^u(m,n)$.

$\{A_i\}_{i > o}$ is recursively independent iff for each $u \geq o$, the set $A u$ is not recursive in the function $\lambda m n \mid A(v(m), n)$.

The sequence of sets is simultaneously recursively enumerable iff every set in the sequence is recursively enumerable.

Corollary 2 : There exists a sequence of recursively independent, simultaneously recursively enumerable sets.

Proof : By our construction of A_o, A_1, ... we know that $\{A_i\}_{i>o}$ is a sequence of simultaneously recursively enumerable sets. In order to show independence, we must prove that A_u is not recursive in any finite number of the other sets $A_i (i \geq 0), (u \neq i)$.

Suppose A_u was recursive in some other sets of the sequence, then the characteristic function for A_u would be $\lambda n A(u,n) = U(\mu y((j)(k)\ j,k,< y f^u(j,k)$ is defined and $T_1^2(\tilde{f}^u(y,y), e,n,y)))$. Let us define the function

$$\{e\}^{f^u}(n) = U(\mu y((j)(k)\ j,k,< y f^u(j,k) \text{ is defined and } T_1^2(\tilde{f}^u(y,y),e,n,y))).$$

We can state that $\{e\}^{f^u}$ is the characteristic function of some set of natural numbers, since if it is not, then it is not the characteristic function for A_u. If for all e, there is at least one n such that $\{e\}^{f^u}(n) \neq A(u,n)$ then A_u is not recursive in the other sets of the sequence. Let $w = 2^u \cdot 3^e$ and proceed as in Corollary 1.

If R_w is a requirement, either it is or is not t-dense.

If R_w is t-dense, then from Theorem I, T meets R_w and there is an s with the following three properties :

i) $g(s) = w = 2^u \cdot 3^e$. Since $g(s) \neq 0$, Case 1 must be true for this s and there is a r and a y

$y \leqslant s$ & $T_1^2(\tilde{f}^u(y,y), e, p_e^r, y)$ and $U(y) = 1$, EQ.2.

$F^s = \{p_u^{1+p_e^r}\}$; and

$H^s = \{p_{v(m)}^{1+n} \mid f_s^u(m,n) = 1 \; m \leqslant s \;\&\; n \leqslant s\}$.

ii) $F^s \subseteq T$. Thus $p_e^r \in A_u = \{n \mid p_u^{1+n} \in T\}$, or $A(u, p_e^r) = 0$.

iii) $H^s \cap T = 0$. Consequently if any element $p_{v(m)}^{1+n} \in H^s$, it is not in T and n is not in A_u, $f^u(m,n) = 1$. From the definition of H^s : $f_s^u(m,n) = 1$ for all m, $n \leqslant s$, so $f^u(m,n) = f_s^u(m,n)$ for all m, $n, \leqslant s$. Since $y \leqslant s$, it follows from EQ.2 that

$\{e\}^{f^u}(p_e^r) = U(\mu y((j)(k)\; j,k, < y\; f^u(j,k)$ is defined &

$T_1^2(\tilde{f}^u(y,y), e, p_e^r, y))) = 1$.

So for all e there is an r whereby $1 = \{e\}^{f^u}(p_e^r) \neq A(u, p_e^r) = 0$.

If R_w is t-dense, then there is a finite set, L, and for all $s > 0$:

i) $H^s \cap T_{s-1} = 0$,

ii) $L \cap F^s = 0$, and

iii) $g(s) = w$ together

imply iv) $F^s \subseteq T_{s-1}$.

We know (i) is true by Remark 5.

Since L is finite, there is a $b > 0$ assuring that $p_u^{1+p_e^b}$ is greater than all members of L. As F^s is either empty or $\{p_u^{1+p_e^b}\}$, $F^s \cap L = 0$; (ii) holds.

Let p_e^b be in A_u.

Then $p_u^{1+p_e^b}$ is in T, and for some $s > 0$, $p_u^{1+p_e^b}$ is in $T_s - T_{s-1}$. We can be sure that $T_s = T_{s-1} \cup F^s \neq T_{s-1}$, so : F^s is non-empty and Case 2 is not true. Case 1 is true whereby $g(s) = w$, so (iii) is verified for this s.

It follows from $T_{s-1} \cup F^s \neq T_{s-1}$ that $F^s \not\subseteq T_{s-1}$. When p_e^b is in A_u, there is an s for which (i), (ii), and (iii) hold but (iv) is false. Hence p_e^b is not in A_u, $A(u, p_e^b) = 1$.

It is now necessary to show that $\{e\}^{f^u}(p_e^b) = 0$. Suppose the converse :

$$\{e\}^{f^u}(p_e^b) = 1 = U(\mu y((j)(k)\ j,k, < y\ f^u(j,k) \text{ is defined } \&\ T_1^2(\tilde{f}^u(y,y),\ e,\ p_e^b,\ y))).$$

In particular there is a y, $f^u(j,k)$ is defined for all j, k, $<$ y. Since $f_s^u(j,k) = f^u(j,k)$ for all sufficiently large s, choose a suitable s such that $(s)_o = u$ and $(s)_1 = e$. Case 1 is true for this s.

This means $F^s = \{p_u^{1+p_e^r}\}$ where $r \geq b$, so $p_u^{1+p_e^r}$ is greater than or equal to $p_u^{1+p_e^b}$ which is greater than all members of L, so $F^s \cap L = 0$. (ii) is true.

At this stage s, as Case 1 is true, $g(s) = w$, so (iii) holds.

From our earlier argument, we know p_e^b is not in A_u for all $b > 0$ such that $p_u^{1+p_e^b}$ is greater than all members of L. Certainly then for $r \geqslant b$, p_e^r is not in A_u, and $p_u^{1+p_e^r}$ is not in T ; F^s is not a subset of T and, a fortiori, $F^s \nsubseteq T_{s-1}$.

Therefore $\{e\}^{f^u}(p_e^b) = 0$ and $A(u, p_e^b) = 1$.

A_u is not recursive in A_i $(i \neq u)$.

Since the degrees of unsolvability of recursively enumerable sets are partially ordered, the degrees of any sequence of recursively independent, simultaneously recursively enumerable sets can be imbedded in the upper semi-lattice of degrees of recursively enumerable sets as a result of the next corollary.

Corollary 3 : If P is a countable, partially ordered set, then P is imbeddable in the upper semi-lattice of degrees of recursively enumerable sets.

Proof : It will be sufficient to show that N_R is imbeddable in the upper semi-lattice. We show that N_R is imbeddable in the upper semi-lattice of degrees of recursively enumerable sets by showing that if $m, n \in N$ and C_m, C_n are recursively enumerable sets then $m \leqslant_R n$ iff C_m is recursive in C_n. In Sacks' terms this is an order-isomorphism ; a monotonic property, an order preservation. We must construct a suitable sequence of sets $\{C_i\}$ $i \in N_R$.

Take $\{A_i\}$ $i > 0$ to be a sequence of recursively independent, simultaneously recursively enumerable sets. For each $m \geqslant 0$, let $B_m = \{p_m^{n+1} \mid n \in A_m\}$: each B_m is recursive in A_m, so the $\{B_i\}$ $i \geqslant 0$ are simultaneously recursively enumerable, and recursively independent. In addition the B_i are pairwise disjoint since B_i contains only powers of the i^{th} prime. Let $C_m = \bigcup \{B_r \mid r \leqslant_R m\}$, C_m is recursively enumerable, since the B_i's are simultaneously recursively enumerable and $\leqslant_R$ is recursive.

It is now necessary to show that C_u is recursive in C_v if and only if $u \leqslant_R v$, where u, v are in N_R.

Suppose C_u is recursive in C_v, and $u \nleqslant_R v$: Then $B_u \cap C_v = 0$ since $C_v = \bigcup_{r \leqslant_R v} B_r$, $(u \nleqslant_R v)$ and $\{B_i\}$ are disjoint, and C_v is recursive in the sets of the sequence $\{B_i\}$ $i \geqslant 0$ $i \neq u$, since x is in C_v if and only if there is an m and an n such that $x = p_m^{n+1}$ and $m \leqslant_R u$ and x is in B_m. We know that the $\{B_i\}$ are independent, so B_u is not recursive in C_v as it is recursive in the other $B_i (i \neq u)$, but C_u is recursive in B_u, and therefore not recursive in C_v. This contradicts the given, and it follows that C_u recursive in C_v implies $u \leqslant_R v$.

If $u \leqslant_R v$, we must show that C_u is recursive in C_v. From $u \leqslant_R v$, $C_u \subseteq C_v$ so : x is in C_u if and only if there is an m and an n such that $x = p_m^{n+1}$ where $m \leqslant_R u$ and x is in B_m. We saw that $\leqslant_R$ was a transitive relation so $m \leqslant_R u$ and $u \leqslant_R v$ imply $m \leqslant_R v$; by definition $C = \bigcup B_r (r \leqslant_R v)$, so $B_m \subseteq C$. Since the B_i's are disjoint, x is in B_m if and only if x is in C_v ; or x is in C_u, is equivalent to $x \in C_v$; or C_u is recursive in C_v.

CHAPTER IV

FRIEDBERG'S SOLUTION

In 1956 Richard M. Friedberg[2] discovered a solution to Post's problem. His original argument went as follows :

If f_1 is recursive in f_2, then there is an e, for which $f_1(x_1^a(e)) = U(\mu y T_1^{f_2}(e, x_1^a(e), y)$ for all numbers $x_1^a(e)$, defined for this e and all a.

In order to show that f_1 is not recursive in f_2 :

i) define a number $x_1^a(e)$ for every pair of integers, a, e ;

ii) If at some stage $a'(\geqslant a)$, $T_1^{f_2^{a'-1}}(e, x_1^a(e), y)$ and $U(y) = 1$, then we set $f_1^{a'}(x_1^{a'}(e)) = 0$.

iii) We ensure (as nearly as possible) that $T_1^{f_2^{a'}}(e, x_1^a(e), y)$ and $U(y) = 1$ will be true for all $a'' \geqslant a'$, (see Remark F.5).

iv) $f_i(x) = \begin{cases} 0 (Eaf_i^a(x) = 0)\ (i = 1, 2) \\ 1 \text{ otherwise} \end{cases}$

It is necessary to define f_1 and f_2 and show that they are partial recursive functions. Since they will be (0, 1) valued, they can be considered as representing functions of recursively enumerable sets.

<u>TO DEFINE</u> : x_i^a and f_i^a ; $(i = 0, 1)$:

$$a = 0 \quad f_1^o(x) = f_2^o(x) = 1 \quad \text{for all } x$$

$$x_1^o(e) = x_2^o(e) = 2^e \quad \text{for all } e.$$

Case 1 : (a odd)

$a = 2b + 1$ Let e_a be the number of different prime divisors of b.

i) $f_2^a(x) = f_2^{a-1}(x)$ for all x.

ii) $x_1^a(e) = x_1^{a-1}(e)$ for all e.

Subcase 1.1 : If $f_1^{a-1}(x_1^{a-1}(e_a)) = 1$ and $(Ey<a)T_1^{f_2^{a-1}}(e_a, x_1^{a-1}(e_a), y)$ and $U(y) = 1$, then

iii) .1 $f_1^a(x_1^{a-1}(e_a)) = 0$; $f_1^a(x) = f_1^{a-1}(x)$ for all $x \neq x_1^{a-1}(e_a)$,

iv) .1 $x_2^a(e) = 2^e(2a + 1)$ all $e \geq e_a$,

$x_2^a(e) = x_2^{a-1}(e)$ all $e < e_a$.

Subcase 1.2 : Otherwise,

iii) .2 $f_1^a(x) = f_1^{a-1}(x)$ for all x,

iv) .2 $x_2^a(e) = x_2^{a-1}(e)$ for all e.

Case 2 : (a even)

$a = 2b + 2$, where e_a is the number of prime divisors of b.

i) $f_1^a(x) = f_1^{a-1}(x)$ for all x,

ii) $x_2^a(e) = x_2^{a-1}(e)$ for all e.

Subcase 2.1 : If $f_2^{a-1}(x_2^{a-1}(e)) = 1$ and $(Ey<a)T_1^{f_1^{a-1}}(e_a, x_2^{a-1}(e_a), y)$ and $U(y) = 1$, then

iii) .1 $f_2^a(x_2^{a-1}(e)) = 0$; $f_2^a(x) = f_2^{a-1}(x)$ for all $x \neq x_2^{a-1}(e_a)$,

iv) .1 $x_1^a(e) = 2^e(2a + 1)$ for all $e \geq e_a$

$$x_1^a(e) = x_1^{a-1}(e) \text{ for all } e \leq e_a.$$

Subcase 2.2: Otherwise,

iii) .2 $f_2^a(x) = f_2^{a-1}(x)$ for all x,

iv) .2 $x_1^a(e) = x_1^{a-1}(e)$ for all e.

$$\text{Define } f_i(x) = \begin{cases} 0 & \text{if } Ea\, f_i^a(x) = 0 \ (i = 1, 2) \\ 1 & \text{Otherwise.} \end{cases}$$

Then f_1 and f_2 represent recursively enumerable sets.

Remark F.1. This result is used in Friedberg's Lemma I and corresponds to Sacks' Lemma I.

If $f_i^a(x)$ changes value at stages $a = r$ and $a = s$ $(r < s)$ from 1 to 0, then there is a u, $r < u < s$, such that $f_i^u(x)$ changes value from 0 to 1 at stage u.

Proof: Given $f_i^{r-1}(x) = 1$, $f_i^r(x) = 0$

$f_i^{s-1}(x) = 1$, $f_i^s(x) = 0$.

Here $f_i^r(x) = f_i^s(x) = 0$, $f_i^{s-1}(x) = 1$, and $r - s = k$, some finite number. If $f_i^{s-i}(x) = f_i^{(s-i-1)}(x)$ (for $i = 1, \ldots, k-1$), then $f_i^{s-1}(x) = f_i^{s-k}(x) = f_i^r(x)$ which is false. So for some stage a $r < a < s$ $f_i^a(x) = 1$. Since there are a finite number of stages between r and s, there must be a first one at which this occurs. Call this stage "u".

Remark F.2. If $x_i^a(e)$ changes value infinitely often, $f_i^a(x_i^a(e))$ changes value infinitely often, since at any stage a at which x_i^a changes, f_i^a does not.

If $f_i^a(x)$ changes value from 1 to 0 at n stages, it must change value from 0 to 1 at at least n − 1 stages and at at most n + 1 stages, and at each of these stages $x_i^a(e)$ changes value. In particular, if $f_i^a(x)$ changes value from 1 to 0 at infinitely many stages there are infinitely many stages at which it changes back from 0 to 1.

Lemma I : For any given e, $x_1^a(e)$ changes only finitely often as a increases through the natural numbers.

Proof : Suppose this lemma does not hold. There is some e such that $x_1^a(e)$ changes infinitely often as a increases through the natural numbers, and therefore there is a least e, $\bar{e}$, such that $x_1^a(\bar{e})$ changes value infinitely often as a increases.

If $x_1^a(\bar{e}) \neq x_1^{a-1}(\bar{e})$, then

i) Subcase 2.1 is true ; and

ii) there is an $\bar{e}' = e_a < \bar{e}$ for which $f_2^{a-1}(x_2^{a-1}(\bar{e}')) = 1$ and, as $x_2^{a-1}(e) = x_2(e)$ for all e, $f_2^a(x_2^{a-1}(\bar{e}')) = f_2^a(x_2^a(\bar{e}')) = 0$.

When this occurs at stages a', a", we know by Remark F.1, that there is a u $a' < u < a''$ such that $f_2^{u-1}(x_2^{u-1}(\bar{e}')) = 0$ and $f_2^u(x_2^u(\bar{e}')) = 1$. At stage u f_2 changes value from 0 to 1 and therefore Subcase 2.1 is not true and $f_2^{u-1}(x) = f_2^u(x)$ for all x, so if $f_2^{u-1}(x_2^{u-1}(\bar{e}')) \neq f_2^u(x_2^u(\bar{e}'))$, then $x_2^{u-1}(\bar{e}') \neq x_2^u(\bar{e}')$. This is true for the infinitely many stages u ($a' < u < a''$) where x_1 changes value at a' and a".

x_2^u changing value implies that

i) Subcase 1.1 is true at stage u.

ii) $e_u = e \leqslant \bar{e}'$.

iii) $f_1^{u-1}(x_1^{u-1}(e)) = 1$.

iv) $f_1^u(x_1^u(e)) = 0$.

Again between any two stages of u at which $f_1^u(x)$ changes value from 1 to 0, there is a stage u' such that $f_1^{u'-1}(x_1^{u'-1}(e)) = 0$ and $f_1^{u'}(x_1^{u'}(e)) = 1$. Since at this stage f_1 does not change value, $x_1^{u'-1}(e) \neq x_1^{u'}(e)$ for the infinitely many u' at which f_1 changes back from 0 to 1.

Therefore we have exhibited an $e(=e_u < \bar{e})$ such that $x_1^u(e)$ changes infinitely often, contradicting our hypothesis that $\bar{e}$ was the least such number. The lemma must hold.

In the same manner for any given e, $x_2^a(e)$ changes only a finite number of times as a increases through the natural numbers.

Remark F.3. If $f_i(x)$ is the characteristic function of a finite set, S_i, it is equal to zero for only a finite number of values of x.

Since $x_1^a(e)$ changes only a finite number of times, for any given e, there is a stage a' for each e such that $x_i^{a''}(e) = x_i^{a'}(e)$ for all $a'' \geqslant a'$; i.e. after a certain finite stage, not constructively defined ; $x_i^a(e)$ is a constant, say $z_i(e)$.

Since $f_i^a(x)$ can change at most once more than $x_i^a(e)$, for each given e, there is a stage a^* such that $f_i^{a^*}(x) = f_i^{a^{**}}(x)$ for all $a^{**} \geqslant a^*$; i.e. after a certain finite stage, not constructively defined, $f_i^a(x)$ is a constant.

Remark F.4. There is an a, for all x, after which $f_i^a(x)$ does not change and this last value is the same as $f_i(x)$.

Since $x_i^a(e)$ has a last value, $z_i(e)$, and $f_i^{a'}(x_1^a(e))$ can change at most once more for some $a' > a$, $f_i^{a'}(z_i(e))$ has a last value.

Suppose $f_i(z_i(e)) = 1$, then there is no a such that $f_i^a(z_i(e)) = 0$ or $f_i^a(z_i(e)) = 1$ for all a. That is, $f_i^a(z_i(e)) = f_i(z_i(e))$ for all a.

Suppose $f_i(z_i(e)) = 0$. There is a stage a such that $f_i^a(z_i(e)) = 0$. Since this a is finite, there is a least stage at which this happens, say a'. $f_i^{a'}(z_i(e)) = 0$ for the first time ; or $f_i^{a^*}(z_i(e)) = 1$ for all $a^* > a'$. This a' must be greater than or equal to the stage a at which $x_1^a(e)$ takes the value $z_i(e)$; or $x_i(e)$ changes last, so $x_i^{a''}(e) = z_i(e)$ for all $a'' \geqslant a'$. If $f_i^{a''}(z_i(e))$ were to change back to 1 at stage a", from Lemma I we know $x_i^{a''}(e)$ would have to change value and this does not happen at any stage $a'' \geqslant a'$. Therefore $f_i^{a''}(z_i(e)) = f_i(z_i(e))$ for all $a'' \geqslant a'$.

Remark F.5. If $T_1^{f_2^a}(e_a, x_1^a(e_a), y)$ is true at some stage a, but for $a' \geqslant a$ $T_1^{f_2^{a'}}(e_a, x_1(e_a), y)$ is false, then at stage a', $f_2^{a'}$ has changed value. Subcase 2.1 must be true, but $e_a < e_{a'}$, since $x_1^{a'}(e_{a'}) = x_1^{a}(e_a)$. Therefore the only way to falsify $T_1^{f_2^a}(e_a, x_1^a(e_a), y)$ at some later stage a' is by having the simultaneous conditions : Subcase 2.1 is true at stage a', but $e_{a'} > e_a$. Similarly the only way to falsify $T_1^{f_1^a}(e_a, x_2^a(e_a), y)$ is by having Subcase 1.1 true, and $e_{a'} \geqslant e_a$.

<u>Lemma II</u> : Let $z_1(e)$ be the last unchanged value assumed by $x_1^a(e)$ as a increases, then $Ey\ [T_1^{f_2}(e, z_1(e), y) \text{ and } U(y) = 1] \equiv (f_1(z_1(e)) = 0)$.

<u>Proof</u> : If $Ey\ (T_1^{f_2}(e, z_1(e), y) \text{ and } U(y) = 1)$: Choose a stage $a' > a$, such that a' is odd and so large that $f_1^{a'}(z_1(e)) = f_1(z_1(e))$. This can be done by Remark F.4. If $f_1^{a'}(z_1(e)) = 0$, then $f_1(z_1(e)) = 0$ and the lemma is proved.

If $f_1^{a'}(z_1(e)) = 1$, for all a', then $f_1^{a'-1}(x_1^{a'-1}(e)) = 1$ for all $a' > a$. Starting with the formal procedure, E, with f_2 as principal function letter, G.N.(E) = e, there is no finite y, or no finite G.N. of this procedure for the argument $z_1(e)$. This contradicts the given. There must be a stage $a' \geqslant a$ such that $f_1(z_1(e)) = f_1^{a'}(z_1(e)) = 0$.

Now, if $f_1(z_1(e)) = 0$, then E a $f_1^a(z_1(e)) = 0$ and for this a :

i) $x_1^a(e) = z_1(e)$, and

ii) $(Ey < a)T_1^{f_2^{a-1}}(e, z_1(e), y) \text{ and } U(y) = 1$.

Suppose this condition is falsified : This can be true only if $f_2^{a'-1}$ changes value for $a' > a$ and $z_1(e)$ does not. This means that $e_{a'} \ngtr e$ for those $a' > a$. We know for any fixed e, say $e - 1$, there are infinitely many $e_a = e - 1$. At stage a' only finitely many of these e can have been calculated, so there will always be another $e_{a'}$, $a' > a$ such that $e_a = e - 1 < e$, causing $z_1(e)$ to change value, contradicting the given. Therefore $f_2^{a'-1}$ does not change value for any $a' > a$, $f_2^{a'-1} = f_2$ and $Ey\, T_1^{f_2}(e, z_1(e), y)$ and $U(y) = 1$. By interchanging the subscripts for the f's, z's and x's : Let $z_2(e)$ be the last unchanged value assumed by $x_2^a(e)$ as a increases ; then $Ey\left[T_1^{f_1}(e, z_2(e), y) \text{ and } U(y) = 1\right]$ $\equiv (f_2(z_2(e)) = 0)$.

Theorem I : There exist two functions f_1 and f_2, both of which represent recursively enumerable sets and neither of which is recursive in the other.

Proof : In an exact analogy to Sacks' Corollary 1, we have shown that for all e, there is an n, depending on e, such that

$$f_i(n) \neq U\,(\mu y\, T_1^{f_j}(e, n, y))\; i \neq j\; (i, j, = 1, 2).$$

So f_1 is not recursive in f_2 and vice-versa. To define two sets $S_i (i = 1, 2)$: by stages.

$a = 0 \qquad f_i^o(2^e) = 0$ if $e \in A$

$\qquad\qquad\qquad = 1$ if $e \notin A$

$x_i^o(e) = 3 . 2$

$f_i^o(x_i^o(e)) = f_i^o(3 . 2^e) \neq f_i^o(2^{e'})$ for all $e' \in A$.

Therefore, $f_i^o(x) = 0$ for all x.

$a > 0$ $\qquad f_i^a$, and x_i^a are defined as before.

Then $S_i = \{x \mid f_i(x) = 0\}$.

Theorem II : Given a set, A, there exist two sets not recursive in one another, both enumerable by a procedure recursive in A and both of degree higher than that of A.

Proof : We have shown that the recursively defined f_1 and f_2 have incomparable degrees of recursive unsolvability. Since f_i is the characteristic function for $S_i (i = 1, 2)$. S_1 and S_2 have incomparable degrees of unsolvability (neither is recursive in the other), and we have exhibited two recursively enumerable sets which are recursively independent.

CHAPTER V

MUČNIK'S SOLUTION

Mučnik's[5] solution to Post's problem appears to be much more complex than Friedberg's.

Friedberg accepts and uses the notation and constructions of Kleene. In particular the T predicate is used to define recursively enumerable, non-recursive sets and if any function is recursive in a second one, it can be expressed in Kleene's normal form for some Gödel Number, e.

Mučnik defines a class of M problems as any mathematical task whose achievement can be resolved into an infinite sequence of elementary acts whose results can each be characterized by a natural number. An M-problem can be put into one to one correspondence with its set of solving equations and is algorithmic or solvable if at least one of its solving functions is general recursive.

The decision problem for B is reducible to the decision problem for A, if there exists a general recursive operator which maps the characteristic function of A into the characteristic function of the set B.

Mučnik defines and uses the following new concepts :

quasisequence : a finite or infinite sequence of numbers and some new symbol λ used as a symbol for an undefined quantity : The length, $M\{l\}$, of a quasisequence, l, is the number of its components.

mutually consistent : elements a and b are mutually consistent, $a \backsim b$, if $a = b$ or $a = \lambda$, or $b = \lambda$; quasisequences l_1 and l_2 are mutually consistent if $l_1(n) \backsim l_2(n)$ for all $n \leqslant \mathrm{Min}(M\{l_i\})$.

cover : a_1 covers $a_2 (a_1 \geqslant a_2)$ if either $a_1 = a_2$ or $a_2 = \lambda$. $f_1 \geqslant f_2$ means that $f_1(n) \geqslant f_2(n)$ for all $n \leqslant \mathrm{Min}(M\{l_i\})$.

compatible : quasisequences e_1 and e_2 are compatible, $e_1 \backsimeq e_2$, if $e_1(n) \backsim e_2(n)$ for all $n \leqslant \min(M\{l\}_i)$. $e_1 \times e_2$ if they are not compatible.

predicate : a sequence of 1's and zeroes. A quasi-predicate : a sequence of 1's, 0's and λ's.

The concept of a partial recursive operator is a precise form of the notion of an effective mapping of systems of functions into functions. Mučnik then proves that the map from infinite quasisequences into sequences of any partial recursive operator is constructively defined by some primitive recursive function. This is certainly still true if the set of partial recursive operators is defined on a subset of infinite quasisequences, the infinite sequences. In addition if we define t as a characteristic value of the operator P_{x_o} if $x(t) = x_o$, then it can be shown that P_x has only a finite number of characteristic values, t, for any given x_o. This is somewhat similar to Sacks' Lemma that any requirement can be injured at most a finite number of times, and Friedberg's Lemma that $x_i(e)$ can change value only a finite number of times.

Let $d_x |\omega|$ be a finite predicate which gets taken into $d'_x |\omega|$ by the operator P_x.

A function defined on a segment of the natural numbers is orderly. To every orderly function, $\phi_x(\omega)$, there corresponds some sequence of pairs of quasisequences $(d_x|\omega|, d'_x|\omega|)$, with $\phi_x(\omega)$ as its index. This $\phi_x(\omega)$ serves the same purpose as a pairing function. Introduce a parametric representation $z = \phi(x,\omega)$; $x = x(t)$, $\omega = \omega(t)$. $z = z(t)$, where $x(t)$, $\omega(t)$, and $z(t)$ are primitive recursive functions. Denote the predicates $d_{x(t)}|\omega(t)|$ and $d'_{x(t)}|\omega(t)|$ by $\bar{f}|t|$ and $\bar{f}'|t|$, and $f|t|$, $f'|t|$ by the quasisequences derived from $\bar{f}|t|$, $\bar{f}'|t|$ by replacing all occurrences of 0 by λ. Note that $M\{f|t|\} = m(t)$ and $M\{f'|t|\} = m'(t)$.

In order to solve Post's problem two recursively enumerable, non-recursive sets must be constructed. Let these sets be E and G whose characteristic functions are $\bar{e}$ and $\bar{g}$ respectively. We define $\bar{e}$ and $\bar{g}$ recursively as follows :

Construct sequences $\{t_i\}$, $\{e_i\}$, $\{g_i\}$:

$t_o = 0$, $e_o = f|0|$, $g_o = cf'|0|$.

Assume that $t_o, \ldots, t_{2k}$; $e_o, \ldots, e_{2k}$; and $g_o, \ldots, g_{2k}$ are already defined. Put $t_{2k+1} = \mu t$, satisfying 1a) - 1d)

1a) $f|t| \gtrsim g_{2k}$;

1b) $f|t| \eqsim [g_{2k}]_{\alpha(t,2k)}$;

where $\alpha(t,k)$ is the control function defined by :

$\alpha(t,q) = \max\{\bar{\alpha}(t,q), \bar{\alpha}'(t,q)\}$,

$$\bar{\alpha}(t,q) = \max_{i \leqslant j(t,q)} \{m(t_i)\} + 1 ;$$

$$\bar{\alpha}'(t,q) = \max_{i \leqslant j(t,q)} \{m'(t_i)\} + 1$$

$$j(t,q) = \max_{(i \leqslant q) x(t_i) \leqslant x(t)} (i)$$

1c) $(i)_{i<k}((x(t_{2i+1}) = x(t) \,\&\, f \,|\, t_{2i+1} \,| \asymp g_{2k}) \to (f \,|\, t_{2i+1} \,| \asymp e_{2k}))$;

1d) $f' \,|\, t \,|\, x(e_{2k}, f' \,|\, t \,|)_{\alpha(t,2k)}$.

Set $e_{2k+1} = (e_{2k}, f' \,|\, t_{2k} \,|)_{\alpha(t_{2k+1}, 2k)}$; $g_{2k+1} = g_{2k} \cup f \,|\, t_{2k+1} \,|$.

Put $t_{2k+2} = \mu t$, satisfying 2a) - 2d) :

2a) $f \,|\, t \,| \geqslant e_{2k+1}$;

2b) $f \,|\, t \,| \asymp [e_{2k+1}]_{\alpha(t,2k+1)}$

2c) $(i)_{i \leqslant k} (x(t_{2t}) = x(t) \,\&\, f \,|\, t_{2i} \,| \asymp e_{2k+1} \to f' \,|\, t_{2i} \,| \asymp g_{2k+1})$;

2d) $f' \,|\, t \,|\, x(g_{2k+1}, f' \,|\, t \,|)_{\alpha(t, 2k+1)}$

Set $e_{2k+2} = e_{2k+1} \cup f \,|\, t_{2k+2} \,|$; $g_{2k+2} = (g_{2k+1}, f' \,|\, t_{2k+2} \,|)_{\alpha(t_{2k+2}, 2k+1)}$

Set $e^* = \bigcup_i e_i$; $g^* = \bigcup_i g_i$, $e = (e^* \circ \lambda)$ and $g = (g^* \circ \lambda)$.

The predicates $\bar{e}$ and $\bar{g}$ are defined from e and g by replacing all occurrences of λ by 0.

Mučnik defines the idea of strong minimality as : s is strongly minimal with respect to x_o if $r > s$ implies $x(t_r) > x_o \geqslant x(t_s)$.

Mučnik formulates this theorem from which a solution to Post's problem follows : There exist recursively enumerable, non-recursive sets E and G such that, for all x, $P_x(\bar{e}) \neq \bar{g}$ and $P_x(\bar{g}) \neq \bar{e}$, where $\bar{e}$ and $\bar{g}$ are the characteristic functions of E and G respectively, i.e. the recursively enumerable sets E and G are not reducible to each other. This theorem is proved for the two cases : if t_{2i} is defined then t_{2i+1} always exists, and t_{2i} is defined but t_{2i+1} does not exist. In the first case a number s strongly minimal with respect to x_o is chosen. One finds a characteristic value for P satisfying conditions 2a) - 2d). Since we have set $2k+1 \geqslant s$, and assumed $P_{x_o}(\bar{e}) = \bar{g}$, the result that $x(t_{2k+2}) = x_o$ contradicts the strong minimality of s. Therefore $P_{x_o}(\bar{e}) \neq \bar{g}$. In an analogous fashion it is shown that $P_{x_o}(\bar{g}) \neq \bar{e}$. From this it follows that the sets E and G are not recursive and not reducible to one another.

Mučnik also states the theorem that there is a recursive sequence of recursively enumerable sets $\{E\}_n$ such that the decision problem for each of them is not reducible to recursive conjunction $\bigwedge_{m \neq n} A_{E_m}$ (A_{E_m} is the decision problem for E_m). He omits the proof as it is much more complicated. The details are given in here in Chapter II, Corollary 2.

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