REDUCTION OF ENVELOPING ALGEBRAS
OF LOW-RANK GROUPS

BY

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A mes parents
Albert et Madeleine

Abstract

We find the generating function for group tensors contained in the enveloping algebra of each simple compact group of rank three or less. The generating function depends on dummy variables which carry, as exponents, the degrees and representation labels of the tensors; it suggests an integrity basis, a finite number of elementary tensors, in terms of which all can be expressed as stretched tensor products. We show how the generating functions for tensors in the enveloping algebra of SO(5) and SU(3) reduce when the tensors are acting on the basis of representations for which one of the Cartan labels vanish. The missing label problem in the reduction $SO(5) \supset SO(3)$ restricted to SO(5) representations of the type $(0,\nu)$ is considered; the eigenvalues and eigenvectors of a missing label operator are found up to (including) representation (0,12).

Sommaire

La fonction génératrice des tenseurs contenus dans l'algèbre enveloppante est évaluée pour chaque groupe simple et compact de rang ≤ 3. Cette fonction génératrice dépend d'un certain nombre de variables dont les exposants correspondent aux degrés et aux étiquettes de représentation des tenseurs; cette fonction génératrice peut être interprétée en terme d'un ensemble fini de tenseurs élémentaires à partir desquels tous les tenseurs peuvent être obtenus. Nous évaluons la fonction génératrice réduite des groupes SO(5) et SU(3) lorsque les tenseurs opèrent sur les bases de représentations pour lesquelles une étiquette de Cartan est égale à zéro. Le problème d'étiquetage lors de la réduction $SO(5) \supset SO(3)$ limitée aux représentations de type $(0,\nu)$ est solutionné en terme d'un opérateur d'étiquetage pour lequel nous évaluons ses valeurs propres et vecteurs propres jusqu'à et incluant la représentation (0,12).

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Contribution to original knowledge

Chapters I and II as well as appendix A and C contains no new material. All the material presented in chapters III to VI and in appendix B is original unless specified. More specifically, the main results are

(1) The generating functions giving a basis for all tensors in the enveloping algebra of each simple compact group of rank ≤ 3 derived in chapter III.

(2) The group-subgroup characteristic function introduced in chapter III and defined in appendix B.

(3) The polynomial expressions given in chapter V corresponding to the highest components of the elementary tensors for the groups SU(3) and SO(5).

(4) The approach(chapter V) to the problem of finding how the generating functions for tensors in the enveloping algebra reduce when the tensors are acting on the basis of representations for which one or more Cartan labels vanish .The reduced forms of the generating functions for SU(3) and SO(5).

(5) The technique by which one obtains a generating function for subgroup scalars in the enveloping algebra of a group from the generating function for group tensors.

(6) Part of the missing labels and all eigenvectors obtained in chapter VI concerning the missing label problem in the reduction $SO(5)\supset SO(3)$.

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CHAPTER I

INTRODUCTION

The enveloping algebra of a Lie group has proven to be a useful concept in theoretical physics; its structure has been the subject of many investigations.

The problem of labelling states in terms of a complete set of commuting operators, in the case where one uses a non-canonical chain of groups (the missing label problem) has motivated the search for subgroup scalars in the enveloping algebra of a group. Solutions[1,2,3], which were given for various group-subgroup combinations, have a common characteristic: all subgroup scalars are finitely generated, i.e., there exists a finite set of elementary subgroup scalars from which all may be constructed.

Other elements of the enveloping algebra of Lie groups have been the subject of many studies. In particle physics, a knowledge of a basis for all vector operators (tensors that transform by the adjoint representation) turns out to be important in the derivation of mass formulas; good examples of such calculations (apart from the well known Gell-Mann-Okubo SU(3) mass formula) are Okubo's SU(4) and SU(8) mass formulas[4]. A complete description[5] (degree and explicit algebraic forms) of a basis for vector operators in the enveloping algebras of A_ℓ, B_ℓ, C_ℓ, D_ℓ and G_2, in any given irreducible representation, was given by Okubo.

Shift operators which raise and lower irreducible representations of a subgroup H of a group G are polynomials in the generators of G, i.e., elements of the enveloping algebra of the group; they have proven useful for the classification and analysis of irreducible representations of various groups. Shift operator techniques have been developed by several authors[6-17]. Nagel and Moshinsky's operators which raise and lower the representations of U(n-1) contained in a representation of U(n) and Hughes and Yadegar's O(3) shift operators are good examples.

The Casimir operators, whose eigenvalues label the representations were probably the most studied elements of the enveloping algebras of Lie groups. It has long been known[18,19] that for a group G of rank ℓ there are just ℓ functionally independent polynomials in the generators, or Casimir invariants. The degrees of these invariants for all simple Lie algebras have been given by Racah. For example he showed[20] that for A_ℓ, B_ℓ, C_ℓ, D_ℓ and G_2 the degrees of the Casimirs were as follows

$$\begin{aligned} A_\ell &: I^2, I^3, \ldots, I^{\ell+1} \\ B_\ell &: I^2, I^4, \ldots, I^{2\ell} \\ C_\ell &: I^2, I^4, \ldots, I^{2\ell} \\ D_\ell &: I^2, I^4, \ldots I^{2\ell-2}, I^\ell \\ G_2 &: I^2, I^6. \end{aligned} \tag{1.1}$$

As mentioned by Kostant[21], it follows from a theorem of Chevalley that for a group G of rank ℓ, there are as many Casimirs as linearly independent vector operators (not counting those obtained through multiplication by invariants) in the enveloping algebra of G, and the relationship between the degree v_i and c_i of the i^{th} vector and Casimir operator is given by the following formula

$$c_i = v_i + 1 \qquad i = 1,2, \ldots \ell \quad . \tag{1.2}$$

The problem of constructing the Casimirs and then computing their eigenvalue spectrum has been studied on many occasions[22-28]. Recently, simplified derivations of their eigenvalues have been given by Okubo[5] and Edwards[29].

Now Casimirs, subgroup scalars and vector operators are only three out of an infinite number of types (we are referring here to their transformation properties under the group) of tensors that can be found in the enveloping algebra of a group. More general results on the global structure of enveloping algebras has been obtained by Kostant[21] as early as 1963. He showed that the number p_λ of linearly independent λ-tensors (tensors that transform by the (λ) representation of the group where λ are the Cartan labels), in the enveloping algebra is equal (not counting those obtained through multiplication by invariants in the enveloping algebra; we shall refer to this as "modulo multiplication by Casimirs") to the number of states of zero weight in the representation (λ) (here (λ) is any irreducible representation other than the scalar one); he also proved that the highest degree of a λ-tensor (modulo multiplication by Casimirs) is the sum of the coefficients of the simple roots in the highest weight of (λ).

So far we have been talking about the irreducible tensors into which the enveloping algebra decomposes under the action of the group or subgroup. A problem which is of interest in theoretical physics is that of the action of these operators on the basis of some given irreducible representation. It has long been known that the enveloping algebra of SU(3) contains two (modulo multiplication by Casimirs) linearly independent vector operators respectively of degrees one and two. Do they remain linearly independent (here I refer the reader to the first two paragraphs of section 2 of chapter V; there we discuss of two definitions of linear independence) when acting on the basis of representations in which one of the Cartan labels vanish ? Concerning this problem, Okubo[5] proved for simple Lie algebras that the number $n_\nu(\nu)$ of linearly independent vector operators in a representation (ν) is given by the following formula

$$n_\nu(\nu) = n - n_0(\nu) \qquad (1.3)$$

where n is the number of fundamental irreducible representations of the algebra and $n_0(\nu)$ is the number of Cartan labels specifying the representation (ν) which are equal to zero. In the case of SU(3), $n = 2$ so that when acting on a general representation ($n_0(\nu) = 0$) there are two linearly independent vector operators and only one in the case of $(\nu,0)$ or $(0,\nu)$ representations. More general results were obtained by Kostant[21] for all tensors in the enveloping algebra of Lie groups; he showed that the multiplicity of a λ-tensor when acting on the basis of a representation (ν) is equal to the multiplicity of (ν) in the Clebsch-Gordan series of $(\lambda) \times (\nu)$. The fact, that the number of linearly independent tensor operators of a given type depends on the representation on which it is acting, may be understood in terms of certain identities among the generators; these identities have been studied by various authors[5,30-34].

Although much work has been done on the structure of enveloping algebras, so far no complete description (except for SU(2)) has been given of what is available in terms of irreducible group tensors in the enveloping algebra of any Lie group, in other words the problem of finding a basis for all the irreducible tensors into which the enveloping algebra decomposes under the action of the group has never been solved. In the case of SU(2), Kostant[21] showed that the enveloping algebra decomposes into λ-tensors (modulo multiplication by Casimirs) of dimension $d = 2j + 1$ where each has multiplicity one and degree m_j given as follows ($\lambda = 2j$)

$$m_j = j \qquad \text{with } j = 1, 2, \ldots \infty \ . \tag{1.4}$$

Now many groups of rank ≤ 3 have proven to be useful in physics and a complete description of the structure of their enveloping algebra would be of interest for present and future applications. For instance in atomic physics orbital states have been classified according to several chain of groups; long ago Racah[35] labelled the orbital states of the f-shell by considering the chain

$$SU(7) \supset SO(7) \supset G_2 \supset SO(3).$$

In nuclear physics we have the well known shell model of Elliot[36] where states are classified according to the $SU(3) \supset SO(3)$ chain. In the Wigner[37] super-multiplet model, the many nucleon spin-isospin states are classified according to the irreducible representations of SU(4) in a SU(2)xSU(2) basis. More recent models in nuclear physics such as the interacting boson model of Arima and Iachello[38] make use of a chain such as

$$U(6) \supset U(5) \supset SO(5) \supset SO(3).$$

The symplectic group Sp(6) was shown by Rowe and Rosensteel[39] to be important in relating the collective and independent particle models.

The object of this thesis may be divided into three parts

(1) Establish a basis for all tensors in the enveloping algebras of simple compact Lie groups of rank ≤ 3.

(2) Discuss the existence of these bases when acting on general and degenerate representations (one or more Cartan labels being zero). We shall consider in detail the case of SU(3) and SO(5).

(3) Consider the missing label problem in the reduction $SO(5)\supset SO(3)$ restricted to SO(5) representations of the type $(0,\nu)$; the eigenvalues and eigenvectors of a missing label operator are found up to (including) representation (0,12).

The thesis is divided as follows : in chap II, after a few definitions, we discuss in some detail the generating function concept as applied to continuous groups; in chapter III the problem of the reduction of enveloping algebras is considered and answers are given for the above mentioned groups; results of chapter III have been subjected to many tests, these are discussed in chapter IV; in chapter V we discuss of a method of constructing tensors in the enveloping algebra of a group and consider the problem of the existence of these tensors when acting on the basis of general and degenerate representations; the missing label problem is discussed in chapter VI and solved in the case of $SO(5)\supset SO(3)$ in terms of a missing label operator. The thesis ends with a brief conclusion.

CHAPTER II

FORMALISM

2.1 Definitions

We begin by a brief description of the Lie groups of interest (all are connected, compact and simple) and their corresponding Lie algebras. The variables in real n dimensional space R^n will be designated $\bar{X} \equiv (x_1, \ldots, x_n)$ and in the complex space C^n as $\bar{Z} \equiv (z_1, \ldots, z_n)$.

The special unitary group SU(n) is the (n^2-1)-parameter group of complex unitary unimodular (determinant equal to + 1) n dimensional matrices which leave invariant the Hermitian form $\sum_{i=1}^{n} z_i z_i^*$. Its Lie algebra has n^2-1 generators (the order r of the group) and is the real compact form of A_{n-1}. The rank ℓ (the maximum number of mutually commuting generators or the number of representation labels) of the group is n-1.

The real special orthogonal group SO(n) is the $\{\frac{n(n-1)}{2}\}$-parameter group of real orthogonal n dimensional matrices of determinant + 1 which leaves invariant the real quadratic form $\sum_{i=1}^{n} x_i^2$. Its algebra has $\frac{n(n-1)}{2}$ generators and is the real compact form of $B_{\frac{(n-1)}{2}}$ for n odd and of $D_{n/2}$ for n even. The rank of the group is $\frac{(n-1)}{2}$ for n odd and $\frac{n}{2}$ for n even.

The unitary symplectic group Sp(2n) is the {n(2n+1)}-parameter group of 2n dimensional real unitary matrices which leave invariant the nondegenerate skew-symmetric bilinear form $\sum_{i=1}^{n} (x_i y_i' - x_i' y_i)$ of two vectors, $\bar{X} \equiv (x_1, \ldots, x_n, x_1', \ldots x_n')$ and $\bar{Y} \equiv (y_1, \ldots, y_n, y_1', \ldots y_n')$. Its algebra has n(2n+1) generators and is the real compact form of C_n. Its rank is n.

The exceptional group G_2 is a 14 parameter group of 7 dimensional matrices. Its algebra has 14 generators and is of rank 2.

Due to the following isomorphisms (we designate the Lie algebra associated with a given Lie group by the same letter as for the group, but in lower case)

$$su(2) \sim so(3) \sim sp(2), \; so(5) \sim sp(4), \; su(4) \sim so(6),$$

our study of the structure of enveloping algebras of compact simple groups of rank ≤ 3 may be reduced to that of the following groups : SU(2), SU(3), SO(5), G_2, SU(4), Sp(6) and SO(7). We complete this section with the definitions of enveloping algebra, tensor and tensor operator.

Let L be the Lie algebra of some Lie group G with basis (generators) $X_1, \ldots X_r$ and commutation rules $[X_i, X_j] = C_{ij}^k X_k$, where C_{ij}^k are the structure constants. The linear associative (associative multiplication) algebra composed of all possible products of generators of L, taken in all possible orders, plus an identity element and where the commutator $X_i X_j - X_j X_i$ is identified with the corresponding Lie product $[X_i, X_j]$, is known as the universal enveloping algebra of G.

A set $\Gamma(\lambda)$ of linearly independent basis vectors that transform among themselves under an irreducible representation $(\lambda) \equiv (\lambda_1, \ldots, \lambda_\ell)$ of a group, will be referred to as an irreducible tensor $\Gamma(\lambda)$; the λ_i are the Cartan labels. Individual basis vectors (the components of $\Gamma(\lambda)$) will be denoted $\Gamma(\lambda;k)$ where k designates all distinguishing labels (weights and others). This transformation property of a tensor for finite and infinitesimal transformations of the group may be written as follows

$$R\,|\lambda;\,k\rangle = \sum_{k'} |\lambda;\,k'\rangle\langle\lambda;k'\,|R|\,\lambda;\,k\rangle\,,$$

$$X_\sigma\,|\,\lambda;\,k\rangle = \sum_{k'} |\,\lambda;k'\rangle\langle\lambda;k'\,|\,X_\sigma\,|\,\lambda;\,k\rangle\,,$$

with the identification $|\lambda;\,k\rangle \equiv \Gamma(\lambda;\,k)$; R represents a finite transformation and X_σ $(\sigma=1, r)$ a generator of the algebra.

A set $T(\lambda)$ of linearly independent operators $T(\lambda;\,k)$, that transform among themselves under the operations of a group according to an irreducible representation (λ), is said to form an irreducible tensor operator $T(\lambda)$; that is, under finite and infinitesimal transformations of the group, $T(\lambda;k)$ transforms as follows

$$R\,T(\lambda;\,k)\,R^{-1} = \sum_{k'} T\,(\lambda;\,k')\,\langle\lambda;\,k'\,|\,R\,|\,\lambda;\,k\rangle\,,$$

$$\left[X_\sigma,\,T(\lambda\,;k)\right] = \sum_{k'} T\,(\lambda;\,k')\,\langle\lambda;\,k'\,|\,X_\sigma\,|\lambda;\,k\rangle\,.$$

Tensors in the enveloping algebra of a group are tensor operators whose components are polynomials in the generators of the group.

2.2 Generating function concept for continuous groups

Introduced in 1897 by Molien[40] in the study of invariants of a finite group of complex matrices, the generating function (GF) concept has proven, in the past few years, to be a useful tool in the representation theory of finite[41-48] and continuous groups. Our discussion will be restricted to the case of continuous groups.

There are several types (the word type refers to the information carried by these functions) of GF's but all those calculated recently have the same general format : they are functions of several variables v_n, which when expanded into a power series contain but positive terms; they are fractions, or sums of fractions, whose denominator factors have the form (1-X), the X's and the numerator terms Y being algebraic expressions of the form $v_1^{n_1}\ v_2^{n_2} \ldots v_n^{n_n}$ with $n_1, \ldots, n_n$ being integers. An example of such a function would be

$$G\,(v_1, v_2, v_3) = (1 + v_1 v_2 v_3^2)\,\{(1-v_1^2)\,(1-v_1 v_2) \times (1-v_1 v_3^2)\,(1-v_2 v_3^2)\}^{-1}. \tag{2.1}$$

There is one known exception to this, i.e., a GF whose power series expansion contains negative terms; we shall discuss such a function in a later chapter. Not only are GF's a compact way of presenting certain results, but they allow through residue calculations, the manipulation of a large amount of information; they may be added, subtracted, and in certain case, coupled and substituted one into another. Since we make extensive use of this recently revived concept in the present thesis we will in what follows discuss several types of GF's by actually giving examples and analyzing the information carried by them. As a complement to this discussion, we illustrate in appendix A certain GF techniques.

(a) Generating function for polynomial irreducible tensors

It will be shown in chapter III that the study of the structure of the enveloping algebra of a simple compact Lie group G may be reduced to the problem of finding a basis for all irreducible tensors, with respect to G, whose components are polynomials in the components of a certain tensor Γ; in other words, our problem will be that of finding a basis for the irreducible tensors that can be obtained from the following tensor products

$$(\Gamma)^u \equiv \{\Gamma \times \Gamma \ldots \times \Gamma\} \; u \text{ identical copies, } u = 1, \infty. \tag{2.2}$$

All these tensors, of which there are an infinite number, may be presented in a compact way in terms of a GF (referred to as the GF for tensors based on Γ). For example, if Γ is a $j = 2$ ($\lambda=4$) SU(2) tensor, the GF[49] giving a basis for all irreducible SU(2) tensors based on Γ is

$$G(U,\Lambda) = (1+U^3\Lambda^6)\{(1-U\Lambda^4)(1-U^2\Lambda^4)(1-U^2)(1-U^3)\}^{-1} \tag{2.3}$$

where U carries the degree in Γ and Λ the representation label (unless specified all representation labels in this thesis are Cartan labels) as exponents. A term $C_{u\lambda}\, U^u\, \Lambda^\lambda$ in the expansion of (2.3) informs us that the number of linearly independent irreducible SU(2) tensors, which transform as (λ) and whose components are polynomials of degree u in the components of Γ, is $C_{u\lambda}$. The GF (2.3) does not only enumerate all tensors but also suggests an integrity basis, i.e., a finite set of elementary tensors in terms of which all may be obtained as stretched tensor products. The stretched tensor product of two tensors Γ_1 and Γ_2 is a tensor Γ_3 whose highest weight is the sum of the highest weights of Γ_1 and Γ_2, which implies that representation labels in such a product are additive. Denoting by (u,λ) the tensors enumerated in (2.3), where u is the degree and λ the representation label, the integrity basis for tensors whose components are polynomials in the $j = 2$ SU(2) tensor is

$$(1,4),\ (2,4),\ (2,0),\ (3,0) \text{ and } (3,6).$$

The structure of the GF (2.3) tells us that all linearly independent tensors may be obtained by the following stretched tensor products of powers of elementary ones

$$(1,4)^a.\ (2,4)^b.\ (2,0)^c.\ (3,0)^d.\ (3,6)^e$$

where a,b,c and d may take values (integer) from 0 to ∞ while e may only be 0 or 1. For example the choice a=3, b=2, c=0, d and e = 1 gives the tensor (13,26), i.e., a tensor of degree 13 that transforms by the $\lambda = 26$ ($j=13$) representation. That (3,6) appears in the numerator of G (U,Λ) implies that the stretched square of it is a linear combination of stretched products of the other elementary tensors and hence redundant, i.e.,

$$(3,6)^2 = A\ (2,4)^3 + B\ (1,4)^2.\ (2,4).(2,0) + C\ (1,4)^3.\ (3,0).$$

The interpretation of these functions in terms of an integrity basis is not always trivial; a glance at the GF (3.53) should convince the reader.

The products (2.2) are a particular case of the following tensor products

$$\{\Gamma \times \Gamma \ldots \times \Gamma\}\ u \text{ \underline{independent} copies},\ u = 1, \infty. \qquad (2.4)$$

The irreducible tensors arising from (2.4) are characterized not only by their degree in Γ and their representation labels but also by their exchange symmetries; this problem, known as the calculation of plethysms, is of particular interest in the study of many identical particle systems in quantum physics. Here again the results of such calculations may be expressed in terms of a GF[46], but will not be discussed here. The polynomial tensors obtained from the products (2.2) constitute the symmetric plethysm.

(b) Generating function for weights corresponding to a generating function for tensors based on a tensor Γ

Since weights are additive in the tensor product of representations, the products (2.2) may be realized in ℓ-dimensional (ℓ being the rank of the group) weight space by a weight GF W giving all weights arising from such products. W is a fraction whose numerator is unity, with a one to one correspondence between its denominator factors (1-X) and weights of Γ. W will be referred to as the GF for weights corresponding to a GF for tensors based on Γ. For example the GF for weights corresponding to the GF (2.3) for SU(2) tensors based on a j = 2 tensor Γ is

$$W(U,\eta) = \{(1-U\eta^4)(1-U\eta^2)(1-U)(1-U\eta^{-2})(1-U\eta^{-4})\}^{-1} \qquad (2.5)$$

where U carries the degree in Γ and η the weight as exponents. A term $[c_{uw} U^u \eta^w]$ in the expansion of (2.5), indicates the presence of a weight w with multiplicity c_{uw}, in the tensor product

$\{\Gamma(j=2) \times \ldots \times \Gamma(j=2)\}$ u identical copies.

(c) Group-subgroup generating functions - generating functions for branching rules

It has been shown[50-53] in the case of compact groups, for various group-subgroup combinations, that the reduction of the irreducible representations (IR's) of a group into irreducible representations (multiplets) of a subgroup, i.e. the branching rules, may be defined in terms of stretched products of powers of a finite set of elementary factors denoted by

$(\lambda_1, \ldots, \lambda_{\ell_G}; n_1, \ldots n_{\ell_H})$ where λ_i and n_i are respectively the Cartan labels of the IR and the multiplet to which it belongs. Weitzenböck[54] proved that this set is finite for all semisimple Lie groups.

This suggests that the branching rules may be expressed in terms of a GF whose terms X and Y, as defined earlier, would correspond to elementary factors or (in the case of Y) to stretched products of powers of them (Cartan labels additive). For example, the branching rules for SU(3) ⊃ SO(3) is given by the following stretched products of elementary factors (SO(3) labels are doubled : $n = 2j$).

$$(10;2)^a.\ (01;2)^b.\ (20;0)^c.\ (02;0)^d.\ (11;2)^f \quad (2.6)$$

where a,b, c and d can take integer values from 0 to ∞ and $f = 0,1$ only. The GF[49] corresponding to (2.6) is

$$F(\Lambda_1,\Lambda_2;N) = (1+\Lambda_1\Lambda_2N^2)\{(1-\Lambda_1N^2)(1-\Lambda_1^2)(1-\Lambda_2N^2)(1-\Lambda_2^2)\}^{-1} \quad (2.7)$$

Λ_1,Λ_2 and N carry respectively the representation labels of SU(3) and SO(3) as exponents. Isolating the term in $\Lambda_1^2\Lambda_2^2$ in (2.7)

$$\Lambda_1^2\ \ \Lambda_2^2\ (N^8+N^6+2N^4+1),$$

indicates the following reduction

$$(22) \supset (8) + (6) + 2(4) + (0).$$

This type of GF will be referred to as the group-subgroup GF or the GF for branching rules. This function not only gives the branching rules, but when interpreted in terms of an integrity basis (the elementary permissible diagrams of Devi and Moskinsky), i.e. a finite set of elementary factors ((2.6) in the case of (2.7)), gives a solution to the corresponding internal (state) labelling problem[55-61]. A GF for tensors based on a tensor Γ may be reduced to its corresponding GF for weights by making use of one or more group-subgroup GF's; this technique which consists in substituting one GF into another is discussed in section 4 of appendix A.

(d) Generating function for Clebsch-Gordan series

Just as in the case of the branching rules where we make use of a complete set of elementary factors, we may express all couplings by the stretched products of powers of a complete set of elementary couplings with certain combinations of them being considered as redundant. A set of elementary couplings has been found for various groups[62-67] such as SU(2), SU(3), SU(4), SU(5) and SO(5).

These couplings are actually a consequence of the interpretation of the general invariant of these groups as products of powers of certain elementary scalars in three representations. Denoting the elementary couplings of SU(2) by $(\lambda_1,\lambda_2,\lambda)$ where λ_1 and λ_2 are the Cartan labels of the IR's being coupled and λ the Cartan label of the product representation (the Cartan labels being twice the angular momentum value), the integrity basis consists of

$$(0,1,1),\ (1,0,1),\ (1,1,0)$$

with no redundant combination ; this actually means that the Clebsch-Gordan series of any coupling of two SU(2) representations may be obtained from the following stretched products (Cartan labels additive).

$$(0,1,1)^a.\ (1,0,1)^b.\ (1,1,0)^c \qquad (2.8)$$

with a, b, and c being any integers from 0 to ∞. The above products may be expressed in terms of the following GF

$$C(\Lambda_1,\Lambda_2,\Lambda) = \{(1-\Lambda_2\Lambda)\ (1-\Lambda_1\Lambda)\ (1-\Lambda_1\Lambda_2)\}^{-1} \qquad (2.9)$$

where Λ_1, Λ_2 carry the Cartan labels of the IR's being coupled as exponents and Λ that of the product IR. Splitting off the term in $\Lambda_1^3 \ \Lambda_2^3$ in (2.9) we get

$$\Lambda_1^3 \ \Lambda_2^3 \ (\Lambda^6 + \Lambda^4 + \Lambda^2 + \Lambda^0)$$

which in terms of the angular momentum coupling is interpreted as follows

$$3/_2 \times 3/_2 = 3 + 2 + 1 + 0.$$

This type of GF will be referred to as the GF for the Clebsch-Gordan series. Not only does it give all couplings but can be used to couple two GF's for tensors; a coupling technique by means of residue calculations is discussed in section 4 of appendix A.

It will be noticed that all GF's described above have the following common characteristic : all numerators contain only positive terms and every denominator factor is such that, if expanded in a power series, it contains but positive terms. Only from GF's having such a property can one deduce information about the existence of an integrity basis. Bringing GF's to such a format may often be a difficult task. We now turn to the problem of the reduction of enveloping algebras of simple compact Lie groups of rank $\leqslant 3$.

CHAPTER III

REDUCTION OF ENVELOPING ALGEBRAS OF LOW-RANK GROUPS[68]

3.1 Our approach to the reduction problem

Let us first recall our objective : it is to find, to enumerate a basis for all tensors in the enveloping algebras of simple compact groups of rank $\leqslant 3$. In order to give a complete description of a basis for tensors in the enveloping algebra U of a group G, one must answer the following questions concerning any of its elements

(1) By which representation does it transform ?

(2) What is its degree (its components being polynomials in the generators) ?

(3) What is its multiplicity ?

(4) How do we construct it ?

The set of all symmetric homogeneous polynomials in the generators forms a basis of U (Poincaré-Birkhoff-Witt Theorem[69]); furthermore the order in a product of generators does not affect its transformation properties under G. Hence the basis for (symmetric) tensors in U corresponds precisely to the basis for tensors whose components are polynomials in the components of a tensor Γ_{α} which transforms by the adjoint representation of G (we will refer to such tensors as polynomial tensors).

Answering the above four questions for polynomial tensors is therefore answering them for tensors in U since once one knows the algebraic form of a polynomial tensor, the corresponding tensor in U is obtained through symmetrization (with respect to order).

The problem of the reduction of enveloping algebras of compact groups is therefore reduced to that of constructing a GF for polynomial irreducible tensors based on a tensor Γ_A.

3.2 Methods of constructing generating functions for polynomial tensors

A possible approach to this problem is the one proposed by Gaskell, Peccia and Sharp[49] (see appendix A) which consists in starting with a weight GF. Unfortunately, the tedium of the method increases rapidly with the number of generators. Two alternative approaches are considered in this thesis, each of more or less general applicability. One is basically the elementary multiplet method discussed in appendix A; however, instead of using the chain $SU(r) \supset G$(r being the order of the group G, the number of generators of its algebra) we insert in it, whenever G is not maximal, one or more intermediate groups and consider chains such as $SU(r) \supset G' \supset G$; one finds the GF for $SU(r) \supset G'$ and for $G' \supset G$ and substitutes the latter in the former. One can also make use of a new type of relation between two semisimple groups and consider chains of the type $SU(r) \supset G'' > G$ where $G'' > G$ means that G is subjoined to G''. The above techniques, first proposed by Patera and Sharp[46] in the evaluation of GF's for general plethysms, simplifies the calculations considerably; they will be discussed in greater detail in section 3 of this chapter.

The other approach, which we believe to be novel, is to work through a subgroup H of G. The tensor Γ_A is a reducible tensor of H; denoting by $\Gamma_H^1, \Gamma_H^2, \ldots, \Gamma_H^n$ the n irreducible subgroup tensors into which Γ_A reduces, it may be relatively easy to construct the GF for H tensors based on the n tensors Γ_H^u (u = 1, n). Under certain circumstances it may be possible to convert that subgroup GF into the corresponding GF for G tensors based on Γ_A. A necessary tool in doing this conversion is the group-subgroup characteristic function; in the remainder of this section we shall give examples of such a function and show how it can be useful for our purpose.

Let G be a simple compact group and H its semisimple or reductive subgroup. The subgroup content of an irreducible representation (λ) of G may be written

$$\chi_\lambda^H(N) = \sum_\nu c_{\lambda\nu} N^\nu, \qquad N^\nu = \prod_{i=1}^{\ell_H} N_i^{\nu_i}. \tag{3.1}$$

$(\nu) = (\nu_1, \ldots, \nu_{\ell_H})$ are the representation labels of H, and N are dummy variables carrying those labels as exponents; $c_{\lambda\nu}$ is the multiplicity of the subgroup representation (ν) in the group representation $(\lambda) = (\lambda_1, \ldots, \lambda_{\ell_G})$.

In appendix B, the group-subgroup characteristic $\xi_\lambda^H(N)$ is defined and it is proved that in terms of it, the subgroup content may be written

$$\chi_\lambda^H(N) = \xi_\lambda^H(N) \,/\, \xi_0^H(N). \tag{3.2}$$

As (3.2) suggests, the group-subgroup characteristic function is a generalization of Weyl's characteristic function (see A.8) to which it reduces when the subgroup is the Cartan subgroup (U(1) X ... X U(1) (ℓ times), whose representation labels are the components of the weight. $\xi_0^H(N)$ may be evaluated straightforwardly from its definition (B.1) in each case of interest. In principle ξ_λ^H (N) may also be evaluated from the definition; however, it may be easier to evaluate χ_λ^H (N) in the form (3.2) from the appropriate GF for G ⊃ H branching rules, if it is known, and read off ξ_λ^H (N). We give explicit examples.

The group-subgroup GF for SU(3) ⊃ SU(2) x U(1) is known to be[50]

$$F(\Lambda_1,\Lambda_2;N_1,N_2) = \{(1-\Lambda_1N_1N_2)(1-\Lambda_1N_2^{-2})(1-\Lambda_2N_1N_2^{-1})(1-\Lambda_2N_2^{2})\}^{-1}, \quad (3.3)$$

where Λ_1, Λ_2 carry the SU(3) representation labels as exponents, N_1 carries twice the isospin, N_2 three times the hypercharge. The SU(2) x U(1) content of the SU(3) representation (λ_1,λ_2) is the coefficient of $\Lambda_1^{\lambda_1}\Lambda_2^{\lambda_2}$ in the expansion of (3.3); this can be evaluated by taking appropriate residues of (3.3) :

$$\chi^H_{\lambda_1\lambda_2}(N_1,N_2) = \Sigma\, \mathrm{Res}_{\Lambda_1\Lambda_2}\Lambda_1^{\lambda_1-1}\Lambda_2^{\lambda_2-1} F(\Lambda_1^{-1},\Lambda_2^{-1};N_1,N_2) \quad (3.4)$$

This method of obtaining $\chi^H_{\lambda_1\lambda_2}(N_1,N_2)$ by the above residue calculations, implies a power series expansion of the denominator factors (1-Z) and therefore one must impose certain restrictions on the norms of the variables $\Lambda_1,\Lambda_2,N_1,N_2$ (see discussion leading to (A.4)); the above consideration remains true for all residue calculations in this thesis and will therefore not be repeated. In (3.4) we make the following choice of norms : $|N_1| = |N_2| = 1$ and the Λ_1,Λ_2 residues are those at poles inside circles whose radii are a little greater than unity. (3.4) may be written

$$\Sigma\, \mathrm{Res}_{\Lambda_1\Lambda_2} \frac{\Lambda_1^{\lambda_1+1}\Lambda_2^{\lambda_2+1}}{(\Lambda_1-N_1N_2)(\Lambda_1-N_2^{-2})(\Lambda_2-N_1N_2^{-1})(\Lambda_2-N_2^{2})}. \quad (3.5)$$

The residue at poles $\Lambda_1=N_1N_2$ and $\Lambda_2=N_1N_2^{-1}$ is

$$\frac{N_1^{\lambda_1+\lambda_2+2}\, N_2^{\lambda_1-\lambda_2}}{(N_1N_2-N_2^{-2})\,(N_1N_2^{-1}-N_2^{2})}. \tag{3.6}$$

The residue at poles $\Lambda_1=N_1N_2$ and $\Lambda_2=N_2^2$ is

$$\frac{N_1^{\lambda_1+1}\, N_2^{2\lambda_2+\lambda_1+3}}{(N_1N_2-N_2^{-2})\,(N_2^{2}-N_1N_2^{-1})}. \tag{3.7}$$

The residue at poles $\Lambda_1=N_2^{-2}$ and $\Lambda_2=N_1N_2^{-1}$ is

$$\frac{N_1^{\lambda_2+1}\, N_2^{-2\lambda_1-\lambda_2-3}}{(N_2^{-2}-N_1N_2)\,(N_1N_2^{-1}-N_2^{2})}. \tag{3.8}$$

The residue at poles $\Lambda_1=N_2^{-2}$ and $\Lambda_2=N_2^{2}$ is

$$\frac{N_2^{-2\lambda_1+2\lambda_2}}{(N_2^{-2}-N_1N_2)\,(N_2^{2}-N_1N_2^{-1})}. \tag{3.9}$$

Adding (3.6), (3.7), (3.8) and (3.9) we get

$$\chi^{H}_{\lambda_1\lambda_2}(N_1,N_2)=\frac{N_1^{\lambda_2+1}N_2^{-2\lambda_1-\lambda_2-3}+N_1^{\lambda_1+1}N_2^{\lambda_1+2\lambda_2+3}-N_1^{\lambda_1+\lambda_2+2}N_2^{\lambda_1-\lambda_2}-N_2^{-2\lambda_1+2\lambda_2}}{(N_1N_2-N_2^{-2})(N_2^{2}-N_1N_2^{-1})}. \tag{3.10}$$

From the group-subgroup characteristic's definition (3.2) we therefore have that

$$\xi^{H}_{\lambda_1\lambda_2}(N_1,N_2)=N_1^{\lambda_2+1}N_2^{-2\lambda_1-\lambda_2-3}+N_1^{\lambda_1+1}N_2^{\lambda_1+2\lambda_2+3}-N_1^{\lambda_1+\lambda_2+2}N_2^{\lambda_1-\lambda_2}-N_2^{2\lambda_2-2\lambda_1}, \tag{3.11a}$$

and

$$\xi^{H}_{00}(N_1,N_2)=(N_1N_2-N_2^{-2})\,(N_2^{2}-N_1N_2^{-1}). \tag{3.11b}$$

The group-subgroup GF for the chain $SU(3)\supset SO(3)$ is[49]

$$F(\Lambda_1,\Lambda_2;N)=(1+\Lambda_1\Lambda_2N)\,\{(1-\Lambda_1N)\,(1-\Lambda_1^2)\,(1-\Lambda_2N)\,(1-\Lambda_2^2)\}^{-1},$$

where Λ_1, Λ_2 carry as exponents the SU(3) representation labels and N the angular momentum. $\chi^{H}_{\lambda_1\lambda_2}(N)$ is obtained from the following sum of residues

$$\chi^{H}_{\lambda_1\lambda_2}(N) = \Sigma\,\mathrm{Res}_{\Lambda_1\Lambda_2}\,\Lambda_1^{\lambda_1-1}\,\Lambda_2^{\lambda_2-1}\,F(\Lambda_1^{-1},\Lambda_2^{-1};N)$$

$$= \Sigma\,\mathrm{Res}_{\Lambda_1\Lambda_2}\,\frac{\Lambda_1^{\lambda_1-1}\,\Lambda_2^{\lambda_2-1}\,(1+\Lambda_1^{-1}\Lambda_2^{-1}N)}{(1-\Lambda_1^{-1}N)\,(1-\Lambda_1^{-2})\,(1-\Lambda_2^{-1}N)\,(1-\Lambda_2^{-2})} \tag{3.12}$$

with $|N|\leq 1$ and residues at poles inside unit circles; from the above calculations we get the $SU(3)\supset SO(3)$ characteristic function

$$\xi^{H}_{\lambda_1\lambda_2}(N) = (1-N^{\lambda_1+1})\,(1-N^{\lambda_2+1}) + N^{-1}\,(1-N)^2\,\delta(\lambda_1,\lambda_2), \tag{3.13a}$$

and

$$\xi^{H}_{00} = (1-N)^2\,(1+N^{-1}). \tag{3.13b}$$

where $\delta(\lambda_1,\lambda_2)$ is unity if λ_1,λ_2 are both even, and zero otherwise.

The GF giving the branching rules for the case $SO(5)\supset SU(2)\times SU(2)$ is[50]

$$F(\Lambda_1,\Lambda_2;N_1,N_2) = \{(1-\Lambda_1N_1)\,(1-\Lambda_1N_2)\,(1-\Lambda_2)\,(1-\Lambda_2N_1N_2)\}^{-1};$$

where Λ_1,Λ_2 carry the $SO(5)$ representation labels as exponents. N_1,N_2 carry those of the subgroup (all subgroup labels are doubled). Taking the appropriate residues we get

$$\xi^{H}_{\lambda_1\lambda_2}(N_1,N_2) = N_1^{\lambda_1+\lambda_2+2}\,N_2^{\lambda_2+1} - N_1^{\lambda_2+1}N_2^{\lambda_1+\lambda_2+2} + N_2^{\lambda_1+1} - N_1^{\lambda_1+1}, \tag{3.14a}$$

$$\xi^{H}_{00}(N_1,N_2) = (N_1N_2-1)\,(N_1-N_2). \tag{3.14b}$$

From the $G_2\supset SU(3)$ GF[50]

$$F(\Lambda_1,\Lambda_2;N_1,N_2) = \{(1-\Lambda_1N_1)\,(1-\Lambda_1N_2)\,(1-\Lambda_2N_1)\,(1-\Lambda_2N_2)\}^{-1}$$
$$\times\,\{(1-\Lambda_1)^{-1} + \Lambda_2N_1N_2(1-\Lambda_2N_1N_2)^{-1}\} \tag{3.15}$$

we find the $G_2\supset SU(3)$ characteristic function

$$\xi^H_{\lambda_1\lambda_2}(N_1,N_2) = N_1^{\lambda_2+1}N_2^{\lambda_1+\lambda_2+2} - N_1^{\lambda_1+\lambda_2+2}N_2^{\lambda_2+1} + N_2^{\lambda_2+1} - N_1^{\lambda_2+1} + N_1^{\lambda_1+\lambda_2+2} - N_2^{\lambda_1+\lambda_2+2} \tag{3.16a}$$

$$\xi^H_{00}(N_1,N_2) = (N_2-N_1)(N_1-1)(N_2-1). \tag{3.16b}$$

The group-subgroup characteristic function $\xi^H_\lambda(N)$ is a linear combination of terms $\prod_i N_i^{p_i}$ whose exponents p_i depend linearly on the λ_j; the dummies N_i are defined so that the coefficients of the λ_j are all integers. As for Weyl's characteristic function (see figure 9), it is instructive to represent each term of $\xi^H_\lambda(N)$ for a given (λ) by a point in ℓ_H dimensional space whose cartesian coordinates are its exponents p_i. When we let λ_i take all possible values each term of the group-subgroup characteristic function defines (is mapped into) a sector in ℓ_H dimensional space; however contrary to Weyl's function, certain sectors may overlap . In figure 1 we illustrate the case $G_2 \supset SU(3)$; notice that the sectors corresponding to the third and sixth term of (3.16a) overlap as well as those of the fourth and fifth term. A term of $\xi^H_\lambda(N)$ will be called <u>distinctive</u> if it satisfies the following two criteria : its sector should not be overlapped by the sector of any other term, and its exponents $\bar{p}_i$ must determine the representation labels λ_j. A sector corresponding to such a term will be called a <u>distinctive sector</u>. The first two terms of (3.16a) are distinctive terms and so are the first terms of (3.11a), (3.14a). When $\ell_H \neq \ell_G$, $\xi^H_\lambda(N)$ contains no distinctive terms; this is the case for $SU(3) \supset SO(3)$; but for all group-subgroup pairs which we have examined with $\ell_H = \ell_G$, including $SU(3) \supset SU(2) \times U(1)$, $SO(5) \supset SU(2) \times SU(2)$, $SU(4) \supset SU(3) \times U(1)$, $G_2 \supset SU(3)$, $G_2 \supset SO(4)$, the group-subgroup characteristic contains at least one distinctive term. We conjecture that this is the case whenever $\ell_H = \ell_G$.

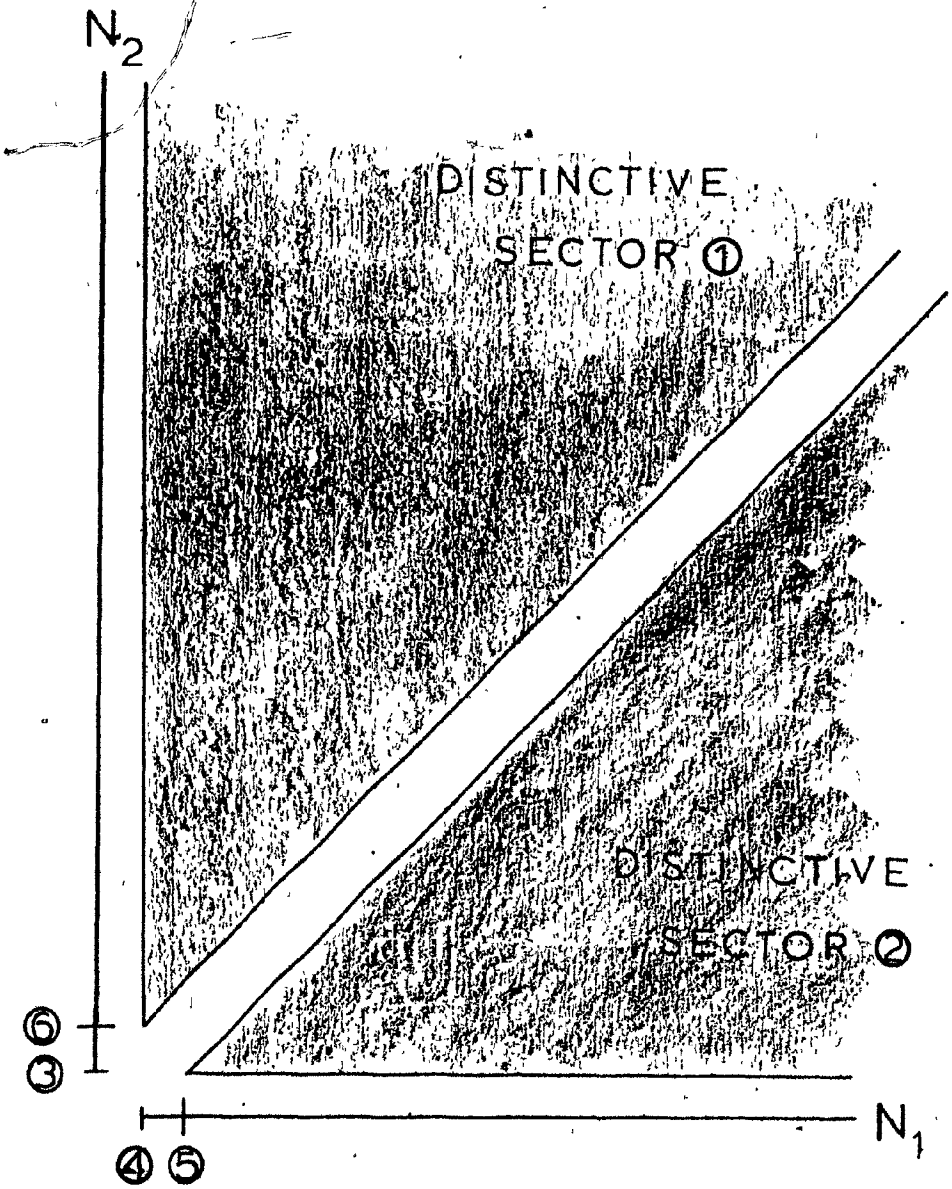


Figure 1 . The distinctive sectors of the group-subgroup characteristic function of $G_2 \supset SU(3)$

We now turn to the problem of converting a GF for subgroup tensors into the corresponding GF for group tensors. Consider a GF F(N) for tensors of the subgroup H of a group G. We assume that the tensors are those contained in complete tensors of G and write

$$F(N) = \sum_{\lambda} \chi_{\lambda}^{H}(N)\, c_{\lambda} \tag{3.17}$$

where $\chi_{\lambda}^{H}(N)$ is the subgroup content of the irreducible representation (λ) of G (see equation(3,1)); c_{λ} is the multiplicity of (λ) in F, and may depend on other dummy variables such as U in (2.3). Substituting (3.2) into (3.17) gives

$$F(N) = \sum_{\lambda} \frac{\xi_{\lambda}^{H}(N)}{\xi_{0}^{H}(N)}\, c_{\lambda}\,. \tag{3.18}$$

The comparison of equations (A.14) and (3.18) suggests that in the case of $\ell_H = \ell_G$, the group-subgroup characteristic function may play the same role in converting a GF F(N) for subgroup tensors into the corresponding GF for group tensors as that of Weyl's characteristic in transforming a GF for weights $W(\eta)$ into a GF for group tensors. Therefore, adapting to our problem the procedure described in appendix A, we first multiply (3.18) by $\xi_{0}^{H}(N)$ giving

$$\xi_{0}^{H}(N).\ F(N) = \sum_{\lambda} \xi_{\lambda}^{H}(N)\, c_{\lambda}. \tag{3.19}$$

The presence of a λ-tensor in F(N) is indicated by the presence of ξ_{λ}^{H} in the product $\xi_{0}^{H}(N)$. F(N), which in turn (the presence of ξ_{λ}^{H}) may be identified by one of its distinctive terms. Therefore, the conversion is done through the following steps

(1) First pick <u>one</u> of the distinctive terms of $\xi_{\lambda}^{H}(N)$; this term and <u>only</u> it will serve to identify the presence of group tensors in F(N).

(2) Multiply F(N) by $\xi_0^H(N)$.

(3) Each term found in $\xi_0^H(N)$. F(N), that belongs to the chosen distinctive sector is replaced (c_λ wich multiplies the term is kept) by dummy variables carrying the corresponding group representation labels as their exponents. All terms of the product (3.19) not belonging to the choosen sector are dropped.

The resulting expression is the desired GF for group tensors. Formally this is done by the following sum of residues

$$\sum \underset{N_1,\ldots,N_{\ell_H}}{\mathrm{Res}} \{\sum_{\lambda_j} (\prod_i N_i^{-\bar{p}_i-1} \prod_j \Lambda_j^{\lambda_j})\, \xi_0^H(N)\, F(N)\} . \tag{3.20}$$

The sums over the λ_j variables are from 0 to ∞; they are geometric and may be done explicitly. The residues are those at poles of $N_1, \ldots, N_{\ell_H}$ inside unit circles.

Let us now proceed to the evaluation of the GF's for tensors in the enveloping algebra of simple compact groups of rank ≤ 3 using the methods described in this section.

3.3 Generating functions for tensors in the enveloping algebras of low-rank groups

(a) SU(2), SU(3) and SO(5)

Before dealing with the two classical groups of rank two, we dispose of the rank one group SU(2). The generators of SU(2) form a vector ($j = 1$) operator; the GF based on a SU(2) tensor $j = 1$ as been worked out in appendix A where we started with the corresponding weight GF. Therefore, the GF for tensors in the SU(2) enveloping algebra is

$$G(U;\Lambda) = \{(1-U^2)\,(1-U\Lambda)\}^{-1} . \tag{A.7}$$

Its interpretation and integrity basis are found in appendix A; notice that it agrees with Kostant's result given in (1.4).

The SU(3) and SO(5) GF's will be derived by making use of a subgroup; we will first construct the GF's for subgroup tensors and then transform them into GF's for group tensors using the group-subgroup characteristic function. We begin with SU(3) and make use of the SU(2) x U(1) subgroup.

The generators of SU(3) decompose under SU(2) x U(1) into a vector and scalar with U(1) labels 0 and two spinors with U(1) labels ± 3. The GF for SU(2) x U(1) tensors based on the vector and scalar is from (A.7)

$$F_1\,(U;N_1') = \{(1-U)\,(1-U^2)\,(1-UN_1'^{\,2})\}^{-1} . \tag{3.21}$$

The factor $(1-U)^{-1}$ takes account of the scalar. The exponent of N_1', to avoid fractional exponents later, is twice the isospin.

The GF for tensors based on a SU(2) spinor ($I=\frac{1}{2}$) is

$$F_2\,(U;N_1''') = \frac{1}{(1-U\,N_1''')} , \tag{3.22}$$

where U carries the degree in the spinor and N_1''' twice the isospin. Therefore, making use of the GF (2.9) for the SU(2) Clebsch-Gordan series, the GF for tensors based on two spinors with U(1) labels $\pm$ 3 is

$$F_3\,(U;N_1'',N_2) = \Sigma\,\underset{N_1'''\;N_1''''}{\text{Res}}\;\frac{N_1'''^{\,-1}\,N_1''''^{\,-1}}{(1-UN_1'''^{\,-1}N_2^{3})\,(1-UN_1''''^{\,-1}N_2^{-3})}$$
$$\times\,\frac{1}{(1-N_1'''\,N_1'')\,(1-N_1''''\,N_1'')\,(1-N_1''''\,N_1''')} \tag{3.23}$$

where N_2 carries the U(1) label and

$$|N_1'''| = |N_2| = 1 \; ; \; |U| < |N_1''''| < |N_1'''''| < 1$$

which leads to

$$F_3(U;N_1'',N_2) = \{(1-U^2)\,(1-UN_1''N_2^3)\,(1-UN_1''N_2^{-3})\}^{-1} \tag{3.24}$$

The isospins of the GF (3.21) and (3.24) must then be coupled again using the SU(2) Clebsch-Gordan GF (2.9) to obtain the GF for SU(2) x U(1) tensors in the SU(3) enveloping algebra

$$F_4\,(U;N_1,N_2) = \sum \operatorname*{Res}_{N_1' \, N_1''} \{N_1'^{-1} N_1''^{-1} \; F_1(U,N_1'^{-1})\, F_3(U,N_1''^{-1},N_2)$$

$$\times \; C(N_1', N_1'', N_1)\} \tag{3.25}$$

with the following choice of norms

$$|U| < |N_1'| < |N_1''| < 1 \quad ; \quad |N_1| = |N_2| = 1$$

we therefore get

$$F_4(U;N_1,N_2) = \{(1-U)\,(1-U^2)\,(1-U^3N_2^6)\,(1-U^3N_2^{-6})\,(1-UN_1N_2^3)\,(1-UN_1N_2^{-3})\}^{-1}$$

$$\times \; \{(1-UN_1^2)^{-1}\,(1+U^2N_1N_2^3)\,(1+U^2N_1N_2^{-3}) + (1-U)^{-1}U^2\}. \tag{3.26}$$

We now follow the prescription described in section 2 by which a GF for subgroup tensors is transformed into a GF for group tensors. The first term in (3.11a) is a distinctive term so that following (3.20), the GF for tensors in the enveloping algebra of SU(3) is

$$G\,(U;\Lambda_1,\Lambda_2) = \sum_{\lambda_1,\lambda_2=0}^{\infty} \{ \sum \operatorname*{Res}_{N_1N_2} \; N_1^{-\lambda_2-2} N_2^{2\lambda_1+\lambda_2+2} \Lambda_1^{\lambda_1} \Lambda_2^{\lambda_2}$$

$$\times \; (N_1N_2 - N_2^{-2})\,(N_2^2 - N_1N_2^{-1}) \;\; F_4\,(U;\, N_1,N_2)\} .$$

The sums over λ_1 and λ_2 being geometric may be done explicitly so that $G\ (U;\Lambda_1\ \Lambda_2)$ becomes

$$\sum \underset{N_1 N_2}{\mathrm{Res}} \frac{N_1^{-2} N_2^{2}(N_1N_2-N_2^{-2})\ (N_2^{2}-N_1N_2^{-1})\ F_4\ (U,N_1,N_2)}{(1-N_2^{2}\Lambda_1)\ (1-N_1^{-1}N_2\Lambda_2)} \quad . \tag{3.27}$$

The N_1, N_2 residues in (3.27) are those at poles within unit circles; the norms ofthe other variables are considered smaller then unity. The result is

$$G(U;\Lambda_1,\Lambda_2) = \{(1-U^2)\ (1-U^3)\ (1-U\Lambda_1\Lambda_2)\ (1-U^2\Lambda_1\Lambda_2)\}^{-1} \times \left[\frac{1}{(1-U^3\Lambda_1^3)} + \frac{U^3\Lambda_2^3}{(1-U^3\Lambda_2^3)} \right] ; \tag{3.28a}$$

U carries the degree, Λ_1 and Λ_2 the SU(3) representation labels of the tensors. Denoting the elementary tensors by $(u,\ \lambda_1\ \lambda_2)$ where u indicates the degree and λ_1, λ_2 the representation labels, the integrity basis suggested by (3.28a) is (2,00), (3,00), (1,11), (2,11), (3,30), (3,03) with the product (3,30) (3,03) being forbidden. Other forms of $G\ (U;\Lambda_1,\Lambda_2)$ are possible such as[49]

$$G\ (U;\Lambda_1,\Lambda_2) = \{(1-U^2)\ (1-U^3)\ (1-U\Lambda_1\Lambda_2)\ (1-U^3\Lambda_1^3)\ (1-U^3\Lambda_2^3)\}^{-1} \times \{1+U^2\Lambda_1\Lambda_2 + U^4\Lambda_1^2\Lambda_2^2\} \tag{3.28b}$$

which as the same integrity basis as (3.28a) but in this case the forbidden product is $(2,11)^3$. Once one knows the elementary factors and syzygyes it is easy to show that the degree m_i of all independent tensors in the enveloping algebra of SU(3) is

$$m_i = \lambda_> + i \qquad i = 0,\ 1,\ \ldots,\ \lambda_<$$

with the restriction $\lambda\text{-}u = 0 \bmod 3$, where $\lambda_>$ and $\lambda_<$ are respectively the larger and smaller values of the Cartan labels λ_1,λ_2.

We now consider the SO(5) group and will make use of its canonical subgroup SU(2) x SU(2) ~ SO(4). The generators of SO(5) decompose under SU(2) x SU(2) into a (1,1) quartet and the vectors (2,0) and (0,2) respectively denoted by W, Q and V. In order to simplify the upcoming calculations, the representation labels are doubled. First we construct a GF $F_3(W,Q,V,N_1,N_2)$ for SU(2) x SU(2) tensors based on the quartet and the two vectors keeping track of the degree of the individual subgroup tensors. The subgroup GF $F_1(W;N_1',N_2')$ and $F_2(Q,V;N_1'',N_2'')$ respectively based on a quartet[49] and the two vectors Q and V are

$$F_1(W;N_1',N_2') = \{(1-W^2)\ (1-WN_1'N_2')\}^{-1} \tag{3.29}$$

$$F_2(Q,V;N_1'',N_2'') = \{(1-Q^2)\ (1-V^2)\ (1-QN_1''^2)\ (1-VN_2''^2)\}^{-1} \tag{3.30}$$

where W, Q and V keep track of the degrees in the quartet and the two vectors. The N's carry the representation labels. The GF for the SU(2) x SU(2) Clebsch-Gordan series is

$$C(N_1',N_2';N_1'',N_2'';N_1,N_2) = \{(1-N_1'N_1)\ (1-N_1''N_1)\ (1-N_2'N_2)\ (1-N_2''N_2)\ (1-N_1'N_1'') \\ \times\ (1-N_2'N_2''))\}^{-1} \tag{3.31}$$

so that

$$F_3(W,Q,V;N_1,N_2) = \Sigma \operatorname*{Res}_{N_1'N_2'N_1''N_2''} \{N_1'^{-1} N_2'^{-1} N_1''^{-1} N_2''^{-1}\ F_1(W;N_1'^{-1},N_2'^{-1}) \\ \times\ F_2(Q,V;N_1'',N_2'')\ C\ (N_1',N_2';N_1''^{-1},N_2''^{-1};N_1,N_2)\}. \tag{3.32}$$

The N_1'', N_2'' residues are those at poles inside circles a little greater then unity in radius; only poles inside unit circles for the N_1' and N_2' variables are considered; the norms of the other variables are considered smaller then unity. The calculations involved in (3.32) are straightforward but laborious; the answer is

$$F_3(W,Q,V;N_1,N_2) = \{(1-Q^2)(1-V^2)(1-W^2)(1-VN_2^2)(1-QN_1^2)(1-W^2QV)(1-WN_1N_2)\}^{-1}$$

$$\times \left[\frac{1+QVW^2N_1^2+QVWN_1N_2+VWN_1N_2}{(1-W^2VN_1^2)} \right. \quad (3.33)$$

$$\left. + \frac{W^2VQN_2^2+QWN_1N_2+Q^2VW^3N_1N_2^3+QW^2N_2^2}{(1-W^2QN_2^2)} \right].$$

Keeping track only of the degree in the generators, (3.33) becomes

$$F_3(U;N_1,N_2) = \{(1-U^2)^3\ (1-UN_2^2)\ (1-UN_1^2)\ (1-U^4)\ (1-UN_1N_2)\}^{-1}$$

$$\times \left[\frac{1+U^4N_1^2+U^3N_1N_2+U^2N_1N_2}{(1-U^3N_1^2)} \right. \quad (3.34)$$

$$\left. + \frac{U^4N_2^2+U^2N_1N_2+U^6N_1N_2^3+U^3N_2^2}{(1-U^3N_2^2)} \right];$$

the variable Q, V and W in (3.33) have been replaced by U, where U carries the degree in the generators and N_1, N_2 the SU(2) x SU(2) representation labels. The next step is to convert $F_3(U;N_1,N_2)$ to a GF for group tensors following the procedure given in (3.20). The first term of (3.14a) is a distinctive term so that the GF $G(U;\Lambda_1,\Lambda_2)$ for tensors in the enveloping algebra of SO(5) is

$$G(U;\Lambda_1,\Lambda_2) = -\sum \underset{N_1N_2}{\mathrm{Res}} \frac{N_1^{-3}N_2^{-2}(1-N_1N_2)(N_1-N_2)\ F_3(U;N_1,N_2)}{(1-N_1^{-1}\Lambda_1)\ (1-N_2^{-1}N_1^{-1}\Lambda_2)} \quad (3.35)$$

where the N_1, N_2 residues in (3.35) are those at poles within unit circles; the norm of the other variables are considered smaller then unity. The result is

$$G(U;\Lambda_1,\Lambda_2) = \frac{1+U^4\Lambda_1^2\Lambda_2}{(1-U^2)\ (1-U^4)\ (1-U\Lambda_1^2)\ (1-U^2\Lambda_2)\ (1-U^2\Lambda_2^2)\ (1-U^3\Lambda_1^2)} \quad (3.36)$$

where Λ_1 and Λ_2 carry the SO(5) representation labels. Using the same notation as for SU(3), the integrity basis consists of thequadratic and quartic

Casimir invariants (2,00) and (4,00), two decuplets of degree 1 and 3 (1,20) and (3,20), a quintet (2,01) and a 14- plet (2,02) each of degree 2, finally a 35- plet (4,21) of degree 4. The strectched square of the 35- plet is redundant. The degrees m_{ij} as given by (3.36) of the independent tensors in the enveloping algebra of SO(5) are as follows

$$m_{ij}^{(1)} = \frac{\lambda_1}{2} + 2\lambda_2 + 2i - 2j \qquad i = 0, 1, \ldots, \frac{\lambda_1}{2} \text{ (only even } \lambda_1) \qquad j = 0, 1, \ldots, \frac{\lambda_2}{2} \left[\frac{(\lambda_2 - 1)}{2} \text{ for } \lambda_2 \text{ odd}\right]$$

and

$$m_{ij}^{(2)} = \frac{\lambda_1}{2} + 2\lambda_2 + 2i - 2j + 1 \qquad i = 0, 1, \ldots, \frac{\lambda_1}{2} - 1 \text{ (only even } \lambda_1) \qquad j = 0, 1, \ldots, \frac{\lambda_2 - 2}{2} \left(\frac{\lambda_2 - 1}{2} \text{ for } \lambda_2 \text{ odd}\right)$$

The exists no tensors with odd λ_1.

We could also derive (3.36) by the elementary multiplet method, using the chain $SU(10) \supset SU(5) \supset SO(5)$; the embedding is such that (10 ... 0) of SU(10) contains (0100) of SU(5) which contains (20) of SO(5). The $SU(10) \supset SU(5)$ GF for one-rowed representations of SU(10) is $\{(1-UM_2)(1-U^2M_4)\}^{-1}$ where U carries the SU(10) label (the degree) and M_2, M_4 carry the second and fourth SU(5) labels. Hence if $F(M_1, M_2, M_3, M_4; \Lambda_1, \Lambda_2)$ is the GF for $SU(5) \supset SO(5)$ branching rules, we see that

$$G(U;\Lambda_1,\Lambda_2) = F(0, U, 0, U^2; \Lambda_1, \Lambda_2) \tag{3.37}$$

is the desired GF for tensors in the SO(5) enveloping algebra. The GF F for $SU(5) \supset SO(5)$ branching rules is given by Patera and Sharp[46].

An alternative chain is $SU(10) \supset SU(4) \supset SO(5)$ with the embedding $(10 \ldots 0) \supset (200) \supset (20)$. The $SU(10) \supset SU(4)$ GF for one-rowed representations of SU(10) is $\{(1-UM_1^2)(1-U^2M_2^2)(1-U^3M_3^2)(1-U^4)\}^{-1}$ where U carries the SU(10)

label (the degree) and M_1, M_2, M_3 carry the SU(4) labels.
Hence if $F(M_1^2, M_2^2, M_3^2; \Lambda_1, \Lambda_2)$ is the part of the SU(4)⊃ SO(5) GF which is even in all SU(4) labels, we see that
$G(U; \Lambda_1, \Lambda_2) = (I-U^4)^{-1} F(U, U^2, U^3; \Lambda_1, \Lambda_2)$ is the desired GF (3.36). The GF for SU(4)⊃ SO(5) branching rules is given by Patera and Sharp[46].

(b) SU(4) and G_2

The GF for tensors in the enveloping algebra of SU(4) could be evaluated by using the SU(3) x U(1) subgroup and the methods of section 3.3a. We found it easier to use the chain SU(15)⊃ SU(6)⊃ SU(4) with the embedding (10 ... 0)⊃ (01 000)⊃ (101).

The SU(15)⊃ SU(6) generating function, for one-rowed representations of SU(15), is $\{(1-UM_2)(1-U^2M_4)(1-U^3)\}^{-1}$, where U carries the SU(15) label, or degree, and M_2, M_4 carry the second and fourth SU(6) labels. The SU(6)⊃ SU(4) generating function, for representations of SU(6) in which only the second and fourth labels are non-zero is

$$F(M_2, M_4; \Lambda_1, \Lambda_2, \Lambda_3) = \{(1-M_2^2)(1-M_4^2)(1-M_2\Lambda_1\Lambda_3)(1-M_4\Lambda_1\Lambda_3)(1-M_2^2\Lambda_2^2)(1-M_4^2\Lambda_2^2)\}^{-1}$$

$$\times \left[\frac{1+M_2^2M_4\Lambda_1^2\Lambda_2 + M_2M_4^2\Lambda_1^2\Lambda_2 + M_2^2M_4^2\Lambda_1\Lambda_2^2\Lambda_3}{(1-M_2M_4\Lambda_1\Lambda_3)(1-M_2^2M_4^2\Lambda_1^4)} \right.$$

$$+ \frac{M_2^2M_4\Lambda_2\Lambda_3^2 + M_2M_4^2\Lambda_2\Lambda_3^2 + M_2^2M_4^2\Lambda_3^4(1+M_2^2M_4^2\Lambda_1\Lambda_2^2\Lambda_3)}{(1-M_2M_4\Lambda_1\Lambda_3)(1-M_2^2M_4^2\Lambda_3^4)} \tag{3.38}$$

$$+ \frac{M_2M_4\Lambda_1^2\Lambda_2(1+M_2^2M_4\Lambda_1^2\Lambda_2+M_2M_4^2\Lambda_1^2\Lambda_2) + M_2^3M_4^3\Lambda_1^4\Lambda_2^2}{(1-M_2M_4\Lambda_1^2\Lambda_2)(1-M_2^2M_4^2\Lambda_1^4)}$$

$$\left. + \frac{M_2M_4\Lambda_2\Lambda_3^2(1+M_2^2M_4\Lambda_2\Lambda_3^2+M_2M_4^2\Lambda_2\Lambda_3^2) + M_2^3M_4^3\Lambda_2^2\Lambda_3^4}{(1-M_2M_4\Lambda_2\Lambda_3^2)(1-M_2^2M_4^2\Lambda_3^4)} \right].$$

M_2, M_4 carry the second and fourth SU(6) labels, Λ_1, Λ_2, Λ_3 the SU(4) labels. Hence substituting (3.38) into the SU(15)$\supset$ SU(6) GF, we obtain the desired GF for tensors in the SU(4) enveloping algebra

$$G\,(U;\Lambda_1,\Lambda_2,\Lambda_3) = (1-U^3)^{-1}\,F\,(U,\ U^2;\Lambda_1,\Lambda_2,\Lambda_3) \tag{3.39}$$

$$= \{(1-U^2)\,(1-U^3)\,(1-U^4)\,(1-U\Lambda_1\Lambda_3)\,(1-U^2\Lambda_1\Lambda_3)\,(1-U^2\Lambda_2^2)\,(1-U^4\Lambda_2^2)\}^{-1}$$

$$\times\left[\frac{1+U^4\Lambda_1^2\Lambda_2+U^5\Lambda_1^2\Lambda_2+U^6\Lambda_1\Lambda_2^2\Lambda_3}{(1-U^3\Lambda_1\Lambda_3)\,(1-U^6\Lambda_1^4)} + \frac{U^4\Lambda_2\Lambda_3^2+U^5\Lambda_2\Lambda_3^2+U^6\Lambda_3^4(1+U^6\Lambda_1\Lambda_2^2\Lambda_3)}{(1-U^3\Lambda_1\Lambda_3)\,(1-U^6\Lambda_3^4)}\right.$$

$$\left.+\frac{U^3\Lambda_1^2\Lambda_2(1+U^4\Lambda_1^2\Lambda_2+U^5\Lambda_1^2\Lambda_2)+U^9\Lambda_1^4\Lambda_2^2}{(1-U^3\Lambda_1^2\Lambda_2)\,(1-U^6\Lambda_1^4)} + \frac{U^3\Lambda_2\Lambda_3^2(1+U^4\Lambda_2\Lambda_3^2+U^5\Lambda_2\Lambda_3^2)+U^9\Lambda_2^2\Lambda_3^4}{(1-U^3\Lambda_2\Lambda_3^2)\,(1-U^6\Lambda_3^4)}\right].$$

U carries the degree and Λ_1, Λ_2, Λ_3 the SU(4) labels of the tensors.

Inspection of (3.39) suggests an integrity basis with 17 elements (the notation is (pabc) where p is the degree and a,b,c the SU(4) labels) : (2000), (3000), (4000), (1101), (2101), (2020), (4020), (3101), (3210), (3012), (4210), (4012), (5210), (5012), (6121), (6400), (6004). Because of syzygies the following products of elementary tensors should be eliminated : (3101) with (3210), (3012); (3210) with (3012), (4012), (5012), (6121), (6004); (3012) with (4210), (5210), (6121), (6400); (4210) with (4012), (5012), (6121), (6004); (4012) with (5210), (6121), (6400); (5210) with (5012), (6121), (6004); (5012) with (6121), (6400); (6121) with (6400), (6004); (6400) with (6004); the squares of (4210), (4012), (5210), (5012), (6121); the products (3101) (4210) (5210) and (3101) (4012) (5012).

The generating function for tensors in the enveloping algebra of G_2 could be evaluated with the use of the SU(3) subgroup. However it proves simpler to get it by exploiting an interesting relationship between the groups SU(4) and G_2 based on the fact that the subgroup SU(3) is embedded similarly in the two groups.

With the help of the SU(4)$\supset$SU(3) generating function

$$F(\Lambda_1,\Lambda_2,\Lambda_3;N_1,N_2) =$$
$$\{(1-\Lambda_1)\ (1-\Lambda_1 N_1)\ (1-\Lambda_2 N_1)\ (1-\Lambda_2 N_2)\ (1-\Lambda_3 N_2)\ (1-\Lambda_3)\}^{-1} \tag{3.40}$$

(Λ_1, Λ_2, Λ_3 carry the SU(4) labels, N_1, N_2 the SU(3) labels) a generating function $H(\Lambda_1,\Lambda_2,\Lambda_3)$ for SU(4) tensors may be converted into a generating function

$$\begin{aligned} J(N_1,N_2) = \{(1-N_1)\ (1-N_2)\ (N_1-N_2)\}^{-1} \{N_1^2 N_2 H(N_1,N_1,N_2) \\ - N_1 N_2^2 H(N_1,N_2,N_2) - N_1^2\ H\ (N_1,\ N_1,1) + N_2^2\ H(1,N_2,N_2) \\ + N_1\ H\ (1,N_1,1) - N_2\ H(1,N_2,1) + N_1 N_2\ H(N_1,N_2,1) \\ - N_1 N_2\ H(1,N_1,N_2)\} \end{aligned} \tag{3.41}$$

for SU(3) tensors.

Similarly the $G_2 \supset$ SU(3) generating function (3.15) converts a generating function $K(\Lambda_1,\Lambda_2)$ for G_2 tensors into the generating function

$$\begin{aligned} L\ (N_1,N_2) = \{(1-N_1)\ (1-N_2)\ (N_1-N_2)\}^{-1} \{N_2^2 K(N_2,N_2) \\ -N_1^2 K(N_1,N_1) + N_1^2 N_2 K(N_1,N_1 N_2) - N_1 N_2^2 K(N_2,N_1 N_2) \\ + N_1 K(1,N_1) - N_2 K(1,N_2)\} \end{aligned} \tag{3.42}$$

for SU(3) tensors.

Now suppose that the SU(4) generating function $H(\Lambda_1,\Lambda_2,\Lambda_3)$ and the G_2 generating function $K(\Lambda_1,\Lambda_2)$ are related by the fact that they generate the same SU(3) tensors. It follows from (3.41) and (3.42) that they are related by the functional equation

$$N_2^2K(N_2,N_2) - N_1^2K(N_1,N_1) + N_1^2N_2K(N_1,N_1N_2) - N_1N_2^2\ K(N_2,N_1N_2)$$
$$+ N_1K(1,N_1) - N_2\ K(1,N_2) = N_1^2N_2\ H(N_1,N_1,N_2) - N_1N_2^2\ H(N_1,\ N_2,\ N_2)$$
$$- N_1^2\ H(N_1,N_1,1) + N_2^2\ H(1,N_2,N_2) + N_1\ H(1,N_1,1) - N_2\ H(1,N_2,1)$$
$$+ N_1N_2\ H(N_1,N_2,1) - N_1N_2H(1,N_1,N_2). \tag{3.43}$$

Under the assumption that $H\ (\Lambda_1,\Lambda_2,\Lambda_3)$ is symmetric in its first and last arguments, $H\ (\Lambda_1,\Lambda_2,\Lambda_3) = H\ (\Lambda_3,\Lambda_2,\Lambda_1)$, it can be verified that a formal solution of (3.43) for $K\ (\Lambda_1,\Lambda_2)$ is

$$K(\Lambda_1,\Lambda_2) = H\ (\Lambda_1,\Lambda_1,\Lambda_2/\Lambda_1) + \Lambda_1^{-1}\ H\ (\Lambda_1,\Lambda_2/\Lambda_1,1). \tag{3.44}$$

The solution (3.44) suffers from the defect that its expansion contains, in general, negative powers of Λ_1. These can be eliminated by adding to $K\ (\Lambda_1,\Lambda_2)$ an appropriate solution of the homogeneous version of (3.43). It can be verified that, for any λ_1,λ_2, $K'(\Lambda_1,\Lambda_2) = \Lambda_1^{-\lambda_1}\ \Lambda_2^{\lambda_2} + \Lambda_1^{\lambda_1-2}\ \Lambda_2^{\lambda_2-\lambda_1+1}$ satisfies the homogeneous equation. Thus we have the following prescription for the solution of (3.43) for $K\ (\Lambda_1,\Lambda_2)$ which contains no negative powers : expand the right side of (3.44) in powers of Λ_1 and replace each negative power $\Lambda_1^{-\lambda_1}$ ($\lambda_1 \geq 2$) by $-\Lambda_1^{\lambda_1-2}\Lambda_2^{-\lambda_1+1}$; drop terms in Λ_1^{-1}. Because of the form (3.44) this cannot introduce negative powers of Λ_2. The prescription can be formulated in terms of residues :

$$K\ (\Lambda_1,\Lambda_2) = \Sigma\ \mathrm{Res}_{\Lambda_1'}\ \left[\frac{1}{\Lambda_1'-\Lambda_1} - \frac{\Lambda_1'}{\Lambda_2-\Lambda_1\Lambda_1'}\right]\left[H(\Lambda_1',\Lambda_1',\frac{\Lambda_2}{\Lambda_1'}) + \frac{1}{\Lambda_1'}\ H\ (\Lambda_1',\frac{\Lambda_2}{\Lambda_1'},\ 1)\right]. \tag{3.45}$$

The first term $(\Lambda_1'-\Lambda_1)^{-1}$ in the first square bracket picks out the positive power part in Λ_1; the second $\Lambda_1'(\Lambda_2-\Lambda_1\Lambda_1')^{-1}$ replaces negative powers $\Lambda_1^{-\lambda_1}$ by $-\Lambda_1^{\lambda_1-2}\Lambda_2^{-\lambda_1+1}$ and cancels Λ_1^{-1}.

Let us now summarize : given a GF H $(\Lambda_1,\Lambda_2,\Lambda_3)$ for SU(4) tensors and a GF K (Λ_1,Λ_2) for G_2 tensors, and providing the following two conditions are satisfied

(1) H $(\Lambda_1,\Lambda_2,\Lambda_3)$ = H $(\Lambda_3,\Lambda_2,\Lambda_1)$

(2) H $(\Lambda_1,\Lambda_2,\Lambda_3)$ and K (Λ_1,Λ_2) must generate the same SU(3) tensors. Actually, as it will be shown below, the relationship (3.45) may still be useful as long as when reduced under SU(3), H and K differ only by denominator factors which correspond to SU(3) scalars,

the G_2 GF is obtained from the SU(4) one through the relationship (3.45).

As a first application, we consider the case were H and K are respectively based on a (010) and (10) tensor. The SU(4) GF is easily obtained by considering the chain SU(6)⊃SU(4) restricted to the symmetric representations of SU(6) with the embedding (10000)⊃(010); it is

$$H\ (U;\Lambda_1,\Lambda_2,\Lambda_3) = \frac{1}{(1-U^2)\ (1-U\Lambda_2)}\ , \tag{3.46}$$

where U carries the degree in the (010) tensor and Λ_2 the representation label as their exponent. Obviously condition (1) is satisfied; since we have the following reductions under SU(3)

(010)⊃ (01) + (10) and (10)⊃ (10) + (01) + (00),

it follows that, when reduced to SU(3) GF's, K = $(1-U)^{-1}$ H which satisfies condition (2) so that following the prescription given in (3.45),

$$K(\Lambda_1,\Lambda_2) = \{(1-U)\ (1-U^2)\}^{-1}\ \sum \underset{\Lambda_1'}{\mathrm{Res}} \left[\frac{1}{(\Lambda_1'-\Lambda_1)} - \frac{\Lambda_1}{(\Lambda_2-\Lambda_1\Lambda_1')}\right]$$
$$\times \left[\frac{1}{(1-U\Lambda_1')} + \frac{1}{\Lambda_1'(1-\frac{U\Lambda_2}{\Lambda_1'})}\right] \tag{3.47}$$

which gives

$$K(\Lambda_1,\Lambda_2) = \frac{1}{(1-U^2)\ (1-U\Lambda_1)}\ , \tag{3.48}$$

which is the correct G_2 GF for tensors based on a (10) tensor.

One may also obtain (3.48) from the SU(4) GF $H(\Lambda_1,\Lambda_2,\Lambda_3)$ for tensors simultaneously based on the tensors (100) and (001) where

$$H(\Lambda_1,\Lambda_2,\Lambda_3) = \frac{1}{(1-U^2)\ (1-U\Lambda_1)\ (1-U\Lambda_3)} \tag{3.49}$$

which satisfies condition (1); (3.49) is easily obtained by coupling the SU(4) GF's for tensors respectively based on a (100) and a (001) tensor. Since

$$(100) \supset (10) + (00) \text{ and } (001) \supset (01) + (00),$$

we have that $K = (1-U)H$ when K and H are reduced to SU(3) GF's. Consequently

$$K\ (\Lambda_1,\Lambda_2) = (1-U)\ (1-U^2)^{-1}\ \sum \underset{\Lambda_1'}{\text{Res}} \left[\frac{1}{(\Lambda_1'-\Lambda_1)} - \frac{\Lambda_1'}{(\Lambda_2-\Lambda_1\Lambda_1')}\right]$$
$$\times \left[\frac{1}{(1-U\Lambda_1')\ (1-\frac{U\Lambda_2}{\Lambda_1'})} + \frac{1}{\Lambda_1'(1-U\Lambda_1')\ (1-U)}\right]$$
$$= \frac{1}{(1-U^2)\ (1-U\Lambda_1)}\ .$$

The GF for tensors in the G_2 enveloping algebra may be obtained from the SU(4) one given by (3.39) through the relationship (3.45) since condition (1) is satisfied and the generators of G_2 decompose under SU(3) into an octet, a triplet and an antitriplet, the same as SU(4), except that the SU(4) generators contain an additional scalar. It follows that, if $(1-U)\ \times$ $G\ (U;\Lambda_1,\Lambda_2,\Lambda_3)$, where $G\ (U;\Lambda_1,\Lambda_2,\Lambda_3)$ is given by (3.39), is substituted for $H\ (\Lambda_1,\Lambda_2,\Lambda_3)$ in (3.45), the result will be the GF for tensors in the G_2 enveloping algebra :

$$G(U;\Lambda_1,\Lambda_2) = \{(1-U^2)\,(1-U^6)\,(1-U\Lambda_2)\,(1-U^2\Lambda_1^2)\,(1-U^3\Lambda_1)\,(1-U^4\Lambda_2^2)\}^{-1}$$

$$\times\left[\frac{1+U^5\Lambda_1^3\Lambda_2+U^6\Lambda_1^3+U^8\Lambda_1^3\Lambda_2}{(1-U^3\Lambda_1^3)\,(1-U^4\Lambda_1^2)}+\frac{U^5\Lambda_2+U^6\Lambda_1^2\Lambda_2+U^7\Lambda_1\Lambda_2+U^{13}\Lambda_1^3\Lambda_2^2}{(1-U^4\Lambda_1^2)\,(1-U^5\Lambda_2)}\right. \tag{3.50}$$

$$\left.+\frac{U^5\Lambda_1\Lambda_2+U^9\Lambda_1\Lambda_2^2+U^{12}\Lambda_2^3+U^8\Lambda_2^2}{(1-U^5\Lambda_2)\,(1-U^8\Lambda_2^2)}\right] .$$

U carries the degree and Λ_1,Λ_2 the G_2 representation labels of the tensors.

The integrity basis implied by (3.50) consists of 17 elements (the notation is (pab) where p is the degree and a, b the G_2 labels) : (200), (600), (101), (220), (310), (330), (402), (420), (501), (511), (531), (630), (621), (711), (802), (912), (12,03). The following products of elementary tensors should not be used : the square or product of any two of (531), (630), (621), (511), (711), (912), (12,03); the product of (330) with (802), (621), (711), (511), (912), (12,03), of (420) with (802), (511), (912), (12,03), of (501) with (531), (630), of (802) with (531), (630), (621), (711), and the product $(330)^2\,(501)^2$.

(c) <u>Sp(6) and SO(7)</u>

The generating function for tensors in the enveloping algebra of Sp(6) is most easily determined with the help of the chain SU(21) ⊃ SU(6) ⊃ Sp(6) for one-rowed representations of SU(21) with the embedding (10...0) ⊃ (20000) ⊃ (200).

For one-rowed representations of SU(21) it is not hard to show that the SU(21) ⊃ SU(6) branching rules are given by the generating function

$$F\,(U;M_1,M_2,M_3,M_4,M_5) = \{(1-U^6)(1-UM_1^2)(1-U^2M_2^2)(1-U^3M_3^2)(1-U^4M_4^2)(1-U^5M_5^2)\}^{-1}\,; \tag{3.51}$$

U carries the SU(21) label (the degree), and M_1,M_2,M_3,M_4,M_5 carry the SU(6) labels.

The generating function for SU(6) ⊃ Sp(6) branching rules is of some interest in its own right. By examining low-lying representations of SU(6) we are led to the function

$$
\begin{aligned}
&H(M_1, M_2, M_3, M_4, M_5; \Lambda_1, \Lambda_2, \Lambda_3) \\
&= \{(1-M_1\Lambda_1)(1-M_2)(1-M_2\Lambda_2)(1-M_3\Lambda_3)(1-M_4)(1-M_4\Lambda_2)(1-M_5\Lambda_1)\}^{-1} \\
&\times \{\{(1-M_3\Lambda_1)(1-M_1M_3\Lambda_2)(1-M_3M_5\Lambda_2)(1-M_1M_3M_5\Lambda_3)\}^{-1} \\
&+ M_1M_4\Lambda_3\{(1-M_3\Lambda_1)(1-M_1M_3\Lambda_2)(1-M_1M_4\Lambda_3)(1-M_1M_3M_5\Lambda_3)\}^{-1} \\
&+ M_2M_5\Lambda_3\{(1-M_3\Lambda_1)(1-M_3M_5\Lambda_2)(1-M_2M_5\Lambda_3)(1-M_1M_3M_5\Lambda_3)\}^{-1} \\
&+ (M_1M_4\Lambda_3)(M_2M_5\Lambda_3)\{(1-M_3\Lambda_1)(1-M_1M_4\Lambda_3)(1-M_2M_5\Lambda_3)(1-M_1M_3M_5\Lambda_3)\}^{-1} \qquad (3.52) \\
&+ M_1M_5\Lambda_2\{(1-M_1M_3\Lambda_2)(1-M_3M_5\Lambda_2)(1-M_1M_5\Lambda_2)(1-M_1M_3M_5\Lambda_3)\}^{-1} \\
&+ (M_1M_4\Lambda_3)(M_1M_5\Lambda_2)\{(1-M_1M_3\Lambda_2)(1-M_1M_4\Lambda_3)(1-M_1M_5\Lambda_2)(1-M_1M_3M_5\Lambda_3)\}^{-1} \\
&+ (M_2M_5\Lambda_3)(M_1M_5\Lambda_2)\{(1-M_3M_5\Lambda_2)(1-M_2M_5\Lambda_3)(1-M_1M_5\Lambda_2)(1-M_1M_3M_5\Lambda_3)\}^{-1} \\
&+ (M_1M_4\Lambda_3)(M_2M_5\Lambda_3)(M_1M_5\Lambda_2)\{(1-M_1M_4\Lambda_3)(1-M_2M_5\Lambda_3)(1-M_1M_5\Lambda_2)(1-M_1M_3M_5\Lambda_3)\}^{-1} \\
&+ M_2M_4\Lambda_1\Lambda_3\{(1-M_3\Lambda_1)(1-M_1M_4\Lambda_3)(1-M_2M_5\Lambda_3)(1-M_2M_4\Lambda_1\Lambda_3)\}^{-1}\}
\end{aligned}
$$

M_1, M_2, M_3, M_4, M_5 carry the SU(6) and $\Lambda_1, \Lambda_2, \Lambda_3$ carry the Sp(6) representation labels. The integrity basis implied by (3.52) consists of 15 elementary multiplets (the notation $(\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ \lambda_5; V_1\ V_2\ V_3)$ where the λ's are the representation labels for SU(6) and the V's those of Sp(6) :

(10000;100), (01000;010), (01000;000), (00100;001), (00100;100), (00010;010), (00010;000), (00001;100), (10100;010), (00101;010), (01001;001), (10010;001), (10001;010), (01010;101), (10101;001).

The form of equation (3.52) indicates that the following products of elementary multiplets should be discarded, in order to avoid redundant states :

(00100;100) with (10001;010),

(10100;010) with any of (01001;001), (01010;101)

(00101;010) with any of (10010;001), (01010;101)

(10001;010) with (01010;101),

(01010;101) with (10101;001).

This integrity basis defines SU(6) polynomial bases reduced according to the Sp(6) subgroup. While (3.52) has not been derived analytically, we are reasonably sure it is correct. For example, it satisfies the dimension and second order index checks for all SU(6) representations up to those for which two labels are equal to two and all others equal to one {(11221), (21121) ...}; also the GF for tensors in the Sp(6) enveloping algebra derived from it, has been subjected to the checks described in chapter IV.

The GF H of (3.52) must be substituted into F of (3.51) to obtain the desired GF G $(U;\Lambda_1,\Lambda_2,\Lambda_3)$ for tensors in the Sp(6) enveloping algebra. The form of (3.51) indicates that only the part of (3.52) which is even in all SU(6) labels is required. Let $H'(M_1^2,M_2^2,M_3^2,M_4^2,M_5^2;\Lambda_1,\Lambda_2,\Lambda_3)$ be this even part; it is obtained straightforwardly from (3.52). Then the desired GF is

$$
\begin{aligned}
G'&(U;\Lambda_1,\Lambda_2,\Lambda_3)\\
&= (1-U^6)^{-1} H'(U,U^2,U^3,U^4,U^5;\Lambda_1,\Lambda_2,\Lambda_3)\\
&= \{(1-U^2)(1-U^4)(1-U^6)(1-U\Lambda_1^2)(1-U^2\Lambda_2^2)(1-U^3\Lambda_3^2)(1-U^4\Lambda_2^2)\\
&\quad \times (1-U^5\Lambda_1^2)\}^{-1} \{\{(1-U^3\Lambda_1^2)(1-U^2\Lambda_2)(1-U^4\Lambda_2)(1-U^9\Lambda_3^2)\}^{-1}\\
&\quad \times \{1+U^3\Lambda_1\Lambda_3+U^4(\Lambda_1^2\Lambda_2+\Lambda_1\Lambda_2\Lambda_3)+U^8(\Lambda_1^2\Lambda_2+\Lambda_1\Lambda_2\Lambda_3)\\
&\quad + U^9(\Lambda_1^2\Lambda_2^2+\Lambda_1^3\Lambda_3+\Lambda_1^2\Lambda_3^2+2\Lambda_1\Lambda_2\Lambda_3)\\
&\quad + U^{12}(\Lambda_1^3\Lambda_2^2\Lambda_3+2\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3+\Lambda_2^2\Lambda_3^2)\}
\end{aligned}
$$

$$
\begin{aligned}
&+\{(1-U^3\Lambda_1^2)(1-U^2\Lambda_2)(1-U^5\Lambda_3^2)(1-U^9\Lambda_3^2)\}^{-1}\{U^5(\Lambda_3^2+\Lambda_1\Lambda_3+\Lambda_1\Lambda_2\Lambda_3)\\
&+U^8(\Lambda_1\Lambda_3^3+\Lambda_1^2\Lambda_3^2+\Lambda_1\Lambda_2\Lambda_3+\Lambda_2\Lambda_3^2+\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3+\Lambda_2^2\Lambda_3^2)\\
&+U^9(\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_2\Lambda_3^2)+U^{12}\Lambda_1\Lambda_2\Lambda_3^3\\
&+U^{13}(\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_1^2\Lambda_3^2+\Lambda_1\Lambda_3^3+\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2\Lambda_3^3)\\
&+U^{14}(\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_1^3\Lambda_3^3+\Lambda_1^2\Lambda_3^4+\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_1^2\Lambda_2\Lambda_3^2)\\
&+U^{17}(\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_1^3\Lambda_2\Lambda_3^3+\Lambda_1^3\Lambda_2^2\Lambda_3^3)+U^{18}(\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_1^3\Lambda_2\Lambda_3^3+\Lambda_1^2\Lambda_2\Lambda_3^4)\\
&+U^{22}\Lambda_1^2\Lambda_2^2\Lambda_3^4\}+\{(1-U^3\Lambda_1^2)(1-U^4\Lambda_2)(1-U^7\Lambda_3^2)(1-U^9\Lambda_3^2)\}^{-1}\\
&\times\{U^7(\Lambda_3^2+\Lambda_1\Lambda_3+\Lambda_1\Lambda_2\Lambda_3)+U^9\Lambda_2\Lambda_3^2\\
&+U^{10}(\Lambda_1\Lambda_3^3+\Lambda_1^2\Lambda_3^2+\Lambda_1\Lambda_2\Lambda_3+\Lambda_2\Lambda_3^2+\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3+\Lambda_2^2\Lambda_3^2)\\
&+U^{11}(\Lambda_1^2\Lambda_3^2+\Lambda_1\Lambda_3^3+\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2\Lambda_3^3)\\
&+U^{12}\Lambda_1\Lambda_2\Lambda_3^3+U^{15}(\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2\Lambda_3^3)\\
&+U^{16}(\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_1^3\Lambda_3^3+\Lambda_1^2\Lambda_3^4+\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_1^2\Lambda_2\Lambda_3^2)\\
&+U^{17}(\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3^3)+U^{18}(\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_1^3\Lambda_2\Lambda_3^3+\Lambda_1^2\Lambda_2\Lambda_3^4)\\
&+U^{19}(\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_1^3\Lambda_2\Lambda_3^3+\Lambda_1^3\Lambda_2^2\Lambda_3^3)+U^{21}\Lambda_1^2\Lambda_2^2\Lambda_3^4\}\\
&+\{(1-U^3\Lambda_1^2)(1-U^5\Lambda_3^2)(1-U^7\Lambda_3^2)(1-U^9\Lambda_3^2)\}^{-1}\\
&\times\{U^{12}(\Lambda_1^2\Lambda_3^2+2\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1^2\Lambda_2^2\Lambda_3^2+2\Lambda_1\Lambda_3^3+2\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_3^4)\\
&+U^{14}(\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_2\Lambda_3^4)\\
&+U^{15}(\Lambda_1^3\Lambda_3^3+2\Lambda_1^3\Lambda_2\Lambda_3^3+\Lambda_1^3\Lambda_2^2\Lambda_3^3+2\Lambda_1^2\Lambda_3^4+2\Lambda_1^2\Lambda_2\Lambda_3^4\\
&+\Lambda_1\Lambda_3^5+\Lambda_3^4+\Lambda_1\Lambda_3^3+2\Lambda_2\Lambda_3^4+2\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_2^2\Lambda_3^4+\Lambda_1\Lambda_2^2\Lambda_3^3)\\
&+U^{16}(\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_2\Lambda_3^4+\Lambda_1\Lambda_3^5+\Lambda_1^2\Lambda_3^4+\Lambda_1\Lambda_2\Lambda_3^5+\Lambda_1^2\Lambda_2\Lambda_3^4)\\
&+U^{17}(\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_1^2\Lambda_2^2\Lambda_3^4+\Lambda_1\Lambda_2\Lambda_3^5)+U^{18}\Lambda_2^2\Lambda_3^4\\
&+U^{19}(\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_1^2\Lambda_2^2\Lambda_3^4+\Lambda_1\Lambda_2\Lambda_3^5)\\
&+U^{20}(\Lambda_1\Lambda_3^5+\Lambda_1^2\Lambda_3^4+2\Lambda_1\Lambda_2\Lambda_3^5+2\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_1\Lambda_2^2\Lambda_3^5+\Lambda_1^2\Lambda_2^2\Lambda_3^4)\\
&+U^{21}(\Lambda_1\Lambda_2^2\Lambda_3^5+\Lambda_1^2\Lambda_3^5+\Lambda_1^3\Lambda_3^5)\\
&+U^{22}(\Lambda_1\Lambda_2\Lambda_3^5+\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_1\Lambda_2^2\Lambda_3^5+\Lambda_1^2\Lambda_2^2\Lambda_3^4)
\end{aligned}
\qquad (3.53)
$$

$$+ U^{23}(\Lambda_1^2\Lambda_2\Lambda_3^6+\Lambda_1^3\Lambda_2\Lambda_3^5)$$

$$+ U^{25}(\Lambda_1^2\Lambda_2\Lambda_3^6+\Lambda_1^3\Lambda_2\Lambda_3^5)+ U^{27}(\Lambda_1^2\Lambda_2^2\Lambda_3^6+\Lambda_1^3\Lambda_2^2\Lambda_3^5)\}$$

$$+ \{(1-U^2\Lambda_2)(1-U^4\Lambda_2)(1-U^6\Lambda_2^2)(1-U^9\Lambda_3^2)\}^{-1}\ \{U^6(\Lambda_1^2\Lambda_2+\Lambda_2^2)$$

$$+ U^9(2\Lambda_1\Lambda_2^2\Lambda_3+\Lambda_2\Lambda_3^2+\Lambda_2^3) + U^{10}(\Lambda_1\Lambda_2^3\Lambda_3+\Lambda_1\Lambda_2^2\Lambda_3)$$

$$+ U^{14}(\Lambda_1\Lambda_2^3\Lambda_3+\Lambda_1\Lambda_2^2\Lambda_3)+ U^{15}(\Lambda_1^2\Lambda_2^2\Lambda_3^2+2\Lambda_1\Lambda_2^3\Lambda_3+\Lambda_1^2\Lambda_2^4)$$

$$+ U^{18}(\Lambda_1^2\Lambda_2^3\Lambda_3^2+\Lambda_2^4\Lambda_3^2)\}+ \{(1-U^2\Lambda_2)(1-U^5\Lambda_3^2)(1-U^6\Lambda_2^2)(1-U^9\Lambda_3^2)\}^{-1}$$

$$\times \{U^{10}(\Lambda_1\Lambda_2\Lambda_3+\Lambda_1\Lambda_2^2\Lambda_3)+ U^{11}(\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3+\Lambda_1\Lambda_2^3\Lambda_3+\Lambda_2^2\Lambda_3^2)$$

$$+ U^{14}(2\Lambda_2^3\Lambda_3^2+\Lambda_2^2\Lambda_3^2+\Lambda_2^4\Lambda_3^2+\Lambda_1^2\Lambda_2^2\Lambda_3^2$$

$$+ \Lambda_1\Lambda_2\Lambda_3^3+\Lambda_1^2\Lambda_2^3\Lambda_3^2+2\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_2\Lambda_3^4)$$

$$+ U^{15}(\Lambda_1\Lambda_2^3\Lambda_3^3+\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_2^3\Lambda_3^2)+ U^{18}(\Lambda_1\Lambda_2^3\Lambda_3^3+\Lambda_2^2\Lambda_3^4)$$

$$+ U^{19}(\Lambda_1\Lambda_2^2\Lambda_3^3+2\Lambda_1\Lambda_2^3\Lambda_3^3+\Lambda_1\Lambda_2^4\Lambda_3^3)$$

$$+ U^{20}(\Lambda_1^2\Lambda_2^2\Lambda_3^4+\Lambda_1^2\Lambda_2^3\Lambda_3^2+\Lambda_1^2\Lambda_2^4\Lambda_3^2+\Lambda_1\Lambda_2^3\Lambda_3^3)+ U^{24}(\Lambda_1\Lambda_2^4\Lambda_3^3+\Lambda_1^2\Lambda_2^3\Lambda_3^4)\} \qquad (3.53)$$

$$+ \{(1-U^4\Lambda_2)(1-U^6\Lambda_2^2)(1-U^7\Lambda_3^2)(1-U^9\Lambda_3^2)\}^{-1}$$

$$\times \{U^8(\Lambda_1\Lambda_2\Lambda_3+\Lambda_1\Lambda_2^2\Lambda_3)+ U^{13}(\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3+\Lambda_1\Lambda_2^3\Lambda_3+\Lambda_2^2\Lambda_3^2)$$

$$+ U^{15}(\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_2^3\Lambda_3^2)+ U^{16}(2\Lambda_2^3\Lambda_3^2+\Lambda_2^2\Lambda_3^2+\Lambda_2^4\Lambda_3^2+\Lambda_1^2\Lambda_2^2\Lambda_3^2$$

$$+ \Lambda_1\Lambda_2\Lambda_3^3+\Lambda_1^2\Lambda_2^3\Lambda_3^2+2\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_2\Lambda_3^4)$$

$$+ U^{17}(\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_1\Lambda_2^3\Lambda_3^3) + U^{18}(\Lambda_1\Lambda_2^3\Lambda_3^3+\Lambda_2^2\Lambda_3^4)$$

$$+ U^{21}(\Lambda_1\Lambda_2^3\Lambda_3^3+\Lambda_1\Lambda_2^2\Lambda_3^3)+ U^{22}(\Lambda_1^2\Lambda_2^2\Lambda_3^4+\Lambda_1^2\Lambda_2^3\Lambda_3^2+\Lambda_1^2\Lambda_2^4\Lambda_3^2+\Lambda_1\Lambda_2^5\Lambda_3^3)$$

$$+ U^{23}(\Lambda_1\Lambda_2^4\Lambda_3^3+\Lambda_1\Lambda_2^3\Lambda_3^3)+ U^{24}(\Lambda_1^2\Lambda_2^3\Lambda_3^4+\Lambda_1\Lambda_2^4\Lambda_3^3)\}$$

$$+ \{(1-U^5\Lambda_3^2)(1-U^6\Lambda_2^2)(1-U^7\Lambda_3^2)(1-U^9\Lambda_3^2)\}^{-1}$$

$$\times \{U^{12}(\Lambda_2\Lambda_3^2+2\Lambda_2^2\Lambda_3^2+\Lambda_2^3\Lambda_3^2)+ U^{13}(\Lambda_1\Lambda_2\Lambda_3^3+\Lambda_1\Lambda_2^2\Lambda_3^3)$$

$$+ U^{17}(\Lambda_1\Lambda_2\Lambda_3^3+2\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_1\Lambda_2^3\Lambda_3^3)$$

$$+ U^{18}(2\Lambda_1^2\Lambda_2^3\Lambda_3^2+2\Lambda_1\Lambda_2^2\Lambda_3^3+2\Lambda_1\Lambda_2^3\Lambda_3^3$$

$$+ \Lambda_1^2\Lambda_2^4\Lambda_3^2+\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_2^2\Lambda_3^4+\Lambda_1^2\Lambda_2^2\Lambda_3^2)$$

$$+ U^{19}(\Lambda_1\Lambda_2^2\Lambda_3^3+\Lambda_1\Lambda_2^3\Lambda_3^3)+ U^{20}(\Lambda_1\Lambda_2^4\Lambda_3^3+\Lambda_1^2\Lambda_2^2\Lambda_3^4+\Lambda_2^3\Lambda_3^4+\Lambda_1\Lambda_2^3\Lambda_3^3)$$

$$
\begin{aligned}
&+ U^{21}(\Lambda_1^2\Lambda_2\Lambda_3^4+\Lambda_2^2\Lambda_3^4+2\Lambda_1^2\Lambda_2^2\Lambda_3^4+2\Lambda_2^3\Lambda_3^4 \\
&+ 2\Lambda_1\Lambda_2\Lambda_3^5+2\Lambda_1\Lambda_2^2\Lambda_3^5+\Lambda_1^2\Lambda_2^3\Lambda_3^4+\Lambda_2^4\Lambda_3^4+\Lambda_2\Lambda_3^6) \\
&+ U^{22}(\Lambda_1\Lambda_2^3\Lambda_3^3+\Lambda_1\Lambda_2^2\Lambda_3^5+\Lambda_1\Lambda_2^4\Lambda_3^3+\Lambda_1\Lambda_2^3\Lambda_3^5+\Lambda_1^2\Lambda_2^2\Lambda_3^4+\Lambda_2^3\Lambda_3^4) \\
&+ U^{23}(\Lambda_1\Lambda_2^2\Lambda_3^5+\Lambda_1\Lambda_2^3\Lambda_3^5+\Lambda_2^2\Lambda_3^6)+ U^{24}(\Lambda_1^2\Lambda_2^3\Lambda_3^4+\Lambda_2^4\Lambda_3^4) \\
&+ U^{25}(\Lambda_1\Lambda_2^2\Lambda_3^5+\Lambda_1\Lambda_2^3\Lambda_3^5+\Lambda_2^2\Lambda_3^6)+ U^{26}(\Lambda_1\Lambda_2^2\Lambda_3^5+2\Lambda_1\Lambda_2^3\Lambda_3^5+\Lambda_1\Lambda_2^4\Lambda_3^5) \\
&+ U^{27}(\Lambda_2^3\Lambda_3^6+\Lambda_1^2\Lambda_2^2\Lambda_3^6)+ U^{28}(\Lambda_1\Lambda_2^3\Lambda_3^5+\Lambda_1\Lambda_2^4\Lambda_3^5) \\
&+ U^{29}\Lambda_1^2\Lambda_2^3\Lambda_3^6+U^{31}\Lambda_1^2\Lambda_2^3\Lambda_3^6+U^{33}\Lambda_1^2\Lambda_2^4\Lambda_3^6\} \\
&+ \{(1-U^3\Lambda_1^2)(1-U^5\Lambda_3^2)(1-U^7\Lambda_3^2)(1-U^3\Lambda_1\Lambda_3)\}^{-1} \qquad (3.53) \\
&\times \{U^6(\Lambda_1\Lambda_3+2\Lambda_1\Lambda_2\Lambda_3+\Lambda_1^2\Lambda_3^2+\Lambda_1\Lambda_2^2\Lambda_3)+ U^7(\Lambda_1^2\Lambda_3^2+\Lambda_1^2\Lambda_2\Lambda_3^2) \\
&+ U^8\Lambda_1^2\Lambda_2\Lambda_3^2+U^{10}\Lambda_1^2\Lambda_2\Lambda_3^2 \\
&+ U^{11}(\Lambda_1^2\Lambda_3^2+\Lambda_1^3\Lambda_3^3+\Lambda_1^3\Lambda_2\Lambda_3^3+2\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1^2\Lambda_2^2\Lambda_3^2) \\
&+ U^{12}(\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_1^3\Lambda_3^3) \\
&+ U^{13}(\Lambda_1^2\Lambda_2\Lambda_3^2+\Lambda_1^3\Lambda_3^3+2\Lambda_1^3\Lambda_2\Lambda_3^3+\Lambda_1^2\Lambda_2^2\Lambda_3^2+\Lambda_1^3\Lambda_2^2\Lambda_3^3) \\
&+ U^{14}\Lambda_1^3\Lambda_2\Lambda_3^3+U^{16}\Lambda_1^3\Lambda_2\Lambda_3^3+U^{17}(\Lambda_1^3\Lambda_2^2\Lambda_3^3+\Lambda_1^3\Lambda_2\Lambda_3^3) \\
&+ U^{18}(2\Lambda_1^4\Lambda_2\Lambda_3^4+\Lambda_1^3\Lambda_2^2\Lambda_3^3+\Lambda_1^4\Lambda_2^2\Lambda_3^4+\Lambda_1^4\Lambda_3^4)\}\}.
\end{aligned}
$$

The GF for tensors in the enveloping algebra of SO(7) is found with the help of the chain SU(21)⊃ SU(7)⊃SO(7) for one-rowed representations of SU(21) with the embedding (10 ... 0)⊃ (000010)⊃(010). One finds the SU(21)⊃ SU(7) and the SU(7)⊃ SO(7) GF's for branching rules and substitutes the latter in the former. The GF for SU(21)⊃ SU(7) branching rules is

$$F\ (U;K_1,K_3,K_5) = \{(1-U^3K_1)\ (1-U^2K_3)\ (1-UK_5)\}^{-1}. \qquad (3.54)$$

U carries the SU(21) label (the degree) while K_1, K_3, K_5 carry respectively first, third and fifth SU(7) labels. Thus we need the GF for SU(7)⊃ SO(7) branching rules only for SU(7) representations with the second, fourth and sixth labels zero. Solving this probelm by the elementary multiplet method turns out to be very difficult; the main difficulty being that of choosing the right elementary multiplets and syzygies among a large number of possibilities. The difficulty is reduced considerably if one introduces the additional group SU(6) between SU(7) and SO(7); SO(7) is not a subgroup of SU(6) but may be subjoined (SU(6)>SO(7)) to it, so that we consider the chain SU(7)⊃ SU(6)>SO(7).

The subjoining of a group to another group, recently introduced by Patera and Sharp[46,48,70,71], is basically a relationship between their weight systems. The insertion of SU(6) between SU(7) and SO(7) is defined by (000010)⊃ (00010) + (00001) with (00010) > (010) - (100) + (000) and (00001) > (100) -(000); for example, the subjoining (00001)>(100) - (000) means that the weights of the (00001) representation (after projection by some matrix into the SO(7) weight space) of SU(6) are equal to the weights of (100) minus those of (000). In general we say that a group B is subjoined to a group A when the weight diagrams (after projection) of all representations of A may be expressed in terms of sum and differences of weights of representations of B.

The problem of finding a GF for SU(7)⊃ SO(7) branching rules may therefore be solved in two steps : first finding the GF's for the chains SU(7)⊃ SU(6) and SU(6) > SO(7) and then substituting one into the other. The SU(7)⊃ SU(6) GF is (even SU(7) labels zero)

$$H(K_1,K_3,K_5;M_1,M_2,M_3,M_4,M_5) = \{(1-K_1)(1-K_1M_1)(1-K_3M_2)(1-K_3M_3)(1-K_5M_4)(1-K_5M_5)\}^{-1} \quad . \qquad (3.55)$$

M_1, M_2, M_3, M_4, M_5 carry the SO(7) representation labels.

The GF for SU(6) > SO(7) is

$$
\begin{aligned}
J&(M_1,M_2,M_3,M_4,M_5;\Lambda_1,\Lambda_2,\Lambda_3) = \\
&\{(1-M_1\Lambda_1)(1-M_5\Lambda_1)(1+M_1)(1+M_5)(1-M_2\Lambda_2)(1-M_4\Lambda_2)(1-M_2)(1-M_4)(1-M_3\Lambda_3^2)(1+M_3)\}^{-1} \\
&\times \{\{(1+M_2\Lambda_1)(1+M_4\Lambda_1)(1-M_3\Lambda_1)(1+M_3\Lambda_2)\}^{-1} + M_2M_4\Lambda_1\Lambda_3^2\{(1+M_2\Lambda_1) \\
&\times (1+M_4\Lambda_1)(1-M_3\Lambda_1)(1-M_2M_4\Lambda_1\Lambda_3^2)\}^{-1} + M_2M_4\Lambda_2\{(1+M_2\Lambda_1)(1+M_4\Lambda_1) \\
&\times (1+M_3\Lambda_2)(1-M_2M_4\Lambda_2)\}^{-1} - M_2M_4\Lambda_3^2\{(1+M_2\Lambda_1)(1+M_4\Lambda_1)(1-M_2M_4\Lambda_1\Lambda_3^2) \\
&\times (1+M_2M_4\Lambda_3^2)\}^{-1} - (M_2M_4\Lambda_2)(M_2M_4\Lambda_3^2)\{(1+M_2\Lambda_1)(1+M_4\Lambda_1)(1-M_2M_4\Lambda_2) \\
&\times (1+M_2M_4\Lambda_3^2)\}^{-1} + M_3M_5\Lambda_2\{(1+M_2\Lambda_1)(1-M_3\Lambda_1)(1+M_3\Lambda_2)(1-M_3M_5\Lambda_2)\}^{-1} \\
&+ M_2M_5\Lambda_3^2\{(1+M_2\Lambda_1)(1-M_3\Lambda_1)(1-M_3M_5\Lambda_2)(1-M_2M_5\Lambda_3^2)\}^{-1} \\
&+ (M_2M_5\Lambda_3^2)(M_2M_4\Lambda_1\Lambda_3^2)\{(1+M_2\Lambda_1)(1-M_3\Lambda_1)(1-M_2M_5\Lambda_3^2)(1-M_2M_4\Lambda_1\Lambda_3^2)\}^{-1} \\
&- M_2M_5\Lambda_2\{(1+M_2\Lambda_1)(1+M_3\Lambda_2)(1-M_3M_5\Lambda_2)(1+M_2M_5\Lambda_2)\}^{-1} \\
&- (M_2M_5\Lambda_2)(M_2M_4\Lambda_2)\{(1+M_2\Lambda_1)(1+M_3\Lambda_2)(1+M_2M_5\Lambda_2)(1-M_2M_4\Lambda_2)\}^{-1} \\
&+ (M_2M_5\Lambda_3^2)(M_2M_5\Lambda_2)\{(1+M_2\Lambda_1)(1-M_3M_5\Lambda_2)(1-M_2M_5\Lambda_3^2)(1+M_2M_5\Lambda_2)\}^{-1} \\
&+ (M_2M_5\Lambda_2)(M_2M_4\Lambda_3^2)\{(1+M_2\Lambda_1)(1-M_2M_5\Lambda_3^2)(1+M_2M_5\Lambda_2)(1+M_2M_4\Lambda_3^2)\}^{-1} \qquad (3.56) \\
&- (M_2M_5\Lambda_3^2)(M_2M_4\Lambda_3^2)\{(1+M_2\Lambda_1)(1-M_2M_5\Lambda_3^2)(1-M_2M_4\Lambda_1\Lambda_3^2)(1+M_2M_4\Lambda_3^2)\}^{-1} \\
&+ (M_2M_5\Lambda_2)(M_2M_4\Lambda_3^2)(M_2M_4\Lambda_2)\{(1+M_2\Lambda_1)(1+M_2M_5\Lambda_2)(1-M_2M_4\Lambda_2)(1+M_2M_4\Lambda_3^2)\}^{-1} \\
&+ (M_1M_3\Lambda_2)\{(1+M_4\Lambda_1)(1-M_3\Lambda_1)(1+M_3\Lambda_2)(1-M_1M_3\Lambda_2)\}^{-1} \\
&+ (M_1M_4\Lambda_3^2)\{(1+M_4\Lambda_1)(1-M_3\Lambda_1)(1-M_1M_3\Lambda_2)(1-M_1M_4\Lambda_3^2)\}^{-1} \\
&+ (M_1M_4\Lambda_3^2)(M_2M_4\Lambda_1\Lambda_3^2)\{(1+M_4\Lambda_1)(1-M_1M_4\Lambda_3^2)(1-M_2M_4\Lambda_1\Lambda_3^2)(1-M_3\Lambda_1)\}^{-1} \\
&- (M_1M_4\Lambda_2)\{(1+M_4\Lambda_1)(1+M_3\Lambda_2)(1-M_1M_3\Lambda_2)(1+M_1M_4\Lambda_2)\}^{-1} \\
&- (M_1M_4\Lambda_2)(M_2M_4\Lambda_2)\{(1+M_4\Lambda_1)(1+M_3\Lambda_2)(1+M_1M_4\Lambda_2)(1-M_2M_4\Lambda_2)\}^{-1} \\
&- (M_1M_4\Lambda_3^2)(M_1M_4\Lambda_2)\{(1+M_4\Lambda_1)(1-M_1M_3\Lambda_2)(1-M_1M_4\Lambda_3^2)(1+M_1M_4\Lambda_2)\}^{-1} \\
&+ (M_1M_4\Lambda_2)(M_2M_4\Lambda_3^2)\{(1+M_4\Lambda_1)(1-M_1M_4\Lambda_3^2)(1+M_1M_4\Lambda_2)(1+M_2M_4\Lambda_3^2)\}^{-1} \\
&- (M_1M_4\Lambda_3^2)(M_2M_4\Lambda_3^2)\{(1+M_4\Lambda_1)(1-M_1M_4\Lambda_3^2)(1-M_2M_4\Lambda_1\Lambda_3^2)(1+M_2M_4\Lambda_3^2)\}^{-1} \\
&+ (M_1M_4\Lambda_2)(M_2M_4\Lambda_2)(M_2M_4\Lambda_3^2)\{(1+M_4\Lambda_1)(1+M_1M_4\Lambda_2)(1-M_2M_4\Lambda_2)(1+M_2M_4\Lambda_3^2)\}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&+ (M_1M_3\Lambda_2)(M_3M_5\Lambda_2)\{(1-M_3\Lambda_1)(1+M_3\Lambda_2)(1-M_1M_3\Lambda_2)(1-M_3M_5\Lambda_2)\}^{-1} \\
&+ (M_1M_3M_5\Lambda_3^2)\{(1-M_3\Lambda_1)(1-M_1M_3\Lambda_2)(1-M_3M_5\Lambda_2)(1-M_1M_3M_5\Lambda_3^2)\}^{-1} \\
&+ (M_1M_4\Lambda_3^2)(M_1M_3M_5\Lambda_3^2)\{(1-M_3\Lambda_1)(1-M_1M_3\Lambda_2)(1-M_1M_4\Lambda_3^2)(1-M_1M_3M_5\Lambda_3^2)\}^{-1} \\
&+ (M_2M_5\Lambda_3^2)(M_1M_3M_5\Lambda_3^2)\{(1-M_3\Lambda_1)(1-M_3M_5\Lambda_2)(1-M_2M_5\Lambda_3^2)(1-M_1M_3M_5\Lambda_3^2)\}^{-1} \\
&+ (M_1M_4\Lambda_3^2)(M_2M_5\Lambda_3^2)\{(1-M_3\Lambda_1)(1-M_1M_4\Lambda_3^2)(1-M_2M_5\Lambda_3^2)(1-M_2M_4\Lambda_1\Lambda_3^2)\}^{-1} \\
&+ (M_1M_4\Lambda_3^2)(M_2M_5\Lambda_3^2)(M_1M_3M_5\Lambda_3^2)\{(1-M_3\Lambda_1)(1-M_1M_4\Lambda_3^2)(1-M_2M_5\Lambda_3^2)(1-M_1M_3M_5\Lambda_3^2)\}^{-1} \\
&+ (M_1M_5\Lambda_2)\{(1+M_3\Lambda_2)(1-M_1M_3\Lambda_2)(1-M_3M_5\Lambda_2)(1-M_1M_5\Lambda_2)\}^{-1} \\
&- (M_1M_4\Lambda_2)(M_1M_5\Lambda_2)\{(1+M_3\Lambda_2)(1-M_1M_3\Lambda_2)(1+M_1M_4\Lambda_2)(1-M_1M_5\Lambda_2)\}^{-1} \\
&- (M_2M_5\Lambda_2)(M_1M_5\Lambda_2)\{(1+M_3\Lambda_2)(1-M_3M_5\Lambda_2)(1+M_2M_5\Lambda_2)(1-M_1M_5\Lambda_2)\}^{-1} \\
&+ (M_1M_5\Lambda_2)(M_2M_4\Lambda_2)\{(1+M_3\Lambda_2)(1+M_1M_4\Lambda_2)(1-M_1M_5\Lambda_2)(1-M_2M_4\Lambda_2)\}^{-1} \\
&- (M_2M_5\Lambda_2)(M_1M_5\Lambda_2)(M_2M_4\Lambda_2)\{(1+M_3\Lambda_2)(1+M_2M_5\Lambda_2)(1-M_1M_5\Lambda_2)(1-M_2M_4\Lambda_2)\}^{-1} \\
&+ (M_1M_5\Lambda_2)(M_1M_3M_5\Lambda_3^2)\{(1-M_1M_3\Lambda_2)(1-M_3M_5\Lambda_2)(1-M_1M_5\Lambda_2)(1-M_1M_3M_5\Lambda_3^2)\}^{-1} \qquad (3.56) \\
&+ (M_1M_4\Lambda_3^2)(M_1M_5\Lambda_2)\{(1-M_1M_3\Lambda_2)(1-M_1M_4\Lambda_3^2)(1+M_1M_4\Lambda_2)(1-M_1M_5\Lambda_2)\}^{-1} \\
&+ (M_1M_4\Lambda_3^2)(M_1M_5\Lambda_2)(M_1M_3M_5\Lambda_3^2)\{(1-M_1M_3\Lambda_2)(1-M_1M_4\Lambda_3^2)(1-M_1M_5\Lambda_2)(1-M_1M_3M_5\Lambda_3^2)\}^{-1} \\
&+ -(M_2M_5\Lambda_3^2)(M_1M_5\Lambda_2)\{(1-M_3M_5\Lambda_2)(1-M_2M_5\Lambda_3^2)(1+M_2M_5\Lambda_2)(1-M_1M_5\Lambda_2)\}^{-1} \\
&+ (M_2M_5\Lambda_3^2)(M_1M_5\Lambda_2)(M_1M_3M_5\Lambda_3^2)\{(1-M_3M_5\Lambda_2)(1-M_2M_5\Lambda_3^2)(1-M_1M_5\Lambda_2)(1-M_1M_3M_5\Lambda_3^2)\}^{-1} \\
&- (M_1M_5\Lambda_2)(M_2M_4\Lambda_3^2)\{(1-M_1M_4\Lambda_3^2)(1-M_2M_5\Lambda_3^2)(1-M_1M_5\Lambda_2)(1+M_2M_4\Lambda_3^2)\}^{-1} \\
&+ (M_1M_4\Lambda_3^2)(M_2M_5\Lambda_3^2)(M_1M_5\Lambda_2)\{(1-M_1M_4\Lambda_3^2)(1-M_2M_5\Lambda_3^2)(1-M_1M_5\Lambda_2)(1-M_1M_3M_5\Lambda_3^2)\}^{-1} \\
&- (M_1M_4\Lambda_3^2)(M_2M_5\Lambda_3^2)(M_2M_4\Lambda_3^2)\{(1-M_1M_4\Lambda_3^2)(1-M_2M_5\Lambda_3^2)(1-M_2M_4\Lambda_1\Lambda_3^2)(1+M_2M_4\Lambda_3^2)\}^{-1} \\
&+ (M_1M_4\Lambda_2)(M_1M_5\Lambda_2)(M_2M_4\Lambda_3^2)\{(1-M_1M_4\Lambda_3^2)(1+M_1M_4\Lambda_2)(1+M_1M_5\Lambda_2)(1+M_2M_4\Lambda_3^2)\}^{-1} \\
&+ (M_2M_5\Lambda_2)(M_1M_5\Lambda_2)(M_2M_4\Lambda_3^2)\{(1-M_2M_5\Lambda_3^2)(1+M_2M_5\Lambda_2)(1-M_1M_5\Lambda_2)(1+M_2M_4\Lambda_3^2)\}^{-1} \\
&- (M_1M_5\Lambda_2)(M_2M_4\Lambda_2)(M_2M_4\Lambda_3^2)\{(1+M_1M_4\Lambda_2)(1-M_1M_5\Lambda_2)(1-M_2M_4\Lambda_2)(1+M_2M_4\Lambda_3^2)\}^{-1} \\
&+ (M_2M_5\Lambda_2](M_1M_5\Lambda_2)(M_2M_4\Lambda_2)(M_2M_4\Lambda_3^2)\{(1+M_2M_5\Lambda_2)(1-M_1M_5\Lambda_2)(1-M_2M_4\Lambda_2)(1+M_2M_4\Lambda_3^2)\}^{-1}\}
\end{aligned}
$$

The GF's giving the branching rules for the subjoining of a group to another one contains, contrary to the other types of GF's so far discussed, negative signs in the numerator and positive signs in the denominator; when interpreted in terms of an integrity basis we therefore get elements with negative signs (the sign of the product of elementary factors is chosen according to the usual rule for products).

The integrity basis implied by (3.56) consists of 25 elements (the notation is (p_1, p_2, p_3, p_4, p_5; a_1, a_2, a_3) where the p's are the SU(6) representation labels and the a's those of SO(7)) :

a = (10000;100),	b = - (10000;000),
a* = (00001;100),	b* = - (00001;000),
c = (01000;010),	d = (01000;000),
c* = (00010;010),	d* = (00010;000),
f = (00100;002),	g = (00100;100),
j = (10100;010),	j* = (00101;010),
k = (10010;002),	ℓ = - (10010;010),
n = (01010;102),	p = (01010;010),
e = - (01000;100),	e* = - (00010;100),
h = - (00100;010),	i = - (00100;000),
m = (10001;010),	k* = (01001;002),
ℓ* = - (01001;010),	q = - (01010;002),
r = (10101;002).	

The following products of elementary multiplets are redundant

e with any of j, k, ℓ, m and r;

e* with any of j*, k*, ℓ*, m and r;

g with any of ℓ, ℓ*, m, p and q;

h with any of k, k*, n, q and r;

j with any of k*, ℓ*, n, p and q;

j* with any of k, ℓ, n, p and q;

k with ℓ* and p; k* with ℓ and p; ℓ with ℓ*, n and r;

ℓ* with n and r; m with n; n with p and r; p with r;

q with r.

As for the SU(6)⊃ Sp(6) GF for branching rules, (3.56) has not been derived analytically; however it satisfies the dimension and second order index check up to (including) SU(6) representations of the type (11221) and all others obtained through permutations of the labels such as (22111), (21112) and so on; also the SO(7) GF for tensors derived from it has been subjected to the checks described in chapter IV.

The SU(7)⊃ SO(7) GF $K(K_1, K_3, K_5; \Lambda_1, \Lambda_2, \Lambda_3)$ is obtained by substituting (3.56) into (3.55) which gives

$$K(K_1, K_3, K_5; \Lambda_1, \Lambda_2, \Lambda_3) = (1-K_1)^{-1} J(K_1', K_3', K_3, K_5, K_5; \Lambda_1, \Lambda_2', \Lambda_3)$$

After some tedious algebra (in order to obtain a GF whose power series expansion contains only positive terms) the answer is

$$\begin{aligned} K(K_1,K_3,K_5;\Lambda_1,\Lambda_2,\Lambda_3) = \\ \{(1-K_5^2)(1-K_3^2)(1-K_1^2)(1-K_5\Lambda_2)(1-K_5^2\Lambda_1^2)(1-K_3\Lambda_3^2)(1-K_1\Lambda_1)(1-K_3^2\Lambda_2^2)\}^{-1} \\ \times \{\{(1-K_3K_5\Lambda_2)(1-K_3^2\Lambda_1^2)(1-K_1K_3\Lambda_2)(1-K_1K_3K_5\Lambda_3^2)\}^{-1}\{1+K_3^2K_5^2\Lambda_1^2\Lambda_2 \\ + K_1K_3^2\Lambda_1\Lambda_2+K_1K_3^2K_5\Lambda_2\Lambda_3^2 + K_1K_3^2K_5^2\Lambda_1\Lambda_2^2 + K_1K_3^2K_5^2\Lambda_1\Lambda_2\Lambda_3^2 + K_1K_3^3K_5\Lambda_1\Lambda_2\Lambda_3^2 \\ + K_1K_3^3K_5^2\Lambda_1^2\Lambda_2\Lambda_3^2\} + \{(1-K_3^2\Lambda_1^2)(1-K_1K_3\Lambda_2)(1-K_1K_5\Lambda_3^2)(1-K_1K_3K_5\Lambda_3^2)\}^{-1} \\ \times \{K_1K_5\Lambda_3^2 + K_1K_3K_5(\Lambda_1\Lambda_3^2 + \Lambda_2\Lambda_3^2) + K_1K_3^2K_5\Lambda_1\Lambda_2\Lambda_3^2 + K_1K_3^2K_5^2\Lambda_1^2\Lambda_3^2+ K_1^2K_3^2K_5^2\Lambda_1\Lambda_2\Lambda_3^2 \\ + K_1^2K_3^2K_5^3\Lambda_1\Lambda_2\Lambda_3^4 + K_1^2K_3^3K_5^3\Lambda_1^2\Lambda_2\Lambda_3^4\} + \{(1-K_1K_5\Lambda_3^2)(1-K_1K_3\Lambda_2)(1-K_1K_3K_5\Lambda_3^2) \\ \times (1-K_1^2K_5^2\Lambda_2^2)\}^{-1}\{K_1^2K_5^3\Lambda_1\Lambda_2\Lambda_3^2 + K_1^2K_3K_5^2\Lambda_2\Lambda_3^2 + K_1^2K_3K_5^3\Lambda_1\Lambda_2^2\Lambda_3^2 + K_1^3K_5^3\Lambda_2^2\Lambda_3^2 \\ + K_1^3K_3K_5^3\Lambda_2^3\Lambda_3^2 + K_1^3K_3K_5^4\Lambda_1\Lambda_2^2\Lambda_3^2 + K_1^3K_3^2K_5^3\Lambda_2^2\Lambda_3^4 + K_1^4K_3^2K_5^5\Lambda_1\Lambda_2^3\Lambda_3^4\} \end{aligned} \qquad (3.57)$$

$$
\begin{aligned}
&+ \{(1-K_3^2\Lambda_1^2)(1-K_1K_5\Lambda_3^2)(1-K_1K_3K_5\Lambda_3^2)(1-K_3^2K_5^2\Lambda_3^4)\}^{-1}\{K_1K_3K_5^3\Lambda_1\Lambda_3^4+K_1K_3^2K_5^3\Lambda_1\Lambda_2\Lambda_3^4 \\
&+ K_1K_3^2K_5^3\Lambda_3^6 + K_1K_3^3K_5^2\Lambda_1\Lambda_3^4 + K_1^2K_3K_5^3\Lambda_1\Lambda_3^4 + K_1K_3^3K_5^3\Lambda_1\Lambda_3^6 + K_1K_3^3K_5^3\Lambda_2\Lambda_3^6 \\
&+ K_1K_3^4K_5^2\Lambda_1\Lambda_2\Lambda_3^4 + K_1^2K_3^2K_5^3\Lambda_3^6 + K_1K_3^4K_5^3\Lambda_1\Lambda_2\Lambda_3^6 + K_1^2K_3^3K_5^3\Lambda_2\Lambda_3^6 + K_1K_3^4K_5^4\Lambda_1^2\Lambda_3^6 \\
&+ K_1^2K_3^3K_5^4\Lambda_1^2\Lambda_3^6 + K_1K_3^5K_5^4\Lambda_1^2\Lambda_2\Lambda_3^6 + K_1^2K_3^4K_5^4\Lambda_1^2\Lambda_2\Lambda_3^6 + K_1^2K_3^4K_5^5\Lambda_1\Lambda_2\Lambda_3^8\} \\
&+ \{(1-K_3K_5\Lambda_2)(1-K_1K_3K_5\Lambda_3^2)(1-K_3^2K_5^2\Lambda_3^4)(1-K_1^2K_5^2\Lambda_2^2)\}^{-1}\{K_1K_3K_5^2\Lambda_2\Lambda_3^2 \\
&+ K_1K_3^2K_5^2\Lambda_2^2\Lambda_3^2 + K_1K_3^2K_5^4\Lambda_1\Lambda_2\Lambda_3^4 + K_1^2K_3K_5^4\Lambda_1\Lambda_2^2\Lambda_3^2 + K_1^2K_3^2K_5^3\Lambda_2^2\Lambda_3^2 \\
&+ K_1K_3^3K_5^4\Lambda_1\Lambda_2^2\Lambda_3^4 + K_1^2K_3^2K_5^4\Lambda_2^2\Lambda_3^4 + K_1^2K_3^2K_5^4\Lambda_1\Lambda_2^3\Lambda_3^2 + K_1^2K_3^3K_5^4\Lambda_2^3\Lambda_3^4 \\
&+ K_1^2K_3^3K_5^5\Lambda_1\Lambda_2^2\Lambda_3^4 + K_1^3K_3^2K_5^5\Lambda_1\Lambda_2^3\Lambda_3^2 + K_1^2K_3^4K_5^5\Lambda_1\Lambda_2^3\Lambda_3^4 + K_1^3K_3^3K_5^5\Lambda_2^3\Lambda_3^4 \\
&+ K_1^2K_3^3K_5^5\Lambda_2^3\Lambda_3^6 + K_1^3K_3^4K_5^5\Lambda_2^4\Lambda_3^4 + K_1^3K_3^5K_5^7\Lambda_1\Lambda_2^4\Lambda_3^6\} \\
&+ \{(1-K_1K_5\Lambda_3^2)(1-K_1K_3K_5\Lambda_3^2)(1-K_3^2K_5^2\Lambda_3^4)(1-K_1^2K_5^2\Lambda_2^2)\}^{-1}\{K_1^2K_3K_5^3\Lambda_2\Lambda_3^4 \\
&+ K_1^2K_3^2K_5^3\Lambda_2^2\Lambda_3^4 + K_1^2K_3^2K_5^4\Lambda_1\Lambda_2\Lambda_3^4 + K_1^2K_3^2K_5^5\Lambda_1\Lambda_2\Lambda_3^6 + K_1^2K_3^3K_5^4\Lambda_2\Lambda_3^6 \\
&+ K_1^2K_3^3K_5^4\Lambda_1\Lambda_2^2\Lambda_3^4 + K_1^3K_3K_5^5\Lambda_1\Lambda_2^2\Lambda_3^4 + K_1^3K_3^2K_5^4\Lambda_2^2\Lambda_3^4 + K_1^2K_3^3K_5^5\Lambda_1\Lambda_2^2\Lambda_3^6 \qquad (3.57) \\
&+ K_1^3K_3^2K_5^5\Lambda_1\Lambda_2^3\Lambda_3^4 + K_1^2K_3^4K_5^4\Lambda_2^2\Lambda_3^6 + K_1^3K_3^2K_5^5\Lambda_2^2\Lambda_3^6 + K_1^3K_3^3K_5^4\Lambda_2^3\Lambda_3^4 \\
&+ K_1^3K_3^3K_5^5\Lambda_2^3\Lambda_3^6 + K_1^3K_3^3K_5^6\Lambda_1\Lambda_2^2\Lambda_3^6 + K_1^3K_3^4K_5^6\Lambda_1\Lambda_2^3\Lambda_3^6\} \\
&+ \{(1-K_3K_5\Lambda_2)(1-K_1K_3\Lambda_2)(1-K_1K_3K_5\Lambda_3^2)(1-K_1^2K_5^2\Lambda_2^2)\}^{-1}\{K_1K_5^2\Lambda_1\Lambda_2 + K_1^2K_5^2\Lambda_2^2 \\
&+ K_1^2K_3^2K_5^2\Lambda_2^3 + K_1^2K_3^2K_5^2\Lambda_2^2\Lambda_3^2 + K_1^2K_3^2K_5^3\Lambda_1\Lambda_2^2\Lambda_3^2 + K_1^3K_3^2K_5^3\Lambda_2^3\Lambda_3^2 + K_1^3K_3^2K_5^4\Lambda_1\Lambda_2^4 \\
&+ K_1^3K_3^2K_5^4\Lambda_1\Lambda_2^3\Lambda_3^2\} + \{(1-K_3K_5\Lambda_1\Lambda_3^2)(1-K_3^2\Lambda_1^2)(1-K_1K_5\Lambda_3^2)(1-K_3^2K_5^2\Lambda_3^4)\}^{-1} \\
&\{K_3K_5\Lambda_1\Lambda_3^2 + K_3^2K_5\Lambda_1\Lambda_3^2 + K_3^2K_5\Lambda_1\Lambda_2\Lambda_3^2 + K_3^2K_5^3\Lambda_1^2\Lambda_3^4 + K_3^3K_5\Lambda_1\Lambda_2\Lambda_3^2 \\
&+ K_1K_3^2K_5^2\Lambda_1^2\Lambda_3^4 + K_3^3K_5^3\Lambda_1^2\Lambda_3^4 + K_3^3K_5^3\Lambda_1^2\Lambda_2\Lambda_3^4 + K_1K_3^2K_5^3\Lambda_1^2\Lambda_3^4 + K_1K_3^3K_5^2\Lambda_1^2\Lambda_3^4 \\
&+ K_1K_3^3K_5^2\Lambda_1^2\Lambda_2\Lambda_3^4 + K_3^4K_5^3\Lambda_1^2\Lambda_2\Lambda_3^4 + K_1K_3^3K_5^3\Lambda_1^2\Lambda_2\Lambda_3^4 + K_1K_3^4K_5^2\Lambda_1^2\Lambda_2\Lambda_3^4 \\
&+ K_1K_3^4K_5^4\Lambda_1^3\Lambda_3^6 + K_1K_3^5K_5^4\Lambda_1^3\Lambda_2\Lambda_3^6\} + \{(1-K_3K_5\Lambda_2)(1-K_3^2\Lambda_1^2)(1-K_1K_3K_5\Lambda_3^2) \\
&\times (1-K_3^2K_5^2\Lambda_3^4)\}^{-1}\{K_3K_5^2\Lambda_1\Lambda_3^2 + K_3^2K_5^2\Lambda_1\Lambda_2\Lambda_3^2 + K_3^2K_5^2\Lambda_3^4 + K_1K_3K_5^2\Lambda_1\Lambda_3^2 \\
&+ K_1K_3^2K_5\Lambda_1\Lambda_3^2 + K_3^3K_5^2\Lambda_2\Lambda_3^4 + K_3^3K_5^2\Lambda_1\Lambda_2\Lambda_3^2 + K_1K_3^2K_5^2\Lambda_3^4 + K_3^4K_5^2\Lambda_1\Lambda_2^2\Lambda_3^2 \\
&+ K_1K_3^3K_5^2\Lambda_2\Lambda_3^4 + K_1K_3^3K_5^3\Lambda_1^2\Lambda_3^4 + K_3^4K_5^4\Lambda_1^2\Lambda_2\Lambda_3^4 + K_1K_3^4K_5^3\Lambda_1^2\Lambda_2\Lambda_3^4 \\
&+ K_3^5K_5^4\Lambda_1^2\Lambda_2^2\Lambda_3^4 + K_1K_3^4K_5^4\Lambda_1\Lambda_2\Lambda_3^6 + K_1K_3^5K_5^3\Lambda_1\Lambda_2\Lambda_3^6\}\} .
\end{aligned}
$$

Finally substituting (3.57) into (3.54) gives the desired SO(7) GF for tensors in its enveloping algebra

$$
\begin{aligned}
G\,(U,\Lambda_1,\Lambda_2,\Lambda_3) &= K\,(U^3,U^2,U;\Lambda_1,\Lambda_2,\Lambda_3) \\
&= \{(1-U^2)(1-U^4)(1-U^6)(1-U\Lambda_2)(1-U^2\Lambda_1^2)(1-U^2\Lambda_3^2)(1-U^3\Lambda_1) \\
&\times (1-U^4\Lambda_2^2)\}^{-1}\ \{\{(1-U^3\Lambda_2)(1-U^4\Lambda_1^2)(1-U^5\Lambda_2)(1-U^6\Lambda_3^2)\}^{-1} \\
&\times \{1+U^6\Lambda_1^2\Lambda_2 + U^7\Lambda_1\Lambda_2 + U^8\Lambda_2\Lambda_3^2 + U^9(\Lambda_1\Lambda_2^2 + \Lambda_1\Lambda_2\Lambda_3^2) \\
&+ U^{10}\Lambda_1\Lambda_2\Lambda_3^2 + U^{11}\Lambda_1^2\Lambda_2\Lambda_3^2\} + \{(1-U^4\Lambda_1^2)(1-U^5\Lambda_2)(1-U^4\Lambda_3^2) \\
&\times (1-U^6\Lambda_3^2)\}^{-1}\ \{U^4\Lambda_3^2 + U^6(\Lambda_1\Lambda_3^2 + \Lambda_2\Lambda_3^2) + U^8\Lambda_1\Lambda_2\Lambda_3^2 \\
&+ U^9\Lambda_1^2\Lambda_3^2 + U^{12}\Lambda_1\Lambda_2\Lambda_3^2 + U^{13}\Lambda_1\Lambda_2\Lambda_3^4 + U^{15}\Lambda_1^2\Lambda_2\Lambda_3^4\} \\
&+ \{(1-U^4\Lambda_3^2)(1-U^5\Lambda_2)(1-U^6\Lambda_3^2)(1-U^8\Lambda_2^2)\}^{-1} \\
&\times \{U^9\Lambda_1\Lambda_2\Lambda_3^2 + U^{10}\Lambda_2\Lambda_3^2 + U^{11}\Lambda_1\Lambda_2^2\Lambda_3^2 + U^{12}\Lambda_2^2\Lambda_3^2 \\
&+ U^{14}\Lambda_2^3\Lambda_3^2 + U^{15}\Lambda_1\Lambda_2^2\Lambda_3^2 + U^{16}\Lambda_2^2\Lambda_3^4 + U^{21}\Lambda_1\Lambda_2^3\Lambda_3^4\} \\
&+ \{(1-U^4\Lambda_1^2)(1-U^4\Lambda_3^2)(1-U^6\Lambda_3^2)(1-U^6\Lambda_3^4)\}^{-1} \\
&\times \{U^8\overline{\Lambda_1\Lambda_3^4} + U^{10}(\Lambda_1\Lambda_2\Lambda_3^4 + \Lambda_3^6) + 2U^{11}\Lambda_1\Lambda_3^4 + U^{12}(\Lambda_1\Lambda_3^6 + \Lambda_2\Lambda_3^6) \\
&+ U^{13}(\Lambda_1\Lambda_2\Lambda_3^4 + \Lambda_3^6) + U^{14}\Lambda_1\Lambda_2\Lambda_3^6 + U^{15}(\Lambda_2\Lambda_3^6 + \Lambda_1^2\Lambda_3^6) + U^{16}\Lambda_1^2\Lambda_3^6 \\
&+ U^{17}\Lambda_1^2\Lambda_2\Lambda_3^6 + U^{18}\Lambda_1^2\Lambda_2\Lambda_3^6 + U^{19}\Lambda_1\Lambda_2\Lambda_3^8\} \\
&+ \{(1-U^3\Lambda_2)(1-U^6\Lambda_3^2)(1-U^6\Lambda_3^4)(1-U^8\Lambda_2^2)\}^{-1} \qquad (3.58) \\
&\times \{U^7\Lambda_2\Lambda_3^2 + U^9\Lambda_2^2\Lambda_3^2 + U^{11}\Lambda_1\Lambda_2\Lambda_3^4 + U^{12}\Lambda_1\Lambda_2^2\Lambda_3^2 \\
&+ U^{13}(\Lambda_2^2\Lambda_3^2 + \Lambda_1\Lambda_2^2\Lambda_3^4) + U^{14}(\Lambda_2^2\Lambda_3^4 + \Lambda_1\Lambda_2^3\Lambda_3^2) + U^{16}\Lambda_2^3\Lambda_3^4 \\
&+ U^{17}\Lambda_1\Lambda_2^2\Lambda_3^4 + U^{18}\Lambda_1\Lambda_2^3\Lambda_3^2 + U^{19}\Lambda_1\Lambda_2^3\Lambda_3^4 + U^{20}\Lambda_2^3\Lambda_3^4 + U^{21}\Lambda_2^3\Lambda_3^6 \\
&+ U^{22}\Lambda_2^4\Lambda_3^4 + U^{26}\Lambda_1\Lambda_2^4\Lambda_3^6\} + \{(1-U^4\Lambda_3^2)(1-U^6\Lambda_3^2)(1-U^6\Lambda_3^4) \\
&\times (1-U^8\Lambda_2^2)\}^{-1}\{U^{11}\Lambda_2\Lambda_3^4 + U^{13}\Lambda_2^2\Lambda_3^4 + U^{14}\Lambda_1\Lambda_2\Lambda_3^4 \\
&+ U^{15}\Lambda_1\Lambda_2\Lambda_3^6 + U^{16}(\Lambda_2\Lambda_3^6 + 2\Lambda_1\Lambda_2^2\Lambda_3^4) + U^{17}(\Lambda_2^2\Lambda_3^4 + \Lambda_1\Lambda_2^2\Lambda_3^6) \\
&+ U^{18}(\Lambda_1\Lambda_2^3\Lambda_3^4 + 2\Lambda_2^2\Lambda_3^6) + U^{19}\Lambda_2^3\Lambda_3^4 + U^{20}\Lambda_2^3\Lambda_3^6 + U^{21}\Lambda_1\Lambda_2^2\Lambda_3^6 \\
&+ U^{23}\Lambda_1\Lambda_2^3\Lambda_3^6\} + \{(1-U^3\Lambda_2)(1-U^5\Lambda_2)(1-U^6\Lambda_3^2)(1-U^8\Lambda_2^2)\}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\times\{U^5\Lambda_1\Lambda_2 + U^8\Lambda_2^2 + U^{12}(\Lambda_2^3 + \Lambda_2^2\Lambda_3^2) + U^{13}\Lambda_1\Lambda_2^2\Lambda_3^2 + U^{16}\Lambda_2^3\Lambda_3^2 \\
&+ U^{17}(\Lambda_1\Lambda_2^4 + \Lambda_1\Lambda_2^3\Lambda_3^2)\} + \{(1-U^3\Lambda_1\Lambda_3^2)(1-U^4\Lambda_1^2)(1-U^4\Lambda_3^2) \\
&\times (1-U^6\Lambda_3^4))\}^{-1} \{U^3\Lambda_1\Lambda_3^2 + U^5(\Lambda_1\Lambda_3^2 + \Lambda_1\Lambda_2\Lambda_3^2) + U^7(\Lambda_1^2\Lambda_3^4 + \Lambda_1\Lambda_2\Lambda_3^2) \\
&+ U^9(2\Lambda_1^2\Lambda_3^4 + \Lambda_1^2\Lambda_2\Lambda_3^4) + U^{10}\Lambda_1^2\Lambda_3^4 + U^{11}(\Lambda_1^2\Lambda_3^4 + 2\Lambda_1^2\Lambda_2\Lambda_3^4) \\
&+ U^{12}\Lambda_1^2\Lambda_2\Lambda_3^4 + U^{13}\Lambda_1^2\Lambda_2\Lambda_3^4 + U^{15}\Lambda_1^3\Lambda_3^6 + U^{17}\Lambda_1^3\Lambda_2\Lambda_3^6\} \qquad (3.58) \\
&+ \{(1-U^3\Lambda_2)(1-U^4\Lambda_1^2)(1-U^6\Lambda_3^2)(1-U^6\Lambda_3^4))\}^{-1} \\
&\times \{U^4\Lambda_1\Lambda_3^2 + U^6(\Lambda_1\Lambda_2\Lambda_3^2 + \Lambda_3^4) + U^7\Lambda_1\Lambda_3^2 \\
&+ U^8(\Lambda_1\Lambda_3^2 + \Lambda_2\Lambda_3^4 + \Lambda_1\Lambda_2\Lambda_3^2) + U^9\Lambda_3^4 + U^{10}\Lambda_1\Lambda_2^2\Lambda_3^2 + U^{11}\Lambda_2\Lambda_3^4 \\
&+ U^{12}(\Lambda_1^2\Lambda_3^4 + \Lambda_1^2\Lambda_2\Lambda_3^4) + U^{14}(\Lambda_1^2\Lambda_2\Lambda_3^4 + \Lambda_1^2\Lambda_2^2\Lambda_3^4) \\
&+ U^{15}\Lambda_1\Lambda_2\Lambda_3^6 + U^{16}\Lambda_1\Lambda_2\Lambda_3^6\}\} \ .
\end{aligned}
$$

3.4 A computer program for the elementary multiplet method

As discussed in appendix A, the elementary multiplet method consists of finding a finite set of elementary factors and syzygies. Unfortunately the problems are not always as easy as the example given in the appendix; actually the search for elementary multiplets and syzygies may turn out in certain cases to be quite a formidable problem. The GF's for tensors in the enveloping algebra of Sp(6) and SO(7), which were obtained following this approach, are good examples. For instance in the case of SO(7), we had to work out the GF of SU(6) > SO(7); this implied finding 25 elementary multiplets and 44 syzygies. What makes these problems difficult is that there is much guess work involved (no selection rules are known); for example, our choice of a forbidden product of two elementary factors at a low dimensional SU(6) representation (in the case of SU(6) > SO(7)) proved to be wrong only at the (12211) representation whose dimension is 672,000 and where there are over

2600 possible products giving (12211); one must therefore keep track of all possible products.

We wrote a computer program that does most of the work leaving to the user the problem of guessing the right elementary factors and syzygies. The program could be used at the very beginning of the search of elementary factors but usually those belonging to representations such as $(\lambda_1, 0, \ldots, 0)$, $(0, \lambda_2, 0, \ldots 0)$, ..., $(0, 0, \ldots, \lambda_\ell)$ are easily work out by hand so that the results are then fed in the program as data along with the syzygies; we then proceed to representations where more then one Cartan label is different from zero; the program gives a listing of all possible products for a given representation (of SU(6) in the case of SU(6) > SO(7)), their composition (in terms of elementary factors), what subgroup (or subjoining) representation a particular product gives, its dimension and second order index, indicates if the product is forbidden and finally at the end of the listing adds the dimension and second order index of all allowed products. If a particular choice of syzygy turns out to be wrong, a new choice is made by changing only one line of the program; if an extra elementary multiplet is needed to balance dimension and index, one card is added to the data. The second order index check is particularly useful in problems as complex as the SO(6) and SO(7) ones, since it may happen that a choice of elementary multiplet and syzygy satisfies to total dimension but fails the index check.

3.5 Other uses of the generating function for tensors in the enveloping algebra of a group

Apart from its primary purpose, that is to decompose the enveloping algebra of a group G, these GF's have other uses. We have seen that they suggest a means of constructing these tensors (a detailed discussion of this topic is given in chapter V). Without the denominator factors which correspond to Casimir invariants, and with U=1, it is a GF for the number of states of zero weight in representations of G. We turn to the question of subgroup scalars in the enveloping algebra of a group. Besides the Casimir operators of a group and subgroup, there are $r_G - \ell_G - r_H - \ell_H$ functionally independent subgroup scalars, or missing label operators (twice the number actually needed to resolve the labelling problem[72]); r_G, r_H, ℓ_G, ℓ_H are the order and rank of group and subgroup. The GF G (U; Λ_1, Λ_2, ...) for tensors in the enveloping algebra contains information about subgroup scalars. Substitute into G the GF F(Λ_1, Λ_2, ...) for subgroup scalars in representations of the group; there results the GF H(U) for subgroup scalars in the enveloping algebra. This substitution is often very simple to make.

As a first example consider SU(3)$\supset$SO(3); the GF for SO(3) scalars in the SU(3) representations is

$$F(\Lambda_1, \Lambda_2) = \{(1-\Lambda_1^2)(1-\Lambda_2^2)\}^{-1} \tag{3.59}$$

obtained by setting equal to zero the dummy which carries the SO(3) representation label in the SU(3)$\supset$ SO(3) branching rules GF given earlier in this chapter. (3.59) states that each even-even representation of SU(3) contains one SO(3) scalar. Substitution of (3.59) into (3.28) means keeping the part of (3.28) even in Λ_1 and in Λ_2 and then setting $\Lambda_1=\Lambda_2=1$. The result is

$$H(U) = (1+U^{6})\{(1-U^{2})^{2}(1-U^{3})^{2}(1-U^{4})\}^{-1} \quad (3.60)$$

which agrees with the GF of Judd et al[1] when their dummy variables D and P are set equal to U.

Similarly, the GF for SU(3) scalars in the G_2 enveloping algebra is obtained from (3.50) by setting $\Lambda_2 = 0$, $\Lambda_1 = 1$; that for SU(2) x SU(2) scalars in the SU(4) enveloping algebra is obtained from (3.39) by keeping the part even in Λ_1, Λ_2 and Λ_3 and setting $\Lambda_1 = \Lambda_2 = \Lambda_3 = 1$; that for SU(2) x U(1) scalars in the SO(5) enveloping algebra is found from (3.39) by keeping the part even in Λ_1 and in Λ_2 and setting $\Lambda_1 = \Lambda_2 = 1$; that for G_2 scalars in the SO(7) enveloping algebra is obtained from (3.58) by setting $\Lambda_1 = \Lambda_2 = 0$, $\Lambda_3 = 1$. The resulting GF's can be compared with the GF's or integrity basis given, variously, by Quesne[2], Sharp[3] and Gaskell et al[49]. Many new GF for subgroup scalars could be found in this way. Of particular interest, because of its importance in nuclear physics[39], is the group-subgroup $Sp(6) \supset SU(3) \times U(1)$; the GF for subgroup scalars is obtained from (3.53) by retaining the part even in Λ_1, in Λ_2 and in Λ_3 and setting $\Lambda_1 = \Lambda_2 = \Lambda_3 = 1$.

CHAPTER IV

TESTING THE RESULTS

In chapter III we discussed two methods of obtaining the GF for tensors in the enveloping algebra of a group. The first (working through a subgroup) is an analytical derivation and constitutes a rigourous mathematical proof; this approach was used for SU(3) and SO(5). The second approach, which makes use of a larger group, involves finding a set of elementary multiplets (the integrity basis) and relations among them (syzygies). It does not constitute a proof since one is not sure that all elementary multiplets and relations have been found; in this case the result must be checked. In this chapter, we discuss several checks to which our results have been subjected.

The number of labels needed to specify a particular term in the enveloping algebra is r, the order (number of generators) of the group. Subtracting $\frac{1}{2}(r-\ell)$, the number of internal group labels we get $\frac{1}{2}(r+\ell)$ as the number of functionally independent elementary tensors (ℓ is the rank of the group); $\frac{1}{2}(r+\ell)$ is thus the (maximum) number of denominator factors in each term of the GF; our GF's satisfy this requirement.

Okubo[5] has proven the following results concerning the number of linearly independent vector operators and their degrees d in the enveloping algebras of A_ℓ, B_ℓ, C_ℓ, D_ℓ and G_2 : A_ℓ has ℓ vector operators with $1 \leq d \leq \ell$; C_ℓ and B_ℓ have ℓ vector operators with $d = 2q + 1$ and $0 \leq q \leq \ell-1$; D_ℓ has ℓ vector operators of which $\ell-1$ have degrees $1,3,\ldots,2\ell-3$ and one with degree $\ell-1$; G_2 has two vector operators of degrees 1 and 5. Our results agree with the above. Our results also agree with (1.1) and (1.2).

One can also check if the tensors enumerated in these GF's have the correct congruence number c. The congruence numbers for these groups are[73] (G_2 isn't characterized by such a number)

$$\begin{aligned} &SU(2) : c = \lambda_1 \bmod 2;\ SU(3) : c = \lambda_1 + 2\lambda_2 \bmod 3; \\ &SU(4) : c = \lambda_1 + 2\lambda_2 + 3\lambda_3 \bmod 4;\ SO(5) : c = \lambda_1 \bmod 2; \\ &SO(7) : c = \lambda_3 \bmod 2;\ Sp(6) : c = \lambda_1 + \lambda_3 \bmod 2; \end{aligned} \tag{4.1}$$

λ_i being the Cartan labels. Now the congruence number of the adjoint representation of these groups is according to (4.1)

$$\begin{aligned} &SU(2) : c = 0 \bmod 2;\ SU(3) : c = 0 \bmod 3; \\ &SU(4) : c = 0 \bmod 4;\ SO(5),\ SO(7),\ Sp(6) : c = 0 \bmod 2. \end{aligned}$$

Consequently, since c is additive in the tensor product of representations, the Cartan labels of the tensors enumerated in these GF's must satisfy the following rules :

$$\begin{aligned} &SU(2) : \lambda \text{ even}; \\ &SU(3) : \lambda_1 + 2\lambda_2 = 0 \bmod 3;\ SU(4) : \lambda_1 + 2\lambda_2 + 3\lambda_3 = 0 \bmod 4; \\ &SO(5) : \lambda_1 \text{ even};\ SO(7) : \lambda_3 \text{ even};\ Sp(6) : \lambda_1 + \lambda_3 \text{ even}. \end{aligned}$$

Our results agree with the above rules.

We also checked if our GF's satisfy Kostant's theorem[21] concerning the highest degree with which any tensor appears; as stated in the introduction, Kostant showed that the highest degree of a λ-tensor (modulo multiplying it by Casimirs) is the sum of the coefficients of the simple roots in the highest weight of (λ). Now the highest weight $\bar{\Lambda}(\lambda)$ of an irreducible representation (λ) may be written in terms of the highest weights $\bar{w}_i$ of the ℓ fundamental irreducible representations of the group :

$$\bar{\Lambda}(\lambda) = \sum_{i=1}^{\ell} \lambda_i \bar{w}_i \tag{4.2}$$

where λ_i are the Cartan labels. The highest weights $\bar{w}_i$ for the algebras of rank two and three are written in terms of their simple roots as follows[74]

$$\begin{aligned}
A_2 &: \bar{w}_1 = \frac{1}{3}(2\bar{\alpha}_1 + \bar{\alpha}_2), \quad \bar{w}_2 = \frac{1}{3}(\bar{\alpha}_1 + 2\bar{\alpha}_2);\\
B_2 &: \bar{w}_1 = \bar{\alpha}_1 + \bar{\alpha}_2, \quad \bar{w}_2 = \frac{1}{2}\bar{\alpha}_1 + \bar{\alpha}_2;\\
G_2 &: \bar{w}_1 = 2\bar{\alpha}_1 + 3\bar{\alpha}_2; \quad \bar{w}_2 = \bar{\alpha}_1 + 2\bar{\alpha}_2;\\
A_3 &: \bar{w}_1 = \frac{1}{4}(3\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3), \quad \bar{w}_2 = \frac{1}{2}(\bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3),\\
&\quad \bar{w}_3 = \frac{1}{4}(\bar{\alpha}_1 + 2\bar{\alpha}_2 + 3\bar{\alpha}_3);\\
B_3 &: \bar{w}_1 = \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3, \quad \bar{w}_2 = \bar{\alpha}_1 + 2\bar{\alpha}_2 + 2\bar{\alpha}_3,\\
&\quad \bar{w}_3 = \frac{1}{2}(\bar{\alpha}_1 + 2\bar{\alpha}_2 + 3\bar{\alpha}_3);\\
C_3 &: \bar{w}_1 = \bar{\alpha}_1 + \bar{\alpha}_2 + \frac{1}{2}\bar{\alpha}_3, \quad \bar{w}_2 = \bar{\alpha}_1 + 2\bar{\alpha}_2 + \bar{\alpha}_3,\\
&\quad \bar{w}_3 = \bar{\alpha}_1 + 2\bar{\alpha}_2 + \frac{3\bar{\alpha}_3}{2}.
\end{aligned} \tag{4.3}$$

In terms of simple roots

$$\bar{\Lambda}(\lambda) = \sum_{i=1}^{\ell} c_i(\lambda)\,\bar{\alpha}_i \quad . \tag{4.4}$$

According to Kostant's theorem, the highest degree $D(\lambda)$ of a λ-tensor is

$$D(\lambda) = \sum_{i=1}^{\ell} c_i(\lambda) \quad . \tag{4.5}$$

Therefore for any given algebra, we substitute the $\bar{w}_i$ given in (4.3) into (4.2) and read off the $c_i(\lambda)$; we then obtain $D(\lambda)$ by means of (4.5). For example let us consider the algebra C_3 which corresponds to the group Sp(6); substituting (4.3) into (4.2) we get

$$\begin{aligned} \bar{H}(\lambda) &= \lambda_1 \bar{w}_1 + \lambda_2 \bar{w}_2 + \lambda_3 \bar{w}_3 \\ &= (\lambda_1+\lambda_2+\lambda_3)\,\bar{\alpha}_1 + (\lambda_1+2\lambda_2+2\lambda_3)\,\bar{\alpha}_2 \\ &\quad + \left(\frac{\lambda_1}{2} + \lambda_2 + \frac{3}{2}\lambda_3\right)\bar{\alpha}_3 \end{aligned}$$

which implies that for Sp(6)

$$\begin{aligned} &c_1(\lambda) = \lambda_1+\lambda_2+\lambda_3, \quad c_2(\lambda) = \lambda_1+2\lambda_2+2\lambda_3, \\ &c_3(\lambda) = \frac{\lambda_1}{2} + \lambda_2 + \frac{3}{2}\lambda_3. \end{aligned} \tag{4.6}$$

Inserting (4.6) into (4.5) gives us the highest degree $D(\lambda)$ at which a λ-tensor appears in the enveloping algebra of Sp(6)

$$D(\lambda) = \frac{5}{2}\lambda_1 + 4\lambda_2 + \frac{9}{2}\lambda_3.$$

The highest degree for each group is obtained in the same way

$$\begin{aligned} &SU(3) : \lambda_1 + \lambda_2, \quad SO(5) : \frac{3}{2}\lambda_1 + 2\lambda_2, \\ &G_2 : 3\lambda_1 + 5\lambda_2, \quad SU(4) : \frac{3}{2}\lambda_1 + 2\lambda_2 + \frac{3}{2}\lambda_3, \\ &Sp(6) : \frac{5}{2}\lambda_1 + 4\lambda_2 + \frac{9}{2}\lambda_3, \quad SO(7) : 3\lambda_1 + 5\lambda_2 + 3\lambda_3. \end{aligned} \tag{4.7}$$

Keeping only terms of highest degree for a given λ-tensor, the GF for tensors in the enveloping algebra reduces to the following

$$\begin{aligned}
SU(3) &: G'(U;\Lambda_1,\Lambda_2) = \frac{1+U^2\Lambda_1\Lambda_2 + U^4\Lambda_1^2\Lambda_2^2}{(1-U^3\Lambda_1^3)(1-U^3\Lambda_2^3)} \,, \\
SO(5) &: G'(U;\Lambda_1,\Lambda_2) = \frac{1}{(1-U^2\Lambda_2)(1-U^3\Lambda_1^2)} \,, \\
G_2 &: G'(U;\Lambda_1,\Lambda_2) = \frac{1}{(1-U^3\Lambda_1)(1-U^5\Lambda_2)} \,, \\
SU(4) &: G'(U;\Lambda_1,\Lambda_2) = \frac{1}{(1-U^3\Lambda_1\Lambda_3)(1-U^4\Lambda_2^2)} \times \\
&\times \left[\frac{1+U^5\Lambda_1^2\Lambda_2}{(1-U^6\Lambda_1^4)} + \frac{U^5\Lambda_2\Lambda_3^2 + U^6\Lambda_3^4}{(1-U^6\Lambda_3^4)} \right] \,, \\
Sp(6) &: G'(U;\Lambda_1,\Lambda_2,\Lambda_3) = \frac{1 + U^7\Lambda_1\Lambda_3}{(1-U^6\Lambda_1^2)(1-U^4\Lambda_2)(1-U^9\Lambda_3^2)} \,, \\
SO(7) &: G'(U;\Lambda_1,\Lambda_2,\Lambda_3) = \frac{1}{(1-U^3\Lambda_1)(1-U^5\Lambda_2)(1-U^6\Lambda_3^2)} \,.
\end{aligned} \tag{4.8}$$

By inspection we see that (4.8) does satisfy the highest degree given in (4.7). More formally we may obtain the formula for the highest degree by taking the following residues

$$\Sigma \{ \operatorname{Res}_{\Lambda_1,\ldots,\Lambda_\ell} \left(\prod_{i=1}^{\ell} \Lambda_i^{\lambda_i - 1} \right) G\,(U;\Lambda_1^{-1},\ldots \Lambda_\ell^{-1}) \}.$$

A simple example is G_2

$$\Sigma \operatorname{Res}_{\Lambda_1\Lambda_2} \frac{\Lambda_1^{\lambda_1-1}\Lambda_2^{\lambda_2-1}}{(1-U^3\Lambda_1^{-1})(1-U^5\Lambda_2^{-1})} = U^{3\lambda_1+5\lambda_2}$$

which checks with the result stated in (4.7).

Each GF is consistent with known GF's for subgroup scalars in the enveloping algebra. For example, let us consider the chain $SU(4) \supset SU(2) \times SU(2)$. The integrity basis for the branching rules of the above chain of groups indicates that only SU(4) representations with even Cartan labels contains SU(2) x SU(2) scalars; therefore, retaining only such terms in (3.39) and setting $\Lambda_1=\Lambda_2=\Lambda_3=1$ we get the GF for SU(2) x SU(2) scalars in the enveloping algebra of SU(4)

$$\{(1-U^2)^3(1-U^3)^2(1-U^4)^3(1-U^6)\}^{-1} \times \{1+U^4+U^5+3U^6+2U^7+2U^8+3U^9 +U^{10}+U^{11}+U^{15}\} ,$$

where U carries the degree in the generators as exponent. This GF agrees with that of Gaskell et al[49] and of Quesne[2]. Similar tests have been done using the following chain of groups : $G_2 \supset SU(3)$, $G_2 \supset SU(2) \times SU(2)$, $Sp(6) \supset Sp(4) \times SU(2)$, $SO(7) \supset G_2$.

Since the elementary multiplet method involves a great deal of guessing, the above checks were often helpful in guiding our choices of the integrity basis and syzygies; however, they constitute only partial tests. The most conclusive check which we apply, is the reduction of the GF for tensors in the enveloping algebra to the corresponding GF for weights, to which the remainder of this chapter is devoted.

A GF for tensors may be reduced to that for weights by substituting the character generator of the group; unfortunately the character generator is not known in general, so one must work through a chain of subgroups. One must utilize a chain of groups of equal rank at each stage, otherwise information is lost. For example, working through the chain $SU(3) \supset SO(3) \supset U(1)$ the GF for tensors based on a (10) tensor reduces to the following GF for weights

$$\frac{1}{(1-U)(1-U\eta)(1-U\eta^{-1})} \qquad (4.9)$$

where U carries the degree and η the weight as exponent. However, this cannot be considered a conclusive check since the GF based on a (01) tensor would also reduce to (4.9). An appropriate chain would be

$$SU(3) \supset SU(2) \times U(1) \supset U(1) \times U(1)$$

in which case, the GF's based on (10) and (01) tensors reduces respectively to

$$\frac{1}{(1-U\eta_1\eta_2)(1-U\eta_2^{-2})(1-U\eta_2\eta_1^{-1})}$$

and

$$\frac{1}{(1-U\eta_2^2)(1-U\eta_1\eta_2^{-1})(1-U\eta_1^{-1}\eta_2^{-1})} .$$

We now proceed to the reduction of the GF's of SU(4), G_2, Sp(6) and SO(7).

For the group SU(4), we proceed through the following chain

$$SU(4) \supset SU(3) \times U(1) \supset SU(2) \times U(1) \times U(1) \supset U(1) \times U(1) \times U(1).$$

The GF for the branching rules of $SU(4) \supset SU(3) \times U(1)$ is[50]

$$\frac{1}{(1-\Lambda_1 N_1\eta_3)(1-\Lambda_1\eta_3^{-3})(1-\Lambda_2 N_2\eta_3^2)(1-\Lambda_2 N_1\eta_3^{-2})(1-\Lambda_3\eta_3^3)(1-\Lambda_3 N_2\eta_3^{-1})} \tag{4.10}$$

where $\Lambda_1,\Lambda_2,\Lambda_3$ carry the SU(4) representation labels, N_1 and N_2 those of SU(3) and η_3 that of U(1). The first stage of reduction is done by substituting (4.10) into (3.39); this is done by the following sum of residues

$$F_1(U;N_1,N_2,\eta_3) = \sum \underset{\Lambda_1\Lambda_2\Lambda_3}{\mathrm{Res}} \left[\frac{G(U;\Lambda_1,\Lambda_2,\Lambda_3)\Lambda_1^{-1}\Lambda_2^{-1}\Lambda_3^{-1}}{(1-\Lambda_1^{-1}N_1\eta_3)(1-\Lambda_1^{-1}\eta_3^{-3})(1-\Lambda_2^{-1}N_2\eta_3^2)} \cdot \frac{1}{(1-\Lambda_2^{-1}N_1\eta_3^{-2})(1-\Lambda_3^{-1}\eta_3^3)(1-\Lambda_3^{-1}N_2\eta_3^{-1})}\right].$$

The calculation of residues is done with the following choice of norms : $|U|<1, |N_1|<1, |N_2|<1, |\Lambda_2| = |\eta_3| = 1$, $|\Lambda_3|$ and $|\Lambda_1|$ a little greater then unity; the result is the GF for SU(3) x U(1) tensors in the enveloping algebra of SU(4)

$$\begin{aligned}F_1(U;N_1,N_2,\eta_3) = &\{(\eta_3^{-3}-N_1\eta_3)(N_2\eta_3^2-N_1\eta_3^{-2})(\eta_3^3-N_2\eta_3^{-1})\}^{-1}\\ &\times\{G(U;\eta_3^{-3},N_2\eta_3^2,\eta_3^3)N_2\eta_3^2-G(U;\eta_3^{-3},N_2\eta_3^2,N_2\eta_3^{-1})N_2^2\eta_3^{-2}\\ &-G(U;\eta_3^{-3},N_1\eta_3^{-2},\eta_3^3)N_1\eta_3^{-2}+G(U;\eta_3^{-3},N_1\eta_3^{-2},N_2\eta_3^{-1})N_1N_2\eta_3^{-6}\\ &-G(U;N_1\eta_3,N_2\eta_3^2,\eta_3^3)N_1N_2\eta_3^6+G(U;N_1\eta_3,N_2\eta_3^2,N_2\eta_3^{-1})N_1N_2^2\eta_3^2\\ &+G(U;N_1\eta_3,N_1\eta_3^{-2},\eta_3^3)N_1^2\eta_3^3-G(U;N_1\eta_3,N_1\eta_3^{-2},N_2\eta_3^{-1})N_1^2N_2\eta_3^{-2}\}\end{aligned} \tag{4.11}$$

Making use of the GF (3.3) for the branching rules of SU(3)⊃ SU(2) x U(1) the reduction from SU(3) x U(1) to SU(2) x U(1) x U(1) is done through the following sum of residues

$$F_2(U;N_3,\eta_2,\eta_3) = \sum \underset{N_1N_2}{\mathrm{Res}}\left[\frac{F_1(U;N_1,N_2,\eta_3)N_1^{-1}N_2^{-1}}{(1-N_1^{-1}N_3\eta_2)(1-N_1^{-1}\eta_2^{-2})(1-N_2^{-1}N_3\eta_2^{-1})} \times \frac{1}{(1-N_2^{-1}\eta_2^2)}\right]$$

where N_3 carries the SU(2) label and η_2 that of U(1). The sum of residues is done with the following choice of norms : $|N_3|<1, |\eta_2| = 1$ and $|N_1|, |N_2|$ slightly larger then one; the result is

$$\begin{aligned}F_2(U;N_3,\eta_2,\eta_3) = &\{(N_3-\eta_2^3)(N_3-\eta_2^{-3})\}^{-1}\times\{F_1(U;N_3\eta_2,N_3\eta_2^{-1},\eta_3)N_3^3\\ &-F_1(U;N_3\eta_2,\eta_2^2,\eta_3)N_3\eta_2^3-F_1(U;\eta_2^{-2},N_3\eta_2^{-1},\eta_3)N_3\eta_2^{-3}\\ &+F_1(U;\eta_2^{-2},\eta_2^2,\eta_3)\}\,.\end{aligned} \tag{4.12}$$

Finally in the last stage of reduction we make use of the SU(2)⊃ U(1) GF[50]

$$\{(1-N_3\eta_1)(1-N_3\eta_1^{-1})\}^{-1}\,. \tag{4.13}$$

The GFfor weights is

$$W(U;\eta_1,\eta_2,\eta_3) = \sum \underset{N_3}{\mathrm{Res}}\,\frac{F_2(U;N_3,\eta_2,\eta_3)N_3^{-1}}{(1-N_3^{-1}\eta_1)(1-N_3^{-1}\eta_1^{-1})}\,. \tag{4.14}$$

where η_1 carries the $U(1)$ label. With the following choice of norms $|\eta_1| = 1$ and $|N_3|$ slightly greater then unity, (4.14) reduces to

$$W(U;\eta_1,\eta_2,\eta_3) = (\eta_1-\eta_1^{-1})^{-1} \{\eta_1\ F_2(U;\eta_1,\eta_2,\eta_3)-\eta_1^{-1}\ F_2(U;\eta_1^{-1},\eta_2,\eta_3)\}. \quad (4.15)$$

In principle an explicit expression for the weight GF $W(U;\eta_1,\eta_2,\eta_3)$ could be obtained analytically by substituting (3.39) into (4.11), the result into (4.12) and finally that result into (4.15), and after some algebra compare the answer with the actual weight GF

$$\begin{aligned} W'(U;\eta_1,\eta_2,\eta_3) = \ & (1-U\eta_1\eta_2\eta_3^4)(1-U\eta_1^{-1}\eta_2\eta_3^4)(1-U\eta_2^{-2}\eta_3^4)(1-U\eta_2^2\eta_3^{-4}) \\ & \times (1-U\eta_1\eta_2^{-1}\eta_3^{-4})(1-U_1^{-1}\eta_2^{+1}\eta_3^{-4})(1-U\eta_1\eta_2^3)(1-U\eta_1^{-1}\eta_2^3)(1-U\eta_1^2)(1-U)^3 \\ & \times (1-U\eta_1^{-2})(1-U\eta_1\eta_2^{-3})(1-U\eta_1^{-1}\eta_2^{-3})\}^{-1}. \end{aligned} \quad (4.16)$$

Each factor in (4.16) corresponds to a weight in the adjoint representation of $SU(4)$. In practice, the algebra soon gets out of hand. We have written a computer program which performs the substitutions numerically in double precision (15 significant figures) and compares the result with (4.16). In table I we show a sample of the results for random values of U,η_1,η_2,η_3; W' stands for the actual GF and W for the weitht GF obtained from the reduction process. $\Delta = (W-W')\,(W')^{-1} \times 100$ is the percentage of error. To check the efficacy of the numerical comparison, we made numerical changes in the GF being tested, such as altering by unity a coefficient or exponent; such a change increases the relative error by many orders of magnitude. The effect of such minimal changes are shown in table II and III. Many more were done.

In the case of G_2, we proceed through the chain

$$G_2 \supset SU(3) \supset SU(2) \times U(1) \supset U(1) \times U(1).$$

The substitution of the group-subgroup GF (3.15) for $G_2 \supset SU(3)$ in (3.50) gives the GF for SU(3) tensors in the enveloping algebra of G_2

$$\begin{aligned} F(U;N_1,N_2) &= \{(N_1-N_2)(N_1-1)(N_2-1)\}^{-1} \times \{ -G(U;N_1,N_1)N_1^2 \\ &+ G(U;N_1,N_1N_2)N_1^2N_2 + G(U;N_2,N_2)N_2^2 - G(U;N_2,N_1N_2)N_2^2N_1 \\ &+ G(U;1,N_1)N_1 - G(U;1,N_2)N_2\}, \end{aligned} \tag{4.17}$$

U carries the degree in the G_2 generators and N_1,N_2 the SU(3) representation labels. The remainder of the reduction is done exactly the same way as that of SU(4) with $\eta_3=1$. The final answer is then compared with the actual G_2 weight GF.

$$\begin{aligned} W'(U;\eta_1,\eta_2) &= \{(1-U\eta_1\eta_2)(1-U\eta_1^{-1}\eta_2)(1-U\eta_2^{-2})(1-U\eta_2^2)(1-U\eta_1\eta_2^{-1}) \\ &\times(1-U\eta_1^{-1}\eta_2^{-1})(1-U\eta_1\eta_2^3)(1-U\eta_1^{-1}\eta_2^3)(1-U\eta_1^2)(1-U)^2 \\ &\times (1-U\eta_1^{-2})(1-U\eta_1\eta_2^{-3})(1-U\eta_1^{-1}\eta_2^{-3})\}^{-1}. \end{aligned} \tag{4.18}$$

Proceeding through the same numerical checks as in the case of SU(4), the relative error Δ was of the order of 10^{-12}.

For Sp(6) the following chain is convenient

$$Sp(6) \supset Sp(4) \times SU(2) \supset SU(2) \times U(1) \times SU(2) \supset U(1) \times U(1) \times U(1).$$

The GF for branching rules for $Sp(6) \supset SP(4) \times SU(2)$ is[52]

$$\begin{aligned} &\{(1-\Lambda_1N_1)(1-\Lambda_1N_3)(1-\Lambda_2N_2)(1-\Lambda_2)(1-\Lambda_3N_1)(1-\Lambda_3N_2N_3)\}^{-1} \\ &\times \{(1-\Lambda_2N_1N_3)^{-1} + \Lambda_1\Lambda_3N_2\,(1-\Lambda_1\Lambda_3N_2)^{-1}\} \end{aligned} \tag{4.19}$$

where $\Lambda_1,\Lambda_2,\Lambda_3$ carry the Sp(6) labels, N_1, N_2 the Sp(4) labels and N_3 the SU(2) label (the dimension of the SU(2) representation (ν) is $\nu+1$). The substitution of (4.19) into (3.53) gives the GF for Sp(4) x SU(2) tensors in the enveloping algebra of Sp(6) t

$$
\begin{aligned}
F_1\ (U;N_1,N_2,N_3) &= N_2\ \{(1-N_2)(N_1-N_3)(N_1-N_2N_3)(N_2-N_1N_3)\}^{-1} \\
&\times \{-N_1^2N_2G(U;N_1,N_2,N_1) + N_1N_3N_2^2G\ (U;N_1,N_2,N_2N_3) \\
&\quad -N_2^2N_3^2G(U;N_3,N_2,N_2N_3) + N_1N_3\ G(U;N_3,1,N_1)\} \\
&+ N_1N_3\{(N_2-N_1N_3)(1-N_1N_3)(N_1-N_3)(N_1-N_2N_3)\}^{-1} \\
&\times \{N_1^2N_3\ G(U;N_1,N_1N_3,N_1) - N_1^2N_2N_3^2\ G(U;N_1,N_1N_3,N_2N_3) \\
&\quad - N_1^2N_3^2\ G(U;N_3,N_1N_3,N_1) + N_1N_2N_3^2\ G(U;N_3,N_1N_3,N_2N_3)\} \\
&+ \{(1-N_2)(N_1-N_3)(N_1-N_2N_3)(1-N_1N_3)\}^{-1} \times \{N_1^2\ G(U;N_1,1,N_1) \\
&\quad - N_1N_3\ G(U;N_3,1,N_1) + N_2N_3^2\ G(U;N_3,1,N_2N_3) - N_1N_2^2N_3 \\
&\quad \times G(U;N_1,N_2,N_2N_3)\} \\
&+ N_1N_2\{(1-N_2)(N_2-N_1^2)(N_2-N_1N_3)(N_1-N_2N_3)\}^{-1} \\
&\times \{N_2\ G(U;N_2N_1^{-1},1,N_1) - N_2^2\ G(U;N_1^{-1}N_2,N_2,N_1)\} \\
&+ N_3\ \{(1-N_2)(1-N_1N_3)(1-N_3^2)(N_1-N_2N_3)\}^{-1} \\
&\times \{N_2^2\ G(U;N_3^{-1},N_2,N_2N_3) - N_2\ G(U;N_3^{-1},1,N_2N_3)\} \\
&+ N_2\{(1-N_2)(N_1-N_3)(N_1-N_2N_3)(N_2-N_1^2)\}^{-1} \\
&\times \{N_2N_1^2\ G(U;N_1,N_2,N_1) - N_1^2\ G(U;N_1,1,N_1)\} \\
&+ \{(1-N_2)(N_1-N_3)(N_1-N_2N_3)(1-N_3^2)\}^{-1} \\
&\times \{N_2^2N_3^2\ G(U;N_3,N_2,N_2N_3) - N_2N_3^2\ G(U;N_3,1,N_2N_3)\}.
\end{aligned}
\tag{4.20}
$$

The GF for branching rules of $Sp(4) \supset SU(2) \times U(1)$ is[50]

$$
\begin{aligned}
&\{(1-N_1N_4\eta_3)(1-N_1N_4\eta_3^{-1})(1-N_2\eta_3^2)(1-N_2\eta_3^{-2})\}^{-1} \\
&\times \{(1-N_1^2)^{-1} + N_2N_4^2(1-N_2N_4^2)^{-1}\}
\end{aligned}
\tag{4.21}
$$

where N_4 carries the SU(2) label and η_3 the U(1) label. Substituting (4.21) into (4.20) gives us the GF for $SU(2) \times U(1) \times SU(2)$ tensors in the enveloping algebra of Sp(6)

$$\begin{aligned} F_2(U;N_3,N_4,\eta_3) &= N_4^2\{(1-\eta_3^2)(N_4^2-\eta_3^2)(1-N_4^2\eta_3^2)\}^{-1} \\ &\times\{-\eta_3^6 F_1(U;N_4\eta_3,\eta_3^2,N_3)+\eta_3^4N_4^2F_1(U;N_4\eta_3,N_4^2,N_3) \\ &\quad + F_1(U;N_4\eta_3^{-1},\eta_3^{-2},N_3)-N_4^2\eta_3^2F_1(U;N_4\eta_3^{-1},N_4^2,N_3)\} \\ &+\{2(1-N_4\eta_3)(1-N_4\eta_3^{-1})(\eta_3^2-\eta_3^{-2})\}^{-1} \\ &\times\{\eta_3^2F_1(U;1,\eta_3^2,N_3)-\eta_3^{-2}F_1(U;1,\eta_3^{-2},N_3)\} \\ &+\{2(1+N_4\eta_3)(1+N_4\eta_3^{-1})(\eta_3^2-\eta_3^{-2})\}^{-1} \\ &\times\{\eta_3^2F_1(U;-1,\eta_3^2,N_3)-\eta_3^{-2}F_1(U;-1,\eta_3^{-2},N_3)\}. \end{aligned} \tag{4.22}$$

Finally, the GF for weights is obtained by substituting the SU(2)$\supset$U(1) GF (4.13) into (4.22) giving

$$\begin{aligned} W(U;\eta_1,\eta_2,\eta_3) &= \{\eta_1\eta_2F_2(U;\eta_1,\eta_2,\eta_3)-\eta_1^{-1}\eta_2F_2(U;\eta_1^{-1},\eta_2,\eta_3) \\ &-\eta_1\eta_2^{-1}F_2(U;\eta_1,\eta_2^{-1},\eta_3)+\eta_1^{-1}\eta_2^{-1}F_2(U;\eta_1^{-1},\eta_2^{-1},\eta_3)\}\times \\ &\{(\eta_2-\eta_2^{-1})(\eta_1-\eta_1^{-1})\}^{-1} \end{aligned} \tag{4.23}$$

which must then be compared with the actual GF for weights of Sp(6)

$$\begin{aligned} W'(U;\eta_1,\eta_2,\eta_3) &= \{(1-U\eta_2^2\eta_3^2)(1-U\eta_3^2)(1-U\eta_2^{-2}\eta_3^2)(1-U\eta_2^2\eta_3^{-2}) \\ &\times(1-U\eta_2^{-2}\eta_3^{-2})(1-U\eta_3^{-2})(1-U\eta_1^2)(1-U\eta_1^{-2})(1-U\eta_2^2)(1-U\eta_2^{-2})(1-U\eta_1\eta_2\eta_3) \\ &\times(1-U\eta_1\eta_2^{-1}\eta_3)(1-U\eta_1^{-1}\eta_2\eta_3)(1-U\eta_1^{-1}\eta_2^{-1}\eta_3)(1-U\eta_1\eta_2\eta_3^{-1}) \\ &\times(1-U\eta_1\eta_2^{-1}\eta_3^{-1})(1-U\eta_1^{-1}\eta_2\eta_3^{-1})(1-U\eta_1^{-1}\eta_2^{-1}\eta_3^{-1})(1-U)^3\}^{-1}. \end{aligned} \tag{4.24}$$

Due to the complexity of the Sp(6) GF, the numerical comparison between the actual weight GF (4.24) and the one obtained from the reduction was done with quadruple precision (33 significant figures); Δ was of the order 10^{-23}.

For SO(7) we followed the chain

SO(7)$\supset$SO(6) $\sim$ SU(4)$\supset$SU(3) x U(1)$\supset$SU(2) x U(1) x U(1)$\supset$U(1) x U(1) x U(1)

The SO(7)$\supset$SO(6) GF is[52]

$$\{(1-\Lambda_1N_2)(1-\Lambda_1)(1-\Lambda_3N_1)(1-\Lambda_3N_3)(1-\Lambda_2N_1N_3)(1-\Lambda_2N_2)\}^{-1} \tag{4.25}$$

where $\Lambda_1\Lambda_2,\Lambda_3$ carry the SO(7) representation labels and N_1, N_2, N_3 those of SU(4). The substitution of (4.25) into (3.58) gives the GF for SU(4) tensors in the enveloping algebra of SO(7)

$$\begin{aligned}
F\,(U;N_1,N_2,N_3) = {} & \{(1-N_2)(N_1-N_3)(N_2-N_1N_3)\}^{-1} \\
& \times \{N_2N_3N_1^2\, G\,(U;N_2,N_1N_3,N_1) - N_1N_2^2\, G\,(U;N_2,N_2,N_1) \\
& - N_1N_2N_3^2\, G\,(U;N_2,N_1N_3,N_3) + N_2^2N_3\, G(U;N_2,N_2,N_3) \\
& - N_1^2N_3\, G\,(U;1,N_1N_3,N_1) + N_1N_2\, G\,(U;1,N_2,N_1) \\
& + N_1N_3^2\, G\,(U;1,N_1N_3,N_3) - N_2N_3\, G\,(U;1,N_2,N_3)\}
\end{aligned} \tag{4.26}$$

The remainder of the reduction is done following that of SU(4). The final answer is compared with the actual GF for weights

$$\begin{aligned}
W'(U;\eta_1,\eta_2,\eta_3) = {} & \{(1-U\eta_1\eta_2\eta_3^4)(1-U\eta_1^{-1}\eta_2\eta_3^4)(1-U\eta_2^{-2}\eta_3^4) \\
& \times (1-U\eta_2^2\eta_3^{-4})(1-U\eta_1\eta_2^{-1}\eta_3^{-4})(1-U\eta_1\eta_2^3)(1-U\eta_1^{-1}\eta_2^3)(1-U\eta_1^2)(1-U)^3 \\
& \times (1-U\eta_1^{-2})(1-U\eta_1\eta_2^{-3})(1-U\eta_1^{-1}\eta_2^{-3})(1-U\eta_1^{-1}\eta_2^{-1}\eta_3^{-4})(1-U\eta_1^{-1}\eta_2\eta_3^{-2}) \\
& \times (1-U\eta_1\eta_2\eta_3^{-2})(1-U\eta_2^{-2}\eta_3^{-2})(1-U\eta_2^2\eta_3^2)(1-U\eta_3^2\eta_1^{-1}\eta_2^{-1}) \times \\
& (1-U\eta_1\eta_2^{-1}\eta_3^2)\}^{-1}
\end{aligned}$$

Numerical checks similar to that of Sp(6) were done giving a Δ of the order of 10^{-23}..

CHAPTER V

CONSTRUCTION OF TENSORS;THEIR ACTION ON DEGENERATE REPRESENTATIONS

5.1 A method of constructing tensors in the enveloping algebra of a group

As we mentioned in chapter II, not only do the GF's for tensors in the enveloping algebra U of a group G enumerate a basis for all tensors, but they also suggest an integrity basis from which all may be obtained therefore reducing the problem of constructing tensors in U to that of constructing a finite set of low degree ones. Now since there is a one to one correspondence (see section 3.1) between the basis for symmetric tensors T (λ) in U and the basis for tensors $\Gamma(\lambda)$ whose components are polynomials in the components of a tensor Γ_A that transforms by the adjoint representation of G (polynomial tensors) the prescription we follow to construct an irreducible tensor $T(\lambda)$ in the enveloping algebra may be divided into three steps :

(1) Construct the highest component (highest weight) of its corresponding polynomial tensor $\Gamma(\lambda)$ (all other components are obtained by cranking with the appropriate generators); this component will be the product of the highest components of some of the elementary tensors.

(2) Symmetrize the expression with respect to order.

(3) In the symmetrized expression, substitute for the components of Γ_A the corresponding components of the vector operator of degree one (the basic vector operator).

Step (1) assumes that one knows the highest components of the elementary polynomial tensors; these are found by the following prescription

(1) Write the most general polynomial in the components of Γ_A with the appropriate degree and weight (highest).

(2) Find the coefficients by requiring that the generators which correspond to the simple roots annihilate the polynomial constructed in (1). It might happen that there is a composite tensor (a tensor which is the product of powers of the elementary ones) of the same degree and transformation properties (that is, it transforms by the same representation) as the elementary one being determined; there might also be two or more elementary tensors of the same degree and transformation properties as the one being determined; in all those cases, the elementary tensor being determined must be chosen (its coefficients) such that it is linearly independent of all others mentioned above.

In what follows we consider the problem of constructing the elementary tensors for the groups SU(3) and SO(5)

(a) <u>SU(3)</u>

The components of Γ_A (which is an octet) are shown in figure 2. Throughout these calculations (for SU(3) and SO(5)) the component of a tensor in U will be designated by the same symbol as that of the component of its corresponding polynomial tensor but with a subscript "op"; for instance, in the case of SU(3), the highest component of the tensor in U corresponding to the polynomial tensor Γ_A will be denoted α_{op}. All calculations for the

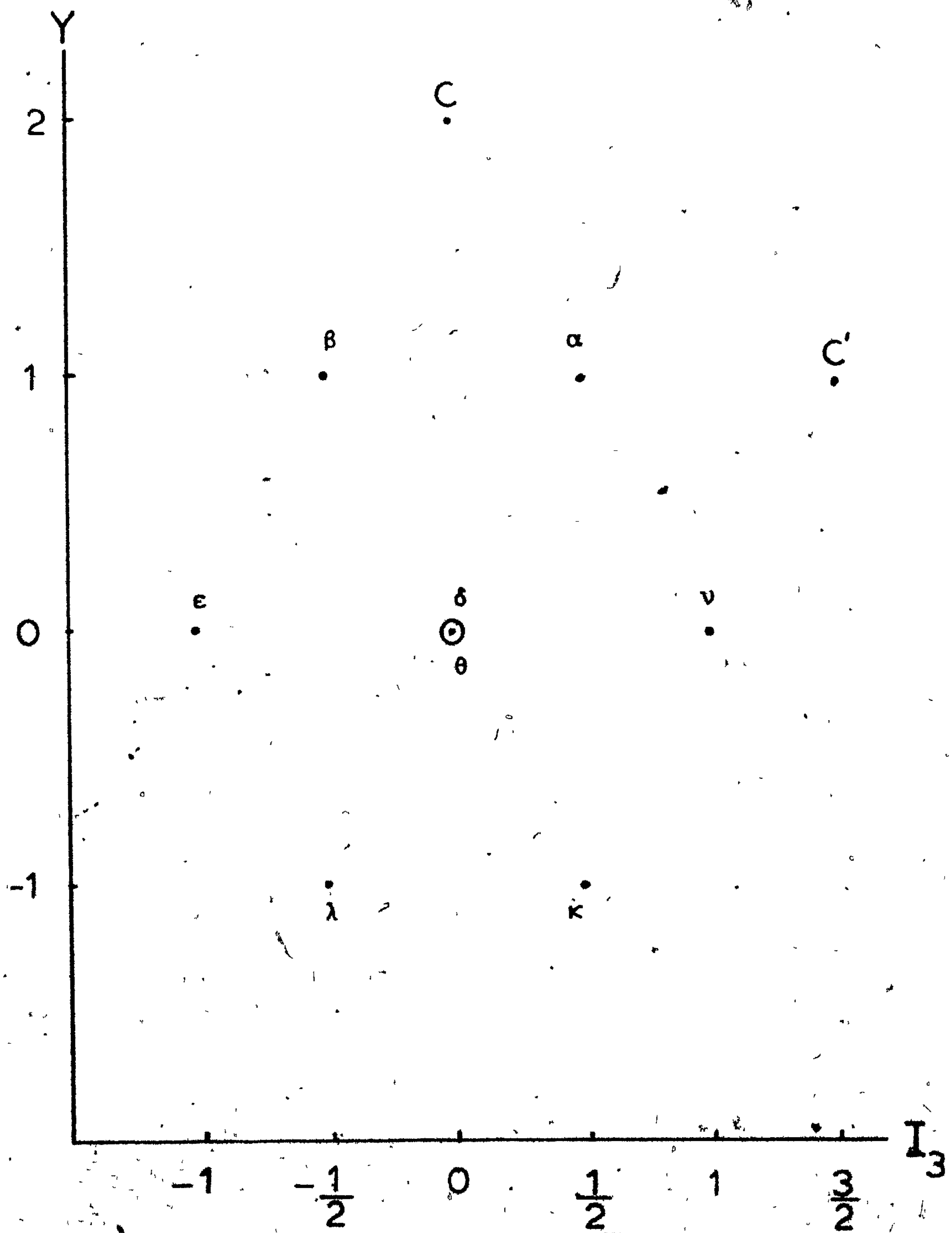


Figure 2. Components of the degree one SU(3) octet. Highest weights (I_3,Y) of the tensors (1,11) and (2,11) are situated at point α; those of (3,03) and (3,30) at points C and C′.

SU(3) group are based on Gel'fand and Zetlin's matrix elements[75]; in their notation the infinitesimal generators $E_{\mu\nu}$ of SU(3) are given by the following formula

$$E_{\mu\nu} = A_{\mu\nu} - \frac{1}{3}\delta_{\mu\nu} \sum_{\lambda=1}^{3} A_{\lambda\lambda} \qquad \mu,\nu = 1,3$$

where the $A_{\mu\nu}$ are the generators of the U(3) group satisfying the Lie commutation relations

$$[A_{\mu\nu}, A_{\alpha\beta}] = \delta_{\nu\alpha} A_{\mu\beta} - \delta_{\mu\beta} A_{\alpha\nu} \qquad (5.1)$$

with all indices assuming values from 1 to 3. The non diagonal generators are represented in figure 3. A particular realization of the generators is the following

$$E_{21} = \xi\, \partial_\eta + \eta^*\partial_{\xi^*}, \quad E_{12} = \eta\, \partial_\xi + \xi^*\partial_{\eta^*},$$
$$E_{23} = \xi\, \partial_\zeta + \zeta^*\partial_{\xi^*}, \quad E_{32} = \zeta\,\partial_\xi + \xi^*\partial_{\zeta^*},$$
$$E_{13} = \eta\, \partial_\zeta + \zeta^*\partial_{\eta^*}, \quad E_{31} = \zeta\,\partial_\eta + \eta^*\partial_{\zeta^*}, \qquad (5.2)$$

and

$$E_{11} = A_{11} - \frac{1}{3}(A_{11} + A_{22} + A_{33})$$
$$E_{22} = A_{22} - \frac{1}{3}(A_{11} + A_{22} + A_{33})$$
$$E_{33} = A_{33} - \frac{1}{3}(A_{11} + A_{22} + A_{33})$$

with

$$A_{11} = \eta\partial_\eta + \xi^*\partial_{\xi^*} + \zeta^*\partial_{\zeta^*}$$
$$A_{22} = \xi\partial_\xi + \zeta^*\partial_{\zeta^*} + \eta^*\partial_{\eta^*}$$
$$A_{33} = \zeta\partial_\zeta + \xi^*\partial_{\xi^*} + \eta^*\partial_{\eta^*} .$$

The variables η, ζ, ξ and η^*, ζ^*, ξ^* are the bases of the fundamental irreducible representations of SU(3), namely (1,0) and (0,1); these are shown

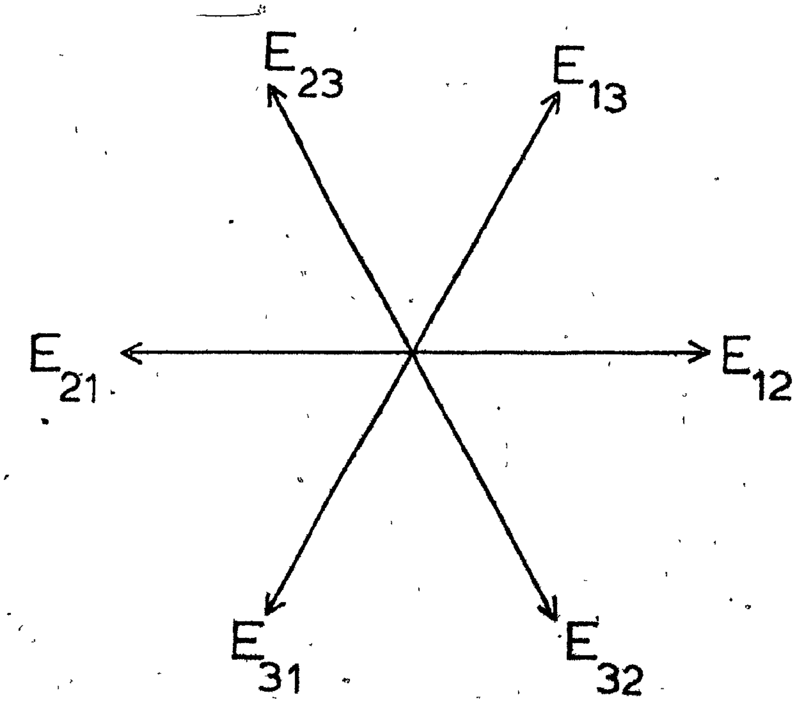


Figure 3. Non diagonal generators of SU(3).

in figure 4. The action on the components of the octet Γ_A by the generators E_{12} and E_{23} corresponding to the simple roots is best described by writing these generators as differential operators which are functions of the components of Γ_A; we have

$$\begin{aligned} E_{12} &= \alpha\partial_\beta + \sqrt{2}\,(\delta\partial_\varepsilon + \nu\partial_\delta) + \kappa\partial_\lambda \\ E_{23} &= \alpha\partial_\nu + \varepsilon\partial_\lambda + \frac{1}{\sqrt{2}}\,(\beta\partial_\delta + \delta\partial_\kappa) + \sqrt{\frac{3}{2}}\,(\beta\partial_\theta + \theta\partial_\kappa) \end{aligned} \tag{5.3}$$

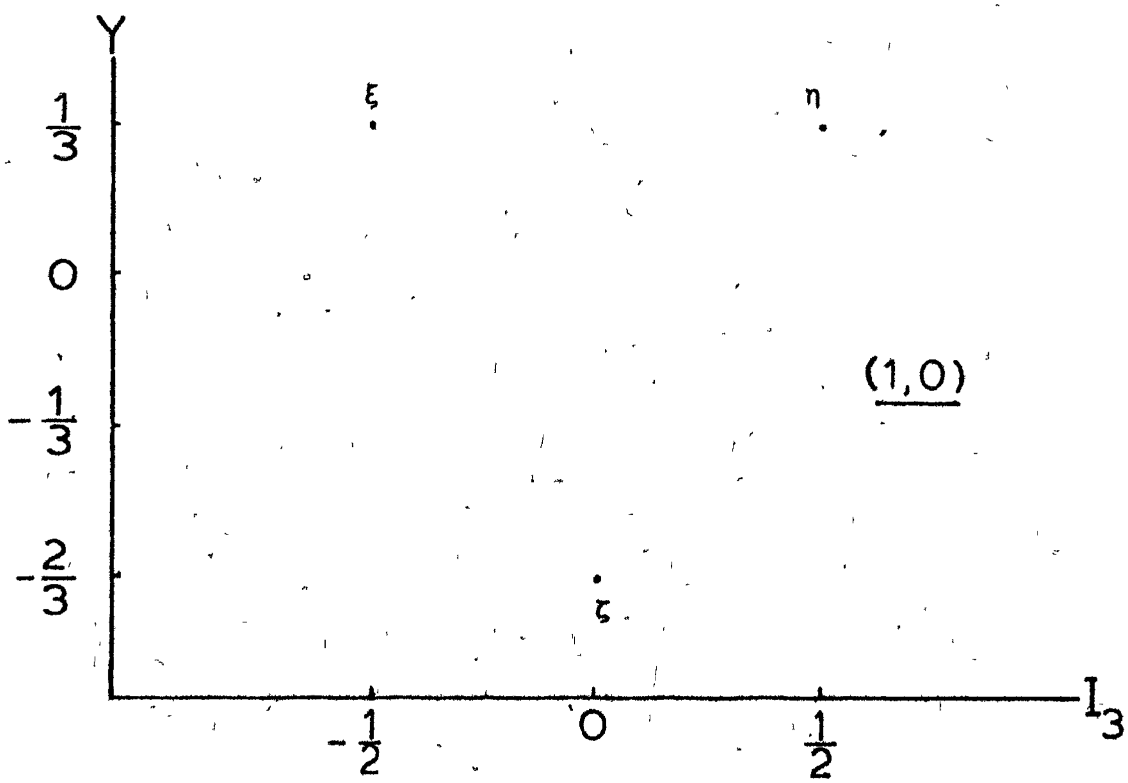
Now since $E_{12}^\dagger = E_{21}$ and $E_{23}^\dagger = E_{32}$ we also have

$$\begin{aligned} E_{21} &= \beta\partial_\alpha + \sqrt{2}\,(\varepsilon\partial_\delta + \delta\partial_\nu) + \lambda\partial_\kappa \\ E_{32} &= \nu\partial_\alpha + \lambda\partial_\varepsilon + \frac{1}{\sqrt{2}}\,(\delta\partial_\beta + \kappa\partial_\delta) + \sqrt{\frac{3}{2}}\,(\theta\partial_\beta + \kappa\partial_\theta) \end{aligned} \tag{5.4}$$

We now turn to the problem of constructing the highest components of the elementary polynomial tensors that is, the highest components of (1,11), (2,00), (2,11), (3,00), (3,30), (3,03). As an example, we will construct the highest component of the degree two polynomial tensor (2,11), i.e., the degree two octet; we shall denote its components in the same way as those of the degree one octet (Γ_A) but with a superscript "(2)"; for example its highest component will be denoted $\alpha^{(2)}$ and so on. Following the prescription given at the beginning of this section, we must first construct the most general second degree polynomial in the components of Γ_A with the necessary weight (see figure 2); this second degree polynomial is

$$\alpha^{(2)} = a_1\,\alpha\delta + a_2\,\alpha\theta + a_3\,\beta\nu \tag{5.5}$$

The second step consists in finding the coefficients a_1, a_2 and a_3; they are found by requiring that the generators E_{12} and E_{23} annihilate $\alpha^{(2)}$. We first apply E_{12} as given in (5.3); we then get



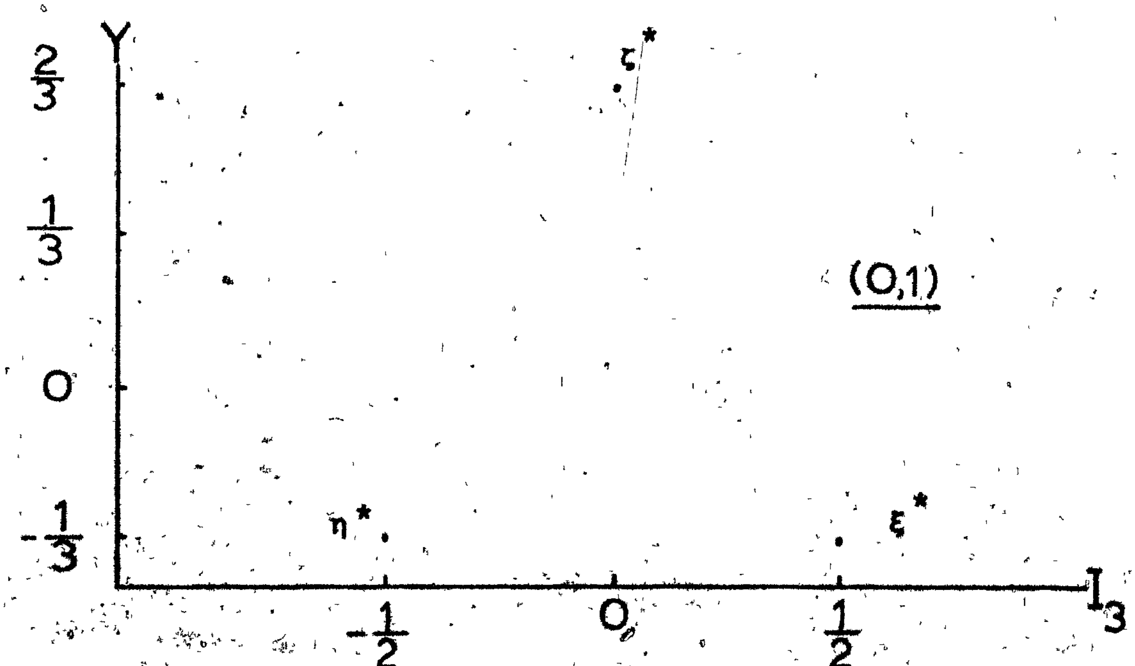


Figure 4. Bases states of the fundamental irreducible representations of SU(3).

$$(a_3 + \sqrt{2}\ a_1)\ \alpha\nu = 0$$

so that

$$a_1 = -\frac{a_3}{\sqrt{2}} .$$

$\alpha^{(2)}$ may therefore be written

$$\alpha^{(2)} = -\frac{a_3}{\sqrt{2}}\alpha\delta + a_2\ \alpha\theta + a_3\ \beta\nu . \tag{5.6}$$

We then apply E_{23} and require that $E_{23}\ \alpha^{(2)} = 0$; we then get

$$\left(\frac{a_3}{2} + \sqrt{\frac{3}{2}}\ a_2 \right) \beta\alpha = 0$$

so that

$$a_3 = -2\sqrt{\frac{3}{2}}\ a_2.$$

Choosing $a_2 = 1$, we have

$$\alpha^{(2)} = \sqrt{3}\ \ \alpha\delta + \alpha\theta\ - \sqrt{6}\ \beta\nu . \tag{5.7}$$

All other elementary polynomial tensors are found in a similar way; their highest components are

$$\begin{aligned}
&(1,11)_h \sim \alpha \\
&(2,00) \sim \alpha\lambda - \beta\kappa - \nu\varepsilon + \frac{1}{2}\delta^2 + \frac{1}{2}\theta^2 \\
&(3,00) \sim \sqrt{3}\ \alpha\delta\lambda + \sqrt{3}\,\beta\delta\kappa - \sqrt{6}\,\beta\nu\lambda - \sqrt{6}\,\alpha\varepsilon\kappa \\
&\qquad + \alpha\theta\lambda - \beta\theta\kappa - \delta^2\theta + 2\ \nu\varepsilon\theta + \frac{1}{3}\theta^3 \\
&(3,30)_h \sim \alpha\nu\delta + \sqrt{3}\,\alpha\nu\theta - \sqrt{2}\,\alpha^2\kappa - \sqrt{2}\,\beta\nu^2 \\
&(3,03)_h \sim \alpha^2\varepsilon + \beta^2\nu - \sqrt{2}\ \alpha\beta\delta .
\end{aligned} \tag{5.8}$$

Later in our discussion we shall require the eighth component of (2,11), i.e., $\theta^{(2)}$; in order to get $\theta^{(2)}$ we first apply E_{32} has given by (5.4) on $\alpha^{(2)}$:

$$\nu^{(2)} = E_{32}\ \alpha^{(2)} = \sqrt{6}\ \alpha\kappa - 2\ \nu\theta.$$

$\beta^{(2)}$ may be obtained by the action of E_{21} on $\alpha^{(2)}$

$$\beta^{(2)} = E_{21}\,\alpha^{(2)} = -\sqrt{3}\,\beta\delta + \sqrt{6}\,\alpha\varepsilon + \beta\theta.$$

$\delta^{(2)}$ is obtained by the action of E_{21} on $\nu^{(2)}$

$$\delta^{(2)} = \frac{1}{\sqrt{2}}\,E_{21}\,\nu^{(2)} = \sqrt{3}\,(\alpha\lambda + \beta\kappa) - 2\delta\,\theta.$$

Finally since

$$E_{32}\,\beta^{(2)} = \sqrt{\frac{3}{2}}\,\theta^{(2)} + \sqrt{\frac{1}{2}}\,\delta^{(2)}$$

we have

$$\theta^{(2)} = 2\nu\varepsilon + \alpha\lambda - \beta\kappa - \delta^2 + \theta^2. \tag{5.9}$$

As a final example we now proceed to the construction of the highest component of the composite tensor (5,03). This tensor is obtained by the stretched tensor product of (2,00) with (3,03) so that the algebraic expression for the highest state of (5,03) is

$$(5,03)_h \sim (\alpha\lambda - \beta\kappa - \nu\varepsilon + \frac{1}{2}\delta^2 + \frac{1}{2}\theta^2).\,(\alpha^2\varepsilon + \beta^2\nu - \sqrt{2}\,\alpha\beta\delta).$$

Once we know the algebraic expression of a polynomial tensor we must, in order to construct its corresponding tensor in the enveloping algebra, know the components of the basic vector operator, i.e., α_{op}, β_{op} etc. . Following Gel'fand and Zetlin's matrix elements (the coefficient of E_{13} has been choosen to be + 1) we have

$$\alpha_{op} = E_{13},\ \beta_{op} = E_{23},\ \nu_{op} = -E_{12},\ \delta_{op} = \sqrt{\frac{1}{2}}\,(A_{11} - A_{22}) \tag{5.10}$$

$$\varepsilon_{op} = E_{21},\ \theta_{op} = \frac{-1}{\sqrt{6}}\,(A_{11} + A_{22} - 2A_{33}),\ \kappa_{op} = -E_{32},$$

$$\lambda_{op} = E_{31}.$$

We now proceed to construct $\alpha_{op}^{(2)}$, i.e., the highest component of the second degree vector operator following the prescription given at the beginning of this section. Symmetrizing (5.7) with respect to order and substituting for the components of Γ_A the corresponding components of the basic vector operator we get

$$\alpha_{op}^{(2)} = \sqrt{\frac{3}{2}}\,(\alpha_{op}\delta_{op} + \delta_{op}\alpha_{op}) + \frac{1}{2}\,(\alpha_{op}\theta_{op} + \theta_{op}\alpha_{op}) - \frac{\sqrt{6}}{2}\,(\beta_{op}\nu_{op} + \nu_{op}\beta_{op}). \quad (5.11)$$

With the help of (5.10), (5.11) may be written in terms of the generators and after some manipulation we finally get

$$\alpha_{op}^{(2)} = \frac{2}{\sqrt{6}}\,E_{13}A_{11} + \sqrt{6}\,E_{23}E_{12} - \frac{4}{\sqrt{6}}\,E_{13}A_{22} + \frac{2}{\sqrt{6}}\,E_{13}A_{33} + \frac{\sqrt{6}}{2}\,E_{13}. \quad (5.12)$$

We now have all the necessary tools to give a simple derivation of the well known Gell-Mann-Okubo mass formula[76]. The basic assumption is that the mass operator M_{op} consists of two terms, one of which is an SU(3) scalar while the second transforms like the eighth component of an SU(3) octet (and therefore an SU(2) x U(1) invariant). Our GF (3.28) informs us that there are only two linearly independent vector operators in the enveloping algebra of SU(3), one of which is degree one and the other degree two. We therefore have

$$M_{op} = \text{scalar} + \theta_{op} + \theta_{op}^{(2)} \quad (5.13)$$

where I recall that θ_{op} and $\theta_{op}^{(2)}$ are respectively the eighth component of the vector operators of degree one and two. Our aim is therefore to express θ_{op} and $\theta_{op}^{(2)}$ in terms of isospin and hypercharge. From (5.10) we have that

$$\theta_{op} = \frac{-1}{\sqrt{6}}\,(A_{11} + A_{22} - 2A_{33})$$

so that in terms of differential operators (see (5.2))

$$\theta_{op} = \frac{-1}{\sqrt{6}} (\eta\partial_\eta + \xi\partial_\xi - 2\zeta\partial_\zeta + 2\zeta^*\partial_{\zeta^*} - \xi^*\partial_{\xi^*} - \eta^*\partial_{\eta^*}).$$

But the hypercharge operator Y_{op} is

$$Y_{op} = \frac{1}{3} (\eta\partial_\eta + \xi\partial_\xi - 2\zeta\partial_\zeta + 2\zeta^*\partial_{\zeta^*} - \xi^*\partial_{\xi^*} - \eta^*\partial_{\eta^*})$$

so that

$$\theta_{op} = -\sqrt{\frac{3}{2}}\; Y_{op} \quad . \tag{5.14}$$

In the case of $\theta^{(2)}_{op}$ it turns out to be easier to work with the corresponding component $\theta^{(2)}$ of the polynomial tensor (2,11). The polynomial tensor $C^{(2)}$ corresponding to the Casimir of degree two is (see (5.8))

$$C^{(2)} = \alpha\lambda - \beta\kappa - \nu\varepsilon + \frac{1}{2}\delta^2 + \frac{1}{2}\theta^2 \cdot$$

so that $\theta^{(2)}$ as given by (5.9) may be written

$$\theta^{(2)} = 3\nu\varepsilon - \frac{3}{2}\delta^2 + \frac{1}{2}\theta^2 + C^{(2)} \tag{5.15}$$

We have that

$$I^2_{op} = \frac{1}{2}(I_+I_- + I_-I_+) + I^2_{3op}$$

where I_{op} is the isopin operator and I_{3op} its third component; I_+ and I_- are the SU(2) raising and lowering operators which in the Gel'fand-Zetlin notation are E_{12} and E_{21}. Writing δ_{op} (see (5.10)) in terms of differential operators we get

$$\delta_{op} = \frac{1}{\sqrt{2}} (\eta\partial_\eta - \xi\partial_\xi + \xi^*\partial_{\xi^*} - \eta^*\partial_{\eta^*})$$

but

$$I_{3op} = \frac{1}{2} (\eta\partial_\eta - \xi\partial_\xi + \xi^*\partial_{\xi^*} - \eta^*\partial_{\eta^*})$$

so that

$$I_{3op} = \frac{1}{\sqrt{2}} \delta op . \tag{5.16}$$

The algebraic form of I_3 is therefore

$$I_3 = \frac{1}{\sqrt{2}} \delta$$

so that the polynomial tensor I^2 corresponding to I^2_{op} is

$$I^2 = \frac{1}{2} \delta^2 - \nu\varepsilon . \tag{5.17}$$

$\theta^{(2)}$ given in (5.15) may therefore be wrriten

$$\theta^{(2)} = - 3I^2 + \frac{1}{2} \theta^2 + C^{(2)}$$

so that returning to the operator formulation, we have

$$\theta^{(2)}_{op} = -3 I^2_{op} + \frac{3}{4} Y^2_{op} + C^{(2)}_{op} . \tag{5.18}$$

Substituting (5.14) and (5.18) into (5.13) we get

$$M_{op} = \text{scalar} + A\, Y_{op} + B\, (Y^2_{op} - 4\, I^2_{op}) + C^{(2)}_{op}$$

where A and B are arbitrary constants. Denoting the eigenvalue of M_{op} by M we have

$$M = \text{constant} + AY + B\{Y^2 - 4\, I\, (I+1)\}$$

which is the desired formula. This technique has the advantage of requiring no coupling coefficients; it could be applied to any group.

(b) SO(5)

In the case of SO(5) we adopt Sharp and Pieper's[77] notation and matrix elements. The components of the tensor Γ_A (the tensor that transforms by the adjoint) are shown in figure 5; the infinitesemal generators of the group are shown in figure 6. When acting on representations of the type

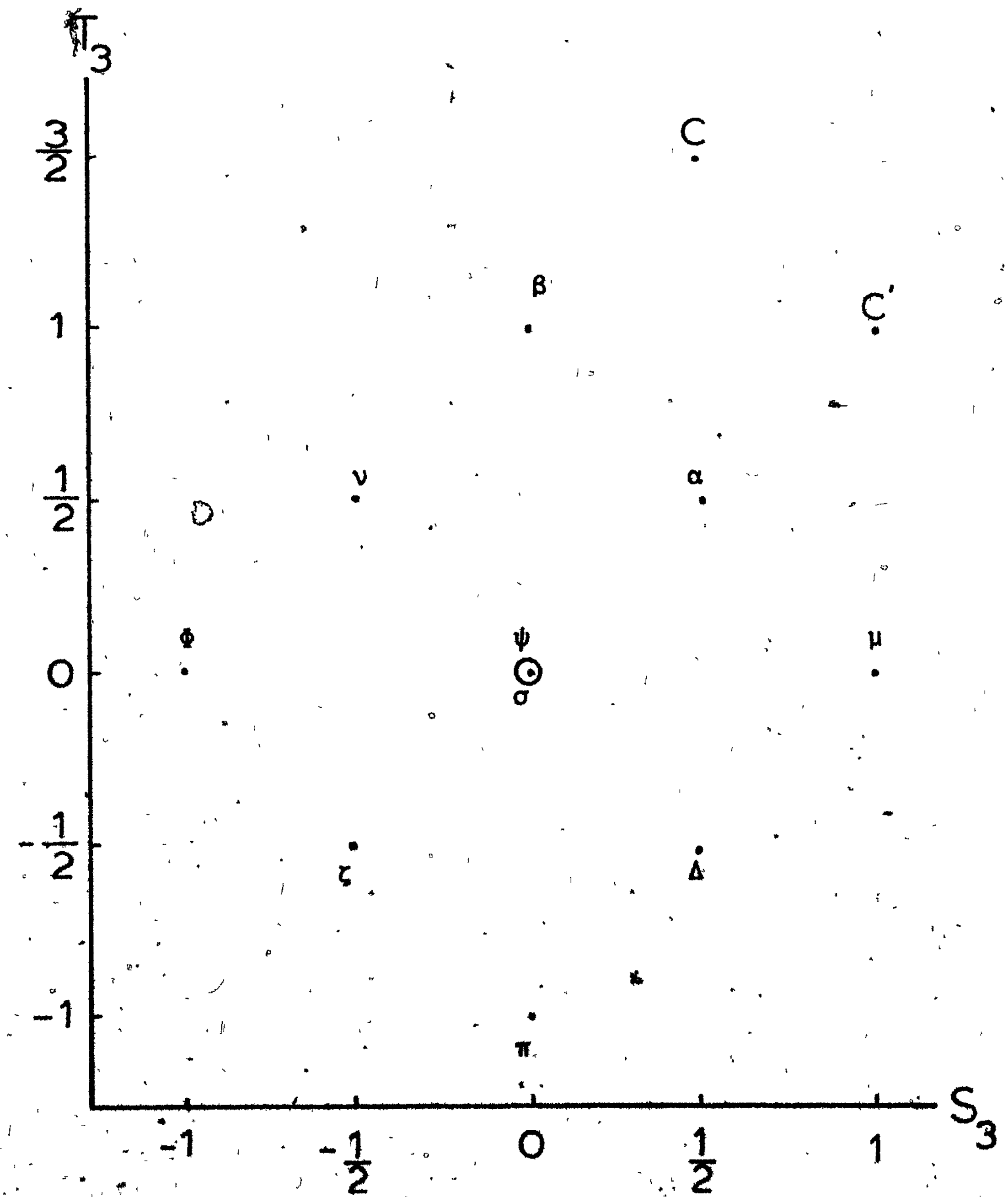


Figure 5. The components of the SO(5) tensor (1,20). The highest components of the (4,21), (2,02), (1,20) and (2,01) tensors are located at points C, C', β and α.

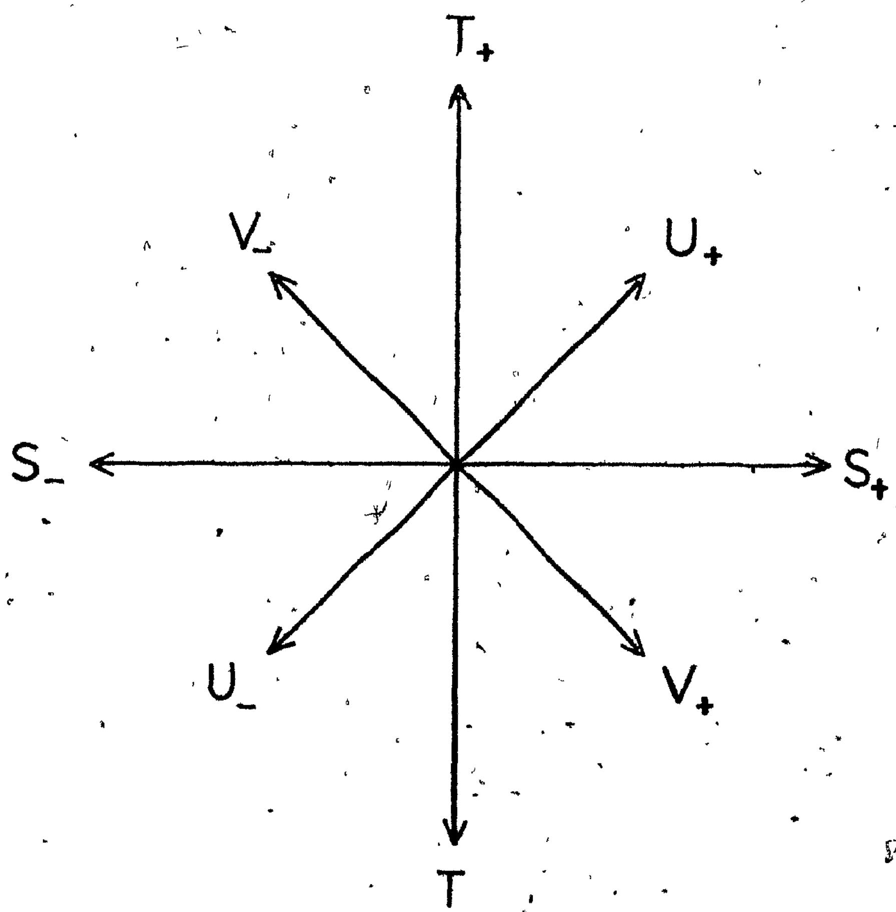


Figure 6. Non diagonal generators of SO(5).

$(0,\nu)$, the generators may be represented by the following differential operators

$$S_3 = \frac{1}{2}(\eta\partial_\eta + \theta\partial_\theta - \xi\partial_\xi - \delta\partial_\delta),$$
$$T_3 = \frac{1}{2}(\eta\partial_\eta + \xi\partial_\xi - \theta\partial_\theta - \delta\partial_\delta),$$
$$S_+ = \eta\partial_\xi + \theta\partial_\delta \quad , \quad S_- = \xi\partial_\eta + \delta\partial_\theta \;,$$
$$T_+ = \eta\partial_\theta + \xi\partial_\delta, \quad T_- = \theta\partial_\eta + \delta\partial_\xi, \tag{5.19}$$
$$U_+ = (2)^{1/2}(\lambda\partial_\delta - \eta\partial_\lambda), \; U_- = (2)^{1/2}(\delta\partial_\lambda - \lambda\partial_\eta)$$
$$V_+ = (2)^{1/2}(\theta\partial_\lambda + \lambda\partial_\xi), \; V_- = (2)^{1/2}(\lambda\partial_\theta + \xi\partial_\lambda)$$

where the variables η, ξ, θ, δ and λ are the basis of the fundamental irreducible representation (01) of SO(5); they are shown in figure 7. The generators S_+ and V_- corresponding to the simple roots may be written as differential operators which are function of the components of the decuplet Γ_A (see figure 5); we have

$$S_+ = \sqrt{2}\,(\mu\partial_\psi + \psi\partial_\phi) + \alpha\partial_\nu + \Delta\partial_\zeta \tag{5.20}$$
$$V_- = (\sigma-\psi)\,\partial_\Delta + \nu\,(\partial_\psi - \partial_\sigma) + \sqrt{2}\,(\beta\partial_\alpha + \alpha\partial_\mu - \phi\partial_\zeta - \zeta\partial_\pi) \;.$$

The components of the basic vector operator (degree one) whose corresponding polynomial tensor is $\Gamma_A \equiv (1,20)$, are as follows

$$\mu_{op} = -S_+, \; \psi_{op} = \sqrt{2}\; S_3, \; \phi_{op} = S_-, \; \nu_{op} = \frac{1}{\sqrt{2}}\; V_-,$$
$$\alpha_{op} = \frac{-1}{\sqrt{2}}\; U_+, \; \beta_{op} = -T_+, \; \sigma_{op} = \sqrt{2}\, T_3, \; \pi_{op} = T_-, \tag{5.21}$$
$$\zeta_{op} = \frac{1}{\sqrt{2}}\, U_-, \; \Delta_{op} = \frac{1}{\sqrt{2}}\; V_+ \;.$$

The elementary polynomial tensors were given in chapter III; they are : (1,20), (2,00), (2,01), (2,02), (3,20), (4,00) and (4,21). The algebraic form of the highest components of these polynomial tensors are found

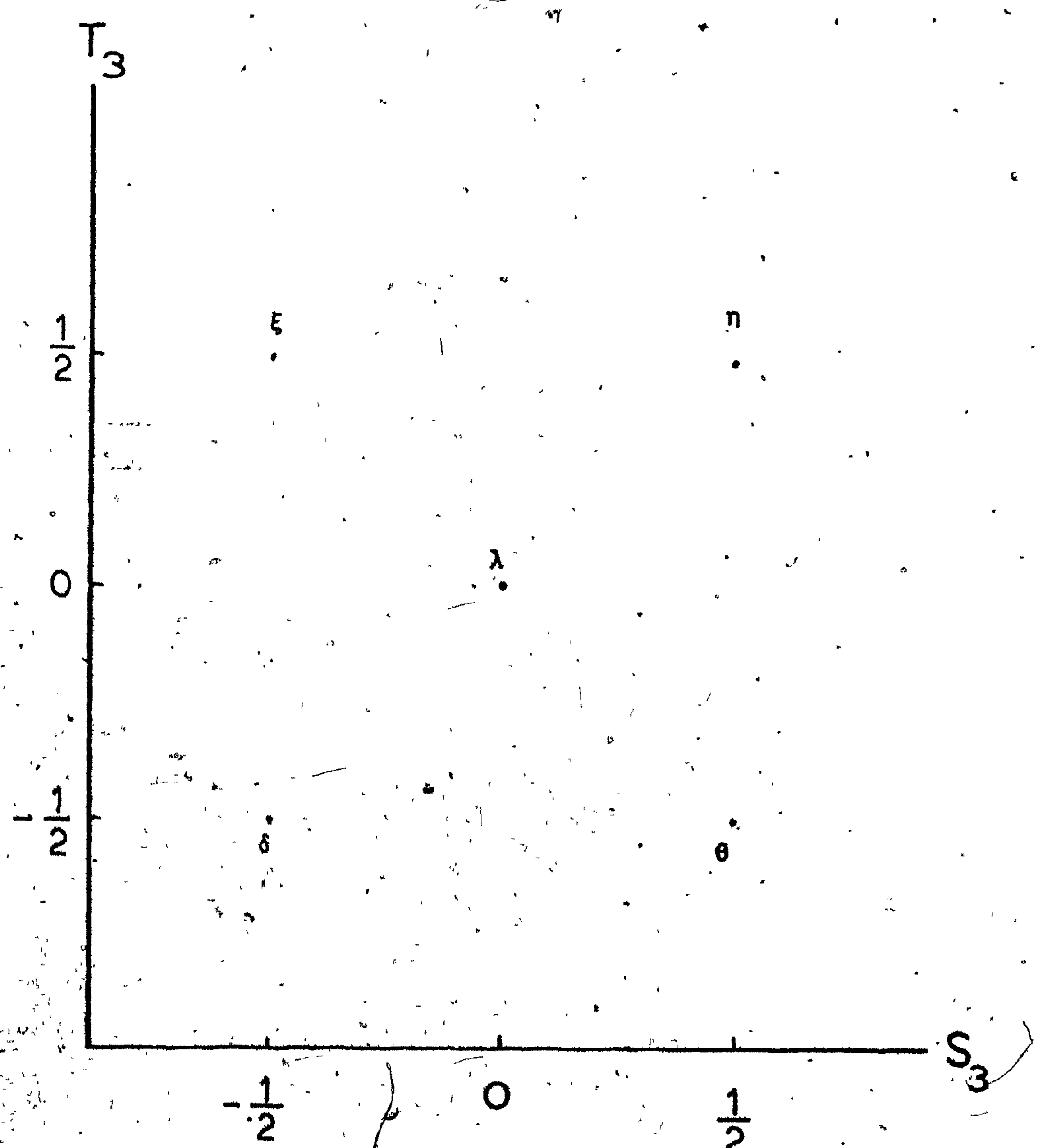


Figure 7. Basis states of the fundamental irreducible representation (0,1) of SO(5).

following the prescription given at the beginning of this section; we get

$$
\begin{aligned}
&(1,20)_h \sim \beta \\
&(2,00) \sim \beta\pi - \nu\Delta + \Phi\mu + \zeta\alpha - \tfrac{1}{2}\sigma^2 - \tfrac{1}{2}\psi^2 \\
&(2,01)_h \sim \alpha\sigma - \alpha\psi + \sqrt{2}\,\nu\mu - \sqrt{2}\,\beta\Delta \\
&(2,02)_h \sim \beta\mu - \tfrac{1}{2}\alpha^2 \\
&(3,20)_h \sim 4\Phi\mu\beta - 2\nu^2\mu - 2\Phi\alpha^2 - 2\beta\psi^2 + 2\sqrt{2}\,\nu\psi\alpha \\
&(4,00) \sim \psi^2\sigma^2 - 2\alpha\zeta\psi\sigma - 2\mu\Phi\sigma^2 - 2\Delta\nu\psi\sigma - 2\beta\pi\psi^2 \\
&\quad + 2\sqrt{2}\,\alpha\Delta\Phi\sigma + 2\sqrt{2}\,\nu\zeta\mu\sigma + 2\sqrt{2}\,\zeta\Delta\beta\psi + 2\sqrt{2}\,\nu\alpha\pi\psi \\
&\quad - 2\Delta^2\beta\Phi - 2\zeta^2\beta\mu - 2\nu^2\pi\mu - 2\alpha^2\pi\Phi + \alpha^2\zeta^2 + \Delta^2\nu^2 \\
&\quad - 2\alpha\nu\zeta\Delta + 4\beta\mu\pi\Phi \\
&(4,21)_h \sim \beta\alpha\nu\Delta - \beta\alpha^2\zeta + 2\beta\alpha\Phi\mu + \beta\alpha\sigma\psi - \beta\alpha\psi^2 \\
&\quad - \sqrt{2}\,\beta\nu\sigma\mu + 2\beta^2\zeta\mu - \tfrac{2}{\sqrt{2}}\beta^2\Delta\psi - \alpha^3\Phi + \sqrt{2}\,\nu\alpha^2\psi \\
&\quad - \nu^2\alpha\mu \, .
\end{aligned}
\tag{5.22}
$$

We conclude this section by recalling some interesting aspects of this method of constructing tensors in the enveloping algebra U of a group :

(1) The problem of constructing any tensor in U is reduced to to that of constructing a finite set of low degree tensors (the elementary tensors).

(2) Assuming that the matrix elements of the generators are known (they have been calculated for most groups of interest in physics), the construction of the elementary tensors are done without making use of any coupling coefficient.

5.2 The action of the tensors in the enveloping algebra on the bases of representations of the group

In this section we shall often refer to the concept of linear independence of tensors so that it is important at this point to make a distinction between two definitions of linear independence.

A set of tensors is of course linearly independent when no linear combination of them vanishes identically unless the coefficients all vanish. In one definition the coefficients are numerical constants. This is the definition we had in mind when we said that the GF's for tensors derived in chapter III gave a basis for all tensors in the enveloping algebra of simple compact groups. In the other definition the coefficients are only required to be group scalars. This is the definition Okubo used in his study of a basis for all vector operators in the enveloping algebra of simple Lie algebras in any given irreducible representation, therefore making no distinction (he counts only one of them) between two vector operators in the enveloping algebra which differ by some group scalar factor (that is, one of the vector operator is equal to the other when multiplied by the group scalar in question; from now on when two operators differ by some group scalar factor we shall say that these two operators are equal modulo multiplication by a group scalar). In what follows we shall be interested in keeping track of all λ-tensors in the enveloping algebra of a group which remain linearly independent (according to the first definition of linear independence) when acting on the bases of certain representations; keeping track of all λ-tensors, even those which are equal modulo some group scalar, turns out to be important when one wants to establish a basis for all subgroup scalars in the enveloping algebra starting from a GF for tensors. When adopting the first definition of linear independence, one must keep in mind in applying Kostant's rule for the multiplicity of a λ-tensor when acting on a representation (ν) that this rule makes no distinction between two tensors (i.e. counts only one of them) which are equal modulo multiplication by some group scalar; note that this scalar may (Casimir) or may not be in the enveloping algebra of the group. The same comment applies to Okubo's result given in (1.3). Let us now return to the object of this section, thas is, the action of the tensors on the basis of a given representation.

Given a GF for tensors in the enveloping algebra of a group, can one find a representation (ν) acting on which any λ-tensor or group of λ-tensors (tensors which have the same transformation properties) enumerated in this GF, exist (are non zero) and are linearly independent ? Kostant[21] has proven that such a representation exists and that actually, there is an infinite number of them for any given λ-tensor or group of λ-tensors. However, this isn't true in all representations; he showed that the multiplicity of a λ-tensor when acting on a representation (ν) is equal to the multiplicity of (ν) in the Clebsch-Gordan series of $(\bar{\lambda}) \times (\nu)$, which implies that the multiplicity of a λ-tensor may vary from one representation to another. Consequently, certain tensors enumerated in these GF's will no longer exist (their components being zero) or be linearly independent when acting on the basis of certain representations.

The problem addressed here, is that of finding the form into which these GF's reduce when the tensors are acting on degenerate representations; by degenerate we mean representations for which one or more Cartan labels vanish. In other words, we want to find GF's giving a basis for tensors in the enveloping algebra U of a group when acting on certain degenerate representations. We shall consider the groups SU(3) and SO(5).

We approach this problem in two ways which are more or less complementary. One consists (as suggested by Moskinsky in a private conversation) of constructing the algebraic expression of the highest component of low degree tensors in U (see for example (5.12)) and then substituting for the generators a certain realization of them (differential operators) proper to the representation on which they are acting (see (5.2) or (5.19)). When one does this, certain relations appear among tensors (some vanish, others are seen to be no longer linearly independent) indicating to us which tensors should be omitted from the GF; the relations among the components of these tensors may be understood in terms of certain identities among the generators, identities which are no longer valid when these tensors are acting on most general representations. Examples of such identities will be given later in our discussion. In certain cases, the above technique (we shall refer to it as the substitution technique) may give sufficient information to write down immediately the reduced form of the GF for tensors (this is the case of SU(3)). However, it may be difficult (if not impossible) to exclude possible higher order relations among tensors only by looking at low degree tensors, which brings us to our second approach.

Kostant's rule for the multiplicity of a λ-tensor in a representation (ν) suggests that one could use Speiser's technique to find that multiplicity.

It turns out that one can actually write down, based on Speiser diagrams and Kostant's rule, a GF giving the multiplicity of all λ-tensors (as discussed earlier, this GF doesn't distinguish two tensors which are equal modulo multiplication by a group scalar) in the enveloping algebra of a group when the tensors are acting on some degenerate representation. Unfortunately, these GF's give us no information on the degree of these tensors; this is why the first approach (substitution technique) complements the second : it gives us information on the degrees. We first consider SU(3).

(a) SU(3) on the $(\nu,0)$ representation

Let us first establish a few facts. As we mentionned earlier, various identities exist between generators in given irreducible representations. Systematic ways of finding such identities have been developed by various authors. In what follows we give a few examples of such identities which will prove useful in our analysis. In general representations we have that

$$E_{13}\, A_{22} \neq E_{23}\, E_{12} \tag{5.23}$$

as can be seen by actually substituting on each side of the inequality (5.23) the realization of these generators given in (5.2); we then get

$$(\eta\partial_\zeta + \zeta^*\partial_{\eta^*})\,(\xi\partial_\xi + \zeta^*\partial_{\zeta^*} + \eta^*\partial_{\eta^*}) \neq$$
$$(\xi\partial_\zeta + \zeta^*\partial_{\xi^*})\,(\eta\partial_\xi + \xi^*\partial_{\eta^*})\,.$$

However, if we restrict ourselves to SU(3) representations of the type $(\nu,0)$ (we drop the star variables) we then get

$$(\eta\partial_\zeta)\,(\xi\partial_\xi) = (\xi\partial_\zeta)\,(\eta\partial_\xi)$$

so that we have the following identity

$$E_{13}\, A_{22} = E_{23}\, E_{12}. \tag{5.24}$$

Another identity is

$$E_{23} A_{11} = E_{13} E_{21} . \qquad (5.25)$$

One can also show that in general representations

$$E_{13} A_{22} \neq E_{23} E_{12} + E_{13}$$

but in representations of the type $(0,\nu)$ we have

$$E_{13} A_{22} = E_{23} E_{12} + E_{13} \qquad (5.26)$$

The identity (5.26) is a particular case of the following identity

$$(A_{\mu\nu} + \delta_{\mu\nu} I) A_{\alpha\beta} = (A_{\mu\beta} + \delta_{\mu\beta} I) A_{\alpha\nu} . \qquad (5.27)$$

The identity (5.27) has been reported by many authors[5,30,31,32,78,79].

Another fact is that $A_{11} + A_{22} + A_{33}$ is an SU(3) scalar in all representations, since in general we have

$$\begin{aligned}\left[A_{\mu\nu}, A_{11} + A_{22} + A_{33}\right] &= \delta_{\nu 1} A_{\mu 1} + \delta_{\nu 2} A_{\mu 2} + \delta_{\nu 3} A_{\mu 3} \\ &\quad - \delta_{\mu 1} A_{1\nu} - \delta_{\mu 2} A_{2\nu} - \delta_{\mu 3} A_{3\nu} \\ &= 0,\end{aligned}$$

and $A_{11} + A_{22} + A_{33}$ is not in the enveloping algebra of SU(3).

Let us now begin our analysis by considering the highest component of the (2,11) tensor given in (5.12) in a $(\nu,0)$ representation. Based on the identity (5.24), (5.12) may be written

$$\alpha_{op}^{(2)} = \frac{2}{\sqrt{6}} (A_{11} + A_{22} + A_{33}) E_{13} + \frac{\sqrt{6}}{2} E_{13}$$

or keeping only the degree two term

$$\alpha_{op}^{(2)} = \frac{2}{\sqrt{6}} (A_{11} + A_{22} + A_{33}) E_{13} . \qquad (5.28)$$

Therefore, the degree one (see 5.10) and two vector operators are equal modulo a group scalar not in the enveloping algebra and according to the definition of linear independence we adopted, both of these tensors should be included in the degenerate (reduced) GF. Had we adopted the second definition, we would have omitted the second degree vector operator from the GF; this is what OKubo did as can be seen from the equation (1.3) in which he predicts only one linearly independent vector operator when the tensors are acting on representations of the type $(\nu,0)$.

Let us now consider the highest component of the (3,03) tensor operator. Given the highest component of its corresponding polynomial tensor (see (5.8)), the highest component of the tensor operator is obtained following the prescription given in section 5.1. Now we could proceed by direct substitution and show that the highest component of (3,03) vanishes. However, in order to see how this (the vanishing of the (3,03) tensor) may be understood in terms of certain identities among generators, the highest component of (3,03) may be written, after some manipulations where we made only use of the commutation rules (5.1), as follows

$$(3,03)_{h_{op}} = \frac{1}{3} \{(E_{13}E_{13}E_{21} - E_{13}E_{23}A_{11}) - (E_{23}E_{23}E_{12} - E_{23}E_{13}A_{22}) + 2\,(E_{13}E_{21}E_{13} - E_{23}A_{11}E_{13}) - 2\,(E_{23}E_{12}E_{23} - E_{13}A_{22}E_{23})\} \qquad (5.29)$$

Now each parenthesis ( ) in (5.29) is equal to zero due to the identities (5.24) or (5.25), which means that the tensor (3,03) does not exist in the representation $(\nu,0)$. In the same way we could show that the tensor (3,30) also vanishes. Therefore, these two tensors should be excluded from the reduced GF.

Following the usual procedure one can show that the Casimir operator of degree two, when acting on representations of the type $(\nu,0)$ may be written

$$C_{op}^{(2)} = (A_{11}+A_{22}+A_{33}) + \frac{1}{3}(A_{11}+A_{22}+A_{33})^2,$$

which in the realization (5.2) is written

$$C_{op}^{(2)} = (\eta\partial_\eta + \xi\partial_\xi + \zeta\partial_\zeta) + \frac{1}{3}(\eta\partial_\eta + \xi\partial_\xi + \zeta\partial_\zeta)^2.$$

Now since the eigenvalue of the operator $A_{11} + A_{22} + A_{33}$ is equal to ν we therefore have that the eigenvalue of $C_{op}^{(2)}$ is $\frac{1}{3}(\nu)(\nu+3)$ which is a well known result[80].

The degree three Casimir is

$$C_{op}^{(3)} = \frac{2}{9}(A_{11}+A_{22}+A_{33})^3 + 2(A_{11}+A_{22}+A_{33})^2 + 4(A_{11}+A_{22}+A_{33}).$$

Finally, we have the following relationship

$$(C_{op}^{(3)})^2 = \frac{4}{3}(C_{op}^{(2)})^3 + 6\, C_{op}^{(2)}\, C_{op}^{(3)} - 8\,(C_{op}^{(2)})^2$$

which informs us that the square of $C_{op}^{(3)}$ should not appear in the degenerate (reduced) GF.

At this point we have sufficient information to write down a GF giving a basis for all tensors in the enveloping algebra of SU(3) in the representation $(\nu,0)$. It is pretty clear in the light of the above results, that a linearly independent set of tensors consists of the stretched products $(1,00)^a.\ (1,11)^b$ where a and b are non negative integers, except that $a = 1$, $b = 0$ is excluded; here $(1,00) \equiv (A_{11}+A_{22}+A_{33})$. In terms of a GF this is written

$$\frac{1 + U^2\Lambda_1\Lambda_2}{(1-U^2)(1-U\Lambda_1\Lambda_2)} + \frac{U^3}{(1-U^2)}. \qquad (5.30)$$

(5.30) is also valid for representations $(0,\nu)$.

(b) SO(5)

We shall first consider the case when these tensors act on representations of the type $(0,\nu)$. From the substitution technique we get the following relations :

$$(2,01)_{hop} = 0, \quad (4,00)_{op} = C_3\,(2,00)^2_{op} + C_4\,(2,00)_{op}$$
$$(3,20)_{hop} = C_1(2,00)_{op}\,(1,20)_{hop} + C_2\,(1,20)_{hop}, \qquad (5.31)$$
$$(4,21)_{hop} = 0.$$

Where C_1, C_2, C_3 and C_4 are constants $\neq 0$.

The tensors $(2,01)_{op}$, $(4,00)_{op}$, $(3,20)_{op}$ and $(4,21)_{op}$ must therefore be excluded from the degenerate GF.

We now show how one can use Speiser's technique to obtain a GF giving the multiplicity of all tensors in the enveloping algebra of SO(5) when the tensors are acting on representations of the type $(0,\nu)$; the reader is referred to figure 8. According to Kostant's rule, the multiplicity of a λ-tensor when acting on a representation (ν) is equal to the multiplicity of (ν) in the Clebsch-Gordan series of $(\lambda) \times (\nu)$; following Speiser's technique, the multiplicity of (ν) is obtained by centering the weight diagram of (λ) on the point (referred to as the origin) in the dominant sector which corresponds to (ν); the multiplicity of (ν) is then equal (after Weyl reflections) to the number of states at the origin. Now one can always choose a representation (ν), for a given λ-tensor, situated far enough from the boundaries (point A) so that the multiplicity of the λ-tensor is equal to the number

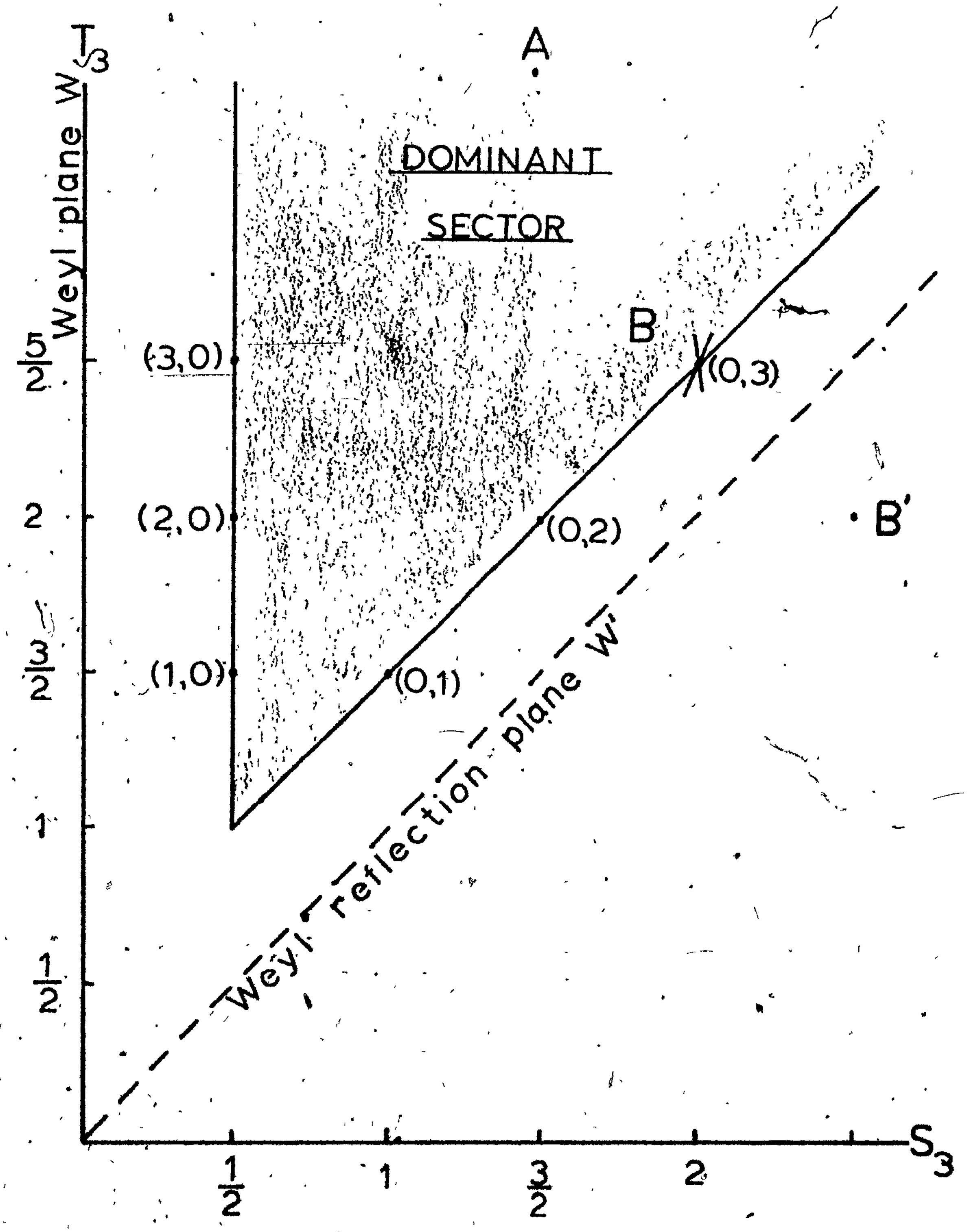


Figure 8. The SO(5) Speiser diagram. The shaded area is the dominant sector (including border lines)

of states of zero weight in the representation (λ) (no cancellation occurs due to Weyl reflections). For the case of interest, i.e., representations of the type (0,ν) whose points lie on the boundary line (such as point B), one can always choose a representation (ν) high enough in weight space (we are interested in the most general case) for any given λ-tensor, such that cancellations at the origin occur only through reflections from Weyl's plane W′; such cancellations occur only if the (λ) representation contains states lying at point B′, i.e., only if it contains SU(2) x SU(2) states with half integer values of S and T. Therefore, the multiplicity of a λ-tensor, when acting on most general representations of the type (0,ν) is equal to the multiplicity of states at the origin of the (λ) representation minus the number of SU(2) x SU(2) states with half integer values of S and T contained in the representation (λ). The GF giving the number of states of zero weight for any (λ) representation may be obtained from the GF giving the branching rules of SO(5)⊃ SU(2) x SU(2), which is[50]

$$F\,(\Lambda_1,\Lambda_2;N_1,N_2) = \{(1-\Lambda_1N_1)\,(1-\Lambda_1N_2)\,(1-\Lambda_2)\,(1-\Lambda_2N_1N_2)\}^{-1} \qquad (5.32)$$

where one keeps only terms with even N_1 (S) and N_2 (T) and then put $N_1 = N_2 = 1$ to get finally

$$\frac{1+\Lambda_1^2\Lambda_2}{(1-\Lambda_1^2)^2\,(1-\Lambda_2)\,(1-\Lambda_2^2)}\,. \qquad (5.33)$$

The GF giving the number of states with half integer values of S and T is also obtained from (5.32) keeping only terms odd in N_1 and N_2 ant then putting $N_1=N_2=1$; we then get

$$\frac{\Lambda_1^2+\Lambda_2}{(1-\Lambda_1^2)^2\,(1-\Lambda_2)\,(1-\Lambda_2^2)}\,. \qquad (5.34)$$

Subtracting (5.34) from (5.33) we get the GF giving the multiplicity of all λ-tensors in the representation $(0,\nu)$, that is

$$\frac{1}{(1-\Lambda_1^2)\ (1-\Lambda_2^2)}\ . \tag{5.35}$$

In the case of SU(3) we had found that two tensors could differ by some group scalar $A_{11}+A_{22}+A_{33}$ not contained in the enveloping algebra. For SO(5) no such scalar exists in the case of representations of the type $(\nu,0)$ (if there were, it would imply that the monomials of degree p in the four basis states of the fundamental irreducible representation (1,0) of SO(5) could contain group scalars; this isn't the case as shown by Sharp and Pieper[77]). In representations of the type $(0,\nu)$ certain tensors could in principle differ by some group scalar not in the enveloping algebra however in the light of the above results (that is (5.31) and (5.35)) this doesn't occur. THe degenerate GF giving a basis for all tensors in the enveloping algebra of SO(5) when the tensors are acting on representations (most general ones) of the type $(0,\nu)$ is

$$G(U;\Lambda_1\Lambda_2) = \frac{1}{(1-U^2)\ (1-U\Lambda_1^2)\ (1-U^2\Lambda_2^2)}\ . \tag{5.36}$$

In the case of representation of the type $(\nu,0)$ it can be shown that (3.36) reduces as follows

$$G(U;\Lambda_1\Lambda_2) = \frac{1+U^2\Lambda_2}{(1-U^2)\ (1-U\Lambda_1^2)\ (1-U^2\Lambda_2^2)}\ . \tag{5.37}$$

In chap IV we discussed how one can check the GF for tensors by reducing it to its corresponding GF for weights; in the case were these tensors are acting on degenerate representations, we couldn't think of any such methods of testing the results.

CHAPTER VI

THE MISSING LABEL PROBLEM IN THE REDUCTION $SO(5) \supset SO(3)$

The general problem addressed in this chapter is that of providing a complete labelling of the basis states of an irreducible representation (IR) of a Lie group G. In section one, we shall briefly discuss possible approaches to this problem; in section two we will consider the particular case of $SO(5) \supset SO(3)$ restricted to representations of the type $(0,\nu)$.

6.1 General discussion of the missing label problem

Depending on the nature of the physical problem under study, one may want to classify the states of the system according to a canonical or non-canonical chain of groups. We define as canonical a reduction of a group G into a subgroup when the subgroup provides enough labels to specify the basis states of G uniquely.

The problem of labelling the states in a canonical way has been completely solved, at least in principle, for Lie groups corresponding to the Cartan algebras A_ℓ, B_ℓ and D_ℓ. (Indeed, the Gel'fand-Tsetlin[75] patterns provides such a complete labelling; in this scheme, the labels are provided by a complete set of commuting operators namely the Casimirs of the group and subgroups (note that the Gel'fand labels are not eigenvalues of the

Casimirs but can be written as function of them). For example, the canonical chain for the group SU(n) is

$$SU(n) \supset SU(n-1) \times U(1) \supset SU(n-2) \times U(1) \times U(1) \supset \ldots\ldots \supset \underbrace{U(1) \times \ldots \times U(1)}_{n-1 \text{ times}}.$$

When one classifies the states of a physical system according to a non-canonical chain of groups, one faces a labelling problem since the subgroup does not provide enough labels to specify uniquely the basis states. Actually as shown by Peccia and Sharp[72], given a semisimple group G and its semisimple subgroup H the number of missing labels n (in the case of general representations) in the reduction $G \supset H$ is

$$n = \frac{1}{2}(r_G - \ell_G - r_H - \ell_H)$$

where r_G, r_H, ℓ_G, ℓ_H are the order and rank of the group and subgroup. For instance in the case of $SU(3) \supset SO(3)$ there is one missing label and two missing labels in the reduction $SO(5) \supset SO(3)$. The missing label problem has given rise to many studies; the solutions proposed may be divided into two classes.

One leads to analytical but non orthogonal bases. For example, in the SU(3) shell model Elliot[36] solved the missing label problem by a projection technique; good SO(3) states are projected from certain intrinsic states, and the missing label is provided by the intrinsic state from which the projection is made. Following a technique which parallels closely Elliot's, the labelling problem in the case of SU(4) states in a SU(2) x SU(2) basis has been solved by Ahmed and Sharp[55] and also by Draayer[81]. Another approach to the missing label problem which also leads to analytical but non orthogonal states consists in defining highest states of subgroup multiplets in terms of products of elementary multiplets (the elementary permissible diagrams of Moshinsky and Devi); it is based on the observation that all subgroup IR's of all group IR's may be defined by the stretched products of powers of a finite set of elementary multiplets (they are also called elementary factors and are the highest states of subgroup IR's belonging to low-lying IR's of the group). The exponents of the power of the elementary multiplets supply the missing (as well as non missing) labels. This technique has been used in many occasions; $SU(3) \supset SO(3)$[82], $SO(5) \supset SU(2)$[49] and $SU(4) \supset SU(2) \times SU(2)$[50] states have been defined in this manner. Our GF (3.52) for $SU(6) \supset Sp(6)$ branching rules also defines such states. In both cases (projection and elementary multiplet techniques) the missing labels are integers.

In the other class of solutions the basis states of the representation are common eigenfunctions of a complete set of commuting Hermitian operators and therefore are orthogonal; as first pointed out by Racah[83], the missing label is not usually an integer. Given a chain of groups $G \supset H$, the labels are provided by the Casimirs of the group and subgroup and the missing label operators. These missing label operators must satisfy the following requirements

(1) They must be hermitian (we want orthogonal states and a missing label which is real).

(2) They must transform every space carrying an irreducible representation of G into itself and therefore should be in the enveloping algebra of the group.

(3) Since they must commute with the Casimirs of H and also since all states within a representation of H must have the same eigenvalue, the missing label operators must be H-scalars.

(4) These operators must provide labels which are independent of those provided by the other labelling operators and therefore should be functionally independent of them (that is not expressible as a function of them only).

(5) If there is more then one missing label, these operators must be constructed such that they mutually commute.

Therefore the missing label operators must be choseh among the subgroup scalars available in the enveloping algebra of the group, which leads us to the problem of establishing a basis for all such scalars. The answer to this problem depends on the type of representation one considers. A basis for all subgroup scalars in the enveloping algebra of a group when the scalars are acting on general representations (that is, representations for which no Cartan labels are zero), may no longer be valid if one restricts itself to certain degenerate representations (one or more Cartan labels being zero). This may be understood (as we discussed for the case of group tensors) in terms of certain identities among the generators, identities which are no longer valid in general representations. Let us first discuss the problem of establishing a basis for subgroup scalars in the enveloping algebra of a group in the case of general representations.

It has been shown for an arbitrary semisimple group G and its semisimple subgroup H that the H-scalars in the enveloping algebra of G are finitely generated[1]; that is, that all subgroup scalars may be expressed as polynomials in a finite set of elementary scalars (integrity basis). Integrity bases have been given[3] in all cases of one missing label and maximal subgroups. Solutions in the case $G_2 \supset SU(2) \times SU(2)$, $SO(5) \supset SU(2)$ and $SU(4) \supset SU(2) \times SU(2)$, in which there are two missing labels, have also been proposed[2,49]. Once the integrity basis is established, the basis for all subgroup scalars may be presented in the form of a GF (one may also derive the GF and then read off the integrity basis). For example, the algebra of SU(3) decomposes under SO(3) into a rank one tensor L and a rank two tensor Q and the integrity basis for SO(3) scalars in the enveloping algebra U of SU(3) is L^2, Q^2, Q^3, L^2Q, L^2Q^2 and L^3Q^3 with $(L^3Q^3)^2$ redundant. The GF giving a basis for all SO(3) scalars in U is therefore equal to (the integrity basis was first conjectured by Racah[83] but first proved by Judd et al[1])

$$\frac{1+L^3Q^3}{(1-L^2)(1-Q^2)(1-Q^3)(1-L^2Q)(1-L^2Q^2)} \qquad (6.1)$$

Once a basis have been established the next step is to choose among all these scalars the missing label operators. Here an important concept is that of functional independence. An operator is said to be functionally independent of a set of operator if no powers of it can be expressed as a function of the elements of the set alone. This property is particularly important for the labelling problem since we are looking for independent labels. A necessary although not sufficient condition for a subgroup scalar to be functionally independent of all other subgroup scalars in the enveloping algebra of the group, is that it must be a member of the integrity basis; note that in the case of SU(3)⊃ SO(3), although L^3Q^3 is a member of the integrity basis it is not functionally independent of all other subgroup scalars since its square may be expressed as a linear combination of other subgroup scalars. Therefore when looking for missing label operators, the simplest solution (the one leading to the lowest degree scalars) is to choose them from the integrity basis. The integrity basis will contain the Casimirs of the group and subgroup and also the lowest degree missing label operators available in the enveloping algebra. A theorem of Peccia and Sharp states that for semisimple groups, in the case of general representations, the number of functionally independent missing label operators available in the enveloping algebra is just twice the number of labels actually missing. Let us summarize the situation : given a semisimple group G and its semisimple subgroup H, techniques have been developed to establish an integrity basis for H-scalars in the enveloping algebra of G and solutions were given for many group-subgroup combinations. This integrity basis contains twice as many missing label operators as there are missing labels and provides the lowest degree missing label operators available in the enveloping algebra; it gives a complete description of these operators that is : degree, multiplicity and composition. In the case of two missing labels or more there is an added difficulty : one must construct out of the 2n operators available n missing label operators that mutually commute. No general approach to this problem has yet been developed.

Unfortunately, in the case of degenerate representations the situation isn't as clear. No systematic approach to find the integrity basis in such cases is known. However, some information may be obtained if one knows the reduced form of the GF for group tensors in the enveloping algebra of the group; indeed, from the GF for group tensors a GF for subgroup scalars may be obtained giving us the multiplicity and degrees (it gives no information on their composition in terms of subgroup tensors). An example of such a GF will be given in the next section.

Having discussed in some detail the missing label problem, let us now consider the case of $SO(5)\supset SO(3)$.

6.2 Missing label problem for $SO(5)\supset SO(3)$ restricted to representations of the type $(0,\nu)$

It is a well known fact [84,85] that the states corresponding to the quadrupole vibrations of the nucleus may be classified according to the totally symmetric irreducible representations of SU(5) in the chain $SU(5)\supset SO(5)\supset SO(3)$ with the embedding $(1000)\supset(01)\supset(2)$. The labels are provided by the Casimirs of SU(5), SO(5) and SO(3) and the third component of angular momentum. There is one missing label at the level $SO(5)\supset SO(3)$. Actually a complete solution to this problem has been given by Chacon et al[85]; their approach consisted in defining the highest SO(3) states in all SO(5) representations in terms of products of elementary multiplets (elementary permissible diagrams) thereby solving the missing label problem and obtaining analytical but non orthogonal states.

In what follows, we propose to solve this missing label problem in terms of a fifth labeling operator whose eigenvalue will provide the missing label; we shall calculate its eigenvalues and eigenvectors up to (including) the SO(5) representation (0,12) where a degeneracy 3 first appears.

Obviously one first wants to know the missing label operators available in the enveloping algebra. An integrity basis for SO(3) scalars in the enveloping algebra of SO(5), in the case of general representations has been given by Gaskell et al[49]; although useful to establish an upper bound, such integrity basis is no longer valid for the case of interest. Some information may be obtained since we know that the reduced form of the GF (3.36) is

$$\{(1-U^2)\ (1-U\Lambda_1^2)\ (1-U^2\Lambda_2^2)\}^{-1}. \qquad (5.36)$$

The GF for the branching rules of SO(5)⊃SO(3) has been given by Gaskell et al[49]; setting equal to zero the dummy variables which carry the SO(3) representation labels in this GF, we obtain the following GF for SO(3) scalars in SO(5) irreducible representations

$$(1+\Lambda_1^6\Lambda_2^3)\ \{(1-\Lambda_2^3)(1-\Lambda_1^4)(1-\Lambda_1^4\Lambda_2^2)\}^{-1}. \qquad (6.2)$$

The GF for SO(3) scalars in the enveloping algebra of SO(5) when the scalars are acting on representations of the type $(0,\nu)$ is obtained by substituting (6.2) into (5.36); we get

$$(1+U^9)\ \{(1-U^2)^2\ (1-U^4)(1-U^6)\}^{-1}. \qquad (6.3)$$

(6.3) informs us that the integrity basis contains five subgroup scalars : two of degree two which are the Casimirs of SO(5) and SO(3), one of degree nine whose square is redundant and two of degree four and six which are the lowest degree missing label operators available in the enveloping algebra of SO(5).

Here, we would like to comment on a result of Vanden Berghe and De Meyer[86]; (6.3) informs us that one can construct only three linearly independent (that is, one cannot be expressed as some linear combination of the other two and Casimirs) SO(3) scalars of degree four in the enveloping algebra U of SO(5) out of which only one will qualify as a missing label operator. Vanden Berghe and De Meyer claim that there are ten SO(3) scalars of degree four in the enveloping algebra of SO(5) which are functionally independent of the Casimirs of SO(5) and SO(3) and therefore qualify as missing label operators. Now assuming that these operators are good missing label operators, (6.3) informs us that if we choose any one of them (we shall denote it X) the other nine are expressible as some linear combination of X with the Casimirs and their products and therefore, they do not provide labels which are independent of the one provided by X. Moreover, the GF proposed by Gaskell et al[49] giving a basis for all SO(3) scalars in U when these scalar are acting on general representations informs us that one can actually construct only seven linearly independent degree four SO(3) scalars in U, and this is including the Casimirs and their products; therefore, when one restricts oneself to degenerate representations of SO(5) the number of linearly independent degree four SO(3) scalars is ≤ 7 and (6.3) tell us that number is actually equal to three.

Returning to our problem, we choose the simplest solution, that is, a degree four missing label operator. How do we construct it ? First we know that the SO(5) algebra decomposes under SO(3) into a rank one tensor L and a rank three tensor Q so that the degree four missing label operator can be chosen between the following possibilities : QL^3, Q^2L^2, Q^3L, Q^4 or any linear combination of them with the Casimirs. It turns out that QL^3 qualifies as a missing label operator and the proof is by direct verification, that is, it is sufficient to show that QL^3 solves the first degeneracy that appears (the first one is at the level (0,6)). We constructed QL^3 in the following way : first coupling Q and one of the L's to give a $j = 2$ tensor and then doing the same thing for the remaining two L tensors. Formally this may be written as follows

$$\{Q\ L\}_m^{j=2} = \sum_{\mu_3\mu_4} Q_{\mu_3} L_{\mu_4} (3,1,\mu_3,\mu_4|\ 2,m) \qquad (6.4)$$
$$\{L\ L\}_m^{j=2} = \sum_{\mu_1\mu_2} L_{\mu_1} L_{\mu_2} (1,1,\mu_1,\mu_2|\ 2,m)$$

where Q_{μ_3} ($\mu_3 = -3,-2, \ldots, +3$) and L_μ ($\mu = -1, 0, +1$) are the components of the tensors Q and L. We then take the scalar product of the two j = 2 tensor given in (6.4); following Racah's[87] definition of scalar product the missing label operator X is

$$X = \sum_{\substack{m \\ \mu_1\mu_3}} \{(-1)^m Q_{\mu_3} L_{m-\mu_3} L_{\mu_1} L_{-m-\mu_1} (3,1,\mu_3,m-\mu_3|2,m) \times (1,1,\mu_1,-m-\mu_1|2,-m)\} \quad (6.5)$$

The components L_{+1} and L_{-1} may be written in terms of the generators of the SO(3) algebra L_+ and L_-; we have

$$L_{+1} = \frac{-1}{\sqrt{2}} L_+ , \; L_{-1} = \frac{1}{\sqrt{2}} L_- . \quad (6.6)$$

The Clebsch-Gordan coefficients are easily calculated so that X may be written as follows

$$\begin{aligned} X = {} & \frac{Q_0 L_0}{\sqrt{14}} \left[3L^2 - 5L_0^2 - 1\right] + \frac{3}{2\sqrt{42}} Q_{-1} L_+ \left[L^2 - 5 L_0 (L_0 + 1) - 2\right] \\ & - \frac{3}{2\sqrt{42}} Q_{+1} L_- \left[L^2 - 5L_0 (L_0 - 1) - 2\right] - \frac{3}{2}\sqrt{\frac{5}{21}} Q_{-2} L_+^2 (L_0 + 1) \\ & - \frac{3}{2}\sqrt{\frac{5}{21}} Q_{+2} L_-^2 (L_0 - 1) - \frac{1}{2}\sqrt{\frac{5}{14}} Q_{-3} L_+^3 + \frac{1}{2}\sqrt{\frac{5}{14}} Q_{+3} L_-^3 \end{aligned} \quad (6.7)$$

where L^2 is the SO(3) Casimir. It is easily shown that X is hermitian. Once we have found an operator statisfying all requirements the next step is to obtain its eigenvalue spectrum and eigenvectors. We intend to do this by actually diagonalizing X in some SO(5) analytical basis; however, before getting into the details of the procedure let us briefly discuss another method of obtaining the eigenvalues of X.

A few years ago Hughes and Yadegar[17] gave an explicit formula for the SO(3) shift operators valid for any Lie group G which has SO(3) as a subgroup. Their shift operators are polynomials in the generators of SO(3) and in the components of a rank j SO(3) tensor Q (for G = SU(3) j = 2; for G = SO(5) j = 3); these operators, denoted by O_ℓ^k, k = -j, ..., j, when acting on SO(3) states with arbitrary m (eigenvalue of L_0 the third component of angular momentum) change the ℓ value (where ℓ(ℓ+1) is the eigenvalue of the SO(3) Casimir L^2) by k without changing m. They are defined as follows

$$O_\ell^k = y_0^k\ (\ell,m)\ R_0 + \sum_{\mu=1}^{j} \left[y_\mu^k\ (\ell,m)\ R_{+\mu} + (-1)^{j+k}\ y_\mu^k\ (\ell,-m)\ R_{-\mu}\right]$$

where for μ = 0, ... j and k ⩾ 0 (6.8)

$$y_\mu^k\ (\ell,m) = (-1)^{j+3\ell-m} \left[\frac{(j+k)!\,(j-k)!\,(2\ell+j+k+1)!\,(\ell-m-\mu)!}{(2j)!\,(2\ell-j+k)!\,(\ell+m+\mu)!\,\{(\ell-m)!\}^2} \times \frac{(\ell+m+k)!\,(\ell-m+k)!}{}\right]^{\frac{1}{2}} \begin{pmatrix} j & \ell & \ell+k \\ \mu & -\mu-m & m \end{pmatrix}$$

and $R_{\pm\mu} = Q_{\mp\mu}\ L_{\pm}^{\mu}$. The other shift operators O_ℓ^{-k} (k⩾0) are obtained through the following relation

$$\begin{aligned} O_\ell^{-k} &= O_{-(\ell+1)}^{k} \\ &= y_0^{-k}\ (\ell,m)\ R_0 + \sum_{\mu=1}^{j} \Big[\delta_\mu^{-k}\ (\ell,m)\ R_{+\mu} + (-1)^{j+k}\delta_\mu^{-k}\ (\ell,-m) \\ &\qquad \times R_{-\mu}\Big] \end{aligned}$$

where

$$\delta_\mu^{-k}\ (\ell,m) = (-1)^{j+3\ell-k-m}.$$
$$\times \left[\frac{(2\ell+j-k+1)!\,(j+k)!\,(j-k)!\,(\ell-m-\mu)!\,\{(\ell+m)!\}^2}{(2j)!\,(2\ell-j-k)!\,(\ell+m+\mu)!\,(\ell-m+k)!\,(\ell+m-k)!}\right]^{\frac{1}{2}}$$
$$\times \begin{pmatrix} j & \ell & \ell-k \\ \mu & -\mu-m & m \end{pmatrix}.$$

The operator O^0_ℓ has interesting properties. It is an SO(3) scalar and is hermitian; moreover, as shown by Hughes in the case where G is SU(3), O^0_ℓ qualifies as a missing label operator and if G is SO(5) it follows from (6.8) that

$$O^0_k = \frac{-2}{\sqrt{5}} Q_0 L_0 (3L^2 - 5L_0^2 - 1) - \sqrt{\frac{3}{5}} Q_{-1} L_+ \{L^2 - 5L_0 (L_0+1) - 2\}$$
$$+\sqrt{\frac{3}{5}} Q_{+1} L_- \{L^2 - 5L_0(L_0-1) - 2\} + \sqrt{6} Q_{-2} L_+^2 (L_0+1) \cdot \tag{6.9}$$
$$+\sqrt{6} Q_{+2} L_-^2 (L_0 - 1) + Q_{-3} L_+^3 - Q_{+3} L_-^3$$

where Q_μ are the components of the j=3 SO(3) tensor into which the SO(5) algebra decomposes under SO(3). Comparing (6.9) and (6.7) we see that

$$O^0_k = -2 \sqrt{\frac{14}{5}} \quad X \tag{6.10}$$

so that O^0_k qualifies as a missing label operator. Hughes[12-16] has developed a method by which the eigenvalue of O^0_k may be calculated without referring to any explicit form of SO(5) basis states and therefore avoiding any diagonalization procedure. His technique is based on the properties of the shift operators and consists mainly in establishing relations between powers of O^0_k and products of shift operators such as $O^{-k}_{\ell+k}\, O^{+k}_\ell$ which are also SO(3) scalars; the weak point about this technique is that for degeneracies greater than two the algebra becomes very laborious.

Vanden Berghe and De Meyer[88] actually used this technique to find the eigenvalues of O^0_k (which they denoted $\alpha_{\nu,\ell}$) for the chain SO(5)$\supset$SO(3) up to (including) representation (0,7) which as a double degeneracy. They gave the following closed formulas for singular eigenvalues (no degeneracy). Starting with the highest ℓ-values SO(3) representation

$$\alpha_{\nu,2\nu} = \frac{2\sqrt{2}}{5}\ \nu(\nu+1)\ (2\nu+1)\ (4\nu+3) \qquad (\nu\geq 1)$$

$$\alpha_{\nu,2\nu-2} = \frac{2\sqrt{2}}{5}\ \nu(2\nu-1)\ (4\nu^2-5\nu-14) \qquad (\nu\geq 2)$$

$$\alpha_{\nu,2\nu-3} = \frac{2\sqrt{2}}{5}\ (4\nu-3)\ (2\nu^3-\nu^2-17\nu+1) \qquad (\nu\geq 3)$$

$$\alpha_{\nu,2\nu-4} = \frac{2\sqrt{2}}{5}\ (\nu-1)\ (8\nu^3-38\nu^2-\nu-60) \qquad (\nu\geq 4)$$

$$\alpha_{\nu,2\nu-5} = \frac{2\sqrt{2}}{5}\ (8\nu^4-42\nu^3-77\nu^2+258\nu-150) \qquad (\nu\geq 5)$$

For the lowest ℓ-value (6.11)

for $\nu = 3z$ $(z = 1,2,\ldots)$

$$\alpha_{\nu,0} = 0$$

$$\alpha_{\nu,3} = -18\sqrt{2}$$

$$\alpha_{\nu,4} = 42\sqrt{2}$$

for $\nu = 3z + 1$ $(z = 0,1,2\ \ldots)$

$$\alpha_{\nu,2} = \frac{6}{5}\sqrt{2}\ (5\nu+9) \qquad (\nu\geq 1)$$

$$\alpha_{\nu,4} = -12\sqrt{2}\ (3\nu+4) \qquad (\nu\geq 4)$$

$$\alpha_{\nu,5} = 18\sqrt{2}\ (3\nu+1) \qquad (\nu\geq 4)$$

$$\alpha_{\nu,6} = 6\sqrt{2}\ (5\nu+36) \qquad (\nu\geq 4)$$

$$\alpha_{\nu,7} = -\frac{6\sqrt{2}}{5}\ (150\nu+401) \qquad (\nu\geq 7)$$

for $\nu = 3z - 1$ $(z = 1,2,\ldots)$

$$\alpha_{\nu,2} = -\frac{6\sqrt{2}}{5}\ (5\nu+6) \qquad (\nu\geq 2)$$

$$\alpha_{\nu,4} = 12\sqrt{2}\ (3\nu+5) \qquad (\nu\geq 2)$$

$$\alpha_{\nu,5} = -18\sqrt{2}\ (3\nu+8) \qquad (\nu\geq 5)$$

$$\alpha_{\nu,6} = -6\sqrt{2}\ (5\nu-21) \qquad (\nu\geq 5)$$

$$\alpha_{\nu,7} = \frac{6\sqrt{2}}{5}\ (150\nu+49) \qquad (\nu\geq 5)$$

These formulas were derived for the case $\nu \leq 7$ but the authors conjectured that they were true for all ν (our results confirms this at least up to (0,12)); as mentioned by them, the eigenvalues $\alpha_{\nu,\ell}$ of O_k^0 for all singular eigenvalues can be written as $\frac{6\sqrt{2}}{5}$-times an integer. The eigenvalue of O_k^0 (more precisely $\frac{5}{6\sqrt{2}}\alpha_{\nu,\ell}$) for $\nu \leq 7$, as given by these authors, is shown in table IV.

Another method of obtaining the eigenvalue spectrum of O_k^0 (we shall use O_k^0 instead of X given in (6.7)), which is the one we will follow, consists in evaluating the matrix elements of O_k^0 in some SO(5) analytical basis and then diagonalizing O_k^0; our approach follows closely that of Judd et al[1] for $SU(3) \supset SO(3)$ (our calculations were under way when Vanden Berghe and De Meyer's paper appeared; in order to compare our results to theirs we multiplied X by the appropriate factor). The analytical basis we shall use are the $SO(5) \supset SU(2) \times SU(2)$ basis states as defined by Sharp and Pieper[77]; we adopt most of their notation. In order to use to above basis states, we must express the components of the tensors Q and L which compose O_k^0 in terms of generators suited for an SU(2) x SU(2) basis. Apart from the generators which compose the Cartan algebra ,that is S_3 and T_3, these generators are shown in figure 6. We have the following commutation relations

$$[L_0, L_\pm] = \pm L_\pm, \quad [L_0, Q_\mu] = \mu Q_\mu,$$
$$[L_+, L_-] = 2L_0, \quad [L_\pm, Q_\mu] = \{(3 \mp \mu)(3 \pm \mu + 1)\}^{\frac{1}{2}} Q_{\mu\pm1} .$$

Let us set

$$L_+ = aT_+ + bV_+ \text{ and } Q_{+3} = S_+$$

where a and b are constants to be determined. From the above commutation rules it is easily shown that a and b must satisfy the following two equations

$$a^2 - b^2 = 1, \quad a^2 - 2b^2 = -2.$$

Solving for a and b we get $a = \pm 2$, $b = \pm\sqrt{3}$ so that

$$L_0 = T_3 + 3S_3 \; . \tag{6.13}$$

Choosing $a = +2$ and $b = +\sqrt{3}$ we have that

$$L_+ = 2T_+ + \sqrt{3}\,V_+ \; , \; L_- = 2T_- + \sqrt{3}\,V_- \; .$$

By making use of the commutation rules of L_- with Q_μ it is easily shown that

$$Q_{+3} = S_+ , \; Q_{+2} = \frac{\sqrt{2}}{2} U_+ \; , \; Q_{+1} = \frac{-1}{\sqrt{5}} V_+ + \sqrt{\frac{3}{5}}\; T_+ \; ,$$

$$Q_0 = \frac{-3}{\sqrt{5}}\; T_3 + \frac{1}{\sqrt{5}} S_3 ; \; Q_{-1} = -\sqrt{\frac{3}{5}}\; T_- + \frac{1}{\sqrt{5}}\; V_- \; , \tag{6.14}$$

$$Q_{-2} = \frac{\sqrt{2}}{2}\; U_- \; , \quad Q_{-3} = -S_- \; .$$

The general $SO(5) \supset SU(2) \times SU(2)$ basis state is denoted $|\lambda\nu;\, st;\, s_3 t_3\rangle$ where λ, ν are the SO(5) representation labels and s, t those of $SU(2) \times SU(2)$. For the representations we are interested in, that is those of the type $(0,\nu)$, $s = t$ and $0 \leqslant s \leqslant \nu/2$ so that the states are denoted $|0\nu;\, ss;\, s_3 t_3\rangle$ (from now on, we shall drop the SO(5) labels). SO(5) states in a SO(3) basis will be denoted $|\ell, m, \alpha\rangle$ where we shall denote the missing label by α. Expanding the basis states $|ss;\, s_3 t_3\rangle$ interms of $|\ell,\, m,\, \alpha\rangle$ we get

$$|\, s\; s;\; s_3\; t_3\rangle = \sum_{\ell,\alpha} a_\ell |\ell,\, m, \alpha\rangle \tag{6.15}$$

where a_ℓ are some coefficients and

$$m = t_3 + 3s_3 \tag{6.16}$$

m, t_3, s_3 being the eigenvalues of L_0, T_3 and S_3. The summation in (6.15) runs over all $\ell \geqslant m$ which occurs in the SO(5) representation $(0,\nu)$. There is no summation over m since both $SO(5) \supset SU(2) \times SU(2)$ and $SO(5) \supset SO(3)$ states are eigenvectors of L_0. When O_k^0 acts on both side of (6.15), we get

$$\sum_{s,s,s_3,t_3} \chi_m (s,s,s_3,t_3)\, |ss;s_3 t_3\rangle = \sum_{\ell,\alpha} a_\ell\, \alpha |\ell, m, \alpha\rangle \tag{6.17}$$

where χ_m (s,s,s_3,t_3) are matrix elements of 0_k^0 between states of $(0,\nu)$ with the same value of m. There will be as many equations (say w) of the type (6.17) as there are $|ss;s_3t_3\rangle$ states satisfying (6.16) for a given m. Substituting (6.15) in the left hand side of these w equations and comparing the coefficients of the linearly independent states $|\ell,m,\alpha\rangle$ we get the following secular equation

$$|\chi_m\ (s,s,s_3,t_3) - \alpha| = 0 \ . \tag{6.18}$$

We have a secular equation for each value of m which occurs in the $(0,\nu)$ representation; the order of the secular equation increases, in general, when the absolute value of m decreases and is maximum when m = 0 or equivalently is equal to the number of SO(3) representations contained in $(0,\nu)$. Now since the eigenvalue α of 0_k^0 is independent of the m value of the state, we can solve (6.18) in the case where m = 0 and get all eigenvalues α at once. The prescription is therefore the following : for each SO(5) IR $(0,\nu)$ evaluate the matrix elements of 0_k^0 between $SO(5) \supset SU(2) \times SU(2)$ states with m = 0 and then diagonalize it; since we only use m = 0 states it is easily shown that the expression for 0_k^0 given in (6.9) simplifies to the following

$$\begin{aligned} 0_k^0 = {} & 2\sqrt{\tfrac{3}{5}}\ Q_{-1}L_+ - 2\sqrt{\tfrac{3}{5}}\ Q_{+1}L_- - \sqrt{\tfrac{3}{5}}\ Q_{-1}L_+L_+L_- \\ & + \sqrt{\tfrac{3}{5}}\ Q_{+1}L_-L_+L_- + \sqrt{6}\ Q_{-2}L_+^2 - \sqrt{6}\ Q_{+2}L_-^2 \\ & + Q_{-3}L_+^3 \quad - Q_{+3}L_-^3 \ . \end{aligned} \tag{6.19}$$

In order to compare our results with those of De Meyer and Vanden Berghe we have diagonalized the operator $X^{(4)}$ defined as follows

$$\chi^{(4)} = \frac{5}{6\sqrt{2}} \left(-\frac{420}{593.97}\right) \; O_k^0 . \tag{6.20}$$

Although we both use the expression (6.9), there is some arbitrariness in the definition of Q_μ as given by (6.14) which explains the factor $\left(-\frac{420}{593.97}\right)$; the other factor is (as they did) to obtain integer values whenever possible.

The diagonalization involves the solution of algebraic equations whose orders are equal to the degeneracies of the SO(3) representation in $(0,\nu)$. Double, threefold and fourfold degeneracies can be treated analytically, at least in principle, by solving quadratic, cubic and quartic equations. Cases of higher degeneracy can be treated only numerically. We wrote a computer program which performs all these calculations (that is, evaluation of matrix elements and diagonalization). Matrix elements which proved useful in writing the program are given in appendix C. Our results are given in table V. Comparing table IV and V we see that our eigenvalues correspond exactly to those given by De Meyer and Vanden Berghe. In table VI we compare the values of $\frac{5}{6\sqrt{2}} \alpha_{\nu,\ell}$ for singular eigenvalues as predicted by the formulas given in (6.11) with those we obtained. As the table shows our values correspond to those conjectured; this was the case for all representations considered. We also calculated the $SO(5) \supset SO(3)$ eigenvectors for which $m = 0$. These eigenvectors are some linear combination of $SO(5) \supset SU(2) \times SU(2)$ states with $t_3 + 3s_3 = 0$; the coefficients are given in table VII.

CHAPTER VII

CONCLUSION

In this thesis we have established a basis for all tensors in the enveloping algebra of simple compact groups of rank ≤ 3 and have discussed in detail for the groups SU(3) and SO(5) how this basis reduces when the tensors are acting on the bases of degenerate representations; we have shown how this collapse of the GF for tensors may be understood in terms of certain identities among the generators and in connexion with the missing label problem, we showed how the GF for tensors and their reduced forms may be useful to obtain a GF for subgroup scalars in the enveloping algebra of a group. A new function, the group-subgroup characteristic function, was introduced in chapter III; it proved to be useful in transforming a GF for subgroup tensors into a GF for group tensors.

Most of the techniques discussed in this thesis could in principle be used for higher rank groups; however, the difficulty of application increases rapidly with the number of generators so that we suggest that one uses the computer whenever possible. In the next few paragraphs we shall point out some of the calculations that could be done by computer.

When one calculates the GF for tensors in the enveloping algebra of a group, an important phase of the calculation (depending on the method used) is that of the testing of the results and, as we discussed in chapter IV, this can be done analytically but for groups of rank > 3 the algebra would soon get out of hand. This reduction can be done quite easily by computer and as shown in chapter IV, the test proved to be very efficient; it is suggested that for high rank groups one uses quadruple precision.

One can also use the computer in connexion with the elementary multiplet method; the program which we discussed in chapter III proved to be very useful, leaving to the user the problem of guessing the elementary factors and syzygies; however, even with the usage of a program, it could be difficult to choose the right elementary multiplets and sysygies since for high rank groups there could be many possibilities. We suggest to divide the problem by making use, whenever possible, of intermediate groups (including the subjoining of a group to another). Usually the insertion of intermediate groups reduces considerably the possibilities.

If one chooses to construct a GF for tensors by making use of a subgroup (as we did for SU(3) and SO(5)) the major difficulty is to eliminate from the final expression all negative terms that appear in the numerator and spurious factors such as (A+B) appearing in the denominator, all these unwanted terms being a consequence of the residue calculations. Here again, one can use a computer to reduce the GF to a useful form; this has been done by Gaskell et al.[49] One could also use to eliminate the unwanted terms mentioned above a program such as the one developed by A.C. Hearn[89] called "Reduce 2", which performs symbolic calculations; however, the usefulness of such a program for large expressions remains to be proved.

A final comment in connexion with the theorem of Peccia and Sharp concerning the number of functionally independent missing label operators available in the enveloping algebra of a group. The theorem states that there are, in the case of general representations of the group, twice as many missing label operators available as there is missing labels. Now in the case of $SO(5) \supset SO(3)$ restricted to SO(5) representations of the type $(0,\nu)$ we showed that there are two functionally independent missing label operators available. In the case of representations of the type $(\nu,0)$, substituting (6.2) into (5.37) we get the following GF for SO(3) scalars in the enveloping algebra of SO(5)

$$\frac{1+U^4 + U^7 + U^9}{(1-U^2)^2\ (1-U^4)\ (1-U^6)} ,$$

so that here again we have twice as many missing label operators available (of degrees four and six) as there is missing labels (only one). The GF for branching rules of $SU(3) \supset SO(3)$ informs us that only SU(3) representations with even Cartan labels contain SO(3) scalars so that keeping only terms with even powers in Λ_1 and Λ_2 in (5.30) and then putting $\Lambda_1 = \Lambda_2 = 1$ we get the following GF for SO(3) scalars in the enveloping algebra of SU(3) in the case of representations of the type $(0,\nu)$ and $(\nu,0)$

$$\frac{1 + U^3}{(1-U^2)^2} + \frac{U^3}{(1-U^2)} .$$

This GF informs us that the integrity basis contains only two functionally independent scalars; these are the Casimirs of SU(3) and SO(3) both of degree two. This was expected since there is no missing label, so that here again the result agree with Peccia and Sharp's theorem. We conjecture that this is the case for all groups, i.e., that their theorem remains valid for all representations of a group.

APPENDIX A

GENERATING FUNCTION TECHNIQUES

In this appendix we discuss certain GF techniques that have been developed in the past few years. In section 1 we illustrate how a GF for tensors may be obtained from a GF for weights. In section 2 we discuss a generalization of this approach which makes use ofthe Weyl characteristic function. The elementary multiplet method which has proven very useful in the evaluation of various types of GF's is described in section 3. Finally, in section 4 we briefly cover the problem of the coupling of two GF's and the substitution of one into another.

1. <u>From a generating function for weights to a generating function for tensors</u>[49]

Let us consider the problem of finding a GF for SU(2) tensors based on a $j = 1$ tensor Γ. The weights associated with the three components of Γ are $w_1 = 1$, $w_2 = 0$ and $w_3 = -1$. The tensor products (2.2) are realized in weight space by the following GF for weights (in the following calculations λ is not a Cartan label and $\lambda = j$)

$$W(\eta) = \frac{1}{(1-U\eta)(1-U)(1-U\eta^{-1})} \qquad \text{(A.1)}$$

where the exponents of U and η are respectively the degree in Γ and the weight. The GF for tensors belonging to the IR(λ) (highest weight λ, lowest weight-λ) is the coefficient $C_{-\lambda}$ of $\eta^{-\lambda}$ in (A.1) minus the coefficient $C_{-\lambda-1}$ of $\eta^{-\lambda-1}$ (the subtraction eliminates contributions from higher tensors). This operation will be done in two steps : first extract from (A.1) the coefficients of $\eta^{-\lambda}$ and $\eta^{-\lambda-1}$ and then make the subtraction. The coefficients may be obtained by making use of the following well known result of the theory of complex variables :

$$\oint \eta^a d\eta = 0 \qquad a \neq -1$$
$$= 2\pi i \qquad a = -1 \tag{A.2}$$

where the integration is done around a circle centered on the origin of complex space and a is an integer. Based on (A.2), we may write

$$C_{-\lambda} = \frac{1}{2\pi i} \oint \frac{\eta^{\lambda-1}\, d\eta}{(1-U\eta)(1-U)(1-U\eta^{-1})} \tag{A.3a}$$

$$C_{-\lambda-1} = \frac{1}{2\pi i} \oint \frac{\eta^{\lambda} d\eta}{(1-U\eta)(1-U)(1-U\eta^{-1})} . \tag{A.3b}$$

The above approach implies a power series expansion for the various fractions $(1-Z)^{-1}$ on the right hand side of (A.3) and therefore imposes the following condition on the norms of the Z's

$$|Z| < 1 \tag{A.4}$$

which in turn imposes certain restrictions on the norms of U and η. Condition (A.4) is satisfied if we choose $|U| < 1$ and integrate η about a unit circle. The GF for tensors belonging to the IR (λ) is therefore

$$C_{-\lambda} - C_{-\lambda-1} = \frac{1}{2\pi i} \oint \frac{(\eta^{\lambda-1} - \eta^{\lambda})\, d\eta}{(1-U\eta)(1-U)(1-U\eta^{-1})}$$
$$= \Sigma \, \mathrm{Res}_{\eta} \frac{(\eta^{\lambda-1} - \eta^{\lambda})}{(1-U\eta)(1-U)(1-U\eta^{-1})} \tag{A.5}$$

where $\Sigma\ \mathrm{Res}_\eta$ represents the sum of residues of poles of the η variable inside the unit circle. Our aim is to obtain a GF for all tensors (all λ) so that multiplying (A.5) by Λ^λ (to keep track of the representations) and summing over λ we get the desired GF for tensors

$$G(U;\Lambda) = \sum_{\lambda=0}^{\infty} (C_{-\lambda} - C_{-\lambda-1})\Lambda^\lambda$$

$$= \sum_{\lambda=0}^{\infty} \Sigma\ \mathrm{Res}_\eta \frac{(\eta^{\lambda-1} - \eta^\lambda)\,\Lambda^\lambda}{(1-U\eta)(1-U)(1-U\eta^{-1})}\ ; \qquad (A.6)$$

the sum over λ is geometric and may be done immediately so that (A.6) becomes

$$G(U;\Lambda) = \Sigma\ \mathrm{Res}_\eta \frac{(\eta^{-1} - 1)}{(1-U\eta)(1-U)(1-U\eta^{-1})(1-\eta\Lambda)}\ ;$$

the sum over residues is easily done so that we finally get

$$G(U;\Lambda) = \frac{1}{(1-U^2)\ (1-U\Lambda)}\ . \qquad (A.7)$$

(A.7) gives us a basis for all irreducible SU(2) tensors whose components are polynomials in the components of the SU(2) vector Γ. U and Λ carry respectively the degree in Γ and the representation label. $G(U;\Lambda)$ may be interpreted in terms of the following set of elementary tensors (μ,λ) where μ stands for the degree in Γ and λ the representation label : (2,0) and (1,1) with no redundant combination. We now turn to a generalization of this approach for all compact semisimple groups.

2. How to use the Weyl characteristic function to obtain a generating function for polynomial tensors [49]

The character χ_λ of a representation (λ) may be written

$$\chi_\lambda = \sum_i N_i \eta^{w_i},\quad \eta^{w_i} = \prod_{j=1}^{\ell_G} \eta_j^{(w_i)j}$$

where η_j carries the j^{th} component of the weight w_i and N_i is the multiplicity of w_i in (λ); ℓ_G is the rank of the group. For example the character function for the representation $j = 1$ $(\lambda=2)$ of SU(2) is

$$\chi_2 = 1 + \eta^2 + \eta^{-2}.$$

Weyl[90] has given an explicit formula for calculating the character of any representation of simple groups, namely

$$\chi_\lambda(\eta) = \frac{\xi_\lambda(\eta)}{\Delta(\eta)}, \quad \xi_\lambda \equiv \sum_S (-1)^S \eta^{(S\bar{R})}, \quad \eta^{\bar{R}} \equiv \prod_i \eta_i^{(\bar{R})_i} \tag{A.8}$$

where the sum is over the Weyl reflections S and $(-1)^S$ is the determinant of the matrix of S; $(-1)^S$ is + 1 if S is a product of an even number of reflections and -1 if it is an odd number. $\bar{R}$ is a vector in ℓ_G dimensional space defined by

$$\bar{R} = R + W_\lambda \tag{A.9}$$

where R is half the sum of the positive roots of G and W_λ is the highest weight of the representation (λ). ξ_λ is therefore a linear combination of terms $\prod_i \eta_i^{p_i}$ whose exponents p_i depend linearly on the representation labels; it is known as the characteristic of the representation (λ). $\Delta(\eta)$ is the characteristic of the scalar representation

$$\Delta(\eta) = \xi_0 = \sum_S (-1)^S \eta^{(SR)}, \quad \eta^R \equiv \prod_i \eta_i^{(R)_i} \tag{A.10}$$

when plotted in ℓ_G dimensional space, ξ_λ corresponds to a set of points equidistant from the origin called a girdle, uniquely characterizing the representation (λ), i.e., each representation corresponds to a different set of points and there is no overlap between the sets. For a given group, the number of points N in a set is independent of the representation. The space is divided in N sectors, which are called defining sectors, each term of ξ_λ belonging to a different one. There is therefore a one to one correspondance between terms belonging to a sector and representations of the group. The sector corresponding to the highest weights is called the dominant sector. For every semisimple compact Lie group, such sectors may be defined. The characteristic function for simple Lie algebras of rank two have been given by Behhends,Dreitlein, Fronsdal and Lee[91]. For example, using their results, but following Weyl's convention concerning highest weights and positive roots and choosing the η_1, η_2 variables of weight space such that the highest weights $\bar{w}_1$ ans $\bar{w}_2$ of the two fundamental irreducible representations of SU(3) that is, (1,0) and (0,1), are respectively $\bar{w}_1$ = (1,1) and $\bar{w}_2$ = (2,0), the characteristic function $\xi_{\lambda_1\lambda_2}$ for SU(3) is

$$\xi_{\lambda_1\lambda_2} = -\eta_1^{-(\lambda_2+1)}\,\eta_2^{-(2\lambda_1+\lambda_2+3)} + \eta_1^{(\lambda_1+1)}\,\eta_2^{(\lambda_1+2\lambda_2+3)} - \eta_1^{-(\lambda_1+1)}\,\eta_2^{(\lambda_1+2\lambda_2+3)} \tag{A.11}$$

$$+ \eta_1^{(\lambda_2+1)}\,\eta_2^{-(2\lambda_1+\lambda_2+3)} - \eta_1^{(\lambda_1+\lambda_2+2)}\,\eta_2^{(\lambda_1-\lambda_2)} + \eta_1^{-(\lambda_1+\lambda_2+2)}\,\eta_2^{(\lambda_1-\lambda_2)}$$

For the scalar and octet representations we get

$$\xi_{11} = -\eta_1^{-2}\,\eta_2^{-6} + \eta_1^2\eta_2^6 - \eta_1^{-2}\,\eta_2^6 + \eta_1^2\eta_2^{-6} - \eta_1^4 + \eta_1^{-4} \tag{A.12}$$

$$\xi_{00} = -\eta_1^{-1}\,\eta_2^{-3} + \eta_1\eta_2^3 - \eta_1^{-1}\,\eta_2^3 + \eta_1\eta_2^{-3} - \eta_1^2 + \eta_1^{-2}$$

The girdles corresponding to ξ_{11} and ξ_{00} are shown in figure 9. In this particular case, the space is divided in six sectors (the shaded areas including border lines). The terms of ξ_{11} are represented by dots and those of ξ_{00} by X's.

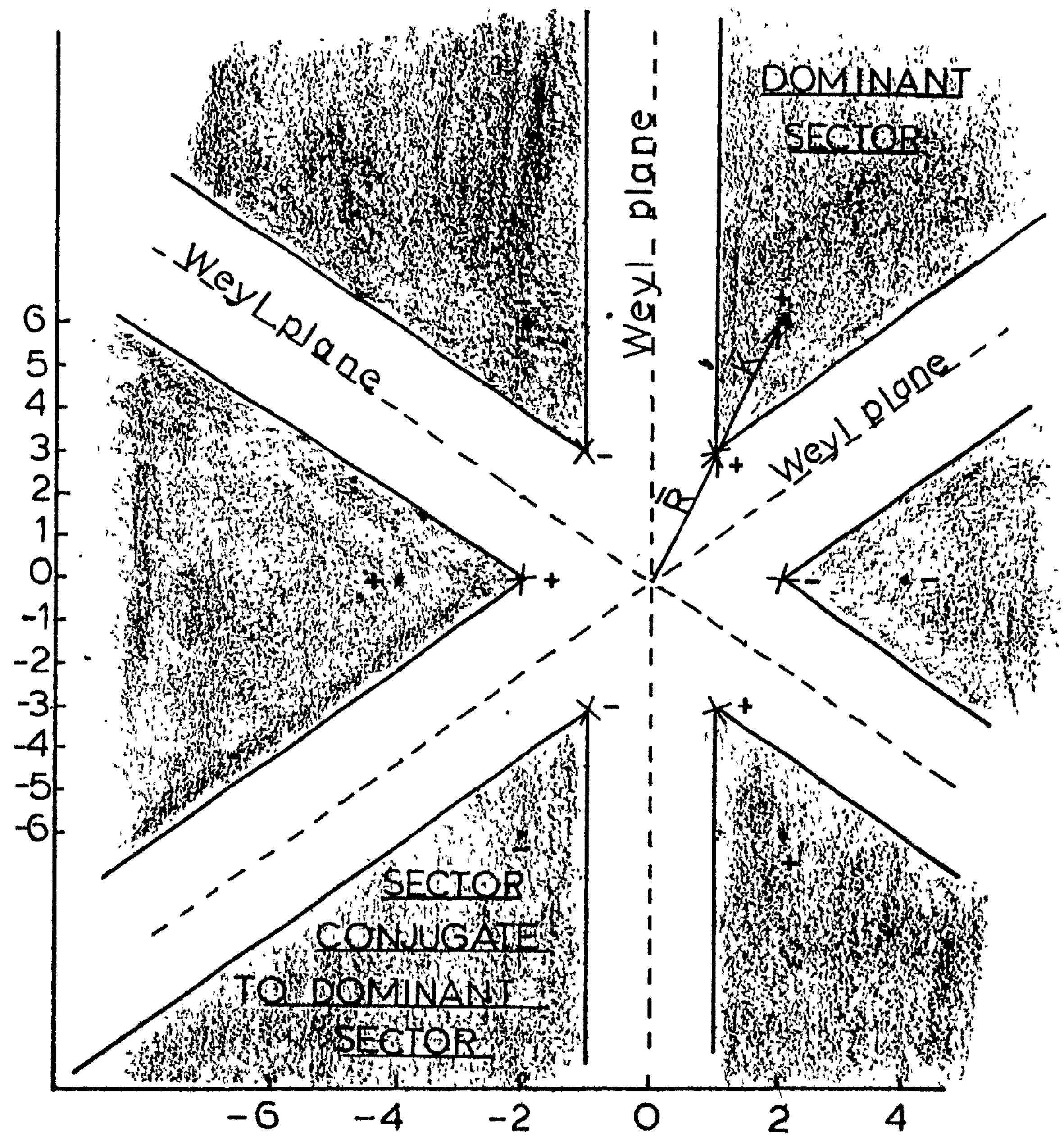


Figure 9. The Weyl characteristic function for the scalar (X) and octet (·) representation of SU(3).

Coming back to our original problem of transforming a weight GF $W(\eta)$ into the corresponding GF for tensors, and assuming that the weights are those of complete IR, we may write

$$W(\eta) = \sum_{\lambda} \chi_{\lambda_1,\ldots,\lambda_{\ell_G}} N_\lambda \tag{A.13}$$

where χ_λ is the character of the IR (λ); N_λ is essentially the multiplicity of λ in $W(\eta)$, and may depend on other dummy variables such as U in (A.1). Inserting (A.8) in (A.13) gives

$$W(\eta) = \sum_{\lambda} \frac{\xi_\lambda}{\Delta} N_\lambda . \tag{A.14}$$

Multiplying (A.14) by Δ we get

$$\Delta.W(\eta) = \sum_{\lambda} \xi_\lambda N_\lambda . \tag{A.15}$$

(A.15) suggests the following : the presence of a IR (λ) in $W(\eta)$ is indicated by the presence of the corresponding ξ_λ in the product $\Delta.W(\eta)$; based on the preceeding discussion of the characteristic function, we see that in order to identify ξ_λ in $\Delta.W(\eta)$ we may limit ourselves to any of the defining sectors (six in the case of SU(3)); i.e., look for terms in the product $\Delta.W(\eta)$ belonging to a given sector (since other sectors give no new information). Therefore the prescription for transforming $W(\eta)$ into a GF for tensors is

(1) Choose a sector

(2) Multiply $W(\eta)$ by Δ

(3) For each term found in $\Delta.W(\eta)$ that belongs to the chosen sector, replace it (keeping N_λ) by dummy variables that carry the representation labels as exponents and drop all other terms in $\Delta.W(\eta)$.

The resulting expression is the desired GF for tensors. Formally this is done by the following sum of residues

$$\Sigma\ \mathrm{Res}_{\eta} \left[\sum_{\lambda} \{ (\prod_{i=1}^{\ell_G} \eta_i^{-p_i-1} \prod_{j=1}^{\ell_G} \Lambda_j^{\lambda_j}) \Delta(\eta)\ W(\eta) \} \right] \qquad (A.16)$$

where the Λ_j variables carry the representation labels as their exponents. $\Sigma\ \mathrm{Res}_\eta$ means the sum of residues of poles of the variables $\eta_1,\ldots,\eta_{\ell_G}$ inside circles of unit radius. The norm of all other variables are considered smaller then unity; the term $\prod_i \eta_i^{p_i}$ belong to the chosen sector. The choice of the sector conjugate to the dominant sector (see figure 9) usually simplifies the calculations. If in principle all GF for tensors may be obtained from the corresponding GF for weights by the above method, in practice, the calculations may soon become out of hand. We now discuss another approach.

3. <u>How to construct a generating function for polynomial tensors using the elementary multiplet method</u>

The problem of finding a basis for all irreducible tensors obtained from the products (2.2) may be looked at in terms of the following reduction problem

$$(\Gamma)^{\mu} = \sum_{\lambda} C_{\mu\lambda}\, T(\mu,\lambda) \qquad \mu=1,\infty \qquad (A.17)$$

where $T(\mu,\lambda)$ is an irreducible tensor of degree μ in the components of Γ, which transforms as (λ) and whose multiplicity is $C_{\mu\lambda}$. Now if Γ is an n dimensional group tensor, $(\Gamma)^{\mu}$ is a $\binom{n+\mu-1}{\mu}$ dimensional reducible tensor; but $\binom{n+\mu-1}{\mu}$ is the dimension of the representation $(\mu,0,\ldots,0)$ of SU(n) which suggests that the reduction problem (A.17) is equivalent to that of finding a GF for the branching rules $SU(n)\supset G$ restricted to the symmetric representations of SU(n). For example if Γ is a $j=2$ SU(2) tensor the chain considered is $SU(5)\supset SU(2)$ with the embedding $(1000)\supset(4)$; (2.3) given in chapter II may therefore be viewed as a group-subgroup GF for the chain

$SU(5) \supset SU(2)$ restricted to the representations $(\mu,0,0,0)$ of SU(5) where U carries the SU(5) label μ as exponent and Λ that of SU(2) or as a GF for SU(2) tensors based on a $j=2$ tensor. The GF for branching rules may be obtained by a method which consists, as discussed in chapter II, of finding a finite set of elementary factors and relations among them. We now discuss this method, that is, the elementary multiplet method.

The elementary factors (elementary multiplets) and relations among them (syzygies) are found by proceeding systematically through the IR's of the group. The subgroup contents of low dimensional representations of the group can often be found in tables like those of McKay and Patera[92], but for higher representations, much guess work is involved. Dimension and second order index checks guide the selection of elementary factors and syzygies. As an example let us consider the problem of obtaining a group-subgroup GF for the chain $G_2 \supset SU(3)$. For (10), we use McKay and Patera's tables

$$(10) \supset (10) + (01) + (00) \quad . \tag{A.18}$$

(A.18) implies the following elementary factors

$$a = (10;10), \quad b = (10;01), \quad c = (10,00). \tag{A.19}$$

We now consider (20). The SU(3) contents of (20) is also given in the tables but in order to illustrate the method we will ignore it. We first take all products of known elementary multiplets (those of (A.19)) that give (20) i.e.

$$\begin{aligned} a.a &= (20;20), \quad a.b = (20;11), \\ a.c &= (20;10), \quad b.b = (20;02), \\ b.c &= (20;01), \quad c.c = (20;00). \end{aligned} \tag{A.20}$$

In order to check if any products are forbidden or if any new elementary multiplets must be added to the list (A.19), we make a dimension and index check (the second order index check being particularly useful in more complicated problems). It turns out that all products (A.20) are necessary and sufficient which implies no new elementary multiplet and no syzygy. We therefore have.

$$(20) \supset (20) + (11) + (10) + (02) + (01) + (00).$$

We then proceed to (30). We have the following possible products

$$\begin{aligned} a.a.a &= (30;30), & a.a.b &= (30;21), \\ a.a.c &= (30;20), & a.b.c &= (30;11), \\ a.b.b &= (30;12), & a.c.c &= (30;10), \\ b.b.b &= (30;03), & b.c.c &= (30;01), \\ b.b.c &= (30;02), & c.c.c &= (30;00). \end{aligned}$$

A dimension and index checks reveals that no products are redundant and no new elementary factors are needed so that

$$(30) \supset (30) + (21) + (20) + (12) + (10) + (03) + (11) + (01) + (02) + (00).$$

We could keep on, but we will assume that the elementary factors given in (A.19) are sufficient for all $(\lambda,0)$ of G_2 and that no products are redundant. We now consider (01) of G_2. From the tables we have that

$$(01) \supset (11) + (01) + (10).$$

Following the same procedure as above, we find that the following elementary factors

$$d = (01;11), \quad e = (01;01), \quad f = (01;10) \tag{A.21}$$

with no redundant combinations are sufficient for all $(0,\lambda)$ of G_2. We then consider (11) and must now form all products of elementary factors that give (11); these are

$$a.d = (11;21), \quad a.e = (11;11), \quad a.f = (11;20),$$
$$b.d = (11;12), \quad b.e = (11,02), \quad b.f = (11;11), \qquad (A.22)$$
$$c.d = (11;11), \quad c.e = (11;01), \quad c.f = (11;10).$$

If all products in (A.12) were allowed, we would have the following decomposition

$$(11)\supset(21) + 3(11) + (20) + (12) + (02) + (01) + (10). \qquad (A.23)$$

A dimension check shows that the right hand side of (A.23) exceeds the dimension by 8 which suggests that one of the following products be chosen as forbidden a.e , b.f or c.d. At this point there is no rule of thumb and often use must work on a trial and error basis. Let us consider each possibility and first declare the product a.e forbidden. We now consider (12); the products leading to (12) are

$$a.f.d = (12;31), \quad a.f.f = (12;30),$$
$$a.d.d = (12;32), \quad c.e.e = (12;02),$$
$$c.e.f = (12;11), \quad c.e.d = (12;12),$$
$$c.f.d = (12;21), \quad c.f.f = (12;20),$$
$$c.d.d = (12;22), \quad b.e.e \quad (12;03),$$
$$b.e.f = (12;12), \quad b.e.d = (12;13),$$
$$b.f.d = (12;22), \quad b.f.f = (12;21),$$
$$b.d.d = (12;23)$$
$$a.e.e = (12;12) \quad \text{forbidden}$$
$$a.e.f = (12;21) \quad \text{forbidden}$$
$$a.e.d = (12;22) \quad \text{forbidden}$$

where the forbidden products have been indicated. Therefore assuming no new elementary multiplet and no new forbidden product, we have that

$$(12)\supset(31) + (30) + (32) + (03) + 2(12) + (13) + 2(22)$$
$$+ 2(21) + (23) + (02) + (11) + (20).$$

A dimension and index check shows us that our assumption is correct. We could proceed through higher representations but we will assume that the set {a,b,c,d,e,f} of elementary factors with the product a.e. forbidden is complete, that is, there is a one to one correspondence between allowed products of elementary factors of that set and SU(3) multiplets contained in G_2 representations. Similar considerations would lead us to the conclusion that the choices b.f or c.d as redundant products are equally valid and introduces no new elementary factor. We therefore have the following set of elementary factors {a,b,c,d,e,f} with a.e or b.f or c.d forbidden. The choice a.e leads to the following GF

$$G_1(\Lambda_1,\Lambda_2;N_1,N_2) = \frac{1}{(1-\Lambda_1)(1-\Lambda_2N_1N_2)(1-\Lambda_1N_2)(1-\Lambda_2N_1)} \times \left[\frac{1}{(1-\Lambda_1N_1)} + \frac{\Lambda_2N_2}{(1-\Lambda_2N_2)}\right];$$

choosing b.f we get

$$G_2(\Lambda_1,\Lambda_2;N_1,N_2) = \frac{1}{(1-\Lambda_1)(1-\Lambda_2N_1N_2)(1-\Lambda_1N_1)(1-\Lambda_2N_2)} \times \left[\frac{1}{(1-\Lambda_1N_2)} + \frac{\Lambda_2N_1}{(1-\Lambda_2N_1)}\right];$$

finally the choice c.d leads to[49]

$$G_3(\Lambda_1,\Lambda_2;N_1,N_2) = \frac{1}{(1-\Lambda_1N_1)(1-\Lambda_1N_2)(1-\Lambda_2N_1)(1-\Lambda_2N_2)} \times \left[\frac{1}{(1-\Lambda_1)} + \frac{\Lambda_2N_1N_2}{(1-\Lambda_2N_1N_2)}\right].$$

where Λ_1,Λ_2 carry the G_2 representation labels as their exponents and N_1, N_2 those of SU(3). It is easily shown that

$$G_1 = G_2 = G_3 \ .$$

In this simple example all choices of syzygies were equally good, but in more complicated problems it is often not the case. Depending on the choice we make, we may have to introduce new elementary factors and forbidden products. For instance we could have choosen b.f and c.d as forbidden and then introduce a new elementary factor (11;11) in order to balance dimension and index, and then proceed to higher representations to see if we have a complete set of elementary factors. Although no rule forbids such a choice, this could lead us to a situation were we must constantly introduce new elementary factors and forbidden products in order to balance dimension and index; this would not prove that this particular choice is wrong, but would certainly suggest that it is not the most efficient way of solving the problem. The set of elementary factors is usually not unique and leads to different forms of GF's although all equivalent.

This approach does not constitute a proof that we do have a complete set and that all syzygies have been found so that the GF obtained by such a method must be checked. The GF for tensors in the enveloping algebra of the groups SU(4), Sp(6) and SO(7) have been found by the elementary multiplet technique and ways of testing the results are discussed in chap IV.

It often occurs in this thesis that we must substitute one GF into another or couple two GF's to obtain a third one; we shall now briefly discuss these two procedures.

4. Substitution and coupling techniques[46]

Given two GF's $G_1(U';\Lambda_1',\ldots,\Lambda_{\ell_G}')$ and $G_2(U'';\Lambda_1'',\ldots,\Lambda_{\ell_G}'')$ respectively based on group G tensors Γ_1 and Γ_2, our aim is to get a GF $G_3(U',U'';\Lambda_1,\ldots,\Lambda_{\ell_G})$

based on Γ_1 and Γ_2. If the GF for the Clebsch-Gordan series $C(\Lambda'_1,\ldots,\Lambda'_{\ell_G};\Lambda''_1,\ldots,\Lambda''_{\ell_G};\Lambda_1,\ldots,\Lambda_{\ell_G})$ is known, the answer is (ℓ_G is equal to the rank of G)

$$G_3(U',U'';\Lambda_1,\ldots,\Lambda_{\ell_G}) = \Sigma \operatorname{Res}_{\Lambda'\Lambda''} \{G_1(U';\Lambda_1'^{-1},\ldots,\Lambda_{\ell_G}'^{-1}) \tag{A.24a}$$

$$\times G_2(U'';\Lambda_1''^{-1},\ldots,\Lambda_{\ell_G}''^{-1})\, C\,(\Lambda'_1,\ldots,\Lambda'_{\ell_G};\Lambda''_1,\ldots,\Lambda''_{\ell_G};\Lambda_1,\ldots,\Lambda_{\ell_G})$$

$$\times \prod_{i=1}^{\ell_G} (\Lambda'_i)^{-1} \prod_{j=1}^{\ell_G} (\Lambda''_j)^{-1}\}$$

or

$$G_3(U',U'';\Lambda_1,\ldots,\Lambda_{\ell_G}) = \Sigma \operatorname{Res}_{\Lambda'\Lambda''} \{G_1(U';\Lambda'_1,\ldots,\Lambda'_{\ell_G}) \tag{A.24b}$$

$$\times G_2(U'';\Lambda''_1,\ldots,\Lambda''_{\ell_G})\, C\,(\Lambda_1'^{-1},\ldots,\Lambda_{\ell_G}'^{-1};\Lambda_1''^{-1},\ldots,\Lambda_{\ell_G}''^{-1};\Lambda_1,\ldots,\Lambda_{\ell_G})$$

$$\times \prod_{i=1}^{\ell_G} (\Lambda'_i)^{-1} \prod_{j=1}^{\ell_G} (\Lambda''_j)^{-1}\}$$

where $\Sigma \operatorname{Res}_{\Lambda'\Lambda''}$ means the sum of residues of poles of the variales Λ'_i and Λ''_j inside circles of a certain radius. The radius of these circles and the norms of the other variables (such as U',U'',Λ_j) are chosen following arguments similar to those which led to condition (A.4). There are several other ways of coupling G_1 and G_2 which are actually a mixture of (A.24a) and (A.24b); in those cases, only certain Λ'_i and Λ''_j of G_1 and G_2 are replaced by their reciprocals, the others are replaced by their reciprocals in the GF for the Clebsch-Gordan series. In any case, we use whichever is easier to evaluate. Examples of such calculations are given in chapter III when evaluating the GF for tensors in the enveloping algebra of groups SU(3) and SO(5).

The procedure by which we substitute one GF into another has proven very useful in testing certain results of this thesis; this technique may serve many other purposes such as converting an integrity basis for group tensors into the corresponding one for subgroup tensors. The problem of interest to us is the following : given two GF's for branching rules, $G_1(\Lambda_1'',\ldots,\Lambda_{\ell_{G''}}'';\Lambda_1',\ldots,\Lambda_{\ell_{G'}}')$ for the chain or groups $G''\supset G'$ and $G_2(\Lambda_1',\ldots,\Lambda_{\ell_{G'}}';\Lambda_1,\ldots,\Lambda_{\ell_G})$ for the chain $G'\supset G$ we want to evaluate the GF $G_3(\Lambda_1'',\ldots,\Lambda_{\ell_{G''}}'';\Lambda_1,\ldots,\Lambda_{\ell_G})$ for the branching rules of the chain $G''\supset G$. G_3 is obtained by substituting G_2 into G_1; this is done by the following residue calculations

$$G_3(\Lambda_1'',\ldots,\Lambda_{\ell_{G''}}'';\Lambda_1,\ldots,\Lambda_{\ell_G}) = \Sigma \operatorname{Res}_{\Lambda'} \{ G_1(\Lambda_1'',\ldots,\Lambda_{\ell_{G''}}'';\Lambda_1'^{-1},\ldots,\Lambda_{\ell_{G'}}'^{-1}) \tag{A.25a}$$

$$\times\, G_2(\Lambda_1',\ldots,\Lambda_{\ell_{G'}}';\Lambda_1,\ldots,\Lambda_{\ell_G}) \prod_{i=1}^{\ell_{G'}} (\Lambda_i')^{-1}\}$$

or

$$G_3(\Lambda_1'',\ldots,\Lambda_{\ell_{G''}}'';\Lambda_1,\ldots,\Lambda_{\ell_G}) = \Sigma \operatorname{Res}_{\Lambda'} \{ G_1(\Lambda_1'',\ldots,\Lambda_{\ell_{G''}}'';\Lambda_1',\ldots,\Lambda_{\ell_{G'}}') \tag{A.25b}$$

$$\times\, G_2(\Lambda_1'^{-1},\ldots,\Lambda_{\ell_{G'}}'^{-1};\Lambda_1,\ldots,\Lambda_{\ell_G}) \prod_{i=1}^{\ell_{G'}} (\Lambda_i')^{-1}\}.$$

As in the case of (A.24), the norms of the variables are chosen such that G_1 and G_2 on the right hand side of (A.25) be expandable in a power series. Here again, G_3 may be also obtained from a mixture (in the sense given above) of (A.25a) and (A.25b). Examples of such substituons are given in chapter IV where we discuss methods of testing results.

APPENDIX B

GROUP-SUBGROUP CHARACTERISTIC FUNCTION

We define the group-subgroup characteristic function for the group-subgroup $G \supset H$ as

$$\xi_\lambda^H (\eta) = (\Delta(\eta)/\Delta'(\eta)) \sum_\nu c_{\lambda\nu} \eta^{K_\nu}. \tag{B.1}$$

The symbols in (B.1) must now be defined.

$c_{\lambda\nu}$ is the multiplicity of the H representation ν in the G representation λ. η^{K_ν} means $\prod_i \eta_i^{(K_\nu)_i}$, where $(K_\nu)_i$ is the i^{th} component of the vector

$$\bar{K}_\nu = \bar{R} + \bar{W}_\nu; \tag{B.2}$$

$\bar{R}$ is half the sum of the positive roots of H, and $\bar{W}_\nu$ is the highest weight of the representation ν.

$$\Delta'(\eta) = \sum_S (-1)^S \eta^{SR} \tag{B.3}$$

is Weyl's characterictic function for the scalar representation of H; the sum is over Weyl reflections S, and $(-1)^S$ is the determinant of the matrix of S.

Similarly $\Delta(\eta)$ is Weyl's characteristic function for the scalar representation of G in which (if necessary) a projection onto the weight space of H has been effected by substituting for the variables in terms of the η appropriate to H.

The constructive definition (B.1) permits the evaluation of $\xi_\lambda^H(\eta)$ for any representation λ for which the branching multiplicities $c_{\lambda\nu}$ are known. In particular, for the scalar representation, $c_{0\nu} = \delta_{0\nu}$ and

$$\xi_0^H(\eta) = \Delta(\eta)\eta^R/\Delta'(\eta). \tag{B.4}$$

Dividing (B.1) by (B.4) we obtain

$$\xi_\lambda^H(\eta)/\xi_0^H(\eta) = \sum_\nu c_{\lambda\nu}\eta^{w_\nu}. \tag{B.5}$$

If one substitutes for the variables η in terms of new variables N so that $\eta^{w_\nu} = N^\nu$, the result is equation (3.2).

We now discuss some properties of the group-subgroup characteristic function. First we sketch a proof that $\xi_\lambda^H(\eta)$ is a sum of monomials. Weyl[93] shows that the vectors $S\bar{R}$ for any compact Lie group are possible weights of that group. After projection onto the weight space of a subgroup, they will be possible subgroup weights. From this it can be shown that $\Delta(\eta)$ is a linear combination of Weyl characteristic functions $\xi_\nu(\eta)$ of H, each of which is, of course, divisible by $\Delta'(\eta)$. Since $\Delta(\eta)/\Delta'(\eta)$ is a sum of monomials, it follows from (B.1) that $\xi_\lambda^H(\eta)$ is also a sum of monomials.

We can say something about the distribution of the terms of $\xi_\lambda^H(\eta)$ in weight space. From the form of (B.1) it is clear that they lie in or near the dominant sector of H weight space, the sector of highest weights of H representations (the terms of Δ/Δ' are independent of λ and cannot shift them far). Weyl's[93] characteristic function for the subgroup H is

$$\xi_\nu(\eta) = \sum_S (-1)^S \eta^{SK_\nu}. \tag{B.6}$$

In terms of it the character function is

$$\chi_\nu(n) = \xi_\nu(n)/\Delta'(n). \tag{B.7}$$

The symbols are defined as in (B.2) and (B.3). There are similar equations for the group G. From the additivity of the characters under the reduction G to H.

$$\chi_\lambda(n) = \sum_\nu c_{\lambda\nu}\chi_\nu(n), \tag{B.8}$$

and (B.6), (B.7) we obtain

$$\xi_\lambda(n) = (\Delta/\Delta') \sum_\nu c_{\lambda\nu}\xi_\nu(n), \tag{B.9}$$

which incidentally suggests an efficient way of calculating branching rules (just divide ξ_λ by Δ/Δ and retain the part of the quotient in the dominant sector of subgroup weight space). Now the number of terms in $\xi_\lambda(n)$ is fixed (independent of λ), and they all lie equidistant from the origin of weight space, at least before projection onto H weight space; much cancellation occurs between the terms on the right-hand side of (B.9). Now our group-subgroup characteristic function (B.1) differs from (B.9) only in that it lacks the sum over Weyl reflections S inplicit in the definition (B.6) of $\xi_\nu(n)$. Hence most of the cancellation in (B.9) persists, since the parts of (B.9) coming from different sectors (under Weyl reflections of H) cannot cancel mutually except near the boundaries of the sectors, because of the small shifts due to Δ/Δ'. We can conclude that the terms of $\xi_\lambda^H(n)$ are either terms from $\xi_\lambda(n)$ which project into the dominant H sector, or else lie on or near the boundaries of the dominant H sector.

APPENDIX C

MATRIX ELEMENTS OF SO(5)⊃SU(2) x SU(2) GENERATORS

Here we give formulas which proved useful in evaluating the matrix elements of the missing label operator between SO(5)⊃SU(2) x SU(2) states with $m = 0$; these were obtained following Sharp and Pieper's[77] general formula for matrix elements.

$$\langle s\ s;\ s_3\ t_3 \mid U_- \mid s+\tfrac{1}{2}\ s+\tfrac{1}{2};\ s_3+\tfrac{1}{2}\ t_3+\tfrac{1}{2} \rangle$$

$$= - \left[\frac{(\nu + 2s + 3)\ (\nu - 2s)\ (s + s_3 + 1)\ (s + t_3 + 1)}{(2s + 2)\ (2s + 1)}\right]^{\frac{1}{2}}$$

$$\langle s\ s;\ s_3\ t_3 \mid U_- \mid s-\tfrac{1}{2}\ s-\tfrac{1}{2};\ s_3+\tfrac{1}{2}\ t_3+\tfrac{1}{2} \rangle$$

$$= \left[\frac{(\nu + 2s + 2)\ (\nu - 2s + 1)\ (s - s_3)\ (s - t_3)}{(2s + 1)\ (2\ s)}\right]^{\frac{1}{2}}$$

$$\langle s\ s;\ s_3\ t_3 \mid U_+ \mid s+\tfrac{1}{2}\ s+\tfrac{1}{2};\ s_3-\tfrac{1}{2}\ t_3-\tfrac{1}{2} \rangle$$

$$= \left[\frac{(\nu + 2\ s + 3)\ (\nu - 2\ s)\ (s - s_3 + 1)\ (s - t_3 + 1)}{(2s + 2)\ (2s + 1)}\right]^{\frac{1}{2}}$$

$$\langle s\ s;\ s_3\ t_3 \mid U_+ \mid s-\tfrac{1}{2}\ s-\tfrac{1}{2};\ s_3-\tfrac{1}{2}\ t_3-\tfrac{1}{2} \rangle$$

$$= - \left[\frac{(\nu + 2s + 2)\ (\nu - 2s + 1)\ (s + s_3)\ (s + t_3)}{(2s + 1)\ (2\ s)}\right]^{\frac{1}{2}}$$

$$\langle s\ s;\ s_3\ t_3 \mid V_- \mid s+\tfrac{1}{2}\ s+\tfrac{1}{2}; s_3+\tfrac{1}{2}\ t_3-\tfrac{1}{2}\rangle$$

$$= \left[\frac{(\nu + 2s + 3)\ (\nu - 2s)\ (s + s_3 + 1)\ (s - t_3 + 1)}{(2s + 2)\ (2s + 1)}\right]^{\frac{1}{2}}$$

$$\langle s\ s;\ s_3\ t_3 \mid V_- \mid s-\tfrac{1}{2}\ s-\tfrac{1}{2};\ s_3+\tfrac{1}{2}\ t_3-\tfrac{1}{2}\rangle$$

$$= \left[\frac{(\nu + 2s + 2)\ (\nu - 2s + 1)\ (s - s_3)\ (s + t_3)}{(2s + 1)\ (2s)}\right]^{\frac{1}{2}}$$

$$\langle s\ s;\ s_3\ t_3 \mid V_+ \mid s+\tfrac{1}{2}\ s+\tfrac{1}{2};\ s_3-\tfrac{1}{2}\ t_3+\tfrac{1}{2}\rangle$$

$$= \left[\frac{(\nu + 2s + 3)\ (\nu - 2s)\ (s - s_3 + 1)\ (s + t_3 + 1)}{(2s + 2)\ (2s + 1)}\right]^{\frac{1}{2}}$$

$$\langle s\ s;\ s_3\ t_3 \mid V_+ \mid s-\tfrac{1}{2}\ s-\tfrac{1}{2};\ s_3-\tfrac{1}{2}\ t_3+\tfrac{1}{2}\rangle$$

$$= \left[\frac{(\nu + 2s + 2)\ (\nu - 2s + 1)\ (s + s_3)\ (s - t_3)}{(2s + 1)\ (2s)}\right]^{\frac{1}{2}}$$

$$\langle s\ s;\ s_3\ t_3 \mid S_3 \mid s\ s;\ s_3\ t_3\rangle = s_3$$

$$\langle s\ s;\ s_3\ t_3 \mid T_3 \mid s\ s;\ s_3\ t_3\rangle = t_3$$

$$\langle s\ s;\ s_3\ t_3 \mid T_+ \mid s\ s\ ;\ s_3\ t_3 - 1\rangle = \{s(s+1) - t_3(t_3-1)\}^{\frac{1}{2}}$$

$$\langle s\ s;\ s_3\ t_3 \mid T_- \mid s\ s\ ;\ s_3\ t_3 + 1\rangle = \{s(s+1) - t_3(t_3+1)\}^{\frac{1}{2}}$$

$$\langle s\ s;\ s_3\ t_3 \mid S_+ \mid s\ s;\ s_3-1\ t_3\rangle = \{s(s+1) - s_3(s_3-1)\}^{\frac{1}{2}}$$

$$\langle s\ s;\ s_3\ t_3 \mid S_- \mid s\ s\ ;\ s_3+1\ t_3\rangle = \{s(s+1) - s_3(s_3+1)\}^{\frac{1}{2}}$$

All other matrix elements are zero.

The subroutine which diagonalizes the missing label operator gives us the eigenvalues and their corresponding eigenvectors. The following formula (which is inserted in the program) proves useful in determining the ℓ-value of these eigenvectors (note that since we are working with states for which $m = 0$, L_+L_- plays the role of L^2)

$$L_+ L_- \,|s,s;\, s_3,t_3\rangle$$

$$= \left[4\,\{s(s+1) - t_3(t_3-1)\} + \frac{3(\nu+2s+3)\,(\nu-2s)\,(s-s_3+1)\,(s+t_3+1)}{(2s+2)\,(2s+1)}\right.$$

$$\left. + \frac{3(\nu+2s+2)\,(\nu-2s+1)\,(s+s_3)\,(s-t_3)}{(2s+1)\,(2s)}\right]|\, s,s;\, s_3,t_3\rangle$$

$$+ \frac{2\sqrt{3}}{} \left[\frac{(\nu+2s+3)\,(\nu-2s)\,(s-s_3+1)\,(s+t_3+1)\,\{(s+\frac{1}{2})\,(s+3/2) - (t_3+\frac{1}{2})\,(t_3+3/2)\}}{(2s+2)\,(2s+1)}\right]^{\frac{1}{2}}$$

$$\times |\, s+\tfrac{1}{2},\, s+\tfrac{1}{2};\, s_3-\tfrac{1}{2},\, t_3+3/2\rangle$$

$$+ \frac{2\sqrt{3}}{} \left[\frac{(\nu+2s+2)\,(\nu-2s+1)\,(s+s_3)\,(s-t_3)\,\{(s-\frac{1}{2})\,(s+\frac{1}{2}) - (t_3+\frac{1}{2})\,(t_3+3/2)\}}{(2s+1)\,(2s)}\right]^{\frac{1}{2}}$$

$$\times |\, s-\tfrac{1}{2},\, s-\tfrac{1}{2};\, s_3-\tfrac{1}{2},\, t_3+3/2\rangle$$

$$+ \frac{2\sqrt{3}}{} \left[\frac{(\nu+2s+3)\,(\nu-2s)\,(s+s_3+1)\,(s-t_3+1)\,\{s(s+1) - (t_3(t_3-1)\}}{(2s+2)\,(2s+1)}\right]^{\frac{1}{2}}$$

$$\times |\, s+\tfrac{1}{2},\, s+\tfrac{1}{2};\, s_3+\tfrac{1}{2},\, t_3-3/2\rangle$$

$$+ \frac{2\sqrt{3}}{} \left[\frac{(\nu+2s+2)\,(\nu-2s+1)\,(s-s_3)\,(s+t_3-1)\,\{s(s+1) - t_3(t_3-1)\}}{(2s+1)\,(2s)}\right]^{\frac{1}{2}}$$

$$\times |\, s-\tfrac{1}{2},\, s-\tfrac{1}{2};\, s_3+\tfrac{1}{2},\, t_3-3/2\rangle$$

$$+ \frac{3}{} \left[\frac{(\nu+2s+3)\,(\nu-2s)\,(s-s_3+1)\,(s+t_3+1)\,(\nu+2s+4)\,(\nu-2s-1)\,(s+s_3+1)\,(s-t_3+1)}{(2s+2)\,(2s+1)\,(2s+3)\,(2s+2)}\right]^{\frac{1}{2}}$$

$$\times |\, s+1,\, s+1;\, s_3,\, t_3\rangle$$

$$+ \frac{3}{} \left[\frac{(\nu+2s+2)\,(\nu-2s+1)\,(s+s_3)\,(s-t_3)\,(\nu+2s+1)\,(\nu-2s+2)\,(s-s_3)\,(s+t_3)}{(2s+1)\,(2s)\,(2s)\,(2s-1)}\right]^{\frac{1}{2}}$$

$$\times |\, s-1,\, s-1;\, s_3,\, t_3\rangle$$

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W	W′	Δ(%)
-2309.484068875199	-2309.484068875113	.3719030023035223D-11
313042895500.4796	313042895500.4765	.1013863649556988D-11
1.055114959312798	1.055114959312738	.5724128163148050D-11
-100.0999749270177	-100.0999749270186	-.9156846740459312D-12
184.9379039252563	184.9379039252960	-.2150017419316956D-10
1201.053151960406	1201.053151960351	.4515089236866764D-11
416.0048053711031	416.0048053711025	.1366412554059324D-12
1.032619851988636	1.032619851772765	.2090512130576292D-07
6206.995445595122	6206.995445595216	-.1509231883665873D-11
-82547643.51478626	-82547643.51478282	.4165404121880938D-11
-1582563.091041461	-1582563.091041470	-.5737777630416737D-12
-2309.484068875199	-2309.484068875113	.3719030023035223D-11
-.1516398347541531D-13	-.1516398347592923D-13	-.3389067503392475D-08
-.3105453583625961D-11	-.3105453583625945D-11	.5137388957062386D-12
15.14119215715013	15.14119215714856	.1038424075961431D-10
.2117247651688286D-11	.2117247651744527D-11	-.2656364703426607D-08
.1067052044827843D-06	.1067052044831091D-06	-.3044000215976704D-09
-.8695499688602893D-11	-.8695499688599503D-11	.3898227677177255D-10
-11632862.03656917	-11632862.03656920	-.2621952725199887D-12
1828.278360079416	1828.278360078226	.6507083053189714D-10
-.3990720660610659D-06	-.3990720660608968D-06	.4236721424145948D-10
.4941656992375776D-07	.4941656992374197D-07	.3194792595995623D-10
-.1227428317306038D-08	-.1227428317306031D-08	.6317939820473566D-12
.2189230634819145D-06	.2189230634834007D-06	-.6788922241481284D-09
1.055114959312798	1.055114959312738	.5724128163148050D-11

Table I. Numerical check for the SU(4) generating function (3.39).

W	W'	$\Delta(\%)$
54195.34685533947	-2309.484068875113	-2446.643026714471
313656132582.6574	313042895500.4765	.1958955437083288
1.055114959312798	1.055114959312738	.5724128163148050D-11
-130.7503916532193	-100.0999749270186	30.61980459889975
-3783.996510877058	184.9379039252960	-2146.090298722953
1186.797721759425	1201.053151960351	-1.186910852168222
438.8075162660938	416.0048053711025	5.481357571014067
1.032619851988636	1.032619851772765	.2090512130576292D-07
6209.384346261750	6206.995445595216	.3848723085868070D-01
-119104732.8649652	-82547643.51478282	44.28604838808710
-1591986.283640401	-1582563.091041470	.5954386685923198
54195.34685533947	-2309.484068875113	-2446.643026714471
-.1075509404687543D-13	-.1516398347592923D-13	-29.06814977773306
-.3112263087842610D-11	-.3105453583625945D-11	.2192756720811713
15.14152800949274	15.14119215714856	.2218136727258800D-02
.2020463063746160D-11	.2117247651744527D-11	-4.571245499723244
-.4711358860163867D-05	.1067052044831091D-06	-4515.303717364275
-.4599270726315955D-11	-.8695499688599503D-11	-47.10745913376354
-11685110.76236847	-11632862.03656920	.4491476442771772
1832.606225130771	1828.278360078226	.2367180592981104
-.3317189476360613D-06	-.3990720660608968D-06	-16.87743246217530
-.1000194551024831D-06	.4941656992374197D-07	-302.4006426525147
-.1233661669319536D-08	-.1227428317306031D-08	.5078383743978266
-.5515196131196960D-04	.2189230634834007D-06	-25292.39427514742
1.055114959312798	1.055114959312738	.5724128163148050D-11

Table II. Checking the efficacy of the numerical check of table I. The term $U^9\Lambda_1^4\Lambda_2^2$ in the numerator of (3.39) is changed to $U^8\Lambda_1^4\Lambda_2^2$. The same values of n_1, n_2, n_3 and U are used.

W	W'	$\Delta(\%)$
-12938.17911095308	-2309.484068875113	460.2194570346145
449245399364.5065	313042895500.4765	43.50953362039250
1.055115410655322	1.055114959312738	.4277662641724743D-04
-121.6852455709903	-100.0999749270186	21.56371233829898
21575.94669207450	184.9379039252960	11566.58983049246
1169.184260803378	1201.053151960351	-2.653412224509551
-227.8466569546498	416.0048053711025	-154.7701983277323
1.032620043744396	1.032619851772765	.1859073602783352D-04
5707.797167165739	6206.995445595216	-8.042510789720849
-232225804.3373866	-82547643.51478282	181.3233600009418
-2060669.515724919	-1582563.091041470	30.21089189997543
-12938.17911095308	-2309.484068875113	460.2194570346145
-.1467336500226262D-10	-.1516398347592923D-13	96664.58053093105
-.3421801346588707D-11	-.3105453583625945D-11	10.18684563925739
15.56748663420891	15.14119215714856	2.815461772335310
.3635805119685548D-11	.2117247651744527D-11	71.72318583939811
-.4665483590030117D-04	.1067052044831091D-06	-43823.11184473331
.2576158161998350D-09	-.8695499688599503D-11	-3062.633838485325
-15345347.80111239	-11632862.03656920	31.91377799266058
2095.236162839603	1828.278360078226	14.60159506290684
-.8564117760130305D-06	-.3990720660608968D-06	114.6007823765709
-.3128275446888926D-03	.4941656992374197D-07	-633141.8019130784
-.1416087407175220D-08	-.1227428317306031D-08	15.37027353933466
.9873097841234259D-04	.2189230634834007D-06	44998.48201527135
1.055115410655322	1.055114959312738	.4277662641724743D-04

Table III. Checking the efficacy of the numerical check of table I. The term $U^4\Lambda_2^2$ in the denominator of (3.39) is changed to $U^3\Lambda_2^2$. The same values of η_1, η_2, η_3 and of U are used.

ℓ \ ν	1	2	3	4	5	6	7
0			0			0	
2	14	-16		29	-31		44
3			-15			-15	
4		110	35	-160	200	35	-250
5				195	-345		330
6			420	280	-20	+708.02 -783.02	355
7					799	-26	-1451
8				1140	915	490	+1825.78 - 480.78
9						2065	895
10					2530	2200	1630
11							4325
12						4914	4459
14							8680

Table IV. Missing label up to SO(5) representation (0,7) as given by De Meyer and Vanden Berghe.

SO(5) IR's						
(0,0)	ℓ	0				
	α	.00				
(0,1)	ℓ	2				
	α	14.00				
(0,2)	ℓ	2	4			
	α	-16.00	110.00			
(0,3)	ℓ	0	3	4	6	
	α	.00	-15.00	35.00	420.00	
(0,4)	ℓ	2	4	5	6	8
	α	29.00	-160.00	195.00	280.00	1140.00
(0,5)	ℓ	2	4	5	6	7
	α	-31.00	200.00	-345.00	-20.00	799.00
	ℓ	8	10			
	α	915.00	2530.00			
(0,6)	ℓ	0	3	4	6	6
	α	.00	-15.00	35.00	-783.02	708.02
	ℓ	7	8	9	10	12
	α	-26.00	490.00	2065.00	2200.00	4914.00

Table V Missing label (α) spectrum for the reduction $SO(5) \supset SO(3)$ restricted to representations of the type $(0,\nu)$. The SO(3) representations are denoted by ℓ (angular momentum).

(0,7)	ℓ	2	4	5	6	7
	α	44.00	-250.00	330.00	355.00	-1451.00
	ℓ	8	8	9	10	11
	α	-480.78	1825.78	895.00	1630.00	4325.00
	ℓ	12	14			
	α	4459.00	8680.00			
(0,8)	ℓ	2	4	5	6	7
	α	-46.00	290.00	-480.00	-95.00	1249.00
	ℓ	8	8	9	10	10
	α	1170.69	-2525.69	-995.00	447.95	3897.04
	ℓ	11	12	13	14	16
	α	2750.00	3724.00	7975.00	8080.00	14279.99
(0,9)	ℓ	0	3	4	6	6
	α	.00	-15.00	35.00	999.59	-1074.59
	ℓ	7	8	9	9	10
	α	-26.00	490.00	-4079.64	3084.64	2800.90
	ℓ	10	11	12	12	13
	α	-2025.90	335.00	7331.09	2321.90	5935.00
	ℓ	14	15	16	18	
	α	7160.00	13474.99	13514.99	22229.99	
(0,10)	ℓ	2	4	5	6	7
	α	59.00	-340.00	465.00	430.00	-1901.00
	ℓ	8	8	9	10	10
	α	-733.39	2528.39	1210.00	1797.70	-6272.69
	ℓ	11	11	12	12	13
	α	-3461.57	6246.57	-548.94	5641.94	2935.00

Table V (continued)

	ℓ	14	14	15	16	17
	α	12600.19	5524.80	10909.99	12389.99	21348.99
	ℓ	18	20			
	α	21279.99	33109.98			
(0,11)	ℓ	2	4	5	6	7
	α	-61.00	380.00	-615.00	-170.00	1699.00
	ℓ	8	8	9	10	10
	α	1430.05	-3235.05	-1310.00	5307.09	297.90
	ℓ	11	11	12	12	13
	α	3626.09	-9241.08	-5580.99	4193.99	11210.23
	ℓ	13	14	14	15	16
	α	-1625.23	10165.36	2289.64	7265.00	20239.17
	ℓ	16	17	18	19	20
	α	10505.82	18198.99	19929.99	32184.98	31954.98
	ℓ	22				
	α	47563.98				
(0,12)	ℓ	0	3	4	6	6
	α	.00	-15.00	35.00	-1367.37	1292.37
	ℓ	7	8	9	9	10
	α	-26.00	490.00	4105.27	-5100.27	3422.35
	ℓ	10	11	12	12	12
	α	-2647.35	335.00	2356.13	9828.66	-13157.79
	ℓ	13	13	14	14	15
	α	7689.41	-8454.41	8119.74	-3494.74	18515.99
	ℓ	15	16	16	17	18
	α	1889.00	6931.27	16913.71	13848.99	30845.75

<u>Table V</u> (continued)

ℓ	18	19	20	21	22
α	17779.23	28389.99	30359.98	46634.98	46183.98
ℓ	24				
α	66299.97				

Table V (continued)

ℓ	α (our values)	$(5/(6\sqrt{2}))\ \alpha_{12,\ell}$ (De Meyer)
0	.00	.00
3	-15.00	-15
4	35.00	35
19	28389.99	28390
20	30359.98	30360
21	46634.98	46635
22	46183.98	46184
24	66299.97	66300

Table VI. In this table we compare the singular eigenvalues of (0,12) we obtained with those conjectured by De Meyer and Vanden Berghe.

$(0,\nu;s$	$,s_3$	$,t_3)$				
(0,0;0	,0	,0)	1 (0)			
(0,1;0	,0	,0)	1 (2)			
(0,2;0	,0	,0)	-.598 (2)	-.802 (4)		
(1	,0	,0)	.802	-.598		
(0,3;0	,0	,0)	-.316 (0)	.000 (3)	-.701 (4)	-.640 (6)
(1	,0	,0)	.621	.000	.357	-.698
(3/2	,1/2	,-3/2)	-.507	-.707	.437	-.228
(3/2	,-1/2	,3/2)	-.507	.707	.437	-.228

Table VII. Here are the eigenvectors corresponding to the eigenvalues given in table V;in this table we give the coefficients of the expansion of $SO(5)\supset SO(3)$ states (with m=0) in terms of $SO(5)\supset SU(2)\times SU(2)$ states;these eigenvectors are read column wise and follow the order of the missing labels given in table V.The first column gives the $SU(2)\times SU(2)$ states and the other columns the coefficients.The angular momentum of the eigenvector is indicated in parenthesis.

$(0,\nu;s$	$,s_3$	$,t_3)$	(2)	(4)	(5)	(6)	(8)
(0,4;0	,0	,0)	.426	-.251	.000	.701	-.514
(1	,0	,0)	-.684	.246	.000	.000	-.687
(3/2	,1/2	,-3/2)	.419	.329	-.707	-.357	-.299
(3/2	,-1/2	,3/2)	.419	.329	.707	-.357	-.299
(2	,0	,0)	.000	-.813	.000	-.505	-.291

$(0,\nu;s$	$,s_3$	$,t_3)$	(2)	(4)	(5)	(6)	(7)	(8)	(10)
(0,5;0	,0	,0)	-.332	.367	.000	-.380	.000	-.661	-.417
(1	,0	,0)	.545	-.447	.000	.208	.000	-.242	-.634
(3/2	,1/2	,-3/2)	-.172	.339	.291	.453	-.645	.216	-.321
(3/2	,-1/2	,3/2)	-.172	.339	-.291	.453	.645	.216	-.321
(2	,0	,0)	-.493	-.445	.000	-.384	.000	.485	-.421
(5/2	,1/2	,-3/2)	.382	.345	-.645	-.357	-.291	.296	-.145
(5/2	,-1/2	,3/2)	.382	.345	.645	-.357	.291	.296	-.145

$(0,\nu;s$	$,s_3$	$,t_3)$	(0)	(3)	(4)	(6)
(0,6;0	,0	,0)	.189	.000	.428	-.081
(1	,0	,0)	-.340	.000	-.580	.071
(3/2	,1/2	,-3/2)	.118	.095	.144	.190
(3/2	,-1/2	,3/2)	.118	-.095	.144	.190
(2	,0	,0)	.397	.000	.233	-.244
(5/2	,1/2	,-3/2)	-.418	-.337	.076	-.234
(5/2	,-1/2	,3/2)	-.418	.337	.076	-.234
(3	,1	,-3)	.382	.615	-.416	-.285
(3	,0	,0)	.149	.000	-.161	.765
(3	,-1	,3)	.382	-.615	-.416	-.283

<u>Table VII.</u>(continued)

$(0,\nu;s\ ,s_3\ ,t_3\ )$

$(0,\nu;s\ ,s_3\ ,t_3\ )$	(6)	(7)	(8)	(9)
(0,6)	-.283	.000	.459	.000
	.245	.000	-.092	.000
	-.320	.416	-.469	.564
	-.320	-.416	-.469	-.564
	.612	.000	.044	.000
	-.236	-.472	.199	.405
	-.236	.472	.199	-.405
	-.288	-.323	.091	.135
	-.057	.000	.493	.000
	-.288	.323	.091	-.135

	(10)	(12)
(0,6)	.607	-.340
	.391	-.567
	-.085	-.316
	-.085	-.316
	-.341	-.485
	-.343	-.227
	-.343	-.227
	-.087	-.038
	-.313	-.149
	-.087	-.038

	(2)	(4)	(5)	(6)
(0,7;0 ,0 ,0)	.271	.187	.000	.442
(1 ,0 ,0)	-.456	-.270	.000	-.473
(3/2 ,1/2 ,-3/2)	.148	-.017	.153	.144
(3/2 ,-1/2 ,3/2)	.148	-.017	-.153	.144
(2 ,0 ,0)	.446	.272	.000	-.135
(5/2 ,1/2 ,-3/2)	-.400	.122	-.414	.218
(5/2 ,-1/2 ,3/2)	-.400	.122	.414	.218
(3 ,1 ,-3)	.268	-.245	.553	-.314
(3 ,0 ,0)	.104	-.594	.000	.244
(3 ,-1 ,3)	.267	-.245	-.553	-.314
(7/2 ,1/2 ,-3/2)	.000	.397	.000	-.291
(7/2 ,-1/2 ,3/2)	.000	.397	.000	-.291

Table VII. (continued)

$(0,\nu;s$, s_3, $t_3)$				
(0,7)	.000 (7)	.153 (8)	-.207 (8)	.000 (9)
	.000	-.087	.116	.000
	-.129	-.313	-.308	-.476
	.129	-.313	-.308	.476
	.000	.236	.587	.000
	.229	.295	-.147	.265
	-.229	.295	-.147	-.265
	.207	.334	-.371	.331
	.000	-.356	.177	.000
	-.207	.334	-.371	.331
	-.623	-.312	-.176	.306
	.623	-.312	-.176	-.306
	-.502 (10)	.000 (11)	.548 (12)	.278 (14)
	-.043	.000	.474	.499
	.419	.483	.021	.297
	.419	-.483	.021	.297
	.154	.000	-.168	.507
	.001	.454	-.301	.280
	.001	-.454	-.301	.280
	-.058	.206	-.116	.063
	-.438	.000	-.406	.251
	-.058	-.206	-.116	.063
	-.294	.135	-.196	.083
	-.294	-.135	-.196	.083
(0,8;0 ,0 ,0)	-.229 (2)	-.236 (4)	.000 (5)	.297 (6)
(1 ,0 ,0)	.388	.355	.000	-.359
(3/2 ,1/2 ,-3/2)	-.087	-.142	-.090	-.045
(3/2 ,-1/2 ,3/2)	-.087	-.142	.090	-.045
(2 ,0 ,0)	-.436	-.211	.000	.202
(5/2 ,1/2 ,-3/2)	.242	.273	.250	.278
(5/2 ,-1/2 ,3/2)	.242	.273	-.250	.278
(3 ,1 ,-3)	-.089	-.199	-.185	-.315
(3 ,0 ,0)	.277	-.380	.000	-.398
(3 ,-1 ,3)	-.089	-.199	.185	-.315
(7/2 ,1/2 ,-3/2)	-.341	.331	-.352	-.035
(7/2 ,-1/2 ,3/2)	-.341	.331	.352	-.035
(4 ,1 ,-3)	.252	-.245	.521	.308
(4 ,0 ,0)	.152	-.148	.000	.186
(4 ,-1 ,3)	.252	-.245	-.521	.308

Table VII. (continued)

(0,8)	.000 (7)	.410 (8)	-.025 (8)	.000 (9)
	.000	-.331	.020	.000
	-.179	.178	.085	.222
	.179	.178	.085	-.222
	.000	-.400	-.078	.000
	.378	.190	-.114	-.274
	-.378	.190	-.114	.274
	-.475	-.180	-.178	-.331
	.000	.352	.202	.000
	.475	-.180	-.178	.331
	.177	-.063	.197	.401
	-.177	-.063	.197	-.401
	-.261	-.320	.354	.323
	.000	-.159	-.720	.000
	.261	-.320	.354	-.324
	-.226 (10)	.149 (10)	.000 (11)	.519 (12)
	.067	-.044	.000	.167
	.391	.284	-.494	-.339
	.391	.284	.494	-.339
	-.145	-.485	.000	-.231
	-.250	.110	.076	-.155
	-.250	.110	-.076	-.155
	-.330	.400	.263	.002
	.022	-.356	.000	.251
	-.330	.400	-.263	.002
	.152	.137	.366	.300
	.152	.137	-.366	.300
	.028	.187	.219	.109
	.477	.039	.000	.325
	.028	.187	-.219	.109
	.000 (13)	.489 (14)	-.230 (16)	
	.000	.513	-.433	
	.408	.098	-.272	
	-.408	.098	-.272	
	.000	-.007	-.500	
	.462	-.218	-.308	
	-.462	-.218	-.308	
	.249	-.116	-.083	
	.000	-.398	-.327	
	-.249	-.116	-.083	
	.224	-.276	-.149	
	-.224	-.276	-.149	
	.088	-.074	-.029	
	.000	-.189	-.078	
	-.088	-.074	-.029	

<u>Table VII</u>. (continued)

$(0,\nu;s$, s_3 , $t_3)$	(0)	(3)	(4)	(6)	(6)	(7)	(8)	(9)
(0,9;0 ,0 ,0)	.135	.000	-.307	-.176	-.075	.000	.365	.000
(1 ,0 ,0)	-.238	.000	.475	.231	.099	.000	-.358	.000
(3/2 ,1/2 ,-3/2)	.056	-.032	-.097	-.141	.059	-.155	-.047	.054
(3/2 ,-1/2 ,3/2)	.056	.032	-.097	-.141	.059	.155	-.047	-.054
(2 ,0 ,0)	.294	.000	-.422	-.012	-.118	.000	.007	.000
(5/2 ,1/2 ,-3/2)	-.177	.102	.185	.217	-.160	.360	.333	-.089
(5/2 ,-1/2 ,3/2)	-.177	-.102	.185	.217	-.160	-.360	.333	.089
(3 ,1 ,-3)	.073	-.083	-.043	-.197	.090	-.226	-.307	-.129
(3 ,0 ,0)	-.227	.000	.134	-.518	.319	.000	-.072	.000
(3 ,-1 ,3)	.073	.083	-.043	-.197	.090	.226	-.307	.129
(7/2 ,1/2 ,-3/2)	.332	-.189	-.021	.349	.070	-.323	-.115	.204
(7/2 ,-1/2 ,3/2)	.332	.189	-.021	.349	.070	.323	-.115	-.204
(4 ,1 ,-3)	-.338	.385	-.181	-.053	-.055	.222	.122	.213
(4 ,0 ,0)	-.204	.000	-.109	-.075	-.568	.000	-.245	.000
(4 ,-1 ,3)	-.338	-.385	-.181	-.053	-.055	-.222	.122	-.213
(9/2 ,3/2 ,-9/2)	.319	-.544	.383	-.297	-.310	.376	.115	.134
(9/2 ,1/2 ,-3/2)	.085	-.049	.102	-.043	.363	.034	.296	-.606
(9/2 ,-1/2 ,3/2)	.085	.049	.102	-.043	.363	-.034	.296	.606
(9/2 ,-3/2 ,9/2)	.319	.544	.383	-.297	-.310	-.376	.115	-.134

<u>Table VII.</u> (continued)

(0,9)	.000	(9)	-.357	(10)	-.053	(10)	.000	(11)
	.000		.202		.030		.000	
	-.181		-.221		.163		-.295	
	.181		-.221		.163		.295	
	.000		.525		-.106		.000	
	.299		-.121		-.172		.243	
	-.299		-.121		-.172		-.243	
	-.414		.065		-.288		.392	
	.000		-.229		.183		.000	
	.414		.065		-.288		-.392	
	.310		-.105		.241		-.173	
	-.310		-.105		.241		.173	
	-.281		.319		.353		-.255	
	.000		-.086		-.344		.000	
	.281		.319		.353		.255	
	-.177		.147		.162		-.090	
	-.006		.213		-.262		-.313	
	.006		.213		-.262		.313	
	.177		.147		.162		.090	
	.108	(12)	.288	(12)	.000	(13)	-.517	(14)
	-.007		-.019		.000		-.270	
	.253		-.426		-.484		.248	
	.253		-.426		.484		.248	
	-.374		.035		.000		.225	
	.102		.145		-.076		.245	
	.102		.145		.076		.245	
	.402		.283		.170		.059	
	-.436		.164		.000		-.054	
	.402		.283		-.170		.059	
	.074		.039		.332		-.203	
	.074		.039		-.332		-.203	
	.269		.042		.299		-.123	
	-.069		-.406		.000		-.392	
	.269		.042		-.299		-.123	
	.076		.012		.070		-.024	
	.085		-.281		.164		-.210	
	.085		-.281		-.164		-.210	
	.076		.012		-.070		-.024	

Table VII. (continued)

(0,ν;s ,s_3 ,t_3)

(0,9)				.000	(15)	.434	(16)	-.191	(18)		
				.000		.520		-.374			
				.344		.151		-.245			
				-.344		.151		-.245			
				.000		.126		-.475			
				.445		-.123		-.317			
				-.445		-.123		-.317			
				.268		-.100		-.096			
				.000		-.327		-.376			
				-.268		-.100		-.096			
				.287		-.300		-.205			
				-.287		-.300		-.205			
				.156		-.116		-.056			
				.000		-.294		-.148			
				-.156		-.116		-.056			
				.026		-.017		-.006			
				.067		-.119		-.047			
				-.067		-.119		-.047			
				-.026		-.017		-.006			
(0,10;	0	,0	,0)	-.199	(2)	.146	(4)	.000	(5)	.320	(6)
	(1	,0	,0)	.339		-.231		.000		-.441	
	(3/2	,1/2	,-3/2)	-.077		.013		.059		.108	
	(3/2	,-1/2	,3/2)	-.077		.013		-.059		.108	
	(2	,0	,0)	-.388		.248		.000		.247	
	(5/2	,1/2	,-3/2)	.219		-.011		-.166		-.136	
	(5/2	,-1/2	,3/2)	.219		-.011		.166		-.136	
	(3	,1	,-3)	-.084		-.034		.126		.035	
	(3	,0	,0)	.259		-.264		.000		.185	
	(3	,-1	,3)	-.084		-.034		-.126		.035	
	(7/2	,1/2	,-3/2)	-.337		.015		.255		-.201	
	(7/2	,-1/2	,3/2)	-.337		.015		-.255		-.201	
	(4	,1	,-3)	.288		.161		-.434		.247	
	(4	,0	,0)	.172		.411		.000		-.098	
	(4	,-1	,3)	.288		.161		.434		.247	
	(9/2	,3/2	,-9/2)	-.197		-.193		.445		-.267	
	(9/2	,1/2	,-3/2)	-.053		-.412		.040		.214	
	(9/2	,-1/2	,3/2)	-.053		-.412		-.040		.214	
	(9/2	,-3/2	,9/2)	-.197		-.193		-.445		-.267	
	(5	,1	,-3)	.000		.246		.000		-.195	
	(5	,0	,0)	.000		.190		.000		-.150	
	(5	,-1	,3)	.000		.246		.000		-.195	

<u>Table VII.</u> (continued)

(0,10)	.000 (7)	-.150 (8)	-.122 (8)	.000 (9)
	.000	.166	.136	.000
	-.060	.109	-.138	.201
	.060	.109	-.138	-.201
	.000	-.129	.088	.000
	.148	-.283	.186	-.387
	-.148	-.283	.186	.387
	-.036	.134	-.207	.237
	.000	.326	-.470	.000
	.036	.134	-.207	-.237
	-.225	.166	.277	.164
	.225	.166	.277	-.164
	.039	-.029	.076	-.018
	.000	-.379	-.207	.000
	-.039	-.029	.076	.018
	.206	-.336	-.362	-.362
	.376	-.052	.139	.161
	-.376	-.052	.139	-.161
	-.206	-.336	-.362	.362
	-.488	.260	-.119	-.265
	.000	.201	-.092	.000
	.488	.260	-.119	.265
	-.396 (10)	-.007 (10)	.000 (11)	.000 (11)
	.304	.006	.000	.000
	.019	.033	.107	.169
	.019	.033	-.107	-.169
	.199	-.025	.000	.000
	-.298	-.048	-.136	-.215
	-.298	-.048	.136	.215
	.272	-.096	-.232	.369
	-.145	.063	.000	.000
	.272	-.096	.232	-.369
	.006	.093	.226	-.371
	.006	.093	-.226	.371
	.033	.179	.307	.251
	.377	-.185	.000	.000
	.033	.179	-.307	-.251
	-.067	.114	.204	.258
	-.048	-.169	-.364	-.084
	-.048	-.169	.364	.084
	-.067	.114	-.204	-.258
	-.288	-.391	-.311	.122
	-.200	.683	.000	.000
	-.288	-.391	.311	-.122

<u>**Table VII.**</u> (continued)

(0,10)	.088 (12)	.298 (12)	.000 (13)	-.078 (14)
	-.031	-.105	.000	-.011
	-.236	.256	.346	-.219
	-.236	.256	-.346	-.219
	.103	-.544	.000	.279
	.185	.073	-.171	-.105
	.185	.073	.171	-.105
	.354	.029	-.397	-.388
	-.091	.023	.000	.443
	.354	.029	.397	-.388
	-.192	.154	-.006	-.029
	-.192	.154	.006	-.029
	-.286	-.246	.107	-.316
	.039	.271	.000	.189
	-.286	-.246	-.107	-.316
	-.186	-.194	.081	-.128
	.114	-.114	.341	-.076
	.114	-.114	-.341	-.076
	-.186	-.194	-.081	-.128
	-.036	-.228	.244	-.102
	.453	-.117	.000	-.020
	-.036	-.228	-.244	-.102
	-.339 (14)	.000 (15)	.503 (16)	.000 (17)
	-.046	.000	.349	.000
	.417	.459	-.161	.289
	.417	-.459	-.161	-.289
	.056	.000	-.170	.000
	-.021	.189	-.278	.415
	-.021	-.189	-.278	-.415
	-.211	-.077	-.101	.271
	-.216	.000	-.099	.000
	-.211	.077	-.101	-.271
	-.167	-.249	.073	.327
	-.167	.249	.073	-.327
	-.121	-.310	.093	.212
	.200	.000	.336	.000
	-.121	.310	.093	-.212
	-.036	-.105	.030	.049
	.266	-.246	.280	.123
	.266	.246	.280	-.123
	-.036	.105	.030	-.049
	.095	-.137	.092	.051
	.330	.000	.211	.000
	.095	.137	.092	-.051

Table VII. (continued)

(0,10)	-.384	(18)	-.159	(20)
	-.509		-.322	
	-.184		-.217	
	-.184		-.217	
	-.227		-.442	
	.033		-.313	
	.033		-.313	
	.074		-.103	
	.226		-.405	
	.074		-.103	
	.278		-.247	
	.278		-.247	
	.137		-.080	
	.345		-.213	
	.137		-.080	
	.028		-.012	
	.200		-.093	
	.200		-.093	
	.028		-.012	
	.053		-.019	
	.112		-.042	
	.053		-.019	

$(0,\nu\ ;s$	$,s_3$	$,t_3\ )$				
(0,11;0	,0	,0)	-.175 (2)	-.174 (4)	.000 (5)	-.238 (6)
(1	,0	,0)	.300	.278	.000	.342
(3/2	,1/2	,-3/2)	-.052	-.075	-.039	-.012
(3/2	,-1/2	,3/2)	-.052	-.075	.039	-.012
(2	,0	,0)	-.360	-.263	.000	-.286
(5/2	,1/2	,-3/2)	.150	.185	.113	-.036
(5/2	,-1/2	,3/2)	.150	.185	-.113	-.036
(3	,1	,-3)	-.041	-.079	-.062	.065
(3	,0	,0)	.319	.050	.000	.203
(3	,-1	,3)	-.041	-.079	.062	.065
(7/2	,1/2	,-3/2)	-.267	-.176	-.201	.146
(7/2	,-1/2	,3/2)	-.267	-.176	.201	.146
(4	,1	,-3)	.146	.189	.221	-.289
(4	,0	,0)	-.110	.315	.000	-.361
(4	,-1	,3)	.146	.189	-.221	-.289
(9/2	,3/2	,-9/2)	-.057	-.135	-.129	.230
(9/2	,1/2	,-3/2)	.258	-.312	.195	.126
(9/2	,-1/2	,3/2)	.258	-.312	-.195	.126
(9/2	,-3/2	,9/2)	-.057	-.135	.129	.230
(5	,1	,-3)	-.258	.261	-.390	.139
(5	,0	,0)	-.199	.202	.000	.108
(5	,-1	,3)	-.258	.261	.390	.139
(11/2	,3/2	,-9/2)	.188	-.190	.426	-.261
(11/2	,1/2	,-3/2)	.087	-.088	.066	-.121
(11/2	,-1/2	,3/2)	.087	-.088	-.066	-.121
(11/2	,-3/2	,9/2)	.188	-.190	-.426	-.261

Table VII. (continued)

(0,11)	.000 (7)	.292 (8)	.026 (8)	.000 (9)
	.000	-.353	-.032	.000
	-.074	.128	-.042	-.115
	.074	.128	-.042	.115
	.000	.052	.046	.000
	.188	-.135	.098	.244
	-.188	-.135	.098	-.244
	-.145	.058	-.005	-.044
	.000	.427	-.137	.000
	.145	.058	-.005	.044
	-.222	-.242	-.097	-.275
	.222	-.242	-.097	.275
	.381	.159	-.062	-.050
	.000	-.211	.271	.000
	-.381	.159	-.062	.050
	-.358	-.143	.162	.312
	-.142	.149	.078	.321
	.142	.149	.078	-.321
	.358	-.143	.162	-.312
	.208	.097	.029	-.135
	.000	.045	-.550	.000
	-.208	.097	.029	.135
	-.227	-.304	-.367	-.354
	-.035	-.115	.330	-.055
	.035	-.115	.330	.055
	.227	-.304	-.367	.354

Table VII. (continued)

(0,11)	.083 (10)	.222 (10)	.000 (11)	.000 (11)
	-.076	-.204	.000	.000
	.129	-.142	-.225	.021
	.129	-.142	.225	-.021
	-.117	.060	.000	.000
	-.154	.350	.358	-.034
	-.154	.350	-.358	.034
	.211	-.159	-.246	-.067
	.345	-.194	.000	.000
	.211	-.159	.244	.067
	-.222	-.161	.020	.076
	-.222	-.161	-.020	-.076
	-.127	-.029	-.080	.120
	.345	.036	.000	.000
	-.127	-.029	.080	-.120
	.374	.314	.276	.102
	-.198	.135	-.271	-.187
	-.198	.135	.271	.187
	.374	.314	-.276	-.102
	.002	-.105	.182	-.207
	.063	.228	.000	.000
	.002	-.105	-.182	.207
	.189	-.041	.268	-.190
	.034	-.289	.032	.591
	.034	-.289	-.032	-.591
	.189	-.041	-.268	.190

Table VII. (continued)

(0,11)	-.018 (12)	-.400 (12)	.000 (13)	.000 (13)
	.010	.227	.000	.000
	.074	-.031	-.150	-.160
	.074	-.031	.150	.160
	-.042	.359	.000	.000
	-.083	-.223	.143	.152
	-.083	-.223	-.143	-.152
	-.182	.228	-.333	.314
	.077	-.202	.000	.000
	-.182	.228	.333	-.314
	.135	-.129	.375	-.178
	.135	-.129	-.375	.178
	.261	.110	-.215	-.312
	-.164	.255	.000	.000
	.261	.110	.215	.312
	.203	.011	-.302	-.245
	-.198	.121	.185	.130
	-.198	.121	-.185	-.130
	.203	.011	.302	.245
	-.349	-.223	-.142	.206
	.328	.060	.000	.000
	-.349	-.223	.142	-.206
	-.210	-.157	-.116	.074
	-.225	-.236	-.005	.315
	.225	-.236	.005	-.315
	-.210	-.157	.116	-.074

Table VII. (continued)

(0,11)	.241 (14)	-.129 (14)	.000 (15)	-.057 (16)
	-.039	.021	.000	-.018
	.275	.297	.378	-.185
	.275	.297	-.378	-.185
	-.498	-.075	.000	.204
	.055	-.158	-.081	-.108
	.055	-.158	.081	-.108
	.108	-.374	-.362	-.363
	-.166	-.012	.000	.410
	.108	-.374	.362	-.363
	.126	.093	-.120	-.007
	.126	.093	.120	-.007
	-.160	.177	-.044	-.339
	.310	.126	.000	.280
	-.160	.177	.044	-.339
	-.197	.163	.037	-.167
	.017	.058	.268	-.042
	.017	.058	-.268	-.042
	-.197	.163	-.037	-.167
	-.277	.118	.296	-.163
	.009	-.386	.000	.028
	-.277	.118	-.296	-.163
	-.135	.066	.102	-.057
	-.137	-.254	.180	-.040
	-.137	-.254	-.180	-.040
	-.135	.066	-.102	-.057

Table VII. (continued)

(0,11)	.376 (16)	.000 (17)	-.480 (18)	.000 (19)
	.117	.000	-.405	.000
	-.384	.425	.082	-.241
	-.384	-.425	.082	.241
	-.114	.000	.091	.000
	-.091	.266	.268	-.377
	-.091	-.266	.268	.377
	.131	.004	.125	-.264
	.178	.000	.194	.000
	.133	-.004	.125	.264
	.213	-.149	.049	-.346
	.213	.149	.049	.346
	.176	-.273	-.040	-.252
	.003	.000	-.216	.000
	.176	.273	-.040	.252
	.060	-.120	-.025	-.069
	-.144	-.277	-.267	-.175
	-.144	.277	-.267	.175
	.060	.120	-.025	.069
	-.087	-.223	-.133	-.102
	-.379	.000	-.310	.000
	-.087	.223	-.133	.102
	-.020	-.060	-.028	-.021
	-.216	-.089	-.141	-.031
	-.216	.089	-.141	.031
	-.020	.060	-.028	.021

Table VII. (continued)

(0,11)	.338 (20)	-.132 (22)
	.485	-.277
	.202	-.192
	.202	-.192
	.298	-.404
	.045	-.301
	.045	-.301
	-.046	-.105
	-.117	-.415
	-.046	-.105
	-.226	-.275
	-.226	-.275
	-.139	-.100
	-.347	-.268
	-.139	-.100
	-.036	-.018
	-.252	-.139
	-.252	-.139
	-.036	-.018
	-.094	-.040
	-.199	-.087
	-094	-.040
	-.016	-.006
	-.073	-.026
	-.073	-.026
	-.016	-.006

Table VII. (continued)

$(0,\nu;s$	$,s_3$	$,t_3)$				
(0,12;0	,0	,0)	.105 (0)	.000 (3)	-.239 (4)	-.066 (6)
(1	,0	,0)	-.184	.000	.388	.098
(3/2	,1/2	,-3/2)	.033	-.014	-.064	.022
(3/2	,-1/2	,3/2)	.033	.014	-.064	.022
(2	,0	,0)	.231	.000	-.411	-.107
(5/2	,1/2	,-3/2)	-.101	.044	.156	-.069
(5/2	,-1/2	,3/2)	-.101	-.044	.156	-.069
(3	,1	,-3)	.029	-.026	-.037	.038
(3	,0	,0)	-.225	.000	.286	.163
(3	,-1	,3)	.029	.026	-.037	.038
(7/2	,1/2	,-3/2)	.202	-.089	-.192	.098
(7/2	,-1/2	,3/2)	.202	.089	-.192	.098
(4	,1	,-3)	-.121	.106	.071	-.115
(4	,0	,0)	.091	.000	-.054	-.335
(4	,-1	,3)	-.121	-.106	.071	-.115
(9/2	,3/2	,-9/2)	.053	-.070	-.010	.045
(9/2	,1/2	,-3/2)	-.241	.106	.046	.070
(9/2	,-1/2	,3/2)	-.241	-.106	.046	.070
(9/2	,-3/2	,9/2)	.053	.070	-.010	.045
(5	,1	,-3)	.289	-.255	.073	.060
(5	,0	,0)	.223	.000	.057	.381
(5	,-1	,3)	.289	.255	.073	.060
(11/2	,3/2	,-9/2)	-.291	.385	-.216	.040
(11/2	,1/2	,-3/2)	-.135	.060	-.100	-.386
(11/2	,-1/2	,3/2)	-.135	-.060	-.100	-.386
(11/2	,-3/2	,9/2)	-.291	-.385	-.216	.040
(6	,2	,-6)	.279	-.492	.354	-.307
(6	,1	,-3)	.060	-.053	.075	.202
(6	,0	,0)	.040	.000	.051	.197
(6	,-1	,3)	.060	.053	.075	.202
(6	,-2	,6)	.279	.492	.354	-.307

Table VII. (continued)

(0,12)	.127 (6)	.000 (7)	-.295 (8)	.000 (9)
	-.188	.000	.379	.000
	.074	.073	-.019	-.079
	.074	-.073	-.019	.079
	.123	.000	-.205	.000
	-.161	-.193	-.060	.178
	-.161	.193	-.060	-.178
	.084	.099	.074	-.150
	.125	.000	-.013	.000
	.084	-.099	.074	.150
	.088	.290	.248	-.144
	.088	-.290	.248	.144
	-.173	-.277	-.307	.309
	-.404	.000	.097	.000
	-.173	.277	-.307	-.309
	.138	.131	.208	-.306
	.374	-.199	-.052	-.242
	.374	.199	-.052	.242
	.138	-.131	.208	.306
	-.192	.271	.119	.315
	-.179	.000	-.123	.000
	-.192	-.271	.119	-.315
	-.038	-.068	-.065	-.179
	.019	-.010	.229	-.035
	.019	.010	.229	.035
	-.038	.068	-.065	.179
	.293	-.388	-.124	-.196
	.038	-.041	-.198	-.011
	.020	.000	-.173	.000
	.038	.041	-.198	.011
	.293	.388	-.124	.196

<u>Table VII.</u> (continued)

(0,12)	.000 (9)	-.243 (10)	-.060 (10)	.000 (11)
	.000	.254	.062	.000
	.033	-.151	.088	-.164
	-.033	-.151	.088	.164
	.000	.091	-.066	.000
	-.075	.156	-.187	.300
	.075	.156	-.187	-.300
	-.012	-.094	.005	-.048
	.000	-.509	.185	.000
	.012	-.094	.005	.048
	.118	.203	.160	-.225
	-.118	.203	.160	.225
	.061	-.052	.129	-.132
	.000	.108	-.263	.000
	-.061	-.052	.129	.132
	-.100	.051	-.240	.350
	-.220	.017	-.174	.141
	.220	.017	-.174	-.141
	.100	.051	-.240	-.350
	.003	-.238	-.054	.039
	.000	.038	.389	.000
	.003	-.238	-.054	-.039
	.158	.247	.287	-.232
	.387	-.108	.043	.159
	-.387	-.108	.043	-.159
	-.158	.247	.287	.232
	.173	.183	.213	-.127
	-.461	.153	-.205	-.271
	.000	.129	-.195	.000
	.461	.153	-.205	.271
	-.173	.183	.213	.127

Table VII. (continued)

(0,12)	.282 (12)	-.056 (12)	.002 (12)	.000 (13)
	-.209	.041	.002	.000
	-.149	-.114	.013	-.232
	-.149	-.114	.013	.232
	-.055	.112	-.008	.000
	.353	.119	-.019	.299
	.353	.119	-.019	-.299
	-.175	-.208	-.047	-.253
	-.037	-.222	.020	.000
	-.175	-.208	-.047	.253
	-.065	.190	.038	.170
	-.065	.190	.038	-.170
	-.074	.133	.087	-.109
	-.202	-.401	-.057	.000
	-.074	.133	.087	.109
	.259	-.367	.081	.176
	.027	.184	-.077	-.267
	.027	.184	-.077	.267
	.259	-.367	.081	-.176
	.002	.114	-.175	.055
	.363	-.119	.172	.000
	.002	.114	-.175	-.055
	.033	-.239	-.144	.326
	-.032	.046	.149	-.073
	-.032	.046	.149	.073
	.033	-.239	-.144	-.326
	.014	-.102	-.062	.112
	-.261	-.055	.411	.151
	-.221	-.048	-.653	.000
	-.261	-.055	.411	-.151
	.014	-.102	-.062	-.112

Table VII. (continued)

(0,12)	.000 (13)	.378 (14)	-.032 (14)	.000 (15)
	.000	-.147	.012	.000
	.047	.090	.119	.129
	-.047	.090	.119	-.129
	.000	-.453	-.051	.000
	-.061	.145	-.106	-.087
	.061	.145	-.106	.087
	-.135	-.181	-.260	.301
	.000	.138	.061	.000
	.135	-.181	-.260	-.301
	.109	.211	.138	-.345
	-.109	.211	.138	.345
	.200	-.133	.282	.188
	.000	-.048	-.084	.000
	-.200	-.133	.282	-.188
	.186	-.084	.262	.321
	-.196	-.135	-.145	-.263
	.196	-.135	-.145	.263
	-.186	-.084	.262	-.321
	-.281	.103	-.244	.125
	.000	-.264	.046	.000
	.281	.103	-.244	-.125
	-.250	.166	-.205	.189
	.340	.112	.089	-.038
	-.340	.112	.089	.038
	.250	.166	-.205	-.186
	-.086	.047	-.058	.045
	.294	.231	-.086	.058
	.000	.168	.426	.000
	-.294	.231	-.086	-.058
	.086	.047	-.058	-.045

Table VII. (continued)

(0,12)	.000 (15)	.171 (16)	-.191 (16)	.000 (17)
	.000	.002	.003	.000
	.208	-.340	-.278	.393
	.208	-.340	-.278	-.393
	.000	.034	.424	.000
	-.141	.101	-.060	.010
	.141	.101	-.060	-.010
	-.365	.356	-.171	-.303
	.000	.091	.298	.000
	.365	.356	-.171	.303
	.098	.013	-.074	-.172
	-.098	.013	-.074	.172
	.247	-.056	.077	-.161
	.000	-.160	-.236	.000
	-.247	-.056	.077	.161
	.247	-.114	.172	-.017
	.036	-.162	-.098	.155
	-.036	-.162	-.098	-.155
	-.247	-.114	.172	.017
	-.036	-.184	.259	.255
	.000	.184	-.162	.000
	.036	-.184	.259	-.255
	-.027	-.114	.205	.135
	-.325	.229	.105	.251
	.325	.229	.105	-.251
	.027	-.114	.205	-.135
	-.006	-.023	.042	.024
	-.250	.062	.147	.163
	.000	.325	.072	.000
	.250	.062	.147	-.163
	.006	-.023	.042	-.024

Table VII. (continued)

(0,12)	.042 (18)	.402 (18)	.000 (19)	.453 (20)
	.020	.188	.000	.441
	.155	-.334	.387	-.016
	.155	-.334	-.387	-.016
	-.148	-.136	.000	.004
	.108	-.177	.314	-.229
	.108	-.177	-.314	-.229
	.333	.058	.070	-.133
	-.359	.096	.000	-.234
	.333	.058	-.070	-.133
	.000	.191	-.048	-.141
	.000	.191	.048	-.141
	.347	.194	-.209	-.016
	-.332	.141	.000	.079
	.347	.194	.209	-.016
	.195	.077	-.118	.013
	.006	.001	-.267	.199
	.006	.001	.267	.199
	.195	.077	.118	.013
	.208	-.030	-.273	.138
	-.099	-.293	.000	.332
	.208	-.030	.273	.138
	.106	-.013	-.104	.043
	.041	-.271	-.156	.224
	.041	-.271	.156	.224
	.106	-.013	.104	.043
	.017	.002	-.014	.005
	.053	-.096	-.082	.069
	.009	-.226	.000	.134
	.053	-.096	.082	.069
	.017	.002	.014	.005

Table VII. (continued)

(0,12)	.000 (21)	-.296 (22)	-.111 (24)
	.000	-.454	-.237
	-.201	-.209	-.168
	.201	-.209	-.168
	.000	-.345	-.365
	-.338	-.108	-.282
	.338	-.108	-.282
	-.249	.019	-.104
	.000	.013	-.410
	.249	.019	-.104
	-.349	.159	-.291
	.349	.159	-.291
	-.278	.125	-.116
	.000	.308	-.310
	.278	.125	-.116
	-.086	.039	-.024
	-.218	.271	-.182
	.218	.271	-.182
	.086	.039	-.024
	-.150	.126	-.063
	.000	.266	-.136
	.150	.126	-.063
	-.043	.030	-.012
	-.067	.137	-.056
	.067	.137	-.056
	.043	.030	-.012
	.005	.003	.001
	-.028	.035	-.012
	.000	.066	-.023
	.028	.035	-.012
	.005	.003	.001