A Stable and Efficient Direct Time Integration of the Vector Wave Equation in the Finite-Element Time-Domain Method for Dispersive Media

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***Abstract*—The finite-element time-domain (FETD) solution of the vector wave equation (VWE) is directly extended to doubly dispersive media using the bilinear transform approach. The proposed formulation is quite general and flexible in the sense of material dispersion model in contrast to limitations of existing convolutional-based approaches. In addition, it can be implemented in a very straightforward and efficient manner. A stability analysis is performed that demonstrates the unconditional stability of the original formulation is preserved in the dispersive case. However, in order to avoid the well-known late-time instabilities that arise in this formulation, an alternative formulation is considered and extended to the dispersive case with the same approach. Several numerical examples are simulated to verify the validity and accuracy of the proposed formulations. The late-time stability of the alternative formulation is tested through a numerical example with 20 million time steps. The solution is completely free from late-time growth.**

***Index Terms*—Dispersive media, finite-element time-domain method.**

I. Introduction

In recent years, there has been a growing interest in dispersive materials [1]–[9]. They can be found almost everywhere. For example, applications of electromagnetic waves in biomedical problems requires modelling of human tissues that show frequency-dependent behavior. Soil, water, snow and vegetation have dispersive behavior, which become important in radar clutter signal cancellation or remote sensing. Recently, engineered materials whose parameters usually have strong dependence on the operating frequency have attracted a great deal of attention both in theory and practice. These properties can be approximated by a constant value within the frequency band of interest in a narrow-band simulation in many practical situations; however, this is not the case in broad-band simulations or for materials with highly rapid changes in their properties. So, it is essential to find some effective and efficient approaches in order to model them in an accurate manner.

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A notable number of papers have been devoted to extending a variety of the finite-difference time-domain (FDTD) formulations to dispersive media [1]–[9]. However, FETD formulations have witnessed relatively less attention due to the following perceived difficulties.

- In the case of the mixed formulation [10], it is easy to include the medium dispersion and both conditionally and unconditionally stable (US) formulations have been introduced [11], [12]. However, in order to avoid using an auxiliary (complementary) mesh, a combination of 1-form and 2-form discretizations has to be utilized. Although hierarchical basis functions have been developed for both discretizations [13], [14], the 2-form hierarchical basis have not found popularity and, therefore, the Whitney 2-form (*face*) elements are still usually employed to discretize $\boldsymbol{\mathcal{B}}$, which are lowest-order accurate.
- In the case of the formulation based on the VWE, only 1-form basis functions are needed in the spatial discretization step. The hierarchical form of these basis functions have been extensively studied and widely employed in both frequency- and time-domain simulations [13]. In addition, the previously-developed frequency-domain codes based on the VWE can be readily extended to the time domain; however, the main difficulty arises in modelling dispersive media. All of the formulations that directly discretize the VWE are based on convolutional approaches [15], [16], which are computationally expensive, particularly for high-order models, because they involve complex-valued calculations in the case of the Lorentz model and require two previous values of the field to update the recursive accumulator variable [2]. However, the proposed formulation does not have the above drawbacks and is more efficient. In addition, we believe that it can treat nonlinear dispersive materials in a similar manner. Moreover, the VWE is well-known to support a spurious solution and exhibit late-time instabilities (e.g., in perfectly matched layer formulations based on the VWE [17]) that results in major difficulties particularly in long-time simulations, which has not been treated for the dispersive case.

In this paper, we first extend the existing US VWE[1] to linear dispersive media. The medium dispersion model is quite general and can have an arbitrary linear form. The proposed approach can be implemented in a highly efficient and straightforward manner with minimal modification to the original FETD

[1]Throughout this paper, we always mean the Newmark-β with $\gamma = 1/2$ and $\beta = 1/4$ known to be US.

code. Moreover, we perform a stability analysis which shows that the unconditional stability of the original FETD formulation is preserved through this approach in dispersive media. In order to overcome the late-time instability problem, a recently-developed alternative formulation [18], which has the same stability, accuracy and efficiency, is extended to dispersive media. In addition, a modification of the source term has been introduced which can increase the accuracy in simulations. Both formulations have been tested and verified through several numerical examples including electromagnetic tunneling through a thin layer of metamaterial and calculation of the reflection and transmission coefficients of a dispersive dielectric slab characterized by different models. Highly accurate results have been obtained in all cases and no sign of late-time growth has been detected in the alternative formulation, even after 20 million time steps.

II. Finite-Element Time-Domain Formulations in Dispersive Media

A. Vector Wave Equation

The VWE in a dispersive media can be written as

$$\nabla \times (\mu^{-1}(t) * \nabla \times \boldsymbol{\mathcal{E}}(t)) + \varepsilon(t) * \frac{\partial^2 \boldsymbol{\mathcal{E}}(t)}{\partial t^2} = -\frac{\partial \boldsymbol{\mathcal{J}}_{\mathrm{imp}}(t)}{\partial t} \tag{1}$$

where $*$ denotes convolution in time. $\boldsymbol{\mathcal{E}}(t)$ and $\boldsymbol{\mathcal{J}}_{\mathrm{imp}}(t)$ represent the electric field intensity and the electric impressed current (excitation) in the continuum case. The weak-form representation of (1) is

$$\int_\Omega \left[\nabla \times \boldsymbol{N} \cdot \mu^{-1}(t) * \nabla \times \boldsymbol{\mathcal{E}}(t) + \boldsymbol{N} \cdot \varepsilon(t) * \frac{\partial^2 \boldsymbol{\mathcal{E}}(t)}{\partial t^2}\right] dV = -\int_\Omega \boldsymbol{N} \cdot \frac{\partial \boldsymbol{\mathcal{J}}_{\mathrm{imp}}(t)}{\partial t} dV \tag{2}$$

in which the curl-conforming basis functions are represented by $\boldsymbol{N}$. To seek the finite-element solution of (2), the electric field intensity is expanded in space using curl-conforming basis functions as

$$\boldsymbol{\mathcal{E}}(t) = \sum_{i=0}^{N_{ed}} \boldsymbol{N}_i e_i(t) \tag{3}$$

where $e_i(t)$ and $\boldsymbol{N}_i$ are the time-dependent electric field intensity, to be discretized in time later, and the curl-conforming basis function corresponding to the ith degree of freedom (DOF), respectively. N_{ed} denotes the total number of DOFs (edges). In addition, we consider the whole computational domain Ω to be composed of m distinct regions with constant properties (each subdomain can be a single element). So, permeability and permittivity can be pulled out of the integral and written as

$$\sum_{k=1}^{m} \left\{ [\mathcal{S}_k] \left(\mu_k^{-1}(t) * \{e(t)\}\right) + [\mathcal{M}_k] \left(\varepsilon_k(t) * \frac{\partial^2 \{e(t)\}}{\partial t^2}\right) \right\} = \{f\} \tag{4}$$

where $\{e(t)\} = [e_1(t), e_2(t), \ldots, e_{N_{ed}}(t)]^T$

$$\mathcal{M}_{k,ij} = \int_\Omega \boldsymbol{N}_i \cdot \boldsymbol{N}_j dV \tag{5}$$

$$\mathcal{S}_{k,ij} = \int_\Omega \nabla \times \boldsymbol{N}_i \cdot \nabla \times \boldsymbol{N}_j dV \tag{6}$$

$$f_i(t) = -\int_\Omega \boldsymbol{N}_i \cdot \frac{\partial \boldsymbol{\mathcal{J}}_{\mathrm{imp}}(t)}{\partial t} dV \tag{7}$$

and the $\varepsilon(t)$ and $\mu(t)$ have the general linear form of

$$\varepsilon_k(t) = \frac{\sum_{n=0}^{p_k} a_{n_k} \mathfrak{D}^{\mathrm{n}}}{\sum_{n=0}^{p_k} b_{n_k} \mathfrak{D}^{\mathrm{n}}}, \mu_k(t) = \frac{\sum_{n=0}^{p_k} \alpha_{n_k} \mathfrak{D}^{\mathrm{n}}}{\sum_{n=0}^{p_k} \beta_{n_k} \mathfrak{D}^{\mathrm{n}}} \tag{8}$$

in each region. $\mathfrak{D}^{\mathrm{n}}$ represents nth derivative with respect to time $((\partial^n)/(\partial t^n))$.

Before moving to the next part, we describe a general procedure to include medium dispersion [9]. Consider an equation like $\mathcal{L}(t) = u(t) * x(t)$ where $u(t)$ has a general form similar to (8). Taking the Laplace transform of it and converting it into the z-domain using a bilinear transformation given by

$$s \mapsto \frac{2}{\Delta t} \frac{1 - z^{-1}}{1 + z^{-1}} \tag{9}$$

we obtain $\tilde{\mathcal{L}} = \tilde{u} \cdot \tilde{x}$ where

$$\tilde{u}(z) = \frac{\nu_0 + \nu_1 z^{-1} + \cdots + \nu_p z^{-p}}{1 + \gamma_1 z^{-1} + \cdots + \gamma_p z^{-p}} \tag{10}$$

and $\tilde{x} = \mathcal{Z}\{x(t)\}$. To implement $\tilde{\mathcal{L}} = \tilde{u} \cdot \tilde{x}$ in the time domain in an efficient manner, an implementation approach named the direct transport II is borrowed from digital signal processing community [9], which involves

$$\begin{aligned} \mathcal{W}_\alpha^n &= \nu_\alpha x^n - \gamma_\alpha \mathcal{L}^n + \mathcal{W}_{\alpha+1}^{n-1} \;\; ; \;\; \alpha = 1, 2, \ldots, p-1 \\ \mathcal{W}_\alpha^n &= \nu_\alpha x^n - \gamma_\alpha \mathcal{L}^n \qquad\qquad ; \;\; \alpha = p \end{aligned} \tag{11}$$

where $\mathcal{W}_\alpha; \alpha = 1, 2, \ldots, p$ are auxiliary variables. The variable $\mathcal{L}$ at $t = n\Delta t$ is given by

$$\mathcal{L}^n = \nu_0 x^n + \mathcal{W}_1^{n-1} \tag{12}$$

It should be noted that $\mathcal{L}$ is an auxiliary variable utilized to simplify explanation of the algorithm. One can substitute it in (11) and eliminate it, if it is not desired.

To apply the above-mentioned procedure to (4), we first map (8) to the z-domain, which results in

$$\tilde{\varepsilon}_k(z) = \frac{c_{0_k} + c_{1_k} z^{-1} + \cdots + c_{(p_k)_k} z^{-p_k}}{1 + d_{1_k} z^{-1} + \cdots + d_{(p_k)_k} z^{-p_k}} \tag{13a}$$

$$\tilde{\mu}_k^{-1}(z) = \frac{q_{0_k} + q_{1_k} z^{-1} + \cdots + q_{(p_k)_k} z^{-p_k}}{1 + r_{1_k} z^{-1} + \cdots + r_{(p_k)_k} z^{-p_k}}. \tag{13b}$$

Having defined equations $\{\mathcal{L}_{\varepsilon_k}(t)\} = \varepsilon_k(t) * ([\mathcal{M}_k]\{e(t)\})$ and $\{\mathcal{L}_{\mu_k^{-1}}(t)\} = \mu_k^{-1}(t) * ([\mathcal{S}_k]\{e(t)\})$, the corresponding final update equations can be written similar to (12) as

$$\{\mathcal{L}_{\varepsilon_k}\}^n = c_{0_k} [\mathcal{M}_k]\{e\}^n + \{\mathcal{W}_{1,k}\}^{n-1} \tag{14a}$$

$$\{\mathcal{L}_{\mu_k^{-1}}\}^n = q_{0_k} [\mathcal{S}_k]\{e\}^n + \{\mathcal{G}_{1,k}\}^{n-1} \tag{14b}$$

where $\mathcal{W}_{\alpha,k}$ and $\mathcal{G}_{\alpha,k}$ are αth auxiliary variables in region k for the permittivity and permeability, respectively. The appropriate update process for the auxiliary variables can be obtained in a similar manner as (11).

Discretizing (4) in time using the Newmark-β method [19] and making use of (14a)–(14b), the fully discretized form can be derived as

$$\begin{aligned}&\left\{\frac{1}{4}[\mathcal{S}_t]+\frac{1}{(\Delta t)^2}[\mathcal{M}_t]\right\}\{e\}^{n+1}\\&=-2\left\{\frac{1}{4}[\mathcal{S}_t]-\frac{1}{(\Delta t)^2}[\mathcal{M}_t]\right\}\{e\}^{n}\\&\quad-\left\{\frac{1}{4}[\mathcal{S}_t]+\frac{1}{(\Delta t)^2}[\mathcal{M}_t]\right\}\{e\}^{n-1}\\&\quad-\frac{1}{(\Delta t)^2}\left\{\{\mathcal{W}_t\}^n-2\{\mathcal{W}_t\}^{n-1}+\{\mathcal{W}_t\}^{n-2}\right\}\\&\quad-\frac{1}{4}\{\{\mathcal{G}_t\}^n+2\{\mathcal{G}_t\}^{n-1}+\{\mathcal{G}_t\}^{n-2}\}\\&\quad+\frac{1}{4}\{\{f\}^{n+1}+2\{f\}^n+\{f\}^{n-1}\}\end{aligned}\tag{15}$$

where

$$[\mathcal{S}_t]=\sum_{k=1}^{m}q_{0_k}[\mathcal{S}_k],[\mathcal{M}_t]=\sum_{k=1}^{m}c_{0_k}[\mathcal{M}_k]\tag{16a}$$
$$\{\mathcal{W}_t\}^n=\sum_{k=1}^{m}\{\mathcal{W}_{1,k}\}^n,\{\mathcal{G}_t\}^n=\sum_{k=1}^{m}\{\mathcal{G}_{1,k}\}^n.\tag{16b}$$

B. The Stabilized Vector Wave Equation With Modified Source

Although the FETD based on the VWE is US when the Newmark-β method is utilized in the time-discretization step, it is shown that it supports the nontrivial solution of $\boldsymbol{\mathcal{E}}(t)=-(at+b)\nabla\phi$ where a and b are constants and ϕ is a time-independent scalar function. The first part of $\boldsymbol{\mathcal{E}}(t)$, i.e., $-at\nabla\phi$, causes the solution to drift in time linearly. This becomes problematic in long-time simulations such as those involving high Q-factor structures, for example. Recently, several different approaches to tackle this problem have been proposed [18], [20], [21]. In [18], it has been shown that by integrating the VWE in time such that

$$\nabla\times\left(\mu^{-1}\nabla\times\int_{-\infty}^{t}\boldsymbol{\mathcal{E}}(\tau)d\tau\right)+\varepsilon\frac{\partial\boldsymbol{\mathcal{E}}(t)}{\partial t}=-\boldsymbol{\mathcal{J}}_{\mathrm{imp}}(t)\tag{17}$$

the constant a will be set to zero and the solution does not drift any more. Multiplying (17) by curl-conforming basis functions as testing functions, following a variational procedure, and using the Newmark-β method for time integrating, the final solution can be cast as a set of recurrence relations as

$$\begin{aligned}&\left\{\varepsilon[\mathcal{M}]+\frac{(\Delta t)^2}{4}\mu^{-1}[\mathcal{S}]\right\}\{v\}^{n+1/2}\\&\quad=\{g\}^{n+1/2}-\mu^{-1}[\mathcal{S}]\{w\}^{n+1/2}\end{aligned}\tag{18a}$$
$$\{e\}^{n+1}=\{e\}^n+\Delta t\{v\}^{n+1/2}\tag{18b}$$
$$\{w\}^{n+3/2}=\{w\}^{n+1/2}+\Delta t\{e\}^{n+1}\tag{18c}$$

where

$$g_i(t)=\int_{-\infty}^{t}f_i(\tau)d\tau.\tag{19}$$

Eliminating the auxiliary variables $\{v\}$ and $\{w\}$ from (18) yields the original discretized form of the US VWE with the following source term:

$$\frac{\{g\}^{n+1/2}-\{g\}^{n-1/2}}{\Delta t}\tag{20}$$

this expression is an approximation to $\{f\}^n$ which can reduce accuracy, particularly if only the vector $\{f\}$ is available. To remove this problem, we suggest using the following source term instead of $\{g\}^{n+1/2}$ in (18a):

$$0.25\{g\}^{n+3/2}+0.5\{g\}^{n+1/2}+0.25\{g\}^{n-1/2}\tag{21}$$

where the midpoint rule is utilized to evaluate (19)

$$g_i((n+1/2)\Delta t)=\Delta t\sum_{m=-\infty}^{n}f_i(m\Delta t).\tag{22}$$

In this manner, exactly the same source term as (15) is recovered.

To derive the formulation in dispersive media, we follow the same procedure as in the last part. We first define $\{\mathcal{L}_{v1_k}(t)\}=\varepsilon_k(t)*([\mathcal{M}_k]\{v(t)\})$, $\{\mathcal{L}_{v2_k}(t)\}=\mu_k^{-1}(t)*([\mathcal{S}_k]\{v(t)\})$ and $\{\mathcal{L}_{w_k}(t)\}=\mu_k^{-1}(t)*([\mathcal{S}_k]\{w(t)\})$. The update equations can be derived as

$$\{\mathcal{L}_{v1_k}\}^n=c_{0_k}[\mathcal{M}_k]\{v\}^n+\{\mathcal{A}_{1,k}\}^{n-1}\tag{23a}$$
$$\{\mathcal{L}_{v2_k}\}^n=q_{0_k}[\mathcal{S}_k]\{v\}^n+\{\mathcal{K}_{1,k}\}^{n-1}\tag{23b}$$
$$\{\mathcal{L}_{w_k}\}^n=q_{0_k}[\mathcal{S}_k]\{w\}^n+\{\mathcal{Q}_{1,k}\}^{n-1}\tag{23c}$$

where the update equations for the auxiliary variables can be cast as (11). Having substituted (23) in (18a), we obtain

$$\begin{aligned}&\left\{[\mathcal{M}_t]+\frac{(\Delta t)^2}{4}[\mathcal{S}_t]\right\}\{v\}^{n+1/2}=\{g\}^{n+1/2}-[\mathcal{S}_t]\{w\}^{n+1/2}\\&\quad-\left\{\{\mathcal{A}_t\}^{n-1/2}+\frac{(\Delta t)^2}{4}\{\mathcal{K}_t\}^{n-1/2}+\{\mathcal{Q}_t\}^{n-1/2}\right\}\end{aligned}\tag{24}$$

where $\{\mathcal{A}_t\}^n=\sum_{k=1}^m\{\mathcal{A}_{1,k}\}^n$, $\{\mathcal{K}_t\}^n=\sum_{k=1}^m\{\mathcal{K}_{1,k}\}^n$ and $\{\mathcal{Q}_t\}^n=\sum_{k=1}^m\{\mathcal{Q}_{1,k}\}^n$.

C. Implementation

In order to implement the formulation developed in Section II-A for an inhomogeneous dispersive problem, one should take the following steps.

1) Break the computational domain into m regions with homogeneous permittivity and permeability $(k=1,\ldots,m)$.
2) Given the $\varepsilon_k(t)$ and $\mu_k(t)$, (8), for each region, discretize them using the bilinear transform (9) to reach (13a)–(13b).
3) Assemble the mass and stiffness matrices for each region ($[\mathcal{M}_k]$ and $[\mathcal{S}_k]$) and form the $[\mathcal{M}_t]$ and $[\mathcal{S}_t]$ using (16a) and the coefficients obtained in Step 2.
4) For each region, k, define a set of auxiliary variables $\{\mathcal{L}_{\varepsilon_k}\}$, $\{\mathcal{L}_{\mu_k^{-1}}\}$, $\{\mathcal{W}_{\alpha,k}\}$ and $\{\mathcal{G}_{\alpha,k}\}$ where $\alpha=1,\ldots,p_k$ and p_k is the order of dispersion model in the kth region. These auxiliary variables should be updated similar to (11) and (12).

Once all of the above variables are defined, the update process can be implemented in two steps.

1) Update $\{e\}$ to $(n+1)$ using (15).

2) Update $\{\mathcal{W}_{\alpha,k}\}$ and $\{\mathcal{G}_{\alpha,k}\}$ in each region to $(n+1)$ using equations similar to (11) and (12).

The formulation described in Section II-B can be implemented in a similar manner.

III. Stability Analysis

In order to have a stable formulation, the following two conditions have to be satisfied [22].

1) All eigenvalues of the amplification matrix have to reside inside or on the unit circle (in the z-plane).
2) If there exist some eigenvalues on the unit-circle, they have to be either simple or non-defective. Non-defective eigenvalue is a multiple eigenvalue whose algebraic multiplicity does not exceed its geometric multiplicity.

We prove the first condition (von Neumann criterion) by investigating roots of the characteristic polynomial using the Routh-Hurwitz criterion [23]. The stability analysis is performed just for the original formulation, because the alternative one (18) has the same update process. In addition, the source term has been removed because it has no effect on the stability of the scheme. Moreover, we restrict ourselves to a homogeneous dispersive case for the sake of simplicity. In the case of the Lorentz model, stability analysis for only either electrically or magnetically dispersive media has been performed. However, in the other cases, a doubly dispersive media has been considered, which can embrace electrically or magnetically dispersive media as special cases.

In a source-free and homogeneous dispersive media, we have

$$[\mathcal{S}]\mu^{-1}(t) * \{e(t)\} + [\mathcal{M}]\varepsilon(t) * \frac{\partial^2\{e(t)\}}{\partial t^2} = 0. \tag{25}$$

Applying the Newmark-β scheme and taking the z-transform gives

$$0.25(z^2+2z+1)\tilde{\mu}^{-1}[\mathcal{S}]\{\tilde{e}\} + \frac{z^2-2z+1}{(\Delta t)^2}\tilde{\varepsilon}[\mathcal{M}]\{\tilde{e}\} = 0. \tag{26}$$

Since matrix $[\mathcal{M}]$ is symmetric and positive-definite, we can multiply both sides by $[\mathcal{M}]^{-1}$. Rearranging it as an eigenvalue problem [24] gives

$$(z^2+2z+1)\tilde{\mu}^{-1}\lambda_i\{\tilde{e}\} + (z^2-2z+1)\tilde{\varepsilon}\{\tilde{e}\} = 0 \tag{27}$$

where $\lambda_i = 0.25(\Delta t)^2 eig\{[\mathcal{M}]^{-1}[\mathcal{S}]\} \geq 0$. We analyze stability of the proposed scheme for different media by substituting corresponding permittivity and permeability (transformed to the z-domain) in (27) and check whether the zeros of the resultant polynomial lie on or inside the unit circle or not.

1) Lossless media: (27) reduces to the conventional FETD formulation in lossless media ($\tilde{\mu}(s) = \mu$ and $\tilde{\varepsilon}(s) = \varepsilon$), which has already been proven to be US.
2) Lossy media: the permittivity and permeability in a lossy medium can be described as: $\tilde{\varepsilon}(s) = \varepsilon_\infty + (\sigma_e)/(s)$ and $\tilde{\mu}(s) = \mu_\infty + (\sigma_m)/(s)$ in the Laplace domain. Converting to the z-domain using (9) and substituting it into (27) yields a second-order polynomial with long coefficients. To investigate the location of the roots more easily, we map it to the s-plane using $z = (\mathsf{s}+1)/(\mathsf{s}-1)$, which gives: $a_2\mathsf{s}^2 + a_1\mathsf{s} + a_0 = 0$ where $a_2 = \sigma_e\sigma_m(\Delta t)^2 + 4\lambda_i$, $a_1 = 2\Delta t(\varepsilon_\infty\sigma_m + \mu_\infty\sigma_e)$ and $a_0 = 4\mu_\infty\varepsilon_\infty$. The sufficient condition for a quadratic polynomial to be Hurwitz-stable is to have positive coefficients, which is the case. So, it is stable regardless of the time step size.
3) Debye media: the model used to characterize a Debye material is: $\tilde{\varepsilon}(s) = \varepsilon_\infty + (\varepsilon_s - \varepsilon_\infty)/(1+\tau_e s)$ and $\tilde{\mu}(s) = \mu_\infty + (\mu_s - \mu_\infty)/(1+\tau_m s)$. Following a similar procedure, we obtain a fourth-order polynomial in the s-plane with the following coefficients: $a_4 = \lambda_i(\Delta t)^2$, $a_3 = 2\lambda_i\Delta t(\tau_e + \tau_m)$, $a_2 = 4\lambda_i\tau_e\tau_m + (\Delta t)^2\varepsilon_s\mu_s$, $a_1 = 2\Delta t(\tau_e\varepsilon_\infty\mu_s + \tau_m\varepsilon_s\mu_\infty)$ and $a_0 = 4\varepsilon_\infty\mu_\infty\tau_e\tau_m$. Performing the Routh-Hurwitz test reveals that the scheme is stable if $\mu_s > \mu_\infty$ and $\varepsilon_s > \varepsilon_\infty$. Since these conditions are always satisfied, the scheme is US for Debye media.
4) Lorentz media: the governing model of a Lorentz model is: $\tilde{\varepsilon}(s) = \varepsilon_\infty + ((\varepsilon_s - \varepsilon_\infty)\omega_e^2)/(s^2 + 2\delta_e s + \omega_e^2)$ and $\tilde{\mu}(s) = \mu_\infty + ((\mu_s - \mu_\infty)\omega_m^2)/(s^2 + 2\delta_m s + \omega_m^2)$. For electrically dispersive media, a fourth-order polynomial with the following coefficients can be obtained: $a_4 = \lambda_i\omega_e^2(\Delta t)^2$, $a_3 = 4\lambda_i\delta_e\Delta t$, $a_2 = 4\lambda_i + \varepsilon_\infty\omega_e^2(\Delta t)^2 + (\varepsilon_s - \varepsilon_\infty)\omega_e^2(\Delta t)^2$, $a_1 = 4\delta_e\varepsilon_\infty\Delta t$ and $a_0 = 4\varepsilon_\infty$. The scheme is stable for $\varepsilon_s > \varepsilon_\infty$. A similar stability criteria can be achieved for magnetically dispersive media (i.e., $\mu_s > \mu_\infty$). However, analyzing stability of the doubly dispersive case, which results in a sixth-order polynomial, seems to be very difficult. For an Nth-order medium the characteristic polynomial is of order $2(N+1)$ (in doubly dispersive case), so the stability analysis becomes increasingly more difficult.

Regarding the second condition of stability, since it seems difficult to be demonstrated for dispersive media, we do not provide any proof for it. However, numerous numerical studies, particularly the long-time simulation in Section IV-C, do not reveal any instability. Hence, we believe that the second condition is also satisfied for the stabilized formulation.

IV. Numerical Results

In order to validate the proposed formulations, several numerical examples are considered in this section. The first example is a 2-D tunneling of electromagnetic energy by use of a metamaterial junction. The second example is a 3-D simulation that involves calculation of the reflection and transmission coefficients of a dispersive dielectric slab.

In all examples the time step is $\Delta t = \ell_{min}/c_0$, which is roughly four times the stability limit of the VWE discretized by the central difference scheme; $\ell_{\min}$ represents the minimum edge length in the mesh.

A. Electromagnetic Tunneling

Recently, it has been demonstrated that electromagnetic waves can be squeezed and tunneled through a thin layer of ε-near zero (ENZ) material [25]. The problem under analysis consist of two parallel waveguides with the height of $a = 4$ cm each having been connected to each other through a thin layer (4 mm wide) of ENZ material (see Fig. 1). The permittivity of the slab is defined by

$$\tilde{\varepsilon}(\omega) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega(\omega + j\Gamma)}\right) \tag{28}$$

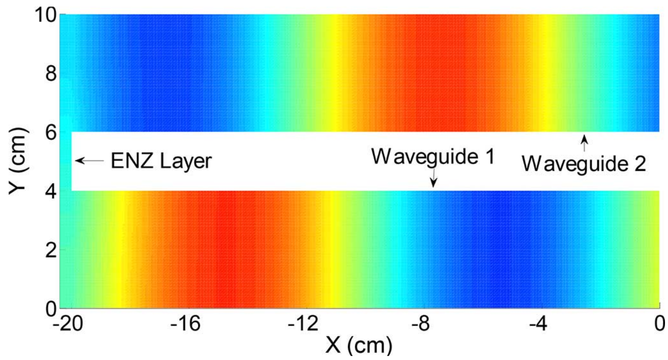

Fig. 1. Magnetic field intensity (H_z) distribution in waveguide 1 and 2 in steady-state.

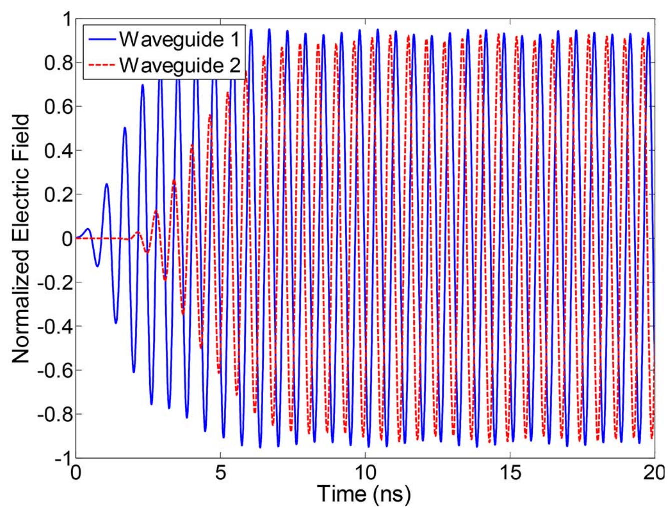

Fig. 2. Normalized electric field intensity (E_y) recorded during simulation in waveguide 1 and 2. After the transient response, the two samples match with each other in amplitude very well, which shows near-perfect tunneling.

where ω_p and Γ are plasma and collision frequencies, respectively. At a frequency around ω_p there should exist a narrow frequency band with a nearly perfect (perfect for $\Gamma = 0$) transmission. The problem is discretized with 32800 right triangles with the average edge length of 1.1 mm. The simulation is performed with $\Delta t = 3.34$ ps. We find the central frequency of this band to be $\omega_0 = 1.605 \times 10^9$ rad/s by numerically calculating the reflection coefficient (with $\Gamma = 0.001\omega_p$ and $\omega_p = 2c_0/(1.5a)$). Afterwards, the lower waveguide (waveguide 1) is excited by a TEM sine source with the frequency of ω_0 and raised cosine ramp [26]. The y-component of the electric field was probed in both waveguides and depicted in Fig. 2. The amplitude of two samples are very close to each other after the transient response, which shows the validity of our formulations. In addition, Fig. 1 shows the distribution of the magnetic field intensity (H_z) in the structure.

B. Reflection and Transmission of the Dispersive Dielectric Slab

The reflection (Γ) and transmission (T) coefficients of a 5 cm wide dispersive dielectric slab for a normally incident plane wave is calculated in this part. This problem can be solved in 1-D; however, we simulate it in three dimensions. The

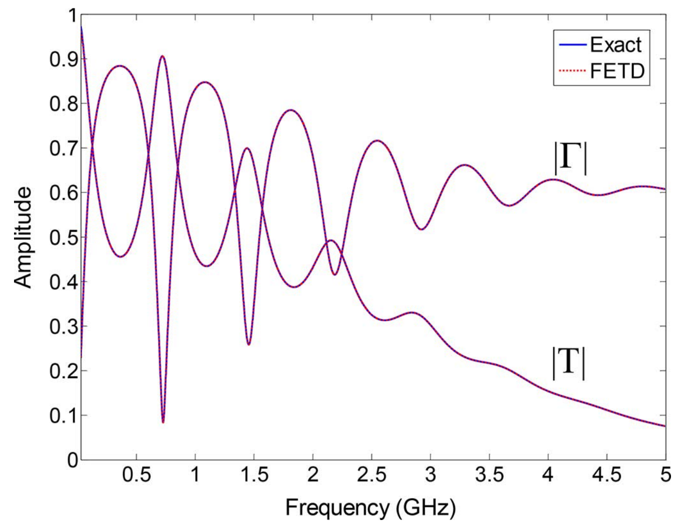

Fig. 3. Numerical and exact values of the reflection and transmission coefficients amplitude of the electrically dispersive dielectric slab.

slab is confined inside a 3-D parallel-plate waveguide excited with TEM mode. The perfect electric and magnetic conductor boundary conditions are imposed on the appropriate walls of the waveguide to support TEM mode. A first-order absorbing boundary condition is utilized to terminate the waveguide end. In all examples the results are plotted within 0.03–5 GHz and the time step is set to $\Delta t = 0.34$ ps. It should be noted that although the material models and parameters utilized in the following examples are physically possible, we are not aware of any specific material with such properties. The material parameters are randomly selected to obtain complex enough reflection and transmission coefficients.

As the first example, an electrically dispersive slab $(\mu = \mu_0)$ with four Debye poles has been selected

$$\tilde{\varepsilon}(s) = \varepsilon_\infty + \sum_{i=1}^{4} \frac{\Delta\varepsilon_i}{1 + \tau_{e_i} s} \tag{29}$$

where the parameters are given by

$$\begin{aligned} \varepsilon_\infty &= 2.4\varepsilon_0, \tau_{e_1} = 10 \text{ ps}, \tau_{e_2} = 6 \text{ ps}, \tau_{e_3} = 12 \text{ ps} \\ \tau_{e_4} &= 5 \text{ ps}, \Delta\varepsilon_1 = 1.8\varepsilon_0, \Delta\varepsilon_2 = 7.3\varepsilon_0 \\ \Delta\varepsilon_3 &= 3.4\varepsilon_0, \Delta\varepsilon_4 = 2.1\varepsilon_0. \end{aligned}$$

The amplitude of T and Γ are computed and plotted in Fig. 3 along with the exact solutions. The absolute error $(|x_{\text{exact}} - x_{\text{FETD}}|)$ of both coefficients are calculated and plotted in Fig. 4, which is less than 0.01 in the given frequency range.

In the next example, a magnetically dispersive $(\varepsilon = \varepsilon_0)$ slab whose dispersion is characterized by one Lorentz pole pair and two Debye poles

$$\tilde{\mu}(s) = \mu_\infty + G_m \frac{(\mu_s - \mu_\infty)\,\omega_m^2}{s^2 + 2\delta_m s + \omega_m^2} + \sum_{i=1}^{2} \frac{\Delta\mu_i}{1 + \tau_{m_i} s} \tag{30}$$

is considered. The parameters are as follows:

$$\begin{aligned} &G_m = 5, \omega_m = 5\pi \times 10^9, \delta_m = 0.005\omega_m, \\ &\mu_s = 4.5\mu_0, \mu_\infty = 3.3\mu_0, \\ &\tau_{m_1} = 6\text{ps}, \tau_{m_2} = 8\text{ps}, \Delta\mu_1 = 3.2\mu_0, \Delta\mu_2 = 4.1\mu_0 \end{aligned}$$

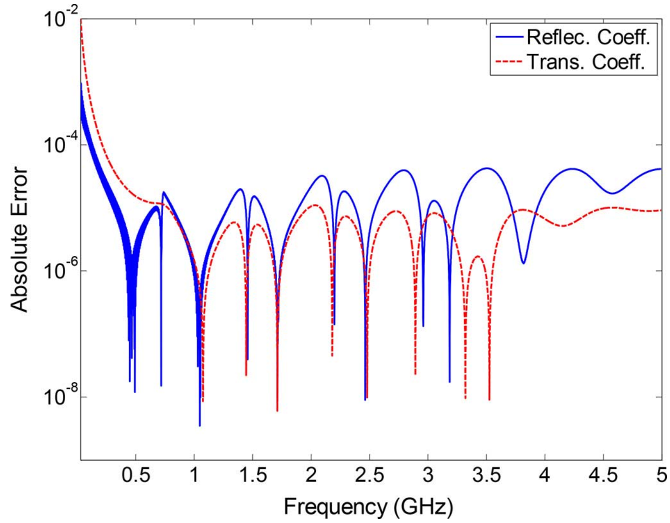

Fig. 4. Absolute error of the reflection and transmission coefficients amplitude of the electrically dispersive dielectric slab.

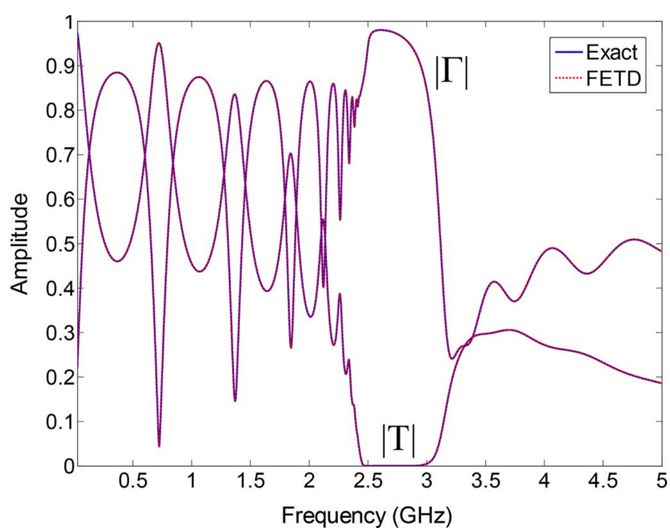

Fig. 5. Numerical and exact values of the reflection and transmission coefficients amplitude of the magnetically dispersive dielectric slab.

the numerical and exact solutions are depicted in Fig. 5. As shown in Fig. 6, the absolute error is less than 0.01 for either coefficients.

The last example deals with a doubly dispersive slab defined by two pairs of Lorentz poles with the following parameters:

$$\begin{aligned}
G_{e_1} &= 0.2, G_{e_2} = 0.4, \delta_{e_1} = 0.025\omega_{e_1}\\
\delta_{e_2} &= 0.01\omega_{e_2}, \varepsilon_s = 5.2\varepsilon_0, \varepsilon_\infty = 3.1\varepsilon_0\\
\omega_{e_1} &= 3.1\pi \times 10^9, \omega_{e_2} = 2.2\pi \times 10^9\\
G_{m_1} &= 0.9, G_{m_2} = 0.5, \delta_{m_1} = 0.03\omega_{m_1}\\
\delta_{m_2} &= 0.015\omega_{m_2}, \mu_s = 3.7\mu_0, \mu_\infty = 1.8\mu_0\\
\omega_{m_1} &= 3.3\pi \times 10^9, \omega_{m_2} = 4.2\pi \times 10^9.
\end{aligned}$$

Fig. 7 shows the obtained results. The obtained absolute error is not more than 0.015 in the frequency range of interest (see Fig. 8).

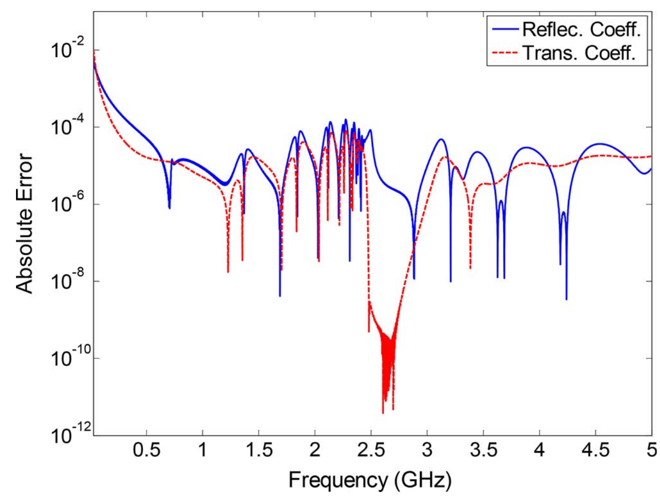

Fig. 6. Absolute error of the reflection and transmission coefficients amplitude of the magnetically dispersive dielectric slab.

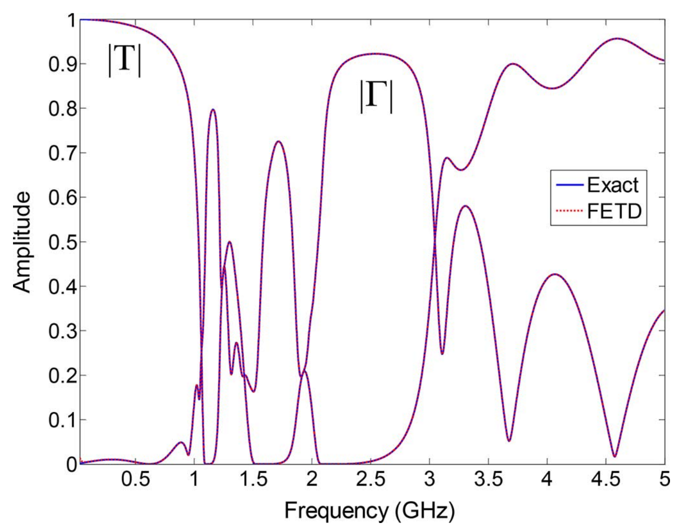

Fig. 7. Numerical and exact values of the reflection and transmission coefficients amplitude of the doubly dispersive dielectric slab.

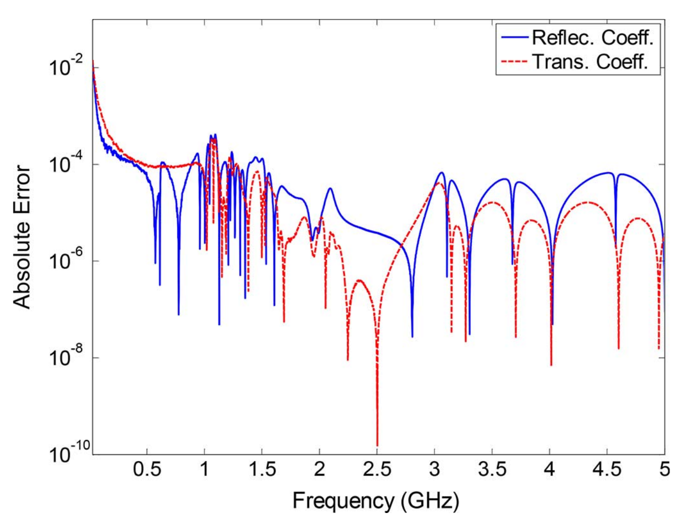

Fig. 8. Absolute error of the reflection and transmission coefficients amplitude of the doubly dispersive dielectric slab.

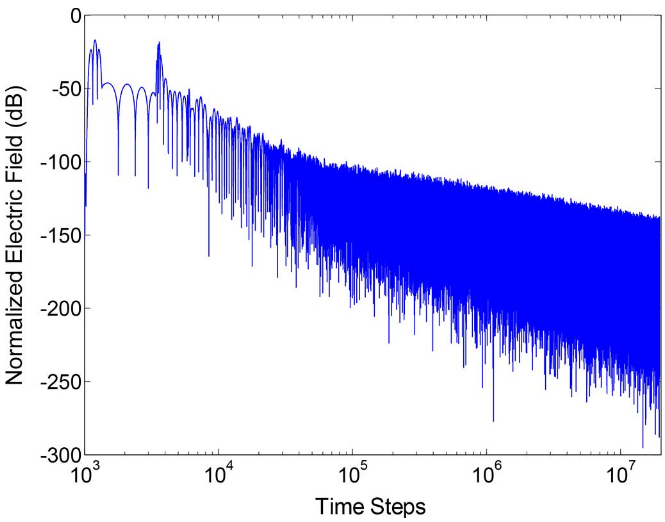

Fig. 9. Normalized electric field intensity recorded inside the waveguide within 20 million time steps.

It should be noted that the mentioned absolute errors are enforced by the low-frequency limit (i.e., 30 MHz), which needs a longer simulation to be improved. Away from that region, the errors drop significantly.

C. Late-Time Stability

In order to study the late-time performance of the proposed formulations, we performed several numerical simulations with different dispersive models, solvers and time steps. The original formulation shows instability almost in all cases as expected, but we couldn't find any sign of instability or late-time growth in the case of the stabilized formulation. In particular, the last example with the fourth-order doubly dispersive slab has been simulated for 20 million time steps. The electric field intensity inside the waveguide has been recorded and plotted in Fig. 9. The probed electric field intensity has been damped-out during time stepping without any instability or late-time growth.

V. Conclusion

By using the bilinear transform method, the FETD formulation based on the second-order VWE has been directly extended to include arbitrary linear dispersive media. Hence, the obtained formulation is more flexible and efficient than the existing formulations based on the convolutional approaches. The unconditional stability of the formulation has been analyzed and verified and a modified formulation has been introduced to avoid excitement of unstable spurious solutions. The accuracy and late-time performance of both formulations have been demonstrated via numerical experiments. Highly-accurate results have been obtained in all cases and the stabilized formulation has shown completely stable behavior in very long time simulations.

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