Reflections Over the Dual Ring—Applications to Kinematic Synthesis

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ABSTRACT

Least-square problems arise in multiple application areas. The numerical algorithm intended to compute offline *the minimum (Euclidian)-norm approximation to an overdetermined system of linear equations, the core of least-squares, is based on* Householder reflections*. It is self-understood, in the application of this algorithm, that the coefficient matrix is dimensionally homogeneous, i.e., all its entries bear the same physical units. Not all applications lead to such matrices, a case in point being parameter identification in mechanical systems involving rigid bodies, whereby both rotation and translation occur, the former dimensionless, the latter bearing units of length. Because of this feature,* dual numbers *have found extensive applications in these fields, as they allow the analyst to include translations within the same relations applicable to rotations, upon* dualization[1] *of the rotation equations, as occurring in the geometric, kinematic or dynamic analyses of mechanical systems. After recalling the basic background on dual numbers and introducing reflection matrices defined over the* dual ring*, we obtain the*

[1]Casually referred to as "putting hats" on all variables involved in rotations, as will be explained in the paper.

dual version of Householder reflections, applicable to the offline *implementation of parameter identification. For the* online *parameter identification* recursive least squares *are to be applied. We provide also the dual version of recursive least squares. Numerical examples are included to illustrate the underlying principles and algorithms.*

1 Introduction

In the realm of kinematic synthesis one can distinguish between *exact synthesis* and *approximate synthesis* [1]. In the former, the number of prescribed pairs of input-output relations over a discrete set equals that of parameters to be determined. In the latter, the number of prescribed pairs exceeds that of linkage parameters available. In practice, the pairs in the discrete sets are prescribed, to a large extent, arbitrarily—the most common practice is to space the pairs at equal intervals of a given parameter, e.g., time. In this light, there is no guarantee that the linkage under synthesis will perform satisfactorily when using this limited set of input-output pairs. There will always be an error between successive pairs of prescribed input-output values. For this reason, a redundant number of prescribed pairs is preferred over the exact number. In fact, the larger the number of prescribed pairs, the smaller the global error in the input-output function, as justified in approximation theory [2] and verified in practice.

Moreover, the measure of the error in a table of value pairs is a norm of the array. Of the various norms available—one is as good as any other because of the monotonicity relation between norms [3]—the *Euclidian norm* is preferred in this paper because i) it is invariant under a change of vector basis and ii) it is infinitely-many-times differentiable. This property enables the application of mathematical-programming techniques based on gradient methods.

Dual numbers are well documented in the literature, beginning with the seminal paper by Clifford [4], then applied to kinematics by Kotel'nikov [5], and later extended to other branches of mechanics by Study [6] and Dimentberg [7]. More recent works on dual numbers include those of Yang and Freudentstein [8] and of Bottema and Roth [9], in the realm of kinematic synthesis. Extensions of dual numbers to Cartesian three-dimensional vectors and 3×3 matrices were reported by Pradeep *et al.* [10], as applied to rigid-body kinematics.

Problems of least squares in which dual numbers show significant benefits in both their formulation and their solution are mainly two: *kinematic synthesis* and *geometric identification*. The problem of kinematic synthesis is most frequently associated with the determination of the geometric parameters of a linkage. In the *analysis problem* associated with *function generation*, for example, all geometric parameters of the linkage are known, the problem being to determine the input-output relation(s). The *synthesis problem* can be considered as an *inverse problem*, i.e., one of *identification*, as the input-output function(s) is(are) given at a discrete set of sample points. The problem then consists in finding the set of geometric parameters that determine the linkage capable of meeting the given set of input-output data with the *least-square error*. This is why the synthesis problem can thus be regarded as one of *identification*. Instances can be mentioned, in biomechanics, for example, in which a prosthetic knee joint is under design. Measurements of the relative *pose*–position and orientation–of the femur and the calf of the healthy limb of the same subject, at various relative poses of femur and calf, are first taken. Then, the geometric parameters of the knee articulation, modeled as a mechanism composed of rigid links coupled by lower and/or higher kinematic pairs, are estimated upon correlating the measurements with the model adopted. A comprehensive review

on this subject is available in the specialized literature [11]. In the foregoing paper the authors cite least squares as a means to identify the geometric parameters involved in the pertinent kinematic models for the *offline* parameter identification. Of course, there is no technical reason why the parameter-identification could not be done online.

On the application of least squares to the *approximate synthesis of linkages*, we can cite Levitskii's pioneer work [12]. Sarkisyan *et al.* [13] applied least-squares to the approximate synthesis of planar four-bar linkages for rigid-body guidance and path generation.

A germane concept, *quaternions*, is well known in the realm of rigid-body rotations, Altmann [14] giving a thorough account of the history of the concept. Upon resorting to *dual quaternions*, nowadays identified as a Clifford algebra [15], the concept can be readily extended to the general kinematics and dynamics of rigid bodies, including both translation and rotation. According to Daniilidis [16], Horaud and Dornaika [17] are to be credited as "the first to apply a simultaneous nonlinear minimization with respect to the rotation quaternion and the translation vector," while acknowledging that "the first simultaneous consideration of rotation and translation in a geometric way was presented by Chen [18], who first introduced the screw theory in the hand-eye calibration."[2] Applications of *dual quaternions*[3] to the same problem were reported by Pennestrì and Stefanelli [19], while providing extensions of dual algebra to n-dimensional vector spaces and their linear transformations. Along the way, the same authors provided algorithms for the porting of the LU-decomposition of square real matrices to their dual counterparts. They also discussed the singular-value decomposition (SVD) of tall rectangular dual matrices and its application to the computation of the least-square solution of overdetermined systems of dual linear equations. Perez-Garcia and McCarthy also applied *dual quaternions* to the synthesis of spatial linkages, including the RPRP chain [20, 21]. More recently, Pennestrì *et al.* proposed a dual version of the Moore-Penrose Generalized Inverse and used it for the synthesis of spatial four-bar linkages [22]. Dual quaternions were also used by Hegedüs *et al.* to factor polynomials in an algorithm for linkage synthesis [23]. Finally, de Falco *et al.* analyzed different generalized inverses of dual matrices and their application to kinematics [24].

A related, although significantly different identification task stems from the *hand-eye problem* [25], as recalled here: Given two coordinate frames defined on instruments mounted on the same rigid link of a robotic end-effector, one camera and one gripper, estimate, from independent pose measurements by each instrument, the constant relative pose—position and orientation—of the two frames. This problem calls naturally for an *online parameter identification* scheme.

1.1 Background on Dual-number Algebra

Dual numbers consist of a primal and a dual part, with different units: those of the latter equal those of the former times units of length. Similar to a complex number, a dual number is represented as an "addition" of these two parts, the dual part pre-multiplied by the dual unit ε, mimicking the role of the imaginary unit in the field of complex numbers. Because of the difference in units in dual numbers, ε is often regarded as a quantity with units of length-inverse, with one property, its square vanishes, the similarity between dual numbers and their complex counterparts ending here. Indeed, a major difference between the two sets of numbers, complex and dual, is to be highlighted: the former constitute a *field*, the latter a *ring*.

[2] The problem is outlined in the paragraph below.

[3] As explained in the sequel, dual quaternions are dual arrays that represent both rotation, as real quaternions do, and translation.

We needn't give a full account of the history of dual numbers, since this can be found in the works cited above. The notation for dual numbers is recalled: a dual number $\hat{x}$ is defined as

$$\hat{x} = x + \varepsilon x_o, \quad x,\, x_o \,\in\, \mathbf{R} \tag{1}$$

where x and x_o are the real *primal* and *dual* parts of $\hat{x}$, respectively, ε being the *dual unit*, with the property

$$\varepsilon^2 = 0 \tag{2}$$

and hence, any power of ε higher than the first one vanishes. For the benefit of the uninitiated, this property can best be understood by thinking of the dual unit ε as a small quantity. Moreover, the zero dual number is naturally defined as the one with *both* its primal and its dual parts equal to zero.

The reason why the set of dual numbers forms a *ring* in the mathematical sense, not a *field*, as the real or complex numbers do, lies in that, as will become apparent in the sequel, there are non-zero dual numbers that do not accept an *inverse under multiplication*, as the real or the complex numbers do.

A dual n-dimensional vector $\hat{\mathbf{a}}$ and a dual $m \times n$ matrix $\hat{\mathbf{A}}$ are defined likewise. Somehow related to the above case of a pure dual number lacking an inverse under multiplication, a dual square matrix is singular if its primal part is, the inverse of its dual part never being needed. Likewise, a rectangular dual matrix is rank-deficient if its primal part is, regardless of whether its dual part is of full rank or not.

For concreteness, let $\hat{x}_i = x_i + \varepsilon x_{io}$, for $i = 1, \ldots, 5$ and $x_i, x_{io} \,\in\, \mathbf{R}$. The sum $\hat{x}_3 = \hat{x}_1 + \hat{x}_2$ is recalled:

$$\hat{x}_3 = x_1 + x_2 + \varepsilon(x_{1o} + x_{2o}) \tag{3}$$

which is obviously a dual number as well. Moreover, the product $\hat{x}_4 = \hat{x}_1 \hat{x}_2$ is obtained as

$$\hat{x}_4 = (x_1 + \varepsilon x_{1o})(x_2 + \varepsilon x_{2o}) = x_1 x_2 + \varepsilon(x_1 x_{2o} + x_2 x_{1o}) \tag{4}$$

the quotient $\hat{x}_5$ of two dual numbers being

$$\hat{x}_5 = \frac{\hat{x}_1}{\hat{x}_2} = \frac{x_1}{x_2} - \varepsilon \frac{x_1 x_{2o} - x_{o1} x_2}{x_2^2} \tag{5}$$

and is, hence, a dual number as well.

It is apparent from eq.(5) that a *non-zero dual number* $\hat{x}$ of the form $\hat{x} = \varepsilon x_o \neq 0$ does not admit an inverse under multiplication, which is a key property of a mathematical *field*. This does not prevent us from defining vectors and matrices over the dual ring, as long as we keep in mind the limitations of a ring. For example, the *Euclidian norm* $\|\hat{\mathbf{a}}\|$ of a dual vector $\hat{\mathbf{a}}$ and its matrix counterpart, the Frobenius norm $\|\hat{\mathbf{A}}\|_F$ of a dual $m \times n$ matrix $\hat{\mathbf{A}}$, for arbitrary natural numbers m and n, are defined in terms of the inner product of the vector or the matrix in question by itself. The inner product of two real vectors of equal dimension is the familiar *dot product*. The counterpart product of two real $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, tall or short, well known in linear algebra, is defined as the trace of the product of the former-transpose times the latter, $\mathrm{tr}(\mathbf{A}^T\mathbf{B})$. It can be readily proven that the foregoing definition of the inner product of two $m \times n$ matrices complies with the properties of the inner product [26]: the product is (a) bilinear-conjugate and (b) positive-definite. In our specific case, limited to dual numbers defined over the real field, property (a) reduces to bilinear.

Based on the above remarks, the Euclidian norm of a dual vector and its matrix counterpart, the Frobenius norm of a dual matrix, are given below:

$$\begin{aligned} \|\hat{\mathbf{a}}\| &= \sqrt{\hat{\mathbf{a}}^T \hat{\mathbf{a}}} = \sqrt{\|\mathbf{a}\|^2 + \varepsilon\, 2\mathbf{a}^T \mathbf{a}_o}, \\ &\mathbf{a} \neq \mathbf{0},\ \mathbf{a}_o \in \mathbf{R}^n \end{aligned} \tag{6a}$$

$$\begin{aligned} \|\hat{\mathbf{A}}\|_F &= \sqrt{\mathrm{tr}(\hat{\mathbf{A}}^T \hat{\mathbf{A}})} \\ &= \sqrt{\mathrm{tr}(\mathbf{A}^T\mathbf{A}) + \varepsilon\, \mathrm{tr}(\mathbf{A}^T\mathbf{A}_o + \mathbf{A}_o^T\mathbf{A})}, \\ &\mathbf{A} \neq \mathbf{O},\ \mathbf{A}_o \in \mathbf{R}^{m\times n} \end{aligned} \tag{6b}$$

where $\mathbf{O}$ denotes the zero $m \times n$ matrix.

To complete the background on dual numbers, recalling the definition of a dual *analytical* function of a dual argument is pertinent. Kotel'nikov [5] defined such a function and its derivative w.r.t. its dual argument:

$$\hat{f}(\hat{x}) = f(x+\varepsilon x_o) = f(x) + \varepsilon x_o \frac{\mathrm{d}f(x)}{\mathrm{dx}} \tag{7a}$$

$$\frac{\mathrm{d}\hat{f}}{\mathrm{d}\hat{x}} = \frac{\mathrm{d}f}{\mathrm{d}x} + \varepsilon \frac{\mathrm{d}f_o}{\mathrm{d}x} \tag{7b}$$

An intuitive explanation of expression (7a) relies on the series expansion of $f(x+\varepsilon x_o)$ about x, while recalling eq.(2).

Now, if relation (7a) is applied to the square-root function $\sqrt{x}$, one obtains

$$\sqrt{\hat{x}} = \sqrt{x} + \varepsilon \frac{x_o}{2\sqrt{x}} \tag{8}$$

Upon applying eq.(8) to the rightmost-hand side of eq.(6a), and invoking the Cayley-Hamilton Theorem [26] in eq.(6b), we obtain

$$\|\hat{\mathbf{a}}\| = \|\mathbf{a}\| + \varepsilon \frac{\mathbf{a}^T \mathbf{a}_o}{\|\mathbf{a}\|}, \quad \mathbf{a} \neq \mathbf{0}, \tag{9a}$$

$$\|\hat{\mathbf{A}}\|_F = \|\mathbf{A}\|_F + \varepsilon \frac{\mathrm{tr}(\mathbf{A}^T \mathbf{A}_o + \mathbf{A}_o^T \mathbf{A})}{2\|\mathbf{A}\|_F}, \quad \mathbf{A} \neq \mathbf{O} \tag{9b}$$

1.2 The Principle of Transference

A key concept in the application of dual numbers to the formulation of problems in mechanics is the *Principle of Transference*, due to Kotel'nikov and Study, according to Fischer [27]: *All identities of ordinary trigonometry hold true for dual angles*. More recent proofs of this principle can also be found [28].

2 Theoretical Issues

The set of dual numbers can be regarded as a two-dimensional set, i.e., as a set of arrays of two real numbers, one representing the primal, the other the dual part of the dual number. Now, arrays are not *ordered sets*, in that a "greater than" (or a "smaller than") relation between two dual numbers cannot be established. The same holds for the more familiar complex numbers. What this means is that a dual scalar function of a set of dual variables cannot be, properly speaking, minimized. However, in the most frequent applications of dual numbers to mechanical systems composed of rigid bodies, the primal part represents a rigid-body rotation, the dual the translation of a landmark point of the body, like its center of mass (c.o.m.). It will be shown in the sequel that a minimization problem over the dual ring is possible by means of an example of

parameter identification drawn from the realm of *linkage synthesis*, in which the problem is known as *approximate synthesis*.

We start by formulating the problem of *dual least squares*: let $\hat{\mathbf{A}}$ be a *known* $m \times n$ *tall* rectangular matrix, i.e., with $m > n$, whose entries are all dual numbers[4]. Further let $\hat{\mathbf{b}}$ be a *known* m-dimensional dual vector and $\hat{\mathbf{x}}$ an *unknown* n-dimensional dual vector. The least-square problem over the dual ring is now formulated as:

Find a dual n-dimensional vector $\hat{\mathbf{x}}$ *that minimizes the Euclidian norm of the error* $\hat{\mathbf{e}}$*, given by*

$$\hat{\mathbf{e}} \equiv \hat{\mathbf{b}} - \hat{\mathbf{A}}\hat{\mathbf{x}} \tag{10}$$

In light of eq.(9a), the primal part of $\|\hat{\mathbf{e}}\|$ is *positive-definite*, while its dual counterpart is sign-indefinite, as the scalar product $\mathbf{e}^T\mathbf{e}_o$ can attain both positive and negative values. Consequently, while the foregoing primal part is bounded from below, and hence, admits a positive minimum, the dual part is unbounded both from above and from below. Therefore, the dual part cannot be, properly speaking, minimized. This is the second theoretical issue within the problem of dual least squares. This issue does not prevent us from formulating a sound least-square problem over the dual ring, as shown in Section 3.

An alternative approach to defining the least-square problem at hand is to look at the primal and the dual parts of the error vector in terms of their mechanical significance: the primal part represents the error in the approximation of the rotation, the dual part that in the approximation of the translation. As shown in Section 3, the dual least-square problem decouples into two real subproblems, one for the primal, one for the dual part of the unknown vector.

3 The Least-Square Problem Over the Dual Ring

Prior to introducing the least-square problem of interest, we look at the problem of solving a *determined* system of n dual linear equations in n dual unknowns:

$$\hat{\mathbf{A}}\hat{\mathbf{x}} = \hat{\mathbf{b}} \tag{11}$$

which, when using the rules for multiplication of dual arrays, leads to two real equations, one for the primal, one for the dual part of the above dual equation, namely,

[4]Some entries may be either real or purely dual, with a primal part equal to 0, which are both special cases of dual numbers.

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{12a}$$

$$\mathbf{A}\mathbf{x}_o = \mathbf{b}_o - \mathbf{A}_o\mathbf{x} \tag{12b}$$

in which we assume that $\mathbf{A}$ is *invertible*, but, apparently, $\mathbf{A}_o$ needn't be so.

Since the equation for $\mathbf{x}$ is decoupled from that for $\mathbf{x}_o$, we first solve for $\mathbf{x}$ from eq.(12a):

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \tag{13}$$

Upon substitution of the foregoing value for $\mathbf{x}$ into eq.(12b), the expression for $\mathbf{x}_o$ is readily derived:

$$\mathbf{x}_o = \mathbf{A}^{-1}\mathbf{b}_o - \mathbf{A}^{-1}\mathbf{A}_o\mathbf{A}^{-1}\mathbf{b} \tag{14}$$

thereby completing the solution of the dual determined system of eq.(11). Two conclusions are derived from the foregoing discussion: (*i*) as long as $\mathbf{A}$ is invertible, the system of eq.(11) admits one solution, and this solution is *unique*, exactly as in the real case; and (*ii*) even in the presence of a singular $\mathbf{A}_o$, its inverse is not needed in the solution of the dual linear system (11). As the reader can readily verify, the expression below provides the dual inverse that verifies eq.(11):

$$\hat{\mathbf{A}}^{-1} = \mathbf{A}^{-1} - \varepsilon\mathbf{A}^{-1}\mathbf{A}_o\mathbf{A}^{-1} \tag{15}$$

a result that can also be derived from relation (7a) as applied to a dual function whose primal part is $f(x) = 1/x$, then recalling the Cayley-Hamilton Theorem [26].

The similarity of the above expression with that for the derivative of $\mathbf{A}^{-1}(t)$ w.r.t. its scalar argument t [29] is to be highlighted[5]. Such a relation is not fortuitous; it can be explained in light of the definition of a dual scalar function of a dual argument, eq.(7a), along with the Cayley-Hamilton Theorem, as stated above.

Now, in connection with the *overdetermined linear system of dual equations*, formally identical to the determined case of eq.(11), we recall the problem over the real field:

[5] $\mathrm{d}[\mathbf{A}^{-1}(t)]/\mathrm{d}t = -\mathbf{A}^{-1}(t)\dot{\mathbf{A}}(t)\mathbf{A}^{-1}(t)$.

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{A} \in \mathbf{R}^{m\times n},\ m > n \tag{16}$$

For the sake of brevity, we assume that $\mathbf{A}$ is of *full rank*[6], i.e., $\text{rank}(\mathbf{A}) = n$. Given the overdeterminacy of the system (16), *the system does not admit a solution*, but, rather, an optimum *approximation* that minimizes the Euclidian norm of the error, with the error $\mathbf{e}$ defined as

$$\mathbf{e} \equiv \mathbf{b} - \mathbf{A}\mathbf{x} \tag{17}$$

Thus, the *optimum solution* sought minimizes *a norm* of $\mathbf{e}$. Of the various norms available, the *Euclidian norm* is the most favored—henceforth, whenever a vector norm is invoked, this means the Euclidian norm; ditto the matrix Frobenius norm—the reason being that this norm is *analytic*, i.e., it admits infinitely-many derivatives with respect to its vector argument, which eases the derivation of the least-square solution $\mathbf{x}_0$, given by

$$\mathbf{x}_0 = \mathbf{A}^L\mathbf{b}, \quad \mathbf{A}^L \equiv (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T, \quad \mathbf{A} \in \mathbf{R}^{m\times n},\ m > n \tag{18}$$

$\mathbf{A}^L$ denoting the *left Moore-Penrose generalized inverse* (LMPGI) of $\mathbf{A}$. It is noteworthy that the above LMPGI formula is of theoretical interest, never to be used for numerical calculations, the reason being that the condition number of $\mathbf{A}^T\mathbf{A}$ is, roughly, the square of that of matrix $\mathbf{A}$ itself [31].

The dual version of the real overdetermined system of linear equations (16) is obtained by simple dualization of the foregoing equation, thereby obtaining

$$\hat{\mathbf{A}}\hat{\mathbf{x}} = \hat{\mathbf{b}}, \quad \hat{\mathbf{A}} \in \mathbb{D}^{m\times n},\ m > n,\ \hat{\mathbf{b}} \in \mathbb{D}^m,\ \hat{\mathbf{x}} \in \mathbb{D}^n \tag{19}$$

where $\mathbb{D}$ denotes the *dual ring*. As in the real case, we assume, for brevity, that $\hat{\mathbf{A}}$ has full rank, i.e., that its primal part does; its dual part needn't be of full rank, as made evident in the sequel.

[6]The case of a rank-deficient matrix is discussed by Golub and Reinsch [30].

The unboundedness of the dual part of the vector norm notwithstanding, we can always equate to zero the derivative of the Euclidian norm-squared of the dual error $\hat{\mathbf{e}} \equiv \hat{\mathbf{b}} - \hat{\mathbf{A}}\hat{\mathbf{x}}$ w.r.t. $\hat{\mathbf{x}}$, which readily leads to

$$\frac{\mathrm{d}\|\hat{\mathbf{e}}\|^2}{\mathrm{d}\hat{\mathbf{x}}} = -2\hat{\mathbf{A}}^T\hat{\mathbf{e}} = \mathbf{0}, \quad \hat{\mathbf{e}} \equiv \hat{\mathbf{b}} - \hat{\mathbf{A}}\hat{\mathbf{x}} \tag{20}$$

Upon expansion into its primal and dual parts, eq.(19) reveals two *decoupled* overdetermined systems of real linear equations, one for the primal, one for the dual part of the unknown $\hat{\mathbf{x}}$, namely,

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{21a}$$

$$\mathbf{A}\mathbf{x}_o = \mathbf{b}_o - \mathbf{A}_o\mathbf{x} \tag{21b}$$

Not surprisingly, the equation for the primal part, eq.(21a), is a restatement of the real linear least-square problem, eq.(16), whose least-square solution is displayed in eq.(18). Moreover, the equation for the dual part, eq.(21b), involves as matrix coefficient of $\mathbf{x}_o$ exactly the same matrix $\mathbf{A}$ of eq.(21a), and hence, while $\mathbf{A}$ must be of full rank for the problem under study to admit a solution, its dual counterpart, $\mathbf{A}_o$, needn't be so.

Now, the least-square solution of eq.(21a) is exactly the same as that pertaining to the well-known real case, as given by eq.(18), the least-square solution of its dual counterpart, eq.(21b), following suit:

$$\mathbf{x}_{o0} = \mathbf{A}^L(\mathbf{b}_o - \mathbf{A}_o\mathbf{A}^L\mathbf{b}) \tag{22}$$

Therefore, the *dual overdetermined linear system* of eq.(19) admits a least-square solution that faithfully mimics that of its real counterpart, namely,

$$\hat{\mathbf{x}}_0 = \hat{\mathbf{A}}^L\hat{\mathbf{b}}, \quad \hat{\mathbf{A}}^L \equiv (\hat{\mathbf{A}}^T\hat{\mathbf{A}})^{-1}\hat{\mathbf{A}}^T, \quad \hat{\mathbf{A}} \in \mathbb{D}^{m\times n},\ m > n \tag{23}$$

which, additionally, is *unique*. The foregoing procedure is to be conducted *offline*, after all data, included in matrix $\hat{\mathbf{A}}$ and in vector $\hat{\mathbf{b}}$, have been collected. For this reason, expression (23) can be fairly termed the *offline solution of the least-square problem*.

Moreover, the *dual left Moore-Penrose generalized inverse* $\hat{\mathbf{A}}^L$ verifies

$$\hat{\mathbf{A}}^L\hat{\mathbf{A}} = \mathbf{1} \in \mathbf{R}^{n\times n} \tag{24}$$

3.1 The Projection Theorem

As pertaining to the real least-square problem, this theorem reads [32]: *The least-square solution of the overdetermined linear system of equations (18) lies in the null space of the coefficient matrix-transpose*. That is,

$$\mathbf{A}^T\mathbf{e}_0 = \mathbf{0}, \quad \mathbf{e}_0 \equiv \mathbf{b} - \mathbf{A}\mathbf{x}_0 \tag{25}$$

This result follows from the *normality conditions* for the constrained minimization problem that consists in finding $\mathbf{x}$ that minimizes $\mathbf{x}^T\mathbf{x}$ subject to eq.(12a). The dual counterpart of the Projection Theorem, i.e., the *Dual Projection Theorem* follows directly from the normality condition of eq.(20), namely,

$$\hat{\mathbf{A}}^T\hat{\mathbf{e}}_0 = \mathbf{0}, \quad \hat{\mathbf{e}}_0 \equiv \hat{\mathbf{b}} - \hat{\mathbf{A}}\hat{\mathbf{x}}_0 \tag{26}$$

with $\hat{\mathbf{x}}_0$ displayed in eq.(23).

4 Online Parameter Identification via Recursive Least Squares

For online identification, *recursive least squares* are to be used. First, it is assumed that $N > n$ observations are available and arranged into an overdetermined system of linear equations, namely,

$$\mathbf{A}_N\mathbf{x}_N = \mathbf{b}_N, \quad \mathbf{A}_N \in \mathbf{R}^{N\times n}, \quad \mathbf{x}_N \in \mathbf{R}^n, \quad \mathbf{b}_N \in \mathbf{R}^N \tag{27}$$

The least-square solution is obtained with the LMPGI of $\mathbf{A}_N$, i.e.

$$\mathbf{x}_N = (\mathbf{A}_N^T\mathbf{A}_N)^{-1}\mathbf{A}_N^T\mathbf{b}_N \tag{28}$$

With one additional observation, leading to an updated vector $\mathbf{x}_{N+1}$, a new system of equations is obtained:

$$\mathbf{A}_{N+1}\mathbf{x}_{N+1} = \mathbf{b}_{N+1} \tag{29}$$

where

$$\mathbf{A}_{N+1} = \begin{bmatrix} \mathbf{A}_N \\ \mathbf{a}_{N+1}^T \end{bmatrix} \quad \text{and} \quad \mathbf{b}_{N+1} = \begin{bmatrix} \mathbf{b}_N \\ b_{N+1} \end{bmatrix} \tag{30}$$

Similar to eq.(28), the least-square approximation with $N+1$ observations is

$$\mathbf{x}_{N+1} = (\mathbf{A}_{N+1}^T\mathbf{A}_{N+1})^{-1}\mathbf{A}_{N+1}^T\mathbf{b}_{N+1} \tag{31}$$

The above expression can be rewritten in terms of the previous approximation to increase the speed of the algorithm. First, by substituting $\mathbf{A}_{N+1}$ and $\mathbf{b}_{N+1}$ with their corresponding expressions found in eq.(30), and by adding, then subtracting the expression of $\mathbf{x}_N$ given in eq.(28), one obtains, after some rearrangement,

$$\begin{aligned} \mathbf{x}_{N+1} =& \mathbf{x}_N[(\mathbf{A}_N^T\mathbf{A}_N - \mathbf{a}_{N+1}\mathbf{a}_{N+1}^T)^{-1} - (\mathbf{A}_N^T\mathbf{A}_N)^{-1}] \\ & \mathbf{A}_N^T\mathbf{b}_N + (\mathbf{A}_N^T\mathbf{A}_N - \mathbf{a}_{N+1}\mathbf{a}_{N+1}^T)^{-1}b_{N+1}\mathbf{a}_{N+1} \end{aligned} \tag{32}$$

Upon factoring out the two inverses within the brackets, the first one to the left, the second to the right, a shorter expression is obtained:

$$\begin{aligned} \mathbf{x}_{N+1} =& \mathbf{x}_N + (\mathbf{A}_N^T\mathbf{A}_N + \mathbf{a}_{N+1}\mathbf{a}_{N+1}^T)^{-1}\mathbf{a}_{N+1} \\ & (b_{N+1} - \mathbf{a}_{N+1}^T\mathbf{x}_N) \end{aligned} \tag{33}$$

If we let

$$\begin{aligned} &\mathbf{P}_N \equiv (\mathbf{A}_N^T\mathbf{A}_N)^{-1}; \quad \mathbf{P}_{N+1} \equiv (\mathbf{A}_N^T\mathbf{A}_N + \mathbf{a}_{N+1}\mathbf{a}_{N+1}^T)^{-1}; \\ &\mathbf{k}_N \equiv \mathbf{P}_{N+1}\mathbf{a}_{N+1} \end{aligned} \tag{34}$$

then we can write

$$\mathbf{x}_{N+1} = (\mathbf{1} - \mathbf{k}_N \mathbf{a}_{N+1}^T)\mathbf{x}_N + b_{N+1}\mathbf{k}_N \tag{35}$$

The next step is to find an expression for $\mathbf{P}_{N+1}$ in terms of $\mathbf{P}_N$, which can be done with the aid of the *matrix-inversion lemma* [33]. If $\mathbf{Q}$ and $\mathbf{R}$ are, respectively, $n \times n$ and $m \times m$ positive-definite matrices, and $\mathbf{M}$ is an arbitrary $n \times m$ matrix, one has

$$\mathbf{Q} - \mathbf{QM}(\mathbf{R} + \mathbf{M}^T\mathbf{QM})^{-1}\mathbf{M}^T\mathbf{Q} = (\mathbf{Q}^{-1} + \mathbf{MR}^{-1}\mathbf{M}^T)^{-1} \tag{36}$$

By setting $\mathbf{Q} = \mathbf{P}_N$, $\mathbf{R} = \mathbf{1}$ and $\mathbf{M} = \mathbf{a}_{N+1}$, the right-hand side of eq.(36) becomes identical to the definition of $\mathbf{P}_{N+1}$ in eq.(34). Thus, one has, with some rearrangement[7],

$$\mathbf{P}_{N+1} = \mathbf{P}_N - \frac{1}{D_N}\mathbf{P}_N\mathbf{a}_{N+1}\mathbf{a}_{N+1}^T\mathbf{P}_N \tag{37}$$

where

$$D_N \equiv 1 + \mathbf{a}_{N+1}^T\mathbf{P}_N\mathbf{a}_{N+1} \tag{38}$$

This new expression allows us to write $\mathbf{k}_N$ in the form

$$\mathbf{k}_N = \frac{1}{D_N}\mathbf{P}_N\mathbf{a}_{N+1} \tag{39}$$

[7] Refer to Bryson and Ho [33] for the details.

4.1 The Online Solution of the Dual Least-Square Problem

If we assume that the first estimate $\hat{\mathbf{x}}_N$ is available after having been computed with $N > n$ observations (with the dual Householder reflection, to be introduced in Subsection 5.1, for instance), then the first value of $\hat{\mathbf{P}}_N$ is obtained as

$$\hat{\mathbf{P}}_N = \mathbf{P}_N + \varepsilon \mathbf{P}_{oN} \tag{40a}$$

$$\mathbf{P_N} \equiv (\mathbf{A}_N^T \mathbf{A}_N)^{-1} \tag{40b}$$

$$\mathbf{P_{oN}} \equiv (\mathbf{A}_N^T \mathbf{A}_N)^{-1} (\mathbf{A}_N^T \mathbf{A}_{oN} + \mathbf{A}_{oN}^T \mathbf{A}_N)(\mathbf{A}_N^T \mathbf{A}_N)^{-1} \tag{40c}$$

The algorithm for online parameter identification hinges on the recursive computation of matrix $\mathbf{Q}$ in eq.(36), as described in Section 4. First, after the dualization of D_N, as given by eq.(38), we expand the latter into its primal and dual parts:

$$\hat{D}_N = D_N + \varepsilon D_{oN} \tag{41a}$$

$$D_N \equiv 1 + \mathbf{a}_{N+1}^T \mathbf{P}_N \mathbf{a}_{N+1} \tag{41b}$$

$$\begin{aligned} D_{oN} \equiv & \mathbf{a}_{N+1}^T \mathbf{P}_N (\mathbf{a}_o)_{N+1} + \mathbf{a}_{N+1}^T \mathbf{P}_{oN} \mathbf{a}_{N+1} \\ & + (\mathbf{a}_o)_{N+1}^T \mathbf{P}_N \mathbf{a}_{N+1} \end{aligned} \tag{41c}$$

With the expressions above, $\hat{\mathbf{k}}_N$ and $\hat{\mathbf{x}}_{N+1}$ can be computed:

$$\hat{\mathbf{k}}_N = \mathbf{k}_N + \varepsilon \mathbf{k}_{oN} \tag{42}$$

whose primal and dual parts are

$$\mathbf{k}_N \equiv \frac{\mathbf{P}_N \mathbf{a}_{N+1}}{D_N} \tag{43a}$$

$$\mathbf{k}_{oN} \equiv \frac{(\mathbf{P}_N \mathbf{a}_{o,N+1} + \mathbf{P}_{oN} \mathbf{a}_{N+1}) D_N - \mathbf{P}_N \mathbf{a}_{N+1} D_{oN}}{D_N^2} \tag{43b}$$

and

$$\hat{\mathbf{x}}_{N+1} = \mathbf{x}_{N+1} + \varepsilon (\mathbf{x}_o)_{N+1} \tag{44}$$

whose primal and dual parts are

$$\mathbf{x}_{N+1} \equiv (\mathbf{1} - \mathbf{k}_N \mathbf{a}_{N+1}^T)\mathbf{x}_N + b_{N+1}\mathbf{k}_N \tag{45a}$$

$$\begin{aligned}(\mathbf{x}_o)_{N+1} \equiv &\mathbf{x}_{oN} - \mathbf{k}_N \mathbf{a}_{N+1}^T \mathbf{x}_{oN} + \mathbf{k}_N (\mathbf{a}_o)_{N+1}^T \mathbf{x}_N \\ &+ \mathbf{k}_{oN} \mathbf{a}_{N+1}^T \mathbf{x}_N + b_{N+1}\mathbf{k}_{oN} + (b_o)_{N+1}\mathbf{k}_N\end{aligned} \tag{45b}$$

Finally,

$$\hat{\mathbf{P}}_{N+1} = \mathbf{P}_{N+1} + \varepsilon(\mathbf{P}_o)_{N+1} \tag{46}$$

with primal and dual parts given below:

$$\mathbf{P}_{N+1} \equiv \mathbf{P}_N - \frac{1}{D_N}\mathbf{P}_N \mathbf{a}_{N+1}\mathbf{a}_{N+1}^T \mathbf{P}_N \tag{47a}$$

$$(\mathbf{P}_o)_{N+1} \equiv \mathbf{P}_{oN} - \frac{\mathbf{U}D_N - \mathbf{P}_N \mathbf{a}_{N+1}\mathbf{a}_{N+1}^T \mathbf{P}_N D_{oN}}{D_N^2} \tag{47b}$$

where

$$\begin{aligned}\mathbf{U} \equiv &(\mathbf{P}_N \mathbf{a}_{N+1}\mathbf{a}_{N+1}^T \mathbf{P}_{oN} + \mathbf{P}_N \mathbf{a}_{N+1}(\mathbf{a}_o)_{N+1}^T \mathbf{P}_N \\ &+ \mathbf{P}_N (\mathbf{a}_o)_{N+1}\mathbf{a}_{N+1}^T \mathbf{P}_N + \mathbf{P}_{oN}\mathbf{a}_{N+1}\mathbf{a}_{N+1}^T \mathbf{P}_N)\end{aligned} \tag{48}$$

Equations (41–48) must be computed with each new observation. As mentioned before, matrix $\mathbf{A}$ is assumed to have full rank. Ideally, it should also be well-conditioned. However, if a large number of new observations are used, the effects of an ill-conditioned $\mathbf{A}$ should be lessened.

5 Numerical Algorithms

The numerical solution to the real overdetermined system of linear equations displayed in eq.(23) can be computed by means of either the *singular-value decomposition* (SVD) or *Householder reflections* (HHR) [34]. In the applications targeted in this paper we focus on HHR, the reason being twofold: (*i*) HHR obviate the SVD and (*ii*) the SVD is iterative by necessity, and hence, unsuitable for online computations. One main purpose of the paper is the development of *dual Householder reflections*, the dual version of HHR.

Briefly, Householder reflections [34] aim to render matrix $\mathbf{A} \in \mathbf{R}^{m\times n}$, $m > n$, into upper-triangular form, similar to what

the LU-decomposition does to a square matrix. In fact, within the LU-decomposition, the solution is preserved upon *arbitrary* linear combinations of the various scalar equations of the determined system in question. This property lies at the foundation of the LU-decomposition. Householder reflections, in turn, aim *to preserve the geometry* of the set of n m-dimensional column vectors of the given tall matrix $\mathbf{A}$, which the LU-decomposition does not. Then again, in a determined system of linear equations, the solution is unique, not depending on the transformation of the original equations by arbitrary linear combinations thereof, as long as these do not lead to *singularity*. That is, the reflections at stake preserve the Euclidian norm of each column and the angle between every pair of the n column vectors. Preservation of the geometric relations between the two sides of eq.(16) is essential in minimizing the Euclidian norm of their difference, i.e., of the approximation error. In the algorithm devised by Householder [35], improper isometries (reflections) are preferred over their proper counterparts, rotations, because of their ease of implementation, as shown in Subsection 5.1.

5.1 Dual Householder Reflections

As in the real case, dual reflections are improper orthogonal matrices, and hence, a $m \times m$ *dual Householder reflection* (DHHR) $\hat{\mathbf{H}}$ satisfies the relations

$$\hat{\mathbf{H}}\hat{\mathbf{H}}^T = \hat{\mathbf{H}}^T\hat{\mathbf{H}} = \mathbf{1} \in \mathbf{R}^{m\times m}, \quad \det(\hat{\mathbf{H}}) = -1 \tag{49}$$

In following Householder, a DHHR $\hat{\mathbf{H}}$ is defined based on an *arbitrary* m-dimensional dual vector $\hat{\mathbf{a}} = \mathbf{a} + \varepsilon\mathbf{a}_o$, where $\mathbf{a} \neq \mathbf{0}$. Upon dividing this vector by its Euclidian norm $\|\hat{\mathbf{a}}\|$, a dual unit vector $\hat{\mathbf{u}}$ is obtained:

$$\hat{\mathbf{u}} \equiv \frac{\hat{\mathbf{a}}}{\|\hat{\mathbf{a}}\|} = \mathbf{u} + \varepsilon\mathbf{u}_o \tag{50a}$$

where

$$\mathbf{u} \equiv \frac{\mathbf{a}}{\|\mathbf{a}\|}, \tag{50b}$$

$$\mathbf{u}_o \equiv -\frac{(\mathbf{a}^T\mathbf{a}_o)\mathbf{a} - \|\mathbf{a}\|^2\mathbf{a}_o}{\|\mathbf{a}\|^3} = -(\mathbf{1} - \mathbf{u}\mathbf{u}^T)\frac{\mathbf{a}_o}{\|\mathbf{a}\|} \tag{50c}$$

which shows that $\mathbf{u}$ is (*i*) a real unit vector and (*ii*) normal to $\mathbf{u}_o$. Now, upon computing $\|\hat{\mathbf{u}}\|$ using the expression in eq.(6a) it is apparent that $\|\hat{\mathbf{u}}\| = 1$.

Furthermore, $\hat{\mathbf{H}}$ follows upon paraphrasing the real case [34], [35]:

$$\hat{\mathbf{H}} = \mathbf{1} - 2\hat{\mathbf{u}}\hat{\mathbf{u}}^T \in \mathbb{D}^{m\times m} \tag{51}$$

As the reader can readily verify, the foregoing expression yields a matrix $\hat{\mathbf{H}}$ that satisfies the first condition of eq.(49). To verify the second condition, the formula for the determinant of a dual matrix is recalled [29]:

$$\det(\hat{\mathbf{H}}) = \det(\mathbf{H})[1 + \varepsilon \mathrm{tr}(\mathbf{H}_o\mathbf{H}^{-1})] \tag{52}$$

However, $\mathrm{tr}(\mathbf{H}_o\mathbf{H}^{-1})$ vanishes. That is, $\mathbf{H}_o$ and $\mathbf{H}^{-1}$ are mutually orthogonal, thereby mimicking the orthogonality between $\mathbf{e}_0$ and $\mathbf{e}_{o0}$ of the least-square error. Hence,

$$\det(\hat{\mathbf{H}}) = -1 \tag{53}$$

A geometric interpretation of $\hat{\mathbf{H}}$ follows: First we look at the three-dimensional dual case. To this end, we recall the concept of dual unit vector defined over $\mathbb{D}^3$. In the same way that a unit real vector $\mathbf{u} \in \mathbf{R}^3$ represents a direction in three-dimensional space, a unit dual vector $\hat{\mathbf{u}} \in \mathbb{D}^3$ represents a line in three-dimensional Euclidian space. Dual Cartesian vectors are thus ideal tools to study the kinematics and dynamics of rigid bodies and systems thereof. Thus, n-dimensional dual vectors can be regarded as representing lines in n-dimensional Euclidian space. The definition of a point in n-dimensional space is straightforward[8]. That of a line in n-dimensional Euclidian space is less so, but can be defined by analogy to the three-dimensional space, as given uniquely by two distinct points in the Euclidian n-space.

A dual unit vector like the one introduced in eqs.(50a–c) represents, in the three-dimensional affine space, a line $\mathscr{L}$ of direction $\mathbf{u}$ and passing through a point P_0 of minimum distance to the origin, of position vector $\mathbf{p}_0$. Below we provide the geometric interpretation of the two m-dimensional vectors $\mathbf{u}$ and $\mathbf{u}_o$ of eqs.(50b & c).

The geometric interpretation of the primal part of $\hat{\mathbf{u}}$ is straightforward: $\mathbf{u}$ is the unit vector obtained upon dividing vector $\mathbf{a}$, assumed to be physically non-dimensional, by its Euclidian norm. The geometric interpretation of the dual part $\mathbf{u}_o$ of the same *dual unit vector* is a bit more elusive. In the case of Cartesian vectors, those in affine three-dimensional space, the dual part of a unit vector $\hat{\mathbf{e}} = \mathbf{e} + \varepsilon\mathbf{e}_o$ is defined as the cross product $\mathbf{e}_o = \mathbf{p}\times\mathbf{e}$, and hence, declared to be the moment of a unit force applied to a rigid body at point P of unit vector $\mathbf{p}$. Thus, while $\mathbf{e}$ represents a unit vector in $\mathbf{R}^3$, $\hat{\mathbf{e}}$ represents a line of direction given by $\mathbf{e}$ and *moment* $\mathbf{e}_o$.

[8] In the same way that a point in three-dimensional space is defined by its three (Cartesian, for example) coordinates, a point in n-dimensional space is defined by n Cartesian coordinates.

Moreover, in Cartesian space, a line is defined by two elements: one point and one direction. Now, let us assume that the dual vector $\hat{\mathbf{a}}$, introduced in Subsection 1.1, is derived from a line $\mathscr{L}$ that (*i*) is parallel to a vector $\mathbf{a}$ *physically non-dimensional* and (*ii*) passes through point A of position vector $\mathbf{a}_o$, the line being offset from the origin. Apparently, then, $\hat{\mathbf{a}}$ represents the line $\mathscr{L}$ described above. To obtain a geometric interpretation of $\mathbf{u}_o$ of eqs.(50a) and (c), let us find the position vector $\mathbf{p}_0$ of point P_0 of $\mathscr{L}$ closest to the origin. To this end, all we have to do is subtract from $\mathbf{a}_o$ its component $(\mathbf{a}_o^T\mathbf{u})\mathbf{u}$ parallel to $\mathbf{a}$. As the reader can readily verify,

$$\mathbf{u}_o = \frac{1}{\|\mathbf{a}\|}\mathbf{p}_0 \tag{54}$$

That is, $\mathbf{u}_o$ is proportional to the position vector $\mathbf{p}_0$ of the point of $\mathscr{L}$ closest to the origin, as opposed to the moment of the same line, which is $\mathbf{p}_0 \times \mathbf{u}$. This now eases the interpretation of $\mathbf{u}_o$ in eqs.(50b & c), as the right-hand side of eq.(54) involves vector operations valid in $\mathbf{R}^m$, for any integer m, as opposed to the cross product, which is defined only for $m = 3$.

5.2 The Computation of the Dual Least-Square Solution

The dual adaptation of Householder's algorithm, the DHHR algorithm, can be used to solve an overdetermined system of m dual linear equations in $n < m$ unknowns, of the form of

$$\hat{\mathbf{A}}\hat{\mathbf{x}} = \hat{\mathbf{b}} \tag{55}$$

or, decomposed into its primal and dual parts,

$$\mathbf{A}\mathbf{x} = \mathbf{b} \qquad \text{and} \qquad \mathbf{A}\mathbf{x_o} + \mathbf{A_o}\mathbf{x} = \mathbf{b_o} \tag{56}$$

First, it is assumed that $i-1$ dual reflections $\hat{\mathbf{H}}_1, \hat{\mathbf{H}}_2, \ldots, \hat{\mathbf{H}}_{i-1}$ have already been applied sequentially to $\hat{\mathbf{A}}$. The matrix $\hat{\mathbf{A}}_{i-1}$, which is *partly*[9] upper triangular, is thus defined as the transformed $\hat{\mathbf{A}}$ under the foregoing sequence of reflections, namely

$$\hat{\mathbf{A}}_{i-1} = \hat{\mathbf{H}}_{i-1} \ldots \hat{\mathbf{H}}_2\hat{\mathbf{H}}_1\hat{\mathbf{A}} \tag{57}$$

[9]The emphasized qualifier means that only the left $m \times (n-i)$ block is upper-triangular.

which has the form

$$\hat{\mathbf{A}}_{i-1} = \begin{bmatrix} \hat{a}^*_{11} & \hat{a}^*_{12} & \dots & \hat{a}^*_{1,i-1} & \hat{a}^*_{1i} & \dots & \hat{a}^*_{1n} \\ 0 & \hat{a}^*_{22} & \dots & \hat{a}^*_{2,i-1} & \hat{a}^*_{2i} & \dots & \hat{a}^*_{2n} \\ 0 & 0 & \dots & \hat{a}^*_{3,i-1} & \hat{a}^*_{3i} & \dots & \hat{a}^*_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \hat{a}^*_{i-1,i-1} & \hat{a}^*_{i-1,i} & \dots & \hat{a}^*_{i-1,n} \\ 0 & 0 & \dots & 0 & \hat{a}^*_{ii} & \dots & \hat{a}^*_{in} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \hat{a}^*_{mi} & \dots & \hat{a}^*_{mn} \end{bmatrix} \tag{58}$$

The above display can also be expressed with its entries expanded into their primal and dual parts, whereby every element $\hat{a}^*_{ij}$ is equal to $a^*_{ij} + \varepsilon a^*_{oij}$. The i^{th} Householder reflection, $\hat{\mathbf{H}}_i = \mathbf{H}_i + \varepsilon \mathbf{H}_{oi}$, is chosen to render the last $m-i$ components of the i^{th} column of $\hat{\mathbf{A}}_i$ equal to zero, while leaving its first $i-1$ columns unchanged. This can be done by setting

$$\hat{\alpha}_i = \alpha_i + \varepsilon \alpha_{oi} \tag{59}$$

with

$$\alpha_i = \operatorname{sgn}(a^*_{ii}) \sqrt{(a^*_{ii})^2 + (a^*_{i+1,i})^2 + \cdots + (a^*_{mi})^2} \tag{60}$$

and

$$\alpha_{oi} = \frac{a^*_{ii} a^*_{oii} + a^*_{i+1,i} a^*_{o,i+1,i} + \cdots + a^*_{mi} a^*_{omi}}{\alpha_i} \tag{61}$$

where $\operatorname{sgn}(x)$, the *signum function* of $x \in \mathbb{R}$, is introduced for numerical robustness i.e., to prevent divisions by quantities with a significantly small absolute value, as made apparent in the sequel. For the production of the i^{th} dual Householder reflection $\hat{\mathbf{H}}_i$, the dual vector $\hat{\mathbf{u}}_i$, defining the unit normal to the corresponding reflection hyperplane in $\mathbb{D}^n$, is given as

$$\hat{\mathbf{u}}_i = \mathbf{u}_i + \varepsilon \mathbf{u}_{oi} \tag{62}$$

where

$$\mathbf{u}_i = \begin{bmatrix} 0 \; 0 \; a^*_{ii} + \alpha_i \; a^*_{i+1,i} \; \cdots \; a^*_{mi} \end{bmatrix}^T \tag{63}$$

and

$$\mathbf{u}_{oi} = \begin{bmatrix} 0 \; 0 \; a^*_{oii} + \alpha_{oi} \; a^*_{o,i+1,i} \; \cdots \; a^*_{omi} \end{bmatrix}^T \tag{64}$$

With $\hat{\mathbf{u}}_i$ available, the matrix $\hat{\mathbf{H}}_i$ is now produced as

$$\hat{\mathbf{H}}_i = \mathbf{1} - \frac{\hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^T}{\|\hat{\mathbf{u}}_i\|^2/2} \tag{65}$$

with

$$\|\hat{\mathbf{u}}_i\|^2/2 \equiv \hat{\gamma}_i = \gamma_i + \varepsilon \gamma_{oi} \tag{66}$$

where

$$\gamma_i = \alpha_i(\alpha_i + a^*_{ii}), \quad \text{and} \quad \gamma_{oi} = 2\alpha_{oi}\alpha_i + a^*_{ii}\alpha_{oi} + a^*_{oii}\alpha_i \tag{67}$$

It is noteworthy that, in the original HHR algorithm, α_i is defined with the sign of a^*_{ii} because γ_i is proportional to $\alpha_i + a^*_{ii}$. Thus, with the *signum function*, it is guaranteed that the absolute value of the resulting sum will always be greater than the absolute value of each term. This is done to avoid roundoff error amplification caused by a division by a small γ_i. In the DHHR, it is still the case for the primal part, as can be seen in eq.(67). However, the dual part α_{oi} of $\hat{\alpha}_i$ does not enter in the

dual HHR algorithm in any divisor; hence, γ_{oi} of eq.(67) need not be kept with a large absolute value. Expression (65) for $\hat{\mathbf{H}}_i$ can be rewritten as

$$\hat{\mathbf{H}}_i = \mathbf{1} - \frac{\mathbf{u}_i \mathbf{u}_i^T}{\gamma_i} + \varepsilon \left(\frac{\gamma_{oi} \mathbf{u}_i \mathbf{u}_i^T}{\gamma_i^2} - \frac{\mathbf{u}_i \mathbf{u}_{oi}^T + \mathbf{u}_{oi} \mathbf{u}_i^T}{\gamma_i} \right) \tag{68}$$

With the matrix $\hat{\mathbf{H}}_i$ now available, one can compute $\hat{\mathbf{A}}_i$, i.e.

$$\hat{\mathbf{A}}_i = \hat{\mathbf{H}}_i \hat{\mathbf{A}}_{i-1} = \mathbf{H}_i \mathbf{A}_{i-1} + \varepsilon (\mathbf{H}_i \mathbf{A}_{o,i-1} + \mathbf{H}_{oi} \mathbf{A}_{i-1}) \tag{69}$$

Upon multiplying both sides of eq.(55) by the product $\hat{\mathbf{H}}_i \hat{\mathbf{H}}_{i-1} \ldots \hat{\mathbf{H}}_1$, a new overdetermined dual linear system is obtained:

$$\hat{\mathbf{A}}_i \hat{\mathbf{x}} = \hat{\mathbf{\Pi}}_i \hat{\mathbf{b}}, \quad \hat{\mathbf{\Pi}}_i \equiv \hat{\mathbf{H}}_i \hat{\mathbf{H}}_{i-1} \ldots \hat{\mathbf{H}}_1 \tag{70}$$

Once n dual Householder reflection matrices have been computed, a new system is obtained:

$$\hat{\mathbf{A}}_n \hat{\mathbf{x}} = \hat{\mathbf{\Pi}}_n \hat{\mathbf{b}} \tag{71}$$

which leads to

$$\mathbf{A}_n \mathbf{x} = \mathbf{\Pi}_n \mathbf{b} \quad \text{and} \quad \mathbf{A}_n \mathbf{x}_o = \mathbf{\Pi}_n \mathbf{b}_o + \mathbf{\Pi}_{on} \mathbf{b} - \mathbf{A}_{on} \mathbf{x} \tag{72}$$

where $\hat{\mathbf{A}}_n$ is upper triangular, while $\mathbf{\Pi}_n$ and $\mathbf{\Pi}_{on}$ are defined as

$$\mathbf{\Pi}_n = \mathbf{H}_n \mathbf{H}_{n-1} \ldots \mathbf{H}_2 \mathbf{H}_1 \tag{73a}$$

$$\mathbf{\Pi}_{on} = \mathbf{H}_n \mathbf{H}_{n-1} \ldots \mathbf{H}_{o1} + \cdots + \mathbf{H}_n (\mathbf{H}_o)_{n-1} \ldots \mathbf{H}_1 + \mathbf{H}_{on} \mathbf{H}_{n-1} \ldots \mathbf{H}_1 \tag{73b}$$

6 Examples

Dual Householder reflections and the recursive least-square algorithm will now be applied to the approximate synthesis of a RCCC spatial four-bar linkage. This example is taken from an earlier paper [36]. The mechanism under synthesis, shown in Fig. 1, is to operate as closely as possible to *homokinetic*[10] in the pair (ψ, ϕ).

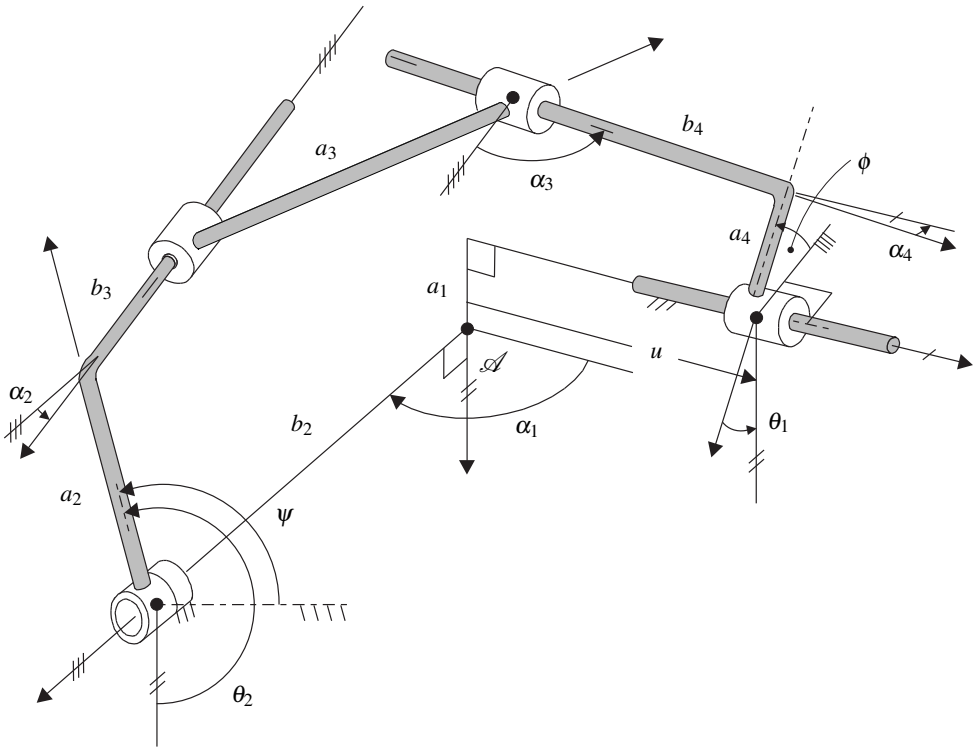

Fig. 1. Generic RCCC spatial four-bar linkage.

For the sake of brevity, the input-output equations are not derived here, as they are available elsewhere [37]. The input is angle $\hat{\psi}$, a "dual angle" with zero dual part; the output is $\hat{\phi} \equiv \phi + \varepsilon u$. One has

$$\begin{aligned} \hat{F}(\hat{\psi}, \hat{\phi}) \equiv & \hat{k}_1 + \hat{k}_2 \cos\hat{\psi} + \hat{k}_3 \cos\hat{\psi}\cos\hat{\phi} - \hat{k}_4 \cos\hat{\phi} \\ & + \sin\hat{\psi}\sin\hat{\phi} = 0 \end{aligned} \tag{74}$$

On the one hand, the primal parts of the *dual Freudenstein parameters* (DFP) $\{\hat{k}_i\}_1^4$ are defined as

$$\begin{aligned} & k_1 \equiv \frac{\lambda_1\lambda_2\lambda_4 - \lambda_3}{\mu_2\mu_4}, \quad k_2 \equiv \frac{\lambda_4\mu_1}{\mu_4}, \quad k_3 \equiv \lambda_1, \\ & k_4 \equiv \frac{\lambda_2\mu_1}{\mu_2} \end{aligned} \tag{75}$$

[10]The *italicized* term means a one-to-one velocity transmission ratio.

their dual counterparts being defined as

$$k_{o1} \equiv -\frac{a_1\lambda_2\lambda_4\mu_1\mu_2\mu_4 + a_2(\lambda_1\lambda_4 - \lambda_2\lambda_3)\mu_4}{\mu_2^2\mu_4^2} - \frac{-a_3\mu_2\mu_3\mu_4 + a_4(\lambda_1\lambda_2 - \lambda_3\lambda_4)\mu_2}{\mu_2^2\mu_4^2}, \tag{76a}$$

$$k_{o2} \equiv \frac{a_1\lambda_1\lambda_4\mu_4 - a_4\mu_1}{\mu_4^2}, \tag{76b}$$

$$k_{o3} \equiv -a_1\mu_1, \tag{76c}$$

$$k_{o4} \equiv \frac{a_1\lambda_1\lambda_2\mu_2 - a_2\mu_1}{\mu_2^2} \tag{76d}$$

The variables λ_i and μ_i are defined, in turn, as:

$$\lambda_i \equiv \cos\alpha_i, \quad \mu_i \equiv \sin\alpha_i \neq 0 \tag{77}$$

6.1 Offline Solution

To formulate the approximate-synthesis problem, two sets of input-output variables are given, $\{\psi_i\}_1^m$ and $\{\phi_i, u_i\}_1^m$, where $m > n$, n being the number of unknown linkage parameters. A system of m dual linear equations in the four DFP is obtained when each pair of input-output values is applied to eq.(74):

$$\hat{\mathbf{S}}\hat{\mathbf{k}} = \hat{\mathbf{b}} \tag{78}$$

with

$$\hat{\mathbf{S}} = \underbrace{\begin{bmatrix} 1 & c\psi_1 & c\psi_1 c\phi_1 & -c\phi_1 \\ 1 & c\psi_2 & c\psi_2 c\phi_2 & -c\phi_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & c\psi_m & c\psi_m c\phi_m & -c\phi_m \end{bmatrix}}_{\mathbf{S}} + \varepsilon \underbrace{\begin{bmatrix} 0 & -b_2 s\psi_1 & -u_1 c\psi_1 s\phi_1 - b_2 s\psi_1 c\phi_1 & u_1 s\phi_1 \\ 0 & -b_2 s\psi_2 & -u_2 c\psi_2 s\phi_2 - b_2 s\psi_2 c\phi_2 & u_2 s\phi_2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & -b_2 s\psi_m & -u_m c\psi_m s\phi_m - b_2 s\psi_m c\phi_m & u_m s\phi_m \end{bmatrix}}_{\mathbf{S}_o} \tag{79}$$

$$\hat{\mathbf{k}} = \underbrace{\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix}}_{\mathbf{k}} + \varepsilon \underbrace{\begin{bmatrix} k_{o1} \\ k_{o2} \\ \vdots \\ k_{om} \end{bmatrix}}_{\mathbf{k}_o} \tag{80}$$

$$-\hat{\mathbf{b}} = \underbrace{\begin{bmatrix} s\psi_1 s\phi_1 \\ s\psi_2 s\phi_2 \\ \vdots \\ s\psi_m s\phi_m \end{bmatrix}}_{-\mathbf{b}} + \varepsilon \underbrace{\begin{bmatrix} u_1 s\psi_1 c\phi_1 + b_2 c\psi_1 s\phi_1 \\ u_2 s\psi_2 c\phi_2 + b_2 c\psi_2 s\phi_2 \\ \vdots \\ u_m s\psi_m c\phi_m + b_2 c\psi_m s\phi_m \end{bmatrix}}_{-\mathbf{b}_o} \tag{81}$$

where $c\psi_i$, $c\phi_i$, $s\psi_i$ and $s\phi_i$ denote, respectively, $\cos\psi_i$, $\cos\phi_i$, $\sin\psi_i$ and $\sin\phi_i$.

For this example, $m = 501$ prescribed data triads are used. Moreover, ψ_i and ϕ_i are, respectively, uniformly distributed in the intervals $86^\circ \leq \psi \leq 206^\circ$ and $-26^\circ \leq \psi \leq 94^\circ$. The values u_i are symmetrically distributed around $u = 0$, with $u_1 = -a_1/10$ and $u_m = a_1/10$. In light of the homokinetic condition imposed on the mechanism, a *cycloidal motion program* for the translation of the output joint is chosen. This motion is defined by a signed distance u_i, calculated from axis $\mathscr{A}$. The

values of u_i are computed with the following distribution:

$$\begin{aligned} u_i =& -\frac{a_1}{10} + U\left(\frac{i-1}{m-1} - \frac{1}{2\pi}\sin\frac{2\pi(i-1)}{m-1}\right), \\ i =& 1,\ldots,m \end{aligned} \tag{82}$$

where U is the amplitude of the motion, set as $a_1/5$. Again, since a homokinetic behavior is desired, the linkage is assumed symmetric, thus reducing the number of unknowns to only two in the primal part, since $k_3 = 0$ and $k_4 = k_2$ under the foregoing constraint. The parameter a_1, based on other design considerations, is chosen to be equal to 240 mm, and so is b_2 by symmetry. The shafts to be coupled being normal to each other, α_1 is equal to 90°. The number of unknowns in the dual part of the system of equations also reduces to two, given that $k_{o3} = -240$ mm as per eq.(76) and $k_{o4} = k_{o2}$, again by symmetry.

In the approximate-synthesis problem, the design error $\hat{\mathbf{e}}$ is minimized with respect to its Euclidian norm, not the structural error $\hat{\mathbf{s}} \equiv (\varphi - \phi) + \varepsilon(\mathbf{v} - \mathbf{u})$, with $\varphi \equiv [\varphi_1 \quad \ldots \quad \varphi_m]^T$, $\phi \equiv [\phi_1 \quad \ldots \quad \phi_m]^T$, $\mathbf{v} \equiv [v_1 \quad \ldots \quad v_m]^T$ and $\mathbf{u} \equiv [u_1 \quad \ldots \quad u_m]^T$[11]. By itself, the design error does not have a geometrical meaning. However, as stated by Hayes *et al.* [38], the structural error can be minimized upon minimizing the design error with a large number of prescribed data-pairs for a planar four-bar linkage. The same applies to the approximate synthesis of spatial four-bar linkages with a large number of data-triads. The two errors converge as m increases. Using the offline algorithm described in Section 5.2, the least-square solution was obtained as

$$\begin{aligned} k_1 =& 1.275,\ k_2 = k_4 = 0.9439,\ k_{o1} = 318.6 \text{ mm}, \\ k_{o2} =& k_{o4} = 144.2 \text{ mm} \end{aligned} \tag{83}$$

The linkage parameters can now be computed with eqs.(75) and (76):

$$\begin{aligned} \alpha_2 =& \alpha_4 = 46.65^\circ,\ \alpha_3 = 132.4^\circ, \\ a_2 =& a_4 = -76.26 \text{ mm},\ a_3 = 249.8 \text{ mm} \end{aligned} \tag{84}$$

The RMS primal and dual errors are, respectively, 0.0194 rad and 29.6156 mm. The dual version of Householder reflections made it possible to obtain recursively k_i and k_{oi} without having to first compute all the components of $\mathbf{k}$. This reduces the computational complexity and enhances the numerical stability.

[11] The terms (φ_i, v_i) are, respectively, the generated values of the outputs ϕ and u attained at $\psi = \psi_i$ and (ϕ_i, u_i) are the prescribed values of the output angle and position at $\psi = \psi_i$.

6.2 Online Solution

To use the online recursive least-square procedure outlined in Subsection 4.1, three data triads are selected from the 501 data triads previously used to initiate the algorithm: (ψ_1, ϕ_1,u_1), (ψ_{251}, ϕ_{251}, u_{251}) and (ψ_{501},ϕ_{501}, u_{501}). Afterwards, one new triad was randomly selected from the 498 remaining and used as a new observation. The results show that with only 154 triads, smaller primal and dual errors were obtained, compared with the offline algorithm. The DFP are slightly different, when compared with those above, in particular their dual parts:

$$\begin{aligned} k_1 &= 1.278,\ k_2 = k_4 = 0.9436,\ k_{o1} = 303.8 \text{ mm}, \\ k_{o2} &= k_{o4} = 133.1 \text{ mm} \end{aligned} \tag{85}$$

The evolution of the primal and dual errors with each new iteration is shown in Fig. 2.

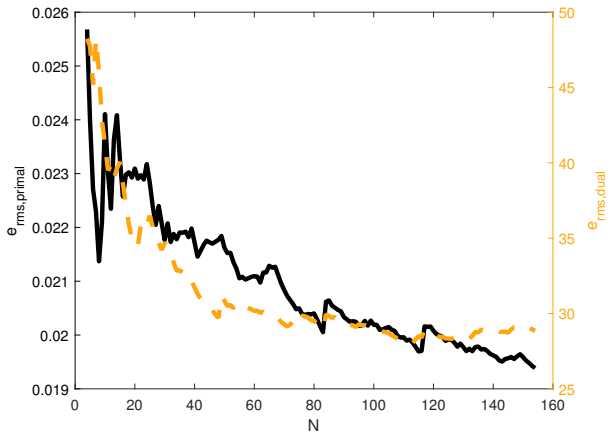

Fig. 2. Evolution of the primal and dual errors.

7 Conclusions

Least-square problems lie at the core of identification techniques arising in a variety of application fields. In connection with mechanisms, in particular, kinematic synthesis problems usually include quantities involving rigid-body rotations and translations. These quantities differ in their units, nondimensional for orientation relations, units of length for their translation counterparts. Quantities derived from these, angular velocities and accelerations, along with their concomitant velocities and accelerations, follow the same pattern. In this context, a favored tool is dual numbers, consisting of a primal, nondimensional part, of a scalar, vector, or matrix nature for the former, and its dual counterpart for the latter. The paper introduced algorithms derived from the classical least-squares domain that are applicable to the two cases. In this connection, both offline and online identification procedures are considered, using the same tool, dual numbers. Applications to the kinematic synthesis of mechanisms are reported in this paper.

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List of figure captions

Fig. 1. Generic RCCC spatial four-bar linkage.

Fig. 2. Evolution of the primal and dual errors.