TURBULENCE IN CLOUDS
AS A FACTOR IN PRECIPITATION

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LIST OF SYMBOLS USED

A	Acceleration of air
C_D	Drag coefficient
D	Drag force in viscous motion
E	Collision efficiency
$G(\omega)$	Spectral density function of air velocity V
g	Gravitational field
$g_{crit.}$	Critical gravitational field, just sufficient to cause collision.
K	Relative velocity parameter in Langmuir's equation $K = \frac{2r^2\rho_s U}{9\eta s}$
m	Mass of sphere
n_r	Number of drops of radius r, per unit volume, per unit radius interval.
Q	Volume swept out in unit time by drop in random motion
Q'	Equivalent swept volume per unit time, allowing for droplet radius and collision efficiency.
$\mathcal{R}e$	Real part of
R	Reynolds number
r	Radius of droplet
s	" " drop.
t	Time
U	Relative velocity of drop and droplet
U_0	$= \frac{U}{K} = \frac{9\eta s}{2r^2\rho_s}$
$U(t)$	Relative velocity as a time function
$U(\omega)$	Frequency spectrum of relative velocity
$\mathbb{U}$	Relative speed of drop and droplet in three dimensions.
V	Velocity of air

$\hat{V}$	Peak velocity of air in sinusoidal motion
$V(t)$	Air velocity as a time function
$V(\omega)$	Frequency spectrum of air velocity
v_r	Velocity of droplet
v_s	Velocity of drop
α	Collision rate integral
ε	Rate of supply of energy per unit mass
η	Viscosity of air ($\eta = 1.72 \times 10^{-4}$ gm cm^{-2} sec^{-1} at 0 C)
ρ	Density of air
ρ_s	Density of sphere (ρ_s = 1 gm cm^{-3} for water)
σ_A	R.m.s. acceleration of air
σ_g	Standard geometric deviation in log-normal distributions
σ_K	R.m.s. value of K in random motion
σ_v	R.m.s. relative velocity of drop and droplet
σ_V	R.m.s. air velocity
τ_r	Time constant of droplet, radius r
τ_s	Time constant of drop, radius s
ω	Angular frequency ($\omega = 2\pi f$)
ω_o	Characteristic angular frequency of air spectrum
ω_H	Angular frequency at upper limit of band spectrum
ω_L	Angular frequency at lower limit of band spectrum

1. INTRODUCTION

Rain and snow are made out of excess water vapour in the atmosphere, but not directly. When conditions are such that the atmosphere can no longer hold all the water it contains as water vapour, clouds appear first, and precipitation may or may not fall out of these clouds.

Clouds can remain relatively unchanged for several hours, they may then evaporate without precipitating at all, or precipitation, in small or large quantities, may form after a shorter or longer time. An important problem in meteorology, then, is to find a precipitation mechanism to account for the fact that clouds do not automatically turn into precipitation, but require favourable circumstances for this.

Bergeron (1933) proposed a theory which, with modifications by Findeisen, is now generally accepted. He suggested that the stability of clouds as colloids is due to the small size of the particles, which cannot grow at one anothers' expense at a sufficient rate to produce precipitation of their own accord. We shall take up this question of colloidal stability again later.

In Bergeron's theory the cloud comes into a region where the ambient temperature is low enough to freeze some of the water droplets. With the help of freezing nuclei this takes place at about -12C. The region between the droplets is saturated by the supercooled water, so that the ice crystals grow rapidly at the expense of the water droplets, and fall through the cloud as snow. If the 0 C isotherm lies above the ground, the snow melts there and reaches the ground as rain.

Recent experimental work has confirmed this model in frontal rain where moderate rainfalls continue for several hours. Radar observations of this type of precipitation have been correlated with meteorological soundings by Austin and Bemis (1950) in the United States, and with aircraft observations by Bowen (1951) in Australia and by R.F. Jones (1951) in England. The characteristic "bright band" seen on radar displays is associated with the melting of the snow.

We turn now to showers, which consist of localised rainfall characterised by vertical development. If the ice process were responsible for these also, they would always have to extend at least above the 0 C isotherm. Experiments on the freezing of small water drops show that even with the most effective nuclei the drops do not freeze until cooled to about -12C. This indicates that clouds which do not penetrate the -12C isotherm would contain no ice particles, though exceptions may be found in two types of situation. Dessens (1950) sampled clouds at the summit of a mountain and found ice particles in 50% of clouds examined between 0C and -12C (and 100% of clouds colder than -12C); he suggested that this result could be due to the proximity of fallen snow on the ground. Dennis (1953), in a study of radar records of showers, has found evidence that some showers whose tops were between -3C and -12C were the result of seeding by snow from higher levels.

There are situations, however, in which it appears to be impossible to account for observed precipitation by the Bergeron process, namely, those in which rain or drizzle is observed to fall from clouds which lie entirely below the 0C isotherm. Mason and Howorth (1952) report many cases of drizzle falling from stratiform clouds whose upper surface is warmer than 0C. E.J. Smith (1951) described three flights in which heavy rain was observed falling from non-freezing clouds.*

Besides the observations of precipitating clouds warmer than 0C in which the Bergeron process is impossible, there are many cases of precipitating clouds warmer than -12C in which the process seems unlikely, and these serve as additional evidence that some other process is at work. (Clouds warmer than -12C are conveniently called "warm").

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* In tropical regions, moderate or heavy rain often falls from clouds warmer than 0C. See Hunt (1949) and Virgo (1950).

Arenberg (1939) discussed the action of turbulence in causing water droplets to combine, but could not reach a definite conclusion. He did not discuss the conditions under which droplets could collide. Gabilly (1949) went further but confined his discussion to sinusoidal motion of the air, rather than true turbulence, which is a random motion. Mason (1952) introduced turbulence to prolong the path travelled by a droplet through a cloud to account for observed drizzle. In Best's (1952) discussion of condensation onto cloud droplets, turbulence served to transport the droplets to and from the centre of the cloud.

In this thesis it will appear that turbulence not only brings droplets together, but ensures sufficient relative velocity for collisions to take place.

2. CLOUDS.

In discussing the conversion of cloud into rain, we shall need to know the initial condition of the cloud, in particular, the size and number of the water droplets in the cloud. Direct measurements of these quantities have been made from aircraft, and described by Diem (1942, 1948) and by aufm Kampe and Weickmann (1952). The droplets are caught in a film of oil and examined under a microscope. This method fails to detect droplets less than 2 or 3 microns in radius, but it is not completely certain that this is due to the measurements. The method appears to be reliable for all droplets larger than this.

Their results show that most types of cloud have J - shaped distributions, in which although most of the droplets have radii less than 20 microns, a small but significant number of droplets have radii in the order of 100 microns. Thus most clouds must be regarded as ones in which some precipitation process has already started, so that rain or drizzle will soon fall (though it may evaporate before reaching the ground).

However, clouds which aufm Kampe and Weickmann classify as Cumulus Humilis or as Cumulus, and Diem as Schönwetter Cumulus, are free from these larger droplets. We shall call these Fair-weather Cumulus, and will take them as our starting point. The drop size spectra of a number of these clouds were replotted on a logarithmico - normal chart (as described by

Kottler (1950, 1951, 1952)) and gave nearly straight lines, showing that the distributions are approximately log-normal. The median radii varied from 6.4 to 7.2 microns and the standard geometric deviation σ_g from 1.2 to 1.7.

It appeared that Fair-weather Cumulus clouds could be represented for computation by the idealised distribution:

$$n_r\,dr = \frac{7\times10^8}{r}\exp\left\{-22\left(\log_{10}\frac{r}{7}\right)^2\right\}dr \qquad (2.1.)$$

where $n_r dr$ is the number of drops per cubic metre whose radii lie between r and $r+dr$ microns.

This is a log-normal distribution with median radius 7 microns and standard geometric deviation $\sigma_g = 1.4$. It is noteworthy that Palmer (1949) found the distribution of rain-drops and snowflake sizes could also be fitted by a log-normal curve, and that the values of σ_g he obtained were close to 1.4.

The idealised distribution of equation (2.1.) is plotted in figure 2.1. It is normalised so that at the median radius of 7 microns it contains 10^8 drops per cubic metre per micron radius interval. This gives it 6.08×10^8 drops per cubic metre altogether.

The important characteristic of the original data for these Fair-weather Cumulus clouds, which is well reproduced in the idealised curve, is the scarcity of drops of radii greater than 16 microns. In contrast, Cumulo-Nimbus distributions are much

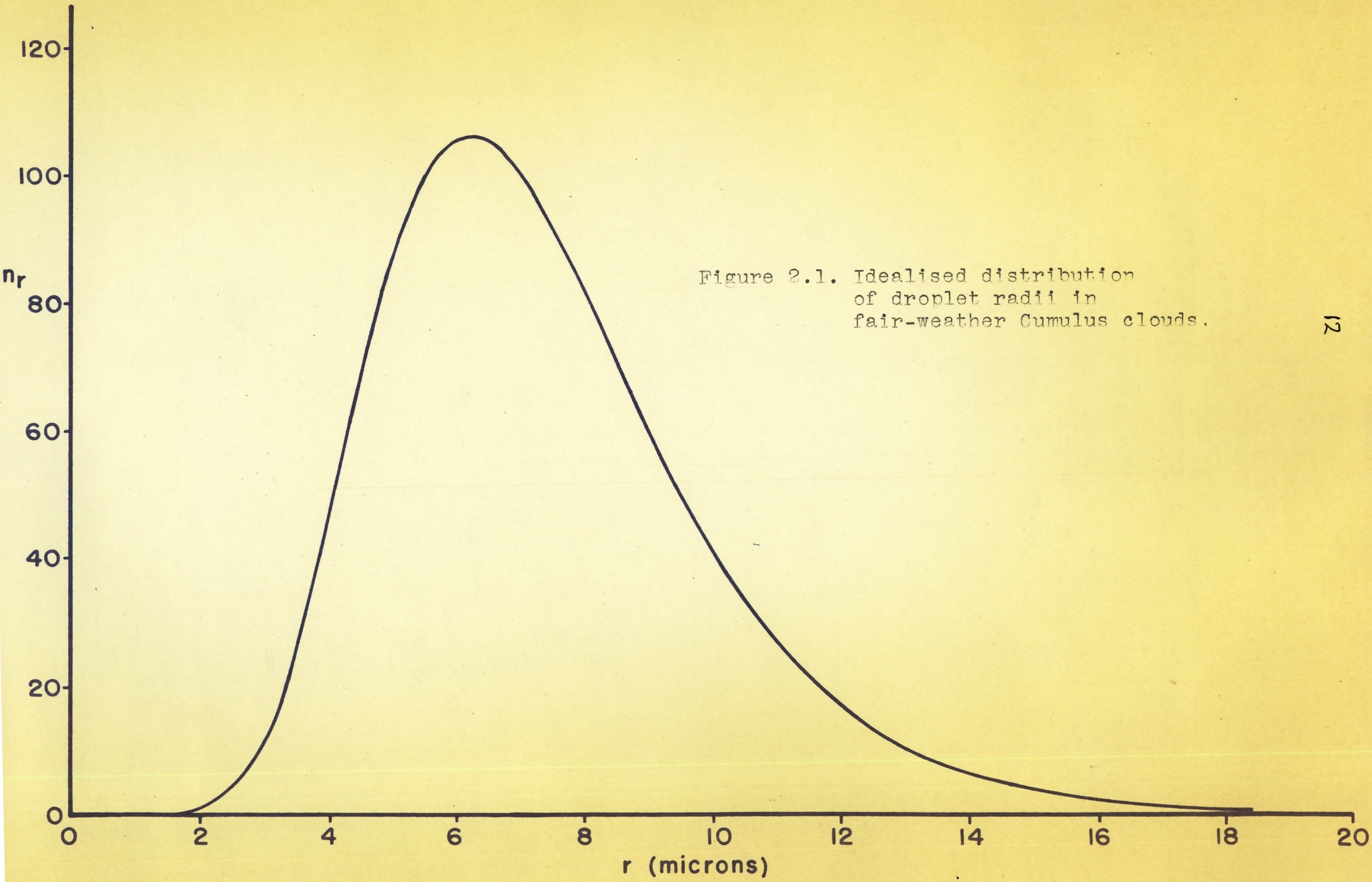

Figure 2.1. Idealised distribution of droplet radii in fair-weather Cumulus clouds.

wider and some are apparently made up of two log-normal distributions with different median radii.

Cumulus clouds differ from layer clouds in having localised updrafts. A freshly-formed cumulus will have just a single central upward current of air with a velocity of a few metres per second. This updraft is maintained by the release of thermal energy by the condensing water vapour, and is prevented from accelerating indefinitely by entrainment of outside air, which makes the flow turbulent. Bowen (1950) has pointed out the importance of the updraft in prolonging the time that a raindrop spends in the cloud in the later stages of growth, and in the following sections we shall discuss the action of turbulence in bringing about the earlier stages of growth.

3. MOTION AND COLLISION OF DROPS

3.1. Motion of droplets in air.

A sphere radius r, density ρ_s, and mass m moves with constant velocity v_r through a fluid.

The drag force

$$D = \pi r^2 . \tfrac{1}{2} \rho v_r^2 . C_D , \tag{3.1.}$$

where C_D is the drag coefficient and is a function of the Reynolds number

$$R = 2 r v_r \rho / \eta . \tag{3.2}$$

Here ρ is the density and η the viscosity of the air.

For values of $R \ll 1$ the relation between C_D and R simplifies to

$$C_D = 24/R , \tag{3.3}$$

giving Stokes' Law

$$D = 6 \pi r \eta v_r . \tag{3.4}$$

Stokes' Law is obeyed sufficiently closely by cloud droplets at their terminal velocity for the purposes of the calculations which follow. Departures from the law will be taken into account where a drop has grown large enough to require it.

When drag is the only force acting on the drop, and Stokes' Law applies, the equation of motion in one dimension is

$$m \frac{dv_r}{dt} = 6 \pi r \eta (V - v_r) , \tag{3.5}$$

where v_r is the velocity of the drop relative to stationary axes and V the air velocity.

We define a "time constant"

$$\tau_r = \frac{m}{6\pi r\eta} = \frac{2r^2\rho_s}{9\eta} . \qquad (3.6)$$

The equation of motion becomes

$$\tau_r \frac{dv_r}{dt} = V - v_r . \qquad (3.7)$$

For example, suppose V to be a step function so that

$$\begin{aligned} V &= 0, & t &\leq 0, \\ V &= V_0, & t &> 0. \end{aligned} \qquad (3.8)$$

Then

$$\begin{aligned} v_r &= 0, & t &\leq 0, \\ v_r &= V_0\left[1 - \exp\left\{-\frac{t}{\tau_r}\right\}\right], & t &> 0. \end{aligned} \qquad (3.9)$$

3.2. Langmuir's Equation.

Langmuir, (1948) considered the problem of a current of air containing very small droplets, flowing past a fixed sphere. If the droplets continued in the same straight line, many would collide with the sphere, the number being proportional to the cross-section of the sphere. The collision cross-section would be πs^2, where s is the radius of the sphere. However, the air flows round the sphere in streamlines, and if the droplets showed no inertia they would be carried round the sphere and never collide with it. Inertia makes some collisions take place, but not as many as with perfectly straight paths. Using a differential analyser, Langmuir computed the paths of the centroids of the droplets to find which would meet the sphere. He defined the "Collision Efficiency" E as the ratio of actual collision cross-section to πs^2.

The results are contained in the empirical equation

$$E = 0, \qquad K \leqslant 1.214,$$

$$E = \left[1 + \frac{\frac{3}{4}\ln(2K)}{K - 1.214}\right]^{-2}, \qquad K > 1.214, \qquad (3.10)$$

where K is a dimensionless quantity given by

$$K = \tau_r \frac{U}{s} = \frac{2r^2\rho_s U}{9\eta s}. \qquad (3.11)$$

Here U is the velocity of the air (and of the droplets before encountering the sphere) relative to the sphere.

The form of equation (3.10) is shown in figure 3.1 where E is plotted against K on a log scale. The main feature is that below a critical value of K no collisions take place at all. This means that for given radii s and r, the relative velocity must be greater than some critical value for collisions to be possible.

Langmuir carried out calculations for a sphere much larger than the droplet, so that $s \gg r$. In the calculations which follow, the radii may be nearly equal in the early stages but for convenience the term "drop" will be used for the larger particle and "droplet" for the smaller. The disparity in sizes allowed Langmuir to represent the droplet by a point mass at its centroid. His values of E, therefore, are lower limits, since a droplet whose centroid would miss the drop by less than r (the droplet radius) will actually collide with the drop. This

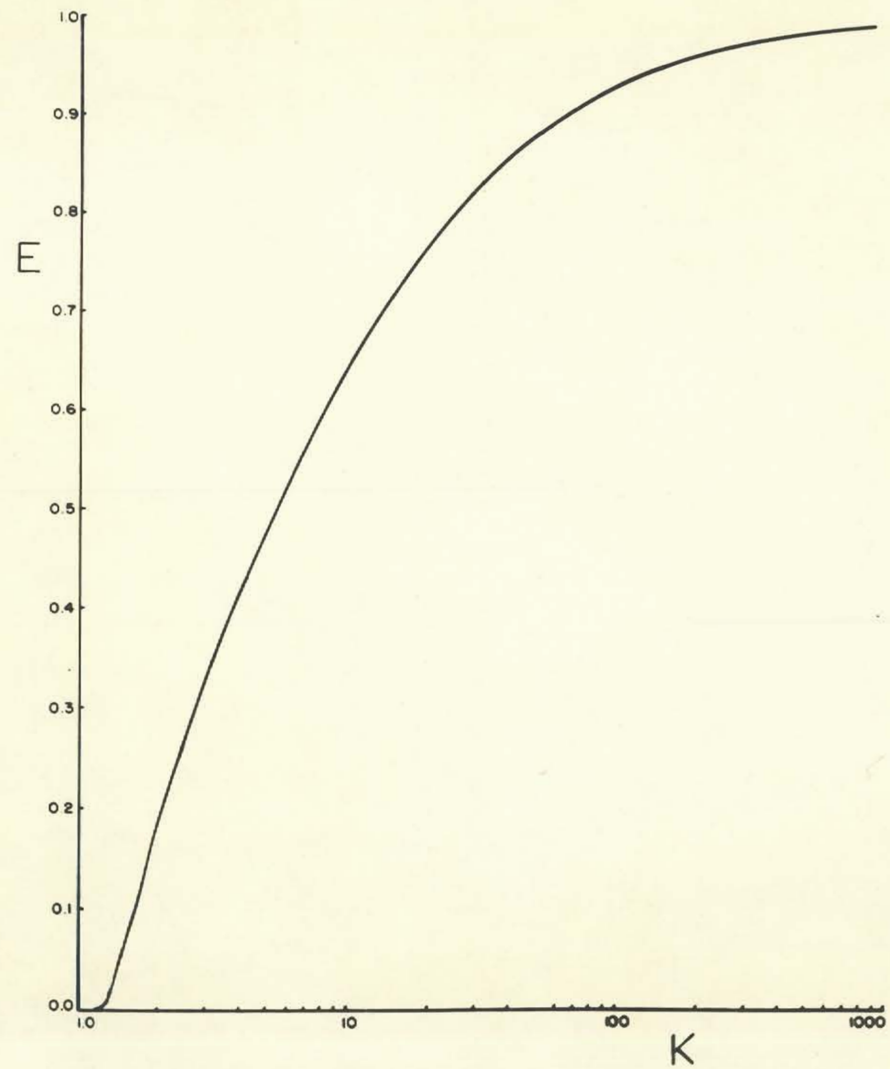

Figure 3.1. Collision Efficiency E versus K (Langmuir's equation).

will be allowed for, to some extent, by taking the capture cross-section for the drop and droplet to be $E\pi(s+r)^2$ instead of $E\pi s^2$.

A further assumption which needs examining, is that the droplet is stationary relative to the air before it comes into the region of the drop. In a gravitational field, both drop and droplet move relative to the air, the droplet more slowly. If the air has just been accelerated, neither particle is stationary relative to it. In such cases U will be put equal to the relative velocity of drop and droplet regardless of motion of the air. The errors arising from this assumption are not expected to be important.

The question remains whether a droplet which touches the drop always combines with it to form a single larger drop, or whether it sometimes "bounces off". Gabilly (1949) observed, under a microscope, coalescence between a captive drop and small droplets blown onto it. Dessens (1950) found by microscopic observations that if droplets are brought into contact, they always coalesce provided the radii are greater than about 1 or 2 microns. Gunn and Hitschfeld (1951) performed quantative experiments which appear to show that a true collision always, or nearly always, results in coalescence, so that collection efficiency and collision efficiency will be assumed to have the same value.

4. COALESCENCE BY GRAVITY.

Consider a drop of radius s and a droplet of radius r in still air in a gravitational field g. If the locale is the earth's troposphere, g will have a value which varies little and is about 980 cm sec^{-2}. By equating the weight of the droplet to the drag force and substituting from equation (3.4) we find that the droplet will fall with velocity

$$v_r = g\frac{2r^2}{9\eta} = g\tau_r, \tag{4.1}$$

and the drop will fall with the greater velocity

$$v_s = g\frac{2s^2}{9\eta} = g\tau_s, \tag{4.2}$$

so that their relative velocity

$$U = \frac{2}{9\eta}\cdot g\left(s^2-r^2\right) = g\left(\tau_s-\tau_r\right). \tag{4.3}$$

If this relative velocity is insufficient, collision between the drops will be impossible even if the drop starts exactly above the droplet. A very small droplet, for instance, would escape collision unless the field were so great that the drop could catch it. Adroplet almost as big as the drop would also escape collision because their velocities are nearly equal, unless the field were great enough to bring up the relative velocity. Intermediate droplet sizes favour capture by the drop, and require less field.

Now $$K = \frac{U}{s}\tau_r = \frac{4\rho_s^2}{81\eta^2}\cdot g\cdot\frac{r^2}{s}\left(s^2-r^2\right). \tag{4.4}$$

The value of g for which collision is just possible is found by putting $K = 1.214$.

$$g_{crit.} = 1.214 \frac{81\eta^2}{4\rho_s^2} \frac{s}{r^2(s^2-r^2)}$$

$$= 7.25 \times 10^5 \frac{s}{r^2(s^2-r^2)} \qquad (4.5)$$

where r, s are in microns and $g_{crit.}$ in cm sec^{-2}.

In figure 4.1., $g_{crit.}$ is plotted against r for $s = 15$ microns. For other values of s, the curve would have the same shape but different scales. Since K is proportional to g (4.2) the value of K corresponding to g other than $g_{crit.}$ can be found for a particular r by reading off $g_{crit.}$ and putting

$$K = 1.214 \frac{g}{g_{crit.}} . \qquad (4.6)$$

For a field of 980 cm sec^{-2}, collisions are only possible over a limited range of radii of the smaller drops, and in this range the collection efficiency is small. At $r = 10$ microns, close to the minimum of the curve, the critical field is 870 cm sec^{-2} and the efficiency, though nearly a maximum, is only about 3%.

A calculation was made of the growth of a drop whose radius s is 15 microns, falling in a cloud which has the idealised characteristic of equation (2.1), (figure 2.1.).

The numerical computation necessary for figure 4.1. table 4.1, and much of that for figure 4.2 were carried out by M. Herschorn.

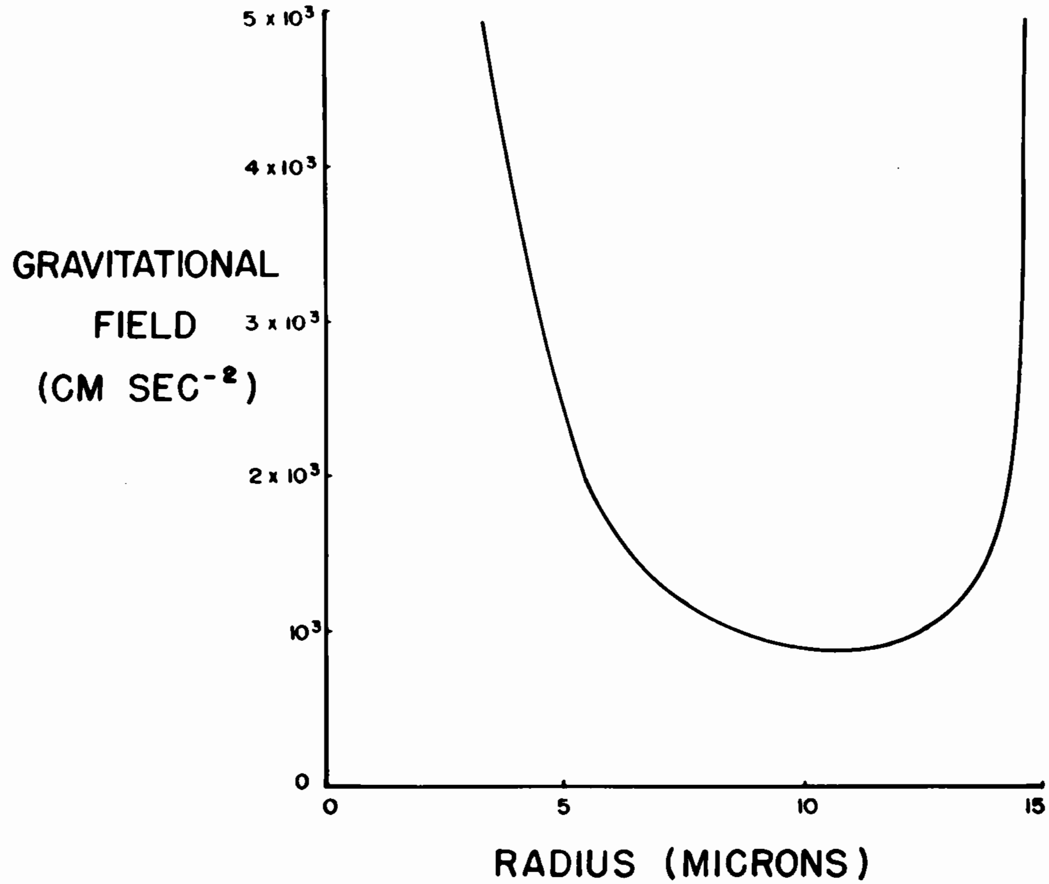

Figure 4.1. Critical gravitational field (for K = 1.214) versus droplet radius, for a drop radius S=15 microns.

It grows by absorbing any droplets with which it collides; it is found to increase in radius by only 0.42×10^{-3} microns per second, so that it would take roughly 40 minutes to increase from 15 to 16 microns radius. Even 18 micron drops (which are scarce in the type of cloud considered) only grow at the rate of 3×10^{-3} microns per second.

The observed stability of Fair-weather Cumulus clouds is explained by these findings. It is evident that the value of g found on the earth is too small to coagulate them.

Similar calculations were carried out for a field twice as great; much faster growth is found. The method of calculation was purely numerical. The results are shown in table 4.1

Table 4.1.

Rate of increase of s
($\times 10^{-3}$ microns per second)

g cm sec^{-2}	$s = 15$	18	microns
980	0.42	3.0	
1960	5.6	15.1	

Doubling the gravitational field has increased the rate of growth of a 15 micron drop by about 13 times, and of an 18 micron drop by about 5 times. The relative velocities have all been doubled, so that the drop falls past twice as many droplets in a given time, but more important still, this has resulted in greatly increased collection efficiencies, so that overtaking results in collision much more often.

To pursue the growth of the drop, calculations were continued by a partly analytical method, in which the function E was approximated by suitably chosen fixed values, and the rate of growth ds/dt found by integration. The resulting graph of ds/dt against s was integrated numerically to obtain s against t. The overall accuracy is estimated to be about 10% in rate of growth. The drop becomes too big for Stokes' Law to apply, so a semi-empirical law for terminal velocity was used. At this stage, efficiency can be taken to be 100% for all significant drop sizes.

Figure 4.2 shows the way that the radius of the drop increases with time, starting from 20 micron radius, (dotted curve). The drop grows quite rapidly, reaching raindrop size in about 25 minutes.

The other curves are copied from Houghton (1950) and are for $g = 980$ cm sec^{-2}. The lowest line is for a rather homogeneous cloud of median radius 7 microns and water content 1 $gm\ m^{-3}$. The other solid line is for a nimbostratus of water content 2 $gm\ m^{-3}$ In both, growth is slow to start with. The dashed curve is for an ice crystal plate in water-saturated air at -5 C.

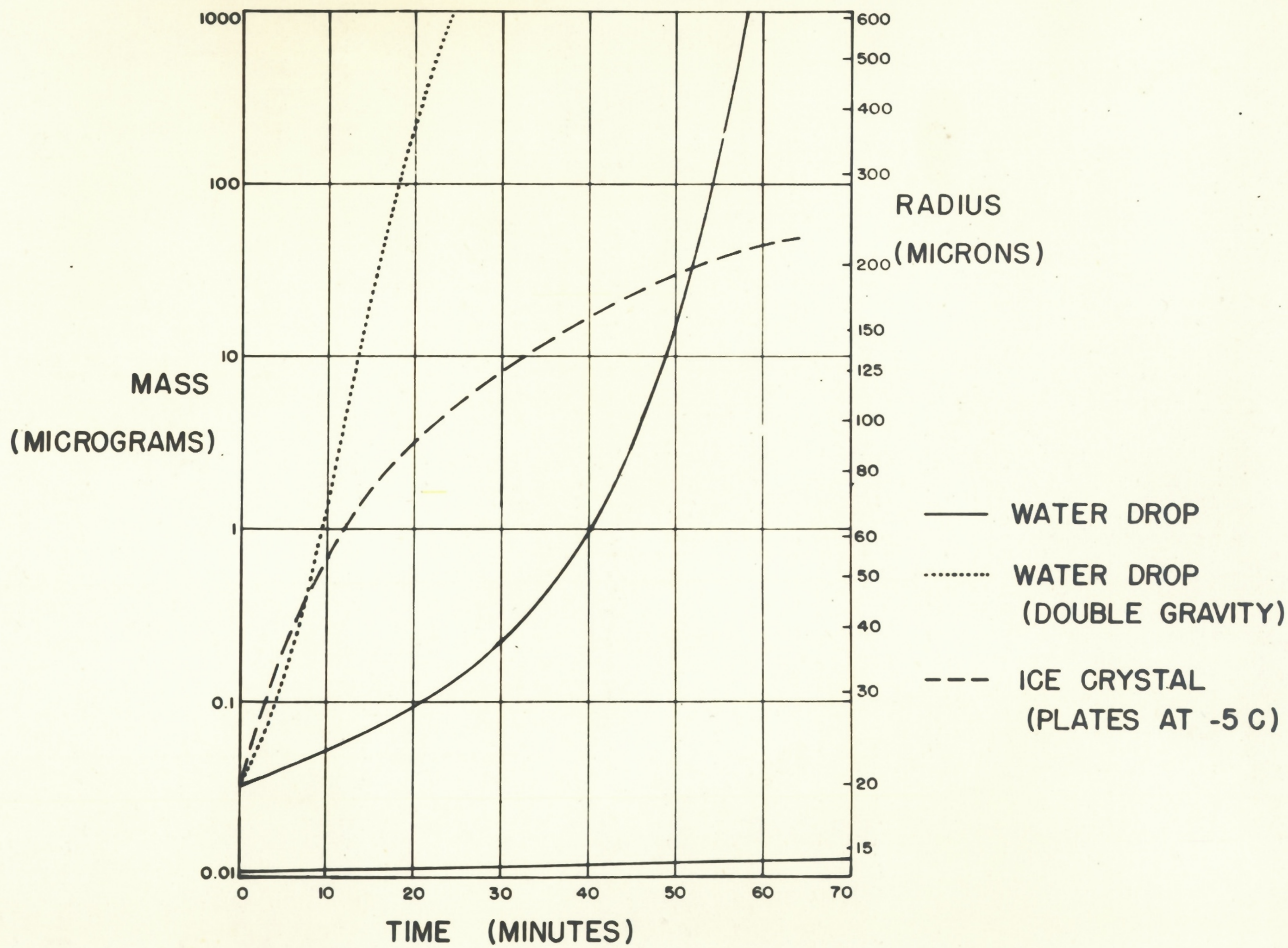

Figure 4.2. Growth of water-drops and ice-crystal in cloud.

Its initial growth is comparable with that for $g = 1960$ cm sec^{-2}.

The gravitational field used for this calculation is not found in the earth's atmosphere, of course. However, a steady acceleration of the air in which a drop and droplet find themselves would produce the same effect as a gravitational field, and could bring about collisions. Any kind of acceleration of the air would produce temporarily the effects of a gravitational field, and could also cause collisions. In the next section we shall consider systematic and random motion of the air in a straight line, as intermediate stages in the approach to the actual motion of turbulent air.

5. ONE - DIMENSIONAL EXCITATION.

5.1. Sinusoidal Excitation.

Consider a parcel of air containing the drop (radius s, time constant τ_s) and the droplet (radius r, time constant τ_r) to be in sinusoidal motion with peak velocity $\hat{V}$ and angular frequency ω. Then the instantaneous air velocity

$$V = \mathcal{R}e\left[\hat{V}\exp(i\omega t)\right]. \tag{5.1}$$

The motion of the droplet is given by substituting for V in equation 3.7, and solving for v_r. The droplet velocity is

$$v_r = \mathcal{R}e\left[\frac{\hat{V}}{1+i\omega\tau_r}\exp(i\omega t)\right].$$

Or in the notation customary in electrical problems, where the symbols $\mathcal{R}e$ and $\exp(i\omega t)$ are omitted

$$v_r = \frac{V}{1+i\omega\tau_r}. \tag{5.2}$$

For the drop

$$v_s = \frac{V}{1+i\omega\tau_s}. \tag{5.3}$$

So the relative velocity

$$U = \frac{V.\,i\omega(\tau_s-\tau_r)}{(1+i\omega\tau_s)(1+i\omega\tau_r)}. \tag{5.4}$$

Taking absolute values only, the relative velocity of the drops (compared to the air velocity) is given by

$$\left|\frac{U}{V}\right|^2 = \frac{\omega^2(\tau_s-\tau_r)^2}{(1+\omega^2\tau_s^2)(1+\omega^2\tau_r^2)}. \tag{5.5}$$

This function is shown in figure 5.1, where $\left|\frac{U}{V}\right|$ is plotted

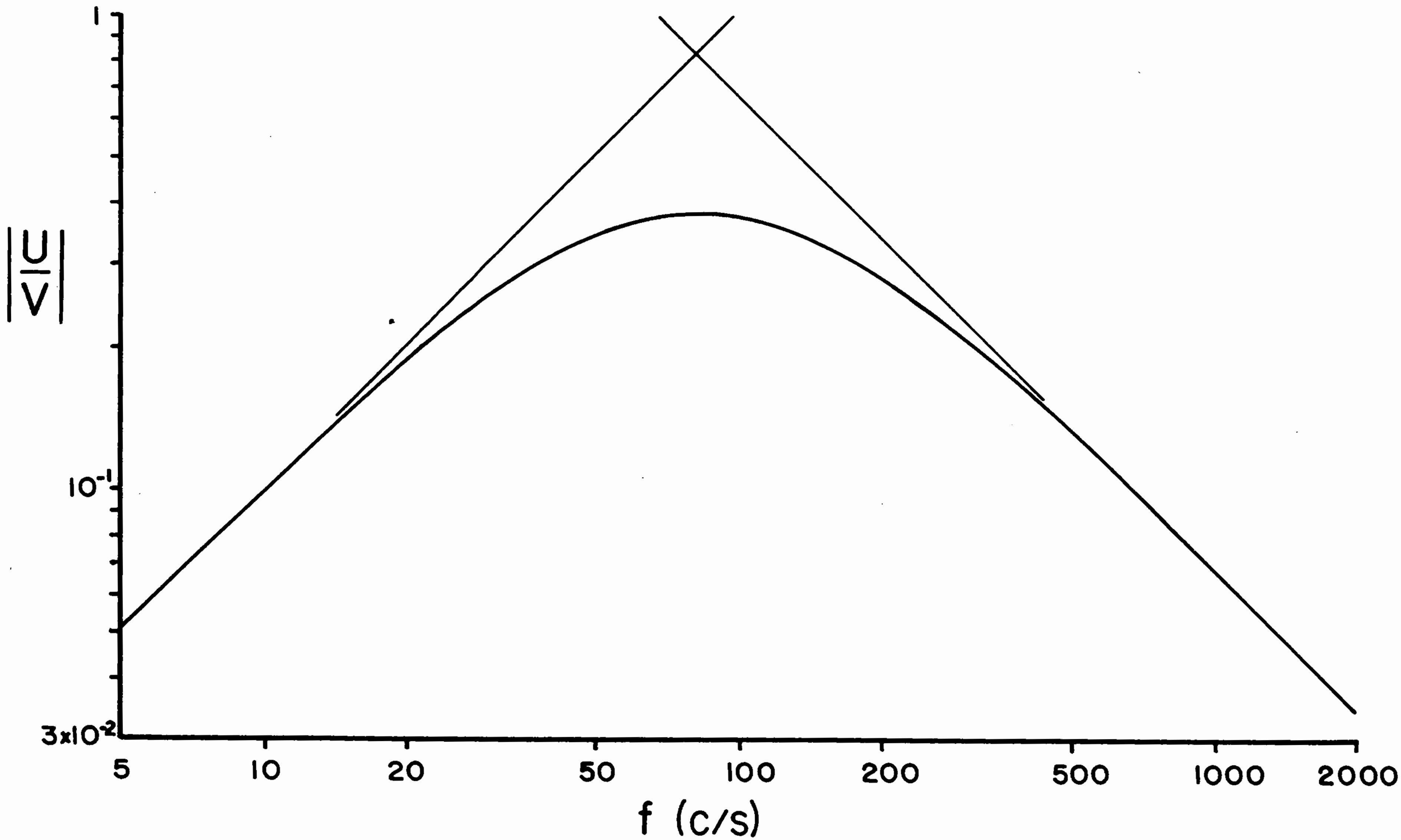

Figure 5.1. Relative velocity of drop and droplet with sinusoidal air velocity of frequency f.

against $f = \omega/2\pi$ for $r = 10$ microns and $s = 15$ microns. At very high frequencies the droplet moves very little and the drop even less; the relative velocity goes as ω^{-1}. At very low frequencies the droplet moves with the air and the drop lags very little, the relative velocity goes as ω. At about 100 c/s (in this case) the relative velocity is a maximum and is about two-fifths of the air velocity. With a greater disparity of drop sizes, a range of frequencies can be found in which the droplet follows the air closely and the drop stays almost still; in this range U almost equals V.

Now consider some fixed frequency, and a fixed drop radius s,(and therefore fixed τ_s) and vary r. The parameter K in Langmuir's Equation is proportional to τ_r so that for very small droplets where τ_r is very small, K is well below the critical value and collisions are impossible, unless the velocities U and V are both extremely large. On the other hand, if the droplet is nearly as big as the drop, τ_r nearly equals τ_s, and equation (5.5) shows that the relative velocity U becomes very small unless the air velocity is again very large. At intermediate values of τ_r , K can exceed the critical value for moderate values of V , and collisions take place. This is just the kind of behaviour found earlier for gravitational excitation.

At low frequencies, where $\omega\tau_s \ll 1$, the similarity becomes an analogy. Equation (5.5) becomes

$$|U|^2 \doteq |V|^2 \omega^2 (\tau_s - \tau_r)^2 . \qquad (5.6)$$

Putting $|V|\omega = A$, the peak acceleration of the air, then the peak relative velocity

$$|U| \doteq A(\tau_s - \tau_r), \qquad (5.7)$$

closely parallel to equation (4.3), but with the peak acceleration A replacing the gravitational field g.

5.2. Turbulence.

In the clear atmosphere turbulence is caused by the wind shear which results from the action of the ground on the wind. In a vertically developed cloud there is a shear between the updrafts and the surrounding air. Entrainment takes place and the motion becomes strongly turbulent.

Although turbulence has been studied theoretically and experimentally for several decades, present-day knowledge is very incomplete. Much work has gone into developing an adequate mathematical description but this is still in an active stage of developement. Experiments "in the field" have mostly been aimed towards learning about diffusion of water vapour or smoke in the clear atmosphere, and none seem to have been carried out inside clouds. Many flights have been made, it is true, in which the accelerations (or "bumps") of the aircraft are measured, but the aircraft travels so far in a short time that this merely reveals the distribution of updrafts along a horizontal line, rather than the variation of the velocity of one region with time.

The most useful results so far are those obtained in wind tunnels. In describing turbulent flow, any steady motion of the air must be subtracted out first so that the mean air velocity over a period of time tends to zero as the period tends to infinity. The work of Simmons and Salter (1934), Townend (1934), and Taylor (1935) and Townsends many published papers show that turbulence in air well clear of any boundaries, such as the walls of the wind tunnel (or in the atmosphere, the ground), can then be expected to have the following properties.

The motion of the air is <u>random</u> and <u>Gaussian</u>, which means that it is not possible to predict the actual velocity at any distant place or time, but only the probability distribution of velocity, and that the distribution is a Gaussian or normal error curve. The motion is also <u>stationary</u>, <u>homogeneous</u> and <u>isotropic</u>, which means that the statistical properties of the motion are independent of time, place, and the direction of the axes of reference.

The velocity of a parcel of air which contains a drop and some droplets can be resolved along one axis, and the occurrence of collisions may be calculated in one dimension. Since the turbulence is isotropic, the situation must be the same in each of the three dimensions. Just how the one-dimensional results can be used to give the rate of occurrence of collisions in a three-dimensional system will be discussed in section 6.

Of course, if the motion were really entirely in one direction, a drop would make all possible collisions in a very few excursions through the droplets, and would then move in a clear channel, making no more collisions. Any transverse motion, however, would ensure that the drop was always encountering a fresh sample of droplets.

5.3. Random Excitation.

The problem of coalescence by one-dimensional excitation in general can be approached in the following way. The air velocity V is a function of time and can be written $V(t)$; similarly, the relative velocity is written $U(t)$. They are linked by two linear differential equations,

$$v_r + \tau_r \frac{dv_r}{dt} = V(t), \tag{5.8}$$

$$v_s + \tau_s \frac{dv_s}{dt} = V(t), \tag{5.9}$$

and the linear equation

$$U(t) = v_r - v_s, \tag{5.10}$$

so that $V(t)$ can be regarded as the cause and $U(t)$ the effect in a linear process.

The straightforward application of Fourier transforms to problems of this kind is well known (for example, see Campbell and Foster (1948) page 24). First, the complex exponential Fourier transform of $V(t)$ is obtained; this is $V(\omega)$, the spectrum of the air velocity. Next, $V(\omega)$ must be multiplied by the frequency response of the process, which is given by equation (5.4). The product is $U(\omega)$, the spectrum of the relative velocity; application of the inverse Fourier transform gives $U(t)$, the relative velocity itself.

If $V(t)$ were actually a definite function (such as a square wave), $U(t)$ would be one also, but in our problem is a random function; and the above method cannot be used as it stands. Since the air velocity $V(t)$ is random and Gaussian, the relative velocity $U(t)$ must also be random and Gaussian. For calculating the rate of collisions, it is necessary to know the probability distribution of the relative velocity, and nothing more. Since we know that the probability distribution is Gaussian, it is completely specified by its mean square value. The problem reduces to one of finding the mean square

relative velocity σ_v^2 in terms of the air velocity $V(t)$. *
The method used is believed to be due originally to N.Wiener; it will be outlined here, without showing how the steps may be justified.

The air velocity $V(t)$ is a random function with stationary properties, so that it can be described by a mean square σ_V^2 and by a spectrum $|V(\omega)|^2$. (The spectrum can be obtained by taking the cosine Fourier transform of the correlation function of $V(t)$). Parseval's theorem states that

$$\int_{-\infty}^{\infty} |V(t)|^2 dt = \int_{-\infty}^{\infty} |V(\omega)|^2 d\omega , \qquad (5.11)$$

and by applying it to the definition of σ_V^2 we find

$$\sigma_V^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |V(t)|^2 dt = \lim_{T\to\infty} \int_{-\infty}^{\infty} |V(\omega)|^2 d\omega. \qquad (5.12)$$

In this way $|V(\omega)|^2$ can be normalised correctly. From here the procedure is similar to that described for definite functions, but the spectra and the frequency response are all in modulus squared form. The air velocity spectrum $|V(\omega)|^2$ must be multiplied by the square of the frequency response, given by (5.5). The product is $|U(\omega)|^2$, the spectrum of the relative velocity. (The cosine Fourier transform of this would give the correlation function of $U(t)$, but this is not needed for

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* The author was fortunate in having the collaboration of Z.A. Melzak for this part of the work, up to Table 5.1.

the present purpose). Applying Parseval's theorem again, this time to the definition of σ_U^2, the mean square of the relative velocity, we find

$$\sigma_U^2 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |U(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-\infty}^{\infty} |U(\omega)|^2 d\omega$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-\infty}^{\infty} |V(\omega)|^2 \left|\frac{U}{V}\right|^2 d\omega \,. \tag{5.13}$$

Thus σ_U^2 is a kind of mean square of the air velocity spectrum weighted by the response spectrum, where σ_V^2 was the unweighted mean square.

It can be seen that σ_U^2 is proportional to σ_V^2, and that the constant of proportionality depends on τ_s and τ_r and on the spectrum of the air velocity V. It can also be seen that a spectrum that is concentrated around 100 c/s, when weighted by the frequency response shown in figure 5.1, would give a larger value of σ_U^2 than one in which the spectral density was low around 100 c/s. Thus, although one does not wish to know anything about the spectrum of U, something must be known about the spectrum of V for the method to give any value at all for σ_V^2.

The foregoing procedure was carried out for a number of trial spectra of the air velocity V, with the results shown in table 5.1.

In this Table, the spectrum of V is given in the second

TABLE 5.1

Air Velocity Spectrum		Mean Square Relative Velocity σ_U^2		Mean Square Acceleration of Air σ_A^2
Name	$G(\omega)$ *	Closed Form *	Infinite Series	
Markoff (Exponential correlation function)	$\sigma_V^2 \frac{2\omega_0}{\pi} \frac{1}{\omega^2+\omega_0^2}$	$\sigma_V^2 \omega_0 \frac{(\tau_s-\tau_r)^2}{\tau_s+\tau_r} \frac{1}{(1+\omega_0\tau_s)(1+\omega_0\tau_r)}$	$= \sigma_V^2 \omega_0 \frac{(\tau_s-\tau_r)^2}{\tau_s+\tau_r}\left[1-\omega_0(\tau_s+\tau_r)+\ldots\right]$	Does not converge
Gaussian (Normal Error curve)	$\sigma_V^2 \sqrt{\frac{2}{\pi}}\, \omega_0 \exp\left(-\frac{\omega^2}{2\omega_0^2}\right)$	$\sigma_V^2 \frac{1}{\omega_0\sqrt{2\pi}} \frac{\tau_s-\tau_r}{\tau_s+\tau_r}\left[\frac{1}{\tau_s}\exp\left(\frac{1}{2\omega_0^2\tau_s^2}\right)\left\{1-\mathrm{erfc}\left(\frac{1}{\sqrt{2}\,\omega_0\tau_s}\right)\right\} - \frac{1}{\tau_r}\exp\left(\frac{1}{2\omega_0^2\tau_r^2}\right)\left\{1-\mathrm{erfc}\left(\frac{1}{\sqrt{2}\,\omega_0\tau_r}\right)\right\}\right]$	$=2\sigma_V^2\omega_0^2(\tau_s-\tau_r)^2\left[1-3\omega_0^2(\tau_s^2+\tau_r^2)+\ldots\ldots\right]$	$2\sigma_V^2\omega_0^2$
Uniform Band ω_L to ω_H	$0 \quad \omega<\omega_L$; $\frac{\sigma_V^2}{\omega_H-\omega_L} \quad \omega_L\leqslant\omega\leqslant\omega_H$; $0 \quad \omega_H<\omega$	$\sigma_V^2 \frac{1}{\omega_H-\omega_L}\frac{\tau_s-\tau_r}{\tau_s+\tau_r}\left[\frac{1}{\tau_s}\arctan\left(\tau_s\frac{\omega_H-\omega_L}{1+\omega_H\omega_L\tau_s^2}\right)-\frac{1}{\tau_r}\arctan\left(\tau_r\frac{\omega_H-\omega_L}{1+\omega_H\omega_L\tau_r^2}\right)\right]$	$=\sigma_V^2\left\{\omega_H\omega_L-\frac{1}{3}(\omega_H-\omega_L)^2\right\}(\tau_s-\tau_r)^2\left[1-\omega_H\omega_L(\tau_s^2+\tau_r^2)+\ldots\right]$	$\sigma_V^2\left\{\omega_H\omega_L-\frac{1}{3}(\omega_H-\omega_L)^2\right\}$
Delta Function (Cisoidal oscillation)	$\sigma_V^2\,\delta(\omega-\omega_0)$	$\sigma_V^2\omega_0^2\frac{(\tau_s-\tau_r)^2}{(1+\omega_0^2\tau_s^2)(1+\omega_0^2\tau_r^2)}$	$=\sigma_V^2\omega_0^2(\tau_s-\tau_r)^2\left[1-\omega_0^2(\tau_s^2+\tau_r^2)+\ldots\right]$	$\sigma_V^2\omega_0^2$

* These two columns of the table and the theorem are the work of Z. A. Melzak.

column in the form of a spectral density function

$$G(\omega) = \lim_{T \to \infty} \frac{1}{T} \langle |V(\omega)|^2 \rangle_{av.} , \qquad (5.14)$$

Where $\langle \ \rangle_{av.}$ signifies the average over a vanishingly small portion of the spectrum around ω. This definition eliminates the operator $\lim_{T \to \infty} \frac{1}{2T}$ from the formulae; the spectrum is now normalised so that

$$\sigma_V^2 = \int_0^\infty G(\omega)\, d\omega , \qquad (5.15)$$

and the mean square relative velocity

$$\sigma_U^2 = \int_0^\infty G(\omega) \left|\frac{U}{V}\right|^2 d\omega . \qquad (5.16)$$

The third column gives the mean square relative velocity calculated by equations (5.16) and 5.5). It shows the dependence on the drop and droplet time constants, on the shape of the spectrum, and on some fixed frequency ω_0 which is characteristic of the spectrum (two fixed frequencies in one case).

The expressions for σ_U^2 were expanded in ascending powers of ω_0 and the first two terms are listed alongside the exact formulae. If the characteristic frequency ω_0 is small enough, all terms except the first can be neglected. More specifically, the condition is

$$\omega_0 \tau_s \ll 1$$
$$(\text{or } \omega_H \tau_s \ll 1) . \qquad (5.17)$$

In all cases except the first, the expressions are very similar, involving $(\tau_s - \tau_r)^2$ and a quantity having the dimensions of mean square acceleration. For comparison, the mean square acceleration of the air motion

$$\sigma_A^2 = \int_0^\infty G(\omega)\,\omega^2\,d\omega \tag{5.18}$$

has been calculated for each spectrum and is listed in the last column.

It is evident that when the condition (5.17) is fulfilled, in any case except the first, the mean square relative velocity and the mean square air acceleration are related by the equation

$$\sigma_U^2 = \sigma_A^2\,(\tau_s - \tau_r)^2\ . \tag{5.19}$$

The analogy with equation (4.3)

$$U = g\,(\tau_s - \tau_r)$$

for a gravitational field is complete.

The result (5.19) applies to any spectrum concentrated at low frequencies, regardless of its form, as the following reasoning shows. A theorem will be used:

> If an air velocity spectrum $G_1(\omega)$ gives rise to a mean square relative velocity σ_{U1}^2, a spectrum $G_2(\omega)$ to σ_{U2}^2 etc., then a spectrum
>
> $$G(\omega) = G_1(\omega) + G_2(\omega) + \ldots\ldots$$
>
> gives rise to a mean square relative velocity
>
> $$\sigma_U^2 = \sigma_{U1}^2 + \sigma_{U2}^2 + \ldots\ldots$$

The air velocity spectrum is divided up into a large number of narrow elements, so that each can be replaced by a uniform band like the one listed in Table 5.1. Let ω_{nH} and ω_{nL} be the upper and lower boundaries of the nth. portion where

$$\omega_{nH} - \omega_{nL} \ll \omega_{nH}$$

and define

$$\omega_n^2 = \omega_{nH}\,\omega_{nL} .$$

Then for all portions of the spectrum for which

$$\sigma_{vn}^2 = \sigma_{An}\,(\tau_s - \tau_r)^2$$

$$= \omega_n^2\, G(\omega_n)(\tau_s - \tau_r)^2 .$$

Then

$$\sigma_v^2 = \sum_{n=1}^{\infty} \sigma_{vn}^2 = (\tau_s - \tau_r)^2 \sum_{n=1}^{\infty} \omega_n^2\, G(\omega_n) .$$

In the limit as the elements become infinitesimally narrow,

$$\sigma_v^2 = (\tau_s - \tau_r)^2 \int_0^{\infty} \omega^2 G(\omega)\, d\omega$$

$$= (\tau_s - \tau_r)^2 \sigma_A^2 .$$

This result does not apply if σ_A^2 receives an appreciable contribution from those parts of the spectrum for which is near to or greater than unity. The Markoff spectrum is of this type, and its formula for σ_v^2 differs considerably from the others; actually, the integral for σ_A^2 does not converge.

Until more is known about the spectrum of turbulence inside clouds, it is not possible to say whether condition (5.17) will be met. If measurements show that it is met, then the simple relationship (5.19) will hold, and only the quantity σ_A^2 need be used. If the condition is not met, then equation (5.16) will have to be applied.

5.4. Collisions by Random Excitation.

There is one difference between a gravitational field and a random one-dimensional acceleration that has to be taken into account; it is that the relative velocity U is random, not steady as it was in the computations of section 4.

The probability of the relative velocity lying between U and $U+dU$ is

$$P(U)\,dU = \frac{1}{\sigma_U\sqrt{2\pi}} \exp\left(-\frac{U^2}{2\sigma_U^2}\right) dU \,. \tag{5.20}$$

The probability density $P(U)$ is normalised so that

$$\int_{-\infty}^{\infty} P(U)\,dU = 1.$$

The mean speed $\overline{|U|}$ is given by the weighted integral

$$\overline{|U|} = \int_{-\infty}^{\infty} |U|\, P(U)\,dU = 2\int_0^{\infty} U\,P(U)\,dU \,, \tag{5.21}$$

since $P(U) = P(-U)$. The volume swept out by a drop radius in unit time is $Q = \pi s^2 \overline{|U|}$. If the collection efficiency were unity, the quantity Q would give the number of droplets per second very simply. Since the collection efficiency is not unity, Q must be replaced by an equivalent swept volume per unit time Q by replacing πs^2 by $E\pi(s+r)^2$, as discussed in section 3.2. Since E depends on U, it must be taken inside the integral. Then the equivalent volume swept per unit time

$$Q' = \pi(s+r)^2 . \, 2\int_0^{\infty} E\,U\,P(U)\,dU \,. \tag{5.22}$$

In the above formula as it stands, ~~and~~ E depends not only on U, but also on s and r, so that the integral would have to be evaluated afresh for every combination of drop and droplet. This can be avoided by putting

$$U = U_0 K \quad \text{and} \quad \sigma_U = U_0 \sigma_K$$

where

$$U_0 = \frac{9\eta s}{2r^2\rho_s} \, ,$$

K is the dimensionless parameter in Langmuir's equation (3.10) and σ_K is the root mean square of K. Then the probability distribution of K is given by

$$P(K)\,dK = \frac{1}{\sigma_K\sqrt{2\pi}} \exp\left(-\frac{K^2}{2\sigma_K^2}\right) = U_0 P(U) \, , \qquad (5.23)$$

and the equivalent swept volume per unit time

$$Q' = \pi(s+r)^2\, 2U_0 \int_0^\infty E(K)\, P(K)\, K\, dK$$

$$= \sqrt{2\pi}\,(s+r)^2 U_0 \int_0^\infty E(K) \frac{K}{\sigma_K} \exp\left(-\frac{K^2}{2\sigma_K^2}\right) dK$$

$$= \sqrt{2\pi}\,(s+r)^2 U_0 \alpha \, , \qquad (5.24)$$

where α stands for the definite integral. The value of α depends on σ_K but not otherwise on s and r; it has been computed for various values of σ_K, and is plotted in figure 5.2.

The main trend of the curve of α versus σ_K is to increase the rate of sweeping up droplets with increase of σ_K. This is similar to the trend shown in figure 3.1, where collection efficiency increases with K. However, there is no critical value of σ_K below which α is zero: α is still finite, however small σ_K becomes. The physical interpretation of this difference

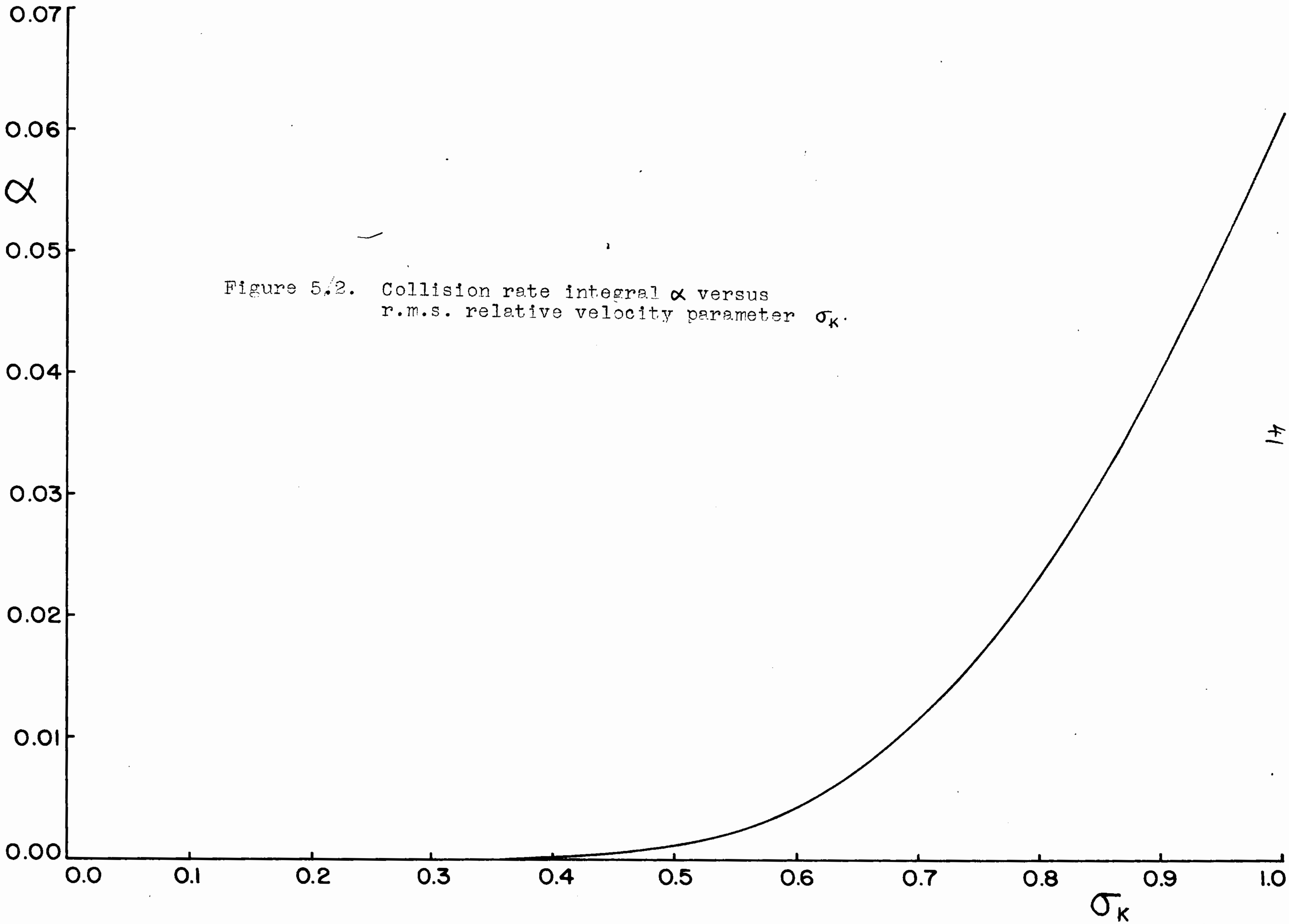

Figure 5.2. Collision rate integral α versus r.m.s. relative velocity parameter σ_K.

is, that even if the root mean square relative velocity σ_U is less than the critical value, U itself will exceed the critical value for some part of the time.

For example, if $K = 1.0$ with gravitational excitation, no collisions occur, because $K < 1.214$; while if $\sigma_K = 1.0$ with random excitation, collisions occur at a rate given by $\alpha = 0.061$, which corresponds to a mean collection efficiency of 4.9%, still quite a large value. On the other hand, the reduction of collision rate with decreasing σ_K is quite rapid; at $\sigma_K = 0.6$, for example, the mean collection efficiency is only 0.7%, so that collisions, although still not impossible, are much rarer.

The position regarding one-dimensional excitation of a cloud can be summarised as follows: qualitatively, the effect is very much the same as gravity, except that the onset of coalescence is more gradual. Quantitatively, given any value for the mean square acceleration of the turbulent air, and provided a certain condition is fulfilled, the rate of growth of a drop in a given cloud can be computed. No measured value for mean square acceleration is available so far. If the condition is not fulfilled, the complete spectrum of the turbulence will be required.

6. THREE-DIMENSIONAL EXCITATION.

6.1. Extension of One-dimensional results.

The derivation of a three-dimensional treatment of turbulence from the one-dimensional results of the previous section can proceed on the following lines. The relationship between the component of relative velocity U_x and the component of air acceleration A_x in one direction is linear and independent of the velocity and acceleration in the other two directions. If we consider an isotropic air motion, for which condition (5.17) is valid, the mean squares of the three components of relative velocity are related to the mean squares of the three components of air acceleration as follows:

$$\sigma_{U_x}^2 = \sigma_{U_y}^2 = \sigma_{U_z}^2 = \sigma_{A_x}^2(\tau_s - \tau_r)^2 = \sigma_{A_y}^2(\tau_s - \tau_r)^2 = \sigma_{A_z}^2(\tau_s - \tau_r)^2. \qquad (6.1)$$

If U is the relative speed, its mean square

$$\sigma_U^2 = \sigma_{U_x}^2 + \sigma_{U_y}^2 + \sigma_{U_z}^2 = 3\sigma_{U_x}^2 . \qquad (6.2)$$

Temporarily, the assumption is made that the components of air motion in the three directions are mutually independent, (as they are for molecular motion in the Kinetic Theory of Gases); the components of relative velocity are then mutually independent also. In such a case, the well known Maxwellian law can be applied to the relative velocity, and the probability distribution of relative speed

$$P(U)\,dU = \sqrt{\frac{2}{\pi}}\,\frac{U^2}{\sigma_{U_x}^2}\exp\left(-\frac{U^2}{2\sigma_{U_x}^2}\right)dU. \qquad (6.3)$$

(See for example Jeans (1948) section 91.) The expression (6.3) is normalised.

The calculations can now proceed as in section 5.4, but with equation (5.20) replaced by the new probability distribution (6.3). The new distribution is narrower than the old, so the result of the procedure would be a curve of rate of collisions resembling figure 5.1 in essence, but differing in detail. In particular, at small values of σ_K, the value of α was achieved by the fact that the relative velocity U considerably exceeded its RMS value for part of the time. In the three dimensions, the narrower distribution of speed U makes such exceptional values of U much less probable, so that the curve would be lower and steeper than the one in figure 5.1. In fact, since the distribution of U is intermediate in width between the broad distribution of U in one-dimensional random motion and the infinitely narrower distribution of U for steady motion, the curve of rate of collision would be intermediate in shape between figute 5.1 and the curve for a gravitational field, which is zero below $K = 1.214$.

For these computations, it would be necessary to know the mean square acceleration of the three-dimensional turbulent motion, so that σ_U^2 can be calculated for each pair of drop and droplet sizes. As before, this quantity has not been measured.

Besides this practical difficulty, however, there is a fundamental objection to the method described. The assumption was made that the components of air motion were mutually independent; this is not true except on the molecular scale (Batchelor (1952)). How this would affect the results is not easy to say. Instead of attempting to modify the procedure to allow for the interaction of velocities in three dimensions, it seems it would be more profitable to start with the more fundamental approach to the nature of turbulent motion that is now becoming available.

6.2. The Modern Theory of Turbulence.

Batchelor (1952) gives a good account of the modern theory of turbulent motion. The velocity field at an instant of time is subjected to a three-dimensional Fourier transform. The result is a spectrum in three-dimensional wave-number space. The distance of a point in wave-number space from the origin is the reciproxal of the size of the eddies which it represents, so that the nearest points belong to the largest eddies.

Heisenberg (1948) has developed the theory by using dimensional arguments. He arrives at a picture in which turbulence is an intermediate stage through which energy passes on its way from a source to eventual dissipation by viscous friction. Eddies of a particular size are always being converted by inertial forces into smaller eddies. The energy from the source goes at first into large eddies (of size comparable to the boundaries of the system), which then hand it down through smaller and smaller eddies. As a certain size is reached, the energy goes into viscous friction instead of into still smaller eddies, and the spectrum does not extend appreciably beyond this size.

This wave number theory is not very closely related to the frequency-spectrum treatment considered previously, chiefly because the wave-number spectrum describes the state of a large region of space at an instant of time, while the frequency spectrum describes the movements in time and space of a single element of the fluid. However, in a semi-quantitative way one can associate the small eddies with high frequencies, and it is possible, at least in principle, to transform equations from one system to another.

The virtue of the new theory is that its results are expressed in terms of very few initial facts. Instead of a mean square velocity or acceleration and a frequency spectrum, neither of which are known, the main quantity which determines the nature of the motion is the rate of supply of energy per unit ~~volume~~ mass, ε. The energy is supplied by thermal processes in the atmosphere, and can be deduced from such facts as the temperature and humidity, which have been measured in typical cases. Braham (1952) has drawn up an energy budget for a thunderstorm, and a similar budget for a fair-weather Cumulus cloud would give the required value of ε.

The vector equation for the acceleration of a parcel of air in non-irrotational flow is

$$\tilde{A} = \frac{\partial}{\partial t}\tilde{V} + (\tilde{V}.\nabla)\tilde{V} \quad . \qquad (6.4)$$

If the new theory of turbulence is to be applied to the problem of collision between drops, it must provide the probability distribution of acceleration, $P(|A|)\,d|A|$. The relationship between $|U|$ and $|A|$ must then be re-established and $P(U)\,dU$ obtained. From here on, the work will follow section 5.4.

Of course, an indirect determination of turbulent motion of this kind cannot replace reliable and accurate direct measurements, but until such measurements can be made, it perhaps offers a means of reaching a conclusion.

7. SUMMARY AND CONCLUSIONS.

The observed generation of moderate or heavy rain in clouds warmer than 0 C makes it necessary to look for an alternative to the Bergeron-Findeison ice-phase mechanism of rain formation. Calculations, based on Langmuir's work on collision between drops, show that gravity brings about too slow a settling for coalescence to take place; this result accounts for the observed stability of fair-weather cumulus clouds.

Increasing the gravitational field by a factor 2 would greatly increase the rate of growth of drops by coalescence, producing raindrops in about 30 minutes. A steady acceleration of the air of the cloud produces the same effect as an extra gravitational field. The updraft in a Cumulus cloud is turbulent, and produces a fluctuating acceleration. A one-dimensional treatment of turbulent motion shows this acceleration to be at least as effective as a steady acceleration equal to its r.m.s. value in bringing about coalescence.

A difficulty of a theoretical nature, and the very practical one that no measured values of turbulent acceleration are available, prevent this treatment from arriving at a definite statement whether turbulence plays an important part in initiation of rain. It may be that a more fundamental approach through energy considerations will give the answer.

If turbulence is the primary cause of rain which forms in warm clouds, it must also be an important factor in similar clouds which are cold enough for the ice phase to exist. Where the Bergeron-Findeison process is predominant, then turbulence will help a falling snowflake to collect supercooled droplets. Where coalescence by turbulence becomes established in the lower part of a cloud, it may inhibit an ice-phase process by cutting off the supply of droplets.

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