MICROWAVE NOISE IN ACCELERATED ELECTRON STREAMS

by

Ross A. McFarlane

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ABSTRACT

Measurements have been made of the noise current fluctuations on the electron beam from a space-charge-limited, diode electron gun, in four sealed-off vacuum tubes. To perform the measurements a synchronous detection system was developed and evaluated. The variation of noise smoothing with changes in the gun anode voltage has been examined at 1,400 Mc/s., 4,250 Mc/s. and 9,520 Mc/s. Values of smoothing calculated from theories which do not take into account the multi-velocity nature of the electron beam in the region immediately beyond the potential minimum are in poor agreement with the experimental measurements.

By using the results of density function calculations to determine input conditions at a plane in the gun beyond which a single valued velocity theory is valid, a method has been developed for calculating the four electron beam noise parameters at the gun anode. A critical examination of the results shows the method exhibits the two invariance properties known to be required for any such theory.

The theory predicts a dependence of anode current fluctuation noise on the square root of the frequency. The present measurements are found to vary with frequency in this manner as did those of earlier workers. The explicit frequency dependence of the anode current fluctuation noise has not previously been reported.

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1. INTRODUCTION

The extremely rapid development of microwave technology and its application for defence purposes which started about 1940, has been followed after the war by its application to new and more complicated radar networks and to trans-continental communication systems capable of handling simultaneously large numbers of independent messages or very detailed visual information. For such purposes a wide range of frequencies must be transmitted and handled with uniform effectiveness. This large bandwidth requirement has made it necessary that such systems employ microwaves for the transmission of the appropriate intelligence. At short wavelengths greater absolute bandwidth is more readily obtained with the attendant greater capacity for transmitting information.

One fundamental phenomenon occurring in all electronic devices or systems of devices is called noise. The noise appears in the form of a fluctuating voltage or current in addition to that of the desired signal. In passive devices, those which are not otherwise sources of additional energy, the noise is the result of random thermal motion of electrons within the conductors, and has been called Johnson Noise. When vacuum tubes are employed, other sources of noise are present which are a consequence of the statistical nature of electron emission from thermionic cathodes. Several distinct noise processes can be identified occurring at or within the cathode itself, in the stream of electrons after emission, or at electrode structures in the path of the electrons. The relative significance of each of the three processes depends upon the frequency of interest. At microwave frequencies the latter two are more important in determining and modifying the noise properties of the electron stream.

When attempts were made to use vacuum tubes of conventional triode or pentode design for microwave applications, it was found that their performance deteriorated as the frequency of operation was increased. Such tubes cannot be used when the transit time of electrons between the various tube elements is comparable to the signal period. For the amplification or production of microwave signals a very general class of devices has been developed, characterized by the use of a longitudinal electron beam. Amplification is achieved by the cumulative interaction of a stream of electrons with an electromagnetic wave propagated along an adjacent guiding structure. The bandwidth of such tubes can be extremely large and for purely practical reasons therefore, they have received extensive study. As with all such devices the lowest signal level at which these tubes are useful is determined by the amount of noise produced within the tube itself that appears at the output terminal. The magnitude of this noise depends upon the noise properties of the electron beam interacting with the slow-wave guiding structure.

To produce the beam of electrons, an electron gun is used, which in most cases operates as a space-charge-limited diode. (Pierce 1949). The gun consists of a thermionic cathode operating in the temperature range of 1000-2500 degrees Kelvin, and a pair of electrodes across which the accelerating voltage is applied. Noise on the emergent beam appears as random fluctuations of the beam current and of the electron velocities at the anode plane, each about some average value. The magnitudes of the fluctuations at the anode are in general not equal to the magnitudes of the corresponding fluctuations on the beam at a plane immediately following emission from the cathode. The initial fluctuations are modified by the presence of electrons in the region between the cathode and anode. The nature of this transformation is such that fluctuations in input velocity and current each produce fluctuations in velocity and current as the electron beam leaves the gun.

From statistical considerations, the mean square fluctuation of current has been calculated for an electron beam emitted randomly from a thermionic cathode. This fluctuation in current is called shot noise. At the anode of a space-charge-limited, diode electron gun, the current fluctuations are no longer completely random and the mean square current fluctuation is less than full shot noise.

The present work is concerned with the experimental and theoretical aspects of the propagation of noise on accelerated streams of electrons. The transformation of noise quantities along a drifting electron beam has received adequate theoretical treatment, and there is good agreement with the experimentally determined noise characteristics. Such is not the case for an electron beam accelerated from a thermionic cathode to the anode of a space-charge-limited electron gun. The several theories reported in the literature for the calculation of anode noise do not simultaneously account for the effect of finite beam diameter and the multivelocity nature of the electron beam in the region of the cathode and the potential minimum. It has not been possible to include these effects in any theory in a simple analytic fashion, and the early results could be considered as only approximate descriptions of the physical processes involved. Recent statistical work (Siegman et al 1957) has given considerable insight into the transportation of noise through the multivelocity region beyond the potential minimum in the gun. To include the effect of finite beam diameter, Kornelsen (1957) and Vessot (1957) have employed an analog computer in the calculation of anode noise.

Very few authors have reported fundamental measurements of anode noise that can be readily compared with theoretical results. The work of Cutler and Quate (1950), and Fried and Smulin (1954) was restricted to a single frequency region. The first measurements over a wide range of frequencies were those of Kornelsen (1957) and Vessot (1957). Although a variation of noise with frequency was observed, no essential dependence of the results on frequency was pointed out. In the very early work, agreement between the theory and measurements of the noise current fluctuations at the anode of a space-charge-limited gun appears to have been the result of the proper selection of frequency or voltage, and no one method of calculation of noise smoothing has been available that is accurate over a wide range of frequency and gun voltage.

The research here reported has had three specific objectives:

1) To develop and evaluate a measuring system for electron beam noise measurements over a wide range of frequencies. Sufficient sensitivity must be available for the detection of current fluctuation noise on electron beams of very low currents.

2) To extend the frequency range of available noise smoothing data to 9,500 Mc/s., and to study the variation of smoothing as a function of frequency and gun voltage. A comparison of the

measurements with predictions from existing theories can indicate the merits and inadequacies of a particular approach.

3) To apply the results of the statistical calculations of Siegman for the more accurate specification of input conditions at a plane beyond the potential minimum, to be used in an analog computer solution of the Kornelsen-Vessot type, and to critically examine the results of this solution.

The extent to which these objectives have been accomplished may be outlined as follows:

1) A receiving system based on the radiometer technique of Dicke (1946) has been developed. The system operates by periodically interrupting the noise signal to be measured and synchronously detecting the receiver output. The method is applicable with equal facility at any frequency and measurements on system performance at 24 Kmc/s. are in excellent agreement with the calculated value of system sensitivity, as determined from Dicke's theory. The limiting sensitivity of the measuring system is approximately 35 db. below the self-noise of the microwave receiver alone, and the noise current fluctuations on an electron beam of 1 μa. are readily detected.

2) Measurements have been made of the noise current fluctuation on the electron beam from a space-charge-limited electron gun. The variation of noise smoothing with changes in anode voltage has been examined at 1,400, 4,250 and 9,520 Mc/s. Smoothing values calculated from theories which do not take into account the multivelocity nature of the electron beam in the region immediately beyond the potential minimum are in poor agreement

with experimental measurements.

3) The statistical results of Siegman et al (1957) have been used to specify the noise characteristics of the electron beam at a plane in front of the potential minimum, beyond which a single valued velocity theory can be applied. The finite-beam, analog-computer solution was then used to study the transformation of the noise along the accelerated electron beam to the gun anode. Expressions for the output noise quantities have been determined, which are linear combinations of input noise quantities. This was made possible by virtue of the linear relation observed between the input and output computer voltages.

A very general formulation of the noise problem by Haus (1955) showed the invariance of two noise parameters under any lossless transformation. The present computer solution exhibits the required invariance properties, which gives considerable confidence as to the validity of the method of calculation. It is believed that the work here reported is the first calculation of anode noise quantities which satisfies the two invariance requirements.

From the calculated value of noise smoothing $\delta_a^{\ 2}$, a distinct dependence on frequency and anode voltage was apparent, which has not before been reported. It was found that the quantity $\delta_a^{\ 2}\ V_{oa}/\omega^{1/2}$ is independent of frequency ω, and essentially independent of anode voltage V_{oa}; decreasing very slightly as V_{oa} is increased. The measurements made reflect the above dependence, as do those of Vessot (1957) and McIntosh (1958). Considering the wide range of frequency over which data has been taken, the variation of smoothing with the square root of frequency is well confirmed.

2. A SYNCHRONOUS DETECTION SYSTEM FOR ELECTRON BEAM NOISE MEASUREMENT

2-I Introduction

The output of a microwave superheterodyne receiver used for noise measurements consists of two parts: that due to the noise under study, and that resulting from sources of noise within the receiver itself. In the microwave region the crystal mixer and the first stage of the intermediate frequency (I.F.) amplifier are the chief contributors to receiver noise. The noisiness of the receiver is characterized by its Noise Figure. With a matched termination on the input, the Noise Figure specifies the ratio of total output noise power to that part of the output power due solely to Johnson noise from the input termination. If S is that C.W. input power capable of producing unity signal to noise ratio at the receiver output, then the Noise Figure is given by

$$F = \frac{S}{k T_r B} \qquad (2\text{-}1)$$

where k is Boltzmann's constant.

T_r is a standard reference temperature 293°K

B is receiver bandwidth

If S represents a white noise source, we can define a source temperature T_s such that

$$S = 2 k T_s B \qquad (2\text{-}2)$$

and the ratio T_s/T_r is simply one-half the receiver Noise Figure. The factor 2 appears because the receiver responds to two noise bands equal in width to the I.F. bandwidth. In microwave beam noise studies, the resonant cavity employed usually restricts the desired signal to one such band and the factor 2 is not present.

The sensitivity of the receiver is limited by output fluctuations when no signal is present. These fluctuations arise from two sources: the statistical fluctuations of a random noise waveform, and spurious gain fluctuations. Output fluctuations due to the first can be reduced by reducing the bandwidth after detection. The fractional output fluctuation due to this source is then the order of

$$\left[\frac{\text{AUDIO BANDWIDTH}}{\text{I.F. BANDWIDTH}}\right]^{1/2}$$

which can be made very small. (Dicke 1946, Selove 1954).

Gain fluctuations are concentrated at low frequencies and their effect can be minimized by modulating the signal at a low audio frequency, and performing the final detection in a synchronous detector operating at this frequency. Dicke employed this technique for the measurement of thermal radiation from space; modulating the signal by periodically terminating the receiver input with a non-reflecting termination. This was accomplished by a rotating attenuator vane moving alternately in and out of the input waveguide. The output of such a system is proportional to the noise temperature difference between the terminating attenuator vane and the applied signal. Sensitivities of the order of 1°K temperature increment can be readily obtained.

The method has been applied to the measurement of noise on electron beams. (Saito 1958) The system to be described, however, avoids the mechanical complexities and problems of impedance matching associated with the signal chopping attenuator, by gating the electron beam with a low frequency square wave. The same square wave is used as the reference signal for the synchronous detector. Measurements over a wide frequency range are possible with equal facility and adequate sensitivity is available for the accurate determination of the shot noise asymptote using very low beam currents.

Dicke's analysis has been applied to a system employing an over-damped output meter circuit, permitting evaluation of the expected sensitivity. Measured values are in excellent agreement with that calculated.

2-II Description and Analysis of the System

A block diagram of the system is shown in Fig. 2-1. The square wave generator drives the electron gun modulator and a resonant cavity couples the current fluctuation noise to the microwave receiver. A bolometer at the receiver output provides a square-law detector, and the audio signal is fed via an amplifier and calibrated attenuator to the synchronous detector. A recording milliammeter is used as an output indicator.

Dicke (1946) has analyzed the radiometer which employs a square-law detector and found the incremental sensitivity to be

$$\frac{\Delta T}{T_r} = \frac{\pi F}{2} \cdot \frac{\left[\int_{-\infty}^{\infty} \left|F(\omega)\right|^4 d\omega \int_{-\infty}^{\infty} \left|S(\omega)\right|^2 d\omega\right]^{1/2}}{S(0) \int_{-\infty}^{\infty} \left|F(\omega)\right|^2 d\omega} \qquad (2\text{-}3)$$

where ΔT is the signal source noise temperature increment sufficient to produce an output voltage increment equal to the R.M.S. output fluctuation.

$S(\omega)$ is the output filter response

$F(\omega)$ is the I.F. amplifier response

F is the receiver Noise Figure

The output filter response must include the effect of the output indicator. Using a simplified circuit to represent the combination of low pass filter and output meter, it was assumed that the output indicator was proportional to the voltage across a condenser in a series R.L.C. circuit; Fig. 2-2.

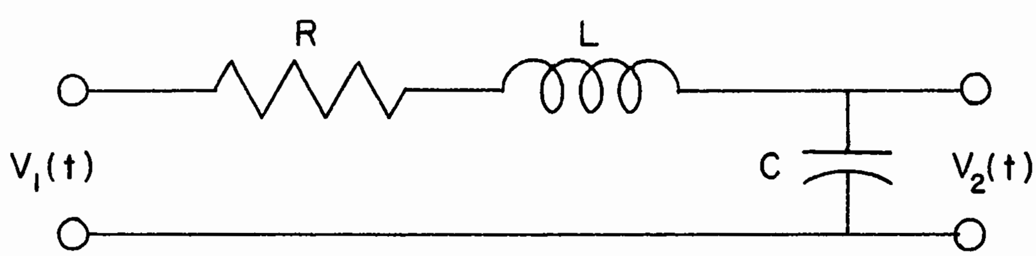


Fig. 2-2

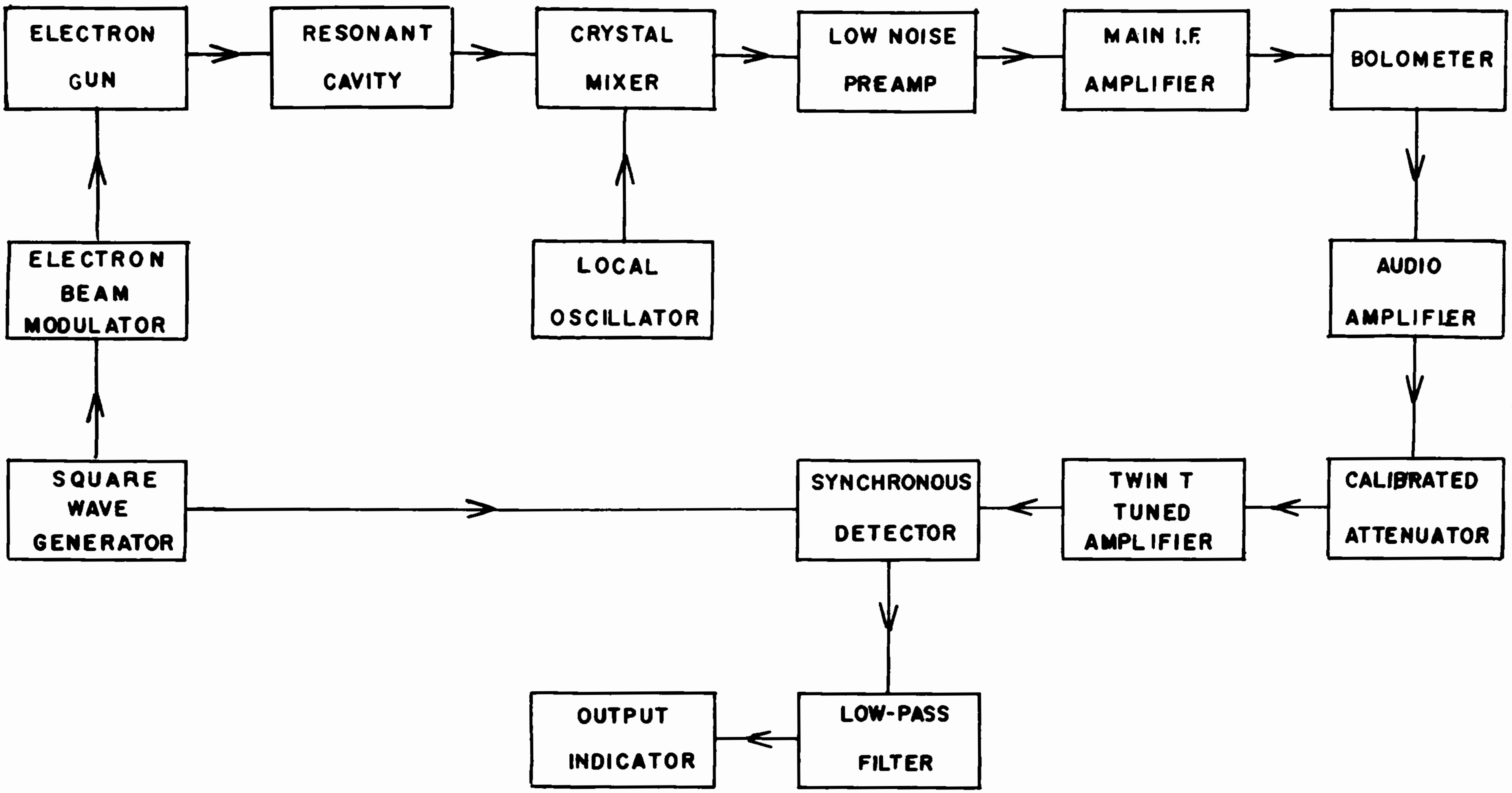


Fig. 2-1 Block Diagram of Noise Measuring System

The present system was over damped and the output response function was written in the form

$$S(\omega) = \frac{-(a^2 - c^2)}{[\omega - j(a + c)][\omega - j(a - c)]} \qquad (2\text{-}4)$$

Noting that S(0) = 1 and using Dicke's values for the integrals over the rectangular I.F. band, the incremental sensitivity is

$$\frac{\Delta T}{T_r} = \frac{\pi^{3/2} F}{4} \cdot \left[\frac{a(1 - c^2/a^2)}{\Delta\omega}\right]^{1/2} \qquad (2\text{-}5)$$

where $\Delta\omega$ represents the I.F. bandwidth.

The case of c = 0 is that for critical damping which is generally used when the value of the noise being observed is changing, and must be recorded as a function of time. The justification for this can be seen by examining a quality factor R, defined as the product of the incremental sensitivity times the time required for the output meter to reach a maximum deflection when an impulse is applied at the filter input.

The output due to an impulse input is the inverse Fourier transform of the steady state response function $S(\omega)$.

$$v_2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{jt\omega}\, d\omega$$

$$= \frac{(a^2 - c^2)}{2c} \left[e^{-(a+c)t} - e^{-(a-c)t}\right] \qquad (2\text{-}6)$$

which reaches a maximum at time

$$t_{max} = \frac{1}{2c} \ln\left[\frac{a + c}{a - c}\right] \qquad (2\text{-}7)$$

and the quality factor becomes

$$R = \frac{1}{2c} \ln\left[\frac{a + c}{a - c}\right] \frac{\pi^{3/2}}{4} \frac{F}{(\Delta\omega)^{1/2}} \left[\frac{a^2 - c^2}{a}\right]^{1/2} \qquad (2\text{-}8)$$

This has its maximum value when $\partial R/\partial c = 0$ or

$$\ln\left[\frac{a + c}{a - c}\right] = \frac{2ac}{a^2 + c^2} \qquad (2\text{-}9)$$

Noting that $c/a, < 1$ the only possible solution for (2-9) is $c = 0$, which corresponds to critical damping.

The present system was overdamped, and the calculation of system performance required the experimental determination of both the filter constants a and c. A quantity was desired to represent the improvement in sensitivity when using the synchronous system. It was defined as the reciprocal of the ratio of the noise power represented by ΔT, to the self noise power of the receiver alone. It becomes:

$$\frac{2}{\pi^{3/2}} \left[\frac{\Delta\omega}{a(1 - c^2/a^2)} \right]^{1/2} \qquad (2\text{-}10)$$

Strictly this is not a measure of the real improvement in sensitivity, as no mention has been made of the incremental sensitivity of the receiver without the synchronous system. It is a factor which can be measured without an absolute determination of noise power.

2-III Description of Apparatus

As the technique is applicable over a wide range of frequencies, no details of a particular microwave receiver will be given.

The square wave generator is shown in Fig. 2-3. It consists of two one-shot multivibrators arranged so that the differentiated and clipped output of one triggers the other. The arrangement permits easy modification where a highly asymmetric waveform is required. With the circuit constants shown, the output is a symmetrical square wave, with provision for frequency and symmetry adjustment. A manual trigger is used to initiate the proper functioning of the unit as the circuit itself is stable. After being clipped by a grid circuit limiter, the square wave is d.c. coupled to a cathode follower which provides the 80 V p.-p. reference signal for the synchronous detector, and a 110 V p.-p. signal to drive the electron beam modulator shown in Fig. 2-4.

Modulation of the electron gun voltage is accomplished by means of a series gate tube. As the grid of the 6C4 gate driver tube is driven positive, the 829B gate tube is cut off by the voltage developed across the 47K, 6C4 plate load resistor. When the grid of the 6C4 swings negative, the current through the tube is cut off and the grid and cathode of the 829B are at the same potential. The series resistance of the gate tube is then very low and the electron gun is effectively connected to the negative supply voltage. The voltage regulator tubes and 100 K series resistors maintain 300 volts for the 6C4 plate supply and permit adjustment of the 6C4 bias to the proper value. A metering circuit is provided for the measurement of peak values of output voltage and current. A monitor point permits examina-

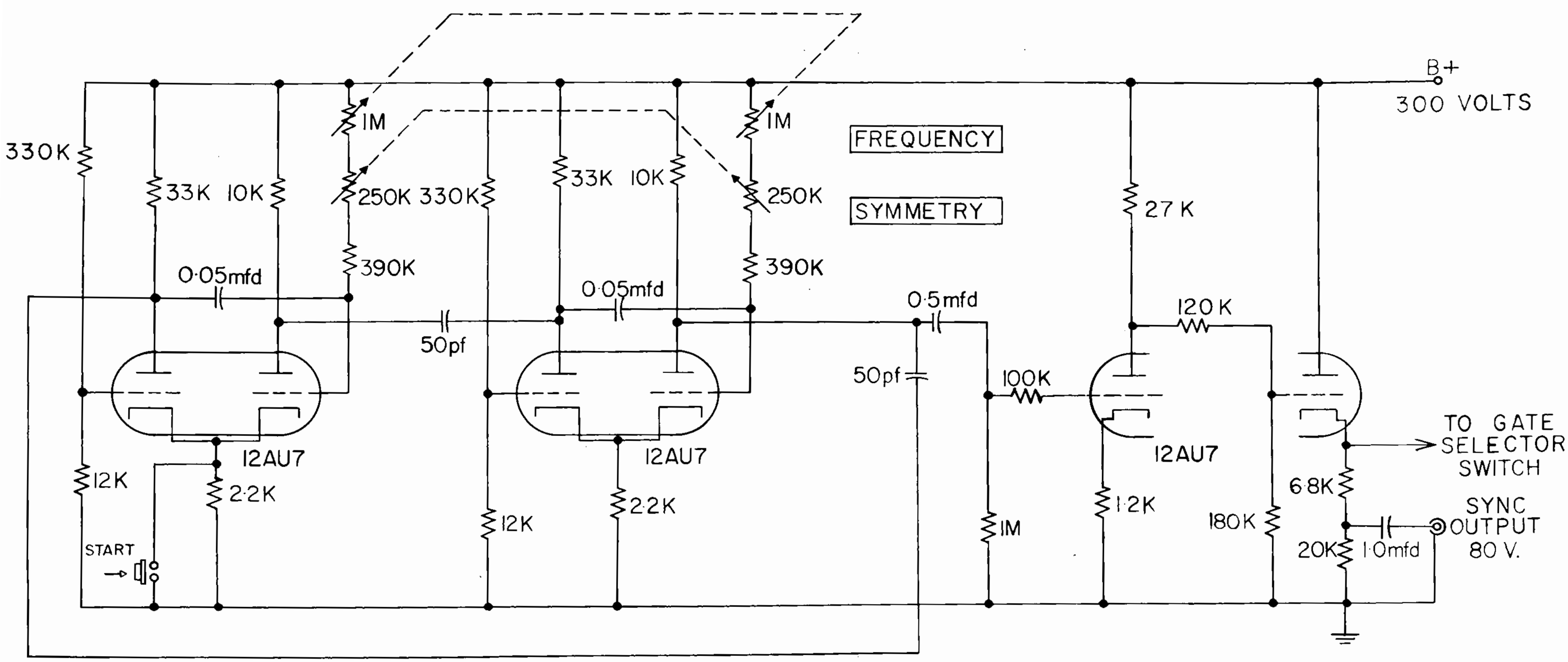


Fig. 2-3 Schematic Diagram of the Square – Wave Generator

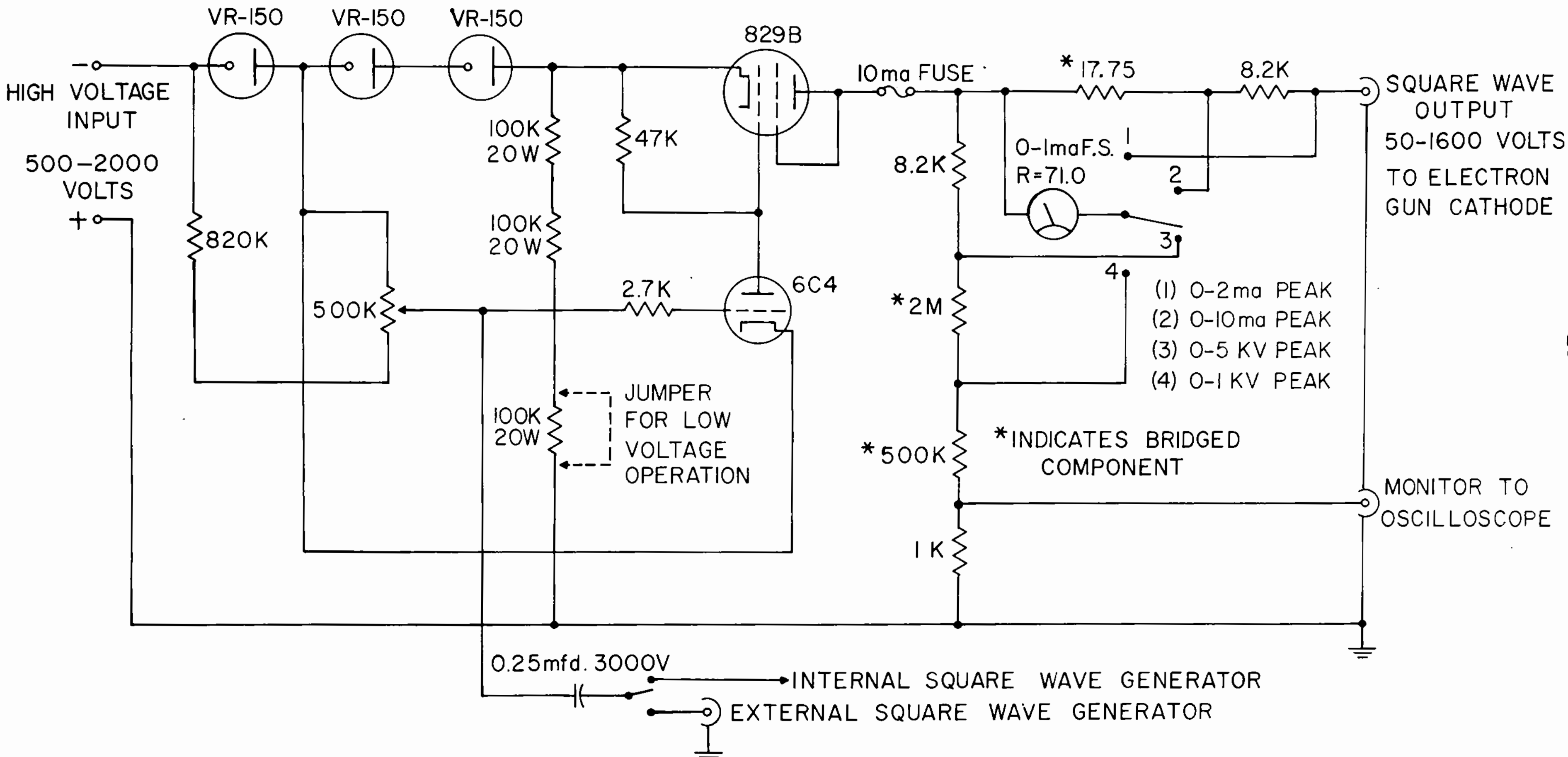


Fig. 2-4 Schematic Diagram of Electron Beam Modulator

tion of the output waveform while the unit is operating.

Examination of the output waveform with an oscilloscope indicates a square wave rise time of approximately 20 μsec. and a somewhat longer decay time. Harmonic content above 50 Kc/s. is therefore very small and will not contribute to signals measured in the microwave region.

The Lock-In amplifier design was given to this laboratory by L. D. Smulin of M.I.T. Slight modifications of the original circuit have been made in co-operation with E. V. Kornelsen and the schematic appears in Fig. 2-5. An Esterline-Angus recording milliammeter is used for an output indicator.

The square wave generator and Lock-In amplifier are powered by voltage stabilized power supplies of conventional design. The circuit constants of the square wave generator are such that a supply voltage change of 100 volts produces only 1/4 c.p.s. change in frequency and supply voltage stability requirements are therefore not stringent. The high voltage for the modulator is obtained from a PRD Type 801 A Universal Klystron Supply.

For the microwave receiver an intermediate frequency of 30 Mc/s. is used. The I.F. Preamplifier employs a low noise cascode input circuit, and has a bandwidth of approximately 15 Mc/s. The main I.F. amplifier limits the overall bandwidth to about 10 Mc/s. and a total gain in excess of 100 db is available. At the output of the I.F. amplifier a 10 ma. Littlefuse is used as a bolometer with about 6 ma. of bias current.

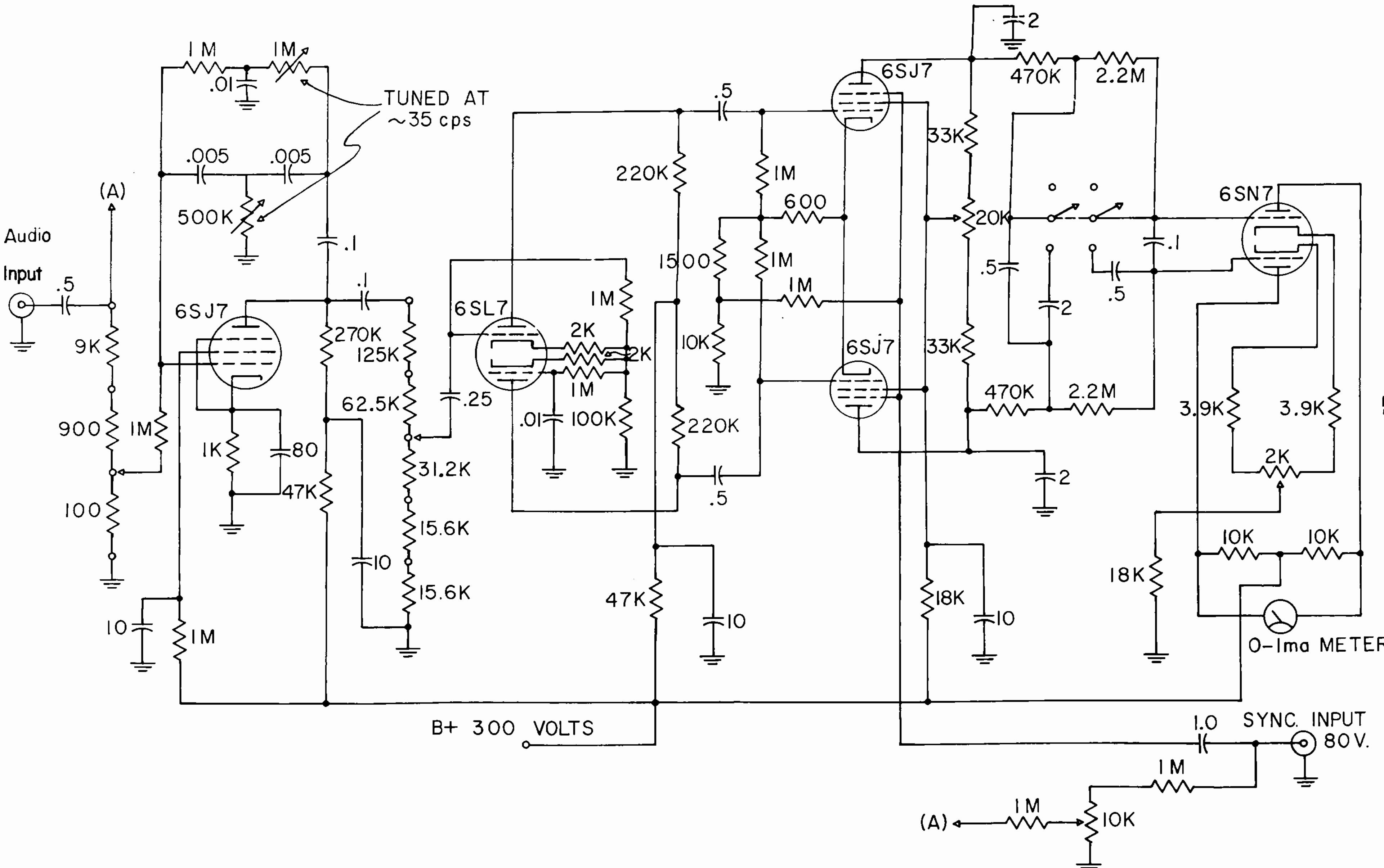


Fig. 2-5 Schematic Diagram of the Lock-In Amplifier

2-IV Measurements and Calculations

The evaluation of the theoretical system performance required knowledge of the output filter response function $S(\omega)$. This was determined as follows. A constant voltage sine wave from a Hewlett-Packard Type 202A Low Frequency Function Generator was applied to the synchronous detector grids. With no square wave reference voltage applied, the amplitude of the recorded sine wave output was measured as a function of frequency. The response is plotted in Fig. 2-6.

By fitting a curve of the theoretical form for agreement at 0, 0.1 and 0.5 c.p.s., the constants a and c were evaluated as

$$a = 2.9980$$

$$c = 2.3306$$

From Fig. 2-6 it is seen that the theoretical curve is a good fit except at frequencies greater than about 0.7 c.p.s. It is of some importance to note that the effective bandwidth of the theoretical curve is somewhat greater than the measurements indicate for the true response.

The I.F. amplifier had an approximately rectangular passband 10 Mc/s. wide. From equation (2-10) the expected improvement in sensitivity when using the synchronous system is

$$\frac{2}{\pi^{3/2}}\left[\frac{2\pi \times 10^7}{1.187}\right]^{1/2} = 2.61 \times 10^3 = 34.2 \text{ db}$$

To compare receiver sensitivity, with and without the synchronous system, a fluorescent noise source (Mumford 1949) was made for operation at 24 Kmc/s. This unit could be operated on d.c. or pulsed by the electron gun modulator.

The relation between receiver noise and the output from the

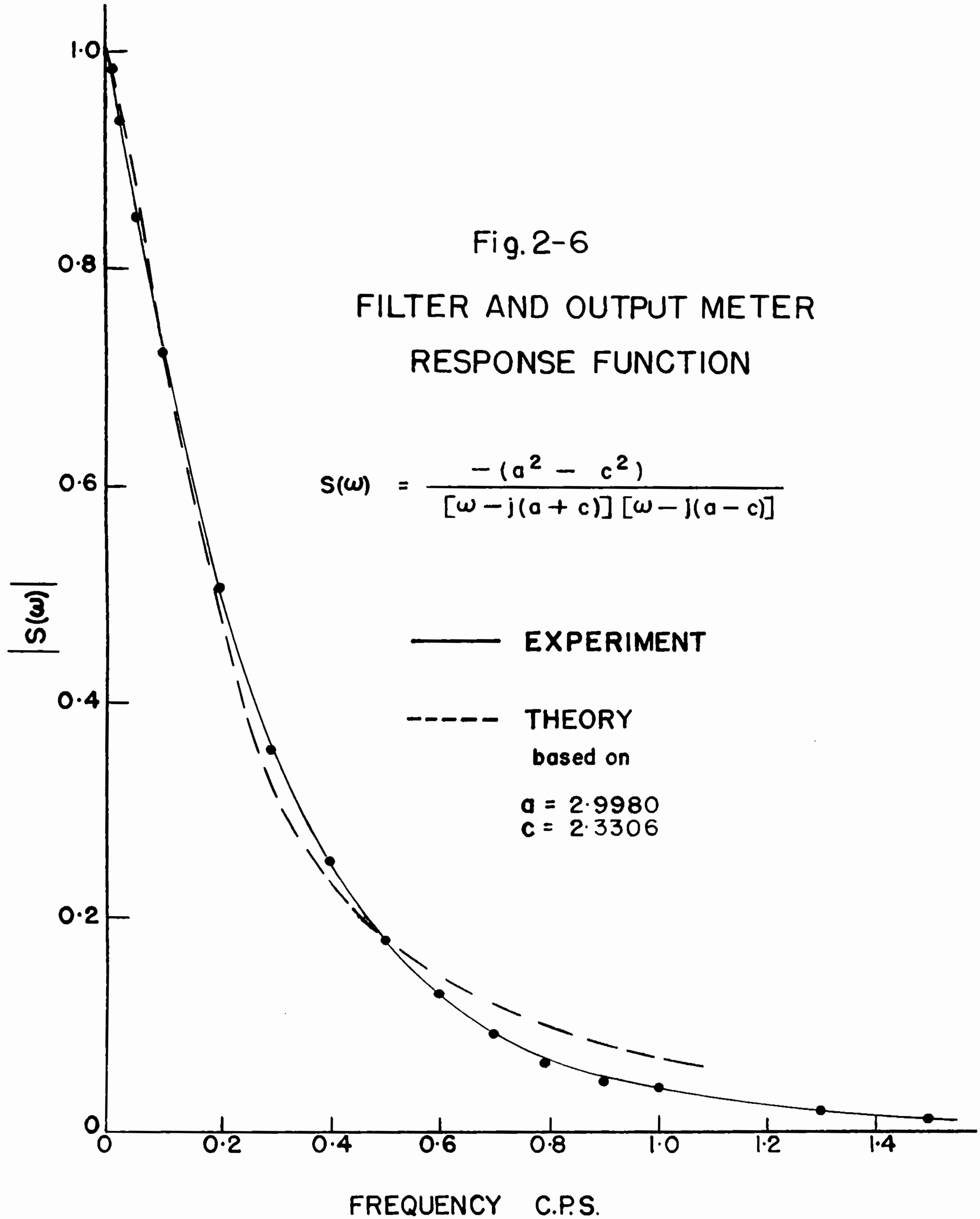


Fig. 2-6
FILTER AND OUTPUT METER
RESPONSE FUNCTION
$S(\omega) = \frac{-(a^2 - c^2)}{[\omega - j(a+c)][\omega - j(a-c)]}$
EXPERIMENT
THEORY
based on
a = 2·9980
c = 2·3306
|S(ω)|
1·0
0·8
0·6
0·4
0·2
0
0·2
0·4
0·6
0·8
1·0
1·2
1·4
FREQUENCY C.P.S.

fluorescent source was obtained by measuring changes in the d.c. voltage across the bolometer as first the preamplifier was turned on and then d.c. applied to strike the fluorescent tube located in the input waveguide. The results of the measurement are summarized below.

Measurement	Bolometer Bias Voltage
Zero signal	3.406
Receiver noise	3.543
Receiver noise plus lamp noise	3.590

Noting that the bolometer is a square-law device, the input signal to noise ratio when the fluorescent lamp was operating was

$$\frac{S}{N} = \frac{3.590 - 3.543}{3.543 - 3.406} = 0.344 = -4.6 \text{ db}$$

The input signal was therefore 4.6 db below that which would produce unity signal to noise ratio at the receiver output.

With the full system operating and the fluorescent noise source gated by the modulator, measurements were made of the output signal to noise ratio as different values of attenuation were used in the input line between the noise source and the receiver. Three determinations were made using values of attenuation of 50 db., 32.2 db., and 29.5 db.; the accuracy of the attenuator calibration being approximately 0.3 db. The first setting provided a negligible amount of signal and the output was essentially system noise. The second and third attenuator settings produced an output signal to noise ratio slightly less and somewhat greater than unity respectively.

The output signal to noise ratio was determined as follows. With sufficient audio gain to produce fluctuations on the output recording milliammeter of approximately one half of full scale, records of the system output were made for a period of eight minutes for each of the three attenuator settings. The records were analysed by counting the number of times the output waveform made a positively directed crossing through a given output level. The results are plotted in Fig. 2-7. The addition of a d.c. term to the noise output is apparent in the presence of signal.

Rice (1944, 1945) has indicated that the expected number of positive crossings per second for a filtered random noise voltage is

$$\frac{N_o}{2} \exp\left(-V^2/2\Psi_o\right) \qquad (2\text{-}11)$$

where N_o is the expected number of zeros per second.

V is the value of the noise voltage.

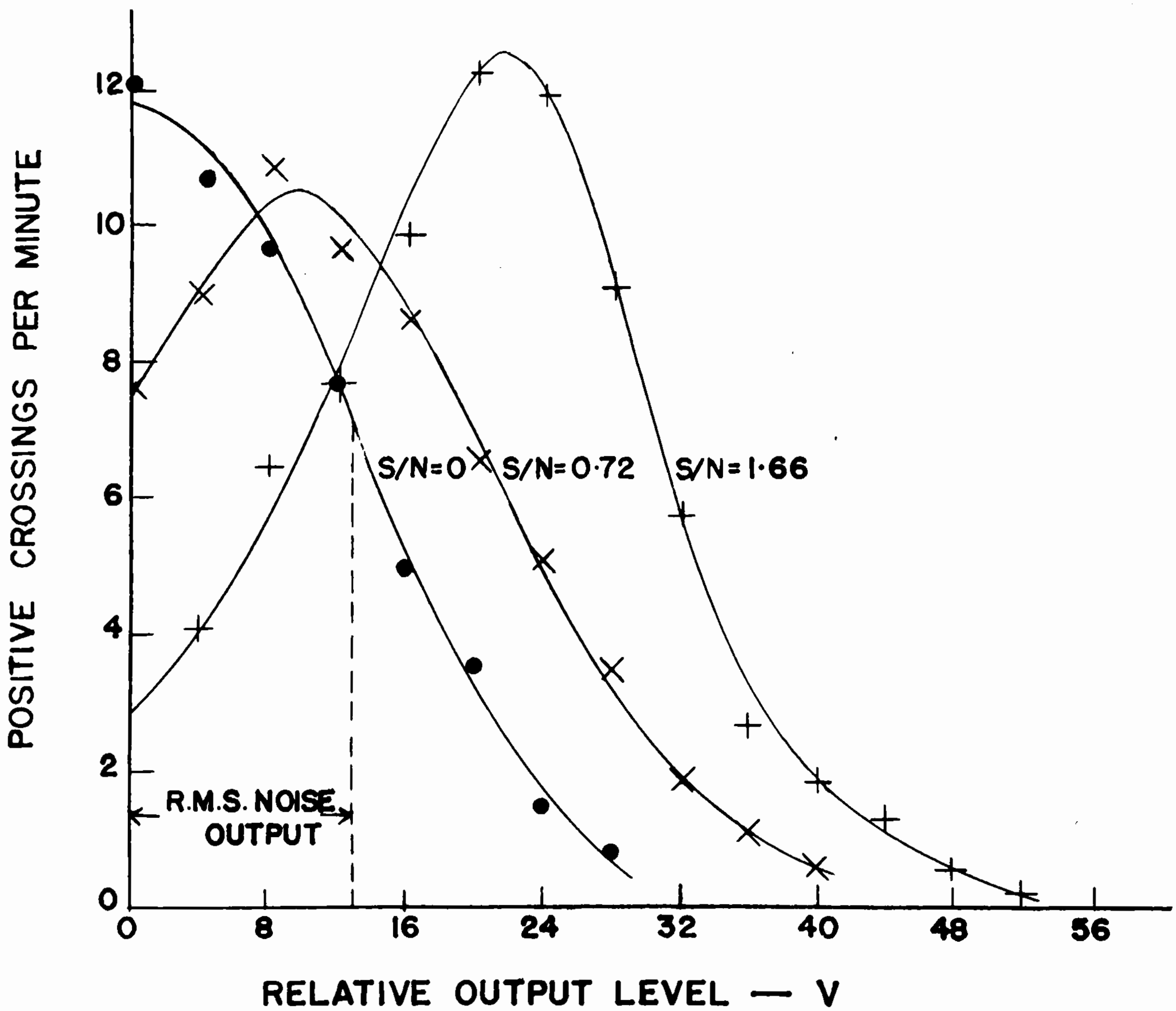
Ψ_o is the mean square noise voltage.

The R.M.S. value of the output voltage is therefore that value for which the number of positive crossing is 0.606 times the number of positive zero crossings. In Fig. 2-7 the R.M.S. output level is 12.8. Table 2-1 summarizes the evaluation of the sensitivity improvement from the measured input and output S/N levels.

TABLE 2-1

Summary of Measured Sensitivity Improvement

Input S/N	Attenuation db.	Output S/N	Output S/N db.	Improvement db.
-4.6 db.	32.2	0.72	-1.4	35.4
-4.6 db.	29.5	1.66	2.2	36.3



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FOR VARIOUS S/N RATIOS

Fig. 2-7

During electron beam noise measurements signals were such as to produce S/N ratios ranging from approximately 5 to 30 db. Using a calibrated signal generator, system linearity was found to be ± 0.1 db. over a greater than 30 db range of input signals. Compression occurred in the audio amplifier at signal levels in excess of -83 dbm. For the accurate measurement of wide band noise, the I.F. amplifier should remain linear at C.W. inputs, 10 db greater than the maximum noise power to be observed. The present system appeared to satisfy this requirement.

2-V Discussion

As has been remarked, the quantity indicated as the improvement in sensitivity is the ratio of the self noise power of the receiver alone, to the minimum detectable noise power using the complete synchronous system. The measured improvement figure is slightly larger than that calculated from theory. This is a consequence of the fact that the effective bandwidth of the theoretical output filter is greater than the measured response. A method exists for confirming this discrepancy. It is based on the number of zeros per second of the output waveform. The expected number of zeros per second is (Rice 1944, 1945)

$$2\left[\frac{\int_0^\infty f^2\, w(f)\, df}{\int_0^\infty w(f)\, df}\right]^{1/2} \qquad (2\text{-}12)$$

where w(f) is the power spectrum of the noise at the output of the filter. The bandwidth of the twin T feedback amplifier in front of the synchronous detector is several cycles and it is assumed that it does not modify the noise centred at the modulation frequency. The filter input noise has therefore a uniform spectral density and except for a possible constant multiplier

$$w(f) = \left|S(\omega)\right|^2 \qquad (2\text{-}13)$$

Using the theoretical form of $S(\omega)$ the expected number of zeros per second is from (2-12)

$$N_o = \frac{1}{\pi}\left[a^2 - c^2\right]^{1/2} \qquad (2\text{-}14)$$

With the values for a and c as determined above, it is found that the expected number of zeros per second is 0.600. The measured value was 0.393. The integral in the numerator of (2-12) depends mainly on the form of $S(\omega)$ at the higher frequencies. Fig. 2-6 shows that it is just in this region that the theoretical value of $S(\omega)$ is too large, resulting in the higher value predicted for N_o.

3 - THEORY OF NOISE IN ELECTRON STREAMS

3-I Introduction

The literature in the field of fluctuation processes in electron beams is sufficiently extensive that an exhaustive survey is not possible in a few pages. It will be valuable however to present a brief outline of the most significant work in order to introduce the fundamental concepts, and comment on the more general aspects of the analytic approaches. A more detailed examination will then be made of the theoretical work applicable to the specific problem of the transport of current and velocity fluctuations on an electron beam from a thermionic cathode, along the accelerated beam to the anode of a space-charge-limited, diode gun. A foundation will thus be indicated on which rests the theoretical contribution of the present work.

As a starting point the input fluctuations on the electron beam at emission are specified. This is most conveniently done in terms of the time average value of the square of the deviation from the average value, which has been called the quadratic content. The mean square current fluctuation of a beam from a thermionic cathode has been determined (Schottky 1937, Thompson, North and Harris 1940) as

$$\overline{J_{sn}^{\;2}} = 2e\, J_o \Delta f \qquad (3\text{-}1)$$

and has been called Shot Noise. J_o is the d.c. beam current density and Δf is the bandwidth over which the measurement is made. Rack (1938) has derived an expression for the quadratic content of velocity for electrons emitted with a Maxwellian distribution of velocities:

$$\overline{v_r^{\;2}} = (4 - \pi)\, \eta \frac{kT_c}{J_o} \Delta f \qquad (3\text{-}2)$$

The use of the current density J_o in (3-1) and (3-2) implies quadratic content per unit beam area. In all noise analysis the assumption is made that no correlation exists between the fluctuations in current and velocity at emission.

Experimentally it is observed that the effect of space-charge, within an electron gun, is to reduce below full shot noise the current fluctuations on the electron beam at the anode of a space-charge-limited diode. At low frequencies the phenomenon is reasonably well understood. The analysis of Thompson North and Harris (1940) postulates that an increase in the number of electrons emitted from the cathode causes a depression of the potential minimum formed in the gun, which is such as to oppose the transmission of the additional current to the anode. At microwave frequencies the effect of a fluctuating potential minimum on the current and velocity fluctuations on the electron beam is less clear. There is good theoretical justification (Tien and Moshman 1956) for assuming that the velocity distribution of the electrons leaving the potential minimum remains Maxwellian. The Monte Carlo calculations of Tien and Moshman (1956) indicated, however, that the current fluctuations on the beam could be greater or less than shot noise depending upon the frequency. A significant result of this calculation was the lack of correlation of the current and velocity fluctuations at the potential minimum.

Beyond the potential minimum, the fluctuation quantities are transformed as a consequence of kinematic and space-charge interaction processes; variations occurring in magnitude and phase as the current and velocity fluctuations propagate along the accelerated electron beam to the anode. The magnitudes are monotonic with distance as contrast to the periodic variations on a drifting beam.

The first analyses of the transformation of microwave noise quantities through a space-charge-limited diode (Pierce 1950) were based on the Llewellyn Peterson Equations (1944). There was the implicit assumption that between the potential minimum and the anode, the electron beam acts as a passive transducer, output current and velocity fluctuations being expressible as a linear combination of input quantities. Under the assumptions of zero initial d.c. velocity and a space-charge factor equal to unity, Pierce found that input shot noise produced no output, the fluctuations in current and velocity appearing on the beam at the anode being entirely a consequence of velocity fluctuations at the potential minimum. Using this theory to predict the amplitude and phase of the noise space-charge wave pattern beyond the anode, Cutler and Quate (1950) obtained reasonable agreement with drift-space noise measurements. However, they did not observe the theoretical zero value predicted for the noise wave minima. F. N. H. Robinson (1952) showed that a second uncorrelated noise wave resulting from shot noise input at same plane in front of the potential minimum,could account for a finite Standing Wave Ratio in the drift region beyond the anode. This plane was specified as one beyond which the beam could be considered single valued in velocity but rather naive and inaccurate input conditions were assumed.

A more sophisticated and much more general analysis was undertaken by Haus (1955); again necessarily starting at a plane in the gun beyond which a single velocity analysis is valid. Included in the formalism are not only terms related to the mean square noise current and the mean square kinetic voltage, which is a counterpart of velocity, but also the real and imaginary parts of their cross product. A real cross power term exists if the current and velocity fluctuations are correlated. No indication was given as to how, if at all, this correlation might arise. It was pointed out that there existed two parameters of the beam noise, which remained invariant under any lossless transformation. A lossless transformation is one in which no electromagnetic power is added to or extracted from the beam by virtue of an interaction with a circuit external to the beam. One of these parameters, called Beam Noisiness S, is related to an earlier study by Pierce (1954) of the noise standing wave in a drift space. The second invariant Π is the real part of the Cross Power Density Spectrum, and is related to the real part of the Kinetic Power carried by the beam. The concept was sufficiently novel to prompt a number of groups to attempt the measurement of Π and contrary to the simple analysis of Pierce (1950), a non zero value of Π was observed.

The beam invariants S and Π are related to the lowest Noise Figure obtainable from a tube employing the particular beam. The Noise Figure decreases with increasing values of Π and for high gain tubes is given as (Haus 1955)

$$F_{min} = 1 + \frac{2\pi}{kT_c}\left(S - \Pi\right) \qquad (3\text{-}3)$$

Measurements of the Noise Figures of low-noise travelling-wave tubes indicated values lower than could be explained by zero values for the real kinetic power. The existence of this real part could be attributed to a correlation between the fluctuations in current and velocity at the potential minimum, but this was in disagreement with the statistical work of Tien and Moshman.

Experimental work by M. R. Currie (1957) with backward-wave amplifiers and theoretical studies by Siegman Watkins and Hsieh, showed that an electron beam can develop real kinetic power when it is drifting in a region where the spread in velocities is comparable to the average velocity. The computations of Siegman et al were based on a digital computer solution of a differential equation describing the Density Function (Watkins 1952) for groups of electrons as they moved from the potential minimum toward the anode in a d.c. field described by the Fry-Langmuir equations. Integrating over all groups of electrons, values were found as a function of distance, for the mean square current, the mean square kinetic voltage and the real and imaginary parts of the cross product of current and kinetic voltage. The results showed that S and Π are not invariant in the low velocity region, but approach constant values as the beam velocity increases. The real part of the cross product is zero at the potential minimum, and increases to a constant value as the beam moves toward the anode.

None of these analyses take into account the finite size of the electron beam in a direction transverse to its motion. Only one attempt has been made to do this (Parzen 1952). The modification of Hutter's (1952) Electronic Equation by the introduction of an effective d.c. current density, proposed by Parzen, has received considerable attention in this laboratory. For an analog computer solution of the Finite Beam Electronic Equation, Kornelsen and Vessot used as input conditions the fluctuations in current and velocity at the potential minimum defined by (3-1) and (3-2). In the region immediately in front of the potential minimum, the single valued velocity assumption used in the derivation of the Electronic Equation is violated. To avoid this difficulty the present work starts the calculation at a plane in front of the potential minimum at which Π has reached its final value; this being considered as indicative of the validity of the small signal, single velocity assumptions. The values of the noise quantities as published by Siegman are used to determine the necessary initial values for the solution of the differential equation by the computer.

3-II The Electronic Equation

Of considerable importance in the analysis of the operation of microwave vacuum tubes is a knowledge of the manner in which disturbances in current and velocity propagate along an electron beam in the absence of any continuous interaction with a radio frequency slow-wave structure. An analysis valid for periodic variations in current and velocity will be applicable to studies of narrow band noise if the d.c. electron velocity is sufficiently large. (Haus 1955).

The Electronic Equation of Hutter (1952) is the result of one such analysis, and a brief outline of the assumptions and method of development follows. In common with most other work the assumptions are made in the interest of mathematical simplicity; more general approaches resulting in a considerable increase in analytical complexity.

1) The electron beam is infinite in a direction perpendicular to beam motion and only one spatial co-ordinate variable is retained.

2) All a.c. variables are assumed to be harmonic functions of time; the dependence being as $e^{j\omega t}$.

3) All electrons passing a beam cross-section have the same velocity.

4) Any a.c. modulation quantities are small compared to the corresponding d.c. values.

If v_ω and J_ω are respectively the amplitudes of the velocity and current density modulations on the electron beam, they are determined from the two basic equations

$$v_\omega = \frac{-1}{j\omega J_o}\left[\frac{\partial Y}{\partial \tau} - \frac{Y}{u_o}\,\frac{\partial u_o}{\partial \tau}\right] e^{-j\omega\tau} \qquad (3\text{-}4)$$

$$\frac{\partial^2 Y}{\partial \tau^2} - \frac{1}{u_o}\left[\frac{\partial^2 u_o}{\partial \tau^2} - \frac{e\,J_o}{m\,\varepsilon_o}\right] Y = \frac{eJ_o}{m\varepsilon_o} I_\omega\, e^{j\omega\tau} \qquad (3\text{-}5)$$

where use has been made of the definitions

$$J_\omega = \frac{Y}{u_o} e^{-j\omega\tau} \qquad \tau = \int_o^z \frac{dz}{u_o} \qquad (3\text{-}6)$$

The result is a particularly general one applicable to regions of an arbitrary axial d.c. potential profile. Where the magnitude of the d.c. current density and the boundary potentials determine this profile, the equations (3-4) and (3-5) are strictly equivalent to the Llewellyn Peterson analysis (1944).

Contrary to the assumption of an electron beam infinite in a direction transverse to its motion, practical devices employ cylindrical beams and consideration must be given to the fact that the forces on an electron within a cylindrical beam are somewhat modified from those for an infinite beam. With the exception of statistical theories it is usual to consider the electrons to be in the form of a fluid in which interactions between individual electrons are neglected, and any given electron is assumed to be in a reasonably regular Coulomb field due to all the other electrons in the beam. When an electron in a plasma of zero net charge density is displaced from its equilibrium position, the restoring forces on it are such as to cause it to oscillate with simple harmonic motion about its mean position. The frequency of oscillation, called the Plasma Frequency, is a function of the electron charge density and has been calculated by Tonks and Langmuir (1929) as

$$\omega_p = \sqrt{\frac{e\rho_o}{m\varepsilon_o}} \qquad (3\text{-}7)$$

Since short range collisions have been neglected, the energy of an oscillating electron is not transmitted to its neighbours and the disturbance is not propagated through the plasma.

The restoring forces on a displaced electron in a cylindrical beam are less than for an infinite beam and the associated plasma frequency is reduced. The ratio of the cylindrical beam plasma frequency ω_q, to that for an infinite beam of the same charge density, is called the plasma frequency reduction factor p.

$$\frac{\omega_q}{\omega_p} = p \tag{3-8}$$

In general p is complex. (Birdsall and Whinnery 1953) For an electron beam drifting in a metallic cylinder the diameter of which is large compared to the beam diameter 2b, the value of p is real and is given by,

$$p = \left[1 + \frac{T^2}{\beta_e^2}\right]^{1/2} \tag{3-9}$$

where T is determined as a function of $\beta_e b$ by the solution of an equation arising from field matching at the electron beam boundary.

$$Tb \frac{J_1 (Tb)}{J_o (Tb)} = \beta_e b \frac{K_1 (\beta_e b)}{K_o (\beta_e b)} \tag{3-10}$$

$$\beta_e = \omega/u_o$$

The value of p is not unique. It is possible to satisfy the requisite boundary conditions with a series of possible field configurations and their related radial profiles of charge and velocity. Each mode is characterized by its own reduction factor. Calculations are usually based on the assumption that only the first order space-charge wave mode is significant, which is valid for a narrow drifting electron beam not enclosed in a metal cylinder of the same diameter. Under rapid accelerations, coupling between modes can occur and as the radial mode functions are in general not orthogonal, the mathematical analysis becomes quite intractable.

In order to make a correction for the finite beam size in the gun region, Kornelsen and Vessot used a modification proposed by Parzen (1952). In the Electronic Equation (3-5) the d.c. current density J_o was replaced by an effective current density J_{oe} defined by

$$J_{oe} = p^2 J_o \tag{3-11}$$

In his analysis Parzen requires the approximations

$$\frac{\omega_p}{\omega} << 1 \qquad \frac{1}{\omega u_o}\frac{\partial u_o}{\partial \tau} = \frac{2}{\omega \tau} << 1 \tag{3-12}$$

where the d.c. velocity has been assumed to be that for a Llewellyn gap in the determination of the equality. Both of the relations break down in the region of the potential minimum and some comment on the first inequality will be made later.

To establish a form of the Electronic Equation, modified for finite beam size, which is applicable to the region between the potential minimum and the electron gun anode, Kornelsen and Vessot assumed that the effective current density at any point in the gun could be characterized by the reduction factor for a drifting beam of the same radius and d.c. velocity. To simplify the solution of the resulting equation the total current I_ω was taken as zero. As the effect of total current is negligibly small for transit angles of greater than 2π radians (Pierce 1950), at the frequencies of interest in the present work, no serious error is expected. The Finite Beam Electronic

Equation becomes from (3-5)

$$\frac{\partial^2 Y}{\partial \tau^2} - \frac{1}{u_o}\left[\frac{\partial^2 u_o}{\partial \tau^2} - p^2 \frac{eJ_o}{m\varepsilon_o}\right] Y = 0 \qquad (3\text{-}13)$$

The d.c. velocity-transit time relation was specified as that for a Llewellyn gap with no initial acceleration and an initial velocity equal to the R.M.S. thermal velocity.

$$u_o = \frac{1}{2}\frac{eJ_o}{m\varepsilon_o}\tau^2 + u_{om} = \sqrt{2\eta V_o} \qquad (3\text{-}14)$$

There is some uncertainty in the proper value to use for u_{om}, and a somewhat lower value than the R.M.S. velocity will be employed later. Equation (3-13) reduces to

$$\frac{\partial^2 Y}{\partial \tau^2} - K\frac{(1 - p^2)}{u_o} Y = 0 \qquad (3\text{-}15)$$

$$K = \frac{eJ_o}{m\varepsilon_o}$$

which is the final form to be solved on the analog computer.

3-III Haus' Analysis

The very general formulation by Haus (1955) treats the electron beam as a four terminal network. The analysis is one dimensional with the usual assumptionsof small signal single velocity theories. With the introduction of a quantity called the kinetic beam voltage,

$$V = -\frac{u_o v}{\eta} \qquad (3\text{-}16)$$

where v is the beam velocity modulation, an analogy between an electron beam and a transmission line is established. The time average power flow for an electron beam, as for a transmission line, is equal to $\frac{1}{2}$ Re VJ^* and this quantity has been called the real kinetic power density. (L. J. Chu 1951). Haus shows that in an electron beam passing through a region in which no electromagnetic power is added to or extracted from the beam, the real kinetic power density is a constant. Such a region is called a lossless beam transducer and the beam excitation at the output, in terms of the kinetic voltage and current modulation, must be linearly related to the input quantities.

Noise in an electron beam is a statistical process and Haus uses statistical methods in its analysis. The starting point is the linear transducer in which the Fourier transforms of the output quantities of current and voltage are linearly related to the Fourier transforms of the input quantities.

It is not possible to determine Fourier Transforms for the input kinetic voltage V_a (t), and the current density modulation J_a (t), that are valid for $-\infty < t < \infty$. The integrals of interest do converge if the time interval is restricted to 2T. By defining

$$
\begin{aligned}
V_{aT}(t) &= V_a(t) && -T < t < T \\
&= 0 && T < |t| \\
J_{aT}(t) &= J_a(t) && -T < t < T \\
&= 0 && T < |t|
\end{aligned}
\qquad (3\text{-}17)
$$

the functions $V_{aT}(t)$ and $J_{aT}(t)$ accurately represent the noise over an interval 2T. These have the Fourier transforms

$$V_{aT}(\omega) = \int_{-\infty}^{\infty} V_{aT}(t) \; e^{-j\omega t} \, dt \qquad (3\text{-}18a)$$

$$J_{aT}(\omega) = \int_{-\infty}^{\infty} J_{aT}(t) \; e^{-j\omega t} \, dt \qquad (3\text{-}18b)$$

The corresponding quantities at the output plane of the linear transducer can be represented as

$$V_{bT}(\omega) = A(\omega)\, V_{aT}(\omega) + B(\omega)\, J_{aT}(\omega) \qquad (3\text{-}19a)$$

$$J_{bT}(\omega) = C(\omega)\, V_{aT}(\omega) + D(\omega)\, J_{aT}(\omega) \qquad (3\text{-}19b)$$

where the associated inverse transforms $V_{bT}(t)$ and $J_{bT}(t)$ represent the true output over most of the interval 2T except for initial transients. If the interval T is allowed to increase to infinity the correspondence of all quantities to the true values becomes exact.

Before taking the limits Haus evaluates the squares of the absolute magnitudes of equations (3-19) and writes the first multiplied by the complex conjugate of the second. Observing that as $T \to \infty$ the quantities approach finite limits Haus defines

$$\lim_{T \to \infty} \frac{\pi}{T} \overline{\left| V_{aT}(\omega) \right|^2} = \Phi_a(\omega)$$ = Self power density spectrum of the kinetic noise voltage modulation. (3-20a)

$$\lim_{T\to\infty} \frac{\pi}{T} \overline{\left| J_{aT}(\omega) \right|^2} = \Psi_a(\omega)$$ = Self power density spectrum of the noise current modulation. (3-20b)

$$\lim_{T\to\infty} \frac{\pi}{T} \overline{V_{aT}(\omega)\, J_{aT}(\omega)^*} = \left[\lim_{T\to\infty} \frac{\pi}{T} \overline{V_{aT}(\omega)^*\, J_{aT}(\omega)} \right]^*$$

$$= \Theta_a(\omega)$$ = Cross power density spectrum between the kinetic voltage and current modulation. (3-20c)

where the bar indicates an ensemble average, and the limits are not only valid for the input plane a as indicated, but also for the output plane b. From the manipulations noted for equations (3-19) there results,

$$\Phi_b = A^2 \Phi_a + B^2 \Psi_a + AB^* \Theta_a + A^* B\, \Theta_a^* \qquad (3\text{-}21a)$$

$$\Psi_b = C^2 \Phi_a + D^2 \Psi_a + CD^* \Theta_a + C^* D\, \Theta_a^* \qquad (3\text{-}21b)$$

$$\Theta_b = AC^* \Phi_a + BD^* \Psi_a + AD^* \Theta_a + A^* D\, \Theta_a^* \qquad (3\text{-}21c)$$

Since the last of these is complex there are required four parameters to characterize the noise in an electron beam at any cross section.

A factor of 4π relates the Self power density spectrum to the more commonly used quantity, the quadratic content per unit bandwidth. For pure shot noise in a beam of current density J_o

$$\Psi = \frac{2eJ_o}{4\pi} = \frac{eJ_o}{2\pi} \qquad (3\text{-}22)$$

By formulating the problem of the transformation of noise quantities by a lossless beam transducer, in terms of matrix algebra, Haus finds two invariants of a noise process in an electron beam.

$$1) \quad \mathrm{Re}\,\Theta = \Pi \qquad (3\text{-}23)$$

$$2) \quad S = \left[\Phi \bar{\Psi} - \Lambda^2 \right]^{1/2} \qquad (3\text{-}24)$$

where the real and imaginary parts of Θ are given by

$$\Theta = \Pi + j\Lambda \qquad (3\text{-}25)$$

From equation (3-20c) it is seen that Θ is proportional to the kinetic power carried by the beam. The invariance of Π signifies that the real part of the kinetic power in a particular frequency band must be conserved through a linear lossless transducer.

Since S and Π are invariant for a lossless transducer, their values can be traced back along the beam to the input plane at which the small signal single velocity approximation first becomes valid. This is true at a plane a few tenths of a volt higher in potential than the potential minimum. The magnitudes of S and Π are therefore entirely determined by conditions in the gun between the cathode and the region immediately beyond the potential minimum.

3-IV Statistical Methods

In the low velocity region of an electron gun, analytical expressions to characterize the noise of the entire beam cannot be developed. It becomes necessary to accept the statistical nature of electron emission and base calculations of fluctuations in current and velocity on the integrated effects of electrons considered individually or in groups. Using a large digital computer and random numbers to specify the velocity and time of emission of each electron, Tien and Moshman (1956) calculated as a function of frequency the mean square current and velocity fluctuations in a particular gun at a plane 1.2 times the distance of the potential minimum from the cathode. As was anticipated by earlier authors, the Maxwellian character of the distribution of electron velocities was unmodified and the fluctuations in current were not correlated with those of velocity. Depending upon the frequency, the current fluctuations were found to be greater or less than shot noise, equality maintaining at frequencies above 5 Kmc/s.

Of more general application is a calculation by Siegman, Watkins and Hsieh (1957) of noise propagation on an accelerated multivelocity electron stream in the region immediately in front of the potential minimum. The basic assumptions of the work are as follows.

1) The analysis is for small signals and the linearized density function equations are used. The density function is defined such that $F(x, u, t)\, dx\, du$ is the amount of smoothed out negative charge in the distance dx with electrons in the velocity range du at time t. The function is separated into a.c. and d.c. parts.

$$F\ (x, u, t) = F_o\ (x, u) + F_1\ (x, u)\ e^{j\omega t} \qquad (3\text{-}26)$$

2) The analysis is one dimensional.

3) The total a.c. current in the stream is zero.

4) Input conditions correspond to full shot noise in each velocity class, the input fluctuations in any one velocity class being uncorrelated with the input fluctuations in any other velocity class.

5) The d.c. potential profile is that of the Fry-Langmuir analysis (1921, 1923) for the space-charge-limited diode with a Maxwellian distribution of initial velocities. The region considered starts at $\xi = 0$, the potential minimum, and extends out toward positive ξ, where ξ is Langmuir's normalized distance parameter.

Using an IBM 650 computer to carry out a numerical solution of a differential equation for $F_1(x, u)$ (Siegman 1957), the noise fluctuations were studied as a function of ξ for three values of a frequency parameter a.

$$a = \frac{\omega}{2\omega_{pm}} \tag{3-27}$$

where ω_{pm} is the plasma frequency at the potential minimum. Values for the total mean square current fluctuation per unit bandwidth, the total mean square kinetic noise voltage fluctuation per unit bandwidth, and the real and imaginary parts of the total correlation between noise current and kinetic voltage were evaluated. The parameters S and Π were found to vary with distance from $\eta = 0$ to $\eta = 4$. The normalized potential, $\eta = eVo/kT_c$, was measured relative to the minimum in potential. The value of S decreased from its initial value of $kT_c/2\pi$ to approximately 0.7 times this value. The real part of the kinetic power Π increased from zero to approximately 0.3 $kT_c/2\pi$. Beyond $\eta \doteq 4$

S and Π were invariant. For the purpose of the present theoretical contribution, this is interpreted as indicating that beyond the plane $\eta = 4$ a single valued velocity theory would accurately characterize the transport of noise through the remaining accelerating region to the anode.

For convenience in later discussion, the following definitions will be made, using quantities calculated by Siegman et al. Noting that $\overline{V_i^{\ 2}}$ and $\overline{J_i^{\ 2}}$ are the total mean square fluctuations per unit bandwidth per unit beam area for the kinetic voltage and current, define:

$$\delta^2 = \frac{\overline{J_i^{\ 2}}}{2eJ_o} = \frac{4\pi \Psi}{2eJ_o} \qquad (3\text{-}28a)$$

$$\mu^2 = \overline{V_i^{\ 2}} \frac{2eJ_o}{4k^2 T_c^{\ 2}} = 4\pi \Phi \frac{2eJ_o}{4k^2 T_c^{\ 2}} \qquad (3\text{-}28b)$$

$$\Pi = \mathrm{Re}\, \frac{\overline{J_i V_i}}{2kT_c} = \frac{4\pi \Pi}{2kT_c} \qquad (3\text{-}28c)$$

$$A = \mathrm{Im}\, \frac{\overline{J_i V_i}}{2kT_c} = \frac{4\pi \Lambda}{2kT_c} \qquad (3\text{-}28d)$$

$$S = \left[\mu^2 \delta^2 - A^2\right]^{1/2} = \frac{4\pi S}{2kT_c} \qquad (3\text{-}28e)$$

The curves presented by Siegman et al provide numerical values for δ^2, μ^2, Π, A and S.

3-V Analog Computer Solution

Two methods have been used to solve the finite beam Electronic Equation.

$$\frac{\partial^2 Y}{\partial \tau^2} - K \frac{(1 - p^2)}{u_o} Y = 0 \qquad (3\text{-}15)$$

H. E. Rowe (1952) has indicated a solution by means of the W.K.B. approximation. Kornelsen and Vessot (1957), by using an analog computer, obtained a more accurate evaluation of beam noise at the electron gun anode. In both cases the plane at which input conditions were specified was taken at the potential minimum. Rowe specified an initial d.c. velocity equal to zero and evaluated the mean square noise current and velocity fluctuations at the anode due only to a velocity fluctuation at the potential minimum. Kornelsen and Vessot took as an initial velocity, the R.M.S. value of the Maxwellian distribution, and determined the mean square output current and velocity fluctuations for each of two uncorrelated inputs: full shot noise and Rack velocity.

The Llewellyn Peterson equations and the modified Electronic Equation which have been considered for calculating the noise properties of an electron beam at the anode of a space-charge-limited, diode gun, involve small signal single velocity assumptions which are unquestionably violated in the region of the potential minimum. It might also be remarked that neither admit of the existence or development of real kinetic power when the input plane is selected at the potential minimum, although the conservation of S is maintained. To remove both of these restrictions, a method of calculation was undertaken which limits the application of the modified Electronic Equation to a region in the gun over which the assumptions made in its derivation can be reasonably presumed to be valid. The statistical calculations of Siegman et al provided the criterion for determining an appropriate input plane and also made it possible to accurately specify the noise input quantities to be used. An analog computer was employed to solve the finite beam Electronic Equation between the input plane and the electron gun anode. There resulted a set of equations of the form of (3-21) for the four anode noise quantities defined by equation (3-28), in terms of the corresponding excitation quantities at the input plane. The method of solution is exactly that described by Kornelsen (1957), and his results are derivable as a special case of the more general formulation. Considerable simplification has resulted from the linear relation observed between the computer voltages used to represent the function Y and its derivative at the input and output planes.

Kornelsen has outlined the details for setting up the computer to solve the finite beam Electronic Equation and these are summarized in Appendix I. A schematic diagram of the computer is shown in Fig. (A-1). The procedure is briefly as follows.

1) The computer is set up to solve an equation corresponding exactly in form to equation (3-15).

2) Two time normalizations are made. If T is the independent variable, t real time on the computer, and τ gun transit time, normalization is such that at a point representing the gun anode

$$T_a = \alpha t_a = \rho \tau_a = 1 \qquad (3\text{-}29)$$

and this serves to define α and ρ.

3) The quantity $(1 - p^2)/u_o$ for a given beam radius and frequency, is determined as a function of time using equation (3-14) and an assumed initial velocity. The function F(T) is the term $(1 - p^2)/u_o$ normalized to its maximum value and this is set on the computer by loading a tapped potentiometer. Kornelsen shows that it is not necessary to change this function for any anode voltage lower than that for which it was calculated, provided the computation is carried to a point beyond the maximum F(T).

4) The necessary circuit constants are determined by equating coefficients of the parallel equations, and the initial values of the noise current and velocity are introduced by setting appropriate values for the voltages representing the initial values of the function and its derivative.

Consideration must be given to the selection of a d.c. velocity to be used at the potential minimum. Kornelsen has used the R.M.S. thermal velocity

$$u_{om}^2 = \frac{2kT_c}{m} \tag{3-30}$$

Siegman et al have compared the results of the statistical calculations for the total mean square current with the results of a similar computation using the Llewellyn Peterson equations. For small values of "a", good agreement is obtained if the charge average velocity is used;

$$u_{om} = \frac{J_o}{\rho_{om}} \tag{3-31}$$

where ρ_{om} is the charge density at the potential minimum. It will be convenient to examine the value of ρ_{om} and ω_{pm} in terms of the Fry-Langmuir dimensionless variables. These are defined by

$$\text{Distance} \quad \xi = \beta \alpha^{3/4} z \tag{3-32a}$$

$$\text{D.C. potential} \quad \eta(\xi) = \frac{eV_o}{kT_c} \tag{3-32b}$$

where $\alpha = \dfrac{m}{2kT_c}$ $\qquad \beta = 2\pi^{1/4}\left[\dfrac{eJ_o}{m\varepsilon_o}\right]^{1/2}$

The charge density at the potential minimum is required to determine u_{om} from equation (3-31). From the Fry-Langmuir analysis it is determined that at the potential minimum (Kleynen 1946)

$$\frac{d^2\eta}{d\xi^2} = 0.5 \tag{3-33}$$

Using the definitions (3-32) and changing to the variables V_o and z, equation (3-33) becomes

$$\frac{d^2\eta}{d\xi^2} = 0.5 = \frac{e}{kT_c} \beta^{-2} \alpha^{-3/2} \frac{d^2V_o}{dz^2} \tag{3-33a}$$

As the analysis in the present use is one dimensional, from Poisson's Equation

$$\nabla^2 V = \frac{d^2V}{dz^2} = \frac{\rho_o}{\varepsilon_o}$$

Solving for the charge density at the potential minimum,

$$\rho_{om} = \frac{1}{2} \frac{kT_c}{e} \beta^2 \alpha^{3/2} \varepsilon_o$$

and using equations (3-31) and (3-7)

$$u_{om}{}^2 = \frac{2}{\pi} \frac{kT_c}{m} \tag{3-34}$$

$$\omega_{pm} = \frac{\alpha^{1/4}}{2} \beta$$

$$= \left[\frac{\pi}{\varepsilon_o{}^2} \frac{e^2}{m} \frac{1}{2kT_c}\right]^{1/4} J_o{}^{1/2} \tag{3-35}$$

The selection of the input plane at which to start the solution of the finite beam Electronic Equation is somewhat of a compromise. The single valued velocity requirement would appear to dictate a large value of ξ beyond which S and Π do not vary. However the plane cannot be advanced beyond the point where finite beam size becomes significant in the transformation of fluctuation quantities. For the present work the reduction factor p is never less than 0.965 between the potential minimum and the input ξ plane, and that only at 1,400 Mc/s being essentially unity at the two higher frequencies studied.

At an input plane where the beam has developed a non-zero value of real kinetic power, there exists correlation between the kinetic voltage and current fluctuations and the computer input conditions are less simple than the two uncorrelated inputs used by Kornelsen and Vessot. As shown by Bloom (1955) and Beam (1955) the noise properties at such a cross section can be defined by three independent sets of input conditions. If $\overline{V_i^2}$ and $\overline{J_i^2}$ represent the total mean square values of kinetic noise voltage and current the three inputs are specified by

$$1)\quad \overline{V_I^2} = (1 - K_1^2)\,\overline{V_i^2} \qquad \overline{J_I^2} = 0 \qquad (3\text{-}36a)$$

$$2)\quad \overline{V_{II}^2} = 0 \qquad \overline{J_{II}^2} = (1-K_2^2)\,\overline{J_i^2} \qquad (3\text{-}36b)$$

$$3)\quad \overline{V_{III}^2} = K_1^2\,\overline{V_i^2} \qquad J_{III}^2 = K_2^2\,\overline{J_i^2} \qquad (3\text{-}36c)$$

$$\text{with}\quad \frac{J_{III}}{V_{III}} = \frac{K_2}{K_1}\,e^{j\theta}\sqrt{\frac{\overline{J_i^2}}{\overline{V_i^2}}} \qquad (3\text{-}36d)$$

The first input consists entirely of a kinetic voltage excitation, the second consists only of a current fluctuation and the third is a completely correlated fluctuation in current and kinetic voltage with a phase difference θ, as yet undetermined. The correlation coefficients are such that

$$0 \leqslant K_1 \leqslant 1$$

$$0 \leqslant K_2 \leqslant 1$$

Using R.M.S. values for the current and kinetic voltage, the kinetic power is given as

$$4\pi \Theta = V_i J_i^* \qquad (3\text{-}37)$$

at the input plane. Noting the definitions (3-28) it is apparent that

$$\Pi = \mu\delta K_1 K_2 \cos\theta \qquad (3\text{-}38a)$$

$$\Lambda = -\mu\delta K_1 K_2 \sin\theta \qquad (3\text{-}38b)$$

The values of μ^2, δ^2, Π and Λ are known from published curves (Siegman et al 1957) and having selected an input plane at a particular ξ, the value of θ and the product $K_1 K_2$ are determined. As will be demonstrated a proper selection of K_2 leads to the conservation of real kinetic power and with K_2 specified, there results an unambiguous determination of all input quantities (3-36). The method of calculating the computer input voltages V_{1i} and V_{2i} (see Fig. A-1) for each of the three inputs is described in Appendix I.

The anode noise quantities corresponding to each input are determined from the appropriate computer output voltages V_{1a} and V_{2a}. The observation that for a given electron gun anode voltage the computer output voltages are linear functions of input voltages has enabled more concise expressions to be written for the beam noise parameters at the gun anode. The computer voltages are related as follows:

$$V_{1a} = a_o V_{1i} - a_1 V_{2i} \qquad (3\text{-}39a)$$

$$V_{2a} = -b_o V_{1i} + b_1 V_{2i} \qquad (3\text{-}39b)$$

The coefficients a_o, a_1, b_o, b_1, are positive and their determination constitutes the complete solution for a given anode voltage. Define

$$c_o = 2\alpha R_2 C_2 a_o - b_o \qquad (3\text{-}40a)$$

$$c_1 = 2\alpha R_2 C_2 a_1 - b_1 \qquad (3\text{-}40b)$$

$$g = \frac{\alpha R_2 C_2 \omega}{\rho} \frac{kT_c}{e} \frac{1}{2V_{oi}} \qquad (3\text{-}40c)$$

$$\lambda = K \tau_s \frac{\alpha}{\rho} \frac{R_2 C_2}{u_{oi}} \qquad (3\text{-}40d)$$

V_{oa} = gun anode voltage.

$V_{oi} = \frac{kT_c}{e} \eta$ where $\eta(\xi)$ specifies the input plane.

$u_{oi}^2 = 2\frac{e}{m} V_{oi}$

τ_s = transit time from the potential minimum to the plane ξ, as determined from equations (3-14) and an assumed value for u_{om}.

After some lengthy but straightforward algebra, summarized in Appendix I, it is found that the noise quantities at the electron gun anode are:

$$\delta_a^{\ 2} = \frac{V_{oi}}{V_{oa}} \Big[g^2 a_1^{\ 2} \mu^2 + a_o^{\ 2} K_2^{\ 2} \delta^2 + (a_1 \lambda + a_o)^2 \delta^2 - 2g\lambda a_1^{\ 2} \mathcal{A} + 2g\, a_o a_1 (\Pi - \mathcal{A}) \Big] \qquad (3\text{-}41a)$$

$$\mu_a^{\ 2} = \frac{V_{oa}}{V_{oi}} \frac{1}{g^2} \Big[g^2 c_1^{\ 2} \mu^2 + c_o^{\ 2} K_2^{\ 2} \delta^2 + (c_1 \lambda + c_o)^2 \delta^2 - 2g\lambda c_1^{\ 2} \mathcal{A} + 2g\, c_o c_1 (\Pi - \mathcal{A}) \Big] \qquad (3\text{-}41b)$$

$$\Pi_a = (a_1c_o - a_oc_1)\left[\Pi + \mathcal{A} - \frac{\lambda}{g} K_2^2 \delta^2\right] \tag{3-41c}$$

$$\mathcal{A}_a = \frac{1}{g}\Big[g^2 a_1 c_1 \mu^2 + a_o c_o K_2^2 \delta^2 + (a_1\lambda + a_o)(c_1\lambda + c_o)\delta^2 - 2g\lambda a_1 c_1 \mathcal{A} + g(a_1c_o + a_oc_1)(\Pi - \mathcal{A})\Big] \tag{3-41d}$$

$$\begin{aligned} \mathcal{S}_a^2 &= \mu_a^2 \delta_a^2 - \mathcal{A}_a^2 \\ &= (a_1c_o - a_oc_1)\Big[(1 + K_2^2)\mu^2\delta^2 - (\Pi - \mathcal{A})^2 \\ &\qquad + \left(\frac{\lambda}{g}\right)^2 K_2^2 \delta^4 - 2\frac{\lambda}{g}\delta^2 (\Pi + K_2^2\mathcal{A})\Big] \end{aligned} \tag{3-41e}$$

In the computations, K_2 is selected so as to conserve Π in equation (3-41c).

The quantity δ^2 represents the ratio of the total mean square current fluctuations to full shot noise. It is this quantity that has been measured experimentally as a function of frequency and voltage at the anode of a space-charge-limited, diode gun.

Equations (3.41) can be reduced to the relations derived by Kornelsen. Noting that at the potential minimum under Kornelsen's assumptions

$$\delta^2 = 1 \tag{3-42a}$$

$$\mu^2 = 4 - \pi \tag{3-42b}$$

$$\Pi = \mathcal{A} = K_1 = K_2 = 0 \tag{3-42c}$$

$$V_{oi} = kT_c/e \tag{3-42d}$$

the relations for the noise quantities at the gun anode become

$$\delta_a^2 = \frac{V_{oi}}{V_{oa}}\left[g^2 a_1^2 (4-\pi) + a_o^2\right] \tag{3-43a}$$

$$\mu_a^{\ 2} = \frac{V_{oa}}{V_{oi}} \ \frac{1}{g^2} \left[g^2 c_1^{\ 2} (4 - \pi) + c_o^{\ 2} \right] \tag{3-43b}$$

$$\mathcal{H}_a = 0 \tag{3-43c}$$

$$\mathcal{A}_a = \frac{1}{g}\left[g^2 a_1 c_1 (4-\pi) + a_o c_o \right] \tag{3-43d}$$

$$\mathcal{S}_a^{\ 2} = (a_1 c_o - a_o c_1) \ \mu^2 \ \delta^2 \tag{3-43e}$$

Equations (3-43a) and (3-43b) are exactly the results found by Kornelsen.

4. THE EXPERIMENTAL VACUUM TUBES

The electron gun used for the measurements reported was based on a design by Pierce (1949). In a Pierce gun designed to produce a cylindrical beam of electrons, the electrodes are arranged in such a manner as to provide a variation of potential along the beam edge that corresponds to the variation with distance computed for a space-charge-limited diode. The equality of potential inside and outside the beam results in a cylindrical stream of electrons which leaves the gun through a circular aperture in the anode. When the noise current fluctuations on the beam are less than full shot noise, additional noise can result if a portion of the beam is intercepted by an electrode in the path of the electrons. The magnitude of the additional beam noise depends upon the current intercepted and the portion of the beam from which it is removed. McIntosh (1958) has determined that aperture interception contributes less noise than a uniform interception across the beam section. For noise studies of the electron beam from a space-charge-limited, diode gun it is therefore desirable to minimize interception noise by employing sufficiently large apertures in the anode and subsequent tube structures.

A structural diagram of the gun studied is shown in Fig. 4-1. It employed an oxide cathode, .090" in diameter, in front of which was a beam forming electrode of the required Pierce shape. An aperture .031" in diameter in the beam forming electrode determined the diameter of the electron beam leaving the cathode. The anode-cathode spacing was 0.156" which includes a .004" retraction of the cathode behind the beam forming electrode, in order to provide thermal isolation. To

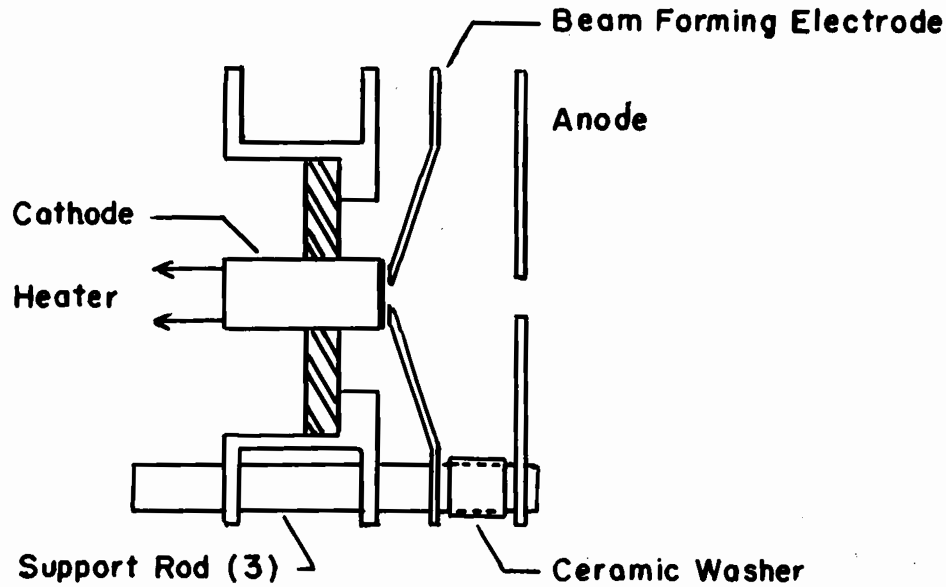

Fig. 4-1 The Electron Gun

simplify construction the anode was approximated by a plane. The two tubes used at 1,400 Mc/s. and 4.250 Mc/s. had an anode aperture .062" diameter and interception of beam current by the anode was small. To improve the coupling of the electron beam to the 9,520 Mc/s. cavity, the anode aperture was reduced to .031" and this resulted in an increase in interception. for the two higher frequency tubes.

To couple the noise power resulting from beam current fluctuations to the microwave receiver, the electron beam was passed through a cavity, resonant at the frequency of interest. By using a short gap it was assured that the output noise power coupled to the receiver would be directly proportional to the total mean square current fluctuation in a frequency interval centred at the resonant frequency of the cavity. A cavity which has a very short gap will produce no output arising from velocity fluctuations on the electron beam. By locating the cavity just beyond the electron gun anode, measurements of noise smoothing were made at 1,400 Mc/s, 4,250 Mc/s, and 9,520 Mc/s.

Two cavity arrangements have been used. The tubes for measurements at 1,400 Mc/s. and 4,250 Mc/s. employed a disc seal structure in which the cavity interaction gap was interior to the glass envelope. The remaining cavity structure was clamped to two copper discs passing through the envelope and by varying the size of the external structure, the different resonant frequencies were obtained. The noise power was coupled from the cavity with a small loop. At 9,520 Mc/s. the small cavity size required that it be placed entirely within the vacuum envelope. A coaxial line through the envelope was used to couple the cavity to the receiver by means of an E field probe in 3 cm. waveguide.

One wall of the 9,520 Mc/s. cavity acted as the electron gun anode. The tubes are shown in Figs. 4-2 and 4-3 and their electrical and mechanical properties are summarized in Table 4-1.

The increase of interception as the anode aperture size was decreased is apparent. It is also noted that the tubes with the highest perveance values had the lowest values of interception. The results are interpreted to mean that the cathodes of tubes D-1 and I-2 were positioned farther behind the beam forming electrode than for optimum operation. This not only resulted in an electron beam of smaller diameter, but this beam was less well focused due to the inaccuracies of potential matching at the beam edge.

All tubes employed long cylindrical electron collectors, biased at 45 volts above the anode potential. This was done to prevent secondary electrons produced upon impact of the primary beam from contributing additional noise at the measurement cavity. As the result of beam spreading, due to space-charge, after drifting beyond the cavity, the effect of elastically reflected primary electrons was also minimized.

Sealed to all of the tubes was an ionization gauge which served two functions. The gauge provided a continuous indication of the pressure within the vacuum envelope. It also acted as a pump to remove from the tube volume any small quantities of gas evolved from metal parts. During operation pressures of 5×10^{-9} mm of Hg were typical.

Fig. 4-2 The Disc Seal Tubes and 4,250 Mc/s. Cavity.

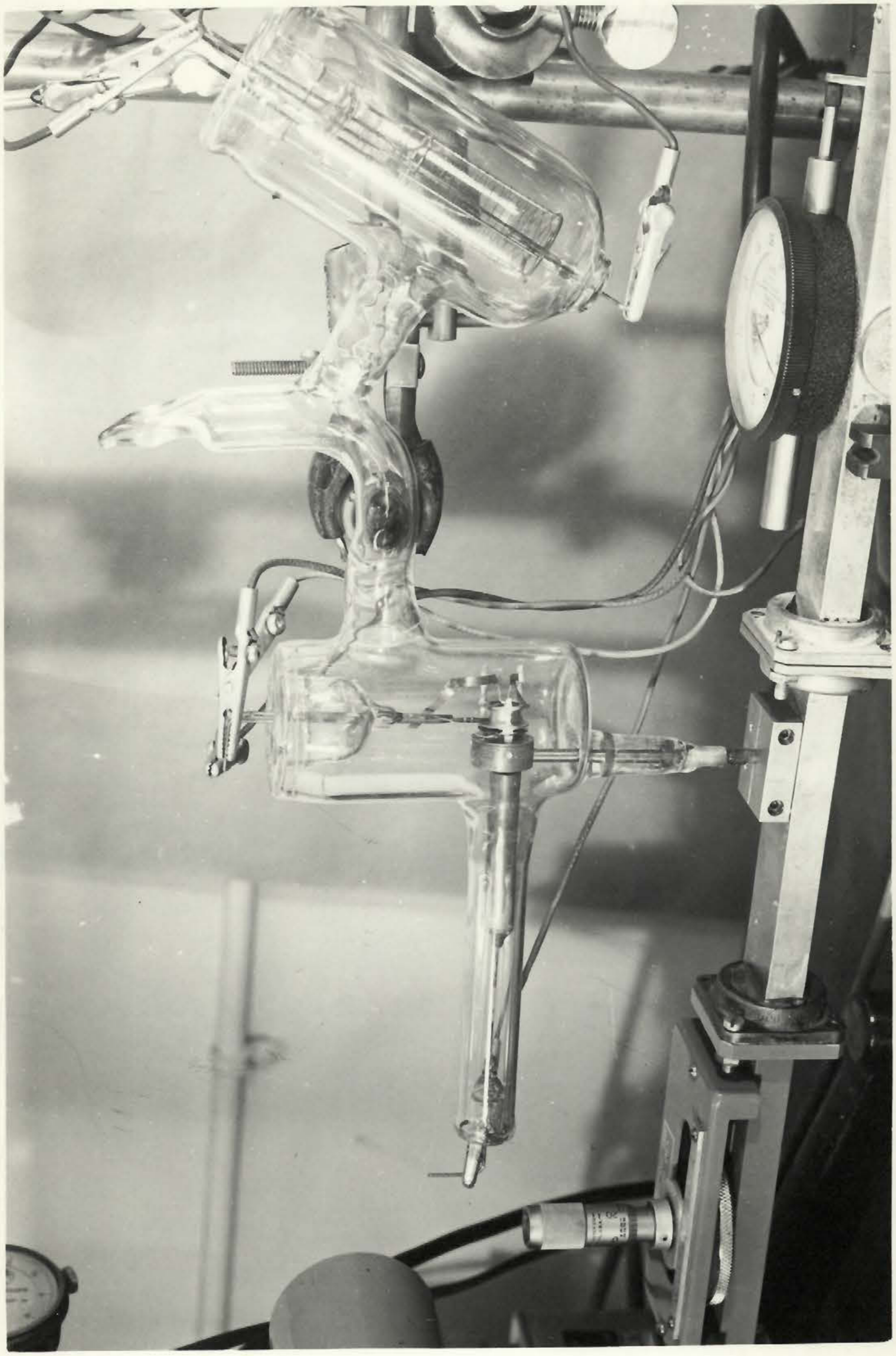

Fig. 4-3 Internal Cavity Tube for 9,520 Mc/s.

TABLE 4-1

Electrical and Mechanical Properties of the Tubes Studied

Tube Number

	D-1	D-2	I-1	I-2
Cavity Type	External	External	Internal	Internal
Anode and Cavity Aperture	.062"	.062"	.031"	.031"
Frequency	4,030 Mc/s. 1,395 Mc/s.	4,250 Mc/s. 1,455 Mc/s.	9,520 Mc/s.	9,520 Mc/s.
Average Perveance Amp/Volts$^{3/2}$	2.46×10^{-8}	3.82×10^{-8}	3.90×10^{-8}	2.43×10^{-8}
Interception	3.8%- 1.5% 200-500volts	0.2%- 0.02% 200-500volts	2.8%	7%

5. MEASUREMENTS OF ANODE NOISE CURRENT FLUCTUATIONS

The electron gun which was studied produced a parallel beam of electrons. No axial magnetic field was used to confine the beam. Devices employing cylindrical electron beams usually require such a field to prevent beam-spreading due to space-charge repulsion. In the present work, noise measurements were made on the beam as it left the anode aperture, and at the resonant cavity gap, no significant increase in the beam diameter was expected.

The experiment consisted of raising the cathode temperature and observing the magnitude of the noise power coupled to the resonant cavity at the gun anode, as the beam current increased to its space-charge-limited value. The form of the variation of noise with d.c. beam current is shown in Fig. 5-1. Two distinct regions of the curve are apparent. At low beam currents the diode gun was temperature-limited, and the initial shot noise fluctuations present on the beam were unmodified at the electron gun anode. As the cathode temperature was raised, the beam current increased to its space-charge-limited value, and the greater charge density within the gun resulted in a decrease of the anode noise current fluctuations. Extrapolation of the linear portion of the curve serves to determine the full shot noise content of the beam for any current value. This line is called the Shot Noise Asymptote. The noise smoothing is defined from the ratio of measured noise power at space-charge limitation, to that value of shot noise indicated by the Shot Noise Asymptote for the space-charge-limited current. This form of normalization is particularly convenient. It obviates the necessity of absolute noise power measurements, which although possible,

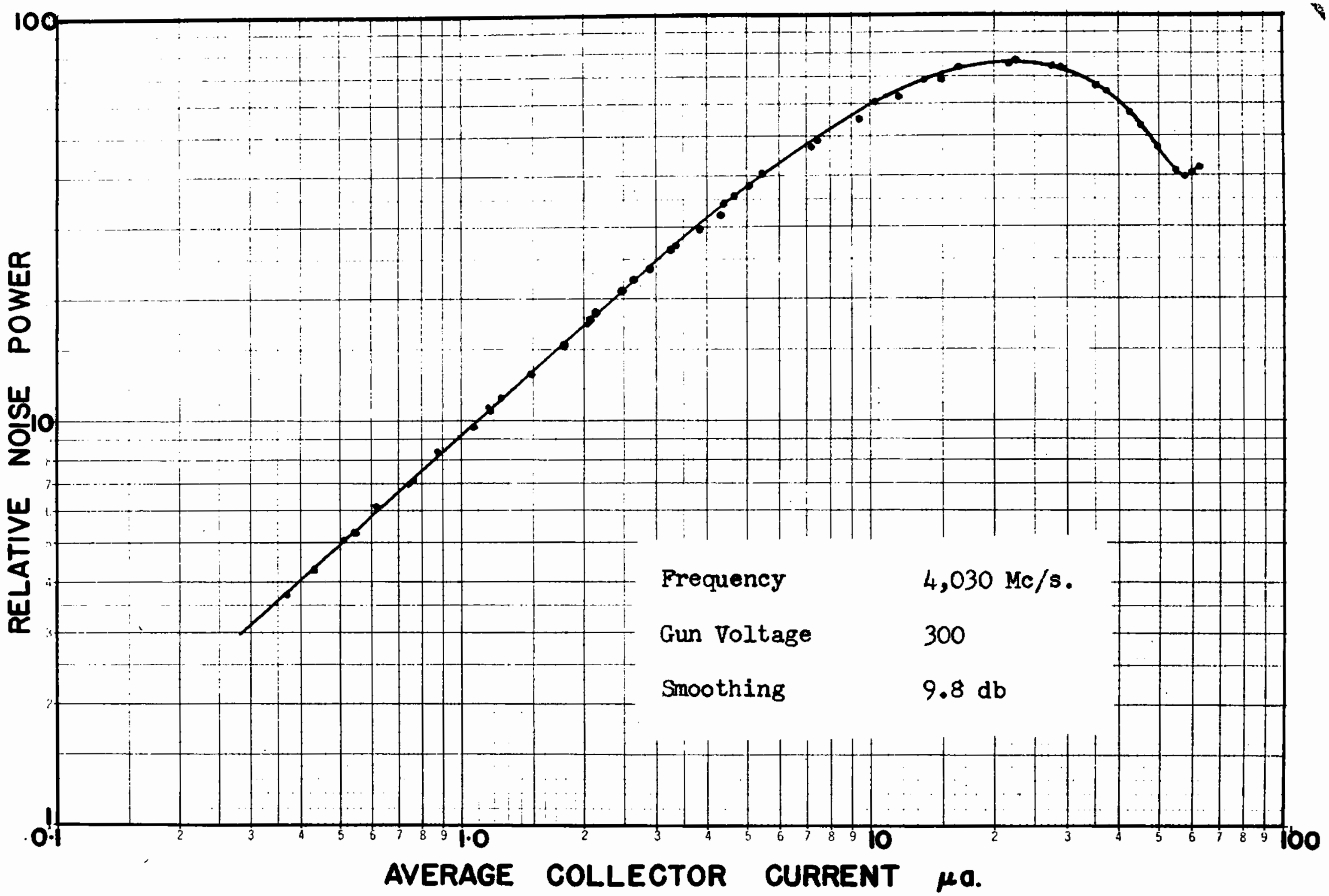

Fig. 5-1 Electron Beam Noise at the Gun Anode as a Function of Current.

require considerably greater experimental complexity.

The quadratic content of shot noise is directly proportional to the d.c. beam current, and it would be expected that when plotted as in Fig. 5-1 the Shot Noise Asymptote would have a unity slope. This was observed at 1,400 Mc/s. and 9,520 Mc/s. At 4,250 Mc/s. however the slope was found to be less than unity. The slope varied regularly from 0.88 with 200 volts applied to the gun to essentially unity at 1000 volts and was consistent to 1% for both disc seal tubes. The smoothing values quoted are relative to this Shot Noise Asymptote. A similar effect has been observed by McIntosh (1958) working with magnetically confined beams at 3,000 Mc/s. It is possible that radial variations in beam current density, or changes in beam diameter as the current increased, resulted in changes in coupling to the resonant cavity.

It is convenient to first summarize the noise smoothing values observed and compare results with theoretical values when these calculations are presented. The experimental results are tabulated in Table (5-1).

TABLE 5-1

Observed Values of Anode Noise Smoothing, db below Shot Noise

Gun Anode Voltage	Tube D-1 Anode Noise Smoothing, db		Tube D-2 Anode Noise Smoothing, db	
	1,395 Mc/s.	4,030 Mc/s.	1,455 Mc/s.	4,250 Mc/s.
100	6.5	-	6.7	-
150	-	-	-	7.5
200	9.7	-	9.6	8.6
300	11.0	9.8	11.0	10.2
400	-	-	12.3	11.1
500	12.9	11.2	13.1	11.4
600	-	11.4	-	-
700	-	12.0	-	12.2
1,000	-	13.0	-	-

Gun Anode Voltage	Tube I-1 Anode Noise Smoothing, db	Tube I-2 Anode Noise Smoothing, db
	9,520 Mc/s.	9,520 Mc/s.
300	7.3	8.5
400	8.0	9.1
500	9.1	9.8
600	9.2	10.6
700	10.0	10.1
1,000	-	10.2

6. CALCULATION OF ANODE NOISE QUANTITIES

Three methods of calculating the smoothing of the noise current fluctuations at the gun anode have been used:

a) The Llewellyn Peterson Equations.

b) A computer solution of the modified Electronic Equation with the input plane at the potential minimum. Full shot noise and Rack velocity fluctuations were assumed as input conditions.

c) A computer solution of the modified Electronic Equation with the input plane beyond the potential minimum. The required noise input conditions were determined from the results of density function calculations (Siegman et al 1957).

In the computer solutions, values for the current density J_o, and the beam diameter 2b, were required. The current density in a space-charge-limited planar diode is given by the well known three-halves power of voltage relation:

$$J_o = 2.33 \times 10^{-6} \frac{V_{oa}}{d^2}^{3/2} \qquad (6\text{-}1)$$

where d is the anode-cathode spacing. In the present work it was assumed that the d.c. relations for a Llewellyn gap were applicable.

$$u_{oa} = \frac{1}{2} \frac{eJ_o}{m\varepsilon_o} \tau^2 + u_{om} \qquad (6\text{-}2)$$

$$d = \frac{1}{6} \frac{eJ_o}{m\varepsilon_o} \tau^3 + u_{om}\tau \qquad (6\text{-}3)$$

Under the assumption $u_{oa} \gg u_{om}$, equation (6-1) follows from (6-2) and (6-3). The input plane for the Llewellyn gap was taken at the potential minimum, and no initial acceleration has been included.

The perveance measurements at different anode voltages made on tubes D-2 and I-1 showed a maximum deviation from the mean of less than ten percent. These tubes had the lower value of interception, each for its own type, and showed superior gun performance. The mean perveance of the two tubes was found to be 3.86×10^{-8} Amp/Volt$^{3/2}$ and this was considered representative of a properly operating gun.

An effective beam diameter was determined by comparing the measured perveance with the theoretical current density. The result was a value of 0.0226", which is 73% of the aperture in the beam forming electrode. As a result of retracting the cathode to provide thermal isolation, the beam area was reduced in the region between the cathode and beam forming electrode.

a) <u>The Llewellyn Peterson Equations</u>

In a Llewellyn diode, two uncorrelated inputs were assumed: full shot noise current fluctuations and Rack velocity fluctuations. The mean square output current for each input was evaluated. The input plane was taken at the potential minimum and an input d.c. velocity equal to the R.M.S. thermal velocity was assumed. Using the coefficients quoted by Pierce (1950) and normalizing the total mean square anode current fluctuations to shot noise, the smoothing is given by,

$$\delta_a^{\ 2} = \frac{V_{oi}}{V_{oa}} + \frac{9}{8}\,\frac{\omega^2 d^2}{V_{oa}^{\ 2}}\,(4-\pi)\,\frac{kT_c}{m} \qquad (6\text{-}4)$$

where V_{oi} is the voltage equivalent of the R.M.S. thermal velocity and is given by $\frac{kT_c}{e}$.

V_{oa} is the gun anode voltage.

d is the cathode-anode spacing.

T_c is the cathode temperature.

k is Boltzmann's constant.

m is the mass of the electron.

ω is the frequency of interest.

With a cathode temperature of 1160°K and the present gun constants, equation (6-4) becomes,

$$\delta_a^2 = \frac{0.1}{V_{oa}} + 8.62 \times 10^{-18} \frac{\omega^2}{V_{oa}^2} \qquad (6\text{-}4a)$$

Calculations were made at 1,400 Mc/s, 4,250 Mc/s, and 9,520 Mc/s. The first term in (6-4a) is usually very small and becomes negligible at very high frequencies.

b) Computer Solution I

The solution of the modified Electronic Equation was carried out between the potential minimum and the electron gun anode. The input conditions were identical to those used for the Llewellyn Peterson calculation. The value of smoothing at the gun anode was determined from equation (3-43a).

c) Computer Solution II

The evaluation of the anode noise quantities required the specification of the plane at which the calculation was to start, of the four fluctuation terms (3-28 a, b, c, d) at this plane, and of the transit time from the potential minimum to the input plane. As noted earlier the choise of an input plane is a compromise. For the present work a plane at $\xi = 5$ was convenient for computer operation. The potential at this plane is 0.359 volts above the minimum of potential in the gun and at this plane the parameters S and Π have reached their final values.

The input noise quantities depend upon the frequency parameter

$a = \omega/2\omega_{pm}$ for which a value was required at each gun voltage and frequency. The plasma frequency at the potential minimum was determined from equations (3-35) and (6-1). As in the case of the other theoretical calculations, the frequencies studied were 1,400 Mc/s., 4,250 Mc/s., and 9,520 Mc/s. Table (6-1) shows the frequency parameter at different gun voltages for these frequencies.

TABLE 6-1

The Frequency Parameter as a Function of Frequency and Gun Voltage

Gun Voltage	Frequency Parameter "a"		
	1,400 Mc/s.	4,250 Mc/s.	9,250 Mc/s.
100	0.845	2.53	5.67
200	0.502	1.50	3.37
300	0.371	1.11	2.49
400	0.299	0.894	2.00
500	0.253	0.756	1.69
600	0.220	0.660	1.48
700	0.196	0.588	1.32
1,000	0.150	0.450	1.01

Siegman et al have published values for the noise quantities as a function of ξ, for frequency parameters a = 1/4, 1/2, and 1. A new definition of kinetic voltage was used for the a = 1/4 computation, which reduces to the conventional definition (3-16) for large d.c. velocities.

In terms of the density function for the electron beam,

$$F(x, u, t) = F_0(x, u) + F_1(x, u)\, e^{j\omega t} \qquad (3\text{-}26)$$

$$V = -\frac{1}{4 J_o \eta} \int \left[W - \bar{W} \right] F_1(x, W)\, dW \qquad (6\text{-}5)$$

where
$$\bar{W} = \frac{1}{2J_o} \int W F_o\ (x, W)\, dW \qquad (6\text{-}6)$$

$$W = u^2 - 2\eta V_o(x) \qquad (6\text{-}7)$$

η = the charge to mass ratio for the electron.

The variable W is related to the excess energy of the electron over its d.c. energy due to acceleration. Examination of equation (6-5) shows that the new definition of kinetic voltage is similar to but not identical with a summation over the individual a.c. velocities of the electrons in the beam. The definition of the kinetic voltage in terms of the density function is such that the kinetic voltage is continuous across a velocity jump in which the beam is rapidly accelerated; the density function definition on the multivelocity side of the jump agreeing with the single velocity theory on the high voltage side. The discrepancy between the two definitions at the ξ = 5 plane was not known and with one exception the published noise quantities were used.

To determine the noise input terms for frequency parameters other than those calculated by Siegman, the input plane was fixed at ξ = 5, and the values of δ^2, μ^2, Π and Λ were plotted as a function of a. These curves permitted interpolation at the required frequency parameters. For the 100 volt 4,250 Mc/s case, and the 300 volt 9,520 Mc/s. case, an extrapolation to a value of a = 2.5 was made. There is considerable doubt as to the accuracy of the noise input parameters obtained, although all calculated quantities based on this extrapolation are reasonable. The exception to the published data was the value for δ^2 at a = 1/4. Siegman quotes a value of δ^2 = 0.133. This did not fall on the straight line plot of δ^2 vs a, which was found for δ^2 values at a = 0, a = 1/2 and a = 1. In addition the 0.133 value was not consistent with the published value of $\mathcal{S}$. For the present work δ^2 at a = 1/4 was taken to be 0.165 as this value minimized the above discrepancies.

The noise input quantities at ξ = 5 as published by Siegman et al (1957) are presented in Table (6-2).

<u>TABLE 6-2</u>

Noise Parameters at the plane ξ = 5 as calculated by Siegman, Watkins and Hsieh (1957).

Frequency Parameter

a =	1/4	1/2	1
δ^2	0.165(0.133)	0.265	0.466
μ^2	8.55	2.91	1.35
Π	0.315	0.275	0.198
$\mathcal{A}$	1.11	0.585	0.394

The determination of the transit time τ_s depends upon the initial d.c. velocity assumed to characterize the beam at the potential minimum. The value used for u_{om} can only approximate conditions for the multi-velocity beam. The larger the value selected for u_{om}, the smaller the transit time τ_s and its associated constant λ, defined in equation (3.40). All other parameters remaining unchanged, the smaller value of λ would result in a decrease of δ_a^2.

Siegman shows that the use of the value of u_{om}^2 defined in equation (3-34), for the calculation of δ^2 using the Llewellyn Peterson equations, gives good agreement at a = 0 and a = 1/4, but predicts values for δ^2

greater than that found from the statistical computations for frequency parameters of 1/2 and 1. The present calculations were based on a value of u_{om}^2 equal to twice that given by equation (3-34). This resulted in a smaller value of λ and it is believed that more accurate determinations of δ_a^2 were obtained at larger values of the frequency parameter.

A method exists for determining λ which is based on the Fry-Langmuir solution and does not require the assumption of a d.c. velocity at the potential minimum. This method had not been developed at the time the computer solution was carried out, and is presented here only to provide a basis on which to criticize the assumptions actually made.

Noting the definition of λ,

$$\lambda = K\tau_s \frac{\alpha}{\rho} \frac{R_2 C_2}{u_{oi}} = \frac{\alpha}{\rho} R_2 C_2 \left. \frac{du_o}{dz} \right|_{u_{oi}} \qquad (6\text{-}8)$$

By changing to the Fry-Langmuir dimensionless variables and carrying out the differention of u_o it is found,

$$\lambda = \frac{\alpha}{\rho} R_2 C_2 \left[\frac{\pi m}{2kT_c}\right]^{1/4} \frac{K^{1/2}}{[\eta(\xi)]^{1/2}} \frac{d\eta}{d\xi} \qquad (6\text{-}9)$$

where the functions of the dimensionless variables are to be evaluated at the input plane. For the present case $\eta(\xi) = 3.59$ and $d\eta/d\xi = 1.2$ (Kleynen 1946). Calculations reveal that λ is independent of the assumed gun voltage. The value of λ used in the computer solution was 0.736 while that determined by equation (6-9) is 0.795. Both of these lie between the values for λ obtained using equations (3-30) and (3-34).

Two restrictions exist on the four noise quantities calculated at the electron gun anode. The transformation of the noise along the accelerated stream must preserve the invariance of S and Π. The values of $\mathcal{S}_a$ and $\mathcal{T}_a$ depend not only on the four input noise quantities, but also on an undetermined correlation coefficient K_2. In the present work the value of K_2 was determined from equation (3-41c) by imposing the necessary invariance of $\mathcal{T}$. The real kinetic power term was selected as it is much more strongly dependent on K_2 than is $\mathcal{S}_a$. For all of the cases studied K_2 was less than its maximum permissible value of unity. The corresponding values of K_1 were also less than unity, with three exceptions. The 200 volt 1400 Mc/s. calculation and the 1,000volt 4,250 Mc/s. calculation indicated a value of K_1 equal to 1.011 and the 500 volt 1,400 Mc/s. calculation resulted in a value of K_1 equal to 1.035. The discrepancies were not considered serious in view of the inaccuracies in the assumed input noise quantities. However, they were interpreted as indicating that the validity of the method of calculation is open to question for values of the frequency parameter $a = \frac{\omega}{2\omega_{pm}}$ less than 0.25. The discrepancy is diminished for lower values of λ; which provides at least an empirical justification for the value assumed. The difficulty can be removed for the lower frequencies by advancing the input plane toward the anode. This appears to imply the necessity of a minimum transit angle between the potential minimum and the plane at which the computation is started.

The computer constants a_o, a_1, c_o, c_1 were determined as described in Appendix I and the solution was complete. The results are summarized in Table (6-3). Both input and anode noise quantities are presented.

Frequency	Gun Voltage	δ^2		μ^2		Π		A		S	
		Input	Anode	Input	Anode	Input	Anode	Input	Anode	Input	Anode
1,400 Mc/s.	100	0.412	1.036×10^{-1}	1.59	151.3	0.220	0.220	0.428	3.894	0.686	0.718
	200	0.265	5.15×10^{-2}	2.91	364.1	0.275	0.275	0.585	4.277	0.655	0.690
	300	0.204	3.34×10^{-2}	4.30	-	0.297	-	0.701	-	0.622	-
	500	0.166	2.00×10^{-2}	8.55	857.1	0.315	0.315	1.00	4.075	0.648	0.717
4,250 Mc/s.	100	0.800	1.593×10^{-1}	0.65	84.0	0.135	0.135	0.282	3.581	0.663	0.751
	300	0.502	5.77×10^{-2}	1.24	461.1	0.182	0.182	0.374	5.099	0.695	0.771
	500	0.377	3.43×10^{-2}	1.81	858.4	0.233	0.233	0.460	5.382	0.685	0.714
	1,000	0.242	1.55×10^{-2}	3.33	1660.4	0.283	0.283	0.624	5.015	0.645	0.710
9,520 Mc/s.	300	0.800	8.81×10^{-2}	0.65	314.4	0.135	0.135	0.282	5.203	0.663	0.784
	1,000	0.466	2.63×10^{-2}	1.35	1944.2	0.198	0.198	0.394	7.114	0.688	0.701

TABLE 6-3

Summary of Theoretical Input and Anode Noise Quantities

7. COMPARISON OF EXPERIMENTAL AND THEORETICAL RESULTS

The experimental values of noise smoothing observed at the gun anode are compared with the results of the three theoretical determinations in Figs. 7-1, 7-2, and 7-3. In Fig. 7-1 the 1,400 Mc/s. measurements on tubes D-1 and D-2 agreed to well within experimental error and they have been shown as one measurement. At 4,250 Mc/s. only slight differences between the tubes were observed, while at 9,520 Mc/s. two distinct sets of measurements appeared.

The predictions of the Llewellyn Peterson equations and the first computer solution are strong functions of frequency, and as would be anticipated, the agreement between them improves as the frequency is increased. Both indicate a more rapid change of smoothing with gun voltage than is observed experimentally.

The second computer solution predicts a variation of smoothing with gun voltage which is in good agreement with that observed. At all frequencies the theoretical smoothing is consistently greater than that observed; the discrepancy being somewhat larger at higher gun voltages. An interesting consistency is observed between the three sets of calculated smoothing curves. They are parallel; the higher frequency curves being displaced toward lower values of smoothing. A better indication of the theoretical dependence of smoothing on frequency and gun voltage can be seen from examination of the quantity $\delta_a^2 V_{oa}/\omega^{1/2}$. The calculated values for this quantity are tabulated in Table (7-1) for all of the frequencies and voltages examined. The experimental values will be presented later.

The quantity $\delta_a^2 V_{oa}/\omega^{1/2}$ appears to be independent of frequency

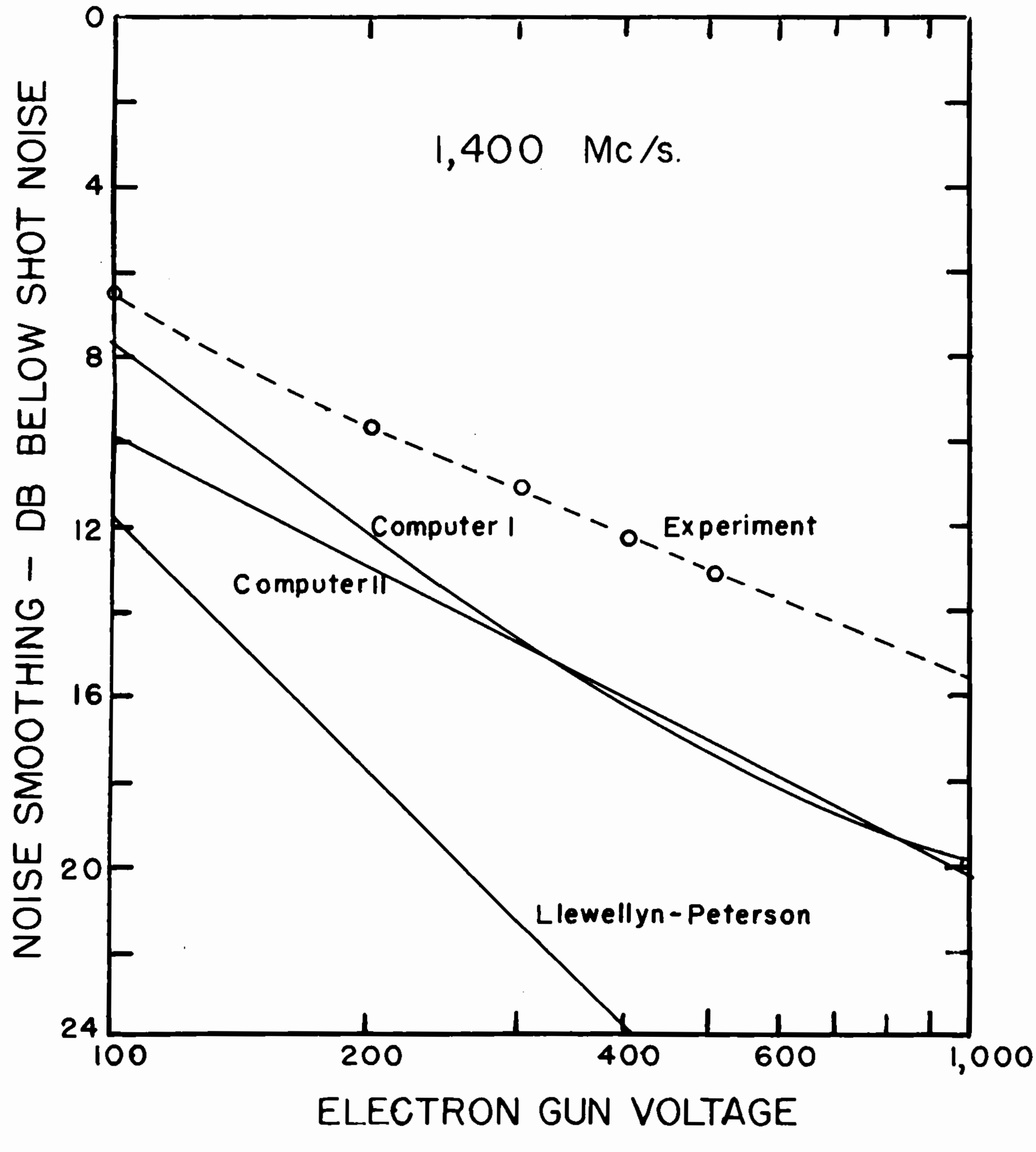

Fig. 7-1 Smoothing as a Function of Gun Voltage

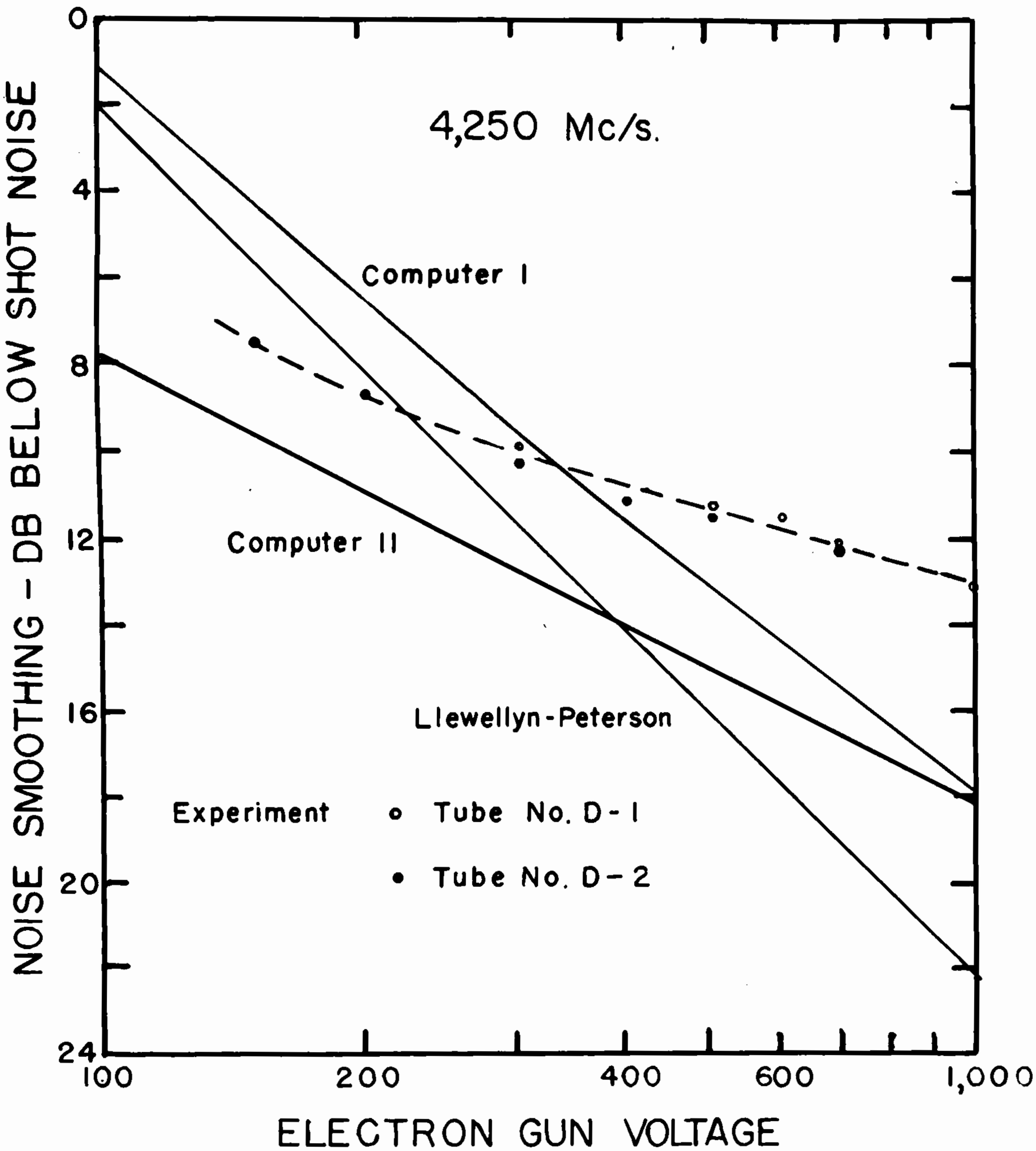

Fig. 7-2 Smoothing as a Function of Gun Voltage

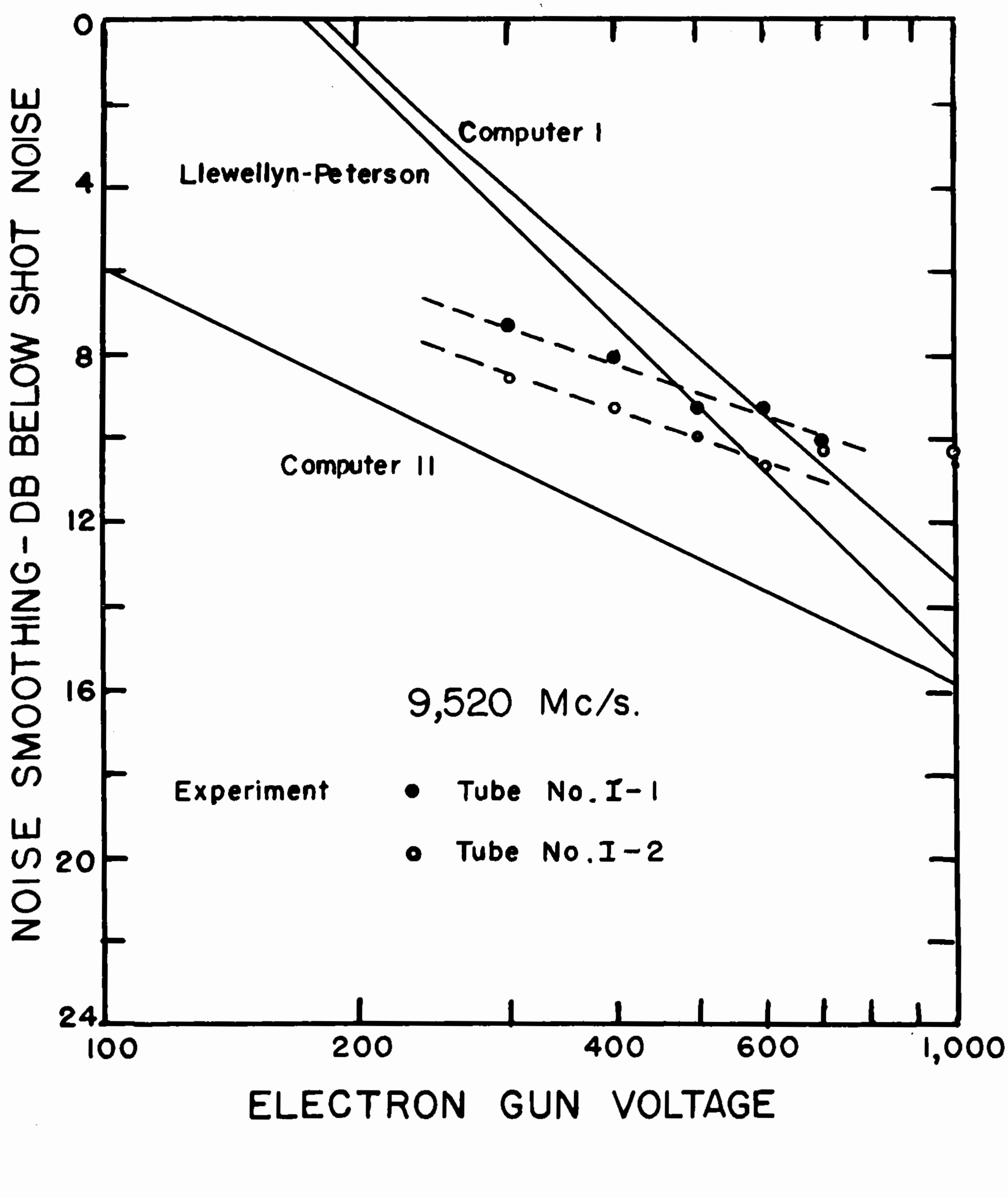

Fig. 7-3 Smoothing as a Function of Gun Voltage

TABLE 7-1

Frequency	Gun Voltage	Theoretical $\delta_a^2 V_{oa}/\omega^{1/2}$
1,400 Mc/s	100	1.1049×10^{-4}
	200	1.0991×10^{-4}
	300	1.0669×10^{-4}
	500	1.0650×10^{-4}
4,250 Mc/s	100	0.9750×10^{-4}
	300	1.0590×10^{-4}
	500	1.0508×10^{-4}
	1,000	0.9457×10^{-4}
9,520 Mc/s	300	1.0804×10^{-4}
	1,000	1.0745×10^{-4}

and to decrease very slightly with increasing voltage. The dependence on gun voltage is somewhat uncertain in view of inaccuracies of the input data, but with the exception of the 100 volt 4,250 Mc/s case, the behaviour is remarkably consistent and is believed to be real.

To examine the experimental measurements for this form of variation of smoothing with frequency and gun voltage, the data of Table(5-1) has been used to evaluate $\delta_a^2 V_{oa}/\omega^{1/2}$, and the results are presented in Table (7-2). The measurements have been made over a frequency range of 6.8:1 and the variation of smoothing with the square root of frequency is apparent. A dependence of the measured smoothing on the square of the frequency, which is predicted by the Llewellyn Peterson equations, would not result in the consistency of values for $\delta_a^2 V_{oa}/\omega^{1/2}$ which is indicated in Table (7-2).

The experimental results of Vessot (1957) and McIntosh (1958) for a higher perveance electron gun of similar design, have been analyzed

TABLE 7-2

Observed Values of $\delta_a^{\,2} V_{oa}/\omega^{1/2}$

Gun Anode Voltage	Tube D-1 $\delta_a^{\,2} V_{oa}/\omega^{1/2}$		Tube D-2 $\delta_a^{\,2} V_{oa}/\omega^{1/2}$	
	1,395 Mc/s.	4,030 Mc/s.	1,455 Mc/s.	4,250 Mc/s.
100	2.41×10^{-4}	-	2.26×10^{-4}	-
150	-	-	-	1.65×10^{-4}
200	2.29×10^{-4}	-	2.28×10^{-4}	1.70×10^{-4}
300	2.54×10^{-4}	2.03×10^{-4}	2.48×10^{-4}	1.76×10^{-4}
400	-	-	2.49×10^{-4}	1.90×10^{-4}
500	2.73×10^{-4}	2.36×10^{-4}	2.58×10^{-4}	2.21×10^{-4}
600	-	2.74×10^{-4}	-	-
700	-	2.76×10^{-4}	-	2.60×10^{-4}
1,000	-	3.14×10^{-4}	-	-

Gun Anode Voltage	Tube I-1 $\delta_a^{\,2} V_{oa}/\omega^{1/2}$	Tube I-2 $\delta_a^{\,2} V_{oa}/\omega^{1/2}$
	9,520 Mc/s.	9,520 Mc/s.
300	2.28×10^{-4}	1.73×10^{-4}
400	2.62×10^{-4}	2.01×10^{-4}
500	2.52×10^{-4}	2.14×10^{-4}
600	2.94×10^{-4}	2.14×10^{-4}
700	2.86×10^{-4}	2.80×10^{-4}
1,000	-	3.90×10^{-4}

to discover if a similar behaviour of the noise was present. The data is presented in Table (7-3).

TABLE 7-3

Values of $\delta_a^2 V_{oa}/\omega^{1/2}$ measured by Vessot and McIntosh

Gun Anode Voltage	$\delta_a^2 V_{oa}/\omega^{1/2}$			
	230 Mc/s.	1010 Mc/s.	3055 Mc/s.	3050 Mc/s.
100	2.34×10^{-4}	1.70×10^{-4}	1.55×10^{-4}	-
200	1.74×10^{-4}	1.41×10^{-4}	2.09×10^{-4}	-
300	1.85×10^{-4}	1.40×10^{-4}	2.07×10^{-4}	1.72×10^{-4}
400	1.67×10^{-4}	1.42×10^{-4}	1.91×10^{-4}	-
500	1.77×10^{-4}	1.38×10^{-4}	-	1.94×10^{-4}
600	1.69×10^{-4}	1.17×10^{-4}	2.07×10^{-4}	-
700	-	-	1.97×10^{-4}	1.92×10^{-4}
900	2.04×10^{-4}	1.38×10^{-4}	2.39×10^{-4}	-

The first three columns are from Vessot's measurements on his Tube No.2, and the final column is based on McIntosh's oxide cathode measurements. No theoretical calculation of $\delta_a^2 V_{oa}/\omega^{1/2}$ has been made for this gun, although the measurements show a variation with frequency in reasonable agreement with the form predicted for the present electron gun.

8. DISCUSSION OF RESULTS

a) Reliability of Measurements

The noise power coupled from the electron stream was considerably greater for the disc seal tubes than that available from the tubes using internal cavities. The measurements made at the two lower frequencies are therefore more accurate than those at 9,520 Mc/s. At 1,400 Mc/s. and 4,250 Mc/s. the Shot Noise Assymptote was determined by a least-squares fit of the low current data to a straight line. A R.M.S. error of fit of .05 db was typical and the major errors in the determination of a smoothing value were therefore associated with the measurement of the space-charge-limited noise power and beam current. The reproducibility of the space-charge-limited noise power measurement was about 0.1 db. Current measurements were made with an estimated accuracy of 2%. Considering receiver or recorder non-linearity of about 0.1 db, the overall accuracy of the normalized noise power measurements at 1,400 Mc/s. and 4,250 Mc/s. is estimated to be ± 0.3 db.

The low noise powers observed at 9,520 Mc/s. required integration periods as long as 20 minutes for the measurement of the noise on beams of very low currents. A complete smoothing determination extended over several hours, during which time the local oscillator in the receiver was required to remain constant in frequency. The oscillator frequency was not stabilized and any drift during the measurement period contributed additional errors. D.C. drift in the Lock-In Amplifier and changes in modulation frequency could result in additional inaccuracies. At 9,520 Mc/s. the normalized noise power measurements are estimated to have an accuracy of ± 0.6 db.

b) Measured Smoothing Values

The determination of accurate smoothing values requires not only measuring instrument accuracy, but also the reduction below a significant level, of the noise contributions of sources within the tube in addition to that being studied. The interception of part of a smoothed beam can result in an increase of the noise content of the remaining beam. It has been observed (McIntosh 1958) that the magnitude of this increase for small amounts of aperture interception, is considerably less than the full shot noise associated with the intercepted beam. No accurate indication of the ratio of the additional noise to the shot noise value is available in the absence of a confining magnetic field. If this ratio is 20%, which is believed to be too high for the present work, errors of less than 0.2 db are introduced for the disc seal tube measurements and less than 0.5 db for the 9,520 Mc/s. measurements.

The observed values for $\delta_a^{\,2}\, V_{oa}/\omega^{1/2}$ showed a definite increase as the gun voltage was raised. This is attributed to the progressively higher cathode temperature required to space-charge-limit the gun. As the cathode temperature was increased the quadratic velocity input was proportionately increased, resulting in a higher value for the anode current fluctuations. The effect would be expected to be more pronounced at higher frequencies. The 1,000 volt measurements at 4,250 Mc/s. and 9,520 Mc/s., as shown in Table (7-2), gave very large values for $\delta_a^{\,2}\, V_{oa}/\omega^{1/2}$.

The measurements of Vessot at 230 Mc/s. and 1010 Mc/s. as shown in Table (7-3) indicated a decrease of $\delta_a^{\,2}\, V_{oa}/\omega^{1/2}$ as the gun voltage was raised. There is some indication of the effect of increased cathode temperature on the 900 volt measurements.

e) The Computer Solution II

The computed values of $\mathcal{S}$ at the electron gun anode exceed the assumed input value by about 10%. This fact, with the conservation of the real kinetic power, gives some confidence in the accuracy of the method of calculation. It was observed that for all of the calculations the quantity

$$a_1c_o - a_oc_1 = a_ob_1 - a_1b_o = 1$$

within the error of the determination of the constants. It is believed that this is a fundamental property of the solution. Examination of equation (3-43e) might have caused this equality to be anticipated.

The present method does not predict the observed (Kornelsen 1957) deviation of δ_a^2 from the one-half power dependence on frequency. This occurs at frequencies lower than those at which calculations were made. In the finite beam equation (Parzen 1952) the restriction was made

$$\frac{\omega_p}{\omega} << 1 \qquad (3\text{-}12)$$

This would appear to imply a minimum frequency parameter below which the description presented is no longer valid. The dependence of the associated critical frequency on anode voltage and cathode temperature can be demonstrated. If a_{min} is the lower limit for the frequency parameter, the critical frequency is

$$\omega_c = 2\omega_{pm}\, a_{min}$$

$$= \text{const.}\ \frac{J_o^{1/2}}{T_c^{1/4}} \qquad \text{from equation} \qquad (3\text{-}35)$$

$$= \text{const.}\ \frac{V_o^{3/4}}{T_c^{1/4}} \qquad (8\text{-}1)$$

for the case of a space-charge-limited diode gun. The deviation occurs over a range of frequencies and some criterion for specifying ω_c from measurements would need to be established before equation (8-1) could be experimentally examined.

The second computer solution incorporates two improvements over the earlier theoretical treatments. The analysis is restricted to a region in the electron gun, over which the small signal, single valued velocity assumptions made in the derivation of the Electronic Equation are not violated. The beam noise parameters at the input plane are specified with some certainty that they accurately characterize the beam conditions. The improved agreement between measured and theoretical smoothing values over a wide range of frequency and gun anode voltage, and the serious inaccuracies of the earlier theories, particularly at the higher frequencies with low anode voltages, are indicative of the importance of the multivelocity region beyond the potential minimum in determining the anode noise quantities.

The experimental values for the quantity $\delta_a^2\, V_{oa}/\omega^{1/2}$ are larger than those predicted from the present theory. This could be attributed to the existence of higher order space-charge modes within the gun (Rowe 1952), which have not been considered in the calculation.

9. CONCLUSIONS

1) A measuring system has been designed and evaluated for studies of noise in electron streams at microwave frequencies. The measured system performance is in excellent agreement with that calculated, and adequate sensitivity is available for noise measurements on electron beams as low as 1 μa.

2) Four sealed off vacuum tubes have been built and used to study the smoothing of shot noise within an electron gun. Data have been taken at 1,400 Mc/s., 4,250 Mc/s. and 9,520 Mc/s. No noise information at the highest of the three frequencies measured has heretofore been published.

3) Various theories for the transport of noise through accelerated electron streams have been studied. Using an electronic computer to solve a differential equation which describes the propagation of signals on electron beams, a method of calculating the four anode noise quantities has been developed which preserves the invariance of S and Π. A formulation of the output quantities has been determined from which earlier results are derivable as a special case.

4) From the theory an essential dependence of anode current noise on the one-half power of the frequency was discovered. This is completely at variance with existing theories not taking account of the multivelocity nature of the electron beam. The expected dependence was found in the measurements reported here, and in those of earlier workers. Some speculation has been made on experimentally examining the inadequacy of the theory at low frequencies. Further theoretical investigation of the effects of higher order space-charge modes is recommended, using a solution of the form presented in this work.

APPENDIX I

Analog Computer Solution of the Electronic Equation

a) Establishment of the Basic Relations

The equation to be solved on the computer is the finite beam Electronic Equation

$$\frac{\partial^2 Y}{\partial \tau^2} - K \frac{(1 - p^2)}{u_o} Y = 0 . \qquad (3\text{-}15)$$

In the gun region both u_o and p are functions of the transit time τ, and a function $f(\tau)$ can be defined

$$f(\tau) = \frac{1 - p^2}{u_o} . \qquad (\text{A1-1})$$

A formal integration of equatio n (3-15), using the definition A1-1, gives

$$Y = K \iint Y f(\tau)\, d\tau^2 + \left.\frac{\partial Y}{\partial \tau}\right|_{\tau=0} \tau + Y\Big|_{\tau=0} . \qquad (\text{A1-2})$$

The computer, set up as in Fig. A-1, solves the equation

$$\frac{d^2 V_1}{dt^2} - \left[\frac{R_d}{R_d + R_x}\right] \cdot \frac{1}{R_1 C_1 R_2 C_2} f(t) V_1 = 0. \qquad (\text{A1-3})$$

For the second integrator there is the relation

$$V_2 = - R_2 C_2 \frac{dV_1}{dt} . \qquad (\text{A1-4})$$

Equation A1-3 may be written in the integral form:

$$V_1 = \frac{R_d}{R_d + R_x} \frac{1}{R_1 C_1 R_2 C_2} \cdot \iint f(t) V_1 dt^2 + \left.\frac{dV_1}{dt}\right|_{t=0} t + V_1\Big|_{t=0} \qquad (\text{A1-5})$$

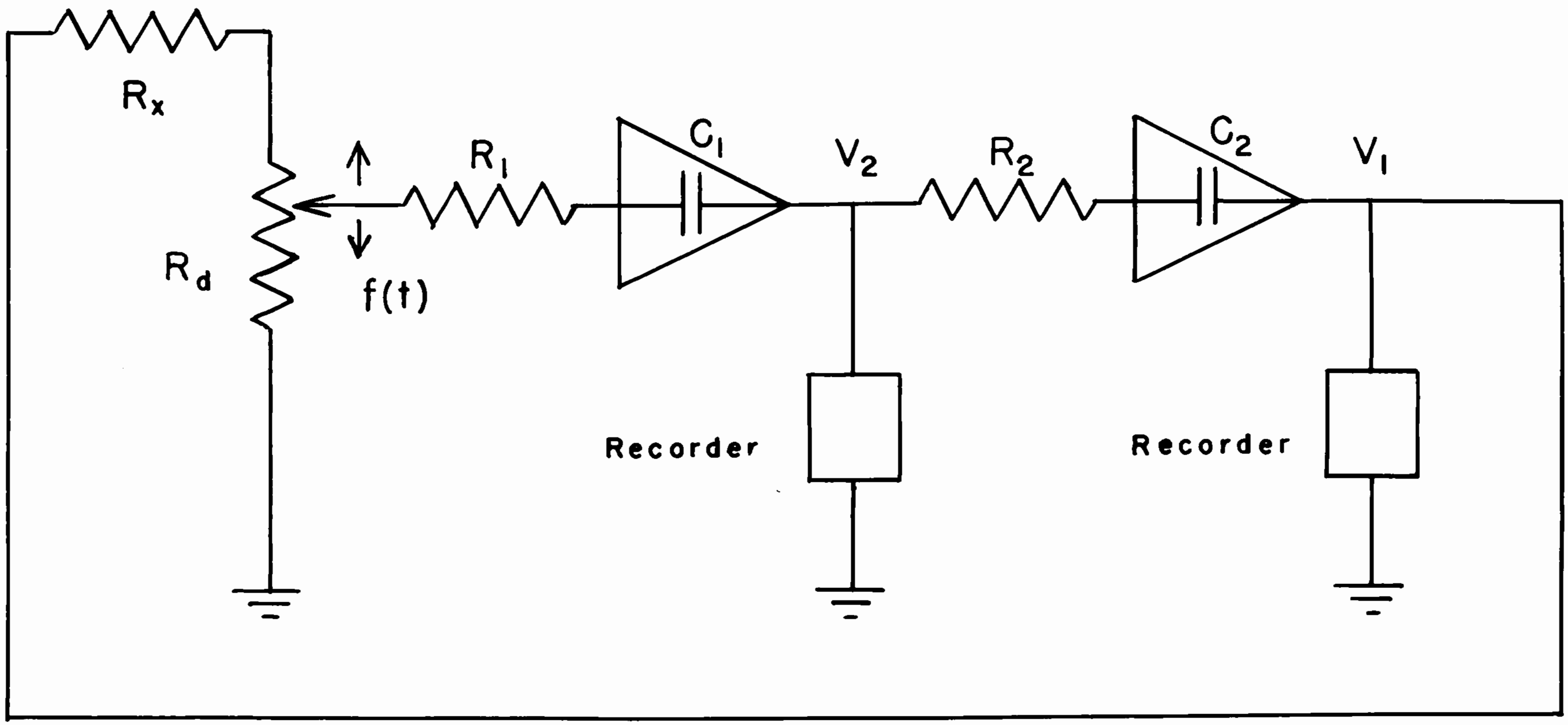

Fig. A-1 Analog Computer Schematic

The function f(t) is obtained by means of a tapped potentiometer, and it is arranged that it have a maximum value of unity.

For convenience in the comparison of equations (A1-2) and (A1-5), the independent variable in each is normalized to have a range of zero to one. Let

$$T = \alpha t = \rho \tau \qquad \text{(A1-6)}$$

such that the time corresponding to the gun anode

$$T_a = \alpha t_a = \rho \tau_a = 1 \qquad \text{(A1-7)}$$

and this equation serves to define α and ρ.

It is also convenient to normalize the function $f(\tau)$ of equation (A1-1). Define

$$F(T) = \sigma f(\tau) \qquad \text{(A1-8)}$$

where $0 \leqslant F(T) \leqslant 1$

and it is apparent that

$$\sigma = \frac{1}{f(\tau)\,\text{max}} \,.$$

The normalized forms of equations (A1-2) and (A1-5) can then be written:

$$Y = \frac{K}{\sigma \rho^2} \iint F(T)\, Y\, dT^2 + \left.\frac{dY}{dT}\right|_{T=0}^{T} + \left. Y \right|_{T=0} , \qquad \text{(A1-9)}$$

$$V_1 = \frac{\dfrac{R_d}{R_d + R_x}}{\alpha^2 R_1 C_1 R_2 C_2} \iint f(T)\, V_1\, dT^2 + \left.\frac{dV_1}{dT}\right|_{T=0}^{T} + \left. V_1 \right|_{T=0} . \qquad \text{(A1-10)}$$

These have the same form provided the function f(T) set on the computer is made identical to F(T). The equations become identical when the coefficients of the first terms are equated and this serves to determine the required value of R_x.

$$R_x = \left[\frac{\sigma \rho^2}{K\alpha^2 R_1 C_1 R_2 C_2} - 1 \right] R_d \qquad \text{(A1-11)}$$

Under the above conditions, Y and V_1 correspond exactly and can be related by a proportionately constant β;

$$Y = \beta V_1 \qquad \text{(A1-12)}$$

The above development is due to Kornelsen (1957) who proceeds to discuss the method of calculation of the normalized $\frac{(1 - p^2)}{u_o}$ function, F(T). The question of assuming an initial velocity again arises. Kornelsen used the R.M.S. thermal velocity, as was done for the first computer solution reported in the present work. The lengthy calculation of F(T) was not repeated for the second computer solution, which employed a value of u_{om} lower than the R.M.S. thermal velocity. The resulting error should be small, as the computation was started at a plane beyond the potential minimum.

b) <u>Calculation of Anode Noise Quantities</u>

The general case is examined, where the three independent noise inputs of equations (3-36) are specified at any plane, and the solution is carried out to determine the noise values at the anode. The algebra is very long, though straightforward, and only the procedure and intermediate results are indicated.

Omitting the phase term of equation (3-4) the input velocity modulation is

$$v_{\omega i} = -\frac{1}{j\omega J_o}\left[\frac{\partial Y}{\partial \tau} - \frac{Y}{u_{oi}}\frac{\partial u_o}{\partial \tau}\right] , \tag{A1-13}$$

$$= -\frac{\rho}{j\omega J_o}\left[\frac{\partial Y}{\partial T} - J_{\omega i}\cdot\frac{K\tau_s}{\rho}\right] . \tag{A1-14}$$

From (A1-4)

$$V_{2i} = -\alpha R_2 C_2 \frac{dV_1}{dT}\bigg|_{input} = -\alpha\frac{R_2 C_2}{\beta}\frac{dY}{dT}\bigg|_{input} \tag{A1-15}$$

where V_{2i} represents the initial value of the computer voltage V_2. Final values are indicated with a subscript a.

Solving for β,

$$\beta = -\frac{\alpha R_2 C_2}{V_{2i}}\left[-\frac{j\omega J_o}{\rho} v_{\omega i} + J_{\omega i}\frac{K\tau_s}{\rho}\right] . \tag{A1-16}$$

If $J_{\omega i}$ represents the input current, a second relation for β exists;

$$\beta = J_{\omega i}\frac{u_{oi}}{V_{1i}} , \tag{A1-17}$$

from which the ratio of the two input voltages is found to be,

$$\frac{V_{2i}}{V_{1i}} = \frac{j\alpha R_2 C_2 \omega}{\rho}\cdot\frac{J_o}{u_{oi}}\cdot\frac{v_{\omega i}}{J_{\omega i}} - \frac{\alpha R_2 C_2}{\rho}\frac{K\tau_s}{u_{oi}} . \tag{A1-18}$$

With equation (A1-18) to define input conditions, the anode noise quantities can be determined for each of the three inputs.

1) <u>Current Input Only</u>

Equation (3-36b) specifies the input value of current and velocity:

$$\overline{V_{II}}^2 = 0 \qquad \overline{J_{II}}^2 = (1-K_2^2)\,\overline{J_i}^2 \tag{3-36b}$$

The problem is solved formally by using the square roots of quantities representing quadratic content. As it is quadratic content that is determined at the gun anode the results are physically meaningfull. Unit bandwidth is assumed.

Noting the definition (3-28a), the input current becomes

$$J_{II} = (1 - K_2^2)^{1/2} \quad \delta\,(2eJ_o)^{1/2} \tag{A1-19}$$

$$= \beta \frac{V_{1i}}{u_{oi}}$$

The current at the anode is therefore

$$J_{\omega aII} = \beta \frac{V_{1a}}{u_{oa}}$$

$$= (1-K_2^2)^{1/2} \quad \delta(2eJ_o)^{1/2}\left(\frac{V_{oi}}{V_{oa}}\right)^{1/2} \frac{V_{1a}}{V_{1i}} . \tag{A1-20}$$

The velocity modulation at the anode is found as

$$v_{\omega a} = -\frac{1}{j\omega J_o}\left[\rho\frac{\partial Y}{\partial T} - \frac{Y}{u_o}\frac{\partial u_o}{\partial \tau}\right]$$

Using (A1-4) and noting that at the gun anode

$$\frac{1}{u_o}\frac{\partial u_o}{\partial \tau} \doteq \frac{2}{\tau} = \frac{2\rho}{T} = 2\rho$$

$$v_{\omega a} = \frac{\rho}{j\omega J_o \alpha R_2 C_2}\left[V_{2a} + 2\alpha\, R_2 C_2\, V_{1a}\right]\beta \tag{A1-21}$$

Converting to Kinetic Voltage and defining

$$g^{*} = \frac{\alpha R_2 C_2 \omega}{\rho} \tag{A1-22}$$

$$d = 2\alpha R_2 C_2 \tag{A1-23}$$

the Kinetic Voltage at the anode is

$$V_{\omega a} = -\frac{u_{oa}}{j\gamma J_o} \frac{1}{g^*} \left[V_{2a} + dV_{1a} \right] \beta \qquad \text{(A1-24)}$$

which for β as defined in equation (A1-19) gives the value

$$V_{\omega aII} = -\frac{u_{oa}u_{oi}}{j\gamma J_o g^*} \delta(2eJ_o)^{1/2} (1-K_2{}^2)^{1/2} \left[\frac{V_{2a}}{V_{1i}} + d\frac{V_{1a}}{V_{1i}} \right] \qquad \text{(A1-25)}$$

From equation (A1-18) the required input ratio is

$$\frac{V_{2i}}{V_{1i}} = -\frac{\alpha R_2 C_2}{\rho} \frac{K\tau_s}{u_{oi}} = -\lambda . \qquad \text{(A1-26)}$$

2) Kinetic Voltage Input Only

$$V_I{}^2 = (1 - K_1{}^2)\, \overline{V_i{}^2} \qquad \overline{J_I{}^2} = 0 \qquad \text{(3-36a)}$$

In an exactly similar manner as for the current input, it is found that

$$J_{\omega aI} = -j\, g^* J_o\, \mu(1 - K_1{}^2)^{1/2} \frac{kT_c}{(V_{oa}V_{oi})^{1/2}} \frac{1}{(2eJ_o)^{1/2}} \frac{V_{1a}}{V_{2i}} \qquad \text{(A1-27)}$$

$$V_{\omega aI} = \mu\, (1 - K_1{}^2)^{1/2} \left(\frac{V_{oa}}{V_{oi}}\right)^{1/2} \frac{2kT_c}{(2eJ_o)^{1/2}} \left[\frac{V_{2a}}{V_{2i}} + d\frac{V_{1a}}{V_{2i}} \right] \qquad \text{(A1-28)}$$

where equation (A1-18) shows the input computer condition to be $V_{1i} = 0$.

3) Correlated Input

Using equations (3-28a,b) and (3-36c, d), and converting to kinetic voltage in equation (A1-18), the computer input for the correlated term becomes,

$$\frac{V_{2i}}{V_{1i}} = -j\, g \frac{K_1}{K_2} \frac{\mu}{\delta} e^{-j\theta} - \lambda . \qquad \text{(A1-29)}$$

The calculation of anode noise requires the determination of real and imaginary components. The related input conditions are:

$$\text{Real Part} \quad \frac{V_{2iR}}{V_{1iR}} = - g \sin \theta \, \frac{K_1}{K_2} \cdot \frac{\mu}{\delta} - \lambda \qquad \text{(A1-30)}$$

$$\text{Imaginary Part} \quad \frac{V_{2iI}}{V_{1iI}} = - g \cos \theta \, \frac{K_1}{K_2} \cdot \frac{\mu}{\delta} \qquad \text{(A1-31)}$$

Noting that V_{1aR} and V_{2aR} are the computer output voltages for the real part of the input, and V_{1aI} and V_{2aI} are the output voltages for the imaginary part of the input, the output current and kinetic voltage become:

$$J_{\omega aIII} = \delta K_2 (2eJ_o)^{1/2} \left(\frac{V_{oi}}{V_{oa}}\right)^{1/2} \left[\frac{V_{1aR}}{V_{1iR}} + j \frac{V_{1aI}}{V_{1iI}}\right] \qquad \text{(A1-32)}$$

$$V_{\omega aIII} = - \frac{u_{oa} u_{oi}}{j\eta J_o g^*} \delta K_2 (2eJ_o)^{1/2} \left\{\left[\frac{V_{2aR}}{V_{1iR}} + d \frac{V_{1aR}}{V_{1iR}}\right] + j \left[\frac{V_{2aI}}{V_{1iI}} + d \frac{V_{1aI}}{V_{1iI}}\right]\right\}$$

(A1-33)

All of the quantities have been determined which are required for the evaluation of the total mean square current and kinetic voltage, and kinetic power at the gun anode.

A linear relation which is observed between the computer voltages, results in considerable simplification.

$$V_{1a} = - a_1 V_{2i} + a_o V_{1i} \qquad \text{(3-39a)}$$

$$V_{2a} = b_1 V_{2i} - b_o V_{1i} \qquad \text{(3-39b)}$$

Using the appropriate computer input terms for each of the three independent noise inputs, evaluating the required computer outputs by equations (3-39a and 3-39b) and noting the relations of equations (3-38a and 3-38b) and the definitions (3-28), equations (3-41) follow after further algebra.

The solution requires the specification of the input quantities and the determination of the four computer constants a_o, a_1, b_o, b_1. These constants must be determined with considerable accuracy. This was done for the present work as follows. Various ratios of V_{2i}/V_{1i} were set on the computer and the computation carried out to determine the corresponding ratios of V_{1a}/V_{1i} and V_{2a}/V_{1i}. A least squares fit of a straight line was made to each set of data, and the slope and intercept provided the required constants.

REFERENCES

Beam, W. R. 1955. R.C.A. Review 16: 458.

Birdsall, C.K. and Whinnery, J.R. 1953. J.A.P. 24: 314.

Bloom, S. 1955. R.C.A. Review 16: 179.

Chu, L. J. 1951. A paper delivered at the I.R.E. Conference PG-ED Durham, N.H.

Currie, M.R. 1957. Proc. I.R.E. 45: 690

Cutler, C.C. and Quate, C. F. 1950 Phys. Rev. 80: 875.

Dicke, R.H. 1946. Rev. Sci. Inst. 17: 268.

Fried, C.E. and Smulin, L.D. 1954. Trans. I.R.E. PGED-2 ED-1: 168.

Fry, T.C. 1921. Phys. Rev. 17: 441.

Haus, H.A. 1955. J.A.P. 26: 560.

Kleynen, P.H.J.A. 1946. Philips Research Reports 1: 91.

Kornelsen, E.V. 1957. Ph.D. Thesis, McGill University, Montreal.

Langmuir, I. 1923. Phys. Rev. 21: 419.

Llewellyn, F.B. and Peterson, L.C. 1944. Proc. I.R.E. 32: 144.

McIntosh, B.A. 1958. Ph.D. Thesis, McGill University, Montreal.

Mumford, W.W. 1949. B.S.T.J. 28: 608.

Parzen, P. 1952. J.A.P. 23: 215.

Pierce, J.R. 1949. Theory and Design of Electron Beams, D. Van Nostrand, First Edition.

1950. Travelling-Wave Tubes, D. Van Nostrand.

1954. J.A.P. 25: 931.

Rack, A.J. 1938. B.S.T.J. 17: 592.

Rice, S.O. 1944. B.S.T.J. 23: 282.

1945. B.S.T.J. 24: 46.

Robinson, F.N.H. 1952. Phil. Mag. 43: 51.

Rowe, H.E. 1952. Technical Report 239, R.L.E., M.I.T.

Saito, S. 1958. Trans. I.R.E. ED-5: 264.

Schottky, W. 1937. Zeit. fur Physik 104: 248.

Siegman, A.E., Watkins, D.A. and Hsieh, H.S. 1957. J.A.P. 28: 1138.

Selove, W. 1954. Rev. Sci. Inst. 25: 120.

Tien, P.K. and Moshman, J. 1956. J.A.P. 27: 1067.

Tonks, L. and Langmuir, I. 1929. Phys. Rev. 33: 195.

Vessot, R.F.C. 1957. Ph.D. Thesis, McGill University, Montreal.

Watkins, D.A. 1952. J.A.P. 23: 568.