ROBUST ADAPTIVE CONTROL

OF

TIME VARYING SYSTEMS

by

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ABSTRACT

This thesis is concerned with the "robust parameter adaptive control problem". It is based upon two papers: "Robust adaptive control: Conditions for global stability" by R. Kosut and B. Friedlander [1] and "Robust adaptive control: Conditions for local stability" by R. Kosut [2].

After a discussion of the input-output representation of an unknown plant which takes into account the presence of unmodeled dynamics, we consider in the first part only time-invariant uncertainty and extend the robust stability results of direct adaptive controllers obtained in [1]. In the second part we introduce time-varying uncertainty and some conditions are derived which guarantee the global L_2 and L_∞ - stability of the adaptive system. Finally the local L_∞ - stability results of [2] are made more precise and applied successfully to the time-varying case. In all cases we assume that an additive disturbance affects the output of the plant.

RESUME

Cette thèse traite du problème de la commande adaptative robuste. Elle généralise les résultats présentés dans les deux articles suivants: "Robust adaptive control: Conditions for global stability" par R. Kosut et B. Friedlander [1] et "Robust adaptive control: Conditions for local stability" par R. Kosut [2].

Après avoir discuté une représentation entrée-sortie d'un procédé inconnu qui tient compte des dynamiques non modélisées, nous considérons dans une première partie un opérateur incertitude ne variant pas dans le temps et étendons les résultats de stabilité robuste obtenus pour des régulateurs avec modèle de référence dans [1]. Dans une deuxième partie, nous introduisons dans la représentation du procédé un opérateur incertitude qui peut varier dans le temps. Des conditions sont obtenues sur les opérateurs incertitude qui garantissent la stabilité L_2 et L_∞ globale du système adaptatif. Finalement les résultats de stabilité L_∞ locale de [2] sont précisés et appliqués avec succès à notre cas. Tout au long de cette étude nous supposons que la sortie du procédé est affectée par une perturbation additive.

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CHAPTER 1

INTRODUCTION

The use of parameter adaptive control techniques is motivated by the need to automatically adjust the parameters of a plant controller when the plant parameters and disturbances are unknown or change with time. Such techniques have been the subject of extensive research during the past twenty years (see e.g. [3], [4]). In principle, adaptive controllers provide a way of handling plant uncertainty. However it is important to determine their robustness with respect to unmodeled dynamics and output disturbances since these may cause instability (see [5], [6]).

Several attempts have been made to deal with this problem. In particular Kosut and Friedlander [1] and [7] and Kosut [2] have obtained some robust stability results for direct adaptive controllers.

In this thesis we extend their results to cover the case of time-varying plants.

Before going further, we need to recall some definitions and concepts.

1.1. - General framework (see James [8], Desoer and Vidyasagar [9])

We adopt the following generic notation:

(a) Let V be given linear space with inner product.

(b) $F = \{ x : \mathbb{R}_+ \rightarrow V \}$ = The set of all functions mapping $\mathbb{R}_+$ into V

Let an inner product $\langle \cdot | \cdot \rangle$ be defined on a subset F' of F .

We need the notion of a truncated function which is defined as follows:

(c) $\forall x \in F \quad x_T : \mathbb{R}_+ \rightarrow V \quad x_T(t) = x(t) \quad \text{if } t \leq T$

0 if $t > T$

(d) Let X denote a normed linear subspace of F ; the norm of $x \in X$ is denoted by $\|x\|$.

(e) The following assumptions will be made concerning X :

(i) If $x \in X$ then $x_T \in X \quad \forall T \in \mathbb{R}_+$.

(ii) $\|x_T\|$ is a nondecreasing function of $T \in \mathbb{R}_+$.

(iii) If $\lim_{T \rightarrow \infty} \|x_T\|$ exists, then $x \in X$ and $\lim_{T \rightarrow \infty} \|x_T\| = \|x\|$.

(f) The extended space X_e is defined as follows:

$$X_e = \{x \in F \text{ s.t. } x_T \in X \quad \forall T \in \mathbb{R}_+\}$$

This is a notion that was introduced by Zames [8].

(a) to (f) are illustrated by the following important example

Example 1.1 :

$X = L_p^n(\mathbb{R}_+) = \{x : \mathbb{R}_+ \rightarrow \mathbb{R}^n \text{ s.t. } \|x\|_p < \infty\}$, then the extension $X_e = L_{pe}^n(\mathbb{R}_+) = \{x : \mathbb{R}_+ \rightarrow \mathbb{R}^n \text{ s.t. } \|x_T\|_p < \infty \quad \forall T \in \mathbb{R}_+\}$ where the norms are defined according to the previous assumptions as follows:

$$\|x\|_p = \lim_{T \rightarrow \infty} \|x_T\|_p$$

where

$$\|x_T\|_p = \max_{i \in \{1, \dots, n\}} \left(\int_0^T |x_{Ti}(t)|^p dt \right)^{\frac{1}{p}} = \max_{i \in \{1, \dots, n\}} \left(\int_0^T |x_i(t)|^p dt \right)^{\frac{1}{p}}$$

for $p \in [1, \infty)$

$x_{Ti}(t)$ is the i^{th} component of $x_T(t)$

and $x_i(t)$ is the i^{th} component of $x(t)$

$$\|x_T\|_\infty = \text{ess sup}_{t \in [0, T]} \|x(t)\|$$

$\|\cdot\|$ is the euclidean norm on $\mathbb{R}^n$

$$\text{ess sup}_{t \in [0, T]} \|x(t)\| = \inf \{M : m \{t \in [0, T] : \|x(t)\| > M\} = 0\}$$

For $p \in [1, \infty)$, $\|x_T\|_p$ is a norm since $\left(\int_0^T |x_{Ti}(t)|^p dt\right)^{\frac{1}{p}}$ is a norm.

$\|x_T\|_\infty$ is clearly a norm (see [9]).

Kosut and Friedlander [1], Desoer and Vidyasagar [9] adopt the following definition for the L_p -norm of x_T when $p \in [1, \infty)$.

$$\|x_T\|_p = \left(\int_0^T \|x(t)\|^p dt \right)^{\frac{1}{p}}$$

where $\|x(t)\| = \left(\sum_{i=1}^n |x_i(t)|^p \right)^{\frac{1}{p}}$.

Let us prove that $\|x_T\|_p$ and $\|x_T\|_\infty$ are equivalent.

We know that $\exists \alpha, \beta \in \mathbb{R}_+ - \{0\}$ s.t.

$$\alpha \left(\sum_{i=1}^n |a_i|^p \right)^{\frac{1}{p}} \leq \max_{i \in \{1, \dots, n\}} |a_i| \leq \beta \left(\sum_{i=1}^n |a_i|^p \right)^{\frac{1}{p}} \quad (1)$$

since all norms are equivalent in $\mathbb{R}^n$.

Let $a_i = \left(\int_0^T |x_i(t)|^p dt \right)^{\frac{1}{p}} \quad \forall i \in \{1, \dots, n\}$

$$(1) \Leftrightarrow \exists \alpha, \beta \in \mathbb{R}_+ - \{0\} \text{ s.t.}$$

$$\alpha \left(\sum_{i=1}^n \int_0^T |x_i(t)|^p dt \right)^{\frac{1}{p}} \leq \max_{i \in \{1, \dots, n\}} \left(\int_0^T |x_i(t)|^p dt \right)^{\frac{1}{p}} \leq \beta \left(\sum_{i=1}^n \int_0^T |x_i(t)|^p dt \right)^{\frac{1}{p}}$$

$$\Leftrightarrow \exists \alpha, \beta \in \mathbb{R}_+ - \{0\} \text{ s.t.}$$

$$\alpha \left(\int_0^T \sum_{i=1}^n |x_i(t)|^p dt \right)^{\frac{1}{p}} \leq \|x_T\|_p \leq \beta \left(\int_0^T \sum_{i=1}^n |x_i(t)|^p dt \right)^{\frac{1}{p}}$$

$$\Leftrightarrow \exists \alpha, \beta \in \mathbb{R}_+ - \{0\} \text{ s.t.}$$

$$\alpha \left(\int_0^T \|x(t)\|^p dt \right)^{\frac{1}{p}} \leq \|x_T\|_p \leq \beta \left(\int_0^T \|x(t)\|^p dt \right)^{\frac{1}{p}}$$

$$\Leftrightarrow \exists \alpha, \beta \in \mathbb{R}_+ - \{0\} \text{ s.t. } \alpha \|x_T\|_p \leq \|x_T\|_p \leq \beta \|x_T\|_p$$

Note: We use the norm $\|\cdot\|_p$ to simplify the proof of Theorem 2.

Finally we introduce the following notion

(g) Let Y be an inner product space with the inner product $\langle \cdot | \cdot \rangle$

$Y \subset F' \subset F$ (see (b))

$$Y \triangleq \{ x \in F \mid \|x\|^2 \triangleq \langle x | x \rangle < \infty \}$$

We require Y to satisfy the assumptions (e). The extended space

$Y_e \triangleq \{ x \in F \mid x_T \in X \ \forall T \in \mathbb{R}_+ \}$ is an inner product space with inner product defined by

$$\forall x, y \in Y_e \quad \forall T \in \mathbb{R}_+ \quad \langle x | y \rangle_T \triangleq \langle x_T | y_T \rangle = \langle x_T | y \rangle = \langle x | y_T \rangle$$

(g) is illustrated by the following important example

Example 1.2:

$$Y = L_2^n(\mathbb{R}_+) = \{ x : \mathbb{R}_+ \rightarrow \mathbb{R}^n \text{ s.t. } \langle x | x \rangle_2 = \int_0^\infty x^T(t)x(t)dt < \infty \}$$

is an inner product space with inner product

$$\langle x | y \rangle_2 = \int_0^\infty x^T(t)y(t)dt \quad \forall x, y \in L_2^n(\mathbb{R}_+)$$

The extension $Y_e = L_{2e}^n(\mathbb{R}_+)$ is the inner product space with inner product

$$\langle x | y \rangle_{T2} = \int_0^T x^T(t)y(t)dt \quad \forall x, y \in L_{2e}^n(\mathbb{R}_+), \quad \forall T \in \mathbb{R}_+$$

In this thesis we consider input-output systems whose mathematical models are operators on a space X_e , the space X_e itself representing

the set of inputs and outputs. In the most general formulation, an operator G on X_e is given by a relation i.e. any subset of the product space $X_e \times X_e$. In this work we will consider only function operators i.e.

$$\forall u \in X_e \exists! y \in X_e \text{ s.t. } y = Gu$$

Unless otherwise stated all the operators considered in this thesis will be causal i.e.

Let $G : X_e \rightarrow X_e$. G is said to be causal iff

$$P_T G P_T = P_T G \quad \forall T \in \mathbb{R}_+$$

where P_T is the truncation operator which chops off the functions at T .

$$(P_T x)(t) = \begin{cases} x(t) & t \leq T \\ 0 & t > T \end{cases} \quad \forall x \in X_e$$

Now we come to our definition of stability:

The system $G : L_{pe}^n \rightarrow L_{pe}^m$ is said to be L_p -stable if:

$$(i) \quad G u \in L_p^m \quad \forall u \in L_p^n$$

(ii) There exist finite constants γ and b such that

$$\|G u\|_p \leq \gamma \|u\|_p + b \quad \forall u \in L_p^n$$

The smallest γ that satisfies the previous inequality is called the L_p -gain of G and is denoted by $\gamma_p(G)$.

However all the systems considered in this work will be taken to have zero bias i.e. $b = 0$.

We recall that, regardless of the non-linearity of the operators involved,

$$\forall p \in [1, \infty], \quad \forall G_1, G_2 : L_{pe}^n \rightarrow L_{pe}^m$$

$$\gamma_p(G_1 + G_2) \leq \gamma_p(G_1) + \gamma_p(G_2)$$

$$\gamma_p(G_1 G_2) \leq \gamma_p(G_1) \gamma_p(G_2)$$

Since we will study questions of local stability, we give the following definition. (see [2])

The system $G: L_{pe}^n \rightarrow L_{pe}^m$ is said to be locally L_p -stable about a subset S of L_p^n if:

$$(i) \quad G u \in L_p^m \quad \forall u \in S$$

(ii) There exist finite constants γ and b such that

$$\|G u\|_p \leq \gamma \|u\|_p + b \quad \forall u \in S.$$

Now let us introduce the notions of passivity and strict passivity.

Let $G: Y_e \rightarrow Y_e$ where Y_e is the extended space of the linear inner product space Y .

Following Zames [8] and Desoer and Vidyasagar [9], we say that

(1) G is passive iff there exists some constant β_G s.t.

$$\langle x | Gx \rangle_T \geq \beta_G \quad \forall x \in Y_e \quad \forall T \in \mathbb{R}_+$$

(2) G is strictly passive iff there exist some constants $\delta_G > 0$

and β_G s.t.

$$\langle x | Gx \rangle_T \geq \delta_G \|x_T\|^2 + \beta_G \quad \forall x \in Y_e, \quad \forall T \in \mathbb{R}_+$$

If G is linear then β_G can be set equal to zero.

Consider a causal linear time invariant operator $G: X_e^n \rightarrow X_e^n$ whose transfer function matrix $G(s)$ exists.

We say that $G(s)$ is proper (strictly proper) iff all elements of $G(s)$ are bounded at ∞ (tend to 0 as $t \rightarrow \infty$).

A rational transfer function matrix $G(s)$ is said to be exponentially stable if every element of $G(s)$ is a proper rational function whose poles have strictly negative real parts.

Four sets of systems are of particular importance for us:

$\underline{G}$: set of operators: $X_0 \rightarrow X_0$ whose transfer function matrix exists and where the elements of the transfer function matrix are rational and proper with real coefficients.

$\underline{G}_0$: subset of $\underline{G}$, where the elements are now strictly proper.

$\underline{H}$: subset of $\underline{G}$, where the elements are now exponentially stable.

$\underline{H}_0$: subset of $\underline{H}$, where the elements are now strictly proper.

If $G \in \underline{H}_0$ then the system G satisfies some properties which are given and proven in Appendix A. (lemmas A_1 and A_2)

Moreover if the operator $G: Y_0 \rightarrow Y_0$ belongs to $\underline{H}$, then:

G is passive if $\mu(G) = \inf_{\omega} \mu(G(j\omega)) \geq 0$

G is strictly passive if for some constant $\delta_G > 0$ $\mu(G) \geq \delta_G$

where $\mu(A)$ is the measure of the complex matrix A , defined as the smallest eigenvalue of $\frac{1}{2}(A + A^*)$. For scalar systems

$\mu(A) = \operatorname{Re}(A)$. (This result is proven in [9] p.174)

Now define the following sets - G is an operator: $Y_0 \rightarrow Y_0$

$G \in \operatorname{SPR}_0 \Leftrightarrow G \in \underline{H}_0$ and $\mu(G) \geq 0$

$G \in \operatorname{SPR}_0^+ \Leftrightarrow G \in \underline{H}_0$ and for some $\delta_G > 0$, $\mu(G) \geq \delta_G$.

This leads to the following terminology:

Systems in $\underline{H}$ which are passive (strictly passive) are called positive real (strictly positive real) respectively.

We consider the case of the scalar linear transfer function

$$G(s) = g \frac{s^m + \alpha_1 s^{m-1} + \dots + \alpha_{m-1} s + \alpha_m}{s^n + \beta_1 s^{n-1} + \dots + \beta_{n-1} s + \beta_n} \quad \text{with } g > 0$$

Let $n^* = n - m$

Then $G(s)$ is strictly passive if:

(1). $G(s)$ is analytic in the closed right half plane

$\operatorname{Re}(s) \geq 0$ i.e. has no poles in $\operatorname{Re}(s) \geq 0$

(2). $\operatorname{Re}(G(j\omega)) = \operatorname{Re}(G(j\omega)) = \frac{1}{2}(G(j\omega) + \overline{G(j\omega)}) > 0 \quad \forall \omega \in \mathbb{R}$

(2) $\Leftrightarrow$ The Nyquist plot of $G(s)$ remains in the open right half plane $\operatorname{Re}(s) > 0 \quad \forall \omega \in \mathbb{R}$.

As $\omega \rightarrow \infty \quad G(j\omega) \sim g \frac{1}{(j\omega)^{n^*}}$, hence n^* has to be ≤ 1 since if

$n^* \geq 2$, $\angle G(j\omega) \rightarrow 180^\circ$ then the phase of $G(j\omega)$ does not lie in the range $(-90^\circ, 90^\circ)$ at all frequencies and so $G(s)$ is not strictly passive. We conclude that $G \in \operatorname{SPR}_0^+ \Rightarrow n^* \in \{0, 1\}$.

1.2. - Problem Statement

The physical process to be controlled (the plant) is assumed to be a time varying infinite dimensional unknown system. The time varying part of the system will represent one part of the uncertainty. In addition, an explicit part of the system model will represent the time invariant uncertainty. Various representations of uncertainty exist and they vary primarily in terms of the amount of structure contained. Following Doyle and Stein [10], we call a set membership statement for the parameters of the parameterized model of a system a structured representation of the uncertainty. This is the approach used for instance by Caines and Chen [11] and Chen and Caines [12] in their analysis of the adaptive control of systems with random time varying parameters.

In their work, Kosut and Friedlander assume that the plant's time invariant uncertainty Δ_1 can be represented by a matrix of rational, proper, exponentially stable transfer functions. Hence this moderately structured representation of uncertainty corresponds to a finite dimensional system. In the first part of our work (chapter 3) we will keep this convention. But in the second part (chapters 4 and 5) we will drop the assumption of rationality and will adopt a less structured representation of plant uncertainties, namely an upper bound on the gain of the linear uncertainty operators Δ_1 and Δ_2 . These operators correspond to infinite dimensional systems; this is much more interesting because it corresponds to the idea that the unmodeled dynamics cannot even in principle be captured by a finite dimensional system.

Among the various adaptive control techniques, the Model Reference Adaptive Controllers (MRAC) are well developed theoretically and will be used in this work.

Let W_p denote the unknown plant to be controlled. It can be viewed as an operator on X_e (extended space of the Banach space X) and in general this will represent an infinite dimensional system.

Let W_Q denote the time invariant part of W_p and let $W_{Q(\alpha)}$ denote a parameterized model of W_Q with parameter $\alpha \in R^k$. In this thesis parameter means always "structured parameter" i.e. an element of a finite dimensional manifold in particular the euclidean space R^k .

$W_{Q(\alpha)}$ is assumed to belong to the set of approximate models $\{W_{Q(\alpha)}, \alpha \in R^k\}$ which are operators $X_e \rightarrow X_e$ whose transfer function matrix exist and are rational. This set is assumed to be given a priori. $W_{Q(\alpha)}$ will often be called an approximate model because we are almost always interested in the case where there is no value of α that yields exact modelling i.e. $W_{Q(\alpha)} \neq W_Q$ for all $\alpha \in R^k$.

Further let W_M denote the operator on X_e representing the stable finite dimensional reference model to be used in the regulator design; we shall denote its transfer function by $W_M(s)$.

Now we know (see e.g. Rudin [13] p.99) that if we consider a linear transformation A from a normed linear space Z_1 into a normed linear space Z_2 , we may define its norm by:

$$|A| = \sup \{ \|Ax\|_{Z_2}, x \in Z_1, \|x\|_{Z_1} = 1 \}$$

If we consider the operator $\epsilon_\alpha: X \rightarrow X_e$ given by

$$u + \varepsilon_\alpha(u) = W_Q u - W_{Q(\alpha)} u$$

we may define $|||\varepsilon_\alpha|||$ as follows:

$$\text{If } \varepsilon_\alpha(u) \in X \quad |||\varepsilon_\alpha||| \triangleq \sup \{ \|\varepsilon_\alpha(u)\|_X, \|u\|_X = 1 \}$$

if, otherwise, $\varepsilon_\alpha(u) \in X_e - X \quad |||\varepsilon_\alpha||| \triangleq \infty$ (recall $X \subset X_e$)

Since $|||\varepsilon_\alpha|||$ may be infinite, it is not a norm as defined by Rudin ([13], p.100). It will be called the extended norm of the operator ε_α .

Then we define $\alpha^* \triangleq \arg \min_{\alpha \in \mathbb{R}^k} \{ |||\varepsilon_\alpha||| \}$ whenever it exists.

$|||\varepsilon_\alpha|||$ clearly has the interpretation of being the extended norm of the modelling error between the time invariant system W_Q and its parameterized model $W_{Q(\alpha)}$, and clearly α^* is the parameter of the best approximating model when such a model exists. We shall call α^* the Prediction Error Estimate of α for the approximate model of the time invariant part of the plant relative to the set of approximate models $\{W_{Q(\alpha)}, \alpha \in \mathbb{R}^k\}$.

$W_{Q^*} \triangleq W_{Q(\alpha^*)}$ is called the Prediction Error approximate model of the time invariant part of the plant.

Let W_Q be described by

$$W_Q = (I + \Delta_1) W_{Q^*} \quad (1.2.1)$$

where Δ_1 represents the linear time invariant uncertainty operator which can be consequently described in the frequency domain. Following Kosut and Friedlander we assume in chapter 3 that Δ_1 lies in the ball $B_{H_0}(r)$ of operators which belong to H_0 and are specified by the frequency function bound $r = \{r(\omega), \omega \in \mathbb{R}\}$ as follows:

$$\Delta_1 \in \mathcal{B}_{\underline{H}_0}(r) = \{G \in \underline{H}_0, \bar{\sigma}(G(j\omega)) \leq r(\omega) \quad \forall \omega \in \mathbb{R}\} \quad (1.2.2)$$

where $\bar{\sigma}(A)$ denotes the maximum singular value of the matrix A defined as the positive square root of the maximum eigenvalue of A^*A , where A^* is the conjugate transpose of A .

In chapters 4 and 5, we drop the assumption of rationality and Δ_1 is characterized by an upper bound α on its Y -gain

$$\gamma_Y(\Delta_1) \leq \alpha \quad (1.2.3)$$

Let the plant to be controlled be described by:

$$W_P = (I + \Delta_1)(I + \Delta_2) W_Q^* \quad (1.2.4)$$

where Δ_2 represents the linear time varying uncertainty operator.

Δ_2 will be characterized by an upper bound α^1 on its Y -gain

$$\gamma_Y(\Delta_2) \leq \alpha^1 \quad (1.2.5)$$

It is crucial at this point for the reader to recognize that in our problem formulation W_P and W_Q are assumed to be fixed and $\{W_Q(\alpha), \alpha \in \mathbb{R}^k\}$ is assumed to be given a priori. Equations (1.2.1) - (1.2.5) constitute hypothesis on the modelling errors Δ_1 and Δ_2 defined via the equations (1.2.1) and (1.2.4). In all the work that follows the efficacy of the adaptation algorithms is predicated upon hypothesis concerning the type and magnitude of the operators Δ_1 and Δ_2 .

The objective of any model reference controller (adaptive or non adaptive) is to minimize the norm $\|e\|_X$ where e is the difference between the output of the plant Y_P and the output of the model Y_M .

In the case of a non-adaptive controller we have the situation

described in the diagram below:

- Fig. 1.1 -

The regulator parameter $\hat{\theta}$ is constant and we shall simply write it as θ . As shown in the diagram $R(\theta)$ denotes the parameterized plant regulator. It is designed in terms of the best approximate plant model i.e. $W_{Q^*} = W_{Q(\alpha^*)}$. It will be assumed to be finitely parameterized and such that the operator representing the plant together with the regulator i.e. the $R(\theta)$ - regulated plant

$W_P^{R(\theta)} \triangleq (I + W_P R(\theta))^{-1} W_P$ is an operator mapping X_e into X_e .

In the case of an adaptive controller we have the situation described in the following diagram:

A: adaptation law

- Fig. 1.2 -

We shall denote the closed - loop adaptive regulated plant by $W_P^{R(A)}$ and let $W_Q^{R(A)}(\alpha)$ denote the closed loop adaptive regulated, α - parameterized model of the time invariant part of the plant.

Now we are going to define the notion of "tuned parameter vector".

In the usual formulation of the adaptive control problem, the unknown plant together with all admissible time invariant feedback regulators is a time invariant system and is assumed to possess a finite dimensional parameterization that is to say it may be described by a rational transfer function

$$W_P^{R(\theta)}(s) = \frac{n(s)}{d(s)} = (1 + W_P(s) R(\theta)(s))^{-1} W_P(s)$$

where $n(s)$, $d(s)$ are polynomials of degree m, n respectively. Then, one designs the adaptive control law in terms of a model described by a rational transfer function

$$W_M(s) = \frac{n_M(s)}{d_M(s)}$$

where $\hat{n}_M(s)$, $\hat{d}_M(s)$ are polynomials of degree $M \leq m$, $N \leq n$ respectively. Hence there exists a constant value of $\hat{\theta}$ in $\mathbb{R}^{n+m+1}$ such that the transfer function of the plant with the controller matches that of the model exactly. This value θ^* is called the tuned parameter vector since then the closed loop system approximates the given model as closely as possible.

This notion is also meaningful when $\theta \in \mathbb{R}^k \subset \mathbb{R}^{n+m+1}$, for in this case one may define:

$$\theta^* \triangleq \arg \min_{\theta \in \mathbb{R}^k} ||| W_M - W_P^{R(\theta)} |||$$

where $|||A|||$ is some specified extended norm of the operator A with transfer function $A(s)$ whenever this minimum exists.

In our case, the plant is both unknown and time varying and so we adopt the following extension of the definition of the tuned parameter vector θ^* given above.

First, following Kosut and Friedlander [1], let us denote by

$e_\theta: X \rightarrow X_\theta$ the operator which maps $\begin{pmatrix} x \\ w \end{pmatrix}$ to $(W_M - W_P^{R(\theta)}) \begin{pmatrix} x \\ w \end{pmatrix} = e_\theta \begin{pmatrix} x \\ w \end{pmatrix}$.

Now define $|||e_\theta|||$ as follows:

If $e_\theta \begin{pmatrix} x \\ w \end{pmatrix} \in X$ $|||e_\theta||| \triangleq |||W_M - W_P^{R(\theta)}||| \triangleq \sup \{ ||e_\theta \begin{pmatrix} x \\ w \end{pmatrix}||_X \mid ||\begin{pmatrix} x \\ w \end{pmatrix}||_X = 1 \}$

if otherwise, $e_\theta \begin{pmatrix} x \\ w \end{pmatrix} \in X_\theta - X$ then $|||e_\theta||| \triangleq \infty$.

Then $\theta^* \triangleq \arg \min_{\theta \in \mathbb{R}^k} |||e_\theta|||$ whenever this minimum exists.

1.3 - Outline of this thesis

This thesis is organized as follows:

In chapter 2 we apply Kosut and Friedlander's approach (see [1]) to the standard model reference adaptive control system (MRAC) proposed in [14], [15] assuming, as already stated, that the plant is unknown and may be represented by the operator $W_p = (I + \Delta_2)(I + \Delta_1) W_{Q^*}$. The tuned signals e^* , z^* as well as the crucial operators H_{ev} and H_{zv} are defined and derived.

In chapter 3, assuming that the time varying uncertainty $\Delta_2 = 0$ and that the time invariant uncertainty Δ_1 may be represented by a rational, proper, exponentially stable transfer function, we recall the main results of [1] and extend them to the case $n^* = 2$ where n^* is the relative degree of W_{Q^*} i.e. the difference between the number of poles and number of zeros of the transfer function $W_{Q^*}(s)$.

In chapter 4, we drop the assumption of rationality of Δ_1 and treat the case where Δ_2 is not necessarily equal to zero. Δ_1 and Δ_2 are now characterized by an upper bound on their γ -gain. We first present (and prove) four passivity lemmas (Lemmas 4.1 - 4.4) and a theorem (Theorem 1) which ensure that the operator H_{ev} is strictly passive. This allows us to state an adaptation theorem (Theorem 2) and the global L_2 -stability theorem (Theorem 3). Then introducing an integrator in the controller we get a stronger global stability theorem (Theorem 4) which ensures, in particular, that the error e tends to the tuned error e^* as $t \rightarrow \infty$. As in [1], some bounds are obtained for certain significant signals.

In chapter 5, we apply Kosut's ideas (see [2]) to the time-varying case. Some conditions on the tuned signals e^* and z^* as well as on the design of the controller are derived which guarantee local L_2 -stability.

CHAPTER 2

THE STRUCTURE OF THE ADAPTIVE SYSTEM

We consider the MRAC system shown in Fig. 2.1 consisting of:

- (i) an unknown scalar plant W_p with a reference input r whose output Y_p is affected by an additive disturbance w .
- (ii) a reference model W_M whose transfer function is $W_M(s)$.
- (iii) a controller which consists (see Narendra and Valavani [15]) of a gain $\hat{c}_0(t)$ and two "auxiliary signal" generators F_1 and F_2

• F_1 contains $(n-1)$ parameters $\hat{c}_i$ $i=1, \dots, n-1$

• F_2 contains n parameters $\hat{d}_j$ $j=0, \dots, n-1$

F_1 and F_2 are described by the following differential equations

$$F_1 \begin{cases} \dot{x}_1(t) = \Lambda x_1(t) + b u(t) \\ x_1'(t) = \hat{c}^T(t) x_1(t) \end{cases} \quad (2.1)$$

$$F_2 \begin{cases} \dot{x}_2(t) = \Lambda x_2(t) + b Y_p(t) \\ x_2'(t) = \hat{d}^T(t) x_2(t) + \hat{d}_0(t) Y_p(t) \end{cases} \quad (2.2)$$

where Λ is an $(n-1 \times n-1)$ asymptotically stable matrix. Let us assume that (Λ, b) is in the controllable canonical form. Then the vector of transfer functions $V(s)$ is

$$V(s) = (sI - \Lambda)^{-1} b = \begin{bmatrix} v_1(s) \\ \vdots \\ v_j(s) \\ \vdots \\ v_{n-1}(s) \end{bmatrix} = \frac{1}{\det(sI - \Lambda)} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-2} \end{bmatrix} \quad (2.3)$$

$$\text{hence } \begin{cases} x_1 = V u \\ x_2 = V Y_p \end{cases} \quad \text{or} \quad \begin{cases} x_1(s) = V(s) u(s) \\ x_2(s) = V(s) Y_p(s) \end{cases} \quad (2.4)$$

where V is the linear time invariant operator whose transfer function is $V(s)$.

Let the parameter vector $\hat{\theta}$ and the signal vector z be defined by

- Fig.2.1 -

$$\hat{\theta}^T = [\hat{c}_0^T, \hat{c}^T, \hat{d}_0^T, \hat{d}^T] \quad (2.5)$$

$$z^T = [r, x_1^T, y_p, x_2^T] \quad (2.6)$$

The input to the plant, as shown in Fig. 2.1 is, using (2.1) - (2.6)

$$u(t) = -\hat{c}_0(t)r(t) - \hat{c}^T(t)x_1(t) - \hat{d}_0(t)y_p(t) - \hat{d}^T(t)x_2(t)$$

$$\text{i.e.} \quad u(t) = - \sum_{i=0}^{2n-1} \hat{\theta}_i(t) z_i(t) \quad (2.7)$$

where $\hat{\theta}_i(t)$ is the i^{th} -component of the $2n$ vector $\hat{\theta}(t)$

and $z_i(t)$ is the i^{th} -component of the $2n$ vector $z(t)$.

$$\text{Define the adaptive gain error: } \tilde{\theta}(t) = \hat{\theta}(t) - \theta^* \quad (2.8)$$

where θ^* is the tuned parameter vector.

The parameters are adjusted using the standard parameter adjustment, or estimation law (see e.g. [15]):

$$\dot{\hat{\theta}}(t) = \Gamma z(t) e(t) \quad \text{where } \Gamma = \Gamma^T > 0 \quad (2.9)$$

$$u(t) = - \sum_{i=0}^{2n-1} \hat{\theta}_i(t) z_i(t) = - \sum_{i=0}^{2n-1} \tilde{\theta}_i(t) z_i(t) - \sum_{i=0}^{2n-1} \theta_i^* z_i(t)$$

where θ_i^* is the i^{th} -component of the $2n$ vector θ^*

$$\theta^* = [c_0^*, c^{*T}, d_0^*, d^{*T}] \quad (2.10)$$

$$\text{Hence } u(t) = -\tilde{\theta}^T(t)z(t) - c_0^* r(t) - c^{*T} x_1(t) - d_0^* y_p(t) - d^{*T} x_2(t)$$

$$\text{Define } v(t) \triangleq \tilde{\theta}^T(t)z(t) \quad \forall t \in \mathbb{R}_+ \quad (2.11)$$

and denote by $v(s)$ the laplace transform of $v(t)$.

Note: From (2.3) and (2.4) it is clear that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a "regression vector" whose entries form a state for the plant based upon derivatives of u and y_p . Hence the vector $v = \tilde{\theta}^T z$ is exactly analogous to the vector $\tilde{\theta}^T \phi$ in [16]. Furthermore, this fact and the form of (2.7) show that the control law has precisely the certainty equivalence inter-

pretation that one replaces the parameter θ in a feedback control law by its current best estimate $\hat{\theta}$.

$$\begin{aligned} \text{We have } u(s) &= -v(s) - c_o^* r(s) - c_o^{*T} x_1(s) - d_o^* Y_P(s) - d_o^{*T} x_2(s) \\ &= -v(s) - c_o^* r(s) - c_o^{*T} V(s) u(s) - d_o^* Y_P(s) - d_o^{*T} V(s) Y_P(s) \end{aligned}$$

$$\text{giving } (1 + c_o^{*T} V(s)) u(s) = -c_o^* r(s) - (d_o^* + d_o^{*T} V(s)) Y_P(s) - v(s)$$

$$\text{Define } C^*(s) \triangleq \frac{c_o^*}{1 + c_o^{*T} V(s)} \quad \text{and} \quad D^*(s) \triangleq \frac{d_o^* + d_o^{*T} V(s)}{1 + c_o^{*T} V(s)} \quad (2.12)$$

$$\text{then } u(s) = -C^*(s) r(s) - D^*(s) Y_P(s) - \frac{C^*(s)}{c_o^*} v(s). \quad (2.13)$$

Let C^* and D^* denote the linear time invariant operators whose transfer functions are $C^*(s)$ and $D^*(s)$.

$$\text{Then (2.13) yields } u = -C^* r - D^* Y_P - \frac{1}{c_o^*} C^* v \quad (2.14)$$

and in this notation we have:

$$\begin{aligned} \text{The output of the plant: } Y_P &= W_P u + w \\ \text{The output of the model: } Y_M &= W_M r \\ \text{The error between the plant and the model: } e &= Y_P - Y_M \end{aligned} \quad (2.15)$$

Define the tuned signals as follows:

$$\begin{aligned} u^* &= u \Big|_{\hat{\theta} = \theta^*}, & Y_P^* &= Y_P \Big|_{\hat{\theta} = \theta^*} \\ e^* &= e \Big|_{\hat{\theta} = \theta^*}, & z^* &= z \Big|_{\hat{\theta} = \theta^*} \end{aligned} \quad (2.16)$$

where θ^* is the tuned parameter vector defined in chapter I.

Clearly $v = 0$ when $\hat{\theta} = \theta^*$.

$$\text{Hence } u^* = -C^* r - D^* Y_P^*$$

$$Y_P^* = W_P u^* + w$$

$$e^* = Y_P^* - Y_M$$

$$z^{*T} = [r, V u^*, Y_P^*, V Y_P^*]$$

(2.17)

(2.17) yields $Y_P^* = W_P (-C^* r - D^* Y_P^*) + w$

$$\text{i.e. } Y_P^* = - (1 + W_P D^*)^{-1} W_P C^* r + (1 + W_P D^*)^{-1} w$$

Therefore $e^* = Y_P^* - Y_M = [- (1 + W_P D^*)^{-1} W_P C^* + W_M] (1 + W_P D^*)^{-1} \begin{bmatrix} r \\ w \end{bmatrix}$

$$u^* = - C^* r - D^* Y_P^* = [-1 + D^* (1 + W_P D^*)^{-1} W_P] C^* r - D^* (1 + W_P D^*)^{-1} w$$

$$\text{and so } z^* = \begin{bmatrix} r \\ Vu^* \\ Y_P^* \\ VY_P^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ V(-1 + D^* (1 + W_P D^*)^{-1} W_P) C^* & -VD^* (1 + W_P D^*)^{-1} \\ - (1 + W_P D^*)^{-1} W_P C^* & (1 + W_P D^*)^{-1} \\ -V(1 + W_P D^*)^{-1} W_P C^* & V(1 + W_P D^*)^{-1} \end{bmatrix} \begin{bmatrix} r \\ w \end{bmatrix} \quad (2.19)$$

(see Figure (2.2))

The operators $H_{ev}, H_{zv} : X_e \rightarrow X_e$, are now defined by the

$$\text{equalities } \begin{cases} e = e^* - H_{ev} v \\ z = z^* - H_{zv} v \end{cases} \quad (2.20)$$

and they may be obtained as follows.

First let us compute $Y_P - Y_P^*$:

$$Y_P - Y_P^* = W_P (u - u^*) = W_P (-D^* (Y_P - Y_P^*) - \frac{1}{c_o^*} C^* v)$$

$$\text{giving } Y_P - Y_P^* = - (1 + W_P D^*)^{-1} W_P \frac{1}{c_o^*} C^* v \quad (2.21)$$

$$\text{Then } e - e^* = Y_P - Y_P^* = - (1 + W_P D^*)^{-1} W_P \frac{1}{c_o^*} C^* v$$

$$\text{giving } H_{ev} = (1 + W_P D^*)^{-1} W_P \frac{1}{c_o^*} C^* \quad (2.22)$$

- Fig. 2.2 -

Let the $2n$ vector z be such that

$$z^T = [z_0, z_1, \dots, z_{n-1}, z_n, z_{n+1}, \dots, z_{2n-1}]$$

where

$$(i) \quad z_0 = r \Delta r - H_{zv_0} v \Rightarrow H_{zv_0} = 0 \quad (2.23a)$$

$$(ii) \quad \text{for } j \in \{1, \dots, n-1\}$$

$$z_j = V_j u = -V_j (C^* r + D^* Y_P + \frac{1}{C_O^*} C^* v) \quad (\text{using (2.14)})$$

where V_j is the j^{th} -component of the operator V . This gives

$$z_j = V_j \underbrace{(-C^* r - D^* Y_P^*)}_{u^*} - V_j D^* (Y_P - Y_P^*) - V_j \frac{1}{C_O^*} C^* v \quad (\text{see 2.17})$$

hence, using (2.21), we get

$$z_j = z_j^* - [-V_j D^* (1+W_P D^*)^{-1} W_P \frac{1}{C_O^*} C^* + V_j \frac{1}{C_O^*} C^*] v$$

$$\text{which implies } \forall j \in \{1, \dots, n-1\} H_{zv_j} = V_j [-D^* (1+W_P D^*)^{-1} W_P + 1] \frac{1}{C_O^*} C^* \quad (2.23b)$$

$$(iii) \quad z_n = Y_P = z_n^* + (Y_P - Y_P^*) \Delta z_n^* - H_{zv_n} v$$

$$\text{which gives } H_{zv_n} = H_{ev} = (1+W_P D^*)^{-1} W_P \frac{1}{C_O^*} C^* \quad (\text{using (2.21)}) \quad (2.23c)$$

$$(iv) \quad \text{for } j \in \{n+1, \dots, 2n-1\}$$

$$z_j = V_{j-n} z_n \Delta V_{j-n} (z_n^* - H_{zv_n} v)$$

$$\text{hence } H_{zv_j} = V_{j-n} H_{zv_n} = V_{j-n} (1+W_P D^*)^{-1} W_P \frac{1}{C_O^*} H_{ev} \quad (2.23d)$$

$$\forall j \in \{n+1, \dots, 2n-1\}$$

Finally

$$H_{zv} = \begin{bmatrix} 0 \\ V[-D^* (1+W_P D^*)^{-1} W_P + 1] \frac{1}{C_O^*} C^* \\ (1+W_P D^*)^{-1} W_P \frac{1}{C_O^*} C^* \\ V (1+W_P D^*)^{-1} W_P \frac{1}{C_O^*} C^* \end{bmatrix} \quad (2.23)$$

We have now set up an adaptive error system (see Fig. 2.3)

$$\begin{cases} \dot{e} = e^* - H_{ev} v \\ \dot{z} = z^* - H_{zv} v \end{cases} \quad (2.24)$$

with the adaptation law $(\Sigma_a) :$

$$\begin{cases} \psi = z e \\ \Sigma_M: \dot{\bar{\theta}} = \Gamma \psi, \quad \Gamma = \Gamma^T > 0 \\ v = \bar{\theta}^T z \end{cases}$$

The interesting point is that the map $\Sigma_M: \psi \rightarrow \bar{\theta}$ is passive; this will allow us (see proof of Theorem 3) to prove that the map $e \rightarrow v$ is passive and therefore to get some global stability results. It is clear that we could take another adaptive error system as long as the map $e \rightarrow v$ remains passive. For instance, we could adopt the following one:

$$\begin{cases} \dot{e} = e^* - H_{ev} v \\ \dot{z} = z^* - H_{zv} v \end{cases} \quad (2.25)$$

- Fig. 23 -

with adaptation law (Σ'_a) :

$$\begin{cases} \psi = \zeta \cdot e \\ \Sigma'_a: \dot{\tilde{\theta}} = \Gamma \psi ; \Gamma = \Gamma^T > 0 \\ v' = \tilde{\theta}^T z + \rho \tilde{\theta}^T \zeta \end{cases}$$

where $\zeta(t) = (1 + \rho \frac{d}{dt})z(t) \quad \forall t \in \mathbb{R}_+$

$(\rho > 0)$

It is shown in the appendix C that the map $e \rightarrow v'$ is still passive and that we obtain similar results to those we obtain with (2.24).

CHAPTER 3

CONDITIONS FOR GLOBAL STABILITY: TIME INVARIANT CASE

3.1 - Review of Kosut and Friedlander's results (see [1], [7])

As already stated in the introduction, we shall assume in this chapter that the time invariant uncertainty Δ_1 can be represented by a matrix of rational, proper, exponentially stable transfer functions. The time varying uncertainty Δ_2 is assumed to be equal to zero. Hence the equation (1.2.4.) becomes $W_p = (I + \Delta_1) W_{Q^*}$.

Since we only consider a scalar plant, this reduces to

$$W_p = (1 + \Delta_1) W_{Q^*} \quad (3.1)$$

where Δ_1 is represented by a rational, proper, exponentially stable transfer function.

Let us present now Kosut and Friedlander's main theorem (see [1]).

A detailed proof can be found in Appendix B. Let C denote the set of continuous functions on R_+ .

Theorem A - Global stability theorem

For the adaptive system (2.15) - (2.17), (2.22), (2.23) with adaptation law (Γ_a) assume that:

either (a0) all inputs $\begin{pmatrix} r \\ w \end{pmatrix}$ in $L_{\infty} \cap C$ produce signals $e, z, \hat{\theta}, \hat{\theta}$ in $L_{\infty} \cap C$

or (a0') all inputs $\begin{pmatrix} r \\ w \end{pmatrix}$ in $L_{\infty} \cap C$ produce signals $e, z, \hat{\theta}$ in $L_{\infty} \cap C$ and the operator $(1 + D^* W_p)^{-1}$ maps C into C

and (a1) $r, w \in L_{\infty} \cap C$

(a2) $H_{ev} \in SFR_0^+$

$$(a3) \quad H_{zv} \in H_0$$

Under these conditions:

If $(e^*, \dot{e}^*) \in L_2 \cap L_\infty$ and $(z^*, \dot{z}^*) \in L_\infty$ then

- (i) $\tilde{\theta} \in L_\infty$, $(\dot{\tilde{\theta}}, \ddot{\tilde{\theta}}) \in L_2 \cap L_\infty$ and $\dot{\tilde{\theta}}(t) \rightarrow 0$ at the rate $e^*(t) \rightarrow 0$ as $t \rightarrow \infty$
- (ii) $(e, \dot{e}) \in L_2 \cap L_\infty$ and $(e - e^*)(t) \rightarrow 0$ exponentially as $t \rightarrow \infty$
- (iii) $v \in L_2 \cap L_\infty$, $\dot{v} \in L_\infty$, $v(t) \rightarrow 0$ as $t \rightarrow \infty$
- (iv) $(z, \dot{z}) \in L_\infty$, $(z - z^*, \dot{z} - \dot{z}^*) \in L_2 \cap L_\infty$ and $(z - z^*)(t) \rightarrow 0$ exponentially as $t \rightarrow \infty$

and the following explicit bounds can be obtained:

$$(v) \quad \|v\|_2 \leq b_1 \triangleq \frac{1}{2\delta_{ev}} (\|e^*\|_2 + (\|e^*\|_2^2 - 4\lambda\delta_{ev})^{\frac{1}{2}})$$

$$\|e\|_2 \leq b_2 \triangleq \|e^*\|_2 + \gamma_2(H_{ev}) b_1$$

$$\|\tilde{\theta}\|_2 \leq b_3 \triangleq \left(\frac{\tilde{\theta}^T(0) \Gamma^{-1} \tilde{\theta}(0) + 2b_1 b_2}{v} \right)^{\frac{1}{2}}$$

$$\|v\|_\infty \leq b_4 \triangleq b_3 (\|z^*\|_\infty + \|h_{zv}\|_2 b_1)$$

$$\|e\|_\infty \leq b_5 \triangleq \|e^*\|_\infty + \gamma_\infty(H_{ev}) b_4$$

where δ_{ev} is the "constant of strict passivity" of H_{ev} .

$$\lambda = -\frac{1}{2} \tilde{\theta}^T(0) \Gamma^{-1} \tilde{\theta}(0)$$

v : a positive constant s.t. $\Gamma^{-1} \geq v I_{2n}$

Note that condition (a0') ensures that $\tilde{\theta} \in C$.

To ensure that conditions (a₂) and (a₃) hold despite the uncertainty Δ_1 we are going to present a modified version of Kocut and

Friedlander's lemma 5.1 (see [1]). Before we need to introduce some new notation and a preliminary lemma: Lemma 3.1

First recall that we are dealing in this chapter only with linear time invariant operators which map X_e into X_e and whose transfer functions exist. Hence we will consider the Laplace domain representations of these operators. The Laplace transform variable "s" will be omitted to make the notations simpler.

$$\text{Set } K^* \underline{\Delta} H_{ev} \Big|_{\Delta_1=0} = (1 + W_{Q^*} D^*)^{-1} W_{Q^*} \frac{C^*}{C_o^*} \quad (\text{using (2.22)}) \quad (3.2)$$

$$\text{and write } T^* \underline{\Delta} (1 + W_{Q^*} D^*)^{-1} W_{Q^*} D^* \bar{V} 1 = S^* \quad (3.3)$$

$$\text{which gives } W_{Q^*} D^* = T^* S^{*-1} \quad (3.4)$$

$$\text{Moreover } 1 + W_P D^* = 1 + (1 + \Delta_1) W_{Q^*} D^* = (1 + \Delta_1 T^*) S^{*-1}$$

using (3.1) and (3.4). Furthermore (3.3) and (3.4) imply that

$$1 + W_{Q^*} D^* = S^{*-1} \quad (3.5)$$

$$\text{Hence (3.2) and (3.5) imply that } W_{Q^*} \frac{C^*}{C_o^*} = K^* S^{*-1}$$

This equality together with (3.2) and (3.5) allows us to say that

$$H_{ev} = K^* (1 + \Delta_1) (1 + \Delta_1 T^*)^{-1} \quad (3.6)$$

Similarly using (2.23) we obtain

$$H_{zv} = \begin{bmatrix} 0 \\ VK^* D^* S^* (1 + \Delta_1 T^*)^{-1} T^{*-1} \\ K^* (1 + \Delta_1) (1 + \Delta_1 T^*)^{-1} \\ VK^* (1 + \Delta_1) (1 + \Delta_1 T^*)^{-1} \end{bmatrix} \quad (3.7)$$

and using (2.18) and (2.19) we obtain

$$e^* = [-(c_o^* K^* (1+\Delta_1) (1+\Delta_1 T^*)^{-1} + W_M) \quad S^* (1+\Delta_1 T^*)^{-1}] \begin{bmatrix} r \\ w \end{bmatrix} \quad (3.8)$$

$$z^* = \begin{bmatrix} 1 & 0 \\ V(-1 + T^* (1+\Delta_1) (1+\Delta_1 T^*)^{-1}) & -VD^* S^* (1+\Delta_1 T^*)^{-1} \\ -c_o^* K^* (1+\Delta_1) (1+\Delta_1 T^*)^{-1} & S^* (1+\Delta_1 T^*)^{-1} \\ -Vc_o^* K^* (1+\Delta_1) (1+\Delta_1 T^*)^{-1} & VS^* (1+\Delta_1 T^*)^{-1} \end{bmatrix} \begin{bmatrix} r \\ w \end{bmatrix} \quad (3.9)$$

Now we shall present lemma 3.1 which is a modified version of Kosut and Friedlander's lemma 4.1 (see [1])

Lemma 3.1 (after Kosut and Friedlander [1])

Let P be given by $P = (I + \Delta) \tilde{P}$. (3.1.1)

Then $P \in \text{SPR}_O^+$ with $\delta_P = \delta_{\tilde{P}}$ if:

(i) $\tilde{P} \in \text{SPR}_O^+$

(ii) $\Delta \in \underline{B}_{H_O}(k)$ where $k(\omega) < k'(\omega) = \frac{\mu(\tilde{P}(j\omega)) - \delta_{\tilde{P}}}{\bar{\partial}(\tilde{P}(j\omega))} \quad \forall \omega \in \mathbb{R}$

($\delta_{\tilde{P}}$ is the constant of strict passivity of $\tilde{P}$)

If case (i) is replaced by $\tilde{P} \in \text{SPR}_O$, then $P \in \text{SPR}_O$ □

Recall that $\underline{B}_{H_O}(k) = \{G \in \underline{H_O} \mid \bar{\partial}(G(j\omega)) \leq k(\omega) \quad \forall \omega \in \mathbb{R}\}$

Proof: let us prove this result in the scalar case where

$$\bar{\partial}(G(j\omega)) = |G(j\omega)|$$

$$\bar{\partial}(G(j\omega)) \text{ and } \mu(G(j\omega)) = \text{Re}(G(j\omega))$$

Clearly (i) and (ii) imply that $P \in \underline{H_O}$.

Let us show now the following fact:

Fact: Let $a = \alpha + j\beta$, $b = \epsilon + j\nu$ be two complex numbers.

$$\text{Then } \text{Re}(a + ba) \geq \text{Re } a - |b| |a|. \quad \square \quad (3.1.2)$$

$$\text{Re}(a + ba) = \alpha + \alpha \epsilon - \beta \nu; \quad \text{Re } a - |b| |a| = \alpha - (\epsilon^2 + \nu^2)^{\frac{1}{2}} (\alpha^2 + \beta^2)^{\frac{1}{2}}$$

Hence (3.1.2) holds iff $\alpha\epsilon - \beta v \geq -(\epsilon^2 + v^2)^{\frac{1}{2}}(\alpha^2 + \beta^2)^{\frac{1}{2}}$ (3.1.3)

If $\alpha\epsilon - \beta v \geq 0$ then (3.1.3) is clearly satisfied

If $\alpha\epsilon - \beta v < 0$, then, since $(\alpha v + \beta\epsilon)^2 \geq 0$ for all $\alpha, \beta, \epsilon, v$,

$$\beta^2 v^2 + \alpha^2 \epsilon^2 - 2\alpha\beta\epsilon v \leq \alpha^2 \epsilon^2 + \beta^2 \epsilon^2 + \alpha^2 v^2 + \beta^2 v^2$$

$$\text{or } (\beta v - \alpha\epsilon)^2 \leq (\alpha^2 + \beta^2)(\epsilon^2 + v^2)$$

$$\text{hence } 0 < \beta v - \alpha\epsilon \leq (\alpha^2 + \beta^2)^{\frac{1}{2}}(\epsilon^2 + v^2)^{\frac{1}{2}}$$

as required. $\square$

$$\begin{aligned} \text{Now } \operatorname{Re}(P(j\omega)) &= \operatorname{Re}(\tilde{P}(j\omega) + \Delta(j\omega)\tilde{P}(j\omega)) \\ &\geq \operatorname{Re}(\tilde{P}(j\omega)) - |\Delta(j\omega)| |\tilde{P}(j\omega)| \quad (\text{using 3.1.2}) \end{aligned}$$

$$\text{By assumption } |\Delta(j\omega)| < \frac{\operatorname{Re}(\tilde{P}(j\omega)) - \delta_{\tilde{P}}}{|\tilde{P}(j\omega)|} \quad \forall \omega \in \mathbb{R}$$

$$\text{Hence } \operatorname{Re}(P(j\omega)) > \delta_{\tilde{P}} \quad \forall \omega \in \mathbb{R}$$

which implies that $P \in \operatorname{SPR}_0^+$ and $\delta_P = \delta_{\tilde{P}}$ (see introduction).

If $\tilde{P} \in \operatorname{SPR}_0$ then $\delta_{\tilde{P}} = 0$ and we obtain $P \in \operatorname{SPR}_0$. $\square$

Now we shall present a slightly modified version of Kosut and Friedlander's lemma 5.1 which ensures that $H_{ev} \in \operatorname{SPR}_0^+$ and $H_{zv} \in \underline{H}_0$ despite the time invariant uncertainty Δ_1 .

Lemma 3.2 (see lemma 5.1 [1])

For the adaptive system (2.15) - (2.17), (2.22), (2.23) with adaptation law (Σ_a) $H_{ev} \in \operatorname{SPR}_0^+$ and $H_{zv} \in \underline{H}_0$ if the following conditions are satisfied.

$$(i) \quad W_{Q^*}(s) = g \frac{s^{n-1} + \beta_1 s^{n-2} + \dots + \beta_{n-1}}{s^n + \alpha_1 s^{n-1} + \dots + \alpha_n} = g \frac{N_Q(s)}{D_Q(s)}$$

(ii) $N_Q(s)$ is a stable monic polynomial

(iii) $g > 0$

(iv) The operator $K^* = (1 + W_{Q^*} D^*)^{-1} W_{Q^*} \frac{1}{C^*}$ $C^* \in \text{SPRO}_O^+$ and its transfer function $K^*(s) = g \frac{K_1(s)}{K_2(s)}$ where $K_1(s)$ and $K_2(s)$ are monic stable polynomials.

(v) $\Delta_1 \in B_{H_O}(\delta)$ where $\delta(\omega) < \delta'(\omega) = \frac{\eta(\omega)}{\eta(\omega) |T^*(j\omega)| + |S^*(j\omega)|}$

In case in (iv) we have $K^* \in \text{SPR}_O^+$ then $H_{ev} \in \text{SPR}_O$ and

$H_{zv} \in H_O$. □

Note: In the time varying case we shall give some explicit conditions on V as well as on θ^* which guarantee that (iv) is satisfied.

(see chapter 4, theorem 1)

The proof of lemma 3.2 is very similar to the proof of Kosut and Friedlander's lemma 5.1. The only difference is that the first part uses the first part of lemma 3.1.

3.2 - Extension to the case $n^* = 2$

We define n^* as the relative degree of the operator W_{Q^*} i.e. the difference between the number of poles and number of zeros of the transfer function $W_{Q^*}(s)$.

We now generalize the analysis to the case $n^* = 2$. Following Kosut and Friedlander [7], instead of considering the adaptive error control system (2.24) with the adaptation law (Σ_a) we use the following adaptive error system (3.10)

$$\left\{ \begin{array}{l} e = e^* - H_{ev_L} v_L \\ z_L = z^*_L - H_{zv} v_L \end{array} \right. \quad \text{with the adaptation law}$$

$$(\Sigma_{a_L}): \left\{ \begin{array}{l} \psi_L = z_L e \\ \dot{\hat{\theta}} = \Gamma \psi_L ; \Gamma = \Gamma^T > 0 \\ v_L = \hat{\theta}^T z_L \end{array} \right.$$

(3.11)

where

$$\left\{ \begin{array}{l} H_{ev_L} \triangleq H_{ev} L \\ z_L \triangleq L^{-1} z \\ z^*_L \triangleq L^{-1} z^* \end{array} \right. \quad (3.12)$$

and L is a linear time invariant operator whose transfer function $L(s) = 1 + \rho s$ is such that $L^{-1}(s)$ is stable i.e. $\rho > 0$.

The new adaptive error system (3.10) with adaptation law (Σ_{a_L}) is depicted in Fig. 3.1.

Now we have to prove that this adaptive error system is realizable and that it allows us to extend Kosut and Friedlander's results to the case $n^* = 2$. Consider the MRAC system depicted in Fig. 3.2.

- Fig. 3.1 -

- Fig. 3.2 -

The input to the plant, as shown in Fig. 3.2, is

$$u(t) = -\dot{\hat{c}}_0(t)r(t) - \rho \dot{\hat{c}}_0 L^{-1}r(t) - \hat{c}^T x_1(t) - \rho \dot{\hat{c}}^T L^{-1}r(t) \\ - \dot{\hat{d}}_0(t)y_p(t) - \rho \dot{\hat{d}}_0 L^{-1}y_p(t) - \hat{d}^T x_2(t) - \rho \dot{\hat{d}}^T L^{-1}x_2(t) .$$

which implies using (2.6) that

$$u(t) = - \sum_{i=0}^{2n-1} \hat{\theta}_i(t) z_i(t) - \rho \sum_{i=0}^{2n-1} \dot{\hat{\theta}}_i(t) L^{-1} z_i(t) \quad (3.13)$$

We know from (2.8) that $\forall i \in \{0, \dots, 2n-1\}$ $\hat{\theta}_i(t) = \bar{\theta}_i(t) + \theta_i^*$ where $\bar{\theta}_i(t)$ is the i^{th} -component of the adaptive gain error $\bar{\theta}(t)$ and θ_i^* is the i^{th} -component of the tuned parameter vector θ^* .

$$(3.13) \Rightarrow u(t) = - \sum_{i=0}^{2n-1} (\bar{\theta}_i(t) z_i(t) + \rho \dot{\bar{\theta}}_i(t) L^{-1} z_i(t)) \\ - c_0^* r(t) - c^{*T} x_1(t) - d_0^* y_p(t) - d^{*T} x_2(t) \quad (3.14)$$

Let us compute $L \bar{\theta}^T z$ where L is the time invariant operator whose transfer function is $L(s) = 1 + \rho s$.

$$L \bar{\theta}^T z_L = L \bar{\theta}^T L^{-1} z = (1 + \rho \frac{d}{dt}) (\bar{\theta}^T L^{-1} z) = \bar{\theta}^T L^{-1} z + \\ \rho (\bar{\theta}^T L^{-1} z + \bar{\theta}^T \frac{d}{dt} (L^{-1} z)) = \bar{\theta}^T (1 + \rho \frac{d}{dt}) L^{-1} z + \rho \bar{\theta}^T z_L$$

$$\text{Hence } L \bar{\theta}^T z_L = \bar{\theta}^T z + \rho \bar{\theta}^T z_L \quad (3.15)$$

which implies that

$$\forall t \in \mathbb{R}_+ (Lv_L)(t) = (L \bar{\theta}^T z_L)(t) = \sum_{i=0}^{2n-1} (\bar{\theta}_i(t) z_i(t) + \rho \dot{\bar{\theta}}_i(t) (L^{-1} z_i)(t))$$

which is a realizable signal.

(3.16)

Using (3.14) it follows that

$$\forall t \in \mathbb{R}_+ u(t) = -(Lv_L)(t) - c_0^* r(t) - c^{*T} x_1(t) - d_0^* y_p(t) - d^{*T} x_2(t)$$

Taking Laplace transforms and using (2.4) we get

$$u(s) = -C^*(s) r(s) - D^*(s) Y_p(s) - \frac{C^*(s)}{c_o^*} L(s) v_L(s) \quad (3.17)$$

$$\text{hence } u = -C^* r - D^* Y_p - \frac{1}{c_o^*} C^* L v_L \quad (3.18)$$

When $\hat{\theta} = \theta^*$, $\tilde{\theta} = 0$ and $v_L = 0$.

As in chapter 2 we define then the tuned signals u^* , Y_p^* , e^* and z_L^* with $z_L^* = L^{-1} z^*$ and compute them (see (2.16) - (2.19)). Then we get the operators H_{ev_L} and H_{zv} defined by (3.10).

$H_{ev_L} = H_{ev} L$ and H_{zv} is unchanged. (see (2.22) and (2.23)).

$$\text{Set } K_L^* \triangleq H_{ev_L}(\Delta_1 = 0) = (1 + W_{Q^*} D^*)^{-1} W_{Q^*} \frac{C^*}{c_o^*} L \quad (3.19)$$

$$\text{and } T^* = (1 + W_{Q^*} D^*)^{-1} W_{Q^*} D^* = 1 - S^* \quad (3.20)$$

$$\text{Using (3.6) we get } H_{ev_L} = K_L^* (1 + \Delta_1) (1 + \Delta_1 T^*)^{-1} \quad (3.21)$$

Clearly $K_L^* = K^* L$. There using (3.7), (3.8) and (3.9) we get

$$H_{ev} = \begin{bmatrix} 0 \\ v K_L^* L^{-1} D^* S^* (1 + \Delta_1 T^*)^{-1} T^{*-1} \\ K_L^* L^{-1} (1 + \Delta_1) (1 + \Delta_1 T^*)^{-1} \\ v K_L^* L^{-1} (1 + \Delta_1) (1 + \Delta_1 T^*)^{-1} \end{bmatrix} \quad (3.22)$$

$$e^* = \begin{bmatrix} - (c_o^* K^* (1 + \Delta_1) (1 + \Delta_1 T^*)^{-1} + W_M) & S^* (1 + \Delta_1 T^*)^{-1} \end{bmatrix} \begin{bmatrix} r \\ w \end{bmatrix} \quad (3.23)$$

$$z_L^* = \begin{bmatrix} L^{-1} & 0 \\ L^{-1} v (-1 + T^* (1 + \Delta_1) (1 + \Delta_1 T^*)^{-1}) & -L^{-1} v D^* S^* (1 + \Delta_1 T^*)^{-1} \\ -L^{-1} c_o^* K^* L^{-1} (1 + \Delta_1 T^*)^{-1} & L^{-1} S^* (1 + \Delta_1 T^*)^{-1} \\ -L^{-1} v c_o^* K^* L^{-1} (1 + \Delta_1) (1 + \Delta_1 T^*)^{-1} & L^{-1} v S^* (1 + \Delta_1 T^*)^{-1} \end{bmatrix} \begin{bmatrix} r \\ w \end{bmatrix} \quad (3.24)$$

Now we present lemma 3.3 which is the analog of lemma 3.2 in the case $n^* = 2$.

Lemma 3.3

For the adaptive system (2.15), (2.17), (3.12), (3.21), (3.22)

$H_{ev_L} \in \text{SPR}_O^+$ and $H_{zv} \in H_O$ if the following conditions are satisfied:

$$(i') \quad W_{Q^*}(s) = g \frac{s^{n-2} + \beta_1 s^{n-3} + \dots + \beta_{n-2}}{s^n + \alpha_1 s^{n-1} + \dots + \alpha_n} = g \frac{N_Q'(s)}{D_Q'(s)}$$

(ii') $N_Q'(s)$ is a stable monic polynomial

(iii') $g > 0$

(iv') The operator $K_L^* \in \text{SPR}_O^+$ and its transfer function

$$K_L^*(s) = g \frac{K_{1L}(s)}{K_{2L}(s)} \quad \text{where } K_{1L}(s) \text{ and } K_{2L}(s) \text{ are monic}$$

stable polynomials.

$$(v') \quad \Delta_1 \in B_{H_O}(\delta) \quad \text{where } \delta(\omega) < \bar{\delta}(\omega) = \frac{\eta(\omega)}{\eta(\omega) |T^*(j\omega)| + |S^*(j\omega)|}$$

(vi') $V(s) = (sI - \Lambda)^{-1}b$ where Λ is an $(n-1 \times n-1)$ asymptotically stable matrix and such that $1 + c^* V(s)$ and $d_O^* + d^* V(s)$ have no zeros in the right half plane $\text{Re}(s) > 0$. $\square$

Before proving this lemma we make the following remarks

$$(3.3.a) \quad \text{From (2.3) we know that } V(s) = \frac{1}{\det(sI - \Lambda)} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-2} \\ s \end{bmatrix}$$

where $\det(sI - \Lambda)$ is a monic polynomial of degree $n-1$ in s .

(vi') implies that $\det(sI - \Lambda)$ has no zeros in the open right half plane $\text{Re}(s) > 0$. Since the denominators of $1 + c^* V(s)$ and of $d_O^* + d^* V(s)$ are both equal to $\det(sI - \Lambda)$, it is clear that

$1 + c^* V(s)$ and $d_o^* + d^* V(s)$ have no poles in the open right half plane $\operatorname{Re}(s) > 0$.

(3.3.b) From (2.3) it is clear that $V_j \in H_o \quad \forall j \in \{1, \dots, n-1\}$

$$(3.3.c) \quad \frac{C^*(s)}{c_o^*} = \frac{1}{1+c^* V(s)} \quad \text{has relative degree } 0$$

$$D^*(s) = \frac{d_o^* + d^* V(s)}{1+c^* V(s)} \quad \text{has relative degree } 0$$

$$\Rightarrow 1 + W_{Q^*}(s) D^*(s) \quad \text{has relative degree } 0$$

$$\text{Hence } K_L^*(s) = (1 + W_{Q^*}(s) D^*(s))^{-1} W_{Q^*}(s) \frac{C^*(s)}{c_o^*} L(s)$$

$$\text{has relative degree } n_K^* = \operatorname{rel deg} W_{Q^*} + \operatorname{rel deg} \frac{C^*}{c_o^*} + \operatorname{rel deg} L$$

$$- \operatorname{rel deg} (1 + W_{Q^*} D^*) = 2 + 0 - 1 - 0 = 1 \in \{0, 1\}$$

hence (iv') is possible according to the last remark of part 1.1.

(3.3.d) As for lemma 3.2, lemma 3.3 still holds when one replaces

$$H_{ev_L}^+ \in \operatorname{SPR}_o^+ \text{ by } H_{ev_L} \in \operatorname{SPR}_o \text{ and } K_L^* \in \operatorname{SPR}_o^+ \text{ by } K_L^* \in \operatorname{SPR}_o.$$

Now we prove lemma 3.3. The proof is similar to Kosut and Friedlander's proof of lemma 3.2. (see [1]). First we state the following fact (see [17])

Fact: If G is any scalar operator in $\underline{G}$ then its transfer function $G(s)$ has a coprime factorization in $\underline{H}$, i.e. there exist operators N, D, A and B in $\underline{H}$ whose respective transfer functions $N(s), D(s), A(s)$ and $B(s)$ are such that:

$$G(s) = \frac{N(s)}{D(s)} \quad \text{and} \quad A(s)N(s) + B(s)D(s) = 1 \quad \square$$

$$(3.3.e): \quad \frac{C^*(s)}{c_o^*} = \frac{1}{1+c^* V(s)} \Rightarrow \frac{1}{c_o^*} C^* \in \underline{G}. \quad \text{Hence it has a coprime}$$

factorization in $\underline{H}$: N_C, D_C, A_C and B_C . The operators N_C and D_C are such that $N_C(s) = 1$ and $D_C(s) = 1 + c^* V(s)$.

(by assumption (vi') $1 + c^*T V(s)$ is stable).

(3.3.f): $D^*(s) = \frac{d_o^* + d^*T V(s)}{1 + c^*T V(s)} \Rightarrow D^* \in \underline{G}$. Hence it has a coprime

factorization in $\underline{H}$: N_D, D_D, A_D and B_D . The operators N_D and D_D are such that $N_D(s) = d_o^* + d^*T V(s)$ and $D_D(s) = 1 + c^*T V(s)$. (by assumption (vi') $d_o^* + d^*T V(s)$ is stable).

(3.3.g): assumption (i') $\Rightarrow W_{Q^*} \in \underline{G}$. Hence it has a coprime factorization in $\underline{H}$: $N_{Q^*}, D_{Q^*}, A_{Q^*}$ and B_{Q^*} . D_{Q^*} is such that $\text{rel deg } D_{Q^*} = 0$.

Lemma B₁ ([1]): The elements of the transfer function matrix (x, w)

into (e^*, z_L^*, y_p, u) all belong to $\underline{H}$ if:

$$(a) \quad Q^{-1} \triangleq (N_{Q^*} N_D + D_{Q^*} D_D)^{-1} \in \underline{H}$$

$$(b) \quad \delta(\omega) |T^*(j\omega)| < 1$$

$$\text{where } T^* = (1 + W_{Q^*} D^*)^{-1} W_{Q^*} D^* = (N_{Q^*} N_D + D_{Q^*} D_D)^{-1} N_{Q^*} N_D$$

□

Let us prove that (a) and (b) are satisfied.

$$(a) \quad K_L^* = (1 + W_{Q^*} D^*)^{-1} W_{Q^*} \frac{C^*}{C_o} L; \text{ then using (3.3.e), (3.3.f) and}$$

$$(3.3.g) \text{ we get } K_L^* = Q^{-1} N_{Q^*} N_C L. \quad (3.25)$$

$$\text{Hence } Q = K_L^{*-1} N_{Q^*} N_C L \quad (3.26)$$

$$(3.3.c) \Rightarrow \text{rel deg } K_L^{*-1} = - \text{rel deg } K_L^* = -1$$

$$(3.3.g) \Rightarrow \text{rel deg } N_{Q^*} = \text{rel deg } W_{Q^*} - \text{rel deg } D_{Q^*} = 2 - 0 = 2$$

$$(3.3.e) \Rightarrow \text{rel deg } N_C = 0$$

$$\text{rel deg } L = -1$$

$$\text{Hence } \text{rel deg } Q = -1 + 2 + 0 - 1 = 0, \text{ therefore } \text{rel deg } Q^{-1} = 0$$

and so Q^{-1} is proper.

(3.27)

From (3.26) we obtain:

$$Q^{-1} = K_L^* \frac{1}{N_{Q^*}} \frac{1}{N_C} L^{-1} = K_L^* \frac{1}{N_{Q^*}} L^{-1} \text{ using (iv')} \text{ and (3.3.e)}$$

(iv') implies $K_L^* \in \text{SPR}_O^+ C \underline{H}_O$ and so K_L^* is exponentially stable

(ii') and (3.3.g) imply $N_{Q^*}^{-1}$ is exponentially stable

$\rho > 0$ implies L^{-1} is exponentially stable

Hence Q^{-1} is exponentially stable.

(3.28)

(3.27) and (3.28) imply that $Q^{-1} \in H$ therefore (a) is satisfied.

(b) Assumption (v') implies $\delta(\omega) |T^*(j\omega)| < \frac{\eta(\omega) |T^*(j\omega)|}{\eta(\omega) |T^*(j\omega)| + |S^*(j\omega)|} < |$

hence (b) is satisfied.

Therefore the conditions (a) and (b) of lemma B1 are satisfied.

Proof of lemma 3.3

We prove first that $H_{ev_L} \in \text{SPR}_O^+$.

$$H_{ev_L} = K_L^* (1 + \Delta_1) (1 + \Delta_1 T^*)^{-1} \quad (\text{see (3.21)})$$

Set $\bar{H}_{ev_L} \triangleq K_L^*$ and $1 + \bar{H}_{ev_L} = (1 + \Delta_1) (1 + \Delta_1 T^*)^{-1}$

$$\Rightarrow \bar{H}_{ev_L} = \Delta_1 S^* (1 + \Delta_1 T^*)^{-1}$$

Applying lemma 3.1 we see that:

Assumption (iv') implies $\bar{H}_{ev_L} \in \text{SPR}_O^+$

(3.29)

Assumption (v') implies $|\Delta_1(j\omega)| < \bar{\delta}(j\omega)$ which in turn implies

$$\eta(\omega) |T^*(j\omega)| |\Delta_1(j\omega)| + |\Delta_1(j\omega)| |S^*(j\omega)| < \eta(\omega)$$

$$\text{and so } \eta(\omega) > \frac{|\Delta_1(j\omega) S^*(j\omega)|}{1 - |\Delta_1(j\omega) T^*(j\omega)|} > \frac{|\Delta_1(j\omega)| |S^*(j\omega)|}{|1 + \Delta_1(j\omega) T^*(j\omega)|}$$

$$\text{since } 1 - |\Delta_1(j\omega)T^*(j\omega)| \leq |1 + \Delta_1(j\omega)T^*(j\omega)|$$

$$\text{Hence } \eta(\omega) > |\overline{H}_{ev_L}(j\omega)| \quad (3.30)$$

Lemma B₁ implies in particular that the second component of the transfer matrix from (r,w) to e* (see (3.23)) namely $S^*(1+\Delta_1 T^*)^{-1} \in \underline{H}$ (3.31)

Moreover assumption (v') implies $\Delta_1 \in B_{\underline{H}_O}(\delta) \subset \underline{H}_O$. Hence

$$\overline{H}_{ev_L} = \Delta_1 S^*(1 + \Delta_1 T^*)^{-1} \in \underline{H}_O \quad (3.32)$$

$$\text{Then (3.30) and (3.32) imply that } \overline{H}_{ev_L} \in B_{\underline{H}_O}(\delta) \quad (3.33)$$

(3.29) and (3.33) together with lemma 3.1 allow us to say that

$$\overline{H}_{ev_L} \in \text{SPR}_O^+ \quad (3.34)$$

Let us prove now that $H_{zv} \in \underline{H}_O$. (see (3.12))

$$\text{for } j = 0 \quad H_{zv_0} = 0 \in \underline{H}_O \quad (3.35)$$

$$\text{for } j \in \{1, \dots, n-1\} \quad H_{zv_j} = V_j (K_L^* L^{-1} D^* T^{*-1}) S^* (1 + \Delta_1 T^*)^{-1}$$

Using (3.3.d), (3.3.e), (3.3.f) and (3.25), we have

$$K_L^* L^{-1} D^* T^{*-1} = (Q^{-1} N_{Q^* D} \frac{N_C}{D_C} L) L^{-1} \frac{N_D}{D_D} Q N_{Q^*}^{-1} N_D^{-1} = \frac{N_C}{D_C}$$

$$(3.3.e) \Rightarrow \text{rel deg } \frac{N_C}{D_C} = 0$$

$$\text{Assumption (vi')} \Rightarrow \frac{N_C}{D_C} = (1 + c^* T V(s))^{-1} \text{ is stable} \quad (3.36)$$

$$\left. \begin{array}{l} (3.36) \Rightarrow K_L^* L^{-1} D^* T^{*-1} \in \underline{H} \\ (3.31) \Rightarrow S^* (1 + \Delta_1 T^*)^{-1} \in \underline{H} \\ (3.3.b) \Rightarrow V_j \in \underline{H}_O \end{array} \right\} \Rightarrow H_{zv_j} \in \underline{H}_O \quad (3.37)$$

$$\text{for } j = n \quad H_{zv_n} = L^{-1} \overline{H}_{ev_L}$$

$$\left. \begin{aligned} (3.34) \Rightarrow H_{ev_L} \in SPR_O^+ \Rightarrow H_{ev_L} \in \underline{H_O} \\ \rho > 0 \Rightarrow L^{-1} \in \underline{H_O} \end{aligned} \right\} \Rightarrow H_{zv_n} \in \underline{H_O} \quad (3.38)$$

$$\text{for } j \in \{n+1, \dots, 2n-1\} \quad H_{zv_j} = v_{j-n} H_{zv_n}$$

$$\left. \begin{aligned} (3.3b) \Rightarrow \forall j \in \{n+1, \dots, 2n-1\} \quad v_{j-n} \in \underline{H_O} \\ (3.38) \Rightarrow H_{zv_n} \in \underline{H_O} \end{aligned} \right\} \Rightarrow H_{zv_j} \in \underline{H_O} \quad (3.39)$$

$$(3.35), (3.37), (3.38) \text{ and } (3.39) \Rightarrow H_{zv} \in \underline{H_O} \quad (3.40)$$

which proves lemma 3.3. $\square$

Hence in the case $n^* = 2$, we have a lemma which ensures that $H_{ev} \in SPR_O^+$ and $H_{zv} \in \underline{H_O}$ despite the time invariant uncertainty Δ_1 . This allows us then to apply the global stability theorem A.

Note that the realizability of the adaptive system (see Fig. 3.1) relies upon the availability of the signal $\dot{\bar{\theta}}$ through the adjustment law $\dot{\bar{\theta}} = \Gamma z_L e$.

For $n^* \geq 2$, we could take $L_{n^*}(s) = 1 + \rho_1 s + \dots + \rho_{n^*-1} s^{n^*-1}$ and adopt the following adaptation law

$$(\Sigma_{L_{n^*}}) = \left\{ \begin{aligned} \psi_{L_{n^*}} &= z_{L_{n^*}} e \\ \Sigma_M: \dot{\bar{\theta}} &= \Gamma \psi_{L_{n^*}}; \quad \Gamma = \Gamma^T > 0 \\ v_{L_{n^*}} &= \bar{\theta}^T z_{L_{n^*}} \end{aligned} \right.$$

But this approach would require higher derivatives of $\bar{\theta}$ which are not available and hence requires the presence of differentiators

in the control system and this, in general, is not realizable.

Therefore this method doesn't work for the case $n^* \geq 2$.

Unfortunately the idea of augmenting the model output by an additional output $y_a = w_M L((L^{-1} \tilde{\theta}^T - \tilde{\theta}^T L^{-1}) z + \alpha e z^T z_L)$ (see [14], [15]) doesn't work either, at least not via an analog of lemmas 3.2 and 3.3.

CHAPTER 4

CONDITIONS FOR GLOBAL STABILITY: TIME VARYING CASE

We shall assume from now on that the two uncertainty operators Δ_1 and Δ_2 are not necessarily equal to zero and characterized by an upper bound on their Y -gain where Y is an inner product space introduced in chapter 1.

4.1 - Four lemmas and a theorem on passive time varying operators.

Let us first present three passivity lemmas. Recall that Y_e is the extended space of the inner product space Y .

Lemma 4.1

Let G be a strictly passive operator $Y_e \rightarrow Y_e$ with finite Y -gain: $\gamma_Y(G)$ which means that:

(4.1.a) For all $u \in Y_e$ and for all $T \in \mathbb{R}_+$ there exist some constants $\delta_G > 0$ and β_G s.t.

$$\langle u | Gu \rangle_T \geq \delta_G \|u_T\|^2 + \beta_G$$

(4.1.b) $\gamma_Y(G) < \infty$

Then the operator $(I + \Delta M)G: Y_e \rightarrow Y_e$ is strictly passive if the Y -gain of the operator $\Delta M: Y_e \rightarrow Y_e$ is such that:

(4.1.c) $\gamma_Y(\Delta M) \leq \frac{\alpha}{\gamma_Y(G)}$ for some constant $\alpha < \delta_G$.

Moreover $\delta_{(I+\Delta M)G} = \delta_G - \alpha$ and $\beta_{(I+\Delta M)G} = \beta_G$. $\square$

Note that G and ΔM may be time varying or time invariant operators.

Proof: For all $u \in Y_e$ and for all $T \in \mathbb{R}_+$, we have

$$\langle u | (I + \Delta M)G u \rangle_T = \langle u | G u \rangle_T + \langle u | \Delta M G u \rangle_T \quad (4.2)$$

Applying the Cauchy Schwarz inequality we get $\forall u \in Y_e \quad \forall T \in \mathbb{R}_+$

$$|\langle u | \Delta M G u \rangle_T| \leq \|u_T\| \|(\Delta M G u)_T\| = \|u_T\| \|(\Delta M G u_T)_T\|$$

since $\Delta M G$ is causal.

$$\leq \|u_T\| \|\Delta M G u_T\| \text{ since truncation reduces the norm}$$

$$\leq \gamma_Y(\Delta M G) \|u_T\|^2 \leq \gamma_Y(\Delta M) \gamma_Y(G) \|u_T\|^2 \quad (4.3)$$

applying the definition and a standard property of the Y -gain.

If $\langle u | \Delta M G u \rangle_T \geq 0$ then (4.1a) and (4.2) imply that

$$\langle u | (I + \Delta M) G u \rangle_T \geq \delta_G \|u_T\|^2 + \beta_G$$

If $\langle u | \Delta M G u \rangle_T < 0$ then

$$\langle u | \Delta M G u \rangle_T = -|\langle u | \Delta M G u \rangle_T| \geq -\gamma_Y(\Delta M) \gamma_Y(G) \|u_T\|^2$$

$$\geq -\alpha \|u_T\|^2 \text{ using (4.1c) and (4.3)}$$

hence $\langle u | (I + \Delta M) G u \rangle_T \geq (\delta_G - \alpha) \|u_T\|^2 + \beta_G$ using (4.1.a) and (4.2)

Therefore $\forall u \in Y_e, \forall T \in \mathbb{R}_+, \langle u | (I + \Delta M) G u \rangle_T \geq (\delta_G - \alpha) \|u_T\|^2 + \beta_G$

□

Lemma 4.2

Let G be a linear strictly passive operator $Y_e \rightarrow Y_e$ with finite Y -gain: $\gamma_Y(G)$ which means that:

(4.4a) For all $u \in Y_e$ and for all $T \in \mathbb{R}_+$ there exists some constant

$$\delta_G > 0 \text{ s.t.}$$

$$\langle u | G u \rangle_T \geq \delta_G \|u_T\|^2$$

(4.4.b) $\gamma_Y(G) < \infty$

Then the operator $G(I + \Delta M): Y_e \rightarrow Y_e$ is strictly passive if the Y -gain of the operator $\Delta M: Y_e \rightarrow Y_e$ is such that:

(4.4.c) $\gamma_Y(\Delta M) \leq \frac{\alpha}{\gamma_Y(G)}$ for some constant $\alpha < \delta_G$.

Moreover $\delta_{G(I+\Delta M)} = \delta_G - \alpha$ and $\beta_{G(I+\Delta M)} = 0$ $\square$

Note that G and ΔM may be time varying or time invariant operators.

Proof: $\forall u \in Y_e, \forall T \in \mathbb{R}_+$

$$\langle u | G(I + \Delta M)u \rangle_T = \langle u | G u \rangle_T + \langle u | G \Delta M u \rangle_T$$

since G is linear. Then we apply the same arguments as in the proof of lemma 4.1 and we get

$$\forall u \in Y_e, \forall T \in \mathbb{R}_+ \quad |\langle u | G \Delta M u \rangle_T| \leq \gamma_Y(G) \gamma_Y(\Delta M) \|u_T\|^2$$

As in the proof of lemma 4.1 we also get:

$$\text{If } \langle u | G \Delta M u \rangle_T \geq 0 \text{ then } \langle u | G(I + \Delta M)u \rangle_T \geq \delta_G \|u_T\|^2$$

$$\text{If } \langle u | G \Delta M u \rangle_T < 0 \text{ then } \langle u | G(I + \Delta M)u \rangle_T \geq (\delta_G - \alpha) \|u_T\|^2$$

$$\text{hence } \forall u \in Y_e, \forall T \in \mathbb{R}_+ \quad \langle u | G(I + \Delta M)u \rangle_T \geq (\delta_G - \alpha) \|u_T\|^2$$

$\square$

Lemma 4.3

$$\text{Consider } e = u - H_2 H_1 e \quad (4.5)$$

where H_1 and H_2 are operators $Y_e \rightarrow Y_e$.

Assume that:

- (4.6.a) for each input u in Y_e , (4.5) has a unique solution for e in Y_e .

(4.6.b) H_1 is linear

(4.6.c) H_1 and H_2 are strictly passive i.e. $\exists \delta_1 > 0, \delta_2 > 0, \beta_2$ s.t.

$$\langle x | H_1 x \rangle_T \geq \delta_1 \|x_T\|^2$$

$$\langle x | H_2 x \rangle_T \geq \delta_2 \|x_T\|^2 + \beta_2 \quad \forall x \in Y_e, \quad \forall T \in \mathbb{R}_+$$

(4.6.d) H_1 and H_2 are bounded i.e. $\gamma_Y(H_1) < \infty$ and $\gamma_Y(H_2) < \infty$

Then the operator $M : Y_e \rightarrow Y_e$

$$u \mapsto Mu \triangleq v = H_1 e$$

(where e is the unique solution of the equation (4.5)) is strictly passive and

$$\begin{cases} \delta = \left(\frac{\delta_1}{\gamma_Y(H_1)^2} + \delta_2 \right) \frac{\delta_1^2}{\gamma_Y(I+H_1 H_2)^2} > 0 \\ \beta = \beta_2 \quad \square \end{cases}$$

Let us first note that:

If H is an operator: $Y_e \rightarrow Y_e$ which is linear and strictly passive then H^{-1} exists.

for: Suppose H^{-1} does not exist, $\exists x \in Y_e - \{0\}$ s.t.

$$Hx = 0 \text{ hence } \langle x | Hx \rangle_T = 0 \geq \delta \|x_T\|^2 \quad \forall T \in \mathbb{R}_+$$

which implies that $x = 0$. Hence we get a contradiction. $\square$

Proof of lemma 4.3:

$$\forall T \in \mathbb{R}_+ \quad \langle u | v \rangle_T = \langle u | H_1 e \rangle_T = \langle u | H_1 u \rangle_T - \langle u | H_1 H_2 v \rangle_T$$

since H_1 is linear and e satisfies (4.5).

$$\Rightarrow \langle u | (I+H_1 H_2) v \rangle_T = \langle u | H_1 u \rangle_T$$

Furthermore $\forall u \in Y_e, \quad \forall T \in \mathbb{R}_+$

$$\delta_1 \|u_T\|^2 \leq \langle u | H_1 u \rangle_T = \langle u | (I+H_1 H_2) v \rangle_T$$

(4.7)

But using the Cauchy-Schwarz inequality, it is clear that

$$\begin{aligned}
 \langle u | (I+H_1 H_2) v \rangle_T &\leq \|u_T\| \|((I+H_1 H_2)v)_T\| \\
 &= \|u_T\| \|((I+H_1 H_2)v)_T\| \quad \text{since } H_1 \text{ and } H_2 \text{ are causal} \\
 &\leq \|u_T\| \|(I+H_1 H_2)v_T\| \quad \text{since truncation reduces the norm} \\
 &\leq \gamma_Y(I+H_1 H_2) \|u_T\| \|v_T\| \quad \text{by definition of the } Y\text{-gain.}
 \end{aligned}$$

This together with (4.7) implies that

$$\forall u \in Y_e, \forall T \in \mathbb{R}_+ \quad \delta_1 \|u_T\|^2 \leq \gamma_Y(I+H_1 H_2) \|u_T\| \|v_T\| \quad (4.8)$$

and so $\gamma_Y(I+H_1 H_2) \neq 0$

$$\text{if } u_T \neq 0 \text{ then (4.8) } \Leftrightarrow \delta_1 \|u_T\| \leq \gamma_Y(I+H_1 H_2) \|v_T\| \quad (4.9)$$

if $u_T = 0$ it is clear that (4.9) still holds

$$\begin{aligned}
 \forall T \in \mathbb{R}_+ \quad \langle u | v \rangle_T &= \langle u - u' | v \rangle_T + \langle u' | v \rangle_T = \langle e | v \rangle_T + \langle u' | v \rangle_T \\
 &= \langle e | H_1 e \rangle_T + \langle H_2 v | v \rangle_T \geq \delta_1 \|e_T\|^2 + \delta_2 \|v_T\|^2 + \beta_2
 \end{aligned}$$

applying (4.6.c)

$$\begin{aligned}
 v = H_1 e \Rightarrow \forall T \in \mathbb{R}_+ \quad \|v_T\| &= \|(H_1 e)_T\| = \|(H_1 e_T)_T\| \leq \|H_1 e_T\| \\
 &\leq \gamma_Y(H_1) \|e_T\|
 \end{aligned}$$

$$\text{Hence } \forall T \in \mathbb{R}_+ \quad \|e_T\| \geq \frac{1}{\gamma_Y(H_1)} \|v_T\|$$

$$\begin{aligned}
 \text{Hence } \forall T \in \mathbb{R}_+ \quad \langle u | v \rangle_T &\geq \left(\frac{\delta_1}{\gamma_Y(H_1)^2} + \delta_2 \right) \|v_T\|^2 + \beta_2 \\
 &\geq \left(\frac{\delta_1}{\gamma_Y(H_1)^2} + \delta_2 \right) \frac{\delta_1^2}{\gamma_Y(I+H_1 H_2)^2} \|u_T\|^2 + \beta_2
 \end{aligned}$$

□

Now we can state a theorem which provides conditions that ensure that the operator $H_{ev}: Y_e \rightarrow Y_e$ is strictly passive.

Theorem 1

For the adaptive system (2.15), (2.17), (2.22) with adaptation law (L_a) assume that:

(c01) The system is well-posed i.e. all inputs $r, w \in L_{\infty} \cap C$ produce signals $e, z, \tilde{\theta}, \dot{\tilde{\theta}} \in L_{\infty} \cap C$.

(c02) $r, w \in L_{\infty} \cap C$

(c1) Let $W_{Q^*}: Y_e \rightarrow Y_e$ be the time invariant operator defined by:

$$W_{Q^*}u = \bar{h} * u. \quad (\text{Recall } W_{Q^*} = W_{Q(a^*)} \in \{W_{Q(a)}, a \in \mathbb{R}^k\})$$

$$\cdot \exists \delta_{Q^*} > 0 \text{ s.t. } \operatorname{Re}(\bar{h}(j\omega)) > \delta_{Q^*} \quad \forall \omega \in \mathbb{R}$$

$$\cdot \gamma_Y(W_{Q^*}) < \infty$$

(c2) The time invariant uncertainty operator Δ_1 is such that its

$$Y\text{-gain } \gamma_Y(\Delta_1) \leq \frac{\alpha_0}{\gamma_Y(W_{Q^*})} \text{ for some constant } \alpha_0 < \delta_{Q^*}$$

(c3) The time varying uncertainty operator Δ_2 is such that its

$$Y\text{-gain } \gamma_Y(\Delta_2) \leq \frac{\alpha_1}{\gamma_Y((1+\Delta_1)W_{Q^*})} \text{ for some constant } \alpha_1 < \delta_{Q^*} - \alpha_0$$

(c4) Let $\Delta M: Y_e \rightarrow Y_e$ be defined by $\Delta M \triangleq 1 + D^*$ and whose

$$Y\text{-gain is s.t. } \gamma_Y(\Delta M) \leq \frac{\alpha_2}{\gamma_Y(W_P)} \text{ for some constant } \alpha_2 < \delta_{Q^*} - \alpha_0 - \alpha_1$$

(c5) Let G_2 and $\Delta M'$ be two operators: $Y_e \rightarrow Y_e$ defined by:

$$G_2 \triangleq (1+W_P D^*)^{-1} W_P D^* \text{ and } \Delta M' \triangleq 1 + D^{*-1} \frac{1}{c_0^2} C^*$$

and whose Y-gains are such that

$$\gamma_Y(G_2) < \infty \text{ and } \gamma_Y(\Delta M') \leq \frac{\alpha_3}{\gamma_Y(G_2)} \text{ for some constant } \alpha_3 < \delta_{G_2}$$

where δ_{G_2} is the constant of strict passivity of G_2 and is given

by

$$\delta_{G_2} = \left(\frac{\delta_{W_P D^*}}{\gamma_Y (W_P D^*)^2} + 1 \right) \frac{(\delta_{W_P D^*})^2}{\gamma_Y (1 + W_P D^*)^2}$$

$$\delta_{W_P D^*} \triangleq \delta_{Q^*} - \alpha_0 - \alpha_1 - \alpha_2$$

(Note that G_2 is necessarily strictly passive by (c1) - (c4))

then the operator $H_{ev} = (I + W_P D^*)^{-1} W_P \frac{1}{c_o^*} D^*$ is strictly passive.

Moreover $\delta_{ev} = \delta_{G_2} - \alpha_3$ and $\beta_{ev} = 0$. $\square$

Before proving this theorem we make the following remarks:

(c1) means that W_{Q^*} is strictly positive real.

(c2) and (c3) impose upper bounds on the Y -gains of Δ_1 and Δ_2 .

(c4) and (c5) are conditions on the operators C^* and D^* i.e.

on the components of the tuned parameter vector θ^* as well as on the operator V and so on the design of the controller.

When $\Delta_2 = 0$, we are back to the time invariant case. Then (c1),

(c4) and (c5) are explicit conditions that guarantee that the operator k^* is strictly passive (see lemma 3.2 condition (iv)).

We now prove the theorem using the three previous passivity lemmas.

Proof: $W_P = (1 + \Delta_2)(1 + \Delta_1) W_{Q^*}$ (see (1.2.4))

Condition (c1) implies that W_{Q^*} is strictly passive with finite Y -gain. Hence lemma 4.1 together with conditions (c1) and (c2) allows us to say that:

$W_Q = (1 + \Delta_1) W_{Q^*}$ is strictly passive and its constant of strict passivity

$\delta_Q = \delta_{Q^*} - \alpha_0 > 0$. Moreover $\beta_Q = \beta_{Q^*} = 0$ (W_{Q^*} is linear)

Applying the standard properties of the γ -gain we have

$$\gamma_Y(W_Q) \leq \gamma_Y(W_{Q^*}) + \gamma_Y(\Delta_1) \gamma_Y(W_{Q^*}) \leq \gamma_Y(W_{Q^*}) + \alpha_0$$

(using (c1) and (c2)). Hence $\gamma_Y(W_Q) < \infty$ and lemma 4.1 together with condition (c3) allows us to say that

$$W_P = (1+\Delta_2)W_{Q^*} \text{ is strictly passive. Moreover} \\ \delta_{W_P} = \delta_Q - \alpha_1 = \delta_{Q^*} - \alpha_0 - \alpha_1 > 0 \text{ and } \beta_{W_P} = \beta_Q = 0.$$

We know that:

$$\gamma_Y(W_P) \leq \gamma_Y(W_Q) + \gamma_Y(\Delta_2) \gamma_Y(W_Q) \leq \gamma_Y(W_Q) + \alpha_1 < \infty.$$

Hence lemma 4.2 together with condition (c4) allows us to say

that $W_{P D^*} = W_P(1+\Delta M)$ is strictly passive. Furthermore

$$\delta_{W_{P D^*}} = \delta_{W_P} - \alpha_2 = \delta_{Q^*} - \alpha_0 - \alpha_1 - \alpha_2 > 0 \text{ and } \beta_{W_{P D^*}} = \beta_{W_P} = 0.$$

(Recall that W_P is linear by assumption).

Clearly $W_{P D^*}$ is linear. Moreover $\gamma_Y(W_{P D^*}) \leq \gamma_Y(W_P) \gamma_Y(D^*)$

$$\leq \gamma_Y(W_P) (1+\gamma_Y(\Delta M)) \leq \gamma_Y(W_P) + \alpha_2 < \infty \text{ (using (c4))}.$$

Since $W_{P D^*}$ is strictly passive, $\gamma_Y(W_{P D^*}) < \infty$ and (c6) is satisfied then lemma 4.3 allows us to say that:

$$G_2 = (1+W_{P D^*})^{-1} W_{P D^*} \text{ is strictly passive with} \\ \delta_{G_2} = \left(\frac{\delta_{W_{P D^*}}}{\gamma_Y(W_{P D^*})^2} + 1 \right) \frac{(\delta_{W_{P D^*}})^2}{\gamma_Y(1+W_{P D^*})^2} > 0 \text{ and } \beta_{G_2} = 0.$$

Finally lemma 4.2 together with (c5) allows us to say that

$$H_{ev} = G_2(1+\Delta M) = (1+W_{P D^*})^{-1} W_P \frac{1}{C^*} C^* \text{ is strictly passive. Moreover}$$

$$\delta_{ev} = \delta_{G_2} - \alpha_3 \text{ and } \beta_{ev} = 0. \quad \square$$

Before presenting the next theorem we need to recall the

following lemma which is an application of a passivity theorem from [9] (see [9] p.181):*

Lemma 4.4

Assuming that for all $u' \in Y$ there exists a unique solution e' to the equation $e' = u' - W_P D^* e'$.

(4.10)

Then if $W_P D^*$ is passive, $u' \in Y$ implies $e^* - e \in Y$ and

$$\|e^* - e\| \leq \|u'\|.$$

□

Proof:

$$\forall T \in \mathbb{R}_+ \quad \langle u' | W_P D^* e' \rangle_T = \langle e' | W_P D^* e' \rangle_T + \langle W_P D^* e' | W_P D^* e' \rangle_T$$

since $u' = e' + W_P D^* e'$ by (4.10)

$\langle e' | W_P D^* e' \rangle_T \geq 0$ since $W_P D^*$ is a passive linear operator by assumption.

$$\langle W_P D^* e' | W_P D^* e' \rangle_T = \| (W_P D^* e')_T \|^2.$$

$$\text{Hence } \langle u' | W_P D^* e' \rangle_T \geq \| (W_P D^* e')_T \|^2 \quad \forall T \in \mathbb{R}_+.$$

Now by the Cauchy-Schwarz inequality,

$$\|u'_T\| \| (W_P D^* e')_T \| \geq \langle u' | W_P D^* e' \rangle_T \geq \| (W_P D^* e')_T \|^2 \quad \forall T \in \mathbb{R}_+.$$

$$\text{Therefore } \|u'_T\| \geq \| (W_P D^* e')_T \| = \| (e^* - e)_T \| \quad \forall T \in \mathbb{R}_+.$$

hence $e^* - e = \lim_{T \rightarrow \infty} (e^* - e)_T \in Y$ and $\|e^* - e\| \leq \|u'\|$. □

4.2 - Two global stability theorems

We now present a theorem that gives conditions under which the operators H_{ev} and $H_{zv}: Y_0 \rightarrow Y_0$ given by (2.22) and (2.23) are Y -stable.

Theorem 2

If conditions (c01) - (c5) of theorem 1 are satisfied and moreover if:

(c6) for each input $u \in Y$, the equation $e' = u' - W_p D^* e'$ has a unique solution $e' \in Y_0$.

(c7) $\gamma_Y(V_j) < \infty$ for all $j \in \{1, \dots, n-1\}$ where V_j is the j^{th} -component of the operator V defined by (2.4)

$$\text{Then } v \in Y \text{ implies } \begin{cases} e^* - e = W_p D^* e' = H_{ev} v \in Y \\ z^* - z = H_{zv} v \in Y \end{cases} \quad (4.11)$$

$$\text{Moreover } \|e^* - e\|_Y \leq N \|v\|_Y \text{ and } \|z^* - z\| \leq N' \|v\|_Y \quad (4.12)$$

where $N \triangleq \gamma_Y(D^*)^{-1} \frac{1}{C_0^*} C^*$

and $N' \triangleq N \max(2 \gamma_Y(D^*) N'', 1, N'')$ and $N'' \triangleq \max_j \gamma_Y(V_j)$

$j \in \{1, \dots, n-1\} \quad \square$

H_{ev} H_{zv}

- Fig. 4.1 -

Proof: $H_{ev} = (1 + W_P D^*)^{-1} W_P \frac{1}{C_O^*} C^*$ (see Fig. 4.1) .

Since conditions (co) - (c5) are satisfied, theorem 1 allows us to say that $W_P D^*$ is strictly passive (see proof of theorem 1) .

Hence since condition (c6) is satisfied, we can apply lemma 4.4 and so $u' \in Y$ implies $e^* - e \in Y$.

Moreover, since $\gamma_Y(D^{*-1} \frac{1}{C_O^*} C^*) = \gamma_Y(1 + \Delta M') \leq 1 + \gamma_Y(\Delta M') < \infty$ using (c4). Hence it is clear that $v \in Y$ implies $u' = D^{*-1} \frac{1}{C_O^*} C^* v \in Y$.

Therefore $e^* - e = H_{ev} v = (1 + W_P D^*)^{-1} W_P D^* u' \in Y$

and $|e^* - e| \leq |u'| \leq \gamma_Y(D^{*-1} \frac{1}{C_O^*} C^*) |v| \triangleq N |v|$ which proves

(4.11) .

From (2.23) we know that

$$H_{zv} = \begin{bmatrix} 0 & & & \\ -V_j D^* (1 + W_P D^*)^{-1} W_P \frac{1}{C_O^*} C^* + V_j \frac{1}{C_O^*} C^* & & & \\ \vdots & & & \\ (1 + W_P D^*)^{-1} W_P \frac{1}{C_O^*} C^* & & & \\ \vdots & & & \\ V_{j-n} (1 + W_P D^*)^{-1} W_P \frac{1}{C_O^*} C^* & & & \end{bmatrix} \begin{matrix} j = 0 \\ j \in \{1, \dots, n-1\} \\ \\ j = n \\ j \in \{n+1, \dots, 2n-1\} \end{matrix}$$

Assume that $v \in Y$

$$(i) \quad j = 0 \quad z_O^* - z_O = H_{zv} v = 0 \in Y \quad (4.13a)$$

$$(ii) \quad j \in \{1, \dots, n-1\}$$

$$\begin{aligned} z_j^* - z_j &= H_{zv_j} v = -V_j D^* H_{zv} v + V_j D^* D^{*-1} \frac{1}{C_O^*} C^* v \\ &= -V_j D^* (e^* - e) + V_j D^* u' \end{aligned}$$

$$\text{then } |z_j^* - z_j| \leq \gamma_Y(V_j) \gamma_Y(D^*) |e^* - e| + \gamma_Y(V_j) \gamma_Y(D^*) |u'|$$

$$\leq 2 \gamma_Y(V_j) \gamma_Y(D^*) \|u^-\| \quad (\text{see proof of lemma 4.4})$$

$$\leq 2 \gamma_Y(V_j) \gamma_Y(D^*) N \|v\| \quad (4.13b)$$

applying the first part of the theorem.

$$(c7) \text{ implies } \gamma_Y(V_j) < \infty \quad \forall j \in \{1, \dots, n-1\}$$

$$(c4) \text{ implies } \gamma_Y(D^*) < \infty$$

$$\text{therefore } \|z_j^* - z_j\| < \infty \quad (4.13c)$$

$$(iii) \quad j = n \quad \|z_n^* - z_n\| = \|H_{ev} v\| = \|e^* - e\| \leq N \|v\| < \infty \quad (4.13d)$$

$$(iv) \quad j \in \{n+1, \dots, 2n-1\}$$

$$z_j^* - z_j = H_{zv_j} v = V_{j-n} H_{ev} v$$

$$\text{hence } \|z_j^* - z_j\| \leq \gamma_Y(V_{j-n}) N \|v\| < \infty \quad (4.13e)$$

(4.13a), (4.13c), (4.13d), (4.13e) imply that $z^* - z \in Y$.

Moreover (4.13a), (4.13b), (4.13d), (4.13e) imply that

$$\forall j \in \{0, \dots, 2n-1\} \quad \|z_j^* - z_j\| \leq N \max(2 \gamma_Y(D^*) N^*, 1, N^*) \|v\|$$

where $N^* = \max_{j \in \{1, \dots, n-1\}} \gamma_Y(V_j)$ which exists by (c7).

$$\text{Hence } \forall j \in \{0, \dots, 2n-1\} \quad \|z_j^* - z_j\| \leq N' \|v\|$$

$$\text{Therefore } \|z^* - z\| \leq \max_{j \in \{0, \dots, 2n-1\}} \|z_j^* - z_j\| \leq N' \|v\|$$

$$\text{Finally } v \in Y \text{ implies } z^* - z \in Y \text{ and } \|z^* - z\| \leq N' \|v\|.$$

□

Now we can present theorem 3 which provides conditions under which the adaptive error system (2.20) with adaptation law (Σ_a) is guaranteed to have global L_2 -stability and performance properties despite the uncertainties Δ_1 and Δ_2 .

Hence we assume $Y = L_2(\mathbb{R}_+) = L_2$ (see example 1.2 in chapter I). Furthermore we will prove that the adaptive gain error $\tilde{\theta}$ belongs to $L_\infty(\mathbb{R}_+) = L_\infty = X$. (see example 1.1 in chapter 1).

Theorem 3

If conditions (c0) - (c7) of theorems 1 and 2 are satisfied and if $e^* \in L_2$ then

(i) $e, v \in L_2$ and

$$\|v\|_2 \leq b'_1 \triangleq \frac{1}{2\delta_{ev}} \left(\|e^*\|_2 + (\|e^*\|_2^2 - 4\lambda\delta_{ev})^{\frac{1}{2}} \right)$$

$$\|e\|_2 \leq b'_2 \triangleq \|e^*\|_2 + \gamma_2(H_{ev}) b'_1$$

$$\|e - e^*\|_2 \leq N_2 \|v\|_2 \leq N_2 b'_1$$

where δ_{ev} is the constant of strict passivity of H_{ev} which is given by theorem 1, $\lambda = -\frac{1}{2} \tilde{\theta}^T(0) \Gamma^{-1} \tilde{\theta}(0)$ and $N_2 \triangleq \gamma_2(D^{*-1} \frac{1}{c_0^*} C^*)$

(ii) $z - z^* \in L_2$ and $\|z - z^*\|_2 \leq N'_2 \|v\|_2 \leq N'_2 b'_1$

where $N'_2 \triangleq N_2 \max(2\gamma_2(D^*) N_2'', 1, N_2'')$

$$N_2'' \triangleq \max_{j \in \{1, \dots, n-1\}} \gamma_2(V_j)$$

(iii) $\tilde{\theta} \in L_\infty$ and $\|\tilde{\theta}\|_\infty \leq b'_3 \triangleq \left(\frac{\tilde{\theta}(0)^T \Gamma^{-1} \tilde{\theta}(0) + 2b'_1 b'_2}{v} \right)^{\frac{1}{2}}$

where v is a positive constant such that $\Gamma^{-1} \geq v I_{2n}$. $\square$

Note: the proof is very similar to Kosut and Friedlander's proof of theorem 1.A (see [1]).

Proof: Since conditions (c01) - (c7) are satisfied, theorems 1 and 2 allow us to say that

H_{ev} is a strictly passive operator: $L_{2e} \rightarrow L_{2e}$ i.e.

$$\exists \delta_{ev} > 0 \text{ s.t. } \langle x | H_{ev} x \rangle_{T2} \geq \delta_{ev} \|x_T\|_2^2 \quad \forall x \in L_{2e}, \quad \forall T \in \mathbb{R}_+ \quad (4.14)$$

$$\text{If } v \in L_2 \text{ then } e - e^* = -H_{ev} v \in L_2 \text{ and } \|e - e^*\|_2 \leq \|u'\|_2$$

$$\text{where } u' = D^{*-1} \frac{1}{c_0^*} C^* v \quad (\text{see fig. 4.1})$$

$$\text{Hence } \gamma_2(H_{ev}) \leq N_2 \leq \gamma_2(D^{*-1} \frac{1}{c_0^*} C^*) < \infty \quad (4.15)$$

We shall prove first that the map: $e \rightarrow v$ is passive

$$\forall T \in \mathbb{R}_+ \quad \langle e | v \rangle_{T2} = \langle e | \tilde{\theta}^T z \rangle_{T2} \quad \text{using (2.11)}$$

$$= \langle e | z^T \tilde{\theta} \rangle_{T2} = \langle z | \tilde{\theta} \rangle_{T2} = \langle \Gamma^{-1} \dot{\tilde{\theta}} | \tilde{\theta} \rangle_{T2} \quad \text{by (2.9)}$$

$$= \int_0^T \dot{\tilde{\theta}}^T \Gamma^{-1} \tilde{\theta} dt = \frac{1}{2} \int_0^T \frac{d}{dt} (\tilde{\theta}^T \Gamma^{-1} \tilde{\theta}) dt$$

since Γ^{-1} is symmetric by assumption

$$\Rightarrow \forall T \in \mathbb{R}_+ \quad \langle e | v \rangle_{T2} = \frac{1}{2} (\tilde{\theta}^T(T) \Gamma^{-1} \tilde{\theta}(T) - \tilde{\theta}^T(0) \Gamma^{-1} \tilde{\theta}(0)) \quad (4.16)$$

$$\Rightarrow \forall T \in \mathbb{R}_+ \quad \langle e | v \rangle_{T2} \geq -\frac{1}{2} \tilde{\theta}^T(0) \Gamma^{-1} \tilde{\theta}(0) = \lambda \quad (4.17)$$

hence $e \rightarrow v$ is passive.

Now let us show that $e, v \in L_2$.

Using (4.14) and (4.17) we get

$$\forall T \in \mathbb{R}_+ \quad \langle e^* | v \rangle_{T2} = \langle e + H_{ev} v | v \rangle_{T2} \geq \lambda + \delta_{ev} \|v_T\|_2^2$$

Using the Cauchy-Schwarz inequality it follows that

$$\lambda + \delta_{ev} \|v_T\|_2^2 \leq \langle e^* | v \rangle_{T2} \leq \|e_T^*\|_2 \|v_T\|_2$$

$$\text{implies } \delta_{ev} \|v_T\|_2^2 - \|e_T^*\|_2 \|v_T\|_2 + \lambda \leq 0$$

$$\text{implies } \|v_T\|_2 \leq \frac{1}{2\delta_{ev}} (\|e_T^*\|_2 + (\|e_T^*\|_2^2 - 4\lambda\delta_{ev})^{\frac{1}{2}}) \quad (4.18)$$

Now (4.15) gives $\gamma_2(H_{ev}) < \infty$. Therefore,

$$\|e_T\|_2 \leq \|e^*_T\|_2 + \gamma_2(H_{ev}) \|v_T\|_2 \quad (4.19)$$

since $e = e^* - H_{ev} v \Rightarrow e_T = e^*_T - H_{ev} v_T \quad \forall T \in \mathbb{R}_+$

knowing that H_{ev} is causal.

By assumption $e^* \in L_2$. Hence when $T \rightarrow \infty$ we get $\|v\|_2 < \infty$ from (4.18) and $\|e\|_2 < \infty$ from (4.19) which implies that $e, v \in L_2$.

The bounds b'_1 and b'_2 follow directly from (4.18) and (4.19). Therefore (i) is proven.

Clearly from theorem 2, $v \in L_2$ implies $z - z^* \in L_2$ and $\|z - z^*\|_2 \leq N'_2 \|v\|_2$ which proves (ii).

Now (4.16) implies that $\forall T \in \mathbb{R}_+ \quad \tilde{\theta}^T(T) \Gamma^{-1} \tilde{\theta}(T) \leq \tilde{\theta}^T(0) \Gamma^{-1} \tilde{\theta}(0) + 2 \langle e | v \rangle_{T2} \leq \tilde{\theta}^T(0) \Gamma^{-1} \tilde{\theta}(0) + 2 b'_1 b'_2$

Since $\Gamma^{-1} > 0$, $\exists v > 0$ s.t. $\Gamma^{-1} \geq v I$ and so

$$\forall T \in \mathbb{R}_+ \quad \|\tilde{\theta}(T)\|^2 \leq \frac{1}{v} (\tilde{\theta}^T(0) \Gamma^{-1} \tilde{\theta}(0) + 2 b'_1 b'_2) = b_3'^2$$

where $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^{2n}$ and hence

$$\|\tilde{\theta}\|_\infty \leq b_3' \text{ which proves (iii). } \quad \square$$

4.3 A stronger result

In this part we are going to present conditions that ensure that the signal $e - e^*$ belongs to L_∞ and tends to zero as $t \rightarrow \infty$.

Consider the adaptive system presented in chapter 2. We now introduce an integrator i.e. a stable linear time invariant operator R whose transfer function $R(s) = \frac{1}{1+\epsilon s}$ where $\epsilon > 0$, in the controller (see fig. 4.2). Therefore the input u to the plant W_p becomes (see (2.11))

$$u = -R(C_O^* r + c^{*T} V u + d_O^* y_p + d^{*T} V y_p) - R v \quad (4.20)$$

(see Chapter 2 for the definitions of θ^* and v)

$$\text{Hence } u(s) = - \frac{1}{1+\epsilon s} (C_O^* r(s) + c^{*T} V(s) u(s) + d_O^* y_p(s) + D^{*T} V(s) y_p(s))$$

$$= - \frac{1}{1+\epsilon s} v(s) \quad (4.21)$$

which gives

$$u(s) = - \frac{C_O^*}{1+\epsilon s + c^{*T} V(s)} r(s) - \frac{d_O^* + d^{*T} V(s)}{1+\epsilon s + c^{*T} V(s)} y_p(s) - \frac{1}{1+\epsilon s + c^{*T} V(s)} v(s) \quad (4.22)$$

- Fig.4.2 -

$$\text{Define } C^*(s) \triangleq \frac{c_o^*}{1+cs+c^*T V(s)} \text{ and } D^*(s) \triangleq \frac{d_o^*+d^*T V(s)}{1+cs+c^*T V(s)} \quad (4.23)$$

Let C^* and D^* denote the linear time invariant operators whose transfer functions are $C^*(s)$ and $D^*(s)$.

Then (4.22) yields

$$u = -C^* r - D^* y_p - \frac{1}{c_o^*} C^* v \quad (4.24)$$

Recall that the signals y_p , y_M , e , z , u^* , y_{p^*} , e^* and z^* are defined as follows:

$$\begin{aligned} y_p &= W_p u + w, \quad y_M = W_M r, \quad e = y_p - y_M, \\ z^T &= [r, Vu, y_p, Vy_p], \quad u^* = u \Big|_{\hat{\theta}=\theta^*}, \quad y_{p^*} = y_p \Big|_{\hat{\theta}=\theta^*} \end{aligned}$$

$$e^* = e \Big|_{\hat{\theta}=\theta^*} \quad \text{and} \quad z^* = z \Big|_{\hat{\theta}=\theta^*} \quad (4.25)$$

Moreover $v = \tilde{\theta}^T z = 0$ when $\hat{\theta} = \theta^*$.

The operators H'_{ev} and H'_{zv} are defined by the equalities

$$\begin{cases} e = e^* = H'_{ev} v \\ z = z^* = H'_{zv} v \end{cases} \quad (4.26)$$

Therefore using a similar computation as in chapter 2 we get

$$H'_{ev} = (1 + W_p D^*)^{-1} W_p \frac{1}{c_o^*} C^* \quad (4.27)$$

$$H'_{zv} = \begin{bmatrix} 0 \\ -VD^* H'_{ev} + V \frac{1}{c_o^*} C^* \\ H'_{ev} \\ V H'_{ev} \end{bmatrix} \quad (4.28)$$

The operator H_{ev}^- can be described by the following diagram.

- Fig. 4.3 -

We now have the following global stability theorem.

Theorem 4

For the adaptive system (4.25), (4.27), (4.28) with adaptation law (Σ_a) assume that:
the conditions (c01), (c02), (c1), (c2), (c3), (c7) of Theorem 3 are satisfied.

. (c4') Let ΔN be the operator: $L_{2e} \rightarrow L_{2e}$ defined by

$\Delta N \triangleq -1 + D^{*'}$ and whose L_2 -gain is such that $\gamma_2(\Delta N) \leq \alpha_3'$ for some constant $\alpha_3' < 1$.

(Note that $D^{*'}$ is then necessarily strictly passive with

$$\delta_{D^{*'}} = 1 - \alpha_3' \text{ and } \beta_{D^{*'}} = 0)$$

. (c5') Let G_2' and $\Delta N'$ be two operators $L_{2e} \rightarrow L_{2e}$ defined by

$G_2' \triangleq (1 + W_P D^{*'})^{-1} W_P$ and $\Delta N' \triangleq -1 + \frac{1}{C_0^{*'}} C^{*'}$ and whose L_2 -gains are such that:

$$\gamma_2(G_2') < \infty \text{ and } \gamma_2(\Delta N') \leq \frac{\alpha_4'}{\gamma_2(G_2')} \text{ for some constant}$$

$\alpha_4' < \delta_{G_2'}$ where $\delta_{G_2'}$ is the constant of strict passivity of G_2'

and is given by

$$\delta_{G_2'} = \left(\frac{\delta_{W_P}}{\gamma_2(W_P)^2} + \delta_{D^{*-}} \right) \frac{(\delta_{W_P})^2}{\gamma_2(1+W_P D^{*-})^2} > 0$$

moreover $\beta_{G_2'} = \beta_{D^{*-}} = 0$

(Note that G_2' is necessarily strictly passive by (c1) - (c3) and (c4')) .

. (c6') for all $u'' \in L_2$ there exists a unique solution $e'' \in L_{2\infty}$ to the equation $e'' = u'' - W_P D^{*-} e''$ (4.29)

. (c8') the operator $\frac{1}{c_0^*} C^{*-} \in \underline{H}_0$

(Note that the transfer function $\frac{C^{*-}(s)}{c_0^*}$ is now rational and strictly proper which was not the case before).

. (c9') $\gamma_{\infty}(W_{Q^*}) < \infty$, $\gamma_{\infty}(\Delta_1) < \infty$, $\gamma_{\infty}(\Delta_2) < \infty$ and $\gamma_{\infty}(D^{*-}) < \frac{1}{\gamma_{\infty}(W_P)}$

(Note that the three first inequalities merely imply that $\gamma_{\infty}(W_P) < \infty$)

. (c10') define $K^{*-} \triangleq \left. \begin{array}{l} H_{ev}' \\ \Delta_1 = \Delta_2 = 0 \end{array} \right| = (1+W_{Q^*} D^{*-})^{-1} W_{Q^*} \frac{1}{c_0^*} C^{*-}$ (4.30)

$K^{*-} \in \underline{H}_0$ and Δ_1, Δ_2 are sufficiently small as operators that the operator $H_{ev}' - K^{*-}$ maps L_2 into the set of functions of t which are in L_{∞} and tend to zero as $t \rightarrow \infty$.

(Note that the time invariant operator K^{*-} has a rational strictly proper transfer function since $(1+W_{Q^*}(s)D^{*-}(s))^{-1}W_{Q^*}(s)$ is rational and proper and $\frac{C^{*-}(s)}{c_0^*}$ is rational and strictly proper.)

Then if $e^* \in L_2 \cap L_{\infty}$

(i) $e, v \in L_2$ and

$$\|v\|_2 \leq \beta_1 \triangleq \frac{1}{2\delta'_{ev}} (\|e^*\|_2 + (\|e^*\|_2^2 - 4\lambda\delta'_{ev})^{\frac{1}{2}})$$

$$\|e\|_2 \leq \beta_2 \triangleq \|e^*\|_2 + \gamma_2(H'_{ev}) \beta_1$$

$$\|e - e^*\|_2 \leq \eta_2 \|v\|_2 \leq \eta_2 \beta_1$$

where δ'_{ev} is the constant of strict passivity of H'_{ev}

$$\delta'_{G_2} = \alpha_4$$

$$\delta'_{ev} = \delta'_{G_2}, \quad \lambda = -\frac{1}{2} \tilde{\theta}^T(0) \Gamma^{-1} \tilde{\theta}(0) \quad \text{and} \quad \eta_2 = -\frac{1}{\delta'_{D^*}} \gamma_2\left(\frac{1}{c_o^*} D^*\right).$$

Moreover $e \in L_\infty$ and $(e - e^*)(t) \rightarrow 0$ as $t \rightarrow \infty$

$$(ii) \quad z - z^* \in L_2 \quad \text{and} \quad \|z - z^*\|_2 \leq \eta'_2 \|v\|_2 \leq \eta'_2 \beta_1$$

where $\eta'_2 \triangleq \max(\eta_2^*, \eta_2, \eta_2 N_2^*)$

$$N_2^* \triangleq \max_{j \in \{1, \dots, n\}} \gamma_2(V_j) \quad \text{and} \quad \eta_2^* \triangleq \max(\gamma_2(D^*) \eta_2, \gamma_2\left(\frac{1}{c_o^*} C^*\right))$$

$$(iii) \quad \tilde{\theta} \in L_\infty \quad \text{and} \quad \|\tilde{\theta}\|_\infty \leq \beta_3 \triangleq \left(\frac{\tilde{\theta}(0)^T \Gamma^{-1} \tilde{\theta}(0) + 2\beta_1 \beta_2}{v} \right)^{\frac{1}{2}}$$

where v is a positive constant such that $\Gamma^{-1} \geq v I_{2n}$ $\square$.

Note that the new factorization of the crucial operator H'_{ev} (see Fig.

4.3) together with the introduction of the integrator R in the

controller allows us to get conditions on the magnitude of the uncertain-

ties Δ_1 and Δ_2 as well as on the operator V and on the tuned

parameter vector θ^* which ensure now that $e - e^* \in L_2 \cap L_\infty$ and

$e(t) \rightarrow e^*$ as $t \rightarrow \infty$; e^* being the tuned error i.e. the "best error"

that we can obtain in our problem formulation.

Proof:

. Conditions (c1) - (c3) ensure that W_p is strictly passive. More-

over $\delta_{W_p} = \delta_{Q^*} - \alpha_o - \alpha_1$ and $\beta_{W_p} = 0$ (see proof of theorem 1)

. Lemma 4.1 together with condition (c4') allows us to say that

$D^{*'} = 1 + \Delta N$ is strictly passive. Furthermore $\delta_{D^{*'}} = 1 - \alpha'_3$ and $\beta_{D^{*'}} = 0$.

. Lemma 4.2 and lemma 4.3 together with condition (c5') allow us to say that $H'_{ev} = G'_2 (1 + \Delta N') = (1 + W_P D^{*'})^{-1} W_P \frac{1}{C_0^*} C^{*'} is strictly passive$

with $\delta'_{ev} = \delta_{G'_2} - \alpha'_4$ and $\beta'_{ev} = \beta_{G'_2} = 0$.

. For all $T \in \mathbb{R}_+$ $\langle u'' | W_P e'' \rangle_{T2} = \langle e'' | W_P e'' \rangle_{T2} + \langle D^{*'}(e^* - e) | (e^* - e) \rangle_{T2}$

applying (4.29) and $e^* - e = W_P e''$ (4.31)

W_P is linear and passive, hence for all $T \in \mathbb{R}_+$ $\langle e'' | W_P e'' \rangle_{T2} \geq 0$

$D^{*'}$ is linear and strictly passive, hence for all $T \in \mathbb{R}_+$

$$\langle u'' | W_P e'' \rangle_{T2} \geq \langle D^{*'}(e^* - e) | (e^* - e) \rangle_{T2} \geq \delta_{D^{*'}} \|(e^* - e)_T\|_2^2$$

Applying the Cauchy-Schwarz inequality we get

$$\|(e^* - e)_T\|_2 \leq \frac{1}{\delta_{D^{*'}}} \|u''\|_2 \leq \frac{1}{\delta_{D^{*'}}} \|u''\|_2 \text{ for all } T \in \mathbb{R}_+$$

Hence if $u'' \in \frac{1}{C_0^*} C^{*'} v \in L_2$, $e^* - e \in L_2$

$$\text{and } \|e^* - e\|_2 \leq \frac{1}{\delta_{D^{*'}}} \|u''\|_2.$$

(c5') implies $\gamma_2 \left(\frac{1}{C_0^*} C^{*'} \right) < \infty$. Therefore if $v \in L_2$ $e^* - e \in L_2$

$$\text{and } \|e^* - e\|_2 \leq \frac{1}{\delta_{D^{*'}}} \gamma_2 \left(\frac{1}{C_0^*} C^{*'} \right) \|v\|_2$$

Now using the same kind of arguments as in the proof of Theorem 3

since H'_{ev} is strictly passive and the map $e \rightarrow v$ is passive we get:

$$e, v \in L_2 \text{ and } \|v\|_2 \leq \beta_1, \|e\|_2 \leq \beta_2 \text{ and } \|\delta\|_2 \leq \beta_3.$$

Furthermore $\|e^* - e\|_2 \leq \eta_2 \|v\|_2 \leq \eta_2 \beta_1$.

Using the expression of H'_{ev} given by (4.28) and the same kind

of arguments as in the proof of theorem 2 we get $z^* - z \in L_2$ and $\|z^* - z\|_2 \leq \eta_2^{-1} \|v\|_2 \leq \eta_2^{-1} \beta_1$.

(c01) and (c02) imply that $v \in C$. Hence $v \in L_2 \cap C$ and since $\frac{1}{c_0^*} C^* \in \underline{H}_0$ (condition (c8')) we can apply lemma A_2 (a).

(see Appendix A) and we obtain $u^* \in L_2 \cap L_\infty$. Then the small gain theorem (see e.g. [8], [9]) together with condition (c9') allows us to say that $(e^* - e) \in L_\infty$ and $\|e^* - e\|_\infty \leq \frac{\gamma_\infty(W_p)}{1 - \gamma_\infty(W_p D^*)} \|u^*\|_\infty$.

Therefore $e^* \in L_\infty$ implies $e \in L_\infty$.

Moreover by (c10') we have

$$(e^* - e)(t) = (K^* v)(t) + (H_{ev}' - K^*)v(t) \rightarrow 0 \text{ as } t \rightarrow \infty$$

since $K^* \in \underline{H}_0$ and $v \in L_2$ imply that $(K^* v)(t) \rightarrow 0$

as $t \rightarrow \infty$ by lemma A_2 (a) (see Appendix A) and (c10') implies that

$$((H_{ev}' - K^*) v)(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

□

CHAPTER 5

CONDITIONS FOR LOCAL STABILITY

In chapters 3 and 4, we have developed conditions that guarantee global stability. One of the conditions is that the operator H_{ev} has to be strictly passive which implies severe conditions on the time invariant and time varying uncertainty operators Δ_1 and Δ_2 . Hence this condition of strict passivity is very difficult to satisfy for an actual system (see [1]). These considerations motivate investigations concerning local stability.

In this chapter, we are going to apply Kosut's results on local stability (see [2]) to our case where Δ_2 is not necessarily equal to zero. Kosut's first theorem is modified and some conditions on the design of the controller as well as on the tuned signals are derived to ensure local L_∞ -stability. Note that the definition of local L_p -stability about a set S has been given in chapter 1.

Let us recall the equations of the adaptive error model with adaptation law (L_a) that has been introduced in chapter 2.

$$e = e^* = H_{ev} v \quad (5.1)$$

$$z = z^* - H_{zv} v \quad (5.2)$$

$$\dot{\tilde{\theta}} = \Gamma z e \quad (5.3)$$

$$v = \tilde{\theta}^T z \quad (5.4)$$

Following Kosut we rewrite (5.3) as:

$$\dot{\tilde{\theta}} = \tilde{\theta}_0 + L' z e \quad \text{where } L' = L^{-T} > 0 \quad (5.3')$$

Clearly the operator L' has a transfer function matrix

$$L'(s) = \frac{1}{s} \Gamma.$$

Starting from the system (Σ) we redefine the local error model of [2] as follows:

$$\begin{aligned} \dot{x} &= \bar{x} - G F(x) & (\Sigma_p) \\ \text{where } x &= \begin{bmatrix} \bar{e} \\ \bar{z} \\ \bar{\theta} \end{bmatrix} \in \mathbb{R}^{4n+1} & F(x) \triangleq \begin{bmatrix} \bar{z}^T \bar{\theta} \\ \bar{z} \\ \bar{e} \end{bmatrix} \in \mathbb{R}^{2n+1} \\ \bar{x} &= \begin{bmatrix} \bar{e} \\ \bar{z} \\ \bar{\theta} \end{bmatrix} \in \mathbb{R}^{4n+1} & \triangleq \begin{bmatrix} -H_{ev} z^{*T} \bar{\theta} \\ -H_{zv} z^{*T} \bar{\theta} \\ (I+LM)^{-1} \bar{\theta}_0 + K z^* e^* \end{bmatrix} \end{aligned} \quad (5.6)$$

$$G \triangleq \begin{bmatrix} H_{ev} [1 - z^{*T} K (z^* H_{ev} + e^* H_{zv})] & H_{ev} z^{*T} K \\ H_{zv} [1 - z^{*T} K (z^* H_{ev} + e^* H_{zv})] & H_{zv} z^{*T} K \\ K (z^* H_{ev} + e^* H_{zv}) & -K \end{bmatrix} \in \mathbb{R}^{(4n+1) \times (2n+1)} \quad (5.7)$$

$$\text{where } M \triangleq z^* H_{ev} z^{*T} + e^* H_{zv} z^{*T} \text{ and } K \triangleq (I + L^* M)^{-1} L^* \quad (5.8)$$

The reader is referred to Appendix D for the derivation of (Σ_p) .

In [2], a theorem (Theorem 4.1) is proposed which provides sufficient conditions for local L_p -stability of the adaptive system (Σ) about a set S . But the proof appears incorrect.

We shall propose now a modified theorem which is based upon the small gain theorem. (see e.g. [8], [9]). This theorem provides conditions that determine the set S about which the adaptive system (Σ) is locally L_p -stable.

We recall that the system (Σ) is locally L_p - stable about a set $S \subset L_p$ if:

(i) for all $(e^*, z^*, \tilde{\theta}_0) \in S$ the system (Σ_ρ) has a unique solution $x \in L_p$.

(ii) there exist finite constants γ and b such that:

$$\|x\|_p \leq \gamma \|(e^*, z^*, \tilde{\theta}_0)^T\|_p + b \quad \forall (e^*, z^*, \tilde{\theta}_0) \in S$$

Then we have the following local stability theorem.

Theorem 5.1

The adaptive system (Σ) is locally L_p - stable about a set S for some $p \in [1, \infty]$ if for all $(e^*, z^*, \tilde{\theta}_0) \in S \subset L_p$

(A₁) there exists $\bar{b} < \infty$ such that $\|\bar{x}\|_p \leq \bar{b}$

(A₂) there exists $g < \infty$ such that $\gamma_p(G) \leq g$

(A₃) for every $\bar{x} \in L_p$ there exists a unique solution to the equation $x = \bar{x} - G F(x)$

(A₄) $\gamma_p(F) < \frac{1}{g}$

In addition, subject to these conditions, one has

$$\|x\|_p \leq (1 - g \gamma_p(F))^{-1} \bar{b}. \quad \square$$

Proof:

(A₁) allows us to say that $(e^*, z^*, \tilde{\theta}_0) \in S \Rightarrow \bar{x} \in L_p$.

Consider now the solution x of $x = \bar{x} - G F x$ which exists by

(A₃). Then for all $T \in \mathbb{R}_+$

$$\begin{aligned} \|x_T\|_p &\leq \|\bar{x}_T\|_p + \|(G F x)_T\|_p \\ &\leq \|\bar{x}_T\|_p + \|(G F x_T)_T\|_p \quad \text{since } G F \text{ is causal} \end{aligned}$$

$$\begin{aligned} & \leq \| \bar{x} \|_p + \| G F x_T \|_p \quad \text{since truncation reduces the norm.} \\ & \leq \| \bar{x} \|_p + \gamma_p (GF) \| x_T \|_p \leq \| \bar{x} \|_p + g \gamma_p (F) \| x_T \|_p \quad \text{by } (A_2) \\ \text{hence } \forall T \in \mathbb{R}_+ \quad \| x_T \|_p & \leq (1 - g \gamma_p (F))^{-1} \| \bar{x} \|_p \leq (1 - g \gamma_p (F))^{-1} \bar{b} \end{aligned}$$

using (A_4) and (A_1) .

$$\text{and so } \| x \|_p \leq (1 - g \gamma_p (F))^{-1} \bar{b} < \infty \Rightarrow x \in L_p.$$

Hence $x \in L_p$ for all $(e^*, z^*, \tilde{\theta}_0) \in S \subset L_p$

$$\begin{aligned} \text{and } \| x \|_p & \leq (1 - g \gamma_p (F))^{-1} \| \bar{x} \|_p \\ & \leq (1 - g \gamma_p (F))^{-1} \gamma' \| (e^*, z^*, \tilde{\theta}_0)^T \|_p \end{aligned}$$

where $\gamma' \in \mathbb{R}_+$.

which proves that the adaptive system (Σ) is locally L_p -stable about the set S for $p \in [1, \infty]$. $\square$

In the case $p = \infty$, it is possible to give explicit conditions which guarantee that (A_1) and (A_2) are satisfied for every

$(e^*, z^*, \tilde{\theta}_0) \in S \subset L_\infty$ and give specifications on the set S . To prepare for this we present lemma 4.1 from [2].

Lemma 5.2

If the operators K, H_{ev}, H_{zv} are locally L_∞ -stable about the set S , then conditions (A_1) and (A_2) are satisfied.

Proof:

(A_1) : $(e^*, z^*, \tilde{\theta}_0) \in S \subset L_\infty$ which implies that $(e^*, z^*, z^* e^*, \tilde{\theta}_0) \in L_\infty$ since $\| z^* e^* \|_\infty \leq \| z^* \|_\infty \| e^* \|_\infty < \infty$.

$$\text{From (5.6) we have } \bar{\theta} = (I + L^* M)^{-1} \tilde{\theta}_0 + K z^* e^*$$

$$= -(I + L^* M)^{-1} L^* M + I \tilde{\theta}_0 + K z^* e^* = -K M \tilde{\theta}_0 + \tilde{\theta}_0 + K z^* e^*.$$

Hence we get the diagram which is shown in Fig. 5.1. From this diagram

it is clear that if $(z^* e^*, \tilde{\theta}_0) \in L_\infty$ and if

- Fig. 5.2 -

K is locally L_∞ - stable then $\bar{\theta} \in L_\infty$. Furthermore if $z^* \in L_\infty$ and the operators H_{ev} and H_{zv} are locally L_∞ - stable then $(\bar{e}, \bar{z}) \in L_\infty$. Hence $\bar{x} \in L_\infty$ and so (A_1) is satisfied.

(A_2) . Using (5.7) we can draw the diagram shown in Fig. 5.2. From this diagram it is clear that if $(e^*, z^*) \in L_\infty$ and if K, H_{ev}, H_{zv} are locally L_∞ - stable then the operator G has a finite L_∞ - gain and so (A_2) is satisfied. $\square$

We shall now present a lemma which provides conditions on the design of the controller i.e. on the operator V and the matrix Γ under which the operators H_{ev} and H_{zv} are L_∞ - stable.

$$\text{Recall that } H_{ev} = (1 + W_P D^*)^{-1} W_P \frac{1}{C^*} C^* \quad (5.9)$$

$$H_{zv} = \begin{bmatrix} 0 \\ -VD^* H_{ev} + V \frac{1}{C^*} C^* \\ H_{ev} \\ V H_{ev} \end{bmatrix} \quad (5.10)$$

Lemma 5.3

If (i) $\gamma_\infty (D^{*-1} \frac{1}{C^*} C^*) < \infty$

(ii) for all $u \in L_\infty$ there exists a unique solution $e' \in L_\infty$ to the equation $e' = u - W_P D^* e'$

(iii) $\gamma_\infty (W_{Q^*}) < \infty, \gamma_\infty (\Delta_1) < \infty, \gamma_\infty (\Delta_2) < \infty$.

(iv) $\gamma_\infty (D^*) < \frac{1}{\gamma(W_P)}$

(v) for all $j \in \{0, \dots, n-1\}$ $\gamma_\infty (V_j) < \infty$

$$(v1) \quad \gamma_{\infty} \left(\frac{1}{C^*} C^* \right) < \infty$$

then the operators H_{ev} and $H_{zv} : L_{\infty} \rightarrow L_{\infty}$ are L_{∞} - stable. $\square$

Note that (iii) merely implies that $\gamma_{\infty}(W_p) < \infty$.

Moreover (i), (ii), (iv), (v), (vi) are conditions on the operator V (see (2.12) for the definition of the operators C^* and D^*)

Proof: From fig. 4.1 and applying the small gain theorem (see e.g. [8], [9]) it is clear that (i), (ii), (iii) and (iv) imply that H_{ev} is L_{∞} - stable. (iv) implies that $\gamma_{\infty}(D^*) < \infty$. Then by (5.10), (v), (vi) it is clear that H_{zv} is L_{∞} - stable. $\square$

To ensure that K is locally L_{∞} - stable Kosut presents two lemmas that provide conditions on the tuned signals e^* and z^* .

Lemma 5.4

Let $M(T)$ and $K(T)$ denote the time varying operators M and K frozen at some time $T \in \mathbb{R}_+$. Thus,

$$M(T) \triangleq z^*(T) H_{ev}(T) z^{*T}(T) + e^*(T) H_{zv}(T) z^{*T}(T)$$

$$K(T) \triangleq (I + L'M(T))^{-1} L' \quad \text{where } L' \text{ is defined by (5.3')}$$

Then K is locally L_{∞} - stable about S if

$$(i') \quad K(T) \text{ is } L_{\infty} \text{ - stable for all } T \in \mathbb{R}_+,$$

$$(ii') \quad H_{ev} \text{ and } H_{zv} \text{ are } L_{\infty} \text{ - stable and}$$

$$(iii') \quad \gamma_{\infty}(K(T)(M(T)-M)) < 1 \quad \text{for all } T \in \mathbb{R}_+. \quad \square$$

Proof: (see [2])

The operator $K: \omega + \eta$ can be described by:

$$\eta = L'\omega - L'M\eta = L'\omega - L'M(T)\eta + L'(M(T)-M)\eta$$

$$\Rightarrow (I + L M(T)) \eta = L' \omega + L' (M(T) - M) \eta$$

$$\Rightarrow \eta = K(T) \omega + K(T) (M(T) - M) \eta$$

(5.1)

If $H_{ev}(t)$ and $H_{zv}(t)$ are locally L_∞ -stable about S ,
 so is $M(T) - M(t) = z^*(T) H_{ev}(T) z^{*T}(T) - z^*(t) H_{ev}(t) z^{*T}(t)$
 $+ e^*(T) H_{zv}(T) z^{*T}(T) - e^*(t) H_{zv}(t) z^{*T}(t)$.

(Recall that $e^*, z^* \in S \subset L_\infty$)

Hence $\gamma_\infty(M(T) - M) < \infty$ - Moreover $\gamma_\infty(K(T)) < \infty \quad \forall T \in \mathbb{R}_+$ (by (i')) .

$$(5.11) \Rightarrow \forall T \in \mathbb{R}_+ \quad \|\eta_T\|_\infty \leq \|K(T) \omega_T\|_\infty + \|(K(T) (M(T) - M) \eta_T)_T\|_\infty$$

since $K(T) (M(T) - M)$ is causal

$$\leq \gamma_\infty(K(T)) \|\omega\|_\infty + \|K(T) (M(T) - M) \eta_T\|_\infty \quad \text{since truncation reduces the norm}$$

$$\leq \gamma_\infty(K(T)) \|\omega\|_\infty + \gamma_\infty(K(T) (M(T) - M)) \|\eta_T\|_\infty$$

$$\Rightarrow \forall T \in \mathbb{R}_+ \quad (1 - \gamma_\infty(K(T) (M(T) - M))) \|\eta_T\|_\infty \leq \gamma_\infty(K(T)) \|\omega\|_\infty$$

Hence using (iii') we obtain

$$\forall T \in \mathbb{R}_+ \quad \|\eta_T\|_\infty \leq (1 - \gamma_\infty(K(T) (M(T) - M)))^{-1} \gamma_\infty(K(T)) \|\omega\|_\infty$$

$\Rightarrow \|\eta_\infty\| < \infty$. Therefore K is locally L_∞ -stable about S . $\square$

Note that (iii') is a condition on the tuned signals e^* and z^* .

Lemma 5.5 (see [2])

Assume that the tuned signal z^* belongs to L_∞ and is

Γ -persistently exciting i.e. there exist positive constants a_1, a_2

and a_3 such that: $a_1 I \leq \int_u^{u+a_3} (z^* z^{*T} \Gamma + \Gamma z^* z^{*T}) dt \leq a_2 I, \quad \forall u \in \mathbb{R}_+$

with $\Gamma = \Gamma^T > 0$ (see [18]). Let us denote the operator Q by

$Q \triangleq \Gamma (z^* z^{*T} I - M)$ where M is defined by (5.8).

Then the operator K is locally L_∞ - stable about S if the L_∞ - gain of Q is sufficiently small. $\square$

The proof is given in [2].

Hence these two mechanisms which work in our case (H_{ev} and H_{zv} are time varying operators) allow us to find conditions on the tuned signals e^* and z^* to guarantee the local L_∞ - stability of the operator K .

To conclude this chapter we may say that lemmas 5.2, 5.3, 5.4, 5.5 provide conditions on the tuned signals e^* and z^* as well as on the operator V which ensure that the conditions (A_1) and (A_2) of theorem 5.1 are satisfied when $p = \infty$. We can view also (A_3) and (A_4) as conditions on V . Hence it turns out that to ensure the L_∞ - local stability of the adaptive system (Σ) the conditions are not as strong as those required to ensure global stability, in particular there is no requirement that H_{ev} be strictly passive and hence no severe restrictions on the gains of the uncertainties Δ_1 and Δ_2 .

CONCLUSION

In this thesis we have extended the robust global and local stability results of a Model Reference Adaptive Control (MRAC) system presented in [1] and [2]. By robust stability we mean stability despite the presence of unmodeled dynamics and the existence at the plant's output of bounded disturbances.

In the first part of our work we have kept Kosut and Friedlander's description of an unknown plant $W_p = (I + \Delta_1) W_{Q^*}$, where Δ_1 is a time invariant uncertainty operator which is assumed to be represented by a rational, proper, exponentially stable transfer function and W_{Q^*} is the "best approximate" parametrized model of the time invariant plant. The main robust global theorem of [1] has been generalized to the case where the relative degree of $W_{Q^*}(s)$ is equal to 2.

In the second part we introduced a time varying uncertainty operator Δ_2 and the time varying plant has been assumed to be represented by $W_p = (I + \Delta_2)(I + \Delta_1)W_{Q^*}$. Moreover we have dropped the strong assumption of the rationality of Δ_1 . Hence we have considered infinite dimensional systems which are much more realistic in their representation via Δ_1 and Δ_2 of the plant's unmodeled dynamics. In this more general situation the global stability of the MRAC system has been established. As in the first part, L_2 and L_∞ bounds have been derived for certain significant signals.

Most of the conditions required for the global stability are actually needed to guarantee the strict passivity of a crucial operator H_{ev} . This has motivated investigations concerning local stability. The results of [2] have been made more precise and applied successfully to our time varying case. In particular there is no longer the requirement that H_{ev} be strictly passive hence the conditions on Δ_1 and Δ_2 are less severe than in the global stability case.

In this thesis we have considered deterministic continuous-time systems. Since we have been using the input / output formulation throughout this work one should be able to easily extend our results to the discrete-time systems. Moreover it would be interesting to consider the stochastic continuous-time and discrete-time systems.

APPENDIX A

Lemma A₁ (see [9] p.59)

Consider an operator $G: X_e^n \rightarrow X_e^n$ which belongs to H_0 .
Then the corresponding impulse response $G(t)$ has the following property:

There are constants $g_m > 0$ and $\alpha > 0$ such that

$$\|G(t)\| \leq g_m e^{-\alpha t} \quad \forall t \in \mathbb{R}_+$$

where $\|G(t)\|$ denotes the norm of the matrix $G(t) \in \mathbb{R}^{n \times n}$ this norm being induced by some chosen norm in $\mathbb{R}^n$. $\square$

Proof:

Let $G(s)$ be the transfer function matrix which corresponds to G .

(It exists because $G \in H_0$)

$G(s) \in \mathbb{R}[s]^{n \times n}$ = class of all $n \times n$ matrices whose elements are in $\mathbb{R}(s)$ (set of polynomials in the complex variable s and with coefficients in $\mathbb{R}$)

Call $G_{ij}(s)$ the element which belongs to the i^{th} row and the j^{th} column of the matrix $G(s)$.

Let $G_{ij}(t)$ denote the inverse Laplace transform of $G_{ij}(s)$.

Since $G(s)$ is exponentially stable and strictly proper,

$$\forall i \in \{1, \dots, n\} = I, \quad \forall j \in I$$

$$G_{ij}(s) = \prod_{k \in K} \frac{c_{ij,k}}{(s - \lambda_k)^{q_k}} = \sum_{k \in K} \frac{d_{ij,k}}{(s - \lambda_k)^{q_k}}$$

where $\text{Re } \lambda_k < 0$, $q_k \geq 1$ and $K \subset I$.

Consequently,

$$g_{ij}(t) = \sum_{k \in K} d_{ij,k} t^{q_k-1} e^{\lambda_k t} \quad \forall t \in \mathbb{R}_+$$

$$\Rightarrow |g_{ij}(t)| \leq \sum_{k \in K} |d_{ij,k}| t^{q_k-1} e^{\operatorname{Re}(\lambda_k)t} \quad \forall t \in \mathbb{R}_+$$

Since $\operatorname{Re}(\lambda_k) < 0$,

$$\exists T > 0 \text{ s.t. } t > T \Rightarrow t^{q_k-1} e^{\frac{\operatorname{Re}(\lambda_k)t}{2}} < 1$$

$$\Rightarrow t^{q_k-1} e^{\operatorname{Re}(\lambda_k)t} \leq e^{\frac{\operatorname{Re}(\lambda_k)t}{2}}$$

On the compact set $[0, T]$, the continuous function $t \mapsto t^{q_k-1} e^{\operatorname{Re}(\lambda_k)t}$ is bounded by a constant M_k .

$$\text{Hence } \forall t \in \mathbb{R}_+ \quad t^{q_k-1} e^{\operatorname{Re}(\lambda_k)t} \leq \max\left(\frac{M_k}{e^{\frac{\operatorname{Re}(\lambda_k)t}{2}}}, 1\right) e^{\frac{\operatorname{Re}(\lambda_k)t}{2}}$$

hence $\forall k \in K \quad \exists \alpha_k > 0$ and $g_{mk} > 0$ s.t.

$$t^{q_k-1} e^{\operatorname{Re}(\lambda_k)t} < g_{mk} e^{-\alpha_k t} \quad \forall t \in \mathbb{R}_+$$

$$\Rightarrow |g_{ij}(t)| \leq \sum_{k \in K} |d_{ij,k}| g_{mk} e^{-\alpha_k t}$$

$$< g_{ij,m} e^{-\alpha_{ij} t} \quad \forall t \in \mathbb{R}_+$$

where $g_{ij,m} = \max_{k \in K} |d_{ij,k}| g_{mk}$ and $\alpha_{ij} = \min_{k \in K} \alpha_k$

Take $g_m = \max_{i,j \in I} g_{ij,m}$ and $\alpha = \min_{i,j \in I} \alpha_{ij}$

then $|G(t)| = \max_{i,j \in I} |g_{ij}(t)| \leq g_m e^{-\alpha t} \quad \forall t \in \mathbb{R}_+ \quad \square$

Lemma A₂ (see [9] , p.59)

Consider an operator $\tilde{G}: X_0 \rightarrow X_0$ which belongs to $\underline{H}_0$.

Let $g(t)$ denote its impulse response. If $y = G u$, we know that

$$\forall t \in \mathbb{R}_+ \quad y(t) = g * u(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

Then if u is continuous on $\mathbb{R}_+$, $y(t)$ exists for all t in $\mathbb{R}_+$.

Moreover

(a) if $u \in L_2(\mathbb{R}_+) \cap C = L_2 \cap C$, then: (a1) $y = Gu \in L_2 \cap L_\infty$

(a2) $\dot{y} \in L_2$

(a3) $y(t) \rightarrow 0$ as $t \rightarrow \infty$

(b) if $u \in L_\infty(\mathbb{R}_+) \cap C = L_\infty \cap C$, then: (b1) $y \in L_\infty$

(b2) $\dot{y} \in L_\infty$

(b3) y is uniformly continuous on $\mathbb{R}$

(c) If $u \in L_\infty$ and if, as $t \rightarrow \infty$, $u(t) \rightarrow u_\infty$ a constant,

then $y(t) \rightarrow G(0) u_\infty$ as $t \rightarrow \infty$ where $G(s) = \int_0^\infty g(t)e^{-st}dt$.

Moreover the convergence is exponential. $\square$

Recall that C is the set of continuous functions on $\mathbb{R}_+$

Proof:

From the proof of lemma A₁, we know that:

$$G \in \underline{H}_0 \Rightarrow g(t) = \sum_{k \in K} d_k t^{q_k-1} e^{\lambda_k t} \quad \forall t \in \mathbb{R}_+$$

where $K \subset \{1, \dots, n\}$, $q_k \geq 1$ and $\text{Re}(\lambda_k) < 0 \quad \forall k \in K$

hence g is a continuous function on $\mathbb{R}_+$.

Let $K' = \{k \in K \text{ s.t. } q_k = 1\}$ $K' \subset K$

$$\dot{g}(t) = \sum_{k \in K - K'} d_k [(q_k - 1)t^{q_k - 2} + \lambda_k t^{q_k - 1}] e^{\lambda_k t} + \sum_{k \in K - K'} d_k \lambda_k e^{\lambda_k t} \quad \forall t \in \mathbb{R}_+ \quad (\text{a01})$$

hence $\dot{g}$ is a continuous function on $\mathbb{R}_+$.

$$\text{We know that } y(t) = \int_0^t g(t-\tau)u(\tau)d\tau \quad \forall t \in \mathbb{R}_+.$$

Since g , $\dot{g}$ and u are continuous functions on $\mathbb{R}_+$, we can apply theorem 12.8 from [19] and say that $\dot{y}$ exists and

$$\dot{y}(t) = g(0_+) u(t) + \int_0^t \dot{g}(t-\tau)u(\tau)d\tau = g(0_+)u(t) + \dot{g} * u(t) \quad (\text{a02})$$

To prove (a) and (b) we recall the following lemma which is proven in [4] p. 241.

Lemma A₂': Let $u, g: \mathbb{R}_+ \rightarrow \mathbb{R}$ and $p \in [1, \infty]$

If $u \in L_p$ and $g \in L_1$, then $\|g * u\|_p \leq \|g\|_1 \|u\|_p$ $\square$

(a1) Since $G \in \underline{H}_0$, $g \in L_1$ (see lemma A₁) then applying lemma A₂'

we get $\|y\|_2 \leq \|g\|_1 \|u\|_2 < \infty$ since $u \in L_2$ hence $y \in L_2$. $\square$

Note that this result can be interpreted in the frequency domain.

$$\forall T \in \mathbb{R}_+ \quad \|y_T\|_2^2 = \int_0^T |y_T(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |g(j\omega)|^2 |u_T(j\omega)|^2 d\omega$$

using Parseval's equality. Moreover $G \in \underline{H}_0 \Rightarrow \exists K > 0$ s.t. $|g(j\omega)| \leq K$

$\forall \omega \in \mathbb{R}$. Hence $\forall T \in \mathbb{R}_+ \quad \|y_T\|_2^2 \leq \frac{K^2}{2\pi} \|u_T\|_2^2 < \infty$ which implies $y \in L_2$. $\square$

$$\forall t \in \mathbb{R}_+ \quad y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^t g(\tau)u(t-\tau)d\tau$$

$$\begin{aligned}
 \text{which implies } \forall t \in \mathbb{R}_+ \quad |y(t)| &\leq \int_0^t |g(\tau)u(t-\tau)| d\tau \\
 &\leq \int_0^\infty |g(\tau)u(t-\tau)| d\tau \leq \|g\|_2 \|u\|_2
 \end{aligned}$$

applying the Cauchy-Schwarz inequality.

Moreover $G \in \underline{H}_0$ implies $\|g\|_2 < \infty$ (see lemma A_1) and so

$$\forall t \in \mathbb{R}_+ \quad |y(t)| < \infty \Rightarrow \|y\|_\infty < \infty \Rightarrow y \in L_\infty \quad \square$$

(a2) Using the same kind of arguments as in the proof of lemma A_1 , we can show that since $\dot{g}$ is given by (a01)

$$\exists g'_m > 0, \exists \alpha' > 0 \text{ s.t. } |\dot{g}(t)| \leq g'_m e^{-\alpha' t} \quad \forall t \in \mathbb{R}_+ \quad (\text{a21})$$

We prove now that $\dot{y} \in L_2$.

$$\begin{aligned}
 \text{from (a02) we have } \|\dot{y}\|_2 &\leq |g(0_+)| \|u\|_2 + \|\dot{g} * u\|_2 \\
 &\leq |g(0_+)| \|u\|_2 + \|\dot{g}\|_1 \|u\|_2 \quad (\text{using lemma } A'_2)
 \end{aligned}$$

(a21) implies $\|\dot{g}\|_1 < \infty$. Moreover $u \in L_2$ by assumption.

Hence $\|\dot{y}\|_2 < \infty$ implies $\dot{y} \in L_2 \quad \square$

(a3) Let us prove that $y(t) \rightarrow 0$ as $t \rightarrow \infty$

$$u \in L_2 \text{ implies } \left(\int_0^\infty |u(s)|^2 ds \right)^{\frac{1}{2}} < \infty$$

First let us prove the following lemma

Lemma A''_2 : If $u \in L_2$ then $\forall \epsilon > 0 \exists A > 0$ s.t. $\int_A^\infty |u(s)|^2 ds < \epsilon \quad \square$

$$\text{Proof: } \forall A > 0 \quad \int_A^\infty |u(s)|^2 ds = \int_0^\infty |u(s)|^2 ds - \int_0^A |u(s)|^2 ds \quad (\text{a31})$$

Let us define $f_n(s) \triangleq 1_{[0,n]} |u(s)|^2 \quad n \geq 1$

$$\forall n \geq 1, |f_n(s)| \leq |u(s)|^2$$

$$\text{and } u \in L_2 \Rightarrow |u(s)|^2 \in L_1$$

Hence, since $\lim_{n \rightarrow \infty} f_n(s) = |u(s)|^2$, by the theorem of dominated

$$\text{convergence } \lim_{n \rightarrow \infty} \int_0^\infty f_n(s) ds = \lim_{n \rightarrow \infty} \int_0^n |u(s)|^2 ds = \int_0^\infty |u(s)|^2 ds$$

$$\Rightarrow \forall \varepsilon > 0 \quad \exists A > 0 \quad n > A \Rightarrow \int_A^\infty |u(s)|^2 ds < \varepsilon \quad (\text{using (a31)})$$

which proves the lemma. $\square$

From lemma A₁, we know that

$$\exists g_m > 0, \exists \alpha > 0 \text{ s.t. } |g(t)| \leq g_m e^{-\alpha t} \quad \forall t \in \mathbb{R}_+$$

$$\text{Fix } \varepsilon > 0 \text{ then there exists } A > 0 \text{ s.t. } \int_A^\infty |u(s)|^2 ds \leq \frac{1}{g_m^2} \frac{\varepsilon^2}{2} \alpha \quad (\text{a32})$$

$$\forall t \geq A \quad y(t) = \int_0^A g(t-\tau) u(\tau) d\tau + \int_A^t g(t-\tau) u(\tau) d\tau \quad \text{then}$$

$$\forall t \geq A \quad |y(t)| \leq \int_0^A |g(t-\tau)| |u(\tau)| d\tau + \int_A^t |g(t-\tau) u(\tau)| d\tau$$

Let us look at the first term:

$$\int_0^A |g(t-\tau)| |u(\tau)| d\tau \leq g_m \int_0^A e^{-\alpha(t-\tau)} |u(\tau)| d\tau$$

$$= g_m e^{-\alpha t} \int_0^A e^{\alpha \tau} |u(\tau)| d\tau$$

$$\text{Since } \alpha > 0 \quad \exists t_0 \in \mathbb{R}_+ \text{ s.t. } t \geq t_0 \Rightarrow \int_0^A |g(t-\tau)| |u(\tau)| d\tau \leq \frac{\varepsilon}{2} \quad (\text{a33})$$

Let us look at the second term:

$$\begin{aligned} \int_A^t |g(t-\tau)u(\tau)| d\tau &\leq g_m \int_A^t |e^{-\alpha(t-\tau)}u(\tau)| d\tau \\ &= g_m \left(\int_A^t e^{-2\alpha(t-\tau)} d\tau \right)^{\frac{1}{2}} \left(\int_A^t |u(\tau)|^2 d\tau \right)^{\frac{1}{2}} \end{aligned}$$

(applying the Cauchy-Schwarz inequality)

$$\begin{aligned} &\leq g_m \left[e^{-2\alpha t} \left(\frac{e^{2\alpha t} - e^{2\alpha A}}{2\alpha} \right) \right]^{\frac{1}{2}} \cdot \left(\int_A^\infty |u(\tau)|^2 d\tau \right)^{\frac{1}{2}} \\ &\leq g_m \left[\frac{1}{2\alpha} (1 - e^{-2\alpha(t-A)}) \right]^{\frac{1}{2}} \cdot \frac{1}{g_m} \frac{\varepsilon}{2} (2\alpha)^{\frac{1}{2}} \end{aligned}$$

$$t > A \text{ implies } 1 - e^{-2\alpha(t-A)} < 1$$

$$\text{Therefore } t > A \Rightarrow \int_A^t |g(t-\tau)u(\tau)| d\tau < \frac{\varepsilon}{2} \quad (\text{a34})$$

Finally, using (a33) and (a34), we can conclude that

$$\forall \varepsilon > 0 \exists t' = \max(A, t_0) \text{ s.t. } t > t' \Rightarrow |y(t)| < \varepsilon$$

which implies $\lim_{t \rightarrow \infty} y(t) = 0$ $\square$

Hence (a) is proven.

(b1) Applying lemma A'2 we have

$$\|y\|_\infty \leq \|g\|_1 \|u\|_\infty < \infty \quad \text{since } g \in L_1 \text{ and } u \in L_\infty$$

hence $y \in L_\infty$. $\square$

(b2) Let us prove that $\dot{y} \in L_\infty$. Using (a02) we have

$$\begin{aligned} \forall t \in \mathbb{R}_+ \quad |\dot{y}(t)| &\leq |g(0_+)| |u(t)| + \int_0^t |\dot{g}(\tau)| |u(t-\tau)| d\tau \\ &\leq |g(0_+)| \|u\|_\infty + \|u\|_\infty \int_0^\infty |\dot{g}(\tau)| d\tau \end{aligned}$$

$$\leq |g(0_+)| \|u\|_{\infty} + \|u\|_{\infty} g_{\infty} \frac{1}{\alpha} < \infty$$

applying (a21)

hence $\|\dot{y}\|_{\infty} < \infty$ and so $\dot{y} \in L_{\infty}$. $\square$

(b3) $\dot{y} \in L_{\infty}$ implies y is uniformly continuous on $\mathbb{R}_+$.

Hence (b) is proven. $\square$

$$\begin{aligned} \text{(c) } \forall t \in \mathbb{R}_+ \quad y(t) &= \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^t g(\tau)u(t-\tau)d\tau \\ &= \int_0^{\infty} g(\tau)u(t-\tau)d\tau \end{aligned}$$

Since $u(t)$ is assumed to be zero when $t < 0$

$$G(0) = \int_0^{\infty} g(\tau)d\tau$$

$$\text{Hence for all } t \in \mathbb{R}_+ \quad y(t) - G(0)u_{\infty} = \int_0^{\infty} g(\tau)(u(t-\tau) - u_{\infty})d\tau$$

By assumption $u(t) \rightarrow u_{\infty}$ as $t \rightarrow \infty$. Therefore

$\forall \varepsilon > 0$ consider $\frac{\varepsilon}{g_{\infty}} \alpha$ then $\exists t_0 > 0$ s.t. $t > t_0 + \tau$

$$\Rightarrow t - \tau > t_0 \Rightarrow |u(t-\tau) - u_{\infty}| < \frac{\varepsilon}{g_{\infty}} \alpha$$

$$\Rightarrow |y(t) - G(0)u_{\infty}| < \frac{\varepsilon}{g_{\infty}} \alpha \int_0^{\infty} |g(\tau)|d\tau \leq \frac{\varepsilon}{g_{\infty}} \alpha g_{\infty} \frac{(-1)}{\alpha} e^{-\alpha t} \Big|_0^{\infty}$$

$$= \varepsilon$$

Hence $y(t) \rightarrow G(0)u_{\infty}$ as $t \rightarrow \infty$. $\square$

The reader is referred to [9] for the proof of exponential convergence.

Lemma A₃

If $u \in L_p(\mathbb{R}_+) = L_p$ $1 \leq p < \infty$ and u is uniformly continuous on $\mathbb{R}_+$ then $u(t) \rightarrow 0$ as $t \rightarrow \infty$ $\square$

Proof

Let us prove this lemma by contradiction.

Suppose $u(t)$ doesn't tend to 0 as $t \rightarrow \infty$

then $\exists \epsilon_0 > 0$ s.t. $\forall t' \in \mathbb{R}_+$ $\exists t > t'$ and $|u(t)| \geq \epsilon_0$ (A31)

Step 1: we form a sequence $\{t_k\}$ as follows:

t_1 is chosen such that $|u(t_1)| \geq \epsilon_0$, then we pick t_2 such that $t_2 > t_1 + 1$ and $|u(t_2)| \geq \epsilon_0$ which is possible using (A31).

We proceed inductively to find t_k for each k .

Step 2: Since u is uniformly continuous on $\mathbb{R}_+$, we have that:

$$\exists \delta(\epsilon_0) > 0 \text{ s.t. } \forall (t, t') \in \mathbb{R}_+^2 \quad |t - t'| < \delta(\epsilon_0) \Rightarrow |u(t) - u(t')| < \frac{\epsilon_0}{2}$$

We define $d \triangleq \min\left(\frac{\delta \epsilon_0}{2}, \frac{1}{3}\right)$ and $A_k = [t_k - d, t_k + d] \quad \forall k \geq 1$

Note that the A_k 's are disjoint by construction.

$$\text{Claim: } |u(t)| \geq \sum_{k=1}^n \left(\frac{\epsilon_0}{2}\right) 1_{A_k} \quad \forall t \in \mathbb{R}_+ \quad (\text{A32})$$

If $t \notin A_k \quad \forall k \geq 1$ Then it is obvious.

If $\exists k' \geq 1 \quad t \in A_{k'}$ (such a k' is unique because the A_k 's are disjoint). Then

$$|t - t_{k'}| \leq d < \delta(\epsilon_0) \Rightarrow |u(t) - u(t_{k'})| < \frac{\epsilon_0}{2}$$

$$\text{But } |u(t_{k'})| - |u(t)| \leq |u(t) - u(t_{k'})| < \frac{\epsilon_0}{2}.$$

$$\text{Hence } |u(t)| \geq |u(t_{k'})| - \frac{\epsilon_0}{2} > \frac{\epsilon_0}{2}$$

$$\text{since } |u(t_{k'})| \geq \epsilon_0 \quad \forall k' \geq 1$$

then (A32) is proven.

$$\int_{-\infty}^{\infty} |u(t)|^p dt \geq \left(\frac{\varepsilon_0}{2}\right)^p \sum_{k=1}^{\infty} 2 \, d \quad \text{which is infinite.}$$

(since the λ_k 's are disjoint)

which implies $u \notin L_p$ Contradiction.

Hence $u(t) \rightarrow 0$ as $t \rightarrow \infty$.

□

APPENDIX B

Proof of Theorem A

(a1) together with (a0) or (a0') implies that $\bar{\theta}$ and z belong to C : the set of continuous functions on R_+ . Hence $v = \bar{\theta}^T z$ (by (2.11)) belongs to C .

We have $H_{ev} \in \text{SPR}_0^+$ (a2)

$H_{zv} \in \underline{H}_0$ (a3)

$(e^*, \dot{e}^*) \in L_2 \cap L_\infty$ (a4)

$(z^*, \dot{z}^*) \in L_\infty$ (a5)

Using the same kind of arguments as in the proof of Theorem 3 we can show that:

$$\left. \begin{aligned} e^* \in L_2 \text{ implies that: } & v \in L_2 \text{ and } \|v\|_2 < b_1 \\ & e \in L_2 \text{ and } \|e\|_2 < b_2 \\ & \bar{\theta} \in L_\infty \text{ and } \|\bar{\theta}\|_\infty < b_3 \end{aligned} \right\} \quad (r1)$$

We have
$$\begin{cases} e - e^* = -H_{ev} v \\ z - z^* = -H_{zv} v \end{cases}$$

Since $v \in L_2 \cap C$ and $H_{ev}, H_{zv} \in \underline{H}_0$ (using (a2) and (a3)), we can apply lemma $\lambda_2(a)$. We get

$$\left\{ \begin{aligned} e - e^* &\in L_2 \cap L_\infty \\ \dot{e} - \dot{e}^* &\in L_2 \\ (e - e^*)(t) &\rightarrow 0 \text{ as } t \rightarrow \infty \end{aligned} \right. \text{ implies using (a4) } \left\{ \begin{aligned} e &\in L_2 \cap L_\infty \\ \dot{e} &\in L_2 \end{aligned} \right. \quad (r2)$$

$$\text{and } \begin{cases} z - z^* \in L_2 \cap L_\infty \\ \dot{z} - \dot{z}^* \in L_2 \\ (z - z^*)(t) \rightarrow 0 \text{ as } t \rightarrow \infty \end{cases} \quad \text{implies using (a5)} \quad \begin{cases} z - z^* \in L_2 \cap L_\infty \\ z \in L_\infty \\ \dot{z} - \dot{z}^* \in L_2^A \end{cases} \quad (r3)$$

Let us prove that $\dot{\theta} \in L_2 \cap L_\infty$. (r4)

We know that $\dot{\theta} = \hat{\theta} - \theta^*$ and $\hat{\theta} = \Gamma z$ (see (2.8) and (2.9)) where $\Gamma = \Gamma^T > 0$. Hence $\dot{\theta} = \Gamma z$ implies that

$$\forall i \in J = \{0, \dots, 2n-1\} \quad \dot{\theta}_i = \sum_{j=0}^{2n-1} \Gamma_{ij} z_j.$$

We know that:

$$\forall j \in J \quad \Gamma_{ij}^2 z_j^2 e^2 \leq \|z\|_\infty^2 \Gamma_{ij}^2 e^2 \quad \text{and}$$

$$\forall j, j' \in J \quad |\Gamma_{ij} z_j e| |\Gamma_{ij'} z_{j'} e| \leq \|z\|_\infty^2 |\Gamma_{ij}| |\Gamma_{ij'}| e^2$$

where $\|z\|_\infty < \infty$ since $z \in L_\infty$ (see (r3)).

$$\text{Hence } \forall i \in J \quad \dot{\theta}_i^2 \leq \sum_{j \in J} \Gamma_{ij}^2 z_j^2 e^2 + 2 \sum_{\substack{j, j' \in J \\ j < j'}} |\Gamma_{ij} z_j e| |\Gamma_{ij'} z_{j'} e|$$

$$\leq \|z\|_\infty^2 \left(\sum_{j \in J} \Gamma_{ij}^2 + 2 \sum_{\substack{j, j' \in J \\ j < j'}} |\Gamma_{ij}| |\Gamma_{ij'}| \right) e^2 \quad (r4a)$$

$\underbrace{\hspace{10em}}_{\alpha_1^2}$

$$\text{hence } \forall i \in J \quad \forall T \in \mathbb{R}_+ \quad \left(\int_0^T \dot{\theta}_i^2 dt \right)^{\frac{1}{2}} \leq \|z\|_\infty |\alpha_1| |\theta|_2 < \infty$$

since $e \in L_2$ (see (r1))

$$\text{Then } \forall i \in J \quad \|\dot{\theta}_i\|_2 = \lim_{T \rightarrow \infty} \|\dot{\theta}_{iT}\|_2 < \infty$$

Therefore $\|\dot{\theta}\|_2 = \max_{i \in J} \|\dot{\theta}_i\|_2 < \infty$ implies $\dot{\theta} \in L_2$

From (r4) we have: $\forall i \in J \quad \dot{\theta}_1^2 < |z|_{\infty}^2 |\alpha_1|^2$

then $(\sum_{i \in J} \dot{\theta}_1^2)^{\frac{1}{2}} < |z|_{\infty} |\alpha_1| (\sum_{i \in J} |\alpha_1|)^{\frac{1}{2}} < \infty$ since $e \in L_{\infty}$
(see (r2))

hence $\|\dot{\theta}\|_{\infty} = \text{ess sup}_{t \in \mathbb{R}_+} [(\sum_{i \in J} \dot{\theta}_1^2(t))^{\frac{1}{2}}] < \infty$

and so $\dot{\theta} \in L_{\infty}$ and (r4) is proven.

Let us prove that $v \in L_{\infty}$

(r5)

From (2.11), we know that $v = z^T \bar{\theta}$.

But $z \in L_{\infty}$ and $\bar{\theta} \in L_{\infty}$ (see (r1) and (r3))

hence $\|v\|_{\infty} \leq \|z\|_{\infty} \|\bar{\theta}\|_{\infty} < \infty$ therefore $v \in L_{\infty}$. $\square$

Since $v \in L_{\infty} \cap C$ and $H_{ev}, H_{zv} \in H_0$ (using (a2) and (a3))

we can apply lemma A_2 (b). We get

$\dot{\theta} - \dot{\theta}^* \in L_{\infty}$ implies using (a4) $\dot{\theta} \in L_{\infty}$

$\dot{z} - \dot{z}^* \in L_{\infty}$ implies using (a5)

$$\left\{ \begin{array}{l} \dot{z} \in L_{\infty} \\ \dot{z} - \dot{z}^* \in L_{\infty} \end{array} \right.$$

(r6)

We know that $v = z^T \bar{\theta}$ implies $\dot{v} = \dot{z}^T \bar{\theta} + z^T \dot{\bar{\theta}} \in L_{\infty}$

(r7)

since $(z, \dot{z}) \in L_{\infty}$ and $(\bar{\theta}, \dot{\bar{\theta}}) \in L_{\infty}$ using (r1), (r3), (r4) and (r6)

Hence v is uniformly continuous on $\mathbb{R}_+$. Moreover $v \in L_2$ (r1)

then lemma A_3 allows us to say that $v(t) \rightarrow 0$ as $t \rightarrow \infty$. (r8)

Lemma A_2 (c) together with (r8) and $v \in L_{\infty} \cap C$ (r5) allows us to say that:

$(e - e^*)(t) \rightarrow -H_{ev}(0) \cdot 0 = 0$ exp. as $t \rightarrow \infty$

$(z - z^*)(t) \rightarrow -H_{zv}(0) \cdot 0 = 0$ exp. as $t \rightarrow \infty$

(r9)

Let us prove that $\ddot{\theta} \in L_2 \cap L_\infty$

(r10)

We know that $\dot{\theta} = \Gamma z e$. Therefore $\ddot{\theta} = \Gamma(\dot{z}e + z\dot{e})$

By assumption $(z, \dot{z}) \in L_\infty$. From the proof of (r4) it follows that

$$\forall i \in J \quad \ddot{\theta}_1^2 \leq |\dot{z}|_\infty^2 \alpha_1^2 e^2 + |z|_\infty^2 \alpha_1^2 \dot{e}^2 \text{ where } \alpha_1 \text{ is defined in (r4a).}$$

Hence $\forall i \in J \quad \forall T \in \mathbb{R}_+$

$$\int_0^T \ddot{\theta}_1^2 dt \leq |\dot{z}|_\infty^2 \alpha_1^2 \int_0^T e^2 dt + |z|_\infty^2 \alpha_1^2 \int_0^T \dot{e}^2 dt$$

$$\text{then } \left(\int_0^T \ddot{\theta}_1^2 dt \right)^{\frac{1}{2}} \leq |\dot{z}|_\infty \alpha_1 \|e\|_2 + |z|_\infty \alpha_1 \|\dot{e}\|_2 < \infty$$

since $e, \dot{e} \in L_2$ ((r1) and (r2))

$$\text{Hence } \|\ddot{\theta}\|_2 = \lim_{T \rightarrow \infty} \|\ddot{\theta}_{1T}\|_2 < \infty \quad \forall i \in J$$

Therefore $\|\ddot{\theta}\|_2 = \max_{i \in J} \|\ddot{\theta}_i\|_2 < \infty$ and so $\ddot{\theta} \in L_2$

$$\text{We also have } \forall i \in J \quad \ddot{\theta}_1^2 \leq |\dot{z}|_\infty^2 \alpha_1^2 |e|_\infty^2 + |z|_\infty^2 \alpha_1^2 |\dot{e}|_\infty^2$$

which implies $\forall i \in J \quad \ddot{\theta}_1^2 < \infty$ since $e, \dot{e} \in L_\infty$ ((r2) and (r6)).

$$\text{Hence } \|\ddot{\theta}\|_\infty = \text{ess sup}_{t \in \mathbb{R}_+} \left[\left(\sum_{i \in J} \ddot{\theta}_i^2(t) \right)^{\frac{1}{2}} \right] < \infty$$

Therefore $\ddot{\theta} \in L_\infty$ and so (r10) is proven. $\square$

$\ddot{\theta} \in L_\infty$ implies $\dot{\theta}$ is uniformly continuous on $\mathbb{R}_+$. Moreover $\dot{\theta} \in L_2$

(see (r4)). Then lemma A_3 allows us to say that $\dot{\theta}(t) \rightarrow 0$ as $t \rightarrow \infty$.

Furthermore:

$$\dot{\theta} = \Gamma' z e = \Gamma(z - z^*) e + \Gamma z^* (e - e^*) + \Gamma z^* e^* \text{ and}$$

$$(z-z^*)(t) \rightarrow 0 \text{ exp. as } t \rightarrow \infty$$

(x9)

$$(e-e^*)(t) \rightarrow 0 \text{ exp. as } t \rightarrow \infty$$

Hence $\dot{\theta}(t) \rightarrow 0$ at the rate $e^*(t) \rightarrow 0$.

(x11)

Let us prove (v).

We know that $v = z^T \theta$ and $\dot{z} = z^* - H_{zv} v$

$$\text{then } \|v\| \leq \|z\| \|\theta\| \leq \|z\| b_3 \quad (\text{see (x1)})$$

$$\text{For all } t \in \mathbb{R}_+ \quad |z(t)| \leq |z^*(t)| + \int_0^t |h_{zv}(t-\tau)v(\tau)| d\tau$$

where h_{zv} is the impulse response of the operator H_{zv} which belongs to H_0 by assumption (a3).

$$\text{Then } \forall t \in \mathbb{R}_+ \quad |z(t)| \leq |z^*(t)| + \left(\int_0^t h_{zv}(t-\tau)^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t v^2(\tau) d\tau \right)^{\frac{1}{2}}$$

(applying the Cauchy-Schwarz inequality)

$$\text{then } \forall t \in \mathbb{R}_+ \quad |z(t)| \leq |z^*|_2 + \|h_{zv}\|_2 \|v\|_2$$

$$\text{and so } \|z\|_2 = \sup_{t \in \mathbb{R}_+} |z(t)| \leq |z^*|_2 + \|h_{zv}\|_2 b_1$$

(see (x1))

$$\text{therefore } \|v\|_2 \leq b_3 (|z^*|_2 + \|h_{zv}\|_2 b_1) \triangleq b_4$$

Moreover $e = e^* - H_{ev} v$

$$\Rightarrow \|e\|_2 \leq \|e^*\|_2 + \gamma_e(H_{ev}) \|v\|_2$$

$$\leq \|e^*\|_2 + \gamma_e(H_{ev}) b_4 \triangleq b_5$$

which proves (v) and theorem A. $\square$

APPENDIX C

We shall consider now the adaptive error system (2.25)

$$\begin{cases} \dot{e} = e^* - H_{ev} v' \\ \dot{z} = z^* - H_{zv} v' \end{cases}$$

with adaptation law (I_a^-) :

$$\begin{cases} \dot{\psi} = \xi e \\ L_M: \dot{\hat{\theta}} = \Gamma \psi; \Gamma = \Gamma^T > 0 \\ v' = \hat{\theta}^T z + \rho \hat{\theta}^T \xi \end{cases}$$

where $\xi(t) = (1 + \rho \frac{d}{dt})z(t) \quad \forall t \in \mathbb{R}_+$

$$(\rho > 0)$$

The input to the plant is: $u = -C^*r - D^*Y_p - \frac{1}{C^*} C^* v'$.

It is identical to the one used in part 3.2. But the signal

$$v' = v_L = \hat{\theta}^T z_L = \hat{\theta}^T \xi.$$

We shall present now a global stability theorem which is very similar to theorem A of Chapter III.

Theorem B

For the adaptive error system (2.25) with adaptation law (I_a^-)

assume that:

- all inputs $\begin{pmatrix} r \\ v \end{pmatrix}$ in $L_{\infty} \cap C$ produce signals $e, v', z, \hat{\theta}, \dot{\hat{\theta}}$ in $L_{\infty} \cap C$.
- $r, v \in L_{\infty} \cap C$
- (a2) and (a3) are satisfied (see Theorem A)

If $(e^*, \dot{e}^*) \in L_2 \cap L_{\infty}$ and $(z^*, \dot{z}^*) \in L_{\infty}$ then

$$(1') \quad \hat{\theta} \in L_{\infty}, (\dot{\hat{\theta}}, \ddot{\hat{\theta}}) \in \tilde{L}_2 \cap L_{\infty} \text{ and } \hat{\theta}(t) \rightarrow 0 \text{ as } t \rightarrow \infty$$

$$(11') \quad (e, \dot{e}) \in L_2 \cap L_{\infty} \text{ and } (e - e^*)(t) \rightarrow 0 \text{ exp. as } t \rightarrow \infty$$

(iii) $v \in L_2 \cap L_\infty$, $\dot{v} \in L_\infty$ and $v(t) \rightarrow 0$ as $t \rightarrow \infty$

(iv) $(z, \dot{z}) \in L_\infty$, $(x-x^*, \dot{x}-\dot{x}^*) \in L_2 \cap L_\infty$ and $(x-x^*)(t) \rightarrow 0$ exp as $t \rightarrow \infty$

(v) $(\xi, \dot{\xi}) \in L_\infty$

and explicit bounds can be obtained.

$$(vi) \quad \|v\|_2 \leq b_1^* \Delta \frac{1}{2\delta_{ev}} (\|e^*\|_2 + (\|e^*\|_2^2 - 4\lambda' \delta_{ev})^{\frac{1}{2}})$$

$$\|e\|_2 \leq b_2^* \Delta \|e^*\|_2 + \gamma_2(H_{ev}) b_1^*$$

where δ_{ev} is the "constant of strict passivity" of H_{ev}

$$\lambda' = \frac{1}{2} (\bar{\theta}^T(T) \Gamma^{-1} \bar{\theta}(T) - \bar{\theta}^T(0) \Gamma^{-1} \bar{\theta}(0)) + \rho (\bar{\theta}^T(T) \Gamma^{-1} \bar{\theta}(T) - \bar{\theta}^T(0) \Gamma^{-1} \bar{\theta}(0))$$

Proof:

Let us prove first that the map $e \rightarrow v$ is passive

$$\forall T \in \mathbb{R}_+ \quad \langle e | v \rangle_{T2} = \langle e | \bar{\theta}^T z \rangle_{T2} + \rho \langle e | \bar{\theta}^T \xi \rangle_{T2}$$

Let us look at the first term

$$\langle e | \bar{\theta}^T z \rangle_{T2} = \langle ze | \bar{\theta} \rangle_{T2} = \langle z(\Gamma \xi)^{-1} \dot{\bar{\theta}} | \bar{\theta} \rangle_{T2} = \langle z \xi^{-1} \Gamma^{-1} \dot{\bar{\theta}} | \bar{\theta} \rangle_{T2}$$

$$= \langle (1 + \rho \frac{d}{dt}) \Gamma^{-1} \dot{\bar{\theta}} | \bar{\theta} \rangle_{T2} = \langle \Gamma^{-1} \dot{\bar{\theta}} | \bar{\theta} \rangle_{T2} + \rho \langle \Gamma^{-1} \dot{\bar{\theta}} | \bar{\theta} \rangle_{T2}$$

$$= \int_0^T \bar{\theta}^T \Gamma^{-1} \dot{\bar{\theta}} dt + \rho \int_0^T \bar{\theta}^T \Gamma^{-1} \dot{\bar{\theta}} dt$$

$$= \frac{1}{2} \int_0^T \frac{d}{dt} (\bar{\theta}^T \Gamma^{-1} \bar{\theta}) dt + \rho (\bar{\theta}^T(T) \Gamma^{-1} \bar{\theta}(T) - \bar{\theta}^T(0) \Gamma^{-1} \bar{\theta}(0)) - \int_0^T \bar{\theta}^T \Gamma^{-1} \dot{\bar{\theta}} dt$$

using integration by parts

$$= \frac{1}{2} (\bar{\theta}^T(T) \Gamma^{-1} \bar{\theta}(T) - \bar{\theta}^T(0) \Gamma^{-1} \bar{\theta}(0)) + \rho (\bar{\theta}^T(T) \Gamma^{-1} \bar{\theta}(T) - \bar{\theta}^T(0) \Gamma^{-1} \bar{\theta}(0))$$

$$= \rho \int_0^T (\xi^T \Gamma e) \Gamma^{-1} (\Gamma \xi e) dt = \lambda' - \rho \int_0^T \xi^T \Gamma \xi e^2 dt$$

Let us look at the second term

$$\rho \langle e | \dot{\xi}^T \xi \rangle_{T_2} = \rho \langle e | \xi^T \Gamma \xi e \rangle_{T_2} = \rho \int_0^T \xi^T \Gamma \xi e^2 dt$$

$$\text{Hence } \forall T \in \mathbb{R}_+ \quad \langle e | v \rangle_T = \lambda^- \quad \square$$

We know that $H_{ev} \in \text{SPR}_0^+(a_2)$; $H_{zv} \in H_0^+(a_3)$;

$$(e^*, \dot{e}^*) \in L_2 \cap L_\infty(a_4); \quad (z^*, \dot{z}^*) \in L_\infty(a_5)$$

Since the map $e \rightarrow v^-$ is passive we can use the same kind of arguments as in the proof of theorem 3 and show that

$$e^* \in L_2 \text{ implies } (v^-, e) \in L_2 \text{ and } \|v^-\|_2 \leq b_1^* \text{ and } \|e\|_2 \leq b_2^*. \quad (s1)$$

$$(2.25) \quad \langle e \rangle \begin{cases} e - e^* = -H_{ev} v^- \\ z - z^* = -H_{zv} v^- \end{cases} \quad \text{As in the proof of theorem A}$$

we can apply lemma A₂ (a) and get

$$e \in L_2 \cap L_\infty, \dot{e} \in L_2, z - z^* \in L_2 \cap L_\infty, z \in L_\infty, \dot{z} - \dot{z}^* \in L_2 \quad (s2)$$

$$\forall t \in \mathbb{R}_+ \quad z(t) = (1 + \rho \frac{d}{dt}) \xi(t) = \xi(t) + \rho \dot{\xi}(t) \Rightarrow$$

$$\forall t \in \mathbb{R}_+ \quad \xi(t) = \xi(0) e^{-\frac{t}{\rho}} + \frac{1}{\rho} \left(\int_0^t e^{-\frac{u}{\rho}} z(u) du \right) e^{-\frac{t}{\rho}}$$

Then $\forall t \in \mathbb{R}_+, \forall i \in \{0, \dots, 2n-1\}$

$$|\xi_1(t)| \leq |\xi_1(0)| e^{-\frac{t}{\rho}} + \frac{1}{\rho} \left| \int_0^t e^{-\frac{u}{\rho}} z_1(u) du \right| e^{-\frac{t}{\rho}}$$

$$\leq |\xi_1(0)| e^{-\frac{t}{\rho}} + \frac{1}{\rho} \left(\int_0^t e^{-\frac{u}{\rho}} |z_1(u)| du \right) e^{-\frac{t}{\rho}}$$

$$\begin{aligned} & \leq |\xi_1(0)| e^{-\frac{t}{\rho}} + \frac{1}{\rho} \|z\|_{\infty} \left(\int_0^t e^{\frac{u}{\rho}} du \right) e^{-\frac{t}{\rho}} \\ & = |\xi_1(0)| e^{-\frac{t}{\rho}} + \|z\|_{\infty} (1 - e^{-\frac{t}{\rho}}) \end{aligned}$$

$$\rho > 0 \Rightarrow \|\xi(t)\| = \left(\sum_{i=0}^{2n-1} \xi_i^2(t) \right)^{\frac{1}{2}} < \infty$$

Hence $\|\xi\|_{\infty} = \text{ess sup}_{t \in \mathbb{R}_+} \|\xi(t)\| < \infty$ and so $\xi \in L_{\infty}$

$$\forall t \in \mathbb{R}_+ \quad \dot{\xi}(t) = z(t) - \rho \xi(t) \quad \text{implies} \quad \dot{\xi} \in L_{\infty}$$

since $(z, \xi) \in L_{\infty}$

We know that $\ddot{\theta} = \Gamma \dot{\xi} e$. Then using a similar calculation as in the proof of theorem A we can show that

$$e \in L_2 \quad \text{and} \quad \xi \in L_{\infty} \quad \text{imply} \quad \dot{\theta} \in L_2$$

$$e \in L_{\infty} \quad \text{and} \quad \xi \in L_{\infty} \quad \text{imply} \quad \dot{\theta} \in L_{\infty}$$

Besides $\ddot{\theta} = \Gamma(\dot{\xi}e + \xi\dot{e})$ and similarly

$$(e, \dot{e}) \in L_2 \quad \text{and} \quad (\xi, \dot{\xi}) \in L_{\infty} \quad \text{imply} \quad \ddot{\theta} \in L_2$$

We have proven that for all $T \in \mathbb{R}_+$

$$\begin{aligned} \rho \dot{\theta}^T(T) \Gamma^{-1} \ddot{\theta}(T) + \frac{1}{2} \ddot{\theta}^T(T) \Gamma^{-1} \ddot{\theta}(T) &= \langle e | v \rangle_{T_2} + \rho \dot{\theta}^T(0) \Gamma^{-1} \ddot{\theta}(0) + \frac{1}{2} \ddot{\theta}^T(0) \Gamma^{-1} \ddot{\theta}(0) \\ &\leq \|e\|_2 \|v\|_2 + \rho \dot{\theta}^T(0) \Gamma^{-1} \ddot{\theta}(0) + \frac{1}{2} \ddot{\theta}^T(0) \Gamma^{-1} \ddot{\theta}(0) \end{aligned}$$

applying the Cauchy Schwarz inequality and since $e, v \in L_2$

$$\text{then } \forall T \in \mathbb{R}_+ \quad \rho \dot{\theta}^T(T) \Gamma^{-1} \ddot{\theta}(T) + \frac{1}{2} \ddot{\theta}^T(T) \Gamma^{-1} \ddot{\theta}(T) \leq M'$$

$$\text{where } M' = \frac{1}{2} \ddot{\theta}^T(0) \Gamma^{-1} \ddot{\theta}(0) + \rho \dot{\theta}^T(0) \Gamma^{-1} \ddot{\theta}(0) + \frac{1}{2} \ddot{\theta}^T(0) \Gamma^{-1} \ddot{\theta}(0)$$

$$\text{hence } \forall T \in \mathbb{R}_+ \quad 2\rho \dot{\theta}^T(T) \Gamma^{-1} \ddot{\theta}(T) + \ddot{\theta}^T(T) \Gamma^{-1} \ddot{\theta}(T) \leq 2 M'$$

therefore $\forall T \in \mathbb{R}_+$ $(\rho \dot{\bar{\theta}}^T(T) + \bar{\theta}(T)) \Gamma^{-1} (\rho \dot{\bar{\theta}}(T) + \bar{\theta}(T)) \leq 2M' + \rho^2 \dot{\bar{\theta}}(T) \Gamma^{-1} \dot{\bar{\theta}}(T)$

since Γ^{-1} is symmetric. Besides $\Gamma^{-1} > 0$ then $\exists \mu > 0$ s.t. $\Gamma^{-1} \geq \mu I_{2n}$

Hence for all $T \in \mathbb{R}_+$

$$\|\rho \dot{\bar{\theta}}(T) + \bar{\theta}(T)\|^2 < \frac{1}{\mu} (2M' + \rho^2 \dot{\bar{\theta}}(T) \Gamma^{-1} \dot{\bar{\theta}}(T)) = L^{-2}$$

where $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^{2n}$.

$$\begin{aligned} \text{Therefore } \forall T \in \mathbb{R}_+ \quad \|\bar{\theta}(T)\| &\leq \|\rho \dot{\bar{\theta}}(T) + \bar{\theta}(T)\| + \|\rho \dot{\bar{\theta}}(T)\| \\ &\leq L' + \rho \|\dot{\bar{\theta}}\|_\infty \triangleq b_3'' \end{aligned}$$

$$\text{and so } \bar{\theta} \in L_\infty \text{ and } \|\bar{\theta}\|_\infty \leq b_3'' \quad (s6)$$

$$\text{We know that } v' = \bar{\theta}^T z + \rho \dot{\bar{\theta}}^T \xi$$

$$(s2), (s3), (s4) \text{ and } (s6) \Rightarrow (\bar{\theta}, z, \dot{\bar{\theta}}, \xi) \in L_\infty \Rightarrow v' \in L_\infty \quad (s7)$$

$$\begin{aligned} \text{Applying lemma A2(b), since } v' \in L_\infty \text{ and using } (a2), (a3), \\ (a4) \text{ and } (a5) \text{ we get } \dot{e} \in L_\infty, \dot{z} \in L_\infty \text{ and } \dot{z} - \dot{z}^* \in L_\infty \end{aligned} \quad (s8)$$

$$(s2), (s3) \text{ and } (s8) \Rightarrow (\xi, \dot{\xi}, e, \dot{e}) \in L_\infty \Rightarrow \ddot{\theta} = \Gamma(\dot{\xi}e + \xi\dot{e}) \in L_\infty \quad (s9)$$

$$\begin{aligned} \text{Hence } \dot{\bar{\theta}}(t) \text{ is uniformly continuous on } \mathbb{R}_+. \text{ This together with } \\ \dot{\bar{\theta}} \in L_2 \text{ implies that } \dot{\bar{\theta}}(t) \rightarrow 0 \text{ at } t \rightarrow \infty \text{ (see lemma A3)} \end{aligned} \quad (s10)$$

$$\begin{aligned} \dot{v}' &= \ddot{\bar{\theta}}^T z + \bar{\theta}^T \dot{z} + \rho \ddot{\bar{\theta}}^T \xi + \rho \dot{\bar{\theta}}^T \dot{\xi} \\ (s2), (s3), (s4), (s6), (s8) \text{ and } (s9) &\Rightarrow (\ddot{\bar{\theta}}, \ddot{\bar{\theta}}, \ddot{\bar{\theta}}, \dot{z}, \dot{z}, \dot{\xi}, \dot{\xi}) \in L_\infty \\ \text{hence } \dot{v}' \in L_\infty \text{ and so } v' \text{ is uniformly continuous on } \mathbb{R}_+. \text{ This} \\ \text{together with } v' \in L_2 \text{ implies } v'(t) \rightarrow 0 \text{ as } t \rightarrow \infty \end{aligned} \quad (s11)$$

Applying lemma A₂(c) and since $v' \in L_2$ and $v'(t) \rightarrow 0$ as $t \rightarrow \infty$,

$$\text{we get } (e - e^*)(t) \rightarrow -H_{ev}(0) \cdot 0 = 0 \text{ exp. as } t \rightarrow \infty$$

$$(z - z^*)(t) \rightarrow -H_{zv}(0) \cdot 0 = 0 \text{ exp. as } t \rightarrow \infty$$

□

APPENDIX D

Let us check the equation of the local error model: $x = \bar{x} - GF(x)$

$$\text{where } x = \begin{bmatrix} \tilde{e} \\ \tilde{z} \\ \tilde{\theta} \end{bmatrix} = \begin{bmatrix} e - e^* \\ z - z^* \\ \hat{\theta} - \theta^* \end{bmatrix}; \quad F(x) = \begin{bmatrix} \tilde{z}^T \tilde{\theta} \\ \tilde{z} \tilde{e} \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} \bar{e} \\ \bar{z} \\ \bar{\theta} \end{bmatrix} = \begin{bmatrix} -H_{ev} z^{*T} \cdot \bar{\theta} \\ -H_{zv} z^{*T} \cdot \bar{\theta} \\ (I + LM)^{-1} \bar{\theta}_0 + K z^* e^* \end{bmatrix}$$

$$G = \begin{bmatrix} H_{ev} [1 - z^{*T} K (z^* H_{ev} + e^* H_{zv})] & H_{ev} z^{*T} K \\ H_{zv} [1 - z^{*T} K (z^* H_{ev} + e^* H_{zv})] & H_{zv} z^{*T} K \\ K (z^* H_{ev} + e^* H_{zv}) & -K \end{bmatrix}$$

where $K \triangleq (I + LM)^{-1} L'$ and $M \triangleq z^* H_{ev} z^{*T} + e^* H_{zv} z^{*T}$

(a) Let us check the third component

We have

$$ze + z^* H_{ev} v + e^* H_{zv} v = ze + z^* (e^* - e) + e^* (z^* - z)$$

$$\text{and } v = z^T \tilde{\theta}$$

$$\Rightarrow ze + z^* H_{ev} z^T \tilde{\theta} + e^* H_{zv} z^T \tilde{\theta} = z^* e^* + (z - z^*) (e - e^*)$$

$$\Rightarrow ze + z^* H_{ev} z^T \tilde{\theta} + e^* H_{zv} z^T \tilde{\theta} = z^* e^* + \tilde{z} \tilde{e} - z^* H_{ev} \tilde{z}^T \tilde{\theta} - e^* H_{zv} \tilde{z}^T \tilde{\theta}$$

$$\Rightarrow L' ze + L' M \tilde{\theta} = L' [z^* e^* - (z^* H_{ev} + e^* H_{zv}) \tilde{z}^T \tilde{\theta} + \tilde{z} \tilde{e}]$$

$$\Rightarrow \tilde{\theta}_0 + L' ze + L' M \tilde{\theta} = (I + LM) \tilde{\theta} = \tilde{\theta}_0 + L' z^* e^* - L' (z^* H_{ev} + e^* H_{zv}) \tilde{z}^T \tilde{\theta} + L' \tilde{z} \tilde{e}$$

$$\Rightarrow \tilde{\theta} = (I + LM)^{-1} \tilde{\theta}_0 + K z^* e^* - [K (z^* H_{ev} + e^* H_{zv}) \tilde{z}^T \tilde{\theta} - K \tilde{z} \tilde{e}]$$

$$\Rightarrow \tilde{\theta} = \underbrace{\tilde{\theta}_0}_{\bar{\theta}} - \underbrace{GF(x)}_{GF(x)_3}$$

(b) Let us check the first component

$$\begin{aligned}
 e - e^* &= -H_{ev} v = -H_{ev} z^T \tilde{\theta} = -H_{ev} (z^{*T} \tilde{\theta} + \tilde{z}^T \tilde{\theta}) \\
 &= -H_{ev} [z^{*T} \tilde{\theta} + z^{*T} K(z^* H_{ev} + e^* H_{zv}) \tilde{z}^T \tilde{\theta} - z^{*T} K \tilde{z} \tilde{e} \\
 &\quad + (1 - z^{*T} K(z^* H_{ev} + e^* H_{zv})) \tilde{z}^T \tilde{\theta} + z^{*T} K \tilde{z} \tilde{e}] \\
 &= -\underbrace{H_{ev} z^{*T} \tilde{\theta}} - H_{ev} [1 - z^{*T} K(z^* H_{ev} + e^* H_{zv})] \tilde{z}^T \tilde{\theta} - H_{ev} z^{*T} K \tilde{z} \tilde{e} \\
 &= \bar{e} + GF(x)_1
 \end{aligned}$$

One uses a similar method to check the second component.

□

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