TURBULENT JETS AND WALL JETS

IN UNIFORM STREAMING FLOW

by

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SUMMARY

For plane turbulent jets and wall jets in uniform streaming flow a simple method, based on a simple empirical formula for the rate of growth which differs from that proposed by Abramovich, is presented. The predictions of length scale, l_0 , and velocity scale, u_0 , are in good agreement with experimental results for all ratios of free stream to jet exit velocity and the eddy viscosity Reynolds number has the correct asymptotic values.

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NOTATION

C	-	constant of proportionality defining the rate of growth (equation (11)).
C_1	-	constant in equation (10).
f	-	function of non-dimensional cross stream co-ordinate η for the variation of mean velocity (equation (1)).
$I_1; I_2; I_3; I'$	-	constants defined by equations (7) and (33).
K	-	constant in equation (8).
l	-	mixing length.
l_o	-	length scale, defined as that value of y at which $U - U_1 = 1/2 u_o$.
Me	-	excess momentum flux constant defined by equation (7).
$\overline{q^2}$	-	$= 1/2 (\overline{u^2} + \overline{v^2} + \overline{w^2})$.
R_T	-	eddy viscosity Reynolds number $(= \frac{u_o l_o}{\nu_T})$.
R_{Tj}	-	eddy viscosity Reynolds number for jet in still air.
$R_{T\omega}$	-	eddy viscosity Reynolds number for a small perturbation jet or wake.
U	-	local mean velocity in the x-direction.
U_J	-	jet velocity at nozzle exit.
U_m	-	maximum velocity in the x-direction.
U_1	-	uniform streaming flow velocity (=constant)
u	-	longitudinal fluctuating component of velocity.
u_o	-	velocity scale $(= (U_m - U_1))$.
v	-	local mean velocity in y-direction.
v_m	-	transverse velocity at y_m .
v	-	transverse fluctuating component of velocity.

(v)

w	-	lateral fluctuating component of velocity.
x	-	downstream distance from slot exit.
x_o	-	distance of hypothetical origin from slot exit.
y	-	transverse co-ordinate.
y_m	-	value of y where $U = U_m$
η	-	non-dimensional cross stream co-ordinate ($= y/l_o$)
ν	-	kinematic viscosity.
ν_T	-	eddy viscosity
ρ	-	density
θ	-	momentum thickness defined by equation (14)
τ	-	turbulent shear stress ($= -\overline{\rho uv}$)

1. INTRODUCTION

The purpose of this investigation is to evaluate a simple integral approach for predicting the streamwise development of turbulent jets and wall jets in streaming flow at constant pressure. Since the mean velocity profiles for these flows, when normalized, are closely similar (for example, see Bradbury (1963): Eskinazi and Kruka (1964): Patel (1962), etc.) the main task then reduces to simply predicting the normalizing length scale l_0 and perturbation velocity scale u_0 .

Although a number of investigations in the past have been undertaken for these particular problems none of the methods proposed have been satisfactory in correlating the existing experimental results. Essentially most of the methods except the one proposed by Wygnanski (1969) are integral methods. Wygnanski's approach is based on a coordinate-type perturbation expansion and the series truncation together with the assumption of constant eddy viscosity across the flow (i.e. ν_T a function of x only). His method requires two experimental constants which are obtained from the limiting self-preserving cases.

All the integral methods, although different in concept assume geometrical similarity of mean velocity profiles. Squire and Trouncer (1944) use a "double integral" technique and introduce Prandtl's mixing length hypothesis, ($\tau = \rho l^2$

$\frac{\partial U}{\partial y} \left| \frac{\partial U}{\partial y} \right|$), to specify shear stress at "half-velocity" point in the shear layer. The mixing length, l , in turn is assumed to be a constant proportion of the width of the shear layer (i.e. in effect $v_T \propto u_o l_o$ and the eddy viscosity Reynolds number, $\frac{u_o l_o}{v_T}$, equals a constant throughout the flow field). Hill (1965) uses integral momentum and moment of momentum equations. The latter has an integral, involving shear stress, which is evaluated from the experimental results of jets in still surroundings. Abramovich (1963) uses integral momentum equation together with an auxiliary equation (see section 2.1) which involves one experimental constant. These integral methods have one serious objection that they do not exhibit expected asymptotic behaviour, i.e. a strong jet degenerates to a small-perturbation jet and both these extreme cases are self-preserving.

Integral methods known as variable eddy Reynolds number methods have been developed to incorporate the expected asymptotic behaviour. This group of methods endeavours to allow for changes in the eddy viscosity Reynolds number, $(u_o l_o / v_T)$, by applying Townsend's (1956) large eddy equilibrium hypothesis using local values of strain-rate ratio to compute local values of the eddy viscosity Reynolds number. Methods of Bradbury (1963) and Gartshore (1965) fall into this category. Indeed they exhibit expected asymptotic behaviour because Townsend used his large eddy equilibrium hypothesis originally to explain differences in the eddy

viscosity Reynolds numbers for the two extreme cases.

In Gartshore's method the eddy viscosity Reynolds number is related to scale of the largest eddies and this scale is assumed to be proportional to the standard deviation of the laminar super-layer. The standard deviation in turn is obtained from measured intermittency distributions. The double integral technique similar to Squire and Trouncer is used and the resulting equations are solved numerically by a four-point Runge Kutta technique. It should be noted that in his method, apart from initial conditions to start numerical calculations, two experimental constants are required. In spite of the sophistication and extra assumptions introduced by Gartshore, his predictions for turbulent jets in uniform streaming flow are not in good agreement with the experimental results of Bradbury.

Bradbury and Riley (1967) have also concluded that simple integral theories using Townsend's large eddy equilibrium hypothesis do not accurately predict this flow. They have, in fact, shown that the application of the large eddy equilibrium hypothesis leads to errors in the predictions of flow development, opposite in sense but equal in order of magnitude to the errors obtained with the constant eddy Reynolds number theories. Thus there is yet no simple method for predicting jets and wall jets in uniform streaming flow.

For interest it is noted that Naudascher (1967) has proposed a method which incorporates a new form of similarity for such flows but his similarity form introduces inconsistency

in the analysis and also has limited experimental verification.

An attempt is therefore made to analyse the development of mean flow characteristics of these flows that is valid over the entire range of flow rather than over asymptotic regions only. The integral momentum equation is used together with an auxiliary equation to provide closed form solutions for both u_0 and l_0 . Only one well established experimental constant which varies from jets to wall jets is required. Even though the investigation is limited to incompressible and isothermal conditions it can be extended to studies of other main characteristics of the flow, e.g. temperature, density and concentration profiles (see for example Abramovich (1963)).

2. SIMPLE ANALYSIS FOR PLANE TURBULENT JET IN UNIFORM STREAMING FLOW

Although the analysis presented here specifically applies to two-dimensional flows it can be extended to axi-symmetric flows (see Newman (1967) for the general equations). Since the approach is to use the integral momentum equation some assumption must be made about the mean velocity profile. Following Townsend (1956) the velocity profile in both jets and wakes to a good degree of accuracy may be written as (Fig. (1)):

$$U = U_1 + u_0 f(\eta) \quad (1)$$

where $f(\eta)$ is a universal function of $\eta = Y/l_0$

U_1 is the velocity of the external irrotational flow

u_0 is a mean velocity scale and l_0 is a characteristic length scale for this layer. Both u_0 and l_0 are functions of x only.

Considering a symmetrical two-dimensional jet (Fig. (1a)) at constant pressure, the time-averaged boundary-layer equation in downstream direction x (for constant density ρ , and using the approximate equation in y -direction) is:-

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = \frac{1}{\rho} \frac{\partial \tau}{\partial y} - \frac{\partial (\overline{u^2} - \overline{v^2})}{\partial x} \quad (2)$$

where U and V are mean velocities in directions x and y respectively; u and v are the associated turbulent fluctuations about

the mean and $\frac{\tau}{\rho} = -\overline{uv} + v\frac{\partial U}{\partial y}$

The time-averaged continuity equation is:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \quad (3)$$

Combining equation (3) with equation (2) and integrating the resulting equation between limits $y = 0$ and $y = y_1 \rightarrow \infty$ (note that at both limits $\tau = 0$),

$$\frac{d}{dx} \int_0^\infty u^2 dy + \left[uv \right]_0^\infty + \frac{d}{dx} \int_0^\infty (\overline{u^2} - \overline{v^2}) dy = 0 \quad (4)$$

Similarly integrating the continuity equation (3) and substituting in equation (4),

$$\frac{d}{dx} \int_0^\infty \left[(u^2 - uv_1) + (\overline{u^2} - \overline{v^2}) \right] dy = 0 \quad (5)$$

If the difference between normal Reynolds stresses (which is quite small compared to the excess momentum flux, except maybe near the edges) can be neglected then equation (5) reduces to:

$$\int_0^\infty (u^2 - uv_1) dy = \text{constant} \quad (6)$$

Equation (6) simply implies that excess momentum flux in x-direction is constant.

Substituting equation (1) into equation (6) gives:

$$l_0(u_0^2 I_2 + u_0 U_1 I_1) = \text{constant} (= Me \text{ (say)}) \quad (7)$$

$$\text{where } I_1 = \int_0^\infty f(\eta) d\eta \text{ and } I_2 = \int_0^\infty f^2(\eta) d\eta$$

There are two unknowns in equation (7), u_0 and l_0 , and hence

another equation giving a connection between u_0 and l_0 is necessary before any prediction regarding their individual variations in x-direction can be made.

2.1 Choice of an Auxiliary Equation

Various methods differ in their choice of the second equation. Whatever the choice may be, the necessary important requirement is that the method must exhibit proper variation of the eddy viscosity Reynolds number. It should be mentioned that this requirement is not sufficient as can be seen from the methods based on Townsend's hypothesis (Bradbury: Gartshore). Hence an additional requirement is that there must be agreement between predictions and experimental results. It should be recalled that invariably experimental results have been in better agreement with predictions when Prandtl's mixing length hypothesis (see Bradbury (1963)) or Abramovich's (1963) method (see Newman (1967)) is adopted.

Abramovich, following Prandtl, introduces an auxiliary equation in the following form:

$$\frac{dl_0}{dx} = K\left(\frac{v}{U}\right) \quad (8)$$

where v is "mean absolute value of transverse pulsation speed ..." and " U is the mean absolute value of the main component of the stream velocity in the direction parallel to the axis of the jet x ". K is a constant.

To reduce equation (8) to a useful form Abramovich then introduces two assumptions, namely:

$$\left. \begin{aligned} (i): \quad v &\propto u_o \\ \text{and } (ii): \quad U &\propto \text{mean velocity of the flow} \\ &= \frac{1}{2} (U_1 + U_M) \text{ where } U_M = U_1 + u_o \end{aligned} \right\} (9)$$

With these assumptions equation (8) reduces to*:

$$\frac{dl_o}{dx} = C_1 \left[\frac{u_o}{U_1 + \frac{u_o}{2}} \right] \quad (10)$$

where C_1 is a constant.

Newman (1967) has derived equation (10) from consideration of a mixing layer in constant pressure. It will be shown later that equation (10) is not in agreement with experimental results.

Examining the assumptions of Abramovich it is reasonable to assume that the spatial rate of growth of the jet will depend on the level of turbulence in the layer, which in turn depends on u_o and hence it is reasonable to replace v by u_o . However, for the second assumption it is proposed that the spatial rate of growth of the jet will also depend on the main component of the flow velocity, i.e. $(U_1 + u_o)$, instead of the mean velocity of the flow as assumed by Abramovich.

Hence the auxiliary equation proposed for this investigation becomes:

$$\frac{dl_o}{dx} = C \left[\frac{u_o}{U_1 + u_o} \right] \quad (11)$$

* Equation (10) can be derived by using Prandtl's mixing length hypothesis, (i.e. $\tau = \rho l(q^2)^{1/2} \frac{\partial U}{\partial y}$), and "restricted form of the energy-length model" of Spalding (1968).

Comparison of equations (10) and (11) indicates that there is hardly any qualitative difference between the two equations except the change in constants, but in effect as will be shown later, one is an approximately constant eddy viscosity Reynolds number hypothesis and the other is a variable eddy viscosity Reynolds number hypothesis with correct asymptotic values.

2.2 Closed-Form Solutions for 1_0 and u_0

The auxiliary equation (11) together with the integrated momentum equation (7) gives:

$$\frac{du_0}{dx} = - \left(\frac{C}{2Me} \right) \left[\frac{u_0^3 (I_2 + I_1 \frac{u_1}{u_0})^2}{(1 + \frac{u_1}{u_0}) (I_2 + \frac{1}{2} I_1 \frac{u_1}{u_0})} \right] \quad (12)$$

Equation (12) can now be integrated in closed form after some manipulations and the solution is:

$$\begin{aligned} & \frac{1}{2} \left(\frac{u_1}{u_0} \right)^2 + \left(\frac{u_1}{u_0} \right) + \left(\frac{I_2}{I_1} \right)^2 \ln \left[\frac{I_2/I_1}{(I_2/I_1 + u_1/u_0)} \right] \\ & + \frac{(I_2/I_1)^2 (1 - I_2/I_1)}{(I_2/I_1 + u_1/u_0)} - \left(\frac{I_2}{I_1} \right) \left(1 - \frac{I_2}{I_1} \right) = \frac{CI_1 u_1^2}{Me} (x - x_0) \end{aligned} \quad (13)$$

where x_0 is introduced as the constant of integration and is such that when $x = x_0$, $u_0 \rightarrow \infty$ and the hypothetical origin of the jet is identified.

Define $\theta = \frac{Me}{u_1^2} = \int_0^\infty \frac{u}{u_1} \left(\frac{u}{u_1} - 1 \right) dy$ (14)

Equations (13) and (7) now reduce to:

$$\begin{aligned} \frac{1}{2} \left(\frac{u_1}{u_o} \right)^2 + \left(\frac{u_1}{u_o} \right) + \left(\frac{I_2}{I_1} \right)^2 \ln \left[\frac{I_2/I_1}{(I_2/I_1 + u_1/u_o)} \right] \\ + \frac{(I_2/I_1)^2 (1 - I_2/I_1)}{(I_2/I_1 + u_1/u_o)} - \left(\frac{I_2}{I_1} \right) \left(1 - \frac{I_2}{I_1} \right) = c I_1 \left(\frac{x-x_o}{\theta} \right) \end{aligned}$$

(15)

and $\frac{l_o}{\theta} = \frac{1}{I_1} \left[\frac{(u_1/u_o)^2}{(I_2/I_1 + u_1/u_o)} \right]$ (16)

These are most convenient expressions and give implicitly the variation of l_o with $(x-x_o)$.

From equations (15) and (16) for a strong jet,

$$u_o \propto (x-x_o)^{-1/2} \quad \text{and} \quad l_o \propto (x-x_o)$$

For a small-perturbation jet equation (15) reduces to,

$$\left(\frac{u_1}{u_o} \right)^2 \propto (x-x_o) \quad (17)$$

and equation (16) then gives:

$$l_o \propto (x-x_o)^{1/2} \quad (18)$$

Hence it is noted that equations (15) and (16) do predict the

expected behaviour of the jet development. Furthermore, note that there is only one experimental constant, C , in these equations. This constant C depends on the definition of l_o . Usually l_o is defined as that value of y at which $(U-U_1) = \frac{1}{2} u_o$ and then the value of C can be obtained experimentally by analyzing data of a jet in still surroundings. Equation (11) for a jet in still surroundings becomes $\frac{dl_o}{dx} = C$.

Newman (1967) has collected values of C from various experimental investigations on jet in still air and gives $C = 0.104$ as an average value. With the same definition of l_o as above the value of C_1 in equation (10) is 0.052 (i.e. $C_1 = C/2$).

The corresponding results are now obtained using Abramovich's equation (10). With equations (10) and (7) the solutions for u_o and l_o are:

$$\begin{aligned} \frac{1}{2} \left(\frac{u_1}{u_o} \right)^2 + \frac{1}{2} \left(\frac{u_1}{u_o} \right) + \left(\frac{I_2}{I_1} \right)^2 \ln \left[\frac{I_{2/I_1}}{(I_{2/I_1} + u_{1/u_o})} \right] \\ + \frac{(I_{2/I_1})^2 \left(\frac{1}{2} - I_{2/I_1} \right)}{(I_{2/I_1} + u_{1/u_o})} - \left(\frac{I_2}{I_1} \right) \left(\frac{1}{2} - \frac{I_2}{I_1} \right) = C_1 I_1 \left(\frac{x-x_o}{\theta} \right) \end{aligned} \quad (19)$$

$$\text{and} \quad \frac{l_o}{\theta} = \frac{1}{I_1} \left[\frac{(u_{1/u_o})^2}{(I_{2/I_1} + u_{1/u_o})} \right] \quad (20)$$

Equation (20) is identical to equation (16) as would be expected however, comparison of equations (19) and (15) indicates that equation (19) for a small-perturbation jet will give

a value of $(\frac{u_1}{u_o})^2$ at a particular downstream station half that obtained from equation (15), e.g. equation (19) for a small-perturbation jet becomes:

$$(\frac{u_1}{u_o})^2 \approx CI_1 (\frac{x-x_o}{\theta}) \quad (21)$$

compared to the one obtained from equation (15),

$$(\frac{u_1}{u_o})^2 \approx 2CI_1 (\frac{x-x_o}{\theta}) \quad (22)$$

2.3 A Test of Usefulness for the Auxiliary Equation (11)

Newman (1967) has collected the available experimental results and, based on a velocity distribution which is Gaussian, concludes that for a small-perturbation jet or wake:

$$\frac{l_o}{(x-x_o)^{1/2} (2\theta)^{1/2}} = 0.316 \quad (23)$$

Also for a Gaussian velocity profile he gives:

$$|2\theta| = 2.13 l_o \left| \frac{u_o}{u_1} \right| \quad (24)$$

The auxiliary equation (11) for a small-perturbation jet reduces to:

$$\frac{dl_o}{dx} \frac{u_1}{u_o} = c \quad (25)$$

From equations (24) and (25) it can be shown that

$$\frac{l_o}{(x-x_o)^{1/2}(2\theta)^{1/2}} = \sqrt{\frac{2C}{2.13}} \quad (26)$$

Equation (26) with $C = 0.104$ reduces to 0.313 compared to the value 0.316 suggested by Newman. However, equation (26) with Abramovich's auxiliary equation will give a value 0.221 (note that Newman's value 0.156 is a misprint). Hence for all practical purposes the choice of the auxiliary equation (11) seems to be consistent with both the rate of growth for a jet in still surroundings and the likely value for a small-perturbation jet or wake.

It should be noted that for a small-perturbation jet or wake the eddy viscosity Reynolds number $u_o l_o / \nu_T$, is related directly to the growth parameter by the equation, (Newman (1967)),

$$\frac{l_o}{(x-x_o)^{1/2}(2\theta)^{1/2}} = \left[\frac{1.30}{R_{T_\omega}} \right]^{1/2} \quad (27)$$

where R_{T_ω} is the eddy viscosity Reynolds number for a small-perturbation jet or wake.

Hence from equations (26) and (27) together with $C = 0.104$, $R_{T_\omega} = 13.3$ which is exactly the same value as that obtained by Newman for both round and two-dimensional wakes from the measurements of Townsend. It can easily be shown that using the auxiliary equation (11) for axisymmetric cases the same value

of $R_{T\omega}$ is obtained. As for example, following Newman, for a small-perturbation axisymmetric jet or wake with a Gaussian velocity profile:

$$\left| 2\theta \right| = 1.20 l_o \left| \frac{u_o}{U_1} \right|^{1/2} \quad (28)$$

and

$$\frac{l_o}{(x-x_o)^{1/3} \left| 2\theta \right|^{2/3}} = \left[\frac{2.88}{R_{T\omega}} \right]^{1/3} \quad (29)$$

Hence from equations (25) and (28),

$$\frac{l_o}{(x-x_o)^{1/3} \left| 2\theta \right|^{2/3}} = \left(\frac{3c}{1.44} \right)^{1/3} \quad (30)$$

Equation (30), with $c = 0.104$, reduces to 0.60 and then from equation (29) $R_{T\omega} = 13.3$.

2.4 The Eddy Viscosity Reynolds Number Variations

In section 2.3 it is shown that the auxiliary equation (11) gives the correct asymptotic value of the eddy viscosity Reynolds number, $R_{T\omega}$. In this section the details of $R_{T\omega}$, $\left(= \frac{u_o l_o}{\nu_T} \right)$, variation from one asymptotic value to another is given and it should be noted that for this purpose either half momentum integral equation or mean energy integral equation may be used. The former will provide $R_{T\omega}$ values at $y = l_o$.

whereas the latter provides an average value of R_T at a particular downstream station. The mean energy integral equation is:

$$\frac{1}{2} \frac{d}{dx} \int_0^{\infty} U (U^2 - U_1^2) dy = - \int_0^{\infty} \frac{\tau}{\rho} \frac{\partial U}{\partial y} dy \quad (31)$$

The turbulent shear stress may be represented by

$$\frac{\tau}{\rho} = - \overline{uv} = \nu_T \frac{\partial U}{\partial y} \quad (32)$$

where ν_T is eddy viscosity.

Following Townsend (1956) it is assumed that ν_T is a function of x only. It is noted that this assumption is not necessary if half momentum integral equation is used. Substituting equations (1) and (32) into equation (31) gives:

$$(I_3 + I_2 \frac{U_1}{u_0}) \frac{dl_0}{dx} + (3I_3 + 2I_2 \frac{U_1}{u_0}) \frac{l_0}{u_0} \frac{du_0}{dx} = - \frac{2I'}{R_T} \quad (33)$$

$$\text{where } I_3 = \int_0^{\infty} f^3(\eta) d\eta$$

$$\text{and } I' = \int_0^{\infty} \left(\frac{df}{d\eta} \right)^2 d\eta$$

Now with equations (7), (11) and (12) equation (33) can be reduced to:

$$\frac{c}{(1 + \frac{u_1}{u_o})} \left[\frac{(I_2 + I_1 \frac{u_1}{u_o})(3I_3 + 2I_2 \frac{u_1}{u_o})}{2(I_2 + \frac{1}{2} I_1 \frac{u_1}{u_o})} - (I_3 + I_2 \frac{u_1}{u_o}) \right]$$

$$= \frac{2I'}{R_T} \quad (34)$$

For a jet in still air equation (34) becomes:

$$R_{T_j} = \frac{4I'}{CI_3} \quad (35)$$

For a small-perturbation jet equation (34) reduces to:

$$R_{T_\omega} = \frac{2I'}{CI_2} \quad (36)$$

It can be seen from equations (35) and (36) that both R_{T_j} and R_{T_ω} are absolute constants throughout the respective flow fields and that their values are indeed different. With $C = 0.104$ and using Bradbury's (1963) integral values (i.e. $I'/I_3 = 0.9125$ and $I'/I_2 = 0.745$) $R_{T_j} = 35.1$ compared to Bradbury's value of $R_{T_j} = 36.5$, the difference being entirely due to the value of C used to calculate R_{T_j} . Bradbury uses $C = 0.10$. Similarly $R_{T_\omega} = 14.35$ compared to 14.7 which Townsend obtained from measurements of the rate of spread of the wake. Note that $R_{T_j} = 32.6$ and $R_{T_\omega} = 13.3$ if the velocity profile is assumed to be Gaussian.

It is easy to show that the corresponding results using Abramovich's equation (10) are:

$$\begin{aligned}
& \frac{c_1}{\left(\frac{1}{2} + \frac{u_1}{u_0}\right)} \left[\frac{(I_2 + I_1 \frac{u_1}{u_0})(3I_3 + 2I_2 \frac{u_1}{u_0})}{2(I_2 + \frac{1}{2} I_1 \frac{u_1}{u_0})} - (I_3 + I_2 \frac{u_1}{u_0}) \right] \\
& = \frac{2I'}{R_T} \tag{37}
\end{aligned}$$

For a jet in still air equation (37) reduces to:

$$R_{T_j} = \frac{2I'}{c_1 I_3} = \frac{4I'}{c I_3} \tag{38}$$

For a small-perturbation jet equation (37) reduces to:

$$R_{T_\omega} = \frac{2I'}{c_1 I_2} = \frac{4I'}{c I_2} \tag{39}$$

It is now obvious that as a consequence of the assumption which Abramovich has made (see equation (9)), his method fails to give the correct value of R_{T_ω} . Finally, as mentioned before, the validity of equation (11) may be confirmed by comparing experimental results with predictions from equations (15) and (16). It should be mentioned that in equations (15) and (16) Bradbury's integral values, i.e. $I_2/I_1 = 0.724$, $I_1 = 1.0125$, are used.

3. COMPARISON WITH EXPERIMENTAL RESULTS

The experimental results of Bradbury and Riley (1967) for plane, symmetrical turbulent jets in uniform streaming flow and collected results of plane turbulent wall jets in uniform streaming flow are compared with the predictions of equations (15) and (16).

3.1 Comparison with the Results of Bradbury and Riley (1967)

The reason for using the results of Bradbury and Riley is that they have given their results in tabulated form and, moreover, their results are considered to be of fairly good quality. One of the serious objections to their tabulated results is that they have been obtained by arbitrarily shifting the curves to correlate the results for all the (U_1/U_j) ratios they investigated. The details are not given except that the shifts are attributed to changes in the hypothetical origin, x_0 , but these are not listed. Fortunately, they have presented some of their raw data in graphical form and these are used for the comparisons in Fig. (3).

Fig. (2) gives the variation in growth of the jet, l_0 , in accordance with equation (16). The tabulated results of Bradbury and Riley are compared with the prediction from equation (16) which is represented by a solid line. Use of their tabulated results is possible here because this presentation does not involve x_0 . The agreement between experimen-

tal results and equation (16) is excellent as would be expected because equation (16) is simply a statement of conservation of excess momentum flux.

Fig. (3) shows the variation of velocity scale u_o with x . In this figure both tabulated and raw data (obtained from their $(\frac{u_1}{u_o})^2$ vs x/θ curves) are given, the former being represented by a broken line. Also included in the figure is a curve representing equation (15) with $2CI_1 = 0.2025$. In replotting the data of Bradbury and Riley for all the cases of $(\frac{u_1}{u_j})$ shown in Fig. (3) it was noted that $(\frac{x_o}{2\theta})$ was small and hence it was neglected. The effect of arbitrary adjustments of the curves by them clearly shows the disagreement between the predicted curve and the dotted line and it also shows that such adjustments are indeed not justified because the raw data and prediction from equation (15) are in agreement.

From Figs. (2) and (3) it may be concluded that the method proposed in this paper does predict satisfactorily the variations in l_o and u_o for jets in uniform streaming flow.

3.2 Wall Jets in Uniform Streaming Flow

Wall jets in uniform streaming flow offer an excellent opportunity to evaluate the analysis presented in this paper because in many investigations of wall jets (Patel (1962), Eskinazi and Kruka (1964), Gartshore (1965)) it was found experimentally that velocity profiles are approximately similar

to half the profile for a jet in uniform streaming flow. If the analysis of section (2) is applied to the outer part of the wall jets additional assumptions have to be made, however. These are: (a) turbulent shear stress at $y = y_m$ is zero and (b) changes in y_m with respect to x are small. Strictly speaking both assumptions are invalid, nevertheless, it is observed that the shear stress at the maximum velocity point is small compared with some characteristic shear stress say at $y = l_0$. As a matter of interest it is noted that changes in y_m with respect to x have compensating effect as seen from the following momentum integral equation: (see Fig. (1b)):

$$\frac{d}{dx} \int_{y_m}^{\infty} (U^2 - UU_1) dy = (\overline{uv})_{y_m} - \left| u_o v_m \right| - u_o U_m \frac{dy_m}{dx}$$

Hence one may expect the previous analysis to apply, at least approximately, to the outer part of the wall jets in uniform flow. Fig. (1b) defines the notations and with these definitions variations in u_o and l_0 are given by equations (15) and (16). The values of I_n ($n = 1, 2, 3$, etc.) are kept the same as before because the non-dimensional velocity profiles are practically the same. However, it is known that the wall on one side affects the growth of the outer part and hence the constant C would be different. (It is known that C for wall jets in still air is about half that for a jet in still air.)

Fig. (4) shows experimental results of Eskinazi and Kruka, Gartshore and Patel plotted in accordance with equation (16). A line representing equation (16) with $I_1 = 1.0125$, (note that

this is the same value of I_1 as used in Figs. (1) and (2)), is drawn. It can be seen from this figure that the collected experimental results for various ratios of (U_1/U_j) are in agreement with the prediction of equation (16).

Fig. (5) shows the variation of velocity scale u_0 with x . It should be noted that for all the wall jet cases considered here the hypothetical origin x_0 is found to be 20 slot widths upstream of the slot (see Fig. (6)). Equation (15) can now be fitted to the experimental results in Fig. (5) by selecting various values of C . Two such curves with values of $CI_1 = 0.062$ and 0.0653 are represented by solid lines. From this figure it may be concluded that the constant C for wall jets is about 0.065 . It is interesting to compare this value of C with that obtained from growth of wall jets in still air, e.g. Schwarz and Cosart (1961) give $C = 0.0678$; Sigalla (1958) gives $C = 0.0664$; Patel (1962) gives $C = 0.065$; Eskinazi and Kruka (1964) give $C = 0.0737$ and Gartshore and Hawaleshka (1964) give $C = 0.065$. Note that the value of C obtained from Fig. (5) is not inconsistent with the values of C obtained from wall jets in still air. It is therefore concluded that the simple method presented in this paper is capable of predicting reasonably well the development of main characteristics of wall jets in uniform streaming flow.

4. CONCLUSIONS

Free shear flows and wall jet flows in uniform streams are successfully predicted by assuming a simple empirical law for the rate of growth of the jet. This is similar to but differs in detail from that proposed by Abramovich. It includes one arbitrary constant which is determined from the relevant growth in still surroundings and then predicts accurately the other extreme case of a small-perturbation jet.

Comparisons with experimental results confirm that the simple method leads to satisfactory predictions of both the length scale l_0 and the velocity scale u_0 . Furthermore, the predictions are valid over the entire range of the flow rather than for asymptotic regions only and also for various free stream to jet exit velocity ratios.

It is noted that the sole experimental constant C involved in this method varies from jets to wall jets as would be expected. It is also noted that as a consequence of Abramovich's (1963) assumption his method fails to give the correct asymptotic value of the eddy viscosity Reynolds number of small-perturbation jets or wakes, $R_{T\omega}$.

It appears that the method proposed in this paper may provide a simple basis for the analysis of flows like jets and wakes in a general stream, axisymmetric jets and wakes in a general stream, wall jets in a general stream and may possibly be extended to more complicated cases like asymmetric

jets. It can also be extended to studies of other main characteristics of the flows, e.g. temperature, density and concentration profiles (see for example Abramovich (1963)).

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Fig. (1a) 2-Dimensional Turbulent Jet in Uniform Streaming Flow

Fig. (1b) 2-Dimensional Turbulent Wall Jet in Uniform Streaming Flow

COMPARISON OF PREDICTED AND EXPERIMENTAL VARIATIONS IN LENGTH
SCALE, 1. PLANE TURBULENT JETS IN UNIFORM STREAMING FLOW

○ - $0.16 \leq u_1/u_j \leq 0.60$: Bradbury and Riley (1967)

COMPARISON OF PREDICTED AND EXPERIMENTAL VARIATIONS IN
PERTURBATION VELOCITY SCALE, u_0 . PLANE TURBULENT JETS

IN UNIFORM STREAMING FLOW

COMPARISON OF PREDICTED AND EXPERIMENTAL VARIATIONS IN LENGTH SCALE, l_o .

WALL JETS IN UNIFORM STREAMING FLOW

COMPARISON OF PREDICTED AND EXPERIMENTAL VARIATIONS IN
PERTURBATION VELOCITY SCALE, u_0 .
WALL JETS IN UNIFORM STREAMING FLOW

HYPOTHETICAL ORIGIN, x_0 FOR WALL JETS IN UNIFORM STREAMING FLOW