INTERNAL CATEGORY THEORY

Sam Melkonian

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Department of Mathematics
McGill University
Montreal

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ABSTRACT

In this thesis , we internalize certain well known notions in category theory , i.e. , given a category $\underline{E}$ with finite limits , we define , for example , a category object (internal category) in $\underline{E}$, in such a way that when $\underline{E}$ is the category of sets , this definition reduces to the ordinary one of a small category .

Chapter I consists of basic definitions , certain formulae involving functors and natural transformations , and triples and simplicial objects induced by category objects . In chapter II , we define and give characterisations of internal functors , while in chapter III , we define limits and colimits , and prove a well known result internally , namely that limit (colimit) is the right (left) adjoint of the "constant" functor Δ .

I wish to thank D. Gildenhuys , my thesis supervisor , for his help and encouragement . I also wish to thank R. Paré , for many interesting discussions .

ABSTRAIT

Dans cette thèse on définit dans un contexte interne certaines notions bien connues dans la théorie des catégories, par example, un objet de catégorie (catégorie interne).

Le premier chapitre consiste en définitions fondamentales, en certaines formules et en des triples et des objets simpliciaux induits par des objets de catégorie. Dans le deuxième chapitre on definit et on donne des caractérisations des foncteurs internes; ensuite dans le troisième chapitre on définit des limites et colimites et on prouve,dans un contexte interne, un résultat qui est bien connu, notamment que la limite (colimite) est l'adjoint à droite (gauche) à Δ, le foncteur "constant".

J'aimerais remercier mon directeur de thèse, D. Gildenhuys, pour son aide et son encouragement. Je voudrais également remercier R. Paré, pour de nombreuses discussions intéressantes que nous avons eues ensemble.

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INTRODUCTION

Internal category theory has its applications in the study of toposes (Diaconescu [1]) and profinite groups (Gildenhuys and Ribes [3]) , and it is believed that several interesting applications may arise through the study of category objects , internal functors , etc. , in an arbitrary category with finite limits .

Throughout this thesis , $\underline{E}$ will be assumed to be a skeletal category with finite limits . Additional conditions , for example cartesian closedness and the existence of coequalizers , are imposed upon $\underline{E}$ in certain parts of chapter III , and are specified in the relevent sections . The words "map" and "morphism" will be used interchangeably , throughout .

I. CATEGORY OBJECTS

1. Category Objects

Definition. A category object C in $\underline{E}$ consists of objects and morphisms

$$\begin{tikzcd} X_2 \arrow[r, "\gamma"] & X_1 \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1"] & X_0 \arrow[l, shift left=3, "\mu"] \end{tikzcd}$$

in $\underline{E}$, such that

(1) $$d_0\mu = d_1\mu = X_0 ,$$

satisfying the following additional conditions :

(2) The square

$$\begin{tikzcd} X_2 \arrow[r, "e_0"] \arrow[d, "e_1"'] & X_1 \arrow[d, "d_0"] \\ X_1 \arrow[r, "d_1"] & X_0 \end{tikzcd}$$

is a pullback , with

(i) $d_0\gamma = d_0e_1$,

(ii) $d_1\gamma = d_1e_0$.

(3) Defining $\phi_0,\phi_1 : X_1 \to X_2$ by the diagrams

$$
\begin{tikzcd}
X_1 \arrow[dr, dashed, "\phi_0"] \arrow[drr, bend left, "X_1"] \arrow[ddr, bend right, "\mu d_0"'] & & \\
& X_2 \arrow[r, "e_0"] \arrow[d, "e_1"] & X_1 \arrow[d, "d_0"] \\
& X_1 \arrow[r, "d_1"] & X_0
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
X_1 \arrow[dr, dashed, "\phi_1"] \arrow[drr, bend left, "\mu d_1"] \arrow[ddr, bend right, "X_1"'] & & \\
& X_2 \arrow[r, "e_0"] \arrow[d, "e_1"] & X_1 \arrow[d, "d_0"] \\
& X_1 \arrow[r, "d_1"] & X_0
\end{tikzcd}
\quad ,
$$

we require that $\gamma\phi_0 = \gamma\phi_1 = X_1$.

(4) Defining $\psi_0, \psi_1 : X_3 \longrightarrow X_2$ by the pullback square

$$
\begin{tikzcd}
X_3 \arrow[r, "\psi_0"] \arrow[d, "\psi_1"] & X_2 \arrow[d, "e_1"] \\
X_2 \arrow[r, "e_0"] & X_1
\end{tikzcd}
\quad ,
$$

and $\alpha_0, \alpha_1 : X_3 \longrightarrow X_2$ by the diagrams

$$
\begin{tikzcd}
X_3 \arrow[dr, dashed, "\alpha_0"] \arrow[drr, bend left, "e_0\psi_0"] \arrow[ddr, bend right, "\gamma\psi_1"'] & & \\
& X_2 \arrow[r, "e_0"] \arrow[d, "e_1"] & X_1 \arrow[d, "d_0"] \\
& X_1 \arrow[r, "d_1"] & X_0
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
X_3 \arrow[dr, dashed, "\alpha_1"] \arrow[drr, bend left, "\gamma\psi_0"] \arrow[ddr, bend right, "e_1\psi_1"'] & & \\
& X_2 \arrow[r, "e_0"] \arrow[d, "e_1"] & X_1 \arrow[d, "d_0"] \\
& X_1 \arrow[r, "d_1"] & X_0
\end{tikzcd}
\quad ,
$$

we require that $\gamma\alpha_0 = \gamma\alpha_1$.

If $\underline{E}$ is the category of sets, then C satisfies the axioms of a small category : X_0 and X_1 are the sets of objects and morphisms (respectively) of C ; d_0 and d_1 are the domain and codomain maps , respectively ; μ assigns to each object $x_0 \epsilon X_0$ the identity map $id_{x_0} \epsilon X_1$. Condition (1) states the requirement that $d_0(id_{x_0}) = d_1(id_{x_0}) = x_0$, for all $x_0 \epsilon X_0$. Condition (2) stipulates that $X_2 = \{(f,g) \epsilon X_1 \times X_1 | d_0(f) = d_1(g)\}$ (which is the set of all composable maps in C) , and that for all $(f,g) \epsilon X_2$, (i) $d_0(fg) = d_0(g)$ and (ii) $d_1(fg) = d_1(f)$. Condition (3) amounts to $f \cdot id_{d_0(f)} = id_{d_1(f)} \cdot f = f$, for all $f \epsilon X_1$ (where "$\cdot$" and juxtaposition both mean composition in C). Condition (4) states the associative law of composition , i.e. that for all (f,g,g') satisfying $(f,g) \epsilon X_2$ and $(g,g') \epsilon X_2$, $(fg)g' = f(gg')$ should hold.

2. Functors

Definition. Given category objects C, C' in $\underline{E}$, a functor $F: C \to C'$ consists of morphisms $F_0 : X_0 \to X_0'$, $F_1 : X_1 \to X_1'$ in $\underline{E}$, subject to the conditions :

(1) $\quad d_0' F_1 = F_0 d_0$;

(2) $\quad d_1' F_1 = F_0 d_1$;

(3) $\quad \mu' F_0 = F_1 \mu$;

(4) $\quad \gamma' F_2 = F_1 \gamma$,

where F_2 is defined by noting that

$$d_0' F_1 e_0 = F_0 d_0 e_0 = F_0 d_1 e_1 = d_1' F_1 e_1 ,$$

and considering the diagram

$$
\begin{tikzcd}
 & & X_2' \arrow[r, "e_0'"] \arrow[d, "e_1'"] & X_1' \arrow[d, "d_0'"] \\
X_2 \arrow[r, "e_0"] \arrow[d, "e_1"] \arrow[urr, dashed, "F_2"] & X_1 \arrow[d, "d_0"] \arrow[urr, "F_1"] & X_1' \arrow[r, "d_1'"] & X_0' \\
X_1 \arrow[r, "d_1"] \arrow[urr, "F_1"] & X_0 \arrow[urr, "F_0"] & &
\end{tikzcd}
$$

We note that all primed symbols above refer to the category object C' .

If $\underline{E}$ is the category of sets , then F is an ordinary functor between two (small) categories , because for all $x_0 \varepsilon X_0$, $F(x_0) = F_0(x_0) \varepsilon X_0'$, i.e. an object of C' ; for all $x_1 \varepsilon X_1$, $F(x_1) = F_1(x_1) \varepsilon X_1'$, i.e. a map in C' ; and conditions (1) to (4) are interpreted as follows :

(1) $\qquad F(d_0(f)) = d_0'(F(f))$,

for all $f \varepsilon X_1$;

(2) $\qquad F(d_1(f)) = d_1'(F(f))$,

for all $f \varepsilon X_1$;

(3) $\qquad F(id_{x_0}) = id_{F(x_0)}$,

for all $x_0 \varepsilon X_0$;

(4) $\qquad F(fg) = F(f)F(g)$,

for all $(f,g) \varepsilon X_2$.

Given category objects and functors

$$C \xrightarrow{F} C' \xrightarrow{F'} C''$$

in $\underline{E}$, best described by the diagram

$$\begin{tikzcd}
X_2 \arrow[r, "\gamma"] \arrow[d, "F_2"] & X_1 \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1"] \arrow[d, "F_1"] & X_0 \arrow[l, shift left=3, "\mu"] \arrow[d, "F_0"] \\
X_2' \arrow[r, "\gamma'"] \arrow[d, "F_2'"] & X_1' \arrow[r, shift left=3, "d_0'"] \arrow[r, "d_1'"] \arrow[d, "F_1'"] & X_0' \arrow[l, shift left=3, "\mu'"] \arrow[d, "F_0'"] \\
X_2'' \arrow[r, "\gamma''"] & X_1'' \arrow[r, shift left=3, "d_0''"] \arrow[r, "d_1''"] & X_0'' \arrow[l, shift left=3, "\mu''"]
\end{tikzcd}$$

we may define the composite $F'F:C\to C''$ by the pair of maps $(F'F)_0 = F_0'F_0:X_0\to X_0''$, $(F'F)_1 = F_1'F_1:X_1\to X_1''$. It is then trivial (but tedious) to verify that conditions (1) to (4) are satisfied, so that $F'F$ is a functor between two category objects.

3. Natural Transformations

<u>Definition</u>. Given category objects and functors $F,F':C\to C'$ in $\underline{E}$,

$$\begin{tikzcd}
X_2 \arrow[r, "\gamma"] \arrow[d, shift right=2, "F_2"'] \arrow[d, shift left=2, "F_2'"] & X_1 \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1"] \arrow[d, shift right=2, "F_1"'] \arrow[d, shift left=2, "F_1'"] & X_0 \arrow[l, shift left=3, "\mu"] \arrow[d, shift right=2, "F_0"'] \arrow[d, shift left=2, "F_0'"] \\
X_2' \arrow[r, "\gamma'"] & X_1' \arrow[r, shift left=3, "d_0'"] \arrow[r, "d_1'"] & X_0' \arrow[l, shift left=3, "\mu'"]
\end{tikzcd}$$

a <u>natural</u> <u>transformation</u> $\tau:F\to F'$ consists of a map $\tau_0:X_0\to X_1'$ in $\underline{E}$, satisfying

(1) $\qquad d_0'\tau_0 = F_0$;

(2) $\qquad d_1'\tau_0 = F_0'$;

(3) $\gamma'\tau_1 = \gamma'\tau_2$,

where $\tau_1, \tau_2 : X_1 \to X_2'$ are defined through the diagrams

$$
\begin{tikzcd}
X_1 \arrow[dr, dashed, "\tau_1"] \arrow[drr, bend left, "\tau_0 d_1"] \arrow[ddr, bend right, "F_1"'] & & \\
& X_2' \arrow[r, "e_0'"] \arrow[d, "e_1'"] & X_1' \arrow[d, "d_0'"] \\
& X_1' \arrow[r, "d_1'"] & X_0'
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
X_1 \arrow[dr, dashed, "\tau_2"] \arrow[drr, bend left, "F_1'"] \arrow[ddr, bend right, "\tau_0 d_0"'] & & \\
& X_2' \arrow[r, "e_0'"] \arrow[d, "e_1'"] & X_1' \arrow[d, "d_0'"] \\
& X_1' \arrow[r, "d_1'"] & X_0'
\end{tikzcd}
\quad ,
$$

noting that $d_0'\tau_0 d_1 = F_0 d_1 = d_1' F_1$ and $d_1'\tau_0 d_0 = F_0' d_0 = d_0' F_1'$.

If $\underline{E}$ is the category of sets , then τ becomes an ordinary natural transformation between functors : for all $x_0 \varepsilon X_0$, i.e. for any object x_0 in C , $\tau_{x_0} = \tau_0(x_0) \varepsilon X_1'$, i.e. τ_{x_0} is a map in C' ; conditions (1) and (2) state , respectively , that $d_0'(\tau_{x_0}) = F(x_0)$ and $d_1'(\tau_{x_0}) = F'(x_0)$; (3) is the naturality condition , i.e. that given a map $f: d_0(f) \to d_1(f)$ in C , the square

$$
\begin{tikzcd}
F(d_0(f)) \arrow[r, "\tau_{d_0(f)}"] \arrow[d, "F(f)"'] & F'(d_0(f)) \arrow[d, "F'(f)"] \\
F(d_1(f)) \arrow[r, "\tau_{d_1(f)}"] & F'(d_1(f))
\end{tikzcd}
$$

should be commutative .

We now wish to define a composition for such natural transformations , but first , a word on notation . Given an object Y in $\underline{E}$ and maps $f,g:Y\to X_1$ satisfying $d_0 f = d_1 g$,

$$\begin{tikzcd}
Y \arrow[dr, dashed, "\xi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& X_2 \arrow[r, "e_0"] \arrow[d, "e_1"] & X_1 \arrow[d, "d_0"] \\
& X_1 \arrow[r, "d_1"] & X_0
\end{tikzcd} ,$$

we shall denote the induced map ξ by $\langle f,g\rangle$. This will greatly facilitate the following discussions , and is not really an abuse of notation , since the above pullback is a product in the comma category $(\underline{E},X_0)$, whose objects consist of all maps in $\underline{E}$ with codomain X_0 . Next , we shall require the following :

<u>Lemma</u>. Given an object Y in $\underline{E}$ and maps $f,g,h:Y\to X_1$ satisfying $d_0 f = d_1 g$ and $d_0 g = d_1 h$, we have

$$\gamma\langle\gamma\langle f,g\rangle,h\rangle = \gamma\langle f,\gamma\langle g,h\rangle\rangle .$$

Proof: Condition (4) in the definition of the category object C in $\underline{E}$ states that $\gamma\alpha_0 = \gamma\alpha_1$; i.e. that $\gamma\langle e_0\psi_0,\gamma\psi_1\rangle = \gamma\langle\gamma\psi_0,e_1\psi_1\rangle$. Define $t:Y\to X_3$ through the diagram

$$\begin{tikzcd}
Y \arrow[dr, dashed, "t"] \arrow[drr, bend left, "{\langle f,g\rangle}"] \arrow[ddr, bend right, "{\langle g,h\rangle}"'] & & \\
& X_3 \arrow[r, "\psi_0"] \arrow[d, "\psi_1"] & X_2 \arrow[d, "e_1"] \\
& X_2 \arrow[r, "e_0"] & X_1
\end{tikzcd} ,$$

we, then have

$$\gamma\langle e_0\psi_0,\gamma\psi_1\rangle t = \gamma\langle\gamma\psi_0,e_1\psi_1\rangle t\ ,$$

$$\Rightarrow\quad \gamma\langle e_0\psi_0 t,\gamma\psi_1 t\rangle = \gamma\langle\gamma\psi_0 t,e_1\psi_1 t\rangle$$

$$\Rightarrow\ \gamma\langle e_0\langle f,g\rangle,\gamma\langle g,h\rangle\rangle = \gamma\langle\gamma\langle f,g\rangle,e_1\langle g,h\rangle\rangle$$

$$\Rightarrow\quad \gamma\langle f,\gamma\langle g,h\rangle\rangle = \gamma\langle\gamma\langle f,g\rangle,h\rangle\ ,$$

which completes the proof.

Now given functors $F,F',F'':C\to C'$, and natural transformations $\tau:F\to F'$, $\tau':F'\to F''$, we define $\tau'\cdot\tau:F\to F''$ by $(\tau'\cdot\tau)_0 = \gamma'\langle\tau_0',\tau_0\rangle:X_0\to X_1'$. It is now required to verify conditions (1) to (3) in the definition of a natural transformation :

(1)
$$\begin{aligned} d_0'(\tau'\cdot\tau)_0 &= d_0'\gamma'\langle\tau_0',\tau_0\rangle \\ &= d_0'e_1'\langle\tau_0',\tau_0\rangle \\ &= d_0'\tau_0 \\ &= F_0\ ; \end{aligned}$$

(2)
$$\begin{aligned} d_1'(\tau'\cdot\tau)_0 &= d_1'\gamma'\langle\tau_0',\tau_0\rangle \\ &= d_1'e_0'\langle\tau_0',\tau_0\rangle \\ &= d_1'\tau_0' \\ &= F_0''\ ; \end{aligned}$$

(3)
$$\begin{aligned} \gamma'(\tau'\cdot\tau)_1 &= \gamma'\langle(\tau'\cdot\tau)_0 d_1,F_1\rangle && \text{(by definition)} \\ &= \gamma'\langle\gamma'\langle\tau_0',\tau_0\rangle d_1,F_1\rangle \\ &= \gamma'\langle\gamma'\langle\tau_0' d_1,\tau_0 d_1\rangle,F_1\rangle \\ &= \gamma'\langle\tau_0' d_1,\gamma'\langle\tau_0 d_1,F_1\rangle\rangle && \text{(by the Lemma)} \\ &= \gamma'\langle\tau_0' d_1,\gamma'\langle F_1',\tau_0 d_0\rangle\rangle && \text{(by the naturality of } \tau) \end{aligned}$$

$$
\begin{aligned}
&= \gamma'\langle\gamma'\langle\tau'_0 d_1, F'_1\rangle, \tau_0 d_0\rangle && \text{(by the Lemma)} \\
&= \gamma'\langle\gamma'\langle F''_1, \tau'_0 d_0\rangle, \tau_0 d_0\rangle && \text{(by the naturality of } \tau'\text{)} \\
&= \gamma'\langle F''_1, \gamma'\langle\tau'_0 d_0, \tau_0 d_0\rangle\rangle && \text{(by the Lemma)} \\
&= \gamma'\langle F''_1, \gamma'\langle\tau'_0, \tau_0\rangle d_0\rangle \\
&= \gamma'\langle F''_1, (\tau'\cdot\tau)_0 d_0\rangle \\
&= \gamma'(\tau'\cdot\tau)_2 \quad ;
\end{aligned}
$$

hence $\tau'\cdot\tau : F \to F''$ is well defined.

Remark. It is now possible to define a 2-category Cat(E) , with 0-cells all category objects in E , 1-cells all functors between such category objects , and 2-cells all natural transformations between such functors .

4. Composition Formulae*

Given category objects C, C', M, M' in E ; functors $F, G : C \to C'$, $V : M \to C$ and $U : C' \to M'$; a natural transformation $\theta : F \to G$; best described by the diagram

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* Godement , Roger : Topologie Algébrique et Théorie des Faisceaux (p. 269) gives rules of composition and interchange of functors and natural transformations , which we shall treat here in our own internal context .

$$
\begin{tikzcd}[row sep=large, column sep=large]
M_2 \arrow[r, "c"] \arrow[d, "V_2"] & M_1 \arrow[r, shift left=2, "\delta_0"] \arrow[r, "\delta_1"] \arrow[d, "V_1"] & M_0 \arrow[l, shift left=2, "m"] \arrow[d, "V_0"] \\
X_2 \arrow[r, "\gamma"] \arrow[d, shift right, "F_2"'] \arrow[d, shift left, "G_2"] & X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] \arrow[d, shift right, "F_1"'] \arrow[d, shift left, "G_1"] & X_0 \arrow[l, shift left=2, "\mu"] \arrow[d, shift right, "F_0"'] \arrow[d, shift left, "G_0"] \arrow[dl, "\theta_0"'] \\
X'_2 \arrow[r, "\gamma'"] \arrow[d, "U_2"] & X'_1 \arrow[r, shift left=2, "d'_0"] \arrow[r, "d'_1"] \arrow[d, "U_1"] & X'_0 \arrow[l, shift left=2, "\mu'"] \arrow[d, "U_0"] \\
M'_2 \arrow[r, "c'"] & M'_1 \arrow[r, shift left=2, "\delta'_0"] \arrow[r, "\delta'_1"] & M'_0 \arrow[l, shift left=2, "m'"]
\end{tikzcd}
,
$$

we define a natural transformation $U{*}\theta{*}V : UFV \to UGV$, by $(U{*}\theta{*}V)_0 = U_1\theta_0V_0 : M_0 \to M'_1$. We must now show that $U{*}\theta{*}V$ satisfies the definition of a natural transformation , i.e. conditions (1) to (3) :

(1)
$$
\begin{aligned}
\delta'_0(U{*}\theta{*}V)_0 &= \delta'_0U_1\theta_0V_0 \\
&= U_0d'_0\theta_0V_0 \\
&= U_0F_0V_0 \\
&= (UFV)_0 \quad ;
\end{aligned}
$$

(2)
$$
\begin{aligned}
\delta'_1(U{*}\theta{*}V)_0 &= \delta'_1U_1\theta_0V_0 \\
&= U_0d'_1\theta_0V_0 \\
&= U_0G_0V_0 \\
&= (UGV)_0 \quad ;
\end{aligned}
$$

(3) It is required to show that

$$c'(U{*}\theta{*}V)_1 = c'(U{*}\theta{*}V)_2 \quad ,$$

where $(U{*}\theta{*}V)_1$ and $(U{*}\theta{*}V)_2$ are defined through the pullback square

$$
\begin{tikzcd}
M'_2 \arrow[r] \arrow[d] & M'_1 \arrow[d, "\delta'_0"] \\
M'_1 \arrow[r, "\delta'_1"] & M'_0
\end{tikzcd}
,
$$

by $(U*\theta*V)_1 = \langle U_1\theta_0V_0\delta_1, U_1F_1V_1\rangle$ and $(U*\theta*V)_2 = \langle U_1G_1V_1, U_1\theta_0V_0\delta_0\rangle$. First , we consider the diagram

$$
\begin{tikzcd}
M_1 \arrow[dr, dashed, "{\langle G_1V_1,\theta_0V_0\delta_0\rangle}"] \arrow[drr, bend left, "G_1V_1"] \arrow[ddddr, bend right, "\theta_0V_0\delta_0"'] & & & M'_2 \arrow[r] \arrow[d] & M'_1 \arrow[d, "\delta'_0"] \\
& X'_2 \arrow[r] \arrow[ddd] \arrow[urr, "U_2"] & X'_1 \arrow[ddd, "d'_0"] \arrow[urr, "U_1"] & M'_1 \arrow[r, "\delta'_1"] & M'_0 \\
& & & & \\
& & & & \\
& X'_1 \arrow[r, "d'_1"] \arrow[uuurr, "U_1"] & X'_0 \arrow[uuurr, "U_0"'] & &
\end{tikzcd}
$$

which shows that $U_2\langle G_1V_1,\theta_0V_0\delta_0\rangle = \langle U_1G_1V_1, U_1\theta_0V_0\delta_0\rangle$. Similarly , we have $U_2\langle\theta_0V_0\delta_1, F_1V_1\rangle = \langle U_1\theta_0V_0\delta_1, U_1F_1V_1\rangle$. Next , the naturality of $\theta: F\to G$ states that

$$\gamma'\langle\theta_0 d_1, F_1\rangle = \gamma'\langle G_1, \theta_0 d_0\rangle \quad ,$$
$$\Rightarrow \quad \gamma'\langle\theta_0 d_1, F_1\rangle V_1 = \gamma'\langle G_1, \theta_0 d_0\rangle V_1$$
$$\Rightarrow \gamma'\langle\theta_0 d_1 V_1, F_1V_1\rangle = \gamma'\langle G_1V_1, \theta_0 d_0 V_1\rangle$$
$$\Rightarrow \gamma'\langle\theta_0 V_0\delta_1, F_1V_1\rangle = \gamma'\langle G_1V_1, \theta_0 V_0\delta_0\rangle \quad ;$$

therefore

$$\begin{aligned} c'(U*\theta*V)_1 &= c'<U_1\theta_0V_0\delta_1, U_1F_1V_1> \\ &= c'U_2<\theta_0V_0\delta_1, F_1V_1> \\ &= U_1\gamma'<\theta_0V_0\delta_1, F_1V_1> \\ &= U_1\gamma'<G_1V_1, \theta_0V_0\delta_0> \\ &= c'U_2<G_1V_1, \theta_0V_0\delta_0> \\ &= c'<U_1G_1V_1, U_1\theta_0V_0\delta_0> \\ &= c'(U*\theta*V)_2 \ . \end{aligned}$$

We have of course, as special cases, $U*\theta: UF \to UG$ and $\theta*V: FV \to GV$, defined by $(U*\theta)_0 = U_1\theta_0$ and $(\theta*V)_0 = \theta_0V_0$.

We now state five rules of composition, keeping in mind that C, C', M, M' will denote category objects in $\underline{E}$, F,G,H,U,V functors, and $\theta, \theta', \phi, \psi$ natural transformations:

(R1) Let $F,G: C \to C'$; $\theta: F \to G$; $V: C' \to M$ and $U: M \to M'$; then

$$(UV)*\theta = U*(V*\theta) \ .$$

(R2) Let $F,G: C \to C'$; $\theta: F \to G$; $V: M \to M'$ and $U: M' \to C$; then

$$\theta*(UV) = (\theta*U)*V \ .$$

(R3) Let $F,G: C \to C'$; $\theta: F \to G$; $V: M \to C$ and $U: C' \to M'$; then

$$(U*\theta)*V = U*(\theta*V) = U*\theta*V \ .$$

(R4) Let $F,G,H: C \to C'$; $\theta: G \to H$; $\theta': F \to G$; $U: C' \to M'$ and $V: M \to C$; then

$$U*(\theta\cdot\theta')*V = (U*\theta*V)\cdot(U*\theta'*V) \ .$$

(R5) Let $F,G: C \to C'$; $U,V: C' \to M$; $\phi: F \to G$ and $\psi: U \to V$; then

$$(\psi*G)\cdot(U*\phi) = (V*\phi)\cdot(\psi*F) \ .$$

Proof:

$$\text{(R1)} \qquad [(UV)*\theta]_0 = (UV)_1\theta_0$$
$$= (U_1V_1)\theta_0$$
$$= U_1(V_1\theta_0)$$
$$= U_1(V*\theta)_0$$
$$= [U*(V*\theta)]_0 \; .$$

$$\text{(R2)} \qquad [\theta*(UV)]_0 = \theta_0(UV)_0$$
$$= \theta_0(U_0V_0)$$
$$= (\theta_0U_0)V_0$$
$$= (\theta*U)_0V_0$$
$$= [(\theta*U)*V]_0 \; .$$

$$\text{(R3)} \qquad (U*\theta*V)_0 = U_1\theta_0V_0$$
$$= (U_1\theta_0)V_0$$
$$= (U*\theta)_0V_0$$
$$= [(U*\theta)*V]_0 \; ;$$

$$(U*\theta*V)_0 = U_1\theta_0V_0$$
$$= U_1(\theta_0V_0)$$
$$= U_1(\theta*V)_0$$
$$= [U*(\theta*V)]_0 \; .$$

$$\text{(R4)} \qquad [U*(\theta\cdot\theta')*V]_0 = U_1(\theta\cdot\theta')_0V_0$$
$$= U_1\gamma'\langle\theta_0,\theta_0'\rangle V_0$$
$$= c'U_2\langle\theta_0,\theta_0'\rangle V_0$$
$$= c'\langle U_1\theta_0,U_1\theta_0'\rangle V_0$$
$$= c'\langle U_1\theta_0V_0,U_1\theta_0'V_0\rangle$$
$$= c'\langle(U*\theta*V)_0,(U*\theta'*V)_0\rangle$$

$$= [(U*\theta*V)\cdot(U*\theta'*V)]_0 \ .$$

(R5) Naturality of $\psi: U \to V$ implies

$$c\langle\psi_0 d_1', U_1\rangle = c\langle V_1, \psi_0 d_0'\rangle \ ,$$

$$\Rightarrow \ c\langle\psi_0 d_1', U_1\rangle\phi_0 = c\langle V_1, \psi_0 d_0'\rangle\phi_0$$

$$\Rightarrow c\langle\psi_0 d_1'\phi_0, U_1\phi_0\rangle = c\langle V_1\phi_0, \psi_0 d_0'\phi_0\rangle$$

$$\Rightarrow \ c\langle\psi_0 G_0, U_1\phi_0\rangle = c\langle V_1\phi_0, \psi_0 F_0\rangle \ ;$$

therefore

$$\begin{aligned}[(\psi*G)\cdot(U*\phi)]_0 &= c\langle(\psi*G)_0, (U*\phi)_0\rangle \\ &= c\langle\psi_0 G_0, U_1\phi_0\rangle \\ &= c\langle V_1\phi_0, \psi_0 F_0\rangle \\ &= c\langle(V*\phi)_0, (\psi*F)_0\rangle \\ &= [(V*\phi)\cdot(\psi*F)]_0 \ .\end{aligned}$$

5. Induced Triples

Given a category object C ,

$$\begin{tikzcd} X_2 \arrow[r, "\gamma"] & X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] & X_0 \arrow[l, shift left=2, "\mu"] \end{tikzcd} \ ,$$

in $\underline{E}$, we wish to define a triple (monad) (T,η,μ) in $(\underline{E},X_0)$; the context in which the symbol μ is used will always make its meaning clear , as it is used to denote two different things.

We define a (covariant) functor $T:(\underline{E},X_0)\to(\underline{E},X_0)$ as the composite

$$(\underline{E},X_0) \xrightarrow{d_1^*} (\underline{E},X_1) \xrightarrow{\Sigma_{d_0}} (\underline{E},X_0) \ ,$$

where d_1^* and Σ_{d_0} are defined as follows : for any object $\alpha:E_0 \to X_0$ in $(\underline{E},X_0)$, the pullback square

$$\begin{array}{ccc} E_1 & \xrightarrow{\pi_1} & E_0 \\ \downarrow{\scriptstyle d_1^*(\alpha)} & & \downarrow{\scriptstyle \alpha} \\ X_1 & \xrightarrow{d_1} & X_0 \end{array}$$

defines $d_1^*(\alpha)$, and for any map $\beta:\alpha \to \alpha'$ in $(\underline{E},X_0)$, i.e. that the diagram

$$\begin{array}{ccc} E_0 & \xrightarrow{\beta} & E_0' \\ & {\scriptstyle \alpha}\searrow & \downarrow{\scriptstyle \alpha'} \\ & & X_0 \end{array}$$

commutes , we let $d_1^*(\beta) = \langle \beta\pi_1, d_1^*(\alpha) \rangle$. For any object $\delta:Y \to X_1$ in $(\underline{E},X_1)$, $\Sigma_{d_0}(\delta) = d_0\delta$. Given a map $\varepsilon:\delta \to \delta'$ in $(\underline{E},X_1)$, we have $\Sigma_{d_0}(\varepsilon) = \varepsilon$ "over X_0" .

We also require natural transformations $\eta:I \to T$, $\mu:T^2 \to T$, such that the conditions

(A) $\qquad \mu(T\mu) = \mu(\mu T)$,

and

(B) $\qquad \mu(T\eta) = \mu(\eta T) = T$

are satisfied , where $T:T \to T$ denotes the identity natural transformation .

To define $\eta : I \to T$, given an object $\alpha : E_0 \to X_0$ in $(\underline{E}, X_0)$, we form the pullback square

$$\begin{tikzcd} E_1 \arrow[r, "\pi_1"] \arrow[d, "d_1^*(\alpha)"'] & E_0 \arrow[d, "\alpha"] \\ X_1 \arrow[r, "d_1"] & X_0 \end{tikzcd} ,$$

and let $\eta_\alpha = \langle E_0, \mu\alpha \rangle$. It is then immediate that $\eta_\alpha : \alpha \to T(\alpha)$ is a map in $(\underline{E}, X_0)$. To verify the naturality of η , we let $\beta : \alpha \to \alpha'$ be a map in $(\underline{E}, X_0)$, so that the outside of the diagram

$$\begin{tikzcd} E_0 \arrow[drr, bend left, "\beta"] \arrow[dr, shift left, "T(\beta)\eta_\alpha"] \arrow[dr, shift right, "\eta_{\alpha'}\beta"'] \arrow[ddr, bend right, "\mu\alpha"'] & & \\ & E_1' \arrow[r, "\pi_1'"] \arrow[d, "d_1^*(\alpha')"] & E_0' \arrow[d, "\alpha'"] \\ & X_1 \arrow[r, "d_1"] & X_0 \end{tikzcd}$$

commutes , then the equations

$$\begin{aligned} d_1^*(\alpha')\eta_{\alpha'}\beta &= \mu\alpha'\beta \\ &= \mu\alpha , \\ \pi_1'\eta_{\alpha'}\beta &= E_0'\beta \\ &= \beta ; \end{aligned}$$

and

$$\begin{aligned} d_1^*(\alpha')T(\beta)\eta_\alpha &= d_1^*(\alpha)\eta_\alpha \quad \text{(by definition of } T(\beta)) \\ &= \mu\alpha \, , \\ \pi_1'T(\beta)\eta_\alpha &= \beta\pi_1\eta_\alpha \quad \text{(by definition of } T(\beta)) \\ &= \beta E_0 \\ &= \beta \; ; \end{aligned}$$

show (by uniqueness) that the square

$$\begin{tikzcd} \alpha \arrow[r, "\eta_\alpha"] \arrow[d, "\beta"] & T(\alpha) \arrow[d, "T(\beta)"] \\ \alpha' \arrow[r, "\eta_{\alpha'}"] & T(\alpha') \end{tikzcd}$$

is commutative , which is precisely the naturality of η .

To define $\mu : T^2 \to T$, given an object $\alpha : E_0 \to X_0$ in $(\underline{E}, X_0)$, we form the pullbacks

$$\begin{tikzcd} E_2 \arrow[r, "\pi_2"] \arrow[d, "p_2"] & E_1 \arrow[r, "\pi_1"] \arrow[d, "d_1^*(\alpha)"] & E_0 \arrow[d, "\alpha"] \\ X_2 \arrow[r, "e_0"] \arrow[d, "e_1"] & X_1 \arrow[r, "d_1"] \arrow[d, "d_0"] & X_0 \\ X_1 \arrow[r, "d_1"] & X_0 & \end{tikzcd}$$

and let $\mu_\alpha = \langle \pi_1\pi_2, \gamma p_2 \rangle$. We then have that

$$\begin{aligned} T(\alpha)\mu_\alpha &= d_0 d_1^*(\alpha)\mu_\alpha \\ &= d_0\gamma p_2 \end{aligned}$$

$$= d_0 e_1 p_2$$
$$= d_0 d_1^*(T(\alpha))$$
$$= T^2(\alpha) \quad ,$$

so that $\mu_\alpha : T^2(\alpha) \to T(\alpha)$ is a map in $(\underline{E}, X_0)$. To show naturality , we require that the diagram

$$\begin{tikzcd}
T^2(\alpha) \arrow[r, "\mu_\alpha"] \arrow[d, "T^2(\beta)"] & T(\alpha) \arrow[d, "T(\beta)"] \\
T^2(\alpha') \arrow[r, "\mu_{\alpha'}"] & T(\alpha')
\end{tikzcd}$$

be commutative , where $\beta : \alpha \to \alpha'$ is a map in $(\underline{E}, X_0)$. First , we consider the diagram

$$\begin{tikzcd}
E_2 \arrow[dr, shift left, "p_2"] \arrow[dr, shift right, "p_2'T^2(\beta)"'] \arrow[drr, bend left, "d_1^*(\alpha)\pi_2"] \arrow[ddr, bend right, "e_1 p_2"'] & & \\
& X_2 \arrow[r, "e_0"] \arrow[d, "e_1"] & X_1 \arrow[d, "d_0"] \\
& X_1 \arrow[r, "d_1"] & X_0
\end{tikzcd} \quad ,$$

and note that

$$e_0 p_2 = d_1^*(\alpha)\pi_2 \ ;$$

and

$$e_0 p_2' T^2(\beta) = d_1^*(\alpha')\pi_2' T^2(\beta)$$

$$
\begin{aligned}
&= d_1^*(\alpha')T(\beta)\pi_2 && \text{(by definition of } T^2(\beta)\text{)} \\
&= d_1^*(\alpha)\pi_2 && \text{(by definition of } T(\beta)\text{)} \ , \\
e_1p_2'T^2(\beta) &= e_1p_2 && \text{(by definition of } T^2(\beta)\text{)} \ ;
\end{aligned}
$$

so that uniqueness implies $p_2'T^2(\beta) = p_2$. Next , we consider

$$
\begin{tikzcd}
E_2 \arrow[r, shift left, "T(\beta)\mu_\alpha"] \arrow[r, shift right, "\mu_{\alpha'}T^2(\beta)"'] \arrow[rr, bend left, "\pi_1'\pi_2'T^2(\beta)"] \arrow[dr, "\gamma p_2'T^2(\beta)"'] & E_1' \arrow[r, "\pi_1'"] \arrow[d, "d_1^*(\alpha')"] & E_0' \arrow[d, "\alpha'"] \\
 & X_1 \arrow[r, "d_1"] & X_0
\end{tikzcd} ,
$$

where

$$
\begin{aligned}
d_1\gamma p_2'T^2(\beta) &= d_1e_0p_2'T^2(\beta) \\
&= d_1d_1^*(\alpha')\pi_2'T^2(\beta) \\
&= \alpha'\pi_1'\pi_2'T^2(\beta) \ ,
\end{aligned}
$$

so that the outside of the diagram commutes ; then the equations

$$
\begin{aligned}
d_1^*(\alpha')\mu_{\alpha'}T^2(\beta) &= \gamma p_2'T^2(\beta) \ , \\
\pi_1'\mu_{\alpha'}T^2(\beta) &= \pi_1'\pi_2'T^2(\beta) \ ;
\end{aligned}
$$

and

$$
\begin{aligned}
d_1^*(\alpha')T(\beta)\mu_\alpha &= d_1^*(\alpha)\mu_\alpha \\
&= \gamma p_2 \\
&= \gamma p_2'T^2(\beta) \ , \\
\pi_1'T(\beta)\mu_\alpha &= \beta\pi_1\mu_\alpha \\
&= \beta\pi_1\pi_2
\end{aligned}
$$

$$= \pi_1' T(\beta)\pi_2$$
$$= \pi_1'\pi_2' T^2(\beta) \;;$$

show (by uniqueness) that $T(\beta)\mu_\alpha = \mu_{\alpha'} T^2(\beta)$.

It remains to verify the triple identities :

(A) We keep in mind the definition of the category object $\mathcal{C}$ in $\underline{E}$, and note that $\mu_\alpha = \langle \pi_1\pi_2, \gamma p_2 \rangle$, $(T\mu)_\alpha = T(\mu_\alpha) = \langle \mu_\alpha \pi_3, d_1^*(d_0\gamma p_2)\rangle$, and $(\mu T)_\alpha = \mu_{T(\alpha)} = \langle \pi_2\pi_3, \gamma\psi_1 p_3\rangle$, where π_3 and p_3 are defined through the pullback square

$$\begin{tikzcd} E_3 \arrow[r, "\pi_3"] \arrow[d, "p_3"] & E_2 \arrow[d, "p_2"] \\ X_3 \arrow[r, "\psi_0"] & X_2 \end{tikzcd} .$$

First , we show that the squares

$$\begin{tikzcd} E_3 \arrow[r, "(\mu T)_\alpha"] \arrow[d, "p_3"] \arrow[dr, phantom, "(1)"] & E_2 \arrow[d, "p_2"] \\ X_3 \arrow[r, "\mu_{d_0}"] & X_2 \end{tikzcd} , \text{ and } \begin{tikzcd} E_3 \arrow[r, "(T\mu)_\alpha"] \arrow[d, "p_3"] \arrow[dr, phantom, "(2)"] & E_2 \arrow[d, "p_2"] \\ X_3 \arrow[r, "T(\gamma)"] & X_2 \end{tikzcd}$$

are commutative . For (1) , we consider the diagram

$$\begin{tikzcd}
E_3 \arrow[dr, "\mu_{d_0}p_3"'] \arrow[dr, bend left, "p_2\mu_{T(\alpha)}"] \arrow[drr, bend left, "e_0\psi_0 p_3"] \arrow[ddr, bend right, "\gamma\psi_1 p_3"'] & & \\
& X_2 \arrow[r, "e_0"] \arrow[d, "e_1"] & X_1 \arrow[d, "d_0"] \\
& X_1 \arrow[r, "d_1"] & X_0
\end{tikzcd}$$

noting that

$$\begin{aligned}
d_1\gamma\psi_1 p_3 &= d_1 e_1 p_2 \mu_{T(\alpha)} \\
&= d_0 e_0 p_2 \mu_{T(\alpha)} \\
&= d_0 d_1^*(\alpha)\pi_2\mu_{T(\alpha)} \\
&= d_0 d_1^*(\alpha)\pi_2\pi_3 \\
&= d_0 e_0 p_2 \pi_3 \\
&= d_0 e_0 \psi_0 p_3 \; ,
\end{aligned}$$

so that the outside of the diagram commutes ; hence the equations

$$\begin{aligned}
e_1\mu_{d_0}p_3 &= \gamma\psi_1 p_3 \; , \\
e_0\mu_{d_0}p_3 &= e_0\psi_0 p_3 \; ;
\end{aligned}$$

and

$$\begin{aligned}
e_1 p_2 \mu_{T(\alpha)} &= \gamma\psi_1 p_3 \; , \\
e_0 p_2 \mu_{T(\alpha)} &= d_1^*(\alpha)\pi_2\mu_{T(\alpha)} \\
&= d_1^*(\alpha)\pi_2\pi_3 \\
&= e_0\psi_0 p_3 \; ;
\end{aligned}$$

imply (by uniqueness) that $\mu_{d_0}p_3 = p_2\mu_{T(\alpha)}$. The commutativity of (2) may be proved similarly , by the uniqueness property of $\langle \gamma\psi_0 p_3, e_1\psi_1 p_3\rangle : E_3 \to X_2$. Next , the outside of the diagram

$$\begin{tikzcd}
E_3 \arrow[dr, shift left, "\mu_\alpha(\mu T)_\alpha"] \arrow[dr, shift right, "\mu_\alpha(T\mu)_\alpha"'] \arrow[drr, bend left, "\pi_1\pi_2\pi_3"] \arrow[ddr, bend right, "\gamma T(\gamma)p_3"'] & & \\
& E_1 \arrow[r, "\pi_1"] \arrow[d, "d_1^*(\alpha)"] & E_0 \arrow[d, "\alpha"] \\
& X_1 \arrow[r, "d_1"] & X_0
\end{tikzcd}$$

commutes , as

$$\begin{aligned}
\alpha\pi_1\pi_2\pi_3 &= d_1 d_1^*(\alpha)\pi_2\pi_3 \\
&= d_1 e_0 p_2 \pi_3 \\
&= d_1 e_0 \psi_0 p_3 \\
&= d_1 \gamma\psi_0 p_3 \\
&= d_1 e_0 T(\gamma) p_3 \\
&= d_1 \gamma T(\gamma) p_3 \ ,
\end{aligned}$$

so that the equations

$$\begin{aligned}
\pi_1\mu_\alpha(\mu T)_\alpha &= \pi_1\pi_2(\mu T)_\alpha \\
&= \pi_1\pi_2\pi_3 \ , \\
d_1^*(\alpha)\mu_\alpha(\mu T)_\alpha &= \gamma p_2(\mu T)_\alpha \\
&= \gamma\mu_{d_0} p_3 \\
&= \gamma T(\gamma) p_3 \quad \text{(by I.1., condition (4), as}
\end{aligned}$$

$\mu_{d_0} = \alpha_0$ and $T(\gamma) = \alpha_1$) ;

and

$$\begin{aligned}
\pi_1\mu_\alpha(T\mu)_\alpha &= \pi_1\pi_2(T\mu)_\alpha \\
&= \pi_1\mu_\alpha\pi_3 \\
&= \pi_1\pi_2\pi_3 \\
d_1^*(\alpha)\mu_\alpha(T\mu)_\alpha &= \gamma p_2(T\mu)_\alpha \\
&= \gamma T(\gamma)p_3 \ ;
\end{aligned}$$

imply (by uniqueness) the required result :

(B) As in (A) , it may be shown , by the universal property of the pullback square

$$\begin{tikzcd}
X_2 \arrow[r, "e_0"] \arrow[d, "e_1"'] & X_1 \arrow[d, "d_0"] \\
X_1 \arrow[r, "d_1"] & X_0
\end{tikzcd} ,$$

that the squares

$$\begin{tikzcd}
E_1 \arrow[r, "(T\eta)_\alpha"] \arrow[d, "d_1^*(\alpha)"'] & E_2 \arrow[d, "p_2"] \\
X_1 \arrow[r, "T(\mu)"] & X_2
\end{tikzcd} \quad , \text{ and } \quad
\begin{tikzcd}
E_1 \arrow[r, "(\eta T)_\alpha"] \arrow[d, "d_1^*(\alpha)"'] & E_2 \arrow[d, "p_2"] \\
X_1 \arrow[r, "\eta_{d_0}"] & X_2
\end{tikzcd}$$

are commutative , so that condition (3) in the definition of the category object C in $\underline{E}$, together with the fact that $T(\mu) = \phi_1$ and $\eta_{d_0} = \phi_0$, implies (by uniqueness) the required result ,upon considering the diagram

$$\begin{tikzcd}
E_1 \arrow[dr, shift left, "\mu_\alpha(T\eta)_\alpha"] \arrow[dr, shift right, "\mu_\alpha(\eta T)_\alpha"'] \arrow[drr, bend left, "\pi_1"] \arrow[ddr, bend right, "d_1^*(\alpha)"'] & & \\
& E_1 \arrow[r, "\pi_1"] \arrow[d, "d_1^*(\alpha)"] & E_0 \arrow[d, "\alpha"] \\
& X_1 \arrow[r, "d_1"] & X_0
\end{tikzcd} .$$

6. Simplicial Objects

Given a category object C ,

$$X_2 \xrightarrow{\gamma} X_1 \underset{\mu}{\overset{d_0,\,d_1}{\rightleftarrows}} X_0 \quad ,$$

in $\underline{E}$, we wish to extend it to a simplicial object in $\underline{E}$.

The limit of the diagram (functor)

$$\begin{tikzcd}[row sep=small]
X_1 \arrow[dr, "d_1"] & \\
& X_0 \\
X_1 \arrow[ur, "d_0"'] \arrow[dr, "d_1"] & \\
& X_0 \\
X_1 \arrow[ur, "d_0"'] & \\
\vdots & \\
X_1 \arrow[dr, "d_1"] & \\
& X_0 \\
X_1 \arrow[ur, "d_0"'] &
\end{tikzcd}$$

where X_1 occurs n times , $(n\geq 2)$, consists of objects and maps

$$\begin{array}{l} X_n \xrightarrow{p_1^n} X_1 \\ X_n \xrightarrow{p_2^n} X_1 \\ X_n \xrightarrow{p_3^n} X_1 \\ \quad\vdots \\ X_n \xrightarrow{p_{n-1}^n} X_1 \\ X_n \xrightarrow{p_n^n} X_1 \end{array}$$

in $\underline{E}$, satisfying

$$d_1 p_i^n = d_0 p_{i+1}^n \ , \ (1\leq i\leq n-1) \ ,$$

with the universal property that for any other objects and maps in $\underline{E}$,

$$\begin{array}{l} Y_n \xrightarrow{q_1^n} X_1 \\ Y_n \xrightarrow{q_2^n} X_1 \\ Y_n \xrightarrow{q_3^n} X_1 \\ \quad\vdots \\ Y_n \xrightarrow{q_{n-1}^n} X_1 \\ Y_n \xrightarrow{q_n^n} X_1 \end{array} ,$$

satisfying

$$d_1 q_i^n = d_0 q_{i+1}^n \ , \quad (1 \leq i \leq n-1) \ ,$$

there exists a unique map $\psi_n : Y_n \to X_n$ in $\underline{E}$, such that

$$p_i^n \psi_n = q_i^n \ , \quad (1 \leq i \leq n) \ .$$

Such a map ψ_n will henceforth be denoted by $\langle q_1^n, q_2^n, \ldots, q_n^n \rangle$. We note that the above definition of the objects X_n agrees , for $n = 2$, with that of X_2 which appears in the definition of C , since pullback is a special case of finite limit .

We define maps $d_i^n : X_n \to X_{n-1}$, $n \geq 1$, $0 \leq i \leq n$, by

$$d_0^1 = d_0 \ ,$$
$$d_1^1 = d_1 \ ,$$
$$d_0^2 = e_1 \ ,$$
$$d_1^2 = \gamma \ ,$$
$$d_2^2 = e_0 \ ,$$

and for $n \geq 3$,

$$d_i^n = \langle p_1^n, p_2^n, \ldots, p_{i-1}^n, \gamma\langle p_i^n, p_{i+1}^n \rangle, p_{i+2}^n, \ldots, p_n^n \rangle \ ,$$
$$(1 \leq i \leq n-1) \ ,$$
$$d_0^n = \langle p_2^n, p_3^n, \ldots, p_n^n \rangle \ ,$$
$$d_n^n = \langle p_1^n, p_2^n, \ldots, p_{n-2}^n, p_{n-1}^n \rangle \ ;$$

and maps $s_i^n : X_n \to X_{n+1}$, $n \geq 0$, $0 \leq i \leq n$, by

$$s_0^0 = \mu \ ,$$
$$s_0^1 = \phi_1 \ ,$$
$$s_1^1 = \phi_0 \ ,$$

and for $n \geq 2$,

$$s_i^n = \langle p_1^n, p_2^n, \ldots, p_i^n, \mu d_1 p_i^n, p_{i+1}^n, \ldots, p_n^n \rangle ,$$
$$(1 \leqq i \leqq n) ,$$
$$s_0^n = \langle \mu d_0 p_1^n, p_1^n, p_2^n, \ldots, p_n^n \rangle .$$

This yields an infinite sequence S of objects and maps in $\underline{E}$, and it remains to prove that the simplicial identities hold , i.e. that

(1) $$d_i^n d_{j+1}^{n+1} = d_j^n d_i^{n+1} , \quad i \leqq j \leqq n ;$$

(2) $$s_{j+1}^n s_i^{n-1} = s_i^n s_j^{n-1} , \quad i \leqq j \leqq n-1 ;$$

(3) $$d_i^n s_j^{n-1} = \left\{ \begin{array}{ll} s_{j-1}^{n-2} d_i^{n-1} , & i<j \\ 1 , & i=j, j+1 \\ s_j^{n-2} d_{i-1}^{n-1} , & j+1<i \leqq n \end{array} \right\} , \quad j \leqq n-1 .$$

We first note that if $\underline{E}$ is the category of sets , then X_n is the set of sequences (of length n) of composable arrows (of the category C) . Given a typical such sequence $\alpha = \{\cdot \to \cdot \to \cdot \cdots \cdots \to \cdot\}$ in X_n , its image under d_i^n , for $1 \leqq i \leqq n-1$, is the sequence derived from α by replacing the i-th and (i+1)-st arrows in α by their composite , hence yielding a sequence of length n-1 ; d_0^n and d_n^n delete the first and last arrows (respectively) in α ; s_i^n , for $1 \leqq i \leqq n$, inserts an identity arrow between the i-th and (i+1)-st arrows in α ; s_0^n inserts an identity arrow before the first arrow in α . With this concrete description , it is now easy to see that the simplicial identities are satisfied (in $\underline{\text{Set}}$) . Next , for any object X in $\underline{E}$, the hom-functor $(X,-):\underline{E}\to\underline{Set}$ associates to any object Y in $\underline{E}$ the set (X,Y) of morphisms (in $\underline{E}$) from X to Y . It associates to any morphism $f:Y\to Z$ in $\underline{E}$ the morphism (in $\underline{Set}$) $(X,f):(X,Y)\to(X,Z)$, such that $(X,f)(g) = fg$ for any $g\varepsilon(X,Y)$. It is well known that this hom-functor preserves limits , so its application to the objects and maps which comprise S (its construction being exclusively through limits) results in a sequence (X,S) of objects and maps in $\underline{Set}$, which is precisely the extension (via the above method) of the category object

$$(X,X_2)\xrightarrow{(X,\gamma)}(X,X_1)\overset{(X,d_0)}{\underset{(X,\mu^{-1})}{\overset{\longrightarrow}{\underset{\longleftarrow}{\xrightarrow{(X,d_1)}}}}}(X,X_0)$$

in $\underline{Set}$. It is now immediate that the simplicial identities hold for (X,S) . This being true for any object X in $\underline{E}$, the Yoneda Lemma implies that S is a simplicial object in $\underline{E}$.

<u>Remark</u>. Given a category object C in $\underline{E}$ and a (covariant) functor $A:\underline{E}\to\underline{Ab}$, where $\underline{Ab}$ denotes the category of Abelian groups , let S be the extension of C to a simplicial object in $\underline{E}$. Then $A(S)$ (with the obvious meaning) is a simplicial object in $\underline{Ab}$, hence we may define maps $d_n:A(X_n)\to A(X_{n-1})$, $n\geq 1$, by

$$d_n = \sum_{i=0}^{n}(-1)^i A(d_i^n) \ ,$$

and it follows easily by (1) of the simplicial identities that $d_n d_{n+1} = 0$, $(n\geq 1)$, so that $\{A(X_n)\}_{n\geq 0}$ and $\{d_n\}_{n\geq 1}$

define a chain complex in Ab . Finally , we define the n-th homology object (group) $H_n(C,A)$ in Ab by

$$H_n(C,A) = \ker(d_n)/\mathrm{im}(d_{n+1}) .$$

The above definition , of course , is possible when Ab is replaced by any Abelian category .

II. INTERNAL FUNCTORS

1. Internal Functors and Natural Transformations

Definition. Given a category object C ,

$$X_2 \xrightarrow{\gamma} X_1 \underset{\mu}{\overset{d_0}{\underset{d_1}{\rightrightarrows}}} X_0 ,$$

in $\underline{E}$, an internal functor $F:C \to \underline{E}$ consists of maps $\pi_0 : E_0 \to X_0$ and $\delta_1 : E_1 \to E_0$ in $\underline{E}$, where

(1) The square

$$\begin{array}{ccc} E_1 & \xrightarrow{\delta_0} & E_0 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ X_1 & \xrightarrow{d_0} & X_0 \end{array}$$

is a pullback and

(2) The square

$$\begin{array}{ccc} E_1 & \xrightarrow{\delta_1} & E_0 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ X_1 & \xrightarrow{d_1} & X_0 \end{array}$$

is commutative , satisfying the following additional conditions :

(3) Let $m = \langle \mu\pi_0, E_0 \rangle : E_0 \to E_1$ be defined through the pullback square in (1) , then we require that $\delta_1 m = E_0$;

(4) Let $\varepsilon_1 : E_2 \to E_1$ be defined by the pullback square

$$\begin{tikzcd} E_2 \arrow[r, "\varepsilon_1"] \arrow[d, "\pi_2"] & E_1 \arrow[d, "\pi_1"] \\ X_2 \arrow[r, "e_1"] & X_1 \end{tikzcd} ,$$

and $c = \langle \gamma\pi_2, \delta_0\varepsilon_1 \rangle$, $\varepsilon_0 = \langle e_0\pi_2, \delta_1\varepsilon_1 \rangle$, then we require that $\delta_1 c = \delta_1 \varepsilon_0$.

If $\underline{E}$ is the category of sets (and so $\mathcal{C}$ is an ordinary category) , then F is a (covariant) set-valued functor : $E_0 = \dot{\bigcup}_{x_0 \varepsilon X_0} \pi_0^{-1}(\{x_0\})$ (disjoint union) ; by condition (1) , $E_1 = \{(f,x) \varepsilon X_1 \times E_0 \mid d_0(f) = \pi_0(x)\} \doteq \{(f,x) \varepsilon X_1 \times E_0 \mid x \varepsilon \pi_0^{-1}(\{d_0(f)\})\}$; for $(f,x) \varepsilon E_1$, $F(f)(x) \equiv \delta_1(f,x)$, so condition (2) states that $\pi_0[F(f)(x)] = d_1(f)$, i.e. that $F(f)(x) \varepsilon \pi_0^{-1}(\{d_1(f)\})$, so that $F(f) : F(d_0(f)) \to F(d_1(f))$; condition (3) requires that for any object x_0 in $\mathcal{C}$, $F(id_{x_0}) = id_{F(x_0)}$; condition (4) states the preservation of composition , because $E_2 = \{((f,g),(h,x)) \varepsilon X_2 \times E_1 \mid g = h\} \simeq \{(f,g,x) \varepsilon X_1 \times X_1 \times E_0 \mid (f,g) \varepsilon X_2, (g,x) \varepsilon E_1\}$, so for $(f,g,x) \varepsilon E_2$, we have $F(fg)(x) = \delta_1(fg,x) = \delta_1 c(f,g,x) = \delta_1 \varepsilon_0(f,g,x) = \delta_1(f, \delta_1(g,x)) = F(f)[F(g)(x)]$.

A contravariant internal functor may be similarly defined, by interchanging δ_0 and δ_1, d_0 and d_1, ε_0 and ε_1, and e_0 and e_1 in the above definition.

Definition. Given two internal functors $F, F' : C \to \underline{E}$, best described by the diagrams

$$\begin{tikzcd}
E_2 \arrow[r, "c"] \arrow[d, "\pi_2"] & E_1 \arrow[r, shift left=2, "\delta_0"] \arrow[r, "\delta_1"] \arrow[d, "\pi_1"] & E_0 \arrow[l, shift left=2, "m"] \arrow[d, "\pi_0"] \\
X_2 \arrow[r, "\gamma"] & X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] & X_0 \arrow[l, shift left=2, "\mu"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
E'_2 \arrow[r, "c'"] \arrow[d, "\pi'_2"] & E'_1 \arrow[r, shift left=2, "\delta'_0"] \arrow[r, "\delta'_1"] \arrow[d, "\pi'_1"] & E'_0 \arrow[l, shift left=2, "m'"] \arrow[d, "\pi'_0"] \\
X_2 \arrow[r, "\gamma"] & X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] & X_0 \arrow[l, shift left=2, "\mu"]
\end{tikzcd}$$

respectively, an internal natural transformation $\tau : F \to F'$ consists of a map $\tau_0 : E_0 \to E'_0$ in $\underline{E}$, with

(1) $$\pi'_0 \tau_0 = \pi_0 ,$$

such that if we let $\tau_1 = \langle \pi_1, \tau_0 \delta_0 \rangle$, the condition

(2) $$\delta'_1 \tau_1 = \tau_0 \delta_1$$

is satisfied.

Whenever there is no danger of ambiguity, we shall omit the word "internal".

If $\underline{E}$ is the category of sets (and so F and F' are ordinary functors $C \to \underline{Set}$), then due to condition (1) , $\tau_0 : \dot{\bigcup}_{x_0 \in X_0} \pi_0^{-1}(\{x_0\}) \to \dot{\bigcup}_{x_0 \in X_0} (\pi_0')^{-1}(\{x_0\})$ has the property that the image of its restriction to $\pi_0^{-1}(\{x_0\})$ is contained in $(\pi_0')^{-1}(\{x_0\})$, so that τ associates to each object x_0 in C a map $\tau_0|F(x_0): F(x_0) \to F'(x_0)$ in $\underline{Set}$; condition (2) states naturality , for given a map $f: d_0(f) \to d_1(f)$ in C and $x \in F(d_0(f))$ (so that $(f,x) \in E_1$) , $F'(f)\tau_0(x) = \delta_1'(f,\tau_0(x)) = \delta_1'\tau_1(f,x) = \tau_0\delta_1(f,x) = \tau_0 F(f)(x)$, which is precisely the commutativity of the diagram

$$\begin{tikzcd}
F(d_0(f)) \arrow[r, "\tau_{d_0(f)}"] \arrow[d, "F(f)"] & F'(d_0(f)) \arrow[d, "F'(f)"] \\
F(d_1(f)) \arrow[r, "\tau_{d_1(f)}"] & F'(d_1(f))
\end{tikzcd}$$

(where $\tau_{d_i(f)} \equiv \tau_0|F(d_i(f))$, $i = 0,1$), i.e. the naturality of τ .

It is immediate from the definition that two natural transformations $\tau, \tau' : F \to F'$ are equal iff $\tau_0 = \tau_0'$.

If $F, F' : C \to \underline{E}$ are contravariant , then the above definition is valid with δ_0 and δ_1 interchanged .

Given (covariant) internal functors $F, F', F'' : C \to \underline{E}$, and natural transformations $\tau : F \to F'$, $\tau' : F' \to F''$, we define the composite $\tau'\tau : F \to F''$ by $(\tau'\tau)_0 = \tau_0'\tau_0$. It can then easily be shown that $(\tau'\tau)_1 = \tau'_1\tau_1$, and that conditions (1) and (2) in the definition are satisfied .

We denote by $\underline{E}^C$ the category which has as objects (covariant) internal functors $F:C\to\underline{E}$, and as morphisms natural transformations of such functors .

2. Characterisation as Discrete Cofibrations

Let $(\underline{Cat}(\underline{E}),C)$ denote the comma category whose objects are maps (functors) with codomain C , of category objects in $\underline{E}$. The morphisms of this category are , of course , maps (functors) of category objects (in $\underline{E}$) "over C" . Let $\underline{DCF}(\underline{E},C)$ denote the full subcategory of $(\underline{Cat}(\underline{E}),C)$ with objects those functors $F:C'\to C$,

$$\begin{tikzcd} X'_2 \arrow[r, "\gamma'"] \arrow[d, "F_2"] & X'_1 \arrow[r, shift left=2, "d'_0"] \arrow[r, "d'_1"] \arrow[d, "F_1"] & X'_0 \arrow[l, shift left=2, "\mu'"] \arrow[d, "F_0"] \\ X_2 \arrow[r, "\gamma"] & X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] & X_0 \arrow[l, shift left=2, "\mu"] \end{tikzcd} ,$$

for which the square

$$\begin{tikzcd} X'_1 \arrow[r, "d'_0"] \arrow[d, "F_1"] & X'_0 \arrow[d, "F_0"] \\ X_1 \arrow[r, "d_0"] & X_0 \end{tikzcd}$$

is a pullback . When $\underline{E}$ is the category of sets , this reduces to the ordinary notion of a discrete cofibration : $F:C' \to C$ becomes an ordinary functor between categories , and the requirement that the square above be a pullback is precisely the condition that for any object C' in C' and a map $\phi: F(C') \to D$ in C , there should exist a unique map $\psi: C' \to D'$ in C' , such that $F(\psi) = \phi$.

We shall now prove that the categories $\underline{E}^{C}$ and $\underline{DCF}(\underline{E},C)$ are equivalent .

Given an internal functor $F:C \to \underline{E}$,

$$
\begin{tikzcd}
E_2 \arrow[r, shift left=2, "c"] \arrow[r, "\varepsilon_0" description] \arrow[r, shift right=2, "\varepsilon_1"'] \arrow[d, "\pi_2"] & E_1 \arrow[r, shift left=2, "\delta_0"] \arrow[r, "\delta_1" description] \arrow[d, "\pi_1"] & E_0 \arrow[l, shift left=2, "m"] \arrow[d, "\pi_0"] \\
X_2 \arrow[r, shift left=2, "\gamma"] \arrow[r, "e_0" description] \arrow[r, shift right=2, "e_1"'] & X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] & X_0 \arrow[l, shift left=2, "\mu"]
\end{tikzcd}
,
$$

we must first show that the objects and maps

$$
\begin{tikzcd}
E_2 \arrow[r, shift left=2, "c"] \arrow[r, "\varepsilon_0" description] \arrow[r, shift right=2, "\varepsilon_1"'] & E_1 \arrow[r, shift left=2, "\delta_0"] \arrow[r, "\delta_1" description] & E_0 \arrow[l, shift left=2, "m"]
\end{tikzcd}
$$

define a category object in $\underline{E}$, i.e. that conditions (1) to (4) in the definition are satisfied (see I.1. , definition of category object) .

(1) $\delta_0 m = \delta_1 m = E_0$ is immediate from condition (3) in the definition of $F:C \to \underline{E}$.

(2) The square

$$\begin{tikzcd} E_2 \arrow[r, "\varepsilon_0"] \arrow[d, "\varepsilon_1"] & E_1 \arrow[d, "\delta_0"] \\ E_1 \arrow[r, "\delta_1"] & E_0 \end{tikzcd}$$

commutes by definition of ε_0 . To show it is a pullback , let $f,g:Y\to E_1$ have the property that $\delta_0 f = \delta_1 g$, then

$$\pi_0\delta_0 f = \pi_0\delta_1 g ,$$

$$\Rightarrow \quad d_0\pi_1 f = d_1\pi_1 g ,$$

so that we have $\langle\pi_1 f,\pi_1 g\rangle:Y\to X_2$; but now $\langle\langle\pi_1 f,\pi_1 g\rangle,g\rangle:Y\to E_2$ satisfies

(a) $$\varepsilon_1\langle\langle\pi_1 f,\pi_1 g\rangle,g\rangle = g ,$$

and

(b) $$\varepsilon_0\langle\langle\pi_1 f,\pi_1 g\rangle,g\rangle = f ;$$

for (a) follows by definition of $\langle\langle\pi_1 f,\pi_1 g\rangle,g\rangle$, and (b) is due to the fact that both f and $\varepsilon_0\langle\langle\pi_1 f,\pi_1 g\rangle,g\rangle$ are equal to $\langle\pi_1 f,\delta_1 g\rangle:Y\to E_1$, the square

$$\begin{tikzcd} E_1 \arrow[r, "\delta_0"] \arrow[d, "\pi_1"] & E_0 \arrow[d, "\pi_0"] \\ X_1 \arrow[r, "d_0"] & X_0 \end{tikzcd}$$

being a pullback . To show that $\langle\langle\pi_1 f,\pi_1 g\rangle,g\rangle$ is unique with respect to the properties (a) and (b) , suppose $\xi : Y \to E_2$ satisfies $\varepsilon_1 \xi = g$ and $\varepsilon_0 \xi = f$; then it is easily seen that $\pi_2 \xi = \langle \pi_1 f, \pi_1 g \rangle$, which shows that $\xi = \langle\langle \pi_1 f, \pi_1 g\rangle, g\rangle$.

(i) $\delta_0 c = \delta_0 \varepsilon_1$ follows from the definition of c ;

(ii) $\delta_1 c = \delta_1 \varepsilon_0$ is condition (4) in the definition of $F : C \to \underline{E}$.

(3) It is required to prove that $c\overline{\phi}_0 = c\overline{\phi}_1 = E_1$, where $\overline{\phi}_0 = \langle E_1, m\delta_0\rangle$ and $\overline{\phi}_1 = \langle m\delta_1, E_1\rangle$ are defined through the pullback square

$$\begin{tikzcd} E_2 \arrow[r, "\varepsilon_0"] \arrow[d, "\varepsilon_1"'] & E_1 \arrow[d, "\delta_0"] \\ E_1 \arrow[r, "\delta_1"] & E_0 \end{tikzcd} .$$

First , the uniqueness property of $\langle \pi_1, \pi_1 m\delta_0\rangle : E_1 \to X_2$, together with the equations

$$\begin{aligned} e_0 \phi_0 \pi_1 &= \pi_1 , \\ e_1 \phi_0 \pi_1 &= \mu d_0 \pi_1 \\ &= \mu \pi_0 \delta_0 \\ &= \pi_1 m \delta_0 ; \end{aligned}$$

and

$$\begin{aligned} e_0 \pi_2 \overline{\phi}_0 &= \pi_1 \varepsilon_0 \overline{\phi}_0 \\ &= \pi_1 , \\ e_1 \pi_2 \overline{\phi}_0 &= \pi_1 \varepsilon_1 \overline{\phi}_0 \\ &= \pi_1 m \delta_0 ; \end{aligned}$$

shows that $\phi_0 \pi_1 = \pi_2 \overline{\phi}_0$. Similarly , we have $\phi_1 \pi_1 = \pi_2 \overline{\phi}_1$.

Next, the diagram

$$\begin{tikzcd}
E_1 \arrow[dr, shift left, "c\overline{\phi}_0"] \arrow[dr, shift right, "c\overline{\phi}_1"'] \arrow[drr, bend left, "\delta_0"] \arrow[ddr, bend right, "\pi_1"'] & & \\
& E_1 \arrow[r, "\delta_0"] \arrow[d, "\pi_1"] & E_0 \arrow[d, "\pi_0"] \\
& X_1 \arrow[r, "d_0"] & X_0
\end{tikzcd},$$

together with the equations

$$\begin{aligned}
\delta_0 c\overline{\phi}_0 &= \delta_0 \varepsilon_1 \overline{\phi}_0 \\
&= \delta_0 m \delta_0 \\
&= \delta_0 , \\
\pi_1 c\overline{\phi}_0 &= \gamma \pi_2 \overline{\phi}_0 \\
&= \gamma \phi_0 \pi_1 \\
&= \pi_1 ;
\end{aligned}$$

and

$$\begin{aligned}
\delta_0 c\overline{\phi}_1 &= \delta_0 \varepsilon_1 \overline{\phi}_1 \\
&= \delta_0 , \\
\pi_1 c\overline{\phi}_1 &= \gamma \pi_2 \overline{\phi}_1 \\
&= \gamma \phi_1 \pi_1 \\
&= \pi_1 ;
\end{aligned}$$

shows (by uniqueness) that $c\overline{\phi}_0 = c\overline{\phi}_1 = E_1$.

(4) Defining $\overline{\alpha}_0 = \langle \varepsilon_0 \overline{\psi}_0, c\overline{\psi}_1 \rangle$ and $\overline{\alpha}_1 = \langle c\overline{\psi}_0, \varepsilon_1 \overline{\psi}_1 \rangle$, where $\overline{\psi}_0$ and $\overline{\psi}_1$ are given by the pullback square

$$\begin{tikzcd} E_3 \arrow[r, "\overline{\psi}_0"] \arrow[d, "\overline{\psi}_1"] & E_2 \arrow[d, "\varepsilon_1"] \\ E_2 \arrow[r, "\varepsilon_0"] & E_1 \end{tikzcd} ,$$

it is required to show that $c\overline{\alpha}_0 = c\overline{\alpha}_1$. First , we let $\pi_3 = \langle\pi_2\overline{\psi}_0, \pi_2\overline{\psi}_1\rangle : E_3 \to X_3$, so that the square

$$\begin{tikzcd} E_3 \arrow[r, "\overline{\alpha}_1"] \arrow[d, "\pi_3"] & E_2 \arrow[d, "\pi_2"] \\ X_3 \arrow[r, "\alpha_1"] & X_2 \end{tikzcd}$$

commutes by the uniqueness property of $\langle\gamma\pi_2\overline{\psi}_0, e_1\pi_2\overline{\psi}_1\rangle : E_3 \to X_2$. Next , the diagram

$$\begin{tikzcd} E_3 \arrow[dr, shift left, "c\overline{\alpha}_0"] \arrow[dr, shift right, "c\overline{\alpha}_1"'] \arrow[drr, bend left, "\delta_0 c\overline{\alpha}_0"] \arrow[ddr, bend right, "\pi_1 c\overline{\alpha}_0"'] & & \\ & E_1 \arrow[r, "\delta_0"] \arrow[d, "\pi_1"] & E_0 \arrow[d, "\pi_0"] \\ & X_1 \arrow[r, "d_0"] & X_0 \end{tikzcd} ,$$

together with the equations

$$\begin{aligned} \delta_0 c\overline{\alpha}_1 &= \delta_0\varepsilon_1\overline{\alpha}_1 \\ &= \delta_0\varepsilon_1\overline{\psi}_1 \end{aligned}$$

$$= \delta_0 c\bar{\psi}_1$$
$$= \delta_0 \varepsilon_1 \bar{\alpha}_0$$
$$= \varepsilon_0 c\bar{\alpha}_0 ,$$

and

$$\pi_1 c\bar{\alpha}_1 = \gamma\pi_2\bar{\alpha}_1$$
$$= \gamma\alpha_1\pi_3$$
$$= \gamma\alpha_0\pi_3$$
$$= \gamma\pi_2\bar{\alpha}_0$$
$$= \pi_1 c\bar{\alpha}_0 ,$$

shows (by uniqueness) that $c\bar{\alpha}_0 = c\bar{\alpha}_1$. This completes the proof that

$$\begin{tikzcd} E_2 \arrow[r, shift left=2, "c"] \arrow[r, "\varepsilon_0"] \arrow[r, shift right=2, "\varepsilon_1"'] & E_1 \arrow[r, shift left=2, "\delta_0"] \arrow[r, "\delta_1"] & E_0 \arrow[l, shift left=2, "m"] \end{tikzcd}$$

defines a category object E in $\underline{E}$, and it follows immediately from the definition of $F:C\to\underline{E}$ that (π_0,π_1,π_2) defines a map (functor) of category objects $E\to C$.

Given two internal functors $F,F':C\to\underline{E}$ and a map $\tau:F\to F'$ in $\underline{E}^C$, i.e. a natural transformation , we show that τ defines a map in $\underline{\text{DCF}}(\underline{E},C)$, i.e. a map in $(\underline{\text{Cat}}(\underline{E}),C)$. τ consists of maps $\tau_0:E_0\to E_0'$ and $\tau_1:E_1\to E_1'$ (where all primed symbols will refer to F') ; and letting $\tau_2 = \langle\pi_2,\tau_1\varepsilon_1\rangle:E_2\to E_2'$, we must verify conditions (1) to (4) (as given in I.2.) , so that (τ_0,τ_1,τ_2) will define a map (functor) of category objects $E\to E'$.

(1) $\delta_0'\tau_1 = \tau_0\delta_0$ follows by definition of τ_1 .

(2) $\delta_1'\tau_1 = \tau_0\delta_1$ is condition (2) in the definition of $\tau:F\to F'$.

(3) $m'\tau_0 = \tau_1 m$ follows by the uniqueness property of $\langle\mu\pi_0'\tau_0,\tau_0\rangle:E_0\to E_1'$.

(4) $c'\tau_2 = \tau_1 c$ follows by the uniqueness property of $\langle\pi_1 c,\tau_0\delta_0 c\rangle:E_2\to E_1'$.

That $(\tau_0,\tau_1,\tau_2):E\to E'$ is "over C" is immediate from condition (1) in the definition of $\tau:F\to F'$, and the definitions of τ_1 and τ_2 .

Conversely , given an object $(\pi_0,\pi_1,\pi_2):C'\to C$,

$$\begin{tikzcd}
X_2' \arrow[r, "\gamma'"] \arrow[d, "\pi_2"] & X_1' \arrow[r, shift left=2, "d_0'"] \arrow[r, "d_1'"] \arrow[d, "\pi_1"] & X_0' \arrow[l, shift left=2, "\mu'"] \arrow[d, "\pi_0"] \\
X_2 \arrow[r, "\gamma"] & X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] & X_0 \arrow[l, shift left=2, "\mu"]
\end{tikzcd} ,$$

in $\underline{DCF}(\underline{E},C)$, we show that (d_1',π_0) defines an internal functor $F:C\to\underline{E}$: conditions (1) and (2) (in the definition of an internal functor) are trivially satisfied . We have $\mu' = \langle X_0',\mu\pi_0\rangle$ and $d_1'\mu' = X_0'$, so that condition (3) is satisfied . To verify (4) , we shall need the following result* :

Consider the commutative diagram

---

* Saunders Mac Lane : Categories for the Working Mathematician , exercise 8(b) on page 72 .

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd};$$

if the outside rectangle and the right hand square are pullbacks , so is the left hand square .

It is first required to show that the square

$$\begin{tikzcd} X_2' \arrow[r, "e_1'"] \arrow[d, "\pi_2"] & X_1' \arrow[d, "\pi_1"] \\ X_2 \arrow[r, "e_1"] & X_1 \end{tikzcd}$$

is a pullback , and this follows by the above , noting that the diagrams

$$\begin{tikzcd} X_2 \arrow[r, "e_1"] \arrow[d, "e_0"] & X_1 \arrow[d, "d_1"] \\ X_1 \arrow[r, "d_0"] & X_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_2' \arrow[r, "e_1'"] \arrow[d, "e_0'"] & X_1' \arrow[d, "d_1'"] \\ X_1' \arrow[r, "d_0'"] \arrow[d, "\pi_1"] & X_0' \arrow[d, "\pi_0"] \\ X_1 \arrow[r, "d_0"] & X_0 \end{tikzcd}$$

are pullbacks , and that $e_0\pi_2 = \pi_1 e_0'$ and $d_1\pi_1 = \pi_0 d_1'$. It is then easily seen that $\gamma' = \langle d_0'e_1', \gamma\pi_2 \rangle$ and $e_0' = \langle d_1'e_1', e_0\pi_2 \rangle$, so that $d_1'\gamma' = d_1'e_0'$ (since C' is a category object) , i.e. that condition (4) is satisfied .

Given a map in $\underline{DCF}(\underline{E}, C)$, it is easy to show that it gives rise to an internal natural transformation .

3. Characterisation as Algebras of a Triple

Given a contravariant internal functor (CIF) $F: C \to \underline{E}$,

$$\begin{tikzcd}
E_2 \arrow[r, "c"] \arrow[d, "\pi_2"] & E_1 \arrow[r, shift left=2, "\delta_0"] \arrow[r, "\delta_1"] \arrow[d, "\pi_1"] & E_0 \arrow[l, shift left=2, "m"] \arrow[d, "\pi_0"] \\
X_2 \arrow[r, "\gamma"] & X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] & X_0 \arrow[l, shift left=2, "\mu"]
\end{tikzcd} ,$$

we claim that (π_0, δ_0) defines an algebra of the triple (T, η, μ) in $(\underline{E}, X_0)$ induced by C (see I.5.) . Conditions (1) and (2) in the definition of a CIF assure us that $\delta_0 : T(\pi_0) \to \pi_0$ is a morphism in $(\underline{E}, X_0)$. The algebra identities

(1) $\qquad \delta_0 \eta_{\pi_0} = id_{\pi_0}$

(2) $\qquad \delta_0 T(\delta_0) = \delta_0 \mu_{\pi_0}$

must now be verified .

(1) It is immediate from the definitions of η_{π_0} and m that $\eta_{\pi_0} = m$, hence condition (3) in the definition of a CIF shows that $\delta_0 \eta_{\pi_0} = id_{\pi_0}$.

(2) Again , the definitions of c , ε_1 , μ_{π_0} and $T(\delta_0)$ show that $\mu_{\pi_0} = c$ and $T(\delta_0) = \varepsilon_1$, so the required result follows by condition (4) in the definition of a CIF .

Now given two CIF's $F,F':C\to\underline{E}$ and a natural transformation $\tau:F\to F'$, we claim that $\tau_0:E_0\to E_0'$ defines a morphism $(\pi_0,\delta_0)\to(\pi_0',\delta_0')$ of the algebras of (T,η,μ) corresponding to F and F' . Indeed , the commutativity of the square

$$\begin{tikzcd} T(\pi_0) \arrow[r, "T(\tau_0)"] \arrow[d, "\delta_0"] & T(\pi_0') \arrow[d, "\delta_0'"] \\ \pi_0 \arrow[r, "\tau_0"] & \pi_0' \end{tikzcd}$$

follows by condition (2) in the definition of a natural transformation (of CIF's) and the fact that $T(\tau_0) = \tau_1$ (clear from their definitions) .

Conversely , given an algebra (π_0,δ_0) of (T,η,μ) , there exist objects E_0 , E_1 in $\underline{E}$ such that the diagram

$$\begin{tikzcd} E_1 \arrow[r, "\delta_0"] \arrow[d, "\pi_1"] & E_0 \arrow[d, "\pi_0"] \\ X_1 \arrow[r, "d_0"] & X_0 \end{tikzcd}$$

is commutative (since $\delta_0:T(\pi_0)\to\pi_0$ is a morphism in $(\underline{E},X_0)$) ;

where the square

$$\begin{tikzcd} E_1 \arrow[r, "\delta_1"] \arrow[d, "\pi_1"] & E_0 \arrow[d, "\pi_0"] \\ X_1 \arrow[r, "d_1"] & X_0 \end{tikzcd}$$

is a pullback , with $\pi_1 = d_1^*(\pi_0)$. It is now very easy to verify that $\pi_0 : E_0 \to X_0$ and $\delta_0 : E_1 \to E_0$ define a CIF $C \to \underline{E}$, so the details will be omitted .

Given two algebras (π_0, δ_0) and (π_0', δ_0') of (T, η, μ) and a morphism $\tau_0 : (\pi_0, \delta_0) \to (\pi_0', \delta_0')$ of these algebras , i.e. that the diagrams

$$\begin{tikzcd} E_0 \arrow[r, "\tau_0"] \arrow[dr, "\pi_0"'] & E_0' \arrow[d, "\pi_0'"] \\ & X_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T(\pi_0) \arrow[r, "T(\tau_0)"] \arrow[d, "\delta_0"] & T(\pi_0') \arrow[d, "\delta_0'"] \\ \pi_0 \arrow[r, "\tau_0"] & \pi_0' \end{tikzcd}$$

are commutative , it is again very easy to verify that τ_0 defines a natural transformation $\tau : F \to F'$, where F and F' are the CIF's corresponding to (π_0, δ_0) and (π_0', δ_0') respectively .

The above discussion amounts to an equivalence of the categories $\underline{E}^{C^{op}}$ and $(\underline{E}, X_0)^{T}$, where $\underline{E}^{C^{op}}$ denotes the category of CIF's $C \to \underline{E}$, and $(\underline{E}, X_0)^{T}$ the category of algebras of the triple $\mathcal{T} = (T, \eta, \mu)$ in $(\underline{E}, X_0)$.

Remark. A category object C in $\underline{E}$ also induces the triple $\mathcal{T}' = (T', \eta', \mu')$ in $(\underline{E}, X_0)$, where $T' = \Sigma_{d_1} d_0^*$. In this case , we get an equivalence of the categories $\underline{E}^C$ and $(\underline{E}, X_0)^{\mathcal{T}'}$. The preference shown for discussing the contravariant case is due to the frequency of occurrence of CIF's (also called (contravariant) internal presheaves) in the above context .

III. LIMITS AND COLIMITS

1. The "Constant" Functor

Given categories $\underline{J}$ and $\underline{C}$ ($\underline{J}$ small), one has the notion of a "constant" functor $\Delta:\underline{C}\to\underline{C}^{\underline{J}}$: for any object C in $\underline{C}$, $\Delta(C)(J) = C$ (for any object J in $\underline{J}$), and if $f:J\to J'$ is a map in $\underline{J}$, then $\Delta(C)(f) = id_C$. If $x:C\to C'$ is a map in $\underline{C}$, then $\Delta(x):\Delta(C)\to\Delta(C')$ is a natural transformation (morphism in $\underline{C}^{\underline{J}}$) which associates to any object J in $\underline{J}$ the map $x:\Delta(C)(J)\to\Delta(C')(J)$.

We shall now treat the internal version of the above.

Let $\mathcal{C}$ (described as in previous chapters) be a category object in $\underline{E}$, and E an object in $\underline{E}$. We claim that the diagram

$$\begin{tikzcd}
X_2\times E \arrow[r, shift left=2, "\gamma\times E"] \arrow[r, "e_0\times E" description] \arrow[r, shift right=2, "e_1\times E"'] \arrow[d, "\rho_2"] & X_1\times E \arrow[r, shift left=2, "d_0\times E"] \arrow[r, "d_1\times E" description] \arrow[d, "\rho_1"] & X_0\times E \arrow[l, shift left=2, "\mu\times E"] \arrow[d, "\rho_0"] \\
X_2 \arrow[r, shift left=2, "\gamma"] \arrow[r, "e_0" description] \arrow[r, shift right=2, "e_1"'] & X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] & X_0 \arrow[l, shift left=2, "\mu"]
\end{tikzcd},$$

where ρ_0, ρ_1 and ρ_2 are the first projections of the products, defines an internal functor $\Delta(E):\mathcal{C}\to\underline{E}$.

Proof: Conditions (1) to (4) in the definition of internal functor must be verified.

(1) The square

$$\begin{tikzcd}
X_1\times E \arrow[r, "d_0\times E"] \arrow[d, "\rho_1"] & X_0\times E \arrow[d, "\rho_0"] \\
X_1 \arrow[r, "d_0"] & X_0
\end{tikzcd}$$

is commutative (by definition of $d_0\times E$) . To show that it is a pullback , let $f:Y\to X_0\times E$ and $g:Y\to X_1$ satisfy $\rho_0 f = d_0 g$, and consider the product diagram

$$\begin{tikzcd}
X_0 & X_0\times E \arrow[l, "\rho_0"'] \arrow[r, "P_0"] & E \\
X_1 \arrow[u, "d_0"] & X_1\times E \arrow[l, "\rho_1"'] \arrow[r, "P_1"] \arrow[u, "d_0\times E"] & E \arrow[u, "E"'] \\
 & Y \arrow[ul, "g"] \arrow[u, dashed, "{<g,P_0f>}"'] \arrow[ur, "P_0f"'] &
\end{tikzcd}$$

from which it is easily seen that $<g,P_0f>:Y\to X_1\times E$ satisfies $(d_0\times E)<g,P_0f> = f$ and $\rho_1<g,P_0f> = g$, and that it is unique with respect to this property . (We note that P_0 and P_1 are the second projections of the products , and that the symbol

$\langle -,- \rangle$ used above has its genuine meaning) .

(2) The square

$$\begin{tikzcd} X_1\times E \arrow[r, "d_1\times E"] \arrow[d, "\rho_1"] & X_0\times E \arrow[d, "\rho_0"] \\ X_1 \arrow[r, "d_1"] & X_0 \end{tikzcd}$$

commutes by definition of $d_1\times E$.

(3) The definition of $\mu\times E$ and the fact that $d_0\mu = X_0$ show that $\mu\times E = \langle \mu\rho_0, X_0\times E\rangle$, so that we have $(d_1\times E)\langle \mu\rho_0, X_0\times E\rangle = (d_1\times E)(\mu\times E) = d_1\mu\times E = X_0\times E$.

(4) The square

$$\begin{tikzcd} X_2\times E \arrow[r, "e_1\times E"] \arrow[d, "\rho_2"] & X_1\times E \arrow[d, "\rho_1"] \\ X_2 \arrow[r, "e_1"] & X_1 \end{tikzcd}$$

is commutative by definition of $e_1\times E$. To show that it is a pullback , let $f:Y\to X_1\times E$ and $g:Y\to X_2$ satisfy $\rho_1 f = e_1 g$, then as in (1) , we have that $\langle g, P_1 f\rangle$ is unique with respect to the properties $\rho_2\langle g, P_1 f\rangle = g$ and $(e_1\times E)\langle g, P_1 f\rangle = f$, where the symbol $\langle -,- \rangle$ again denotes an induced map into a product . To complete the requirements of condition (4) , we notice that the definitions of $\gamma\times E$ and $e_0\times E$, together with the equations $d_0\gamma = d_0e_1$ and $d_0e_0 = d_1e_1$, show that $\gamma\times E = \langle\gamma\rho_2, d_0e_1\times E\rangle$ and $e_0\times E = \langle e_0\rho_2, d_1e_1\times E\rangle$; and that the equation $d_1\gamma = d_1e_0$ shows that $(d_1\times E)(\gamma\times E) = (d_1\times E)(e_0\times E)$.

Now given a map $\alpha:E\to E'$ in $\underline{E}$, we define a map $(\Delta(\alpha))_0:X_0\times E\to X_0\times E'$ in $\underline{E}$ by letting $(\Delta(\alpha))_0 = X_0\times\alpha$. It then follows easily that $\Delta(\alpha):\Delta(E)\to\Delta(E')$ is a natural transformation of internal functors , with $(\Delta(\alpha))_1 = X_1\times\alpha$.

We now see that Δ , as defined above , is a functor $\underline{E}\to\underline{E}^{\mathcal{C}}$, for if $E:E\to E$ is an identity map in $\underline{E}$, then $(\Delta(E))_0 = X_0\times E$, hence $\Delta(E)$ is an identity map $\Delta(E)\to\Delta(E)$ in $\underline{E}^{\mathcal{C}}$, and if $\alpha:E\to E'$, $\alpha':E'\to E''$ is a pair of (composable) maps in $\underline{E}$, then $(\Delta(\alpha')\Delta(\alpha))_0 = (\Delta(\alpha'))_0(\Delta(\alpha))_0 = (X_0\times\alpha')(X_0\times\alpha) = X_0\times\alpha'\alpha = (\Delta(\alpha'\alpha))_0$, so that $\Delta(\alpha')\Delta(\alpha) = \Delta(\alpha'\alpha)$.

If $\underline{E}$ is the category of sets , then for any object x_0 in the category $\mathcal{C}$, $(\Delta(E))(x_0) = \rho_0^{-1}(\{x_0\}) = \{x_0\}\times E \simeq E$; and identifying elements which are isomorphic images of each other , we see that $(\Delta(E))(f) = id_E$ for any map f in $\mathcal{C}$. Given a map $\alpha:E\to E'$ in $\underline{E} = \underline{Set}$, $\Delta(\alpha):\Delta(E)\to\Delta(E')$ associates to any object x_0 in $\mathcal{C}$ the map $(\Delta(\alpha))_0|_{x_0}:\{x_0\}\times E\to\{x_0\}\times E'$, which is essentially $\alpha:E\to E'$ itself .

2. Colimits

<u>Definition</u>. Let $\mathcal{C}$ be a category object in $\underline{E}$, and $F:\mathcal{C}\to\underline{E}$,

$$\begin{tikzcd}
E_2 \arrow[r, "c"] \arrow[d, "\pi_2"] & E_1 \arrow[r, shift left=4, "\delta_0"] \arrow[r, shift left=1, "\delta_1"'] \arrow[d, "\pi_1"] & E_0 \arrow[l, shift left=2, "m"] \arrow[d, "\pi_0"] \\
X_2 \arrow[r, "\gamma"] & X_1 \arrow[r, shift left=4, "d_0"] \arrow[r, shift left=1, "d_1"'] & X_0 \arrow[l, shift left=2, "\mu"]
\end{tikzcd} ,$$

a (covariant) internal functor . Then the colimit of F is the coequalizer of δ_0 and δ_1 :

$$E_1 \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} E_0 \xrightarrow{\text{coeq}} \text{colim}F \quad .$$

If $\underline{E}$ is the category of sets , then colimF = $E_0/\sim$ is the set of $\sim$ equivalence classes of elements of E_0 , $\sim$ being the smallest equivalence relation on E_0 containing R , where R is a relation satisfying xRy iff there exists $z \varepsilon E_1$ such that $x = \delta_0(z)$ and $y = \delta_1(z)$.

We now assume (only for this and the next sections) that $\underline{E}$ has coequalizers of all pairs of morphisms (with common domain and codomain) , so that colimF exists for any object F in $\underline{E}^C$.

Given a natural transformation $\tau:F\to F'$, i.e. a map in $\underline{E}^C$, we define a map $\text{colim}\tau:\text{colim}F\to\text{colim}F'$ in $\underline{E}$ as follows : let $\xi:E_0\to\text{colim}F'$ be given by $\xi = (\text{coeq}')\tau_0$ (where all primed symbols refer to the functor F') , then we have $\xi\delta_1 =$

$(\text{coeq}')\tau_0\delta_1 = (\text{coeq}')\delta_1'\tau_1 = (\text{coeq}')\delta_0'\tau_1 = (\text{coeq}')\tau_0\delta_0 = \xi\delta_0$, so that the universal property of coequalizer shows that there exists a unique map (which we shall denote by $\text{colim}\tau$) $\text{colim}F \to \text{colim}F'$, satisfying $(\text{colim}\tau)(\text{coeq}) = \xi$.

We claim that the above definitions give rise to a functor $\text{colim}:\underline{E}^C \to \underline{E}$.

Proof: Given an identity map $\tau:F\to F$ in $\underline{E}^C$, i.e. that $\tau_0:E_0\to E_0$ is the identity map on E_0 , we have $\xi = (\text{coeq})\tau_0 = \text{coeq}$, so that $\text{colim}\tau$ is the identity map on $\text{colim}F$ (by the uniqueness property of $\text{colim}\tau$) . Given composable maps $\tau:F\to F'$, $\tau':F'\to F''$ in $\underline{E}^C$, we have $\xi = (\text{coeq}')\tau_0$, $\xi' = (\text{coeq}'')\tau_0'$, and $\xi'' = (\text{coeq}'')\tau_0'\tau_0$, so in order to prove that $\text{colim}\tau'\tau = (\text{colim}\tau')(\text{colim}\tau)$, it suffices to show (by uniqueness) that $(\text{colim}\tau')(\text{colim}\tau)(\text{coeq}) = \xi''$, which holds since $(\text{colim}\tau')(\text{colim}\tau)(\text{coeq}) = (\text{colim}\tau')\xi = (\text{colim}\tau')(\text{coeq}')\tau_0 = \xi'\tau_0 = (\text{coeq}'')\tau_0'\tau_0 = \xi''$.

<u>Remark</u>. The <u>internal</u> <u>coproduct</u> of the map $\pi_0:E_0\to X_0$ is the domain of π_0 , namely E_0 . We see that this is a special case of the notion of (internal) colimit , for taking the discrete category object $\mathcal{D}$,

$$\begin{tikzcd} X_0 \arrow[r, "X_0"] & X_0 \arrow[r, shift left=2, "X_0"] \arrow[r, "X_0"] & X_0 \arrow[l, shift left=2, "X_0"] \end{tikzcd} ,$$

in $\underline{E}$, π_0 defines an internal functor $F:\mathcal{D}\to\underline{E}$,

$$
\begin{tikzcd}
E_0 \arrow[r, "E_0"] \arrow[d, "\pi_0"] & E_0 \arrow[r, shift left=2, "E_0"] \arrow[r, "E_0"] \arrow[d, "\pi_0"] & E_0 \arrow[l, shift left=2, "E_0"] \arrow[d, "\pi_0"] \\
X_0 \arrow[r, "X_0"] & X_0 \arrow[r, shift left=2, "X_0"] \arrow[r, "X_0"] & X_0 \arrow[l, shift left=2, "X_0"]
\end{tikzcd}
$$
,

and colimF is precisely E_0 .

3. Colim as a Left Adjoint

We shall prove in this section that colim is the left adjoint of Δ . It suffices to show the existence of a natural transformation (in the ordinary sense) $\eta:I_{\underline{E}}C \to \Delta\text{colim}$ (where $I_{\underline{E}}C$ denotes the identity functor on $\underline{E}^C$) such that each $\eta^F:F \to \Delta\text{colim}F$ is universal among (internal) natural transformations $\phi:F \to \Delta(E)$, E an object of $\underline{E}$; i.e. for every (internal) natural transformation $\phi:F \to \Delta(E)$, there exists a unique map $\psi:\text{colim}F \to E$ in $\underline{E}$, such that $\phi = (\Delta\psi)\eta^F$.

Denote by $1_i:X_i \times \text{colim}F \to X_i$ the first projections of the products , arising from the internal functor $\Delta\text{colim}F$ (as discussed in section 1). , and define $\eta_0^F:E_0 \to X_0 \times \text{colim}F$ by the product diagram

$$\begin{tikzcd}
& X_0\times\mathrm{colim}F \arrow[dl, "l_0"'] \arrow[dr, "L_0"] & \\
X_0 & & \mathrm{colim}F \\
& E_0 \arrow[ul, "\pi_0"] \arrow[ur, "\mathrm{coeq}"'] \arrow[uu, dashed, "\eta_0^F"] &
\end{tikzcd}$$

where L_0 is the second projection map . To show that $\eta^F : F \to \Delta\mathrm{colim}F$ is a map in $\underline{E}^C$, conditions (1) and (2) in the definition of internal natural transformation must be verified .

(1) $l_0\eta_0^F = \pi_0$ is immediate from the definition of η_0^F .

(2) Letting $\eta_1^F = \langle\pi_1, \eta_0^F\delta_0\rangle$, it is required to show that $(d_1\times\mathrm{colim}F)\eta_1^F = \eta_0^F\delta_1$, but this follows by the universal property of products ,

$$\begin{tikzcd}
& X_0\times\mathrm{colim}F \arrow[dl, "l_0"'] \arrow[dr, "L_0"] & \\
X_0 & E_0 \arrow[l, "\pi_0"'] \arrow[r, "\mathrm{coeq}"] \arrow[u, dashed, "\eta_0^F"] & \mathrm{colim}F \\
& E_1 \arrow[ul, "d_1\pi_1"] \arrow[u, "\delta_1"] \arrow[ur, "(\mathrm{coeq})\delta_0"'] &
\end{tikzcd}$$

noting that

$$\begin{aligned}
l_0(d_1\times\mathrm{colim}F)\eta_1^F &= d_1 l_1\eta_1^F \\
&= d_1\pi_1 ,
\end{aligned}$$

and

$$\begin{aligned}
L_0(d_1\times\mathrm{colim}F)\eta_1^F &= L_1\eta_1^F \\
&= L_0\eta_0^F\delta_0 \\
&= L_0\eta_0^F\delta_0
\end{aligned}$$

$$= (\mathrm{coeq})\delta_0 \quad ,$$

where $L_1 : X_1 \times \mathrm{colim}F \to \mathrm{colim}F$ is the second projection map of the product .

To show the universality of $\eta^F : F \to \Delta\mathrm{colim}F$, let $\phi : F \to \Delta(E)$ be a map in $\underline{E}^C$, for some object E in $\underline{E}$. Denote by $\rho_i : X_i \times E \to X_i$ $(0 \leq i \leq 2)$ the first projections of the products , arising from the internal functor $\Delta(E)$ (as in section 1) , with corresponding second projections $P_i : X_i \times E \to E$. Now ϕ consists of a map $\phi_0 : E_0 \to X_0 \times E$, satisfying $\rho_0\phi_0 = \pi_0$ and $(d_1 \times E)\phi_1 = \phi_0\delta_1$, where $\phi_1 = \langle \pi_1, \phi_0\delta_0 \rangle$. We seek a unique map $\psi : \mathrm{colim}F \to E$ in $\underline{E}$ satisfying $\phi = (\Delta\psi)\eta^F$. We first let $\phi' = P_0\phi_0$, and note that $\phi'\delta_1 = P_0\phi_0\delta_1 = P_0(d_1 \times E)\phi_1 = P_1\phi_1 = P_0\phi_0\delta_0 = \phi'\delta_0$, so that the universal property of coequalizers shows that there exists a unique map $\psi : \mathrm{colim}F \to E$, satisfying $\phi' = \psi(\mathrm{coeq})$. Now the diagram

$$\begin{tikzcd}
X_0 & X_0 \times E \arrow[l, "\rho_0"'] \arrow[r, "P_0"] & E \\
X_0 \arrow[u, "X_0"] & X_0 \times \mathrm{colim}F \arrow[l, "l_0"'] \arrow[r, "L_0"] \arrow[u, "X_0 \times \psi"] & \mathrm{colim}F \arrow[u, "\psi"] \\
& E_0 \arrow[ul, "\pi_0"] \arrow[u, "\eta_0^F"] \arrow[ur, "\mathrm{coeq}"'] &
\end{tikzcd}$$

shows , by the universal property of products , that $(X_0\times\psi)\eta_0^F = \phi_0$, since $\rho_0\phi_0 = \pi_0$ and $P_0\phi_0 = \phi' = \psi(\mathrm{coeq})$; therefore $(\Delta\psi)_0\eta_0^F = \phi_0$, and hence $(\Delta\psi)\eta^F = \phi$. The uniqueness of ψ with respect to this property follows by its uniqueness in making the diagram

$$\begin{tikzcd} E_1 \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] & E_0 \arrow[r, "\mathrm{coeq}"] \arrow[dr, "\phi'"'] & \mathrm{colim}F \arrow[d, "\psi"] \\ & & E \end{tikzcd}$$

commute , for if $\psi':\mathrm{colim}F\to E$ satisfies $(\Delta\psi')\eta^F = \phi$, then $(X_0\times\psi')\eta_0^F = \phi_0$, so that $\psi'(\mathrm{coeq}) = P_0(X_0\times\psi')\eta_0^F = P_0\phi_0 = \phi'$, and hence $\psi' = \psi$.

It remains to prove the naturality of $\eta:I_{\underline{E}^C}\to\Delta\mathrm{colim}$. Given a map $\tau:F\to F'$ in $\underline{E}^C$, it is required to show that the square

$$\begin{tikzcd} F \arrow[r, "\eta^F"] \arrow[d, "\tau"'] \arrow[dr, phantom, "(1)"] & \Delta\mathrm{colim}F \arrow[d, "\Delta\mathrm{colim}\tau"] \\ F' \arrow[r, "\eta^{F'}"] & \Delta\mathrm{colim}F' \end{tikzcd}$$

is commutative . Let the internal functor F' be described as usual (with $\pi_i':E_i'\to X_i$, $0\leq i\leq 2$, etc.) , and denote by

$l'_i : X_i \times \mathrm{colim}F' \to X_i$ the first projection maps of the products , arising from the internal functor $\Delta\mathrm{colim}F'$, with corresponding second projections $L'_i : X_i \times \mathrm{colim}F' \to \mathrm{colim}F'$. Then square (1) is commutative iff $(\Delta\mathrm{colim}\tau)_0 \eta_0^F = \eta_0^{F'}\tau_0$, i.e. $(X_0 \times \mathrm{colim}\tau)\eta_0^F = \eta_0^{F'}\tau_0$, and the latter follows by the universal property of products , upon considering the diagram

$$\begin{tikzcd}
X_0 & X_0 \times \mathrm{colim}F' \arrow[l, "l'_0"'] \arrow[r, "L'_0"] & \mathrm{colim}F' \\
X_0 \arrow[u, "X_0"] & X_0 \times \mathrm{colim}F \arrow[l, "l_0"'] \arrow[r, "L_0"] \arrow[u, "X_0 \times \mathrm{colim}\tau"] & \mathrm{colim}F \arrow[u, "\mathrm{colim}\tau"] \\
& E_0 \arrow[ul, "\pi_0"] \arrow[u, "\eta_0^F"] \arrow[ur, "\mathrm{coeq}"'] &
\end{tikzcd}$$

and noting that

$$\begin{aligned} l'_0 \eta_0^{F'} \tau_0 &= \pi'_0 \tau_0 \\ &= \pi_0 \ , \end{aligned}$$

and

$$\begin{aligned} L'_0 \eta_0^{F'} \tau_0 &= (\mathrm{coeq}')\tau_0 \\ &= (\mathrm{colim}\tau)(\mathrm{coeq}) \qquad \text{(by def. of colim}\tau\text{)} \ ; \end{aligned}$$

hence η is a natural transformation , and this completes the proof that colim is the left adjoint of Δ .

<u>Remark</u>. The colimit of a functor $F : \underline{C} \to \underline{E}$ may , of course , be defined as an object colimF , together with a natural

transformation $\eta:F\to\Delta\text{colim}F$ which is universal* among natural transformations $F\to\Delta(E)$ (E an object in $\underline{E}$), and this definition is easily seen to be equivalent to the given one, due to the above discussion.

4. Limits

Definition. Let C be a category object in $\underline{E}$, and $F:C\to\underline{E}$ a (covariant) internal functor (described as usual). Then the limit of F is an object $\lim F$ in $\underline{E}$, together with a natural transformation $\tau:\Delta\lim F\to F$ which is universal† among natural transformations $\Delta(E)\to F$.

We assume, from here on, that $\underline{E}$ is cartesian closed, and construct $\lim F$, noting that the following notation and terminology will be used: given a map $f:A\times B\to C$ in $\underline{E}$, its image under the map $\tau_{B,C}:\underline{E}(A\times B,C)\to\underline{E}(B,C^A)$ (where τ is the natural isomorphism which defines a product exponential adjointness in $\underline{E}$) will be called the transpose of f, and written $\ulcorner f\urcorner:B\to C^A$; the image of a map $g:B\to C^A$ in $\underline{E}$ under $\tau^{-1}_{B,C}$ will also be called the transpose of g, but will be denoted by $\tilde{g}:A\times B\to C$; the transpose of the identity map $A^B\to A^B$ will be called the evaluation map, and denoted by $ev:B\times A^B\to A$; given a map $h:A\to C$, $h^B:A^B\to C^B$ will denote the transpose of the composite $h(ev):B\times A^B\to A\to C$.

We define objects R_0 and R_1 in $\underline{E}$ by the pullback square

*An initial object in the comma category $(F,\underline{E}^C)$.

†A terminal object in the comma category $(\underline{E}^C,F)$.

$$\begin{tikzcd} R_0 \arrow[r, "1_0"] \arrow[d, "\alpha_0"'] & 1 \arrow[d, "\ulcorner X_0 \urcorner"] \\ E_0^{X_0} \arrow[r, "\pi_0^{X_0}"] & X_0^{X_0} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R_1 \arrow[r, "1_1"] \arrow[d, "\alpha_1"'] & 1 \arrow[d, "\ulcorner d_1 \urcorner"] \\ E_0^{X_1} \arrow[r, "\pi_0^{X_1}"] & X_0^{X_1} \end{tikzcd} ,$$

where 1 is a terminal object in $\underline{E}$, and define maps $c_0, c_1 : R_0 \to R_1$ as follows : the commutativity of the first pullback square above implies the commutativity of its "transpose diagram" , namely

$$\begin{tikzcd} X_0 \times R_0 \arrow[r, "X_0 \times \alpha_0"] \arrow[rrd, "\text{proj.}"'] & X_0 \times E_0^{X_0} \arrow[r, "ev"] & E_0 \arrow[d, "\pi_0"] \\ & & X_0 \end{tikzcd} ,$$

which shows that

$$\begin{tikzcd} X_1 \times R_0 \arrow[r, "d_1 \times R_0"] \arrow[d, "X_1 \times 1_0"'] & X_0 \times R_0 \arrow[r, "X_0 \times \alpha_0"] & X_0 \times E_0^{X_0} \arrow[r, "ev"] & E_0 \arrow[d, "\pi_0"] \\ X_1 \times 1 \arrow[r, "\text{proj.}"] & X_1 \arrow[rr, "d_1"] & & X_0 \end{tikzcd}$$

commutes , so letting $\phi : X_1 \times R_0 \to E_0$ be the composite

$$X_1 \times R_0 \xrightarrow{d_1 \times R_0} X_0 \times R_0 \xrightarrow{X_0 \times \alpha_0} X_0 \times E_0^{X_0} \xrightarrow{ev} E_0 ,$$

we see that the outside of the diagram

$$\begin{tikzcd}
R_0 \arrow[dr, dashed, "c_0"] \arrow[drr, bend left, "1_0"] \arrow[ddr, bend right, "\ulcorner\phi\urcorner"'] & & \\
& R_1 \arrow[r, "1_1"] \arrow[d, "\alpha_1"] & 1 \arrow[d, "\ulcorner d_1\urcorner"] \\
& E_0^{X_1} \arrow[r, "\pi_0^{X_1}"] & X_0^{X_1}
\end{tikzcd}$$

commutes , hence inducing $c_0 : R_0 \to R_1$. So $c_1 = \langle \ulcorner \delta_1 \xi \urcorner , 1_0 \rangle$ is defined through the same pullback square , where $\xi : X_1 \times R_0 \to E_1$ is given by

$$\begin{tikzcd}
X_1 \times R_0 \arrow[r, "d_0 \times R_0"] \arrow[dr, dashed, "\xi"] \arrow[ddr, bend right, "\mathrm{proj.}"'] & X_0 \times R_0 \arrow[r, "X_0 \times \alpha_0"] & X_0 \times E_0^{X_0} \arrow[d, "ev"] \\
& E_1 \arrow[r, "\delta_0"] \arrow[d, "\pi_1"] & E_0 \arrow[d, "\pi_0"] \\
& X_1 \arrow[r, "d_0"] & X_0
\end{tikzcd}$$

Finally , we have

$$\begin{tikzcd}
\lim F \arrow[r, "eq"] & R_0 \arrow[r, shift left, "c_0"] \arrow[r, shift right, "c_1"'] & R_1
\end{tikzcd}$$

That this construction agrees with the definition of the limit of $F:C \to \underline{E}$ will be clear by the discussions in the next section .

We give an interpretation of the above construction in the category of sets : $R_0 \simeq \{f \in E_0^{X_0} \mid \pi_0 f = id_{X_0}\} \cong \prod_{x_0 \in X_0} \pi_0^{-1}(\{x_0\})$; $R_1 \simeq \{f \in E_0^{X_1} \mid \pi_0 f = d_1\} \cong \prod_{x_1 \in X_1} \pi_0^{-1}(\{d_1(x_1)\})$; for all $f \in R_0$, for all $g \in X_1$, we have $c_0(f)(g) = f(d_1(g))$ and $c_1(f)(g) = F(g)(f(d_0(g)))$, so

$$\lim F \simeq \{f \in R_0 \mid F(g)(f(d_0(g))) = f(d_1(g)), \text{for all } g \in X_1\} .$$

Now given a map $\tau : F \to F'$ in $\underline{E}^C$, i.e. an internal natural transformation , we define a map $\lim\tau : \lim F \to \lim F'$ in $\underline{E}$ as follows : let $s : \lim F \to (E_0')^{X_0}$ denote the composite

$$\lim F \xrightarrow{eq} R_0 \xrightarrow{\alpha_0} E_0^{X_0} \xrightarrow{\tau_0^{X_0}} (E_0')^{X_0}$$

(where all primed symbols will refer to the functor F') , and define $t : \lim F \to R_0'$ by

$$\begin{tikzcd}
\lim F \arrow[dr, dashed, "t"] \arrow[rrd, bend left, ""] \arrow[rdd, bend right, "s"'] & & \\
& R_0' \arrow[r, "1_0'"] \arrow[d, "\alpha_0'"] & 1 \arrow[d, "\ulcorner x \urcorner_0"] \\
& (E_0')^{X_0} \arrow[r, "(\pi_0')^{X_0}"] & X_0^{X_0}
\end{tikzcd} ;$$

it then follows (although the proof is very tedious , and will be omitted) that $c_0't = c_1't$, so that the universal property of equalizers shows that there exists a unique map (which we shall denote by $\lim\tau$) $\lim F \to \lim F'$ making the diagram

$$\begin{tikzcd} \lim F' \arrow[r, "eq'"] & R_0' \arrow[r, shift left, "c_0'"] \arrow[r, shift right, "c_1'"'] & R_1' \\ \lim F \arrow[u, dashed, "\lim\tau"] \arrow[ur, "t"'] & & \end{tikzcd}$$

commute .

We are now in a position to prove that $\lim:\underline{E}^C \to \underline{E}$ is a functor , for given an identity map $\tau:F\to F$ in $\underline{E}^C$, i.e. that $\tau_0:E_0 \to E_0$ is the identity , we have $s = \alpha_0(eq)$, so that $t = eq$, hence the uniqueness property of $\lim\tau$ shows that $\lim\tau = id_{\lim F}$; given (composable) maps $\tau:F\to F'$, $\tau':F'\to F''$ in $\underline{E}^C$, we have

$$s:\lim F \xrightarrow{eq} R_0 \xrightarrow{\alpha_0} E_0^{X_0} \xrightarrow{\tau_0^{X_0}} (E_0')^{X_0} ,$$

$$s':\lim F' \xrightarrow{eq'} R_0' \xrightarrow{\alpha_0'} (E_0')^{X_0} \xrightarrow{(\tau_0')^{X_0}} (E_0'')^{X_0} ,$$

$$s'':\lim F \xrightarrow{eq} R_0 \xrightarrow{\alpha_0} E_0^{X_0} \xrightarrow{(\tau_0'\tau_0)^{X_0}} (E_0'')^{X_0} ,$$

$$t = \langle s, (\lim F \to 1) \rangle ;$$

$$t' = \langle s', (\lim F' \to 1) \rangle ,$$

$$t'' = \langle s'', (\lim F \to 1) \rangle ,$$

and

$$\begin{aligned} s'_0 \lim\tau &= (\tau'_0)^{X_0}\alpha'_0(eq')\lim\tau \\ &= (\tau'_0)^{X_0}\alpha'_0 t \\ &= (\tau'_0)^{X_0} s \\ &= (\tau'_0)^{X_0}\tau_0^{X_0}\alpha_0(eq) \\ &= (\tau'_0\tau_0)^{X_0}\alpha_0(eq) \\ &= s'' \ , \end{aligned}$$

so that $t'\lim\tau = t''$ (by the uniqueness property of t''), hence $(eq'')(\lim\tau')(\lim\tau) = t'\lim\tau = t''$, and the uniqueness property of $\lim(\tau'\tau)$ now shows that $\lim(\tau'\tau) = (\lim\tau')(\lim\tau)$.

Remark. The internal product of the map $\pi_0 : E_0 \to X_0$ along $X_0 \to 1$ (i.e. $\Pi_{X_0\to 1}(\pi_0)$) is by definition the object R_0 in $\underline{E}$. It is easily seen that this is a special case of the notion of (internal) limit, for taking the discrete category object $\mathcal{D}$ in $\underline{E}$ (as in Section 2), π_0 defines an internal functor $F:\mathcal{D}\to\underline{E}$, and $\lim F$ is precisely R_0 .

5. Lim as a Right Adjoint

We prove in this section that lim is the right adjoint of Δ . It suffices to show the existence of a natural transformation $\varepsilon:\Delta\lim\to I_{\underline{E}^C}$ such that each $\varepsilon^F:\Delta\lim F\to F$ is universal from Δ to F .

Let $F:C\to\underline{E}$ be described as usual, and denote by $l_i : X_i \times \lim F \to X_i$ $(0\leq i\leq 2)$ the first projections of the products, arising from the internal functor $\Delta\lim F$. Define $\varepsilon_0^F : X_0 \times \lim F \to E_0$

by the composite

$$X_0\times\lim F \xrightarrow{X_0\times(eq)} X_0\times R_0 \xrightarrow{X_0\times\alpha_0} X_0\times E_0^{X_0} \xrightarrow{ev} E_0 \ .$$

To show that $\varepsilon^F:\Delta\lim F\to F$ is an internal natural transformation, conditions (1) and (2) in the definition must be verified.

(1) $\pi_0\varepsilon_0^F = 1_0$ iff the outside of the diagram

$$\begin{tikzcd}
X_0\times\lim F \arrow[r, "X_0\times(eq)"] \arrow[d, "1_0"] & X_0\times R_0 \arrow[r, "X_0\times\alpha_0"] \arrow[d, "\mathrm{proj.}"] & X_0\times E_0^{X_0} \arrow[r, "ev"] & E_0 \arrow[dll, "\pi_0"] \\
X_0 \arrow[r, "X_0"] & X_0 & &
\end{tikzcd}$$

commutes, and this is so, since the right hand triangle commutes (as shown in section 4, when defining c_0).

(2) Defining ε_1^F by the diagram

$$\begin{tikzcd}
X_1\times\lim F \arrow[dr, dashed, "\varepsilon_1^F"] \arrow[drr, bend left, "\varepsilon_0^F(d_0\times\lim F)"] \arrow[ddr, bend right, "1_1"'] & & \\
& E_1 \arrow[r, "\delta_0"] \arrow[d, "\pi_1"] & E_0 \arrow[d, "\pi_0"] \\
& X_1 \arrow[r, "d_0"] & X_0
\end{tikzcd},$$

it is required to show that $\delta_1\varepsilon_1^F = \varepsilon_0^F(d_1\times\lim F)$. We recall the construction of limF ; in particular the definition of $\xi : X_1 \times R_0 \to E_1$, and consider the diagram

$$\begin{tikzcd}
X_1 \times \lim F \arrow[dr, "X_1\times(eq)"] \arrow[drr, bend left, "d_0\times(eq)"] \arrow[dddrr, bend right, "1_1"'] & & & \\
& X_1 \times R_0 \arrow[r, "d_0\times R_0"] \arrow[dr, dashed, "\xi"] \arrow[ddr, "\mathrm{proj}"'] & X_0 \times R_0 \arrow[r, "X_0\times\alpha_0"] & X_0 \times E_0^{X_0} \arrow[d, "ev"] \\
& & E_1 \arrow[r, "\delta_0"] \arrow[d, "\pi_1"] & E_0 \arrow[d, "\pi_0"] \\
& & X_1 \arrow[r, "d_0"] & X_0
\end{tikzcd}$$,

after which it follows (by the uniqueness property of ε_1^F) that $\varepsilon_1^F = \xi(X_1\times(eq))$; hence $\delta_1\varepsilon_1^F = \delta_1\xi(X_1\times(eq))$. Now by product exponential adjointness in $\underline{E}$, the diagram

$$\begin{tikzcd}
\underline{E}(X_1\times R_0, E_0) \arrow[r] \arrow[d, "\underline{E}(X_1\times(eq), E_0)"] & \underline{E}(R_0, E_0^{X_1}) \arrow[d, "\underline{E}(eq, E_0^{X_1})"] \\
\underline{E}(X_1\times \lim F, E_0) \arrow[r] & \underline{E}(\lim F, E_0^{X_1})
\end{tikzcd}$$

commutes , so that

$$\ulcorner\delta_1\xi(X_1\times(eq))\urcorner = \ulcorner\delta_1\xi\urcorner(eq)$$

$= \alpha_1 c_1(eq)$ (see section 4 , construction of limF)

$$\begin{aligned} &= \alpha_1 c_0(eq) \qquad \text{(by def. of } eq\text{)} \\ &= \ulcorner\phi\urcorner(eq) \qquad \text{(by def. of } c_0\text{)} \\ &= \ulcorner\phi(X_1\times(eq))\urcorner \quad , \end{aligned}$$

hence

$$\delta_1\xi(X_1\times(eq)) = \phi(X_1\times(eq)) \ .$$

But it follows immediately by the definitions of ε_0^F and ϕ that $\phi(X_1\times(eq)) = \varepsilon_0^F(d_1\times \lim F)$, hence $\delta_1\varepsilon_1^F = \varepsilon_0^F(d_1\times \lim F)$.

To verify the universal property of $\varepsilon^F:\Delta\lim F\to F$, let $\theta:\Delta(E)\to F$ be a map in $\underline{E}^{C}$, i.e. an internal natural transformation , then θ consists of a map $\theta_0:X_0\times E\to E_0$ in $\underline{E}$, satisfying $\pi_0\theta_0 = \rho_0$ and $\delta_1\theta_1 = \theta_0(d_1\times E)$, where $\theta_1 = \langle\rho_1,\theta_0(d_0\times E)\rangle$, and $\rho_i:X_i\times E\to X_i$ $(0\leq i\leq 2)$ denote the first projections of the products , arising from the internal functor $\Delta(E)$. We then require a unique map $\psi:E\to\lim F$ in $\underline{E}$ such that $\varepsilon^F\cdot\Delta\psi = \theta$, and for this , we exploit the universal property of equalizers . We define a map $\psi':E\to R_0$ by the diagram

$$\begin{tikzcd}
E \arrow[dr, dashed, "\psi'"] \arrow[drr, bend left] \arrow[ddr, "\ulcorner\theta_0\urcorner"'] & & \\
& R_0 \arrow[r, "1_0"] \arrow[d, "\alpha_0"] & 1 \arrow[d, "\ulcorner X_0\urcorner"] \\
& E_0^{X_0} \arrow[r, "\pi_0^{X_0}"] & X_0^{X_0}
\end{tikzcd} \quad ,$$

and to show that $c_0\psi' = c_1\psi'$, it suffices to show , by considering the diagram

$$\begin{tikzcd}
E \arrow[dr, "\psi'"] & & & \\
& R_0 \arrow[dr, shift left, "c_0"] \arrow[dr, shift right, "c_1"'] \arrow[rrd, bend left, "1_0"] \arrow[ddr, bend right, shift left, "\ulcorner\phi\urcorner"] \arrow[ddr, bend right=40, shift right, "\ulcorner\delta_1\xi\urcorner"'] & & \\
& & R_1 \arrow[r, "1_1"] \arrow[d, "\alpha_1"] & 1 \arrow[d, "\ulcorner d_1\urcorner"] \\
& & E_0^{X_1} \arrow[r, "\pi_0^{X_1}"] & X_0^{X_1}
\end{tikzcd}$$

that $\ulcorner\phi\urcorner\psi' = \ulcorner\delta_1\xi\urcorner\psi'$ (since $\alpha_1 c_0 = \ulcorner\phi\urcorner$ and $\alpha_1 c_1 = \ulcorner\delta_1\xi\urcorner$) .

Now

$$\alpha_0\psi' = \ulcorner\theta_0\urcorner$$

$$\Rightarrow \quad (ev)(X_0\times\alpha_0)(X_0\times\psi') = \theta_0 \qquad \text{(by transposition)}$$

$$\Rightarrow \quad (ev)(X_0\times\alpha_0)(X_0\times\psi')(d_0\times E) = \theta_0(d_0\times E)$$

$$\Rightarrow \quad (ev)(X_0\times\alpha_0)(d_0\times R_0)(X_1\times\psi') = \theta_0(d_0\times E) \quad ,$$

so that the diagram

$$\begin{tikzcd}
X_1\times E \arrow[dr, "X_1\times\psi'"] \arrow[rrrdd, bend left=40, "\theta_0(d_0\times E)"] \arrow[rrddd, bend right, "\rho_1"'] & & & \\
& X_1\times R_0 \arrow[r, "d_0\times R_0"] \arrow[dr, dashed, "\xi"] & X_0\times R_0 \arrow[r, "X_0\times\alpha_0"] & X_0\times E_0^{X_0} \arrow[d, "ev"] \\
& & E_1 \arrow[r, "\delta_0"] \arrow[d, "\pi_1"] & E_0 \arrow[d, "\pi_0"] \\
& & X_1 \arrow[r, "d_0"] & X_0
\end{tikzcd}$$

shows (by the uniqueness property of θ_1) that $\theta_1 = \xi(X_1\times\psi')$.
Therefore

$$\begin{aligned}
\delta_1\xi(X_1\times\psi') &= \delta_1\theta_1 \\
&= \theta_0(d_1\times E) \\
&= (ev)(X_0\times\ulcorner\theta_0\urcorner)(d_1\times E) \\
&= (ev)(X_0\times\alpha_0\psi')(d_1\times E) \\
&= (ev)(d_1\times\alpha_0\psi') \\
&= (ev)(X_0\times\alpha_0)(d_1\times R_0)(X_1\times\psi') \\
&= \phi(X_1\times\psi') \quad ,
\end{aligned}$$

hence by transposition , we get $\ulcorner\phi(X_1\times\psi')\urcorner = \ulcorner\delta_1\xi(X_1\times\psi')\urcorner$, which shows that $\ulcorner\phi\urcorner\psi' = \ulcorner\delta_1\xi\urcorner\psi'$. So there exists a unique map $\psi : E\to \lim F$ making the diagram

$$\begin{tikzcd}
\lim F \arrow[r, "eq"] & R_0 \arrow[r, shift left, "c_0"] \arrow[r, shift right, "c_1"'] & R_1 \\
E \arrow[u, dashed, "\psi"] \arrow[ur, "\psi'"'] & &
\end{tikzcd}$$

commute . Now

$$\begin{aligned}
\varepsilon_0^F(X_0\times\psi) &= (ev)(X_0\times\alpha_0)(X_0\times eq)(X_0\times\psi) \\
&= (ev)(X_0\times\alpha_0)(X_0\times\psi') \ ,
\end{aligned}$$

hence

$$\begin{aligned}
\ulcorner\varepsilon_0^F(X_0\times\psi)\urcorner &= \alpha_0\psi' \\
&= \ulcorner\theta_0\urcorner \ ,
\end{aligned}$$

therefore $\varepsilon_0^F(X_0\times\psi) = \theta_0$, which shows that $\varepsilon^F\cdot\Delta\psi = \theta$. To show that ψ is unique with respect to this property , assume $\psi'':E\to\lim F$ satisfies $\varepsilon^F\cdot\Delta\psi'' = \theta$, then we have

$$\begin{aligned}\theta_0 &= \varepsilon_0^F(X_0\times\psi'')\\ &= (ev)(X_0\times\alpha_0)(X_0\times eq)(X_0\times\psi'')\\ &= (ev)(X_0\times\alpha_0)(X_0\times(eq)\psi'') ,\end{aligned}$$

which implies that

$$\begin{aligned}&\ulcorner\theta_0\urcorner = \alpha_0(eq)\psi''\\ \Rightarrow\ &(eq)\psi'' = \psi' \qquad \text{(by the uniqueness property of } \psi'\text{)}\\ \Rightarrow\ &\psi'' = \psi \qquad \text{(by the uniqueness property of } \psi\text{)} .\end{aligned}$$

It remains to verify the naturality of $\varepsilon:\Delta\lim\to I_{\underline{E}^C}$, i.e. the commutativity of the diagram

$$\begin{tikzcd}\Delta\lim F \arrow[r, "\varepsilon^F"] \arrow[d, "\Delta\lim\tau"] & F \arrow[d, "\tau"] \\ \lim F' \arrow[r, "\varepsilon^{F'}"] & F'\end{tikzcd} ,$$

where $\tau:F\to F'$ is a map in $\underline{E}^C$. Noting that all primed symbols refer to the internal functor F' , and recalling that $s = (\tau_0^X 0)\alpha_0(eq)$ (as stated when defining lim on maps) , it follows , by the equation

$$(ev')(X_0\times s) = \tilde{s} = \tau_0(ev)(X_0\times\alpha_0)(X_0\times eq) ,$$

that the outside of the diagram

$$\begin{tikzcd}
X_0\times\lim F \arrow[r, "X_0\times(eq)"] \arrow[d, "X_0\times\lim\tau"'] \arrow[dr, "X_0\times t"] \arrow[drr, "X_0\times s"] & X_0\times R_0 \arrow[r, "X_0\times\alpha_0"] & X_0\times E_0^{X_0} \arrow[r, "ev"] & E_0 \arrow[d, "\tau_0"] \\
X_0\times\lim F' \arrow[r, "X_0\times(eq')"] & X_0\times R_0' \arrow[r, "X_0\times\alpha_0'"] & X_0\times(E_0')^{X_0} \arrow[r, "ev'"] & E_0'
\end{tikzcd}$$

commutes , and this is precisely equivalent to the commutativity of the required diagram above . This completes the proof that lim is the right adjoint of Δ .

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