

INTERNAL CATEGORY THEORY

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ABSTRACT

In this thesis , we internalize certain well known notions in category theory , i.e. , given a category \underline{E} with finite limits , we define , for example , a category object (internal category) in \underline{E} , in such a way that when \underline{E} is the category of sets , this definition reduces to the ordinary one of a small category .

Chapter I consists of basic definitions , certain formulae involving functors and natural transformations , and triples and simplicial objects induced by category objects . In chapter II , we define and give characterisations of internal functors , while in chapter III , we define limits and colimits , and prove a well known result internally , namely that limit (colimit) is the right (left) adjoint of the "constant" functor Δ .

I wish to thank D. Gildenhuys , my thesis supervisor , for his help and encouragement . I also wish to thank R. Paré , for many interesting discussions .

ABSTRAIT

Dans cette thèse on définit dans un contexte interne certaines notions bien connues dans la théorie des catégories, par exemple, un objet de catégorie (catégorie interne).

Le premier chapitre consiste en définitions fondamentales, en certaines formules et en des triples et des objets simpliciaux induits par des objets de catégorie. Dans le deuxième chapitre on définit et on donne des caractérisations des foncteurs internes; ensuite dans le troisième chapitre on définit des limites et colimites et on prouve, dans un contexte interne, un résultat qui est bien connu, notamment que la limite (colimite) est l'adjoint à droite (gauche) à Δ , le foncteur "constant".

J'aimerais remercier mon directeur de thèse, D. Gildenhuys, pour son aide et son encouragement. Je voudrais également remercier R. Paré, pour de nombreuses discussions intéressantes que nous avons eues ensemble.

TABLE OF CONTENTS

INTRODUCTION	i
I. CATEGORY OBJECTS	1
1. Category Objects	1
2. Functors	3
3. Natural Transformations	5
4. Composition Formulae	9
5. Induced Triples	14
6. Simplicial Objects	24
II. INTERNAL FUNCTORS	30
1. Internal Functors and Natural Transformations	30
2. Characterisation as Discrete Cofibrations	34
3. Characterisation as Algebras of a Triple	43
III. LIMITS AND COLIMITS	47
1. The "Constant" Functor	47
2. Colimits	50
3. Colim as a Left Adjoint	53
4. Limits	58
5. Lim as a Right Adjoint	63
BIBLIOGRAPHY	71

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INTRODUCTION

Internal category theory has its applications in the study of toposes (Diaconescu [1]) and profinite groups (Gildenhuys and Ribes [3]), and it is believed that several interesting applications may arise through the study of category objects, internal functors, etc., in an arbitrary category with finite limits.

Throughout this thesis, \underline{E} will be assumed to be a skeletal category with finite limits. Additional conditions, for example cartesian closedness and the existence of coequalizers, are imposed upon \underline{E} in certain parts of chapter III, and are specified in the relevant sections. The words "map" and "morphism" will be used interchangeably, throughout.

(1)

I. CATEGORY OBJECTS

1. Category Objects

Definition. A category object C in E consists of objects and morphisms

$$x_2 \xrightarrow{\gamma} x_1 \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xleftarrow{\mu} \end{array} x_0$$

in E , such that

(1) $d_0 \mu = d_1 \mu = x_0$,

satisfying the following additional conditions :

(2) The square

$$\begin{array}{ccc} x_2 & \xrightarrow{e_0} & x_1 \\ \downarrow e_1 & & \downarrow d_0 \\ x_1 & \xrightarrow{d_1} & x_0 \end{array}$$

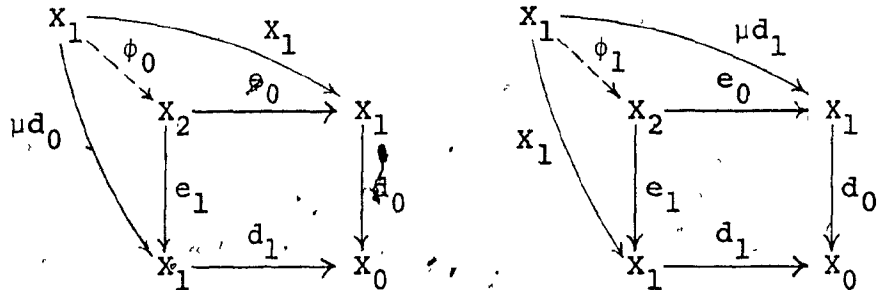
is a pullback, with

(i) $d_0 \gamma = d_0 e_1$,

(ii) $d_1 \gamma = d_1 e_0$.

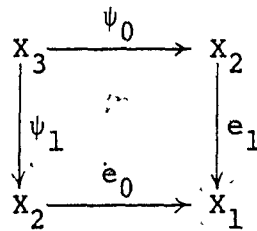
(3) Defining $\phi_0, \phi_1: X_1 \rightarrow X_2$ by the diagrams

(2)

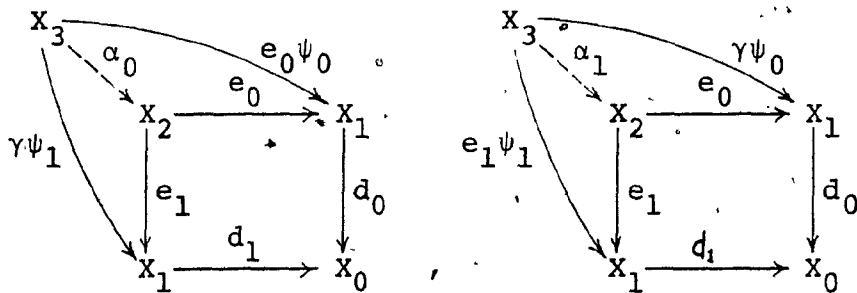


we require that $\gamma\phi_0 = \gamma\phi_1 = X_1$.

(4) Defining $\psi_0, \psi_1: X_3 \longrightarrow X_2$ by the pullback square



and $\alpha_0, \alpha_1: X_3 \longrightarrow X_2$ by the diagrams



we require that $\gamma\alpha_0 = \gamma\alpha_1$.

If \underline{E} is the category of sets, then \mathcal{C} satisfies the axioms of a small category: X_0 and X_1 are the sets of objects

(3)

and morphisms (respectively) of \mathcal{C} ; d_0 and d_1 are the domain and codomain maps, respectively; μ assigns to each object $x_0 \in X_0$ the identity map $\text{id}_{x_0} \in X_1$. Condition (1) states the requirement that $d_0(\text{id}_{x_0}) = d_1(\text{id}_{x_0}) = x_0$, for all $x_0 \in X_0$. Condition (2) stipulates that $X_2 = \{(f, g) \in X_1 \times X_1 \mid d_0(f) = d_1(g)\}$ (which is the set of all composable maps in \mathcal{C}), and that for all $(f, g) \in X_2$, (i) $d_0(fg) = d_0(g)$ and (ii) $d_1(fg) = d_1(f)$. Condition (3) amounts to $f \cdot \text{id}_{d_0(f)} = \text{id}_{d_1(f)} \cdot f = f$, for all $f \in X_1$ (where " \cdot " and juxtaposition both mean composition in \mathcal{C}). Condition (4) states the associative law of composition, i.e. that for all (f, g, g') satisfying $(f, g) \in X_2$ and $(g, g') \in X_2$, $(fg)g' = f(gg')$ should hold.

2. Functors

Definition. Given category objects $\mathcal{C}, \mathcal{C}'$ in \underline{E} , a functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ consists of morphisms $F_0: X_0 \rightarrow X'_0$, $F_1: X_1 \rightarrow X'_1$ in \underline{E} , subject to the conditions:

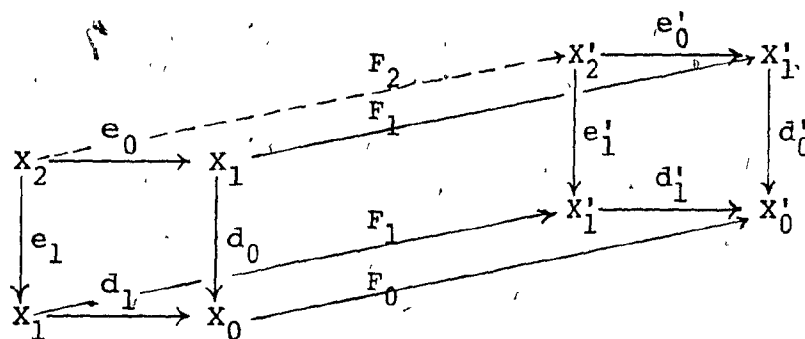
- (1) $d'_0 F_1 = F_0 d_0$;
- (2) $d'_1 F_1 = F_0 d_1$;
- (3) $\mu' F_0 = F_1 \mu$;
- (4) $\gamma' F_2 = F_1 \gamma$,

where F_2 is defined by noting that

$$d'_0 F_1 e_0 = F_0 d_0 e_0 = F_0 d_1 e_1 = d'_1 F_1 e_1,$$

and considering the diagram

(4)



We note that all primed symbols above refer to the category object C' .

If \underline{E} is the category of sets, then F is an ordinary functor between two (small) categories, because for all $x_0 \in X_0$, $F(x_0) = F_0(x_0) \in X'_0$, i.e. an object of C' ; for all $x_1 \in X_1$, $F(x_1) = F_1(x_1) \in X'_1$, i.e. a map in C' ; and conditions (1) to (4) are interpreted as follows:

$$(1) \quad F(d_0(f)) = d'_0(F(f)),$$

for all $f \in X_1$;

$$(2) \quad F(d_1(f)) = d'_1(F(f)),$$

for all $f \in X_1$;

$$(3) \quad F(\text{id}_{x_0}) = \text{id}_{F(x_0)},$$

for all $x_0 \in X_0$;

$$(4) \quad F(fg) = F(f)F(g),$$

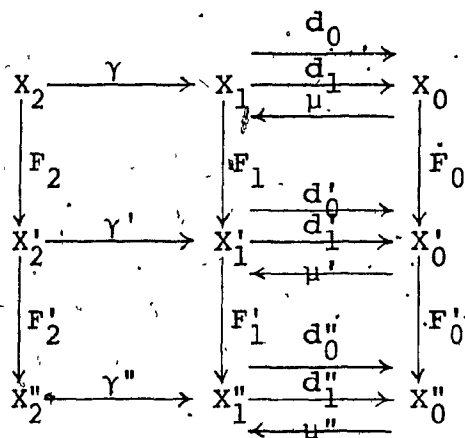
for all $(f, g) \in X_2$.

Given category objects and functors

$$C \xrightarrow{F} C' \xrightarrow{F'} C''$$

in \underline{E} , best described by the diagram

(5)



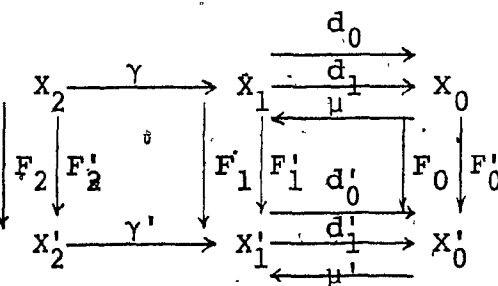
we may define the composite $F'F:C \rightarrow C''$ by the pair of maps

$(F'F)_0 = F'_0 F_0 : X_0 \rightarrow X''_0$, $(F'F)_1 = F'_1 F_1 : X_1 \rightarrow X''_1$. It is then trivial (but tedious) to verify that conditions (1) to (4) are satisfied, so that $F'F$ is a functor between two category objects.

3. Natural Transformations

Definition. Given category objects and functors

$F, F': C \rightarrow C'$ in \underline{E} ,



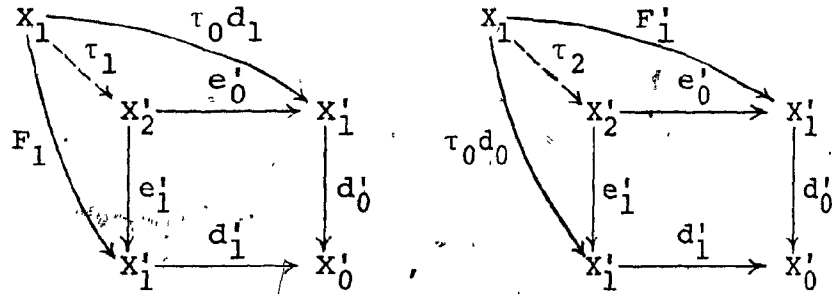
a natural transformation $\tau: F \rightarrow F'$ consists of a map $\tau_0: X_0 \rightarrow X'_0$ in \underline{E} , satisfying

- (1) $d'_0 \tau_0 = F_0$;
- (2) $d'_1 \tau_0 = F'_0$;

(6)

$$(3) \quad \gamma' \tau_1 = \gamma' \tau_2$$

where $\tau_1, \tau_2: X_1 \rightarrow X'_1$ are defined through the diagrams



noting that $d'_0 \tau_0 d_1 = F'_0 d'_1 = d'_1 F_1$ and $d'_1 \tau_0 d_0 = F'_0 d'_0 = d'_0 F'_1$.

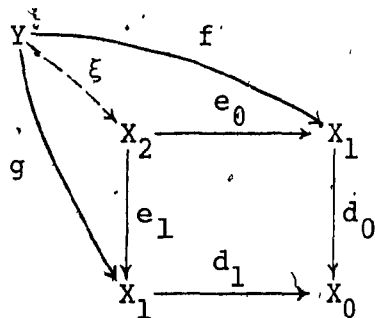
If \underline{E} is the category of sets, then τ becomes an ordinary natural transformation between functors: for all $x_0 \in X_0$, i.e. for any object x_0 in \mathcal{C} , $\tau_{x_0} = \tau_0(x_0) \in X'_1$, i.e. τ_{x_0} is a map in \mathcal{C}' ; conditions (1) and (2) state, respectively, that $d'_0(\tau_{x_0}) = F_0(x_0)$ and $d'_1(\tau_{x_0}) = F'_0(x_0)$; (3) is the naturality condition, i.e. that given a map $f: d_0(f) \rightarrow d_1(f)$ in \mathcal{C} , the square

$$\begin{array}{ccc} F(d_0(f)) & \xrightarrow{\tau_{d_0(f)}} & F'(d_0(f)) \\ \downarrow F(f) & & \downarrow F'(f) \\ F(d_1(f)) & \xrightarrow{\tau_{d_1(f)}} & F'(d_1(f)) \end{array}$$

should be commutative.

We now wish to define a composition for such natural

transformations, but first, a word on notation. Given an object Y in \underline{E} and maps $f, g: Y \rightarrow X_1$ satisfying $d_0 f = d_1 g$,

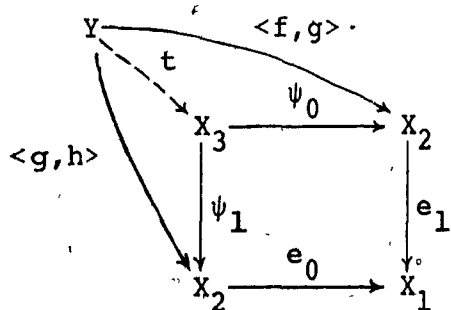


we shall denote the induced map ξ by $\langle f, g \rangle$. This will greatly facilitate the following discussions, and is not really an abuse of notation, since the above pullback is a product in the comma category (\underline{E}, X_0) , whose objects consist of all maps in \underline{E} with codomain X_0 . Next, we shall require the following:

Lemma. Given an object Y in \underline{E} and maps $f, g, h: Y \rightarrow X_1$ satisfying $d_0 f = d_1 g$ and $d_0 g = d_1 h$, we have

$$\gamma \langle \gamma \langle f, g \rangle, h \rangle = \gamma \langle f, \gamma \langle g, h \rangle \rangle.$$

Proof: Condition (4) in the definition of the category object C in \underline{E} states that $\gamma \alpha_0 = \gamma \alpha_1$; i.e. that $\gamma \langle e_0 \psi_0, \gamma \psi_1 \rangle = \gamma \langle \gamma \psi_0, e_1 \psi_1 \rangle$. Define $t: Y \rightarrow X_3$ through the diagram



we, then have

$$\begin{aligned}
 \gamma \langle e_0 \psi_0, \gamma \psi_1 \rangle t &= \gamma \langle \gamma \psi_0, e_1 \psi_1 \rangle t, \\
 \Rightarrow \gamma \langle e_0 \psi_0 t, \gamma \psi_1 t \rangle &= \gamma \langle \gamma \psi_0 t, e_1 \psi_1 t \rangle \\
 \Rightarrow \gamma \langle e_0 \langle f, g \rangle, \gamma \langle g, h \rangle \rangle &= \gamma \langle \gamma \langle f, g \rangle, e_1 \langle g, h \rangle \rangle \\
 \Rightarrow \gamma \langle f, \gamma \langle g, h \rangle \rangle &= \gamma \langle \gamma \langle f, g \rangle, h \rangle,
 \end{aligned}$$

which completes the proof.

Now given functors $F, F', F'' : C \rightarrow C'$, and natural transformations $\tau : F \rightarrow F'$, $\tau' : F' \rightarrow F''$, we define $\tau' \cdot \tau : F \rightarrow F''$ by
 $(\tau' \cdot \tau)_0 = \gamma' \langle \tau'_0, \tau_0 \rangle : X_0 \rightarrow X'_1$. It is now required to verify conditions (1) to (3) in the definition of a natural transformation :

$$\begin{aligned}
 (1) \quad d'_0 (\tau' \cdot \tau)_0 &= d'_0 \gamma' \langle \tau'_0, \tau_0 \rangle \\
 &= d'_0 e'_1 \langle \tau'_0, \tau_0 \rangle \\
 &= d'_0 \tau_0 \\
 &= F_0 ;
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad d'_1 (\tau' \cdot \tau)_0 &= d'_1 \gamma' \langle \tau'_0, \tau_0 \rangle \\
 &= d'_1 e'_0 \langle \tau'_0, \tau_0 \rangle \\
 &= d'_1 \tau'_0 \\
 &= F''_0 ;
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \gamma' (\tau' \cdot \tau)_1 &= \gamma' \langle (\tau' \cdot \tau)_0 d_1, F_1 \rangle && \text{(by definition)} \\
 &= \gamma' \langle \gamma' \langle \tau'_0, \tau_0 \rangle d_1, F_1 \rangle \\
 &= \gamma' \langle \gamma' \langle \tau'_0 d_1, \tau_0 d_1 \rangle, F_1 \rangle \\
 &= \gamma' \langle \tau'_0 d_1, \gamma' \langle \tau_0 d_1, F_1 \rangle \rangle && \text{(by the Lemma)} \\
 &= \gamma' \langle \tau'_0 d_1, \gamma' \langle F'_1, \tau_0 d_0 \rangle \rangle && \text{(by the naturality of } \tau)
 \end{aligned}$$

(9)

$$\begin{aligned}
 &= \gamma' \langle \gamma' \langle \tau'_0 d_1, F'_1 \rangle, \tau_0 d_0 \rangle && \text{(by the Lemma)} \\
 &= \gamma' \langle \gamma' \langle F''_1, \tau'_0 d_0 \rangle, \tau_0 d_0 \rangle && \text{(by the naturality of } \tau') \\
 &= \gamma' \langle F''_1, \gamma' \langle \tau'_0 d_0, \tau_0 d_0 \rangle \rangle && \text{(by the Lemma)} \\
 &= \gamma' \langle F''_1, \gamma' \langle \tau'_0, \tau_0 \rangle d_0 \rangle \\
 &= \gamma' \langle F''_1, (\tau' \cdot \tau)_0 d_0 \rangle \\
 &= \gamma' (\tau' \cdot \tau)_2 ;
 \end{aligned}$$

hence $\tau' \cdot \tau : F \rightarrow F''$ is well defined.

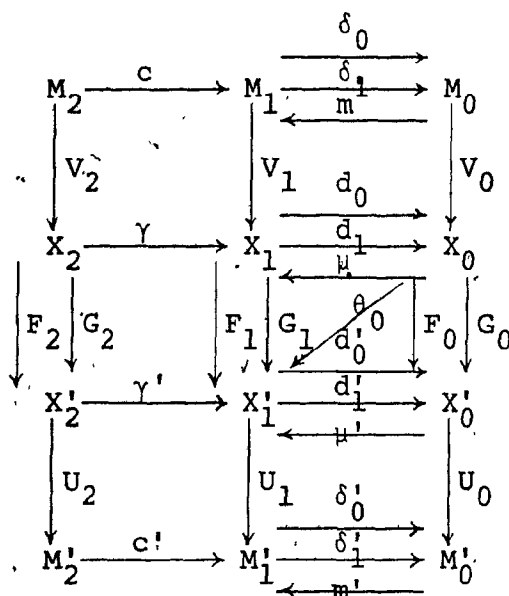
Remark. It is now possible to define a 2-category $\text{Cat}(\underline{E})$, with 0-cells all category objects in \underline{E} , 1-cells all functors between such category objects, and 2-cells all natural transformations between such functors.

4. Composition Formulae*

Given category objects C, C', M, M' in \underline{E} ; functors $F, G: C \rightarrow C'$, $V: M \rightarrow C$ and $U: C' \rightarrow M'$; a natural transformation $\theta: F \rightarrow G$; best described by the diagram

* Godement, Roger : Topologie Algébrique et Théorie des Faisceaux (p. 269) gives rules of composition and interchange of functors and natural transformations, which we shall treat here in our own internal context.

(10)



we define a natural transformation $U \circ \theta \circ V : UFV \rightarrow UGV$, by

$(U \circ \theta \circ V)_0 = U_1 \theta_0 V_0 : M_0 \rightarrow M'_1$. We must now show that $U \circ \theta \circ V$ satisfies the definition of a natural transformation, i.e. conditions

(1) to (3) :

$$\begin{aligned}
 (1) \quad \delta'_0 (U \circ \theta \circ V)_0 &= \delta'_0 U_1 \theta_0 V_0 \\
 &= U_0 d'_0 \theta_0 V_0 \\
 &= U_0 F_0 V_0 \\
 &= (UFV)_0 ;
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \delta'_1 (U \circ \theta \circ V)_0 &= \delta'_1 U_1 \theta_0 V_0 \\
 &= U_0 d'_1 \theta_0 V_0 \\
 &= U_0 G_0 V_0 \\
 &= (UGV)_0 ;
 \end{aligned}$$

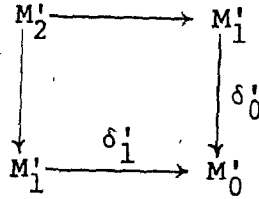
(3) It is required to show that

$$c' (U \circ \theta \circ V)_1 = c' (U \circ \theta \circ V)_2$$

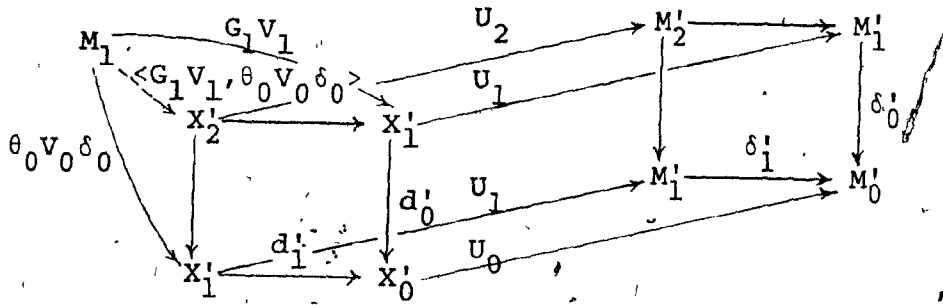
where $(U \circ \theta \circ V)_1$ and $(U \circ \theta \circ V)_2$ are defined through the pullback

(11)

square



by $(U*\theta*V)_1 = \langle U_1\theta_0V_0\delta_1, U_1F_1V_1 \rangle$ and $(U*\theta*V)_2 = \langle U_1G_1V_1, U_1\theta_0V_0\delta_0 \rangle$.
 . First, we consider the diagram



which shows that $U_2\langle G_1V_1, \theta_0V_0\delta_0 \rangle = \langle U_1G_1V_1, U_1\theta_0V_0\delta_0 \rangle$. Similarly, we have $U_2\langle \theta_0V_0\delta_1, F_1V_1 \rangle = \langle U_1\theta_0V_0\delta_1, U_1F_1V_1 \rangle$. Next, the naturality of $\theta:F \rightarrow G$ states that

$$\begin{aligned} \gamma' \langle \theta_0d_1, F_1 \rangle &= \gamma' \langle G_1, \theta_0d_0 \rangle \\ \Rightarrow \gamma' \langle \theta_0d_1, F_1 \rangle V_1 &= \gamma' \langle G_1, \theta_0d_0 \rangle V_1 \\ \Rightarrow \gamma' \langle \theta_0d_1V_1, F_1V_1 \rangle &= \gamma' \langle G_1V_1, \theta_0d_0V_1 \rangle \\ \Rightarrow \gamma' \langle \theta_0V_0\delta_1, F_1V_1 \rangle &= \gamma' \langle G_1V_1, \theta_0V_0\delta_0 \rangle ; \end{aligned}$$

therefore

(12)

$$\begin{aligned}
c'(U*\theta*V)_1 &= c' \langle U_1 \theta_0 V_0 \delta_1, U_1 F_1 V_1 \rangle \\
&= c' U_2 \langle \theta_0 V_0 \delta_1, F_1 V_1 \rangle \\
&= U_1 \gamma' \langle \theta_0 V_0 \delta_1, F_1 V_1 \rangle \\
&= U_1 \gamma' \langle G_1 V_1, \theta_0 V_0 \delta_0 \rangle \\
&= c' U_2 \langle G_1 V_1, \theta_0 V_0 \delta_0 \rangle \\
&= c' \langle U_1 G_1 V_1, U_1 \theta_0 V_0 \delta_0 \rangle \\
&= c'(U*\theta*V)_2 .
\end{aligned}$$

We have of course , as special cases , $U*\theta:UF \rightarrow UG$ and $\theta*V:FV \rightarrow GV$, defined by $(U*\theta)_0 = U_1 \theta_0$ and $(\theta*V)_0 = \theta_0 V_0$.

We now state five rules of composition , keeping in mind that C, C', M, M' will denote category objects in \underline{E} , F, G, H, U, V functors , and $\theta, \theta', \phi, \psi$ natural transformations :

(R1) Let $F, G: C \rightarrow C'$; $\theta: F \rightarrow G$; $V: C' \rightarrow M$ and $U: M \rightarrow M'$; then

$$(UV)*\theta = U*(V*\theta) .$$

(R2) Let $F, G: C \rightarrow C'$; $\theta: F \rightarrow G$; $V: M \rightarrow M'$ and $U: M' \rightarrow C$; then

$$\theta*(UV) = (\theta*U)*V .$$

(R3) Let $F, G: C \rightarrow C'$; $\theta: F \rightarrow G$; $V: M \rightarrow C$ and $U: C' \rightarrow M'$; then

$$(U*\theta)*V = U*(\theta*V) = U*\theta*V .$$

(R4) Let $F, G, H: C \rightarrow C'$; $\theta: G \rightarrow H$; $\theta': F \rightarrow G$; $U: C' \rightarrow M'$ and $V: M \rightarrow C$; then

$$U*(\theta \cdot \theta')*V = (U*\theta*V) \cdot (U*\theta'*V) .$$

(R5) Let $F, G: C \rightarrow C'$; $U, V: C' \rightarrow M$; $\phi: F \rightarrow G$ and $\psi: U \rightarrow V$; then

$$(\psi*G) \cdot (U*\phi) = (V*\phi) \cdot (\psi*F) .$$

Proof:

(13)

$$\begin{aligned}
 (R1) \quad [(UV)*\theta]_0 &= (UV)_1 \theta_0 \\
 &= (U_1 V_1) \theta_0 \\
 &= U_1 (V_1 \theta_0) \\
 &= U_1 (V*\theta)_0 \\
 &= [U*(V*\theta)]_0 .
 \end{aligned}$$

$$\begin{aligned}
 (R2) \quad [\theta*(UV)]_0 &= \theta_0 (UV)_0 \\
 &= \theta_0 (U_0 V_0) \\
 &= (\theta_0 U_0) V_0 \\
 &= (\theta*U)_0 V_0 \\
 &= [(\theta*U)*V]_0 .
 \end{aligned}$$

$$\begin{aligned}
 (R3) \quad (U*\theta*V)_0 &= U_1 \theta_0 V_0 \\
 &= (U_1 \theta_0) V_0 \\
 &= (U*\theta)_0 V_0 \\
 &= [(U*\theta)*V]_0 ;
 \end{aligned}$$

$$\begin{aligned}
 (U*\theta*V)_0 &= U_1 \theta_0 V_0 \\
 &= U_1 (\theta_0 V_0) \\
 &= U_1 (\theta*V)_0 \\
 &= [U*(\theta*V)]_0 .
 \end{aligned}$$

$$\begin{aligned}
 (R4) \quad [U*(\theta*\theta')*V]_0 &= U_1 (\theta*\theta')_0 V_0 \\
 &= U_1 \gamma' \langle \theta_0, \theta'_0 \rangle V_0 \\
 &= c' U_2 \langle \theta_0, \theta'_0 \rangle V_0 \\
 &= c' \langle U_1 \theta_0, U_1 \theta'_0 \rangle V_0 \\
 &= c' \langle U_1 \theta_0 V_0, U_1 \theta'_0 V_0 \rangle \\
 &= c' \langle (U*\theta*V)_0, (U*\theta'*V)_0 \rangle
 \end{aligned}$$

(14)

$$= [(U * \theta * V) \cdot (U * \theta' * V)]_0$$

(R5) Naturality of $\psi: U \rightarrow V$ implies

$$\begin{aligned} c\langle \psi_0 d_1', U_1 \rangle &= c\langle V_1, \psi_0 d_0' \rangle, \\ \Rightarrow c\langle \psi_0 d_1', U_1 \rangle \phi_0 &= c\langle V_1, \psi_0 d_0' \rangle \phi_0 \\ \Rightarrow c\langle \psi_0 d_1' \phi_0, U_1 \phi_0 \rangle &= c\langle V_1 \phi_0, \psi_0 d_0' \phi_0 \rangle \\ \Rightarrow c\langle \psi_0 G_0, U_1 \phi_0 \rangle &= c\langle V_1 \phi_0, \psi_0 F_0 \rangle; \end{aligned}$$

therefore

$$\begin{aligned} [(\psi * G) \cdot (U * \phi)]_0 &= c\langle (\psi * G)_0, (U * \phi)_0 \rangle \\ &= c\langle \psi_0 G_0, U_1 \phi_0 \rangle \\ &= c\langle V_1 \phi_0, \psi_0 F_0 \rangle \\ &= c\langle (V * \phi)_0, (\psi * F)_0 \rangle \\ &= [(V * \phi) \cdot (\psi * F)]_0. \end{aligned}$$

5. Induced Triples

Given a category object C ,

$$x_2 \xrightarrow{\gamma} x_1 \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xleftarrow{\mu} \end{array} x_0,$$

in \underline{E} , we wish to define a triple (monad) (T, η, μ) in (\underline{E}, X_0) ; the context in which the symbol μ is used will always make its meaning clear, as it is used to denote two different things.

We define a (covariant) functor $T: (\underline{E}, X_0) \rightarrow (\underline{E}, X_0)$ as the composite

$$(\underline{E}, X_0) \xrightarrow{d_1^*} (\underline{E}, X_1) \xrightarrow{\Sigma d_0} (\underline{E}, X_0),$$

(15)

where d_1^* and Σ_{d_0} are defined as follows : for any object $\alpha: E_0 \rightarrow X_0$ in (\underline{E}, X_0) , the pullback square

$$\begin{array}{ccc} E_1 & \xrightarrow{\pi_1} & E_0 \\ d_1^*(\alpha) \downarrow & & \downarrow \alpha \\ X_1 & \xrightarrow{d_1} & X_0 \end{array}$$

defines $d_1^*(\alpha)$, and for any map $\beta: \alpha \rightarrow \alpha'$ in (\underline{E}, X_0) , i.e. that the diagram

$$\begin{array}{ccc} E_0 & \xrightarrow{\beta} & E'_0 \\ & \searrow \alpha & \downarrow \alpha' \\ & & X_0 \end{array}$$

commutes , we let $d_1^*(\beta) = \langle \beta \pi_1, d_1^*(\alpha) \rangle$. For any object $\delta: Y \rightarrow X_1$ in (\underline{E}, X_1) , $\Sigma_{d_0}(\delta) = d_0 \delta$. Given a map $\varepsilon: \delta \rightarrow \delta'$ in (\underline{E}, X_1) , we have $\Sigma_{d_0}(\varepsilon) = \varepsilon$ "over X_0 " .

We also require natural transformations $\eta: I \rightarrow T$, $\mu: T^2 \rightarrow T$, such that the conditions

$$(A) \quad \mu(T\mu) = \mu(\mu T) ,$$

and

$$(B) \quad \mu(T\eta) = \mu(\eta T) = T$$

are satisfied , where $T: T \rightarrow T$ denotes the identity natural transformation .

(16)

To define $\eta: I \rightarrow T$, given an object $\alpha: E_0 \rightarrow X_0$ in (\underline{E}, X_0) , we form the pullback square

$$\begin{array}{ccc} E_1 & \xrightarrow{\pi_1} & E_0 \\ \downarrow d_1^*(\alpha) & & \downarrow \alpha \\ X_1 & \xrightarrow{d_1} & X_0 \end{array}$$

and let $\eta_\alpha = \langle E_1, \mu\alpha \rangle$. It is then immediate that $\eta_\alpha: \alpha \rightarrow T(\alpha)$ is a map in (\underline{E}, X_0) . To verify the naturality of η , we let $\beta: \alpha \rightarrow \alpha'$ be a map in (\underline{E}, X_0) , so that the outside of the diagram

$$\begin{array}{ccccc} E_0 & & & & E'_0 \\ & \searrow \beta & & \nearrow T(\beta)\eta_\alpha & \\ & & E_1 & \xrightarrow{\pi'_1} & E'_1 \\ & \searrow \eta_{\alpha', \beta} & \downarrow d_1^*(\alpha') & & \downarrow \alpha' \\ & & X_1 & \xrightarrow{d_1} & X_0 \end{array}$$

$\mu\alpha$ (curved arrow from E_0 to X_1)

commutes, then the equations

$$d_1^*(\alpha')\eta_{\alpha, \beta} = \mu\alpha'\beta$$

$$= \mu\alpha,$$

$$\pi'_1\eta_{\alpha, \beta} = E'_0\beta$$

$$= \beta;$$

and

(17)

$$d_1^*(\alpha') T(\beta) \eta_\alpha = d_1^*(\alpha) \eta_\alpha \quad (\text{by definition of } T(\beta))$$

$$= \mu_\alpha$$

$$\pi_1^! T(\beta) \eta_\alpha = \beta \pi_1 \eta_\alpha \quad (\text{by definition of } T(\beta))$$

$$= \beta E_0$$

$$= \beta ;$$

show (by uniqueness) that the square

$$\begin{array}{ccc} \alpha & \xrightarrow{\eta_\alpha} & T(\alpha) \\ \downarrow \beta & & \downarrow T(\beta) \\ \alpha' & \xrightarrow{\eta_{\alpha'}} & T(\alpha') \end{array}$$

is commutative, which is precisely the naturality of η .

To define $\mu: T^2 \rightarrow T$, given an object $\alpha: E_0 \rightarrow X_0$ in (\underline{E}, X_0) , we form the pullbacks

$$\begin{array}{ccccc} E_2 & \xrightarrow{\pi_2} & E_1 & \xrightarrow{\pi_1} & E_0 \\ \downarrow p_2 & & \downarrow d_1^*(\alpha) & & \downarrow \alpha \\ X_2 & \xrightarrow{e_0} & X_1 & \xrightarrow{d_1} & X_0 \\ \downarrow e_1 & & \downarrow d_0 & & \\ X_1 & \xrightarrow{d_1} & X_0 & & \end{array}$$

and let $\mu_\alpha = \langle \pi_1 \pi_2, \gamma p_2 \rangle$. We then have that

$$\begin{aligned} T(\alpha) \mu_\alpha &= d_0 d_1^*(\alpha) \mu_\alpha \\ &= d_0 \gamma p_2 \end{aligned}$$

(18)

$$\begin{aligned}
 &= d_0 e_1 p_2 \\
 &= d_0 d_1^*(T(\alpha)) \\
 &= T^2(\alpha)
 \end{aligned}$$

so that $\mu_\alpha: T^2(\alpha) \rightarrow T(\alpha)$ is a map in (\underline{E}, X_0) . To show naturality, we require that the diagram

$$\begin{array}{ccc}
 T^2(\alpha) & \xrightarrow{\mu_\alpha} & T(\alpha) \\
 \downarrow T^2(\beta) & & \downarrow T(\beta) \\
 T^2(\alpha') & \xrightarrow{\mu_{\alpha'}} & T(\alpha')
 \end{array}$$

be commutative, where $\beta: \alpha \rightarrow \alpha'$ is a map in (\underline{E}, X_0) . First, we consider the diagram

$$\begin{array}{ccccc}
 E_2 & & & & \\
 \searrow p_2 & \searrow d_1^*(\alpha) \pi_2 & & & \\
 & & X_2 & \xrightarrow{e_0} & X_1 \\
 \searrow p_2' T^2(\beta) & & \downarrow e_1 & & \downarrow d_0 \\
 & & X_1 & \xrightarrow{d_1} & X_0 \\
 \searrow e_1 p_2 & & & &
 \end{array}$$

and note that

$$e_0 p_2 = d_1^*(\alpha) \pi_2 ;$$

and

$$e_0 p_2' T^2(\beta) = d_1^*(\alpha') \pi_2' T^2(\beta)$$

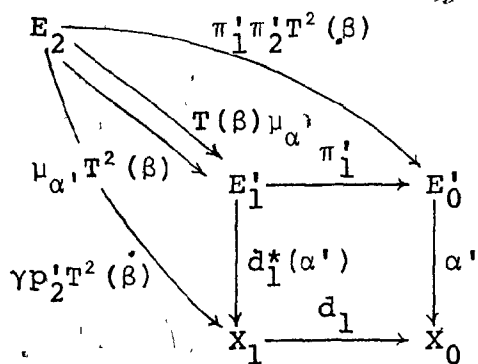
(19)

$$= d_1^*(\alpha') T(\beta) \pi_2 \quad (\text{by definition of } T^2(\beta))$$

$$= d_1^*(\alpha) \pi_2 \quad (\text{by definition of } T(\beta))$$

$$e_1 p_2' T^2(\beta) = e_1 p_2 \quad (\text{by definition of } T^2(\beta)) ;$$

so that uniqueness implies $p_2' T^2(\beta) = p_2$. Next, we consider



where

$$d_1 \gamma p_2' T^2(\beta) = d_1 e_0 p_2' T^2(\beta)$$

$$= d_1 d_1^*(\alpha') \pi_2' T^2(\beta)$$

$$= \alpha' \pi_1' \pi_2' T^2(\beta) ,$$

so that the outside of the diagram commutes ; then the equations

$$d_1^*(\alpha') \mu_{\alpha} T^2(\beta) = \gamma p_2' T^2(\beta) ,$$

$$\pi_1' \mu_{\alpha} T^2(\beta) = \pi_1' \pi_2' T^2(\beta) ;$$

and

$$d_1^*(\alpha') T(\beta) \mu_{\alpha} = d_1^*(\alpha) \mu_{\alpha}$$

$$= \gamma p_2$$

$$= \gamma p_2' T^2(\beta) ,$$

$$\pi_1' T(\beta) \mu_{\alpha} = \beta \pi_1 \mu_{\alpha}$$

$$= \beta \pi_1 \pi_2$$

$$\begin{aligned}
&= \pi_1' T(\beta) \pi_2 \\
&= \pi_1' \pi_2' T^2(\beta) ;
\end{aligned}$$

show (by uniqueness) that $T(\beta)\mu_\alpha = \mu_\alpha T^2(\beta)$.

It remains to verify the triple identities :

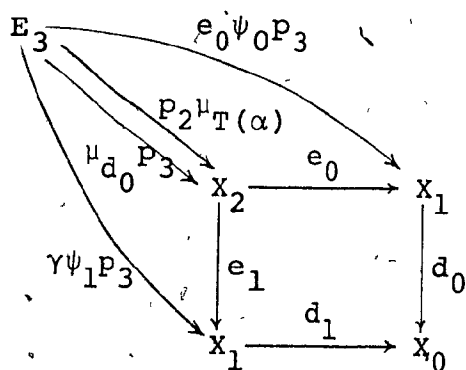
(A) We keep in mind the definition of the category object \tilde{C} in \underline{E} , and note that $\mu_\alpha = \langle \pi_1 \pi_2, \gamma p_2 \rangle$, $(T\mu)_\alpha = T(\mu_\alpha) = \langle \mu_\alpha \pi_3, d_1^*(d_0 \gamma p_2) \rangle$, and $(\mu T)_\alpha = \mu_{T(\alpha)} = \langle \pi_2 \pi_3, \gamma \psi_1 p_3 \rangle$, where π_3 and p_3 are defined through the pullback square

$$\begin{array}{ccc}
E_3 & \xrightarrow{\pi_3} & E_2 \\
\downarrow p_3 & & \downarrow p_2 \\
X_3 & \xrightarrow{\psi_0} & X_2
\end{array}$$

First, we show that the squares

$$\begin{array}{ccc}
E_3 & \xrightarrow{(\mu T)_\alpha} & E_2 \\
\downarrow p_3 & (1) & \downarrow p_2 \\
X_3 & \xrightarrow{\mu d_0} & X_2
\end{array}
, \text{ and }
\begin{array}{ccc}
E_3 & \xrightarrow{(T\mu)_\alpha} & E_2 \\
\downarrow p_3 & (2) & \downarrow p_2 \\
X_3 & \xrightarrow{T(\gamma)} & X_2
\end{array}$$

are commutative. For (1), we consider the diagram



noting that

$$\begin{aligned}
 d_1 \gamma \psi_1 p_3 &= d_1 e_1 p_2 \mu_T(\alpha) \\
 &= d_0 e_0 p_2 \mu_T(\alpha) \\
 &= d_0 d_1^*(\alpha) \pi_2 \mu_T(\alpha) \\
 &= d_0 d_1^*(\alpha) \pi_2 \pi_3 \\
 &= d_0 e_0 p_2 \pi_3 \\
 &= d_0 e_0 \psi_0 p_3,
 \end{aligned}$$

so that the outside of the diagram commutes ; hence the equations

$$\begin{aligned}
 e_1 \mu_{d_0} p_3 &= \gamma \psi_1 p_3, \\
 e_0 \mu_{d_0} p_3 &= e_0 \psi_0 p_3;
 \end{aligned}$$

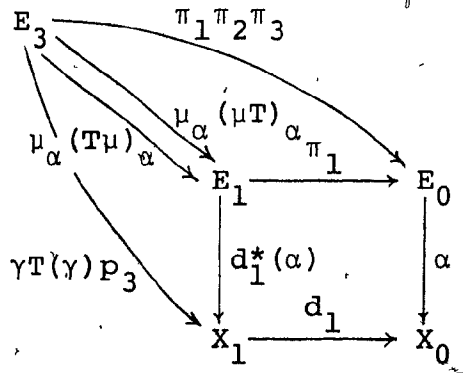
and

$$\begin{aligned}
 e_1 p_2 \mu_T(\alpha) &= \gamma \psi_1 p_3, \\
 e_0 p_2 \mu_T(\alpha) &= d_1^*(\alpha) \pi_2 \mu_T(\alpha) \\
 &= d_1^*(\alpha) \pi_2 \pi_3 \\
 &= e_0 \psi_0 p_3;
 \end{aligned}$$

imply (by uniqueness) that $\mu_{d_0} p_3 = p_2 \mu_T(\alpha)$. The commutativity of (2) may be proved similarly, by the uniqueness property

(22)

of $\langle \gamma \psi_0 p_3, e_1 \psi_1 p_3 \rangle : E_3 \rightarrow X_2$. Next, the outside of the diagram



commutes, as

$$\begin{aligned}
 \alpha \pi_1 \pi_2 \pi_3 &= d_1 d_1^*(\alpha) \pi_2 \pi_3 \\
 &= d_1 e_0 p_2 \pi_3 \\
 &= d_1 e_0 \psi_0 p_3 \\
 &= d_1 \gamma \psi_0 p_3 \\
 &= d_1 e_0 T(\gamma) p_3 \\
 &= d_1 \gamma T(\gamma) p_3,
 \end{aligned}$$

so that the equations

$$\begin{aligned}
 \pi_1 \mu_\alpha (\mu T)_\alpha &= \pi_1 \pi_2 (\mu T)_\alpha \\
 &= \pi_1 \pi_2 \pi_3, \\
 d_1^*(\alpha) \mu_\alpha (\mu T)_\alpha &= \gamma p_2 (\mu T)_\alpha \\
 &= \gamma \mu_{d_0} p_3 \\
 &= \gamma T(\gamma) p_3 \quad (\text{by I.1., condition (4), as}
 \end{aligned}$$

$$\mu_{d_0} = \alpha_0 \text{ and } T(\gamma) = \alpha_1);$$

and

$$\begin{aligned}
\pi_1 \mu_\alpha (T\mu)_\alpha &= \pi_1 \pi_2 (T\mu)_\alpha \\
&= \pi_1 \mu_\alpha \pi_3 \\
&= \pi_1 \pi_2 \pi_3 \\
d_1^*(\alpha) \mu_\alpha (T\mu)_\alpha &= \gamma p_2 (T\mu)_\alpha \\
&= \gamma T(\gamma) p_3 ;
\end{aligned}$$

imply (by uniqueness) the required result :

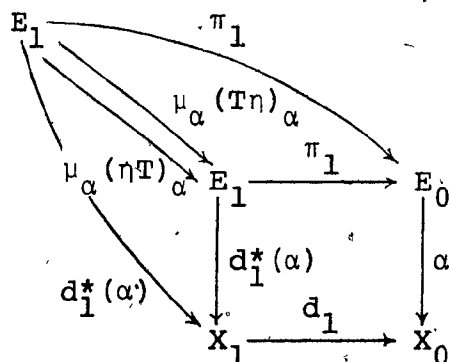
(B) As in (A) , it may be shown , by the universal property of the pullback square

$$\begin{array}{ccc}
x_2 & \xrightarrow{e_0} & x_1 \\
\downarrow e_1 & \searrow d_1 & \downarrow d_0 \\
x_1 & \xrightarrow{\quad} & x_0
\end{array}$$

that the squares

$$\begin{array}{ccc}
E_1 & \xrightarrow{(T\eta)_\alpha} & E_2 \\
\downarrow d_1^*(\alpha) & & \downarrow p_2 \\
X_1 & \xrightarrow{T(\mu)} & X_2
\end{array}
, \text{ and }
\begin{array}{ccc}
E_1 & \xrightarrow{(\eta T)_\alpha} & E_2 \\
\downarrow d_1^*(\alpha) & & \downarrow p_2 \\
X_1 & \xrightarrow{\eta_{d_0}} & X_2
\end{array}$$

are commutative , so that condition (3) in the definition of the category object C in \underline{E} , together with the fact that $T(\mu) = \phi_1$ and $\eta_{d_0} = \phi_0$, implies (by uniqueness) the required result , upon considering the diagram



6. Simplicial Objects

Given a category object C ,

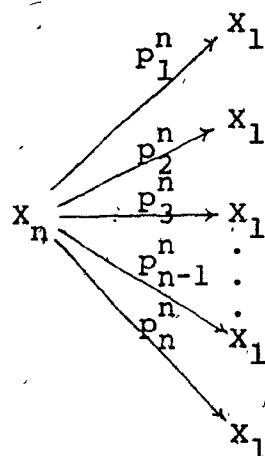
$$x_2 \xrightarrow{\gamma} x_1 \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xleftarrow{\mu} \end{array} x_0,$$

in \underline{E} , we wish to extend it to a simplicial object in \underline{E} .

The limit of the diagram (functor)

$$\begin{array}{ccc} x_1 & \xrightarrow{d_1} & x_0 \\ & \searrow d_0 & \\ x_1 & \xrightarrow{d_1} & x_0 \\ & \searrow d_0 & \\ x_1 & \xrightarrow{d_1} & x_0 \\ & \searrow d_0 & \\ \vdots & & \\ x_1 & \xrightarrow{d_1} & x_0 \\ & \searrow d_0 & \\ x_1 & \xrightarrow{d_1} & x_0 \end{array}$$

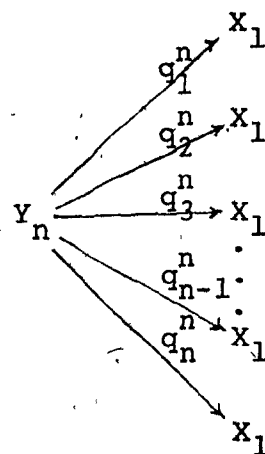
where x_1 occurs n times, ($n \geq 2$), consists of objects and maps



in \underline{E} , satisfying

$$d_1 p_i^n = d_0 p_{i+1}^n, \quad (1 \leq i \leq n-1),$$

with the universal property that for any other objects and maps in \underline{E} ,



satisfying

$$d_1 q_i^n = d_0 q_{i+1}^n, \quad (1 \leq i \leq n-1),$$

there exists a unique map $\psi_n: Y_n \rightarrow X_n$ in \underline{E} , such that

$$p_i^n \psi_n = q_i^n, \quad (1 \leq i \leq n).$$

Such a map ψ_n will henceforth be denoted by $\langle q_1^n, q_2^n, \dots, q_n^n \rangle$.

We note that the above definition of the objects X_n agrees, for $n = 2$, with that of X_2 which appears in the definition of \mathcal{C} , since pullback is a special case of finite limit.

We define maps $d_i^n: X_n \rightarrow X_{n-1}$, $n \geq 1$, $0 \leq i \leq n$, by

$$d_0^1 = d_0,$$

$$d_1^1 = d_1,$$

$$d_0^2 = e_1,$$

$$d_1^2 = \gamma,$$

$$d_2^2 = e_0,$$

and for $n \geq 3$,

$$d_i^n = \langle p_1^n, p_2^n, \dots, p_{i-1}^n, \gamma \langle p_i^n, p_{i+1}^n \rangle, p_{i+2}^n, \dots, p_n^n \rangle, \quad (1 \leq i \leq n-1),$$

$$d_0^n = \langle p_2^n, p_3^n, \dots, p_n^n \rangle,$$

$$d_n^n = \langle p_1^n, p_2^n, \dots, p_{n-2}^n, p_{n-1}^n \rangle;$$

and maps $s_i^n: X_n \rightarrow X_{n+1}$, $n \geq 0$, $0 \leq i \leq n$, by

$$s_0^0 = \mu,$$

$$s_0^1 = \phi_1,$$

$$s_1^1 = \phi_0,$$

and for $n \geq 2$,

$$s_i^n = \langle p_1^n, p_2^n, \dots, p_i^n, \mu d_1 p_i^n, p_{i+1}^n, \dots, p_n^n \rangle, \\ (1 \leq i \leq n),$$

$$s_0^n = \langle \mu d_0 p_1^n, p_1^n, p_2^n, \dots, p_n^n \rangle.$$

This yields an infinite sequence S of objects and maps in \underline{E} , and it remains to prove that the simplicial identities hold, i.e. that

$$(1) \quad d_i^n d_{j+1}^{n+1} = d_j^n d_i^{n+1}, \quad i \leq j \leq n;$$

$$(2) \quad s_{j+1}^n s_i^{n-1} = s_i^n s_j^{n-1}, \quad i \leq j \leq n-1;$$

$$(3) \quad d_i^n s_j^{n-1} = \begin{cases} s_{j-1}^{n-2} d_i^{n-1}, & i < j \\ 1, & i = j, j+1 \\ s_j^{n-2} d_{i-1}^{n-1}, & j+1 < i \leq n \end{cases}, \quad j \leq n-1.$$

We first note that if \underline{E} is the category of sets, then X_n is the set of sequences (of length n) of composable arrows (of the category \mathcal{C}). Given a typical such sequence $\alpha = \{\rightarrow \rightarrow \rightarrow \dots \rightarrow\}$ in X_n , its image under d_i^n , for $1 \leq i \leq n-1$, is the sequence derived from α by replacing the i -th and $(i+1)$ -st arrows in α by their composite, hence yielding a sequence of length $n-1$; d_0^n and d_n^n delete the first and last arrows (respectively) in α ; s_i^n , for $1 \leq i \leq n$, inserts an identity arrow between the i -th and $(i+1)$ -st arrows in α ; s_0^n inserts an identity arrow before the first arrow in α . With this concrete description, it is now easy to see that the simplicial identities are satisfied (in Set). Next, for any

object X in \underline{E} , the hom-functor $(X, -): \underline{E} \rightarrow \underline{\text{Set}}$ associates to any object Y in \underline{E} the set (X, Y) of morphisms (in \underline{E}) from X to Y . It associates to any morphism $f: Y \rightarrow Z$ in \underline{E} the morphism (in $\underline{\text{Set}}$) $(X, f): (X, Y) \rightarrow (X, Z)$, such that $(X, f)(g) = fg$ for any $g \in (X, Y)$. It is well known that this hom-functor preserves limits, so its application to the objects and maps which comprise S (its construction being exclusively through limits) results in a sequence (X, S) of objects and maps in $\underline{\text{Set}}$, which is precisely the extension (via the above method) of the category object

$$(X, X_2) \xrightarrow{(X, \gamma)} (X, X_1) \begin{array}{c} \xrightarrow{(X, d_0)} \\ \xrightarrow{(X, d_1)} \\ \xleftarrow{(X, u_1)} \end{array} (X, X_0)$$

in $\underline{\text{Set}}$. It is now immediate that the simplicial identities hold for (X, S) . This being true for any object X in \underline{E} , the Yoneda Lemma implies that S is a simplicial object in \underline{E} .

Remark. Given a category object C in \underline{E} and a (covariant) functor $A: \underline{E} \rightarrow \underline{\text{Ab}}$, where $\underline{\text{Ab}}$ denotes the category of Abelian groups, let S be the extension of C to a simplicial object in \underline{E} . Then $A(S)$ (with the obvious meaning) is a simplicial object in $\underline{\text{Ab}}$, hence we may define maps $d_n: A(X_n) \rightarrow A(X_{n-1})$, $n \geq 1$, by

$$d_n = \sum_{i=0}^n (-1)^i A(d_i^n),$$

and it follows easily by (1) of the simplicial identities that $d_n d_{n+1} = 0$, ($n \geq 1$), so that $\{A(X_n)\}_{n \geq 0}$ and $\{d_n\}_{n \geq 1}$

(29)

define a chain complex in $\underline{\text{Ab}}$. Finally, we define the n -th homology object (group) $H_n(C, A)$ in $\underline{\text{Ab}}$ by

$$H_n(C, A) = \ker(d_n) / \text{im}(d_{n+1}).$$

The above definition, of course, is possible when $\underline{\text{Ab}}$ is replaced by any Abelian category.

II. INTERNAL FUNCTORS

1. Internal Functors and Natural Transformations

Definition. Given a category object C ,

$$x_2 \xrightarrow{\gamma} x_1 \xrightarrow[\mu]{\begin{matrix} d_0 \\ d_1 \end{matrix}} x_0$$

in \underline{E} , an internal functor $F:C \rightarrow \underline{E}$ consists of maps $\pi_0:E_0 \rightarrow X_0$

and $\delta_1:E_1 \rightarrow E_0$ in \underline{E} , where

(1) The square

$$\begin{array}{ccc} E_1 & \xrightarrow{\delta_0} & E_0 \\ \pi_1 \downarrow & & \downarrow \pi_0 \\ X_1 & \xrightarrow{d_0} & X_0 \end{array}$$

is a pullback and

(2) The square

$$\begin{array}{ccc} E_1 & \xrightarrow{\delta_1} & E_0 \\ \pi_1 \downarrow & & \downarrow \pi_0 \\ X_1 & \xrightarrow{d_1} & X_0 \end{array}$$

is commutative, satisfying the following additional conditions:

(3) Let $m = \langle \mu\pi_0, E_0 \rangle : E_0 \rightarrow E_1$ be defined through the pullback square in (1), then we require that $\delta_1 m = E_0$;

(4) Let $\varepsilon_1 : E_2 \rightarrow E_1$ be defined by the pullback square

$$\begin{array}{ccc} E_2 & \xrightarrow{\varepsilon_1} & E_1 \\ \pi_2 \downarrow & & \downarrow \pi_1 \\ X_2 & \xrightarrow{e_1} & X_1 \end{array}$$

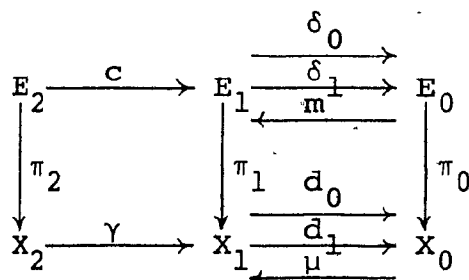
and $c = \langle \gamma\pi_2, \delta_0\varepsilon_1 \rangle$, $\varepsilon_0 = \langle e_0\pi_2, \delta_1\varepsilon_1 \rangle$, then we require that $\delta_1 c = \delta_1 \varepsilon_0$.

If \underline{E} is the category of sets (and so \mathcal{C} is an ordinary category), then F is a (covariant) set-valued functor:

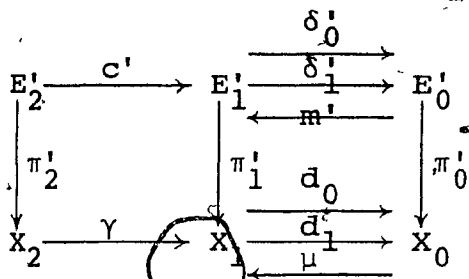
$E_0 = \dot{\bigcup}_{x_0 \in X_0} \pi_0^{-1}(\{x_0\})$ (disjoint union); by condition (1), $E_1 = \{(f, x) \in X_1 \times E_0 \mid d_0(f) = \pi_0(x)\} \doteq \{(f, x) \in X_1 \times E_0 \mid x \in \pi_0^{-1}(\{d_0(f)\})\}$; for $(f, x) \in E_1$, $F(f)(x) \doteq \delta_1(f, x)$, so condition (2) states that $\pi_0[F(f)(x)] = d_1(f)$, i.e. that $F(f)(x) \in \pi_0^{-1}(\{d_1(f)\})$, so that $F(f) : F(d_0(f)) \rightarrow F(d_1(f))$; condition (3) requires that for any object x_0 in \mathcal{C} , $F(\text{id}_{x_0}) = \text{id}_{F(x_0)}$; condition (4) states the preservation of composition, because $E_2 = \{((f, g), (h, x)) \in X_2 \times E_1 \mid g = h\} \doteq \{(f, g, x) \in X_1 \times X_1 \times E_0 \mid (f, g) \in X_2, (g, x) \in E_1\}$, so for $(f, g, x) \in E_2$, we have $F(fg)(x) = \delta_1(fg, x) = \delta_1 c(f, g, x) = \delta_1 \varepsilon_0(f, g, x) = \delta_1(f, \delta_1(g, x)) = F(f)[F(g)(x)]$.

A contravariant internal functor may be similarly defined, by interchanging δ_0 and δ_1 , d_0 and d_1 , ε_0 and ε_1 , and e_0 and e_1 in the above definition.

Definition. Given two internal functors $F, F': C \rightarrow \underline{E}$, best described by the diagrams



and



respectively, an internal natural transformation $\tau: F \rightarrow F'$ consists of a map $\tau_0: E_0 \rightarrow E'_0$ in \underline{E} , with

$$(1) \quad \pi'_0 \tau_0 = \pi_0,$$

such that if we let $\tau_1 = \langle \pi_1, \tau_0 \delta_0 \rangle$, the condition

$$(2) \quad \delta'_1 \tau_1 = \tau_0 \delta_1$$

is satisfied.

Whenever there is no danger of ambiguity, we shall omit the word "internal".

If \underline{E} is the category of sets (and so F and F' are ordinary functors $\underline{C} \rightarrow \underline{Set}$), then due to condition (1), $\tau_0: \bigcup_{x_0 \in X_0} \pi_0^{-1}(\{x_0\}) \rightarrow \bigcup_{x_0 \in X_0} (\pi'_0)^{-1}(\{x_0\})$ has the property that the image of its restriction to $\pi_0^{-1}(\{x_0\})$ is contained in $(\pi'_0)^{-1}(\{x_0\})$, so that τ associates to each object x_0 in \underline{C} a map $\tau_0|_{F(x_0)}: F(x_0) \rightarrow F'(x_0)$ in \underline{Set} ; condition (2) states naturality, for given a map $f: d_0(f) \rightarrow d_1(f)$ in \underline{C} and $x \in F(d_0(f))$ (so that $(f, x) \in E_1$), $F'(f)\tau_0(x) = \delta'_1(f, \tau_0(x)) = \delta'_1\tau_1(f, x) = \tau_0\delta_1(f, x) = \tau_0F(f)(x)$, which is precisely the commutativity of the diagram

$$\begin{array}{ccc} F(d_0(f)) & \xrightarrow{\tau_{d_0(f)}} & F'(d_0(f)) \\ \downarrow F(f) & & \downarrow F'(f) \\ F(d_1(f)) & \xrightarrow{\tau_{d_1(f)}} & F'(d_1(f)) \end{array}$$

(where $\tau_{d_i(f)} \equiv \tau_0|_{F(d_i(f))}$, $i = 0, 1$), i.e. the naturality of τ .

It is immediate from the definition that two natural transformations $\tau, \tau': F \rightarrow F'$ are equal iff $\tau_0 = \tau'_0$.

If $F, F': \underline{C} \rightarrow \underline{E}$ are contravariant, then the above definition is valid with δ_0 and δ_1 interchanged.

Given (covariant) internal functors $F, F', F'': \underline{C} \rightarrow \underline{E}$, and natural transformations $\tau: F \rightarrow F'$, $\tau': F' \rightarrow F''$, we define the composite $\tau'\tau: F \rightarrow F''$ by $(\tau'\tau)_0 = \tau'_0\tau_0$. It can then easily be

shown that $(\tau'\tau)_1 = \tau'_1\tau_1$, and that conditions (1) and (2) in the definition are satisfied.

We denote by \underline{E}^C the category which has as objects (covariant) internal functors $F:C \rightarrow \underline{E}$, and as morphisms natural transformations of such functors.

2. Characterisation as Discrete Cofibrations

Let $(\text{Cat}(\underline{E}), C)$ denote the comma category whose objects are maps (functors) with codomain C , of category objects in \underline{E} . The morphisms of this category are, of course, maps (functors) of category objects (in \underline{E}) "over C ". Let $\text{DCF}(\underline{E}, C)$ denote the full subcategory of $(\text{Cat}(\underline{E}), C)$ with objects those functors $F:C' \rightarrow C$,

$$\begin{array}{ccccc}
 & & d'_0 & & \\
 & & \longrightarrow & & \\
 x'_2 & \xrightarrow{\gamma'} & x'_1 & \xrightarrow{d'_1} & x'_0 \\
 \downarrow F_2 & & \downarrow F_1 & \xleftarrow{\mu'} & \downarrow F_0 \\
 x_2 & \xrightarrow{\gamma} & x_1 & \xrightarrow{d_1} & x_0 \\
 & & \xleftarrow{\mu} & &
 \end{array}$$

for which the square

$$\begin{array}{ccc}
 x'_1 & \xrightarrow{d'_0} & x'_0 \\
 \downarrow F_1 & & \downarrow F_0 \\
 x_1 & \xrightarrow{d_0} & x_0
 \end{array}$$

is a pullback . When \underline{E} is the category of sets , this reduces to the ordinary notion of a discrete cofibration : $F:C' \rightarrow C$ becomes an ordinary functor between categories , and the requirement that the square above be a pullback is precisely the condition that for any object C' in C' and a map $\phi:F(C') \rightarrow D$ in C , there should exist a unique map $\psi:C' \rightarrow D'$ in C' , such that $F(\psi) = \phi$.

We shall now prove that the categories \underline{E}^C and $\underline{DCF}(\underline{E}, C)$ are equivalent .

Given an internal functor $F:C \rightarrow \underline{E}$,

$$\begin{array}{ccccc}
 & \xrightarrow{c} & & \xrightarrow{\delta_0} & \\
 E_2 & \xrightarrow{\varepsilon_0} & E_1 & \xrightarrow{\delta_1} & E_0 \\
 & \xrightarrow{\varepsilon_1} & & \xleftarrow{m} & \\
 \downarrow \pi_2 & \xrightarrow{\gamma} & \downarrow \pi_1 & \xrightarrow{d_0} & \downarrow \pi_0 \\
 X_2 & \xrightarrow{e_0} & X_1 & \xrightarrow{d_1} & X_0 \\
 & \xrightarrow{e_1} & & \xleftarrow{\mu} &
 \end{array}$$

we must first show that the objects and maps

$$\begin{array}{ccccc}
 & \xrightarrow{c} & & \xrightarrow{\delta_0} & \\
 E_2 & \xrightarrow{\varepsilon_0} & E_1 & \xrightarrow{\delta_1} & E_0 \\
 & \xrightarrow{\varepsilon_1} & & \xleftarrow{m} &
 \end{array}$$

define a category object in \underline{E} , i.e. that conditions (1) to (4) in the definition are satisfied (see I.1. , definition of category object) .

(1) $\delta_0 m = \delta_1 m = E_0$ is immediate from condition (3) in the definition of $F:C \rightarrow \underline{E}$.

(2) The square

$$\begin{array}{ccc}
 E_2 & \xrightarrow{\epsilon_0} & E_1 \\
 \downarrow \epsilon_1 & & \downarrow \delta_0 \\
 E_1 & \xrightarrow{\delta_1} & E_0
 \end{array}$$

commutes by definition of ϵ_0 . To show it is a pullback, let $f, g: Y \rightarrow E_1$ have the property that $\delta_0 f = \delta_1 g$, then

$$\begin{aligned}
 \pi_0 \delta_0 f &= \pi_0 \delta_1 g, \\
 \Rightarrow d_0 \pi_1 f &= d_1 \pi_1 g,
 \end{aligned}$$

so that we have $\langle \pi_1 f, \pi_1 g \rangle: Y \rightarrow X_2$; but now $\langle \langle \pi_1 f, \pi_1 g \rangle, g \rangle: Y \rightarrow E_2$ satisfies

$$(a) \quad \epsilon_1 \langle \langle \pi_1 f, \pi_1 g \rangle, g \rangle = g,$$

and

$$(b) \quad \epsilon_0 \langle \langle \pi_1 f, \pi_1 g \rangle, g \rangle = f;$$

for (a) follows by definition of $\langle \langle \pi_1 f, \pi_1 g \rangle, g \rangle$, and (b) is due to the fact that both f and $\epsilon_0 \langle \langle \pi_1 f, \pi_1 g \rangle, g \rangle$ are equal to $\langle \pi_1 f, \delta_1 g \rangle: Y \rightarrow E_1$, the square

$$\begin{array}{ccc}
 E_1 & \xrightarrow{\delta_0} & E_0 \\
 \downarrow \pi_1 & & \downarrow \pi_0 \\
 X_1 & \xrightarrow{d_0} & X_0
 \end{array}$$

being a pullback. To show that $\langle \langle \pi_1 f, \pi_1 g \rangle, g \rangle$ is unique with

respect to the properties (a) and (b) ; suppose $\xi: Y \rightarrow E_2$ satisfies $\varepsilon_1 \xi = g$ and $\varepsilon_0 \xi = f$; then it is easily seen that $\pi_2 \xi = \langle \pi_1 f, \pi_1 g \rangle$, which shows that $\xi = \langle \langle \pi_1 f, \pi_1 g \rangle, g \rangle$.

(i) $\delta_0 c = \delta_0 \varepsilon_1$ follows from the definition of c ;

(ii) $\delta_1 c = \delta_1 \varepsilon_0$ is condition (4) in the definition of $F: C \rightarrow E$.

(3) It is required to prove that $c\bar{\phi}_0 = c\bar{\phi}_1 = E_1$, where $\bar{\phi}_0 = \langle E_1, m\delta_0 \rangle$ and $\bar{\phi}_1 = \langle m\delta_1, E_1 \rangle$ are defined through the pullback square

$$\begin{array}{ccc} E_2 & \xrightarrow{\varepsilon_0} & E_1 \\ \varepsilon_1 \downarrow & & \downarrow \delta_0 \\ E_1 & \xrightarrow{\delta_1} & E_0 \end{array}$$

First , the uniqueness property of $\langle \pi_1, \pi_1 m\delta_0 \rangle: E_1 \rightarrow X_2$, together with the equations

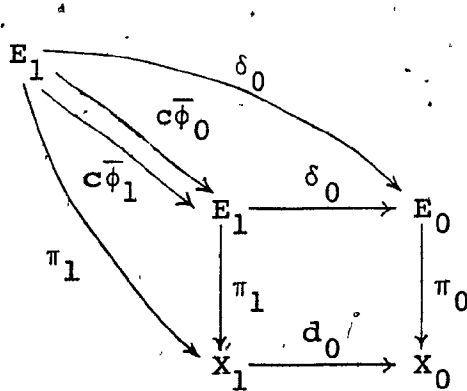
$$\begin{aligned} e_0 \phi_0 \pi_1 &= \pi_1 , \\ e_1 \phi_0 \pi_1 &= m\delta_0 \pi_1 \\ &= \mu \pi_0 \delta_0 \\ &= \pi_1 m\delta_0 ; \end{aligned}$$

and

$$\begin{aligned} e_0 \pi_2 \bar{\phi}_0 &= \pi_1 \varepsilon_0 \bar{\phi}_0 \\ &= \pi_1 , \\ e_1 \pi_2 \bar{\phi}_0 &= \pi_1 \varepsilon_1 \bar{\phi}_0 \\ &= \pi_1 m\delta_0 ; \end{aligned}$$

shows that $\phi_0 \pi_1 = \pi_2 \bar{\phi}_0$. Similarly , we have $\phi_1 \pi_1 = \pi_2 \bar{\phi}_1$.

Next, the diagram



together with the equations

$$\begin{aligned}\delta_0 c\bar{\phi}_0 &= \delta_0 \epsilon_1 \bar{\phi}_0 \\ &= \delta_0 m \delta_0 \\ &= \delta_0 ,\end{aligned}$$

$$\begin{aligned}\pi_1 c\bar{\phi}_0 &= \gamma \pi_2 \bar{\phi}_0 \\ &= \gamma \phi_0 \pi_1 \\ &= \pi_1 ;\end{aligned}$$

and

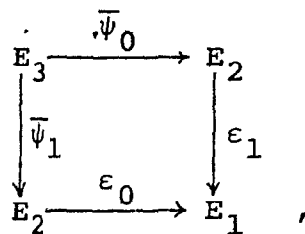
$$\begin{aligned}\delta_0 c\bar{\phi}_1 &= \delta_0 \epsilon_1 \bar{\phi}_1 \\ &= \delta_0 , \\ \pi_1 c\bar{\phi}_1 &= \gamma \pi_2 \bar{\phi}_1 \\ &= \gamma \phi_1 \pi_1 \\ &= \pi_1 ;\end{aligned}$$

shows (by uniqueness) that $c\bar{\phi}_0 = c\bar{\phi}_1 = E_1$.

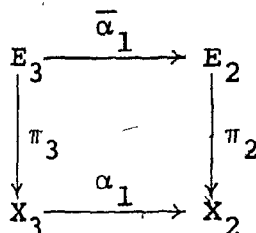
(4) Defining $\bar{\alpha}_0 = \langle \epsilon_0 \bar{\psi}_0, c\bar{\psi}_1 \rangle$ and $\bar{\alpha}_1 = \langle c\bar{\psi}_0, \epsilon_1 \bar{\psi}_1 \rangle$, where

$\bar{\psi}_0$ and $\bar{\psi}_1$ are given by the pullback square

(39)

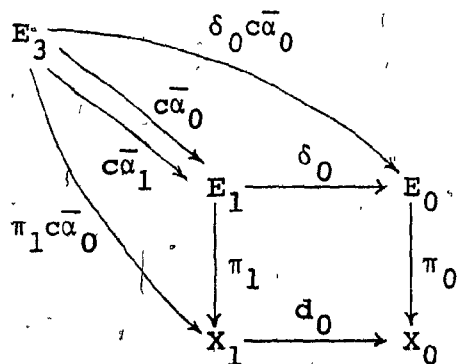


it is required to show that $c\bar{\alpha}_0 = c\bar{\alpha}_1$. First, we let $\pi_3 = \langle \pi_2 \bar{\psi}_0, \pi_2 \bar{\psi}_1 \rangle : E_3 \rightarrow X_3$, so that the square



commutes by the uniqueness property of $\langle \gamma \pi_2 \bar{\psi}_0, e_1 \pi_2 \bar{\psi}_1 \rangle : E_3 \rightarrow X_2$.

Next, the diagram



together with the equations

$$\begin{aligned}
 \delta_0 c\bar{\alpha}_1 &= \delta_0 \epsilon_1 \bar{\alpha}_1 \\
 &= \delta_0 \epsilon_1 \bar{\psi}_1
 \end{aligned}$$

(40)

$$\begin{aligned}
 &= \delta_0 c \bar{\psi}_1 \\
 &= \delta_0 \varepsilon_1 \bar{\alpha}_0 \\
 &= \varepsilon_0 c \bar{\alpha}_0,
 \end{aligned}$$

and

$$\begin{aligned}
 \pi_1 c \bar{\alpha}_1 &= \gamma \pi_2 \bar{\alpha}_1 \\
 &= \gamma \alpha_1 \pi_3 \\
 &= \gamma \alpha_0 \pi_3 \\
 &= \gamma \pi_2 \bar{\alpha}_0 \\
 &= \pi_1 c \bar{\alpha}_0,
 \end{aligned}$$

shows (by uniqueness) that $c \bar{\alpha}_0 = c \bar{\alpha}_1$. This completes the proof that

$$\begin{array}{ccccc}
 & \xrightarrow{c} & & \xrightarrow{\delta_0} & \\
 E_2 & \xrightarrow{\varepsilon_0} & E_1 & \xrightarrow{\delta_1} & E_0 \\
 & \xrightarrow{\varepsilon_1} & & \xleftarrow{m_1} &
 \end{array}$$

defines a category object F in \underline{E} , and it follows immediately from the definition of $F: C \rightarrow \underline{E}$ that (π_0, π_1, π_2) defines a map (functor) of category objects $E \rightarrow C$.

Given two internal functors $F, F': C \rightarrow \underline{E}$ and a map $\tau: F \rightarrow F'$ in \underline{E}^C , i.e. a natural transformation, we show that τ defines a map in $\underline{DCF}(\underline{E}, C)$, i.e. a map in $(\underline{Cat}(\underline{E}), C)$. τ consists of maps $\tau_0: E_0 \rightarrow E'_0$ and $\tau_1: E_1 \rightarrow E'_1$ (where all primed symbols will refer to F'); and letting $\tau_2 = \langle \pi_2, \tau_1 \varepsilon_1 \rangle: E_2 \rightarrow E'_2$, we must verify conditions (1) to (4) (as given in I.2.), so that (τ_0, τ_1, τ_2) will define a map (functor) of category objects $E \rightarrow E'$.

(1) $\delta'_0 \tau_1 = \tau_0 \delta_0$ follows by definition of τ_1 .

(2) $\delta'_1 \tau_1 = \tau_0 \delta_1$ is condition (2) in the definition of $\tau: F \rightarrow F'$.

(3) $m' \tau_0 = \tau_1 m$ follows by the uniqueness property of

$$\langle \mu \pi'_0 \tau_0, \tau_0 \rangle: E_0 \rightarrow E'_1.$$

(4) $c' \tau_2 = \tau_1 c$ follows by the uniqueness property of

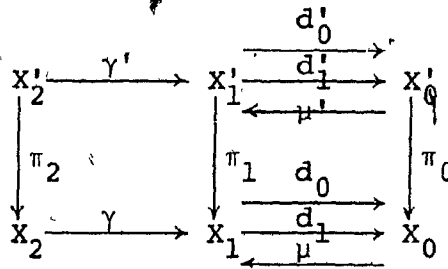
$$\langle \pi_1 c, \tau_0 \delta_0 c \rangle: E_2 \rightarrow E'_1.$$

That $(\tau_0, \tau_1, \tau_2): E \rightarrow E'$ is "over C " is immediate from condition

(1) in the definition of $\tau: F \rightarrow F'$, and the definitions of τ_1

and τ_2 .

Conversely, given an object $(\pi_0, \pi_1, \pi_2): C' \rightarrow C$,



in $\underline{DCF}(\underline{E}, C)$, we show that (d'_1, π_0) defines an internal functor

$F: C \rightarrow E$: conditions (1) and (2) (in the definition of an

internal functor) are trivially satisfied. We have $\mu' =$

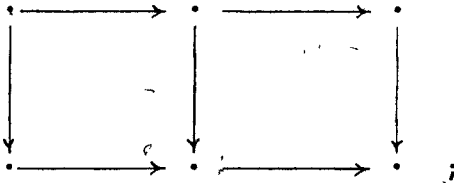
$\langle X'_0, \mu \pi_0 \rangle$ and $d'_1 \mu' = X'_0$, so that condition (3) is satisfied.

To verify (4), we shall need the following result*:

Consider the commutative diagram

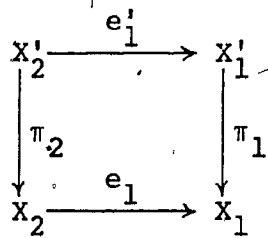
* Saunders Mac Lane: Categories for the Working Mathematician, exercise 8(b) on page 72.

(42)

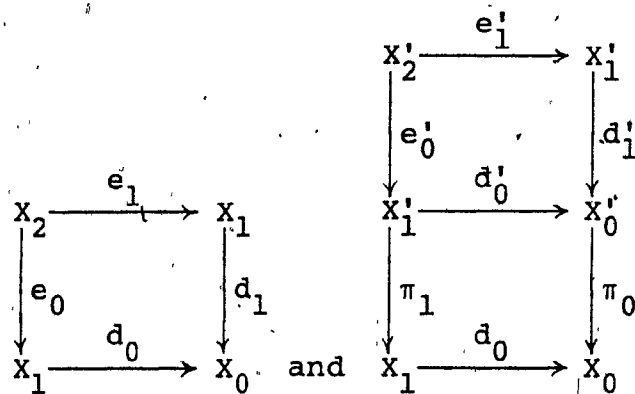


if the outside rectangle and the right hand square are pullbacks , so is the left hand square .

It is first required to show that the square



is a pullback , and this follows by the above , noting that the diagrams



are pullbacks , and that $e_0\pi_2 = \pi_1e'_1$ and $d_1\pi_1 = \pi_0d'_1$. It is

(43)

then easily seen that $\gamma' = \langle d_0' e_1', \gamma \pi_2 \rangle$ and $e_0' = \langle d_1' e_1', e_0' \pi_2 \rangle$, so that $d_1' \gamma' = d_1' e_0'$ (since C' is a category object), i.e. that condition (4) is satisfied.

Given a map in $\underline{DCF}(\underline{E}, C)$, it is easy to show that it gives rise to an internal natural transformation.

3. Characterisation as Algebras of a Triple

Given a contravariant internal functor (CIF) $F: C \rightarrow \underline{E}$,

$$\begin{array}{ccccc}
 E_2 & \xrightarrow{C} & E_1 & \xrightarrow[\eta_1]{\delta_0'} & E_0 \\
 \downarrow \pi_2 & & \downarrow \pi_1 & \xleftarrow{m} & \downarrow \pi_0 \\
 X_2 & \xrightarrow{\gamma} & X_1 & \xrightarrow[\mu]{d_0} & X_0
 \end{array}$$

we claim that (π_0, δ_0) defines an algebra of the triple (T, η, μ) in (\underline{E}, X_0) induced by C , (see I.5.) . Conditions (1) and (2) in the definition of a CIF assure us that $\delta_0: T(\pi_0) \rightarrow \pi_0$ is a morphism in (\underline{E}, X_0) . The algebra identities

$$(1) \quad \delta_0 \eta_{\pi_0} = \text{id}_{\pi_0}$$

$$(2) \quad \delta_0 T(\delta_0) = \delta_0 \mu_{\pi_0}$$

must now be verified.

(1) It is immediate from the definitions of η_{π_0} and m that $\eta_{\pi_0} = m$, hence condition (3) in the definition of a CIF shows that $\delta_0 \eta_{\pi_0} = \text{id}_{\pi_0}$.

(2) Again, the definitions of c , ε_1 , μ_{π_0} and $T(\delta_0)$ show that $\mu_{\pi_0} = c$ and $T(\delta_0) = \varepsilon_1$, so the required result follows by condition (4) in the definition of a CIF.

Now given two CIF's $F, F': C \rightarrow \underline{E}$ and a natural transformation $\tau: F \rightarrow F'$, we claim that $\pi_0: E_0 \rightarrow E'_0$ defines a morphism $(\pi_0, \delta_0) \rightarrow (\pi'_0, \delta'_0)$ of the algebras of (T, η, μ) corresponding to F and F' . Indeed, the commutativity of the square

$$\begin{array}{ccc} T(\pi_0) & \xrightarrow{T(\tau_0)} & T(\pi'_0) \\ \downarrow \delta_0 & & \downarrow \delta'_0 \\ \pi_0 & \xrightarrow{\tau_0} & \pi'_0 \end{array}$$

follows by condition (2) in the definition of a natural transformation (of CIF's) and the fact that $T(\tau_0) = \tau_1$ (clear from their definitions).

Conversely, given an algebra (π_0, δ_0) of (T, η, μ) , there exist objects E_0, E_1 in \underline{E} such that the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\delta_0} & E_0 \\ \downarrow \pi_1 & & \downarrow \pi_0 \\ X_1 & \xrightarrow{d_0} & X_0 \end{array}$$

is commutative (since $\delta_0: T(\pi_0) \rightarrow \pi_0$ is a morphism in (\underline{E}, X_0)),

where the square

$$\begin{array}{ccc} E_1 & \xrightarrow{\delta_1} & E_0 \\ \pi_1 \downarrow & & \downarrow \pi_0 \\ X_1 & \xrightarrow{d_1} & X_0 \end{array}$$

is a pullback, with $\pi_1 = d_1^*(\pi_0)$. It is now very easy to verify that $\pi_0: E_0 \rightarrow X_0$ and $\delta_0: E_1 \rightarrow E_0$ define a CIF $C \rightarrow \underline{E}$, so the details will be omitted.

Given two algebras (π_0, δ_0) and (π'_0, δ'_0) of (T, η, μ) and a morphism $\tau_0: (\pi_0, \delta_0) \rightarrow (\pi'_0, \delta'_0)$ of these algebras, i.e. that the diagrams

$$\begin{array}{ccc} E_0 & \xrightarrow{\tau_0} & E'_0 \\ & \searrow \pi_0 & \downarrow \pi'_0 \\ & & X_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} T(\pi_0) & \xrightarrow{T(\tau_0)} & T(\pi'_0) \\ \downarrow \delta_0 & & \downarrow \delta'_0 \\ \pi_0 & \xrightarrow{\tau_0} & \pi'_0 \end{array}$$

are commutative, it is again very easy to verify that τ_0 defines a natural transformation $\tau: F \rightarrow F'$, where F and F' are the CIF's corresponding to (π_0, δ_0) and (π'_0, δ'_0) respectively.

The above discussion amounts to an equivalence of the categories $\underline{E}^{C^{op}}$ and $(\underline{E}, X_0)^T$, where $\underline{E}^{C^{op}}$ denotes the category of CIF's $C \rightarrow \underline{E}$, and $(\underline{E}, X_0)^T$ the category of algebras

of the triple $T = (T, \eta, \mu)$ in (\underline{E}, X_0) .

Remark. A category object C in \underline{E} also induces the triple $T' = (T', \eta', \mu')$ in (\underline{E}, X_0) where $T' = \Sigma_{d_1} d_0^*$. In this case, we get an equivalence of the categories \underline{E}^C and $(\underline{E}, X_0)^{T'}$. The preference shown for discussing the contravariant case is due to the frequency of occurrence of CIF's (also called (contra)variant internal presheaves) in the above context.

III. LIMITS AND COLIMITS

1. The "Constant" Functor

Given categories \underline{J} and \underline{C} (\underline{J} small), one has the notion of a "constant" functor $\Delta: \underline{C} \rightarrow \underline{C}^{\underline{J}}$: for any object C in \underline{C} , $\Delta(C)(J) = C$ (for any object J in \underline{J}), and if $f: J \rightarrow J'$ is a map in \underline{J} , then $\Delta(C)(f) = \text{id}_C$. If $x: C \rightarrow C'$ is a map in \underline{C} , then $\Delta(x): \Delta(C) \rightarrow \Delta(C')$ is a natural transformation (morphism in $\underline{C}^{\underline{J}}$) which associates to any object J in \underline{J} the map $x: \Delta(C)(J) \rightarrow \Delta(C')(J)$.

We shall now treat the internal version of the above.

Let \underline{C} (described as in previous chapters) be a category object in \underline{E} , and E an object in \underline{E} . We claim that the diagram

$$\begin{array}{ccccc}
 & \xrightarrow{\gamma \times E} & & \xrightarrow{d_0 \times E} & \\
 X_2 \times E & \xrightarrow{e_0 \times E} & X_1 \times E & \xrightarrow{d_1 \times E} & X_0 \times E \\
 & \xrightarrow{e_1 \times E} & & \xleftarrow{\mu \times E} & \\
 \downarrow \rho_2 & & \downarrow \rho_1 & & \downarrow \rho_0 \\
 X_2 & \xrightarrow{\gamma} & X_1 & \xrightarrow{d_0} & X_0 \\
 & \xrightarrow{e_0} & & \xrightarrow{d_1} & \\
 & \xrightarrow{e_1} & & \xleftarrow{\mu} &
 \end{array}$$

where ρ_0 , ρ_1 and ρ_2 are the first projections of the products, defines an internal functor $\Delta(E): \underline{C} \rightarrow \underline{E}$.

Proof: Conditions (1) to (4) in the definition of internal functor must be verified.

(1) The square

$$\begin{array}{ccc}
 X_1 \times E & \xrightarrow{d_0 \times E} & X_0 \times E \\
 \rho_1 \downarrow & & \downarrow \rho_0 \\
 X_1 & \xrightarrow{d_0} & X_0
 \end{array}$$

is commutative (by definition of $d_0 \times E$). To show that it is a pullback, let $f: Y \rightarrow X_0 \times E$ and $g: Y \rightarrow X_1$ satisfy $\rho_0 f = d_0 g$, and consider the product diagram

$$\begin{array}{ccccc}
 X_0 & \xleftarrow{\rho_0} & X_0 \times E & \xrightarrow{P_0} & E \\
 \uparrow d_0 & & \uparrow d_0 \times E & & \uparrow E \\
 X_1 & \xleftarrow{\rho_1} & X_1 \times E & \xrightarrow{P_1} & E \\
 & & \uparrow \langle g, P_0 f \rangle & & \uparrow P_0 f \\
 & & Y & &
 \end{array}$$

g (arrow from Y to X_1)

from which it is easily seen that $\langle g, P_0 f \rangle: Y \rightarrow X_1 \times E$ satisfies $(d_0 \times E) \langle g, P_0 f \rangle = f$ and $\rho_1 \langle g, P_0 f \rangle = g$, and that it is unique with respect to this property. (We note that P_0 and P_1 are the second projections of the products, and that the symbol

$\langle -, - \rangle$ used above has its genuine meaning) .

(2) The square

$$\begin{array}{ccc}
 X_1 \times E & \xrightarrow{d_1 \times E} & X_0 \times E \\
 \downarrow \rho_1 & & \downarrow \rho_0 \\
 X_1 & \xrightarrow{d_1} & X_0
 \end{array}$$

commutes by definition of $d_1 \times E$.

(3) The definition of $\mu \times E$ and the fact that $d_0 \mu = X_0$ show that $\mu \times E = \langle \mu \rho_0, X_0 \times E \rangle$, so that we have $(d_1 \times E) \langle \mu \rho_0, X_0 \times E \rangle = (d_1 \times E) (\mu \times E) = d_1 \mu \times E = X_0 \times E$.

(4) The square

$$\begin{array}{ccc}
 X_2 \times E & \xrightarrow{e_1 \times E} & X_1 \times E \\
 \downarrow \rho_2 & & \downarrow \rho_1 \\
 X_2 & \xrightarrow{e_1} & X_1
 \end{array}$$

is commutative by definition of $e_1 \times E$. To show that it is a pullback , let $f: Y \rightarrow X_1 \times E$ and $g: Y \rightarrow X_2$ satisfy $\rho_1 f = e_1 g$, then as in (1) , we have that $\langle g, P_1 f \rangle$ is unique with respect to the properties $\rho_2 \langle g, P_1 f \rangle = g$ and $(e_1 \times E) \langle g, P_1 f \rangle = f$, where the symbol $\langle -, - \rangle$ again denotes an induced map into a product . To complete the requirements of condition (4) , we notice

that the definitions of $\gamma \times E$ and $e_0 \times E$, together with the equations $d_0 \gamma = d_0 e_1$ and $d_0 e_0 = d_1 e_1$, show that $\gamma \times E = \langle \gamma \rho_2, d_0 e_1 \times E \rangle$ and $e_0 \times E = \langle e_0 \rho_2, d_1 e_1 \times E \rangle$; and that the equation $d_1 \gamma = d_1 e_0$ shows that $(d_1 \times E)(\gamma \times E) = (d_1 \times E)(e_0 \times E)$.

Now given a map $\alpha: E \rightarrow E'$ in \underline{E} , we define a map $(\Delta(\alpha))_0: X_0 \times E \rightarrow X_0 \times E'$ in \underline{E} by letting $(\Delta(\alpha))_0 = X_0 \times \alpha$. It then follows easily that $\Delta(\alpha): \Delta(E) \rightarrow \Delta(E')$ is a natural transformation of internal functors, with $(\Delta(\alpha))_1 = X_1 \times \alpha$.

We now see that Δ , as defined above, is a functor $\underline{E} \rightarrow \underline{E}^C$, for if $E: E \rightarrow E$ is an identity map in \underline{E} , then $(\Delta(E))_0 = X_0 \times E$, hence $\Delta(E)$ is an identity map $\Delta(E) \rightarrow \Delta(E)$ in \underline{E}^C , and if $\alpha: E \rightarrow E'$, $\alpha': E' \rightarrow E''$ is a pair of (composable) maps in \underline{E} , then $(\Delta(\alpha')\Delta(\alpha))_0 = (\Delta(\alpha'))_0(\Delta(\alpha))_0 = (X_0 \times \alpha')(X_0 \times \alpha) = X_0 \times \alpha'\alpha = (\Delta(\alpha'\alpha))_0$, so that $\Delta(\alpha')\Delta(\alpha) = \Delta(\alpha'\alpha)$.

If \underline{E} is the category of sets, then for any object x_0 in the category C , $(\Delta(E))(x_0) = \rho_0^{-1}(\{x_0\}) = \{x_0\} \times E \simeq E$; and identifying elements which are isomorphic images of each other, we see that $(\Delta(E))(f) = \text{id}_E$ for any map f in C . Given a map $\alpha: E \rightarrow E'$ in $\underline{E} = \underline{\text{Set}}$, $\Delta(\alpha): \Delta(E) \rightarrow \Delta(E')$ associates to any object x_0 in C the map $(\Delta(\alpha))_0|_{x_0}: \{x_0\} \times E \rightarrow \{x_0\} \times E'$, which is essentially $\alpha: E \rightarrow E'$ itself.

2. Colimits

Definition. Let C be a category object in \underline{E} , and $F: C \rightarrow \underline{E}$,

$$\begin{array}{ccccc}
 & & \delta_0 & & \\
 & & \longrightarrow & & \\
 E_2 & \xrightarrow{c} & E_1 & \xrightarrow{\delta_1} & E_0 \\
 \downarrow \pi_2 & & \downarrow \pi_1 & \xleftarrow{m} & \downarrow \pi_0 \\
 X_2 & \xrightarrow{\gamma} & X_1 & \xrightarrow{d_0} & X_0 \\
 & & \xleftarrow{\mu} & &
 \end{array}$$

a (covariant) internal functor . Then the colimit of F is the coequalizer of δ_0 and δ_1 :

$$\begin{array}{ccccc}
 & \delta_0 & & & \\
 E_1 & \xrightarrow{\quad} & E_0 & \xrightarrow{\text{coeq}} & \text{colim} F \\
 & \delta_1 & & &
 \end{array}$$

If \underline{E} is the category of sets , then $\text{colim} F = E_0 / \sim$ is the set of \sim equivalence classes of elements of E_0 , \sim being the smallest equivalence relation on E_0 containing R , where R is a relation satisfying xRy iff there exists $z \in E_1$ such that $x = \delta_0(z)$ and $y = \delta_1(z)$.

We now assume (only for this and the next sections) that \underline{E} has coequalizers of all pairs of morphisms (with common domain and codomain) , so that $\text{colim} F$ exists for any object F in \underline{E}^C .

Given a natural transformation $\tau: F \rightarrow F'$, i.e. a map in \underline{E}^C , we define a map $\text{colim} \tau: \text{colim} F \rightarrow \text{colim} F'$ in \underline{E} as follows : let $\xi: E_0 \rightarrow \text{colim} F'$ be given by $\xi = (\text{coeq}') \tau_0$ (where all primed symbols refer to the functor F') , then we have $\xi \delta_1 =$

$(\text{coeq}')\tau_0\delta_1 = (\text{coeq}')\delta_1'\tau_1 = (\text{coeq}')\delta_0'\tau_1 = (\text{coeq}')\tau_0\delta_0 = \xi\delta_0$,
 so that the universal property of coequalizer shows that
 there exists a unique map (which we shall denote by $\text{colim}\tau$)
 $\text{colim}F \rightarrow \text{colim}F'$, satisfying $(\text{colim}\tau)(\text{coeq}) = \xi$.

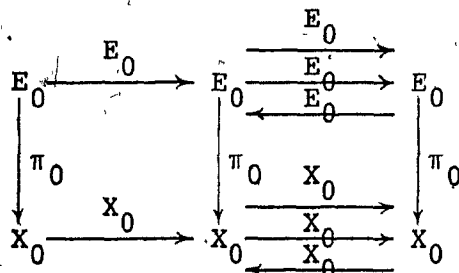
We claim that the above definitions give rise to a
 functor $\text{colim}: \underline{E}^C \rightarrow \underline{E}$.

Proof: Given an identity map $\tau: F \rightarrow F$ in \underline{E}^C , i.e. that
 $\tau_0: E_0 \rightarrow E_0$ is the identity map on E_0 , we have $\xi = (\text{coeq})\tau_0 =$
 coeq , so that $\text{colim}\tau$ is the identity map on $\text{colim}F$ (by the
 uniqueness property of $\text{colim}\tau$). Given composable maps $\tau: F \rightarrow F'$,
 $\tau': F' \rightarrow F''$ in \underline{E}^C , we have $\xi = (\text{coeq}')\tau_0$, $\xi' = (\text{coeq}'')\tau_0'$, and
 $\xi'' = (\text{coeq}'')\tau_0'\tau_0$, so in order to prove that $\text{colim}\tau'\tau =$
 $(\text{colim}\tau')(\text{colim}\tau)$, it suffices to show (by uniqueness) that
 $(\text{colim}\tau')(\text{colim}\tau)(\text{coeq}) = \xi''$, which holds since
 $(\text{colim}\tau')(\text{colim}\tau)(\text{coeq}) = (\text{colim}\tau')\xi = (\text{colim}\tau')(\text{coeq}')\tau_0 =$
 $\xi'\tau_0 = (\text{coeq}'')\tau_0'\tau_0 = \xi''$.

Remark. The internal coproduct of the map $\pi_0: E_0 \rightarrow X_0$
 is the domain of π_0 , namely E_0 . We see that this is a special
 case of the notion of (internal) colimit, for taking the
 discrete category object \mathcal{D} ,

$$\begin{array}{ccccc}
 & & X_0 & & \\
 & & \downarrow & & \\
 X_0 & \xrightarrow{X_0} & X_0 & \begin{array}{c} \xrightarrow{X_0} \\ \xrightarrow{X_0} \\ \xleftarrow{X_0} \end{array} & X_0
 \end{array}$$

in \underline{E} , π_0 defines an internal functor $F: \mathcal{D} \rightarrow \underline{E}$,



and $\text{colim} F$ is precisely E_0 .

3. Colim as a Left Adjoint

We shall prove in this section that colim is the left adjoint of Δ . It suffices to show the existence of a natural transformation (in the ordinary sense) $\eta: I_{\underline{E}^C} \rightarrow \Delta \text{colim}$ (where $I_{\underline{E}^C}$ denotes the identity functor on \underline{E}^C) such that each $\eta^F: F \rightarrow \Delta \text{colim} F$ is universal among (internal) natural transformations $\phi: F \rightarrow \Delta(E)$, E an object of \underline{E} ; i.e. for every (internal) natural transformation $\phi: F \rightarrow \Delta(E)$, there exists a unique map $\psi: \text{colim} F \rightarrow E$ in \underline{E} , such that $\phi = (\Delta \psi) \eta^F$.

Denote by $l_i: X_i \times \text{colim} F \rightarrow X_i$ the first projections of the products, arising from the internal functor $\Delta \text{colim} F$ (as discussed in section 1), and define $\eta_0^F: E_0 \rightarrow X_0 \times \text{colim} F$ by the product diagram

$$\begin{array}{ccccc}
 & & X_0 \times \text{colim} F & & \\
 & \swarrow l_0 & \uparrow \eta_0^F & \searrow L_0 & \\
 X_0 & & E_0 & & \text{colim} F \\
 & \nwarrow \pi_0 & \downarrow \text{coeq} & &
 \end{array}$$

where L_0 is the second projection map. To show that $\eta^F: F \rightarrow \Delta \text{colim} F$ is a map in \underline{E}^C , conditions (1) and (2) in the definition of internal natural transformation must be verified.

- (1) $l_0 \eta_0^F = \pi_0$ is immediate from the definition of η_0^F .
- (2) Letting $\eta_1^F = \langle \pi_1, \eta_0^F \delta_0 \rangle$, it is required to show that $(d_1 \times \text{colim} F) \eta_1^F = \eta_0^F \delta_1$, but this follows by the universal property of products,

$$\begin{array}{ccccc}
 & & X_0 \times \text{colim} F & & \\
 & \swarrow l_0 & \uparrow \eta_0^F & \searrow L_0 & \\
 X_0 & \xrightarrow{\pi_0} & E_0 & \xrightarrow{\text{coeq}} & \text{colim} F \\
 & \nwarrow d_1 \pi_1 & \uparrow \delta_1 & \nearrow (\text{coeq}) \delta_0 & \\
 & & E_1 & &
 \end{array}$$

noting that

$$\begin{aligned}
 l_0 (d_1 \times \text{colim} F) \eta_1^F &= d_1 l_1 \eta_1^F \\
 &= d_1 \pi_1,
 \end{aligned}$$

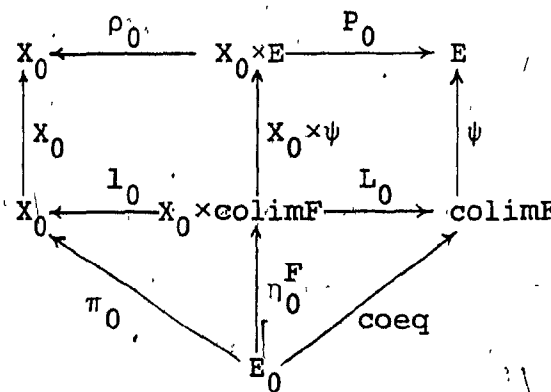
and

$$\begin{aligned}
 L_0 (d_1 \times \text{colim} F) \eta_1^F &= L_1 \eta_1^F \\
 &= L_0 \eta_0^F \delta_0 \\
 &= L_0 \eta_0^F \delta_0
 \end{aligned}$$

$$= (\text{coeq}) \delta_0 ,$$

where $L_1: X_1 \times \text{colim} F \rightarrow \text{colim} F$ is the second projection map of the product .

To show the universality of $\eta^F: F \rightarrow \Delta \text{colim} F$, let $\phi: F \rightarrow \Delta(E)$ be a map in \underline{E}^C , for some object E in \underline{E} . Denote by $\rho_i: X_i \times E \rightarrow X_i$ ($0 \leq i \leq 2$) the first projections of the products , arising from the internal functor $\Delta(E)$ (as in section 1) , with corresponding second projections $P_i: X_i \times E \rightarrow E$. Now ϕ consists of a map $\phi_0: E_0 \rightarrow X_0 \times E$, satisfying $\rho_0 \phi_0 = \pi_0$ and $(d_1 \times E) \phi_1 = \phi_0 \delta_1$, where $\phi_1 = \langle \pi_1, \phi_0 \delta_0 \rangle$. We seek a unique map $\psi: \text{colim} F \rightarrow E$ in \underline{E} satisfying $\phi = (\Delta \psi) \eta^F$. We first let $\phi' = P_0 \phi_0$, and note that $\phi' \delta_1 = P_0 \phi_0 \delta_1 = P_0 (d_1 \times E) \phi_1 = P_1 \phi_1 = P_0 \phi_0 \delta_0 = \phi' \delta_0$, so that the universal property of coequalizers shows that there exists a unique map $\psi: \text{colim} F \rightarrow E$, satisfying $\phi' = \psi(\text{coeq})$. Now the diagram



shows , by the universal property of products , that $(X_0 \times \psi) \eta_0^F = \phi_0$, since $\rho_0 \phi_0 = \pi_0$ and $P_0 \phi_0 = \phi' = \psi(\text{coeq})$; therefore $(\Delta \psi)_0 \eta_0^F = \phi_0$, and hence $(\Delta \psi) \eta^F = \phi$. The uniqueness of ψ with respect to this property follows by its uniqueness in making the diagram

$$\begin{array}{ccccc}
 E_1 & \xrightarrow{\delta_0} & E_0 & \xrightarrow{\text{coeq}} & \text{colim} F \\
 & \xrightarrow{\delta_1} & & & \downarrow \psi \\
 & & & & E \\
 & & \searrow \phi' & &
 \end{array}$$

commute , for if $\psi': \text{colim} F \rightarrow E$ satisfies $(\Delta \psi') \eta^F = \phi$, then $(X_0 \times \psi') \eta_0^F = \phi_0$, so that $\psi'(\text{coeq}) = P_0 (X_0 \times \psi') \eta_0^F = P_0 \phi_0 = \phi'$, and hence $\psi' = \psi$.

It remains to prove the naturality of $\eta: I_{\underline{E}}^C \rightarrow \Delta \text{colim}$. Given a map $\tau: F \rightarrow F'$ in \underline{E}^C , it is required to show that the square

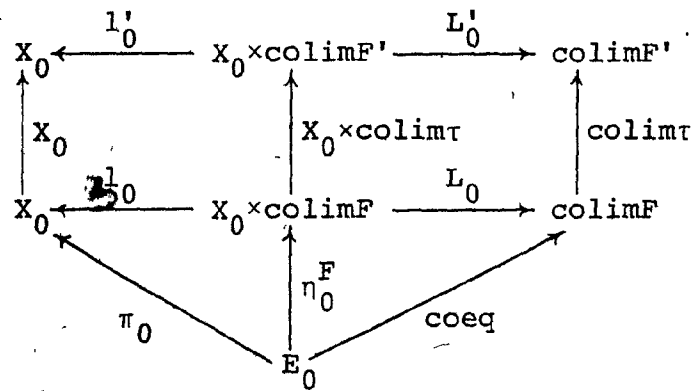
$$\begin{array}{ccc}
 F & \xrightarrow{\eta^F} & \Delta \text{colim} F \\
 \tau \downarrow & (1) & \downarrow \Delta \text{colim} \tau \\
 F' & \xrightarrow{\eta^{F'}} & \Delta \text{colim} F'
 \end{array}$$

is commutative . Let the internal functor F' be described as usual (with $\pi_i^!: E_i^! \rightarrow X_i$, $0 \leq i \leq 2$, etc.) , and denote by

$l'_i: X_i \times \text{colim} F' \rightarrow X_i$ the first projection maps of the products, arising from the internal functor $\Delta \text{colim} F'$, with corresponding second projections $L'_i: X_i \times \text{colim} F' \rightarrow \text{colim} F'$. Then square

(1) is commutative iff $(\Delta \text{colim} \tau)_0 \eta_0^F = \eta_0^{F'} \tau_0$, i.e.

$(X_0 \times \text{colim} \tau) \eta_0^F = \eta_0^{F'} \tau_0$, and the latter follows by the universal property of products, upon considering the diagram



and noting that

$$\begin{aligned} l'_0 \eta_0^{F'} \tau_0 &= \pi'_0 \tau_0 \\ &= \pi_0 \end{aligned}$$

and

$$\begin{aligned} L'_0 \eta_0^{F'} \tau_0 &= (\text{coeq}') \tau_0 \\ &= (\text{colim} \tau) (\text{coeq}) \quad (\text{by def. of colim} \tau); \end{aligned}$$

hence η is a natural transformation, and this completes the proof that colim is the left adjoint of Δ .

Remark. The colimit of a functor $F: C \rightarrow E$ may, of course, be defined as an object $\text{colim} F$, together with a natural

transformation $\eta: F \rightarrow \Delta \text{colim} F$ which is universal* among natural transformations $F \rightarrow \Delta(E)$ (E an object in \underline{E}), and this definition is easily seen to be equivalent to the given one, due to the above discussion.

4. Limits

Definition. Let C be a category object in \underline{E} , and $F: C \rightarrow \underline{E}$ a (covariant) internal functor (described as usual). Then the limit of F is an object $\text{lim} F$ in \underline{E} , together with a natural transformation $\tau: \Delta \text{lim} F \rightarrow F$ which is universal† among natural transformations $\Delta(E) \rightarrow F$.

We assume, from here on, that \underline{E} is cartesian closed, and construct $\text{lim} F$, noting that the following notation and terminology will be used: given a map $f: A \times B \rightarrow C$ in \underline{E} , its image under the map $\tau_{B,C}: \underline{E}(A \times B, C) \rightarrow \underline{E}(B, C^A)$ (where τ is the natural isomorphism which defines a product exponential adjointness in \underline{E}) will be called the transpose of f , and written $(f)^{\sim}: B \rightarrow C^A$; the image of a map $g: B \rightarrow C^A$ in \underline{E} under $\tau_{B,C}^{-1}$ will also be called the transpose of g , but will be denoted by $\tilde{g}: A \times B \rightarrow C$; the transpose of the identity map $A^B \rightarrow A^B$ will be called the evaluation map, and denoted by $\text{ev}: B \times A^B \rightarrow A$; given a map $h: A \rightarrow C$, $h^B: A^B \rightarrow C^B$ will denote the transpose of the composite $h(\text{ev}): B \times A^B \rightarrow C$.

We define objects R_0 and R_1 in \underline{E} by the pullback square

*An initial object in the comma category (F, \underline{E}^C) .

†A terminal object in the comma category (\underline{E}^C, F) .

$$\begin{array}{ccc}
 R_0 & \xrightarrow{l_0} & 1 \\
 \downarrow \alpha_0 & & \downarrow (x'_0) \\
 E_0 \times R_0 & \xrightarrow{\pi_0^{X_0}} & X_0 \times X_0
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 R_1 & \xrightarrow{l_1} & 1 \\
 \downarrow \alpha_1 & & \downarrow (d'_1) \\
 E_0 \times R_1 & \xrightarrow{\pi_0^{X_1}} & X_0 \times X_1
 \end{array}$$

where 1 is a terminal object in \underline{E} , and define maps $c_0, c_1: R_0 \rightarrow R_1$ as follows: the commutativity of the first pullback square above implies the commutativity of its "transpose diagram", namely

$$\begin{array}{ccccc}
 X_0 \times R_0 & \xrightarrow{X_0 \times \alpha_0} & X_0 \times E_0 \times X_0 & \xrightarrow{ev} & E_0 \\
 & \searrow \text{proj.} & & & \downarrow \pi_0 \\
 & & & & X_0
 \end{array}$$

which shows that

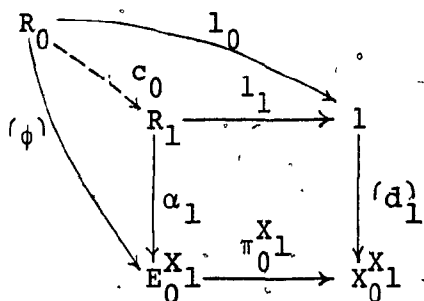
$$\begin{array}{ccccccc}
 X_1 \times R_0 & \xrightarrow{d_1 \times R_0} & X_0 \times R_0 & \xrightarrow{X_0 \times \alpha_0} & X_0 \times E_0 \times X_0 & \xrightarrow{ev} & E_0 \\
 \downarrow X_1 \times l_0 & & & & & & \downarrow \pi_0 \\
 X_1 \times 1 & \xrightarrow{\text{proj.}} & X_1 & \xrightarrow{d_1} & X_0 & & \\
 & & & & \downarrow & & \\
 & & & & X_0 & &
 \end{array}$$

commutes, so letting $\phi: X_1 \times R_0 \rightarrow E_0$ be the composite

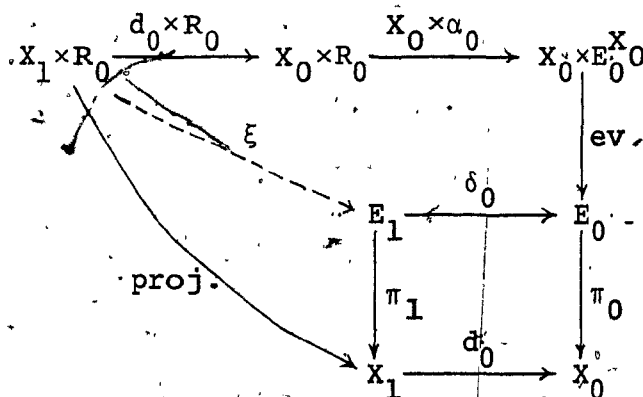
$$X_1 \times R_0 \xrightarrow{d_1 \times R_0} X_0 \times R_0 \xrightarrow{X_0 \times \alpha_0} X_0 \times E_0 \times X_0 \xrightarrow{ev} E_0$$

(60)

we see that the outside of the diagram



commutes, hence inducing $c_0: R_0 \rightarrow R_1$. So $c_1 = \langle (\delta_1 \xi), l_0 \rangle$ is defined through the same pullback square, where $\xi: X_1 \times R_0 \rightarrow E_1$ is given by



Finally, we have

$$\lim F \xrightarrow{\text{eq}} R_0 \xrightleftharpoons[c_1]{c_0} R_1$$

That this construction agrees with the definition

of the limit of $F:C \rightarrow \underline{E}$ will be clear by the discussions in the next section.

We give an interpretation of the above construction in the category of sets : $R_0 = \{f \in E_0^{X_0} \mid \pi_0 f = \text{id}_{X_0}\} = \prod_{x_0 \in X_0} \pi_0^{-1}(\{x_0\})$; $R_1 = \{f \in E_0^{X_1} \mid \pi_0 f = d_1\} = \prod_{x_1 \in X_1} \pi_0^{-1}(\{d_1(x_1)\})$; for all $f \in R_0$, for all $g \in X_1$, we have $c_0(f)(g) = f(d_1(g))$ and $c_1(f)(g) = F(g)(f(d_0(g)))$, so

$$\lim F \approx \{f \in R_0 \mid F(g)(f(d_0(g))) = f(d_1(g)), \text{ for all } g \in X_1\} .$$

Now given a map $\tau:F \rightarrow F'$ in \underline{E}^C , i.e. an internal natural transformation, we define a map $\lim \tau: \lim F \rightarrow \lim F'$ in \underline{E} as follows : let $s: \lim F \rightarrow (E'_0)^{X_0}$ denote the composite

$$\lim F \xrightarrow{\text{eq}} R_0 \xrightarrow{\alpha_0} E_0^{X_0} \xrightarrow{\tau_0^{X_0}} (E'_0)^{X_0}$$

(where all primed symbols will refer to the functor F'), and define $t: \lim F \rightarrow R'_0$ by

$$\begin{array}{ccccc} \lim F & & & & \\ \downarrow s & \searrow t & \xrightarrow{l'_0} & & 1 \\ & R'_0 & & & \\ & \downarrow \alpha'_0 & & & \downarrow (\pi'_0)^{X_0} \\ & (E'_0)^{X_0} & \xrightarrow{(\pi'_0)^{X_0}} & & X_0^{X_0} \end{array}$$

it then follows (although the proof is very tedious, and will

be omitted) that $c'_0 t = c'_1 t$, so that the universal property of equalizers shows that there exists a unique map (which we shall denote by $\lim \tau$) $\lim F \rightarrow \lim F'$ making the diagram

$$\begin{array}{ccc}
 \lim F' & \xrightarrow{eq'} & R'_0 \xrightleftharpoons[c'_1]{c'_0} R'_1 \\
 \uparrow \lim \tau & \nearrow t & \\
 \lim F & &
 \end{array}$$

commute.

We are now in a position to prove that $\lim: \underline{E}^C \rightarrow \underline{E}$ is a functor, for given an identity map $\tau: F \rightarrow F$ in \underline{E}^C , i.e. that $\tau_0: E_0 \rightarrow E_0$ is the identity, we have $s = \alpha_0(eq)$, so that $t = eq$, hence the uniqueness property of $\lim \tau$ shows that $\lim \tau = id_{\lim F}$; given (composable) maps $\tau: F \rightarrow F'$, $\tau': F' \rightarrow F''$ in \underline{E}^C , we have

$$\begin{aligned}
 s: \lim F &\xrightarrow{eq} R_0 \xrightarrow{\alpha_0} E_0^{X_0} \xrightarrow{\tau_0^{X_0}} (E'_0)^{X_0} \\
 s': \lim F' &\xrightarrow{eq'} R'_0 \xrightarrow{\alpha'_0} (E'_0)^{X_0} \xrightarrow{(\tau'_0)^{X_0}} (E''_0)^{X_0} \\
 s'': \lim F &\xrightarrow{eq} R_0 \xrightarrow{\alpha_0} E_0^{X_0} \xrightarrow{(\tau'_0 \tau_0)^{X_0}} (E''_0)^{X_0}
 \end{aligned}$$

$$t = \langle s, (\lim F \rightarrow 1) \rangle$$

$$t' = \langle s', (\lim F' \rightarrow 1) \rangle$$

$$t'' = \langle s'', (\lim F \rightarrow 1) \rangle$$

and

$$\begin{aligned}
 s'_0 \lim \tau &= (\tau'_0)^{X_0} \alpha'_0 (eq') \lim \tau \\
 &= (\tau'_0)^{X_0} \alpha'_0 t \\
 &= (\tau'_0)^{X_0} s \\
 &= (\tau'_0)^{X_0} \tau_0^{X_0} \alpha_0 (eq) \\
 &= (\tau'_0 \tau_0)^{X_0} \alpha_0 (eq) \\
 &= s'' ,
 \end{aligned}$$

so that $t' \lim \tau = t''$ (by the uniqueness property of t''), hence $(eq'')(\lim \tau')(\lim \tau) = t' \lim \tau = t''$, and the uniqueness property of $\lim(\tau' \tau)$ now shows that $\lim(\tau' \tau) = (\lim \tau')(\lim \tau)$.

Remark. The internal product of the map $\pi_0: E_0 \rightarrow X_0$ along $X_0 \rightarrow 1$ (i.e. $X_0 \prod_{X_0 \rightarrow 1} (\pi_0)$) is by definition the object R_0 in \underline{E} . It is easily seen that this is a special case of the notion of (internal) limit, for taking the discrete category object \mathcal{D} in \underline{E} (as in Section 2), π_0 defines an internal functor $F: \mathcal{D} \rightarrow \underline{E}$, and $\lim F$ is precisely R_0 .

5. Lim as a Right Adjoint

We prove in this section that \lim is the right adjoint of Δ . It suffices to show the existence of a natural transformation $\epsilon: \Delta \lim \rightarrow I_{\underline{E}}$, such that each $\epsilon^F: \Delta \lim F \rightarrow F$ is universal from Δ to F .

Let $F: \mathcal{C} \rightarrow \underline{E}$ be described as usual, and denote by $l_i: X_i \times \lim F \rightarrow X_i$ ($0 \leq i \leq 2$) the first projections of the products, arising from the internal functor $\Delta \lim F$. Define $\epsilon^F: X_0 \times \lim F \rightarrow E_0$

by the composite

$$X_0 \times \lim F \xrightarrow{X_0 \times (eq)} X_0 \times R_0 \xrightarrow{X_0 \times \alpha_0} X_0 \times E_0^{X_0} \xrightarrow{ev} E_0$$

To show that $\epsilon^F: \Delta \lim F \rightarrow F$ is an internal natural transformation, conditions (1) and (2) in the definition must be verified.

(1) $\pi_0 \epsilon_0^F = 1_0$ iff the outside of the diagram

$$\begin{array}{ccccc} X_0 \times \lim F & \xrightarrow{X_0 \times (eq)} & X_0 \times R_0 & \xrightarrow{X_0 \times \alpha_0} & X_0 \times E_0^{X_0} & \xrightarrow{ev} & E_0 \\ \downarrow 1_0 & & \downarrow \text{proj.} & & \nearrow \pi_0 & & \\ X_0 & \xrightarrow{X_0} & X_0 & & & & \end{array}$$

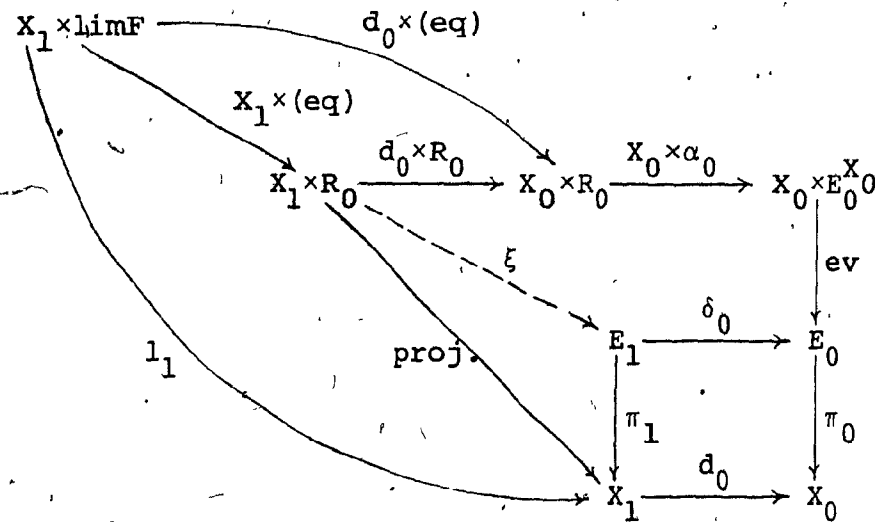
commutes, and this is so, since the right hand triangle commutes (as shown in section 4, when defining c_0).

(2) Defining ϵ_1^F by the diagram

$$\begin{array}{ccccc} X_1 \times \lim F & & & & \epsilon_0^F(d_0 \times \lim F) \\ & \searrow \epsilon_1^F & & \searrow \delta_0 & \\ & & E_1 & \xrightarrow{\delta_0} & E_0 \\ & \searrow 1_1 & \downarrow \pi_1 & & \downarrow \pi_0 \\ & & X_1 & \xrightarrow{d_0} & X_0 \end{array}$$

it is required to show that $\delta_1 \epsilon_1^F = \epsilon_0^F(d_1 \times \lim F)$. We recall

the construction of $\lim F$; in particular the definition of $\xi: X_1 \times R_0 \rightarrow E_1$, and consider the diagram



after which it follows (by the uniqueness property of ϵ_1^F) that $\epsilon_1^F = \xi(X_1 \times (eq))$; hence $\delta_1 \epsilon_1^F = \delta_1 \xi(X_1 \times (eq))$. Now by product exponential adjointness in \underline{E} , the diagram

$$\begin{array}{ccc}
 \underline{E}(X_1 \times R_0, E_0) & \xrightarrow{\quad} & \underline{E}(R_0, E_0^{X_1}) \\
 \downarrow \underline{E}(X_1 \times (eq), E_0) & & \downarrow \underline{E}(eq, E_0^{X_1}) \\
 \underline{E}(X_1 \times \lim F, E_0) & \xrightarrow{\quad} & \underline{E}(\lim F, E_0^{X_1})
 \end{array}$$

commutes, so that

$$\begin{aligned}
 (\delta_1 \xi(X_1 \times (eq))) &= (\delta_1 \xi)(eq) \\
 &= \alpha_1 c_1(eq) \quad (\text{see section 4, construction of } \lim F)
 \end{aligned}$$

$$= \alpha_1 c_0(\text{eq}) \quad (\text{by def. of eq})$$

$$= (\phi)(\text{eq}) \quad (\text{by def. of } c_0)$$

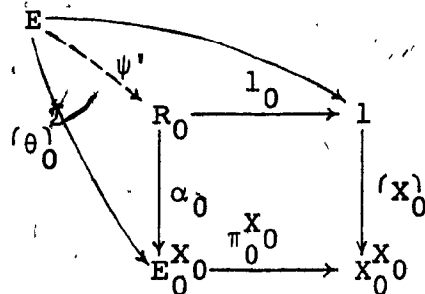
$$= (\phi(X_1 \times (\text{eq})))$$

hence

$$\delta_1 \xi(X_1 \times (\text{eq})) = \phi(X_1 \times (\text{eq})) .$$

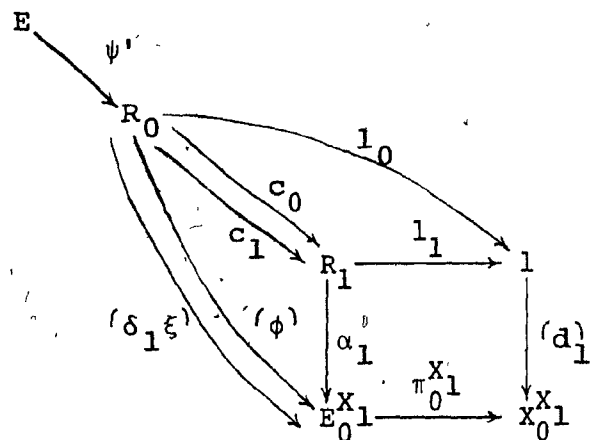
But it follows immediately by the definitions of ε_0^F and ϕ that $\phi(X_1 \times (\text{eq})) = \varepsilon_0^F(d_1 \times \text{lim} F)$, hence $\delta_1 \varepsilon_1^F = \varepsilon_0^F(d_1 \times \text{lim} F)$.

To verify the universal property of $\varepsilon^F: \Delta \text{lim} F \rightarrow F$, let $\theta: \Delta(E) \rightarrow F$ be a map in \underline{E}^C , i.e. an internal natural transformation, then θ consists of a map $\theta_0: X_0 \times E \rightarrow E_0$ in \underline{E} , satisfying $\pi_0 \theta_0 = \rho_0$ and $\delta_1 \theta_1 = \theta_0(d_1 \times E)$, where $\theta_1 = \langle \rho_1, \theta_0(d_0 \times E) \rangle$, and $\rho_i: X_i \times E \rightarrow X_i$ ($0 \leq i \leq 2$) denote the first projections of the products, arising from the internal functor $\Delta(E)$. We then require a unique map $\psi: E \rightarrow \text{lim} F$ in \underline{E} such that $\varepsilon^F \cdot \Delta \psi = \theta$, and for this, we exploit the universal property of equalizers. We define a map $\psi': E \rightarrow R_0$ by the diagram



and to show that $c_0 \psi' = c_1 \psi'$, it suffices to show, by considering the diagram

(67)



that $(\phi)\psi' = (\delta_1 \xi)\psi'$ (since $\alpha_1 c_0 = (\phi)$ and $\alpha_1 c_1 = (\delta_1 \xi)$).

Now

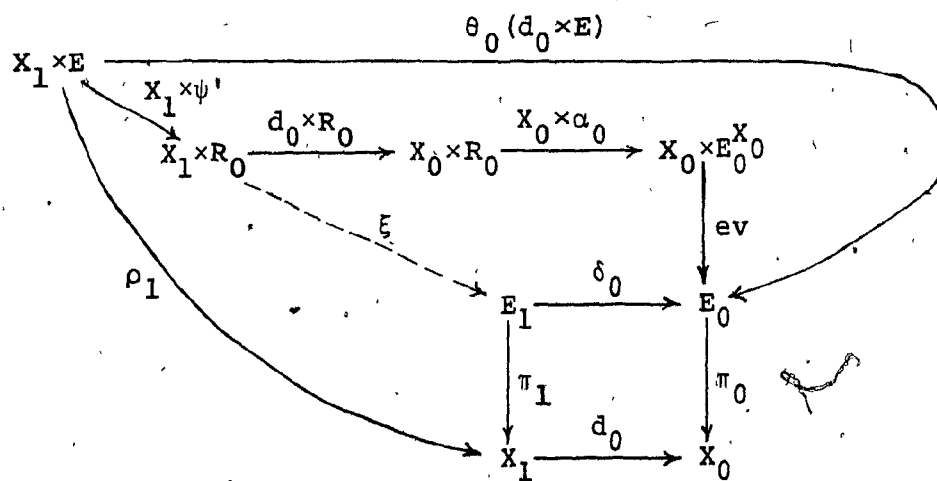
$$\alpha_0 \psi' = (\theta_0)$$

$$\Rightarrow (ev)(X_0 \times \alpha_0)(X_0 \times \psi') = \theta_0 \quad (\text{by transposition})$$

$$\Rightarrow (ev)(X_0 \times \alpha_0)(X_0 \times \psi')(d_0 \times E) = \theta_0(d_0 \times E)$$

$$\Rightarrow (ev)(X_0 \times \alpha_0)(d_0 \times R_0)(X_1 \times \psi') = \theta_0(d_0 \times E)$$

so that the diagram



shows (by the uniqueness property of θ_1) that $\theta_1 = \xi(X_1 \times \psi')$.

Therefore

$$\begin{aligned}
 \delta_1 \xi(X_1 \times \psi') &= \delta_1 \theta_1 \\
 &= \theta_0(d_1 \times E) \\
 &= (ev)(X_0 \times (\theta_0)) (d_1 \times E) \\
 &= (ev)(X_0 \times \alpha_0 \psi') (d_1 \times E) \\
 &= (ev)(d_1 \times \alpha_0 \psi') \\
 &= (ev)(X_0 \times \alpha_0) (d_1 \times R_0) (X_1 \times \psi') \\
 &= \phi(X_1 \times \psi') ,
 \end{aligned}$$

hence by transposition, we get $(\phi(X_1 \times \psi'))^\sim = (\delta_1 \xi(X_1 \times \psi'))^\sim$, which shows that $(\phi)^\sim \psi' = (\delta_1 \xi)^\sim \psi'$. So there exists a unique map $\psi: E \rightarrow \lim F$ making the diagram

$$\begin{array}{ccccc}
 \lim F & \xrightarrow{eq} & R_0 & \begin{array}{c} \xrightarrow{c_0} \\ \xrightarrow{c_1} \end{array} & R_1 \\
 \downarrow \psi & \nearrow \psi' & & & \\
 E & & & &
 \end{array}$$

commute. Now

$$\begin{aligned}
 \varepsilon_0^F(X_0 \times \psi) &= (ev)(X_0 \times \alpha_0) (X_0 \times eq) (X_0 \times \psi) \\
 &= (ev)(X_0 \times \alpha_0) (X_0 \times \psi') ,
 \end{aligned}$$

hence

$$\begin{aligned}
 (\varepsilon_0^F(X_0 \times \psi))^\sim &= \alpha_0 \psi' \\
 &= (\theta_0)^\sim ,
 \end{aligned}$$

therefore $\varepsilon_0^F(X_0 \times \psi) = \theta_0$, which shows that $\varepsilon^F \cdot \Delta \psi = \theta$. To

show that ψ is unique with respect to this property, assume $\psi'' : E \rightarrow \lim F$ satisfies $\varepsilon^F \cdot \Delta \psi'' = \theta$, then we have

$$\begin{aligned}\theta_0 &= \varepsilon_0^F (X_0 \times \psi'') \\ &= (\text{ev}) (X_0 \times \alpha_0) (X_0 \times \text{eq}) (X_0 \times \psi'') \\ &= (\text{ev}) (X_0 \times \alpha_0) (X_0 \times (\text{eq}) \psi'') ,\end{aligned}$$

which implies that

$$\begin{aligned}(\theta)_0 &= \alpha_0 (\text{eq}) \psi'' \\ \Rightarrow (\text{eq}) \psi'' &= \psi' \quad (\text{by the uniqueness property of } \psi') \\ \Rightarrow \psi'' &= \psi \quad (\text{by the uniqueness property of } \psi) .\end{aligned}$$

It remains to verify the naturality of $\varepsilon : \Delta \lim \rightarrow I_{\underline{E}} C$, i.e. the commutativity of the diagram

$$\begin{array}{ccc} \Delta \lim F & \xrightarrow{\varepsilon^F} & F \\ \downarrow \Delta \lim \tau & & \downarrow \tau \\ \lim F' & \xrightarrow{\varepsilon^{F'}} & F' \end{array}$$

where $\tau : F \rightarrow F'$ is a map in \underline{E}^C . Noting that all primed symbols refer to the internal functor F' , and recalling that $s = (\tau_0^{X_0}) \alpha_0 (\text{eq})$ (as stated when defining \lim on maps), it follows, by the equation

$$(\text{ev}') (X_0 \times s) = \tilde{s} = \tau_0 (\text{ev}) (X_0 \times \alpha_0) (X_0 \times \text{eq}) ,$$

that the outside of the diagram

(70)

$$\begin{array}{ccccccc}
 X_0 \times \lim F & \xrightarrow{X_0 \times (eq)} & X_0 \times R_0 & \xrightarrow{X_0 \times \alpha_0} & X_0 \times E_0^{X_0} & \xrightarrow{ev} & E_0 \\
 \downarrow X_0 \times \lim \tau & & & & & & \downarrow \tau_0 \\
 X_0 \times \lim F' & \xrightarrow{X_0 \times (eq')} & X_0 \times R'_0 & \xrightarrow{X_0 \times \alpha'_0} & X_0 \times (E'_0)^{X_0} & \xrightarrow{ev'} & E'_0 \\
 & \nearrow X_0 \times t & & \nearrow X_0 \times s & & & \\
 & & X_0 \times R_0 & & & &
 \end{array}$$

commutes , and this is precisely equivalent to the commutativity of the required diagram above . This completes the proof that \lim is the right adjoint of Δ .

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