ADAPTIVE CONTROL WITH RECURSIVE IDENTIFICATION
FOR
STOCHASTIC LINEAR SYSTEMS

by

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ABSTRACT

This thesis presents a new << adaptive control with recursive identification >> scheme for discrete time stochastic linear systems. Our scheme has the following properties when applied to an unstable system with unknown parameters.

(1) The adaptive control part of our algorithm stabilizes and asymptotically optimizes the system.

(2) The feedback control law is such that it is subject to a random disturbance so that the resulting control signals possess an important "persistency of excitation " property. This results in strongly consistent estimates of the system parameters being produced by the recursive parameter estimation (AML) part of our algorithm.

The above results are subject to an inverse stability assumption on the deterministic part of the system, a positive real condition on the stochastic part, and a hypothesis on the irreducibility of the system representation. Our analysis covers both the scalar (unit delay and general delay) and multivariable (unit delay) cases.

SOMMAIRE

Cette thèse présente et analyse une nouvelle méthode permettant de faire simultanément la commande adaptative et l'identification récursive des systèmes stochastiques linéaires à temps discret. Notre algorithme possède les propriétés suivantes lorsqu'il est appliqué à un système instable dont les paramètres sont inconnus.

(1) Le système est stabilisé et sa performance optimisée par l'algorithme de commande adaptative.

(2) La loi de commande de ce régulateur étant perturbée par l'addition d'un signal aléatoire, les signaux de contrôle qui sont générés possèdent une importante propriété dite "d'excitation persistante". En conséquence, des estimés convergents des paramètres du système sont produits par l'algorithme récursif d'estimation des paramètres (AML).

Ces résultats sont sujets à une hypothèse de phase minimale quant à la partie déterministe du système, une condition de passivité quant à la partie stochastique, et une hypothèse d'irréductibilité en ce qui concerne le modèle du système.

La thèse analyse le cas univariable (délai quelconque) et le cas multivariable (délai unité).

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CHAPTER I

INTRODUCTION

This thesis is concerned with the "adaptive control" problem of linear stochastic discrete time systems. Since the expression "adaptive control" is often employed in different contexts and therefore may lead to some confusion, we specify immediately that we shall consider the "parameter adaptive control problem". This problem consists in the design of a regulator for a plant whose parameters are not precisely known. The customary usage of the term adaptive - which we also adopt - implies that satisfactory regulation would be possible if the parameter values were known. If one's lack of precise knowledge of the plant involved some other information, for instance structural data concerning the plant, then one would have another type of adaptive control problem, in this case a "structural adaptive control problem". Interpreted in this way we see that there is no definitive adaptive control problem, but rather adaptive versions of previously defined control problems.

Parameter adaptive control theory and its applications have been an object of study within control engineering for many years. For a sample of this work we refer the reader to the list of references [1 - 15] drawn from publications which have appeared over the last thirty years. In particular, for the case of discrete time parameter stochastic adaptive control, we mention the seminal paper of Åström and Wittenmark [6] that introduced the self-tuning regulator.

It was not until relatively recently, however, that there began to appear complete analyses of the stability of various adaptive feedback schemes which stood upon acceptable hypotheses. To be specific, during the period 1978-1980 there appeared treatments of the deterministic scalar continuous time parameter case in [16-18], the deterministic scalar and multivariable discrete time parameter cases in [18-19] and [20] respectively, and the stochastic multivariable discrete time parameter case in [21].

In order to proceed we need to be more precise about the results contained in [21]. This paper considers systems of the form $a(z)y = z^d b(z)u + c(z)w$ (assuming for the moment that the reader is familiar with this notation). It establishes the existence of adaptive control laws for such linear stochastic discrete time unstable systems where the parameters and noise variance are unknown. These control laws stabilize the system in the sense that the input process u and the output process y are both sample mean square bounded. Furthermore the system is optimized in the sense that the sample mean square of the deviation of the output from a bounded deterministic demand process y^* converges to the prediction error of the system when this is computed using the true values of the parameters.

In the adaptive control schemes of [21] and the related work [22-27], the control laws analysed employ either a type of stochastic approximation parameter estimation algorithm [21,22,27] or modified least squares algorithms [23-26]. None use the techniques (referred to as monitoring) which project parameter estimates into regions which correspond to a stable system, an operation requiring certain a priori information about the system parameters. The conditions for these control algorithms to perform as described consist of relatively weak assumptions on the system noise process w, an inverse asymptotic stability hypothesis on the deterministic part of the system, and some form of positive real condition on the stochastic part.

Despite some partial results (see e.g. [23,26]) none of the control algorithms described above have been shown to produce consistent parameter estimates.

The contribution of this thesis is to extend the results of Goodwin, Ramadge and Caines [21] by showing that there exist asymptotically optimizing adaptive control algorithms which, in addition, generate consistent estimates of the parameters of the system which is being regulated. To the best of the author's knowledge, it is the first result of this type. The reader is referred to [28,29] for results of a distinct but analogous type concerning the adaptive control and identification of a completely observed discrete time parameter Markov process with finite parameter set.

A new << adaptive control with recursive identification >> scheme is presented. It consists of two recursive algorithms, one for adaptive control and one for system identification, which operate simultaneously on the system being considered. More specifically, the following result is established:

Let the control law of [21] be subject to a white noise disturbance or dither signal resulting in what is termed a "continually disturbed control" (introduced in [30]). Then the resulting system behaviour is such that

(1) The degradation of the asymptotically optimal performance of the system is given by the addition of a term equal to the variance of the dither signal.

(2) The resulting control process is "persistently exciting" in that the limit of the sample covariance of a particular regression vector is positive definite w.p.1.

(3) It is possible to use, in parallel, a second recursive parameter estimation algorithm (called the AML algorithm [31]) which produces strongly consistent estimates of the parameters appearing in the polynomials a(z),b(z),c(z).

The results (1),(2),(3) are, of course, subject to a set of hypotheses which are described in detail in the main body of the thesis. In the author's opinion, the only restrictive hypotheses are, first, that structural information about the system is available in the form of knowledge of the orders of a(z),b(z),c(z) and the delay d, second, that the c(z) polynomial satisfies some form of positive real condition, and third, the inverse stability assumption on the deterministic part of the system. However, these hypotheses reflect the current state-of-the-art in the theoretical analysis of adaptive control and parameter estimation methods. The subject of adaptive control of non-minimum phase systems is still only partially treated in the literature and, at present, positive real conditions are ubiquitous in the analysis of recursive stochastic algorithms (see e.g. [37]).

This thesis is organized as follows:

In Chapter II, we state in a formal way the adaptive control problem under consideration and then briefly recall the main results of [21]. In Chapter III we give a description of the "continually disturbed control actions" method and analyse its effect on the performance of the system, and in Chapter IV we prove the important "persistency of excitation" property of this method. Chapter V contains the unified theorem describing the joint use of the (stabilizing and asymptotically optimizing) control laws together with the AML parameter estimation algorithm in the scalar unit delay case.

Chapter VI extends the previous results to the scalar general delay-colored noise case, and Chapter VII to the multivariable unit delay-colored noise case. The extension of the algorithm of [21] to the multivariable case is done using a technique described in [39]. It seems it is the first time that such a technique is employed in deriving multivariable versions of stochastic adaptive control algorithms.

We also carry out in the same manner the derivation of the multivariable AML algorithm and give, in Appendix D, a complete proof of its almost sure convergence.

CHAPTER II

THE ADAPTIVE CONTROL PROBLEM

2.1 - System model

In this thesis we are concerned with the adaptive control and recursive identification of (scalar) linear time-invariant finite dimensional systems which we represent (see e.g. [39]) in their autoregressive moving average (ARMA) form, i.e. we consider systems of the form

$$a(z)y = z^d b(z)u + c(z)w,$$

with initial conditions given at t = 0, where z is the unit backward shift operator, and where u,w and y denote the input, disturbance and output processes of the system.

In chapters II to V of this thesis, we will restrict our analysis to the unit delay case, i.e. d = 1. Hence, the system S under consideration will be described by:

$$S: a(z)y = zb(z)u + c(z)w \qquad (2.1)$$

The polynomials appearing in (2.1) are defined as follows:

$$a(z) = 1 + a_1 z + \ldots + a_n z^n,$$

$$b(z) = b_0 + b_1 z + \ldots + b_m z^m, \qquad b_0 \neq 0$$

$$c(z) = 1 + c_1 z + \ldots + c_\ell z^\ell .$$

We adopt the following assumptions about the structure of the system S:

(S1) $c(z)$ and $b(z)$ are asymptotically stable polynomials i.e. all zeros lie outside of the closed unit disk.

(S2) $[c(z) - \frac{\bar{a}}{2}]$ is strictly positive real for some $\bar{a} > 0$, i.e., recalling the definition of a strict positive real function (see e.g. [36]):

(i) $[c(z) - \frac{\bar{a}}{2}]$ has no poles in $\{z: |z| \leq 1\}$;

(trivial here since $c(z)$ is a polynomial)

(ii) $c(e^{i\theta}) + c(e^{-i\theta}) - \bar{a} > 0 \quad \forall\, \theta \in [0, 2\pi]$.

Concerning these assumptions we make the following remarks:

(i) If the process $c(z)w$ is wide sense stationary and w is a stationary orthogonal process then $c(z)$ can be taken to be stable without loss of generality.

(ii) It will be seen that it is necessary to assume that $b(z)$ is asymptotically stable because we are going to consider the (asymptotic) minimization of a cost function only involving y, not u.

(iii) The positive reality assumption is indeed a substantive one but, at present, conditions of this type seem inevitable when one wishes to ensure the convergence of recursive schemes (see [37]).

All random variables appearing in this thesis will be defined upon an underlying probability space $(\Omega, \mathcal{B}, P)$.

Let $\bar{n} = \max(n, m+1, \ell)$ and let x_o denote the initial condition

$$\{y_o, \ldots, y_{-\bar{n}} \,;\, u_o, \ldots, u_{-\bar{n}} ;\, w_o, \ldots, w_{-\bar{n}}\} \quad \text{for} \quad (2.1).$$

We take x_o to be a random variable defined on $(\Omega, \mathcal{B}, P)$ and $\{w_t; t \geq 1\}$ to be a stochastic process on $(\Omega, \mathcal{B}, P)$. Let F_o denote the σ-field generated by x_o, and, for $t \geq 1$, let F_t denote that generated by $\{x_o, w_1, w_2, \ldots, w_t\}$; then our initial hypotheses on the disturbance process w are as follows:

(W1) All finite dimensional distributions of x_o and the w process are mutually absolutely continuous with respect to Lebesgue measure.

(W2) $E(w_t | F_{t-1}) = 0$ a.s. $t \geq 1$

(W3) $E(w_t^2 | F_{t-1}) = \gamma^2$ a.s. $t \geq 1$

(W4) $\limsup_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} w_t^2 < \infty$ a.s.

These assumptions imply that w is a martingale difference process of constant (conditional) variance and finite power.

2.2 - Control objective

The control objective is as follows: design a feedback control algorithm for S, with only $\bar{n}$ and observations on y and u as input data, that (i) stabilizes the system, in the sense that u and y shall have a.s. sample mean square bounded trajectories, and (ii) asymptotically optimizes its behaviour, in the sense that given a sequence y^* the sample mean square error between y and y^* is minimized.

The hypothesis on the target sequence y^* is

(T1) y^* is a bounded, deterministic (i.e. $\{\Omega, \phi\}$ measurable) sequence defined on $t \geq 1$.

The loss function we use is $\frac{1}{N}\sum_{t=1}^{N} E(y_t - y_t^*)^2|F_{t-1})$; it is a result of the analysis in [21] that the control law described below yields a limit as $N \to \infty$ for this expression which is equal to the quantity γ^2. We stress that γ^2 is the loss incurred by Åström's minimum variance control algorithm [6] when it is applied to a system of the form (2.1) with $b(z)$ asymptotically stable, w,u,y wide sense stationary and the coefficients of $a(z),b(z),c(z)$ known (i.e., available for the design of the regulator).

The control problem we treat is an adaptive one because $u_t, t \geq 1$, is not permitted to be an explicit function of the coefficients of $a(z),b(z),c(z)$ and γ^2, but only depends on these quantities through the observations $\{y_1,...,y_t\}$ and $\{u_1,...,u_{t-1}\}$.

The specification of our feedback control algorithm is such that it depends only upon n,m,ℓ,d, and the feedback control action u_t, $t \geq 1$, is required to be measurable only with respect to the σ-field of past observations i.e. that generated by $\{y_o\}$, in case $t = 1$, and by $\{y_1,...,y_t\}$ together with $\{u_1,...,u_{t-1}\}$, in case $t > 1$. Hence, reasoning inductively, we see that u_t is measurable with respect to the σ-field generated by $\{y_1,...,y_t\}$, which we denote $F\{y_1,...,y_t\}$, for all $t \geq 1$.

The control algorithm that was proposed and analysed in [21] consisted of two parts: first, a stochastic approximation parameter identification algorithm, and second, an algorithm computing the control action. The parameter θ_o that is estimated in the algorithm is the vector of coefficients appearing on the right hand side of (2.2) below. This expression gives the predicted deviations of y from the sequence y^* computed using the true model (2.1):

$$(\hat{y}_{t+1} - y^*_{t+1}) + c_1(\hat{y}_t - y^*_t) + \ldots + c_{\bar{n}}(\hat{y}_{t-\bar{n}+1} - y^*_{t-\bar{n}+1})$$

$$= \phi^T(t)\theta_o - y^*_{t+1}, \qquad t \geq 1 \qquad (2.2)$$

$$\equiv (y_t,\ldots,y_{t-\bar{n}+1};u_t,\ldots,u_{t-\bar{n}+1};-y^*_t,\ldots,-y^*_{t-\bar{n}+1})\theta_o - y^*_{t+1}$$

where we have used (2.2) to define the vector $\phi(t)$, where $\hat{y}_t$ denotes $E\, y_t | F_{t-1})$ for $t \geq 1$ and where (2.2) is initialized with $x_o \in F_o$ and hence the recursion has constant coefficients. (See the proof of Theorem 3.1 for a detailed derivation of a generalized form of equation (2.2).)

The control action given below is that which would give a minimum variance control action if θ_o were available for the design of the control algorithm. To be specific, the control action defined by (A3) alone, with $\hat{\theta}(t) = \theta_o$, is the minimum variance control action; the adaptive control algorithm operates by estimating θ_o and substituting the estimate into (A3).

ADAPTIVE CONTROL ALGORITHM

Take $\{\hat{\theta}(1),\ldots,\hat{\theta}(\bar{n})\}$ and $\{u_1,\ldots,u_{\bar{n}}\}$ as arbitrary functions of the observations; then set

(A1) $$\hat{\theta}(t) = \hat{\theta}(t-1) + \frac{\bar{a}}{r(t-1)}\phi(t-1)[y_t - \phi(t-1)^T\hat{\theta}(t-1)],$$

$$\bar{a} > 0, \; t \geq \bar{n}+1,$$

(A2) $$r(t) = r(t-1) + \phi^T(t)\phi(t), \; r(1) = \ldots = r(\bar{n}) = 1, \; t \geq \bar{n}+1$$

(A3) $$\phi^T(t)\hat{\theta}(t) = y^*_{t+1} \qquad , \; t \geq \bar{n}+1$$

(Notice that this algorithm starts at $t = \bar{n} + 1$ when all the necessary initial conditions are available). □

Equation (A3) implicity defines the feedback control law which is explicitly given by

$$u_t = -\frac{1}{\hat{\theta}_{n+1}(t)}[\hat{\theta}_1(t)y_t + \ldots + \hat{\theta}_{\bar{n}}(t)y_{t-\bar{n}+1}$$

$$+ \hat{\theta}_{\bar{n}+2}(t)u_{t-1} + \ldots + \hat{\theta}_{2\bar{n}}(t)u_{t-\bar{n}+1}$$

$$- y^*_{t+1} - \hat{\theta}_{2\bar{n}+1}(t)y^*_t \ldots - \hat{\theta}_{3\bar{n}}(t)y^*_{t-\bar{n}+1}]$$

$$t \geq \bar{n}+1, \quad \text{a.s.} \qquad (2.3)$$

By virtue of (W1) division by zero in (2.3) is a zero probability event.

The behaviour of S, subject to the hypotheses stated earlier and the adaptive control algorithm just described, is given by

<u>Theorem 2.1</u> [21]

Let S satisfy the structural assumptions (S1) and (S2), the assumptions (W1)-(W4) concerning the disturbance process w, and let y^* satisfy (T1). Let the control actions u be generated by the control algorithm described by (A1),(A2),(A3).

Then, the specification of the algorithm via (A1),(A2),(A3) requires only the structural data on the ARMA system S given by the integer $\bar{n}$ and u_t is $F\{y_1,\ldots,y_t\}$ measurable for $t \geq 1$.

Further, the input-output sample paths of S when u is generated by (A1),(A2),(A3) satisfy:

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y_t^2 < \infty \quad \text{a.s.} \tag{2.4}$$

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u_t^2 < \infty \quad \text{a.s.} \tag{2.5}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E((y_t - y_t^*)^2 | F_{t-1}) = \gamma^2 \quad \text{a.s.} \tag{2.6}$$

□

CHAPTER III

ADAPTIVE CONTROL USING CONTINUALLY DISTURBED CONTROLS

3.1 Introduction

In general, the parameter estimation parts of the adaptive control algorithms developed to this day do not produce consistent (i.e. convergent) system parameter estimates. This is true for both stochastic approximation type and least squares type algorithms. In fact, to the best of the author's knowledge, in all the results on the convergence of the system and/or predictor parameter estimates, some form of persistency of excitation (see Chapter IV) is assumed to be satisfied; however, nothing guarantees that the inputs generated by adaptive control algorithms are persistently exciting. (See e.g. [26].)

In an initial step towards the analysis of the behaviour of the parameter estimates in the adaptive control scheme of Chapter II, Caines [30] studied the effect of adding disturbances to the control action u by injecting a "dither" signal into the controller part of the algorithm. As in [30], we will call such controls "continually disturbed controls". We will see in Chapter IV that these disturbances play a crucial role in the convergence of the identification part of our scheme.

In this chapter, we restate the main result of [30] and give its (new) complete and detailed proof. More specifically, we provide an analysis of the degradation of the performance of the adaptive control scheme of Chapter II when the controls suffer a disturbance involving an exogenous noise process. We then present a new corollary. These results constitute foundations for the analysis which follows in the next chapters.

3.2 - Modified adaptive control algorithm

The so-called continually disturbed controls result from the injection of an exogenous noise process ε into equation (A 3) and ϕ (the regression vector); we shall use the notation (A^D3) and ϕ^D in the following to denote this disturbed case. The recursive identification part of the algorithm, (A1) and (A2), remains unchanged except that we employ ϕ^D instead of ϕ .

ADAPTIVE CONTROL ALGORITHM WITH CONTINUALLY DISTURBED CONTROLS

Take $\{\hat{\theta}(1),\ldots,\hat{\theta}(\bar{n})\}$ and $\{u_1,\ldots,u_{\bar{n}}\}$ as arbitrary functions of the observations $\{y_1,\ldots,y_{\bar{n}}\}$; then, for $t \geq \bar{n}+1$, set

(A1) $\hat{\theta}(t) = \hat{\theta}(t-1) + \frac{\bar{a}}{r(t-1)}\phi^D(t-1)[y_t - \phi^D(t-1)^T\hat{\theta}(t-1)], \quad \bar{a} > 0$

(A2) $r(t) = r(t-1) + \phi^D(t)^T\phi^D(t), \quad r(1) = \ldots = r(\bar{n}) = 1$

(A^D3) $\phi^D(t)^T\hat{\theta}(t) = y^*_{t+1} + \varepsilon_t$

where

$$\phi^D(t) \triangleq (y_t,\ldots,y_{t-\bar{n}+1},u_t,\ldots,u_{t-\bar{n}+1},$$

$$-(y^*_t + \varepsilon_{t-1}),\ldots,-(y^*_{t-\bar{n}+1} + \varepsilon_{t-\bar{n}}))^T, \quad t \geq \bar{n} \qquad (3.1)$$

and where the process ε will be defined below. □

Thus, u is explicitely given by

$$u_t = -\frac{1}{\hat{\theta}_{\bar{n}+1}(t)}[\hat{\theta}_1(t)y_t + \ldots + \hat{\theta}_{\bar{n}}(t)y_{t-\bar{n}+1} + \hat{\theta}_{\bar{n}+2}(t)u_{t-1} + \ldots$$

$$+ \hat{\theta}_{2\bar{n}}(t)u_{t-\bar{n}+1} - (y^*_{t+1} + \varepsilon_t)$$

$$- \hat{\theta}_{2\bar{n}+1}(t)(y^*_t + \varepsilon_{t-1}) - \ldots$$

$$- \hat{\theta}_{3\bar{n}}(t)(y^*_{t-\bar{n}+1} + \varepsilon_{t-\bar{n}})] \quad \text{a.s.} \qquad (3.2)$$

ε is an exogenous noise process defined upon the underlying probability space $(\Omega,\mathcal{B},P)$. Denoting by $\mathcal{G}_t$ the σ-field generated by $\{x_o,w_1,\ldots,w_t,\varepsilon_1,\ldots,\varepsilon_t\}$, the following assumptions will be made concerning w and ε:

(W^D1) All finite dimensional distributions of x_o and the w and ε processes are mutually absolutely continuous with respect to Lebesgue measure.

(W^D2) $\quad E(w_t|\mathcal{G}_{t-1}) = 0 \quad$ a.s. $\quad t \geq 1$

(W^D3) $\quad E(w_t^2|\mathcal{G}_{t-1}) = \gamma^2 \quad$ a.s. $\quad t \geq 1$

(W^D4) $\quad \limsup_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} w_t^2 < \infty \quad$ a.s. $\quad$ (as before)

(E1) $\quad E(\varepsilon_t|\mathcal{G}_{t-1}) = 0 \quad$ a.s. $\quad t \geq 1$

(E2) $\quad E(\varepsilon_t^2|\mathcal{G}_{t-1}) = \mu^2 \quad$ a.s. $\quad t \geq 1$

(E3) $\quad E(\varepsilon_t^4|\mathcal{G}_{t-1}) < K < \infty \quad$ a.s. $\quad t \geq 1$.

We point out immediately that the last three assumptions (E1),(E2),(E3) imply, by Lemma A.2 in Appendix A, the following ergodic type result for ε :

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \varepsilon_t^2 = \mu^2 \qquad \text{a.s.} \tag{3.3}$$

(the raison d'être of assumption (E3) is solely to ensure the validity of equation (3.3).)

We now present a generalization of Theorem 2.1 in the case of continually disturbed controls. (In fact, the proof of the following Theorem also applies for Theorem 2.1 when the process ε is identically zero.)

Theorem 3.1.

Let S satisfy the structural assumptions (S1) and (S2), the assumptions (W^D1) - (W^D4) concerning the disturbance process w and let y^* satisfy (T1). Let the control actions u be generated by the control algorithm described by (A1),(A2),(A^D3) with the exogenous noise process ε satisfying (E1) - (E3).

Then, the input-output sample paths of S satisfy:

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y_t^2 < \infty \quad \text{a.s.} \tag{3.4}$$

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u_t^2 < \infty \quad \text{a.s.} \tag{3.5}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E((y_t - y_t^*)^2 | G_{t-1}) = \gamma^2 + \mu^2 \quad \text{a.s.} \tag{3.6}$$

□

Remark 3.1: Notation

We shall use the usual mixed notation $\eta_t = \frac{m(z)}{n(z)} \xi_t$ to denote the generation of the output process η from the input process ξ via the ARMA scheme $n_o\eta_t + \ldots + n_n\eta_{t-n} = m_o\xi_t + \ldots + m_m\xi_{t-m}$, $t \geq 0$, (n and m being respectively the degrees of n(z) and m(z)) together with initial conditions at t = 0.

Proof of Theorem 3.1.

Part 1. We begin by motivating the choice of the regression vector and establish an important relation using the filter equation.

The system equation (2.1) can be written in its prediction form:

$$c(z)(y_{t+1}-w_{t+1}) = (c(z)-a(z))y_{t+1}+b(z)u_t, \quad t \geq 1,$$

with initial condition x_o. (Since d = 1 this is just a transposition of (2.1)). Subtracting $c(z)(y^*_{t+1}+\varepsilon_t)$ from each side, we obtain

$$\begin{aligned} c(z)[y_{t+1}-y^*_{t+1}-w_{t+1}-\varepsilon_t] &= (c_1-a_1)y_t+\ldots+(c_{\bar{n}}-a_{\bar{n}})y_{t-\bar{n}+1} \\ &\quad + b_o u_t+\ldots+b_{\bar{n}-1}u_{t-\bar{n}+1} \\ &\quad - c_1(y^*_t+\varepsilon_{t-1})-\ldots-c_{\bar{n}}(y^*_{t-\bar{n}+1}+\varepsilon_{t-\bar{n}}) \\ &\quad - (y^*_{t+1}+\varepsilon_t) \\ &\equiv \phi^D(t)^T\theta_o - (y^*_{t+1}+\varepsilon_t) \end{aligned} \tag{3.7}$$

where the vectors $\phi^D(t)$ and θ_o are defined as follows:

$$\phi^D(t) = (y_t,\ldots,y_{t-\bar{n}+1},u_t,\ldots,u_{t-\bar{n}+1},-(y^*_t+\varepsilon_{t-1}),\ldots,-(y^*_{t-\bar{n}+1}+\varepsilon_{t-\bar{n}}))^T \tag{3.8}$$

$$\theta_o = (c_1-a_1,\ldots,c_{\bar{n}}-a_{\bar{n}}, \; b_o,\ldots,b_{\bar{n}-1}, \; c_1,\ldots,c_{\bar{n}})^T . \tag{3.9}$$

Denoting the control error $y_t - y^*_t$ as e_t, and defining:

$$z_t \triangleq e_{t+1} - w_{t+1} - \varepsilon_t \tag{3.10}$$

we have from (3.7):

$$c(z)z_t = \phi^D(t)^T\theta_o - (y^*_{t+1}+\varepsilon_t), \quad t \geq 1 . \tag{3.11}$$

But, from $(A^D 3)$, $y^*_{t+1} + \varepsilon_t = \phi^D(t)^T \hat{\theta}(t)$; hence, defining $\tilde{\theta}(t) \triangleq \hat{\theta}(t) - \theta_o$ we obtain the following crucial relation:

$$c(z) z_t = - \tilde{\theta}(t)^T \phi^D(t) \qquad t \geq \bar{n} + 1 . \qquad (3.12)$$

Since $x_o \in G_t$ for $t \geq 0$, $w_{t+1} = y_{t+1} - Ey_{t+1} | G_t)$ for $t \geq 0$; therefore, $z_t = - y^*_{t+1} + Ey_{t+1} | G_t) - \varepsilon_t$ and hence z_t is G_t measurable, which we write as $z_t \in G_t$. It follows that $Ez_t | G_t) = z_t$.

Part 2. In this section, we establish an important property of the algorithm. In the analysis to follow we take $t \geq \bar{n} + 1$, and note that all the required initial conditions have been specified.

Substituting $(A^D 3)$ in (A1), we have:

$$\tilde{\theta}(t) = \tilde{\theta}(t-1) + \frac{\bar{a}}{r(t-1)} \phi^D(t-1) [e_t - \varepsilon_{t-1}] . \qquad (3.13)$$

Let $V(t) \triangleq \tilde{\theta}(t)^T \tilde{\theta}(t)$ then

$$V(t) = V(t-1) + \frac{2\bar{a}}{r(t-1)} \tilde{\theta}(t-1)^T \phi^D(t-1) [e_t - w_t - \varepsilon_{t-1}]$$

$$+ \frac{2\bar{a}}{r(t-1)} \tilde{\theta}(t-1)^T \phi^D(t-1) [w_t]$$

$$+ \frac{\bar{a}^2}{r(t-1)^2} \phi^D(t-1)^T \phi(t-1) [(e_t - \varepsilon_{t-1} - w_t)^2$$

$$+ 2 w_t (e_t - w_t - \varepsilon_{t-1}) + w_t^2] \qquad (3.14)$$

Taking conditional expectations and writing $b(t-1) \underline{\Delta} -\tilde{\theta}(t-1)^T \phi^D(t-1)$

$$EV(t)|G_{t-1}) = V(t-1) - \frac{2\bar{a}}{r(t-1)} b(t-1) z_{t-1} + \frac{\bar{a}^2}{r(t-1)^2} \phi^D(t-1)^T \phi^D(t-1) z_{t-1}^2$$

$$+ \frac{\bar{a}^2}{r(t-1)^2} \phi^D(t-1)^T \phi^D(t-1) \gamma^2 \qquad (3.15)$$

from $(W^D 2)$, $(W^D 3)$ and $z_{t-1} \in G_{t-1}$.

So, noting that

$$\frac{\phi^D(t-1)^T \phi^D(t-1)}{r(t-1)} \leq 1,$$

we have

$$EV(t)|G_{t-1}) \leq V(t-1) - \frac{2\bar{a}}{r(t-1)} \{b(t-1) - \frac{(\bar{a}+\rho)}{2} z_{t-1}\} z_{t-1}$$

$$- \rho\bar{a} \frac{z_{t-1}^2}{r(t-1)} + \frac{\bar{a}^2}{r(t-1)^2} \phi^D(t-1)^T \phi^D(t-1) \gamma^2 \text{ a.s.} \qquad (3.16)$$

where ρ is a small positive constant chosen so that

$$[c(z) - \frac{\bar{a}+\rho}{2}]$$

is positive real. The existence of such a ρ is assured by the strict positive real condition (S2).

Now let

$$h(t-1) \underline{\Delta} b(t-1) - \frac{(\bar{a}+\rho)}{2} z_{t-1} \qquad (3.17)$$

Recalling equation (3.12) and the definition of $b(t-1)$, we have

$$h(t-1) = [c(z) - \frac{\bar{a}+\rho}{2}] z_{t-1} \qquad (3.18)$$

Equation (3.16) can now be written as

$$E(V(t)|G_{t-1}) \le V(t-1) - \frac{2\bar{a}}{r(t-1)} h(t-1)z_{t-1} - \frac{\rho\bar{a}z_{t-1}^2}{r(t-1)}$$

$$+ \frac{\bar{a}^2}{r(t-1)^2} \phi^D(t-1)^T\phi^D(t-1)\ \gamma^2 \quad \text{a.s.} \qquad (3.19)$$

Let us define

$$S(t) = 2\bar{a} \sum_{j=\bar{n}+1}^{t} h(j-1)z_{j-1} + K, \quad 0 < K < \infty ,$$

$$t \ge \bar{n} + 1 \qquad (3.20)$$

where K is a positive quantity depending upon x_o that ensures $S(t) \ge 0$ for all $t \ge \bar{n}+1$. The existence of K follows from the positive real condition (S2) (see for instance the proof of the Positive Real Lemma given in Lemma A.4 of Appendix A).

Now define the non-negative random variable

$$Z(t) = V(t) + \frac{S(t)}{r(t-1)} \qquad t \ge \bar{n}+1 \qquad (3.21)$$

So

$$E(Z(t)|G_{t-1}) = E(V(t)|G_{t-1}) + \frac{S(t)}{r(t-1)}$$

$$\le V(t-1) + \frac{S(t-1)}{r(t-1)} - \frac{\rho\bar{a}z_{t-1}^2}{r(t-1)}$$

$$+ \frac{\bar{a}^2}{r(t-1)^2} \phi^D(t-1)^T\phi^D(t-1)\ \gamma^2 \quad \text{a.s.,}$$

where we have used (3.19) and (3.20).

Next, since $r(t-2) \leq r(t-1)$, we obtain the important "near-super-martingale" inequality

$$E\{Z(t) \mid G_{t-1}\} \leq V(t-1) + \frac{S(t-1)}{r(t-2)} - \frac{\rho\bar{a}z_{t-1}^2}{r(t-1)} + \frac{\bar{a}^2}{r(t-1)^2}\phi^D(t-1)^T\phi^D(t-1)\gamma^2$$

$$\leq Z(t-1) - \frac{\rho\bar{a}z_{t-1}^2}{r(t-1)} + \frac{\bar{a}^2}{r(t-1)^2}\phi^D(t-1)^T\phi^D(t-1)\gamma^2 \quad \text{a.s.} \tag{3.22}$$

Since $\{r(t);\ t \geq 1\}$ is a non-decreasing sequence and $\phi^D(t)^T\phi^D(t) + r(t-1) = r(t)$ it follows that

$$\sum_{j=n+1}^{\infty} \frac{\phi^D(j-1)^T\phi^D(j-1)}{r(j-1)^2} \leq 1/r(1) = 1 \quad .$$

Applying an extended form of the convergence theorem for positive super-martingales (see e.g. [21,31]) to (3.22) we obtain

$$Z(t) \to Z(\infty) \quad \text{a.s. with } E\{Z(\infty)\} < \infty \quad ,$$

and

$$\sum_{t=1}^{\infty} \frac{\rho\bar{a}z_t^2}{r(t)} < \infty \quad \text{a.s.}$$

Now since $\rho\bar{a} \neq 0$ we conclude

$$\sum_{t=1}^{\infty} \frac{z_t^2}{r(t)} < \infty \quad . \tag{3.23}$$

Our objective is now to establish the important relation

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} z_t^2 = 0 \quad \text{a.s.} \tag{3.24}$$

We shall consider two cases depending upon the behaviour of $r(t)$ as $t \to \infty$ and will divide the sample space accordingly. (In [21] only the second case was considered).

(i) Let $H = \{\omega \in \Omega: \lim_{t\to\infty} r(t) < \infty\}$; in that case, from the definition of $r(t)$ in (A2), we have $\lim_{t\to\infty} \phi^D(t)^T\phi^D(t) = 0$ which implies that $\lim_{t\to\infty} \phi^D(t) = 0$. We also know, from the definition of $Z(t)$ in (3.21) and the fact that $Z(\infty)$ is finite, that $\limsup_{t\to\infty} V(t) < \infty$ a.s. and so $\limsup_{t\to\infty} \|\tilde{\theta}(t)\| < \infty$ a.s. Hence, recalling equation (3.12) and the stability condition (S1) on $c(z)$, we conclude $\lim_{t\to\infty} z_t = 0$ for almost all $\omega \in H$; consequently

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} z_t^2 = 0 \qquad \text{for almost all } \omega \in H \qquad (3.25)$$

as required.

(ii) Let $H' \triangleq \Omega / H$; in that case, we can apply Kronecker's lemma to (3.23) because $r(t) \to \infty$ as $t \to \infty$ (see e.g. [42]). This yields

$$\lim_{N\to\infty} \frac{1}{r(N)} \sum_{t=1}^{N} z_t^2 = 0 \quad \text{for almost all} \quad \omega \in H' \qquad (3.26)$$

We show in part 3 of this proof that $\liminf_{N\to\infty} \frac{N}{r(N)} \geq \frac{1}{K} > 0$ a.s. on Ω. Then, from (3.26) we have that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} z_t^2 = 0 \quad \text{for almost all} \quad \omega \in H'. \qquad (3.27)$$

In conclusion, (3.24) has been shown to hold a.s. on H and H' and hence holds a.s. on Ω as required.

<u>Part 3.</u> We now prove $\liminf_{N\to\infty} \frac{N}{r(N)} \geq \frac{1}{K} > 0$ a.s on Ω which was needed to establish (3.24) above. At the same time we prove the stability property (3.4) - (3.5) of the algorithm.

First, using assumptions (S1) and (W^D4), it may be shown that (see e.g. Lemma A.5 of [21]) there exists $K_1, K_2(\omega)$ both positive and a.s. finite such that

$$\frac{1}{N} \sum_{t=\bar{n}+1}^{N} u_t^2 \leq \frac{K_1}{N} \sum_{t=\bar{n}}^{N} y_{t+1}^2 + K_2, \quad \forall N \geq \bar{n}+1 . \tag{3.28}$$

Using the definitions of $r(N)$ and $\phi^D(t)$ together with assumption (T1) and equation (3.3) it follows that for $K_3, K_4(\omega)$ positive

$$\frac{r(N)}{N} \leq \frac{N_3}{N} \sum_{t=\bar{n}}^{N} y_{t+1}^2 + K_4, \qquad \forall N \geq \bar{n}+1 . \tag{3.29}$$

We remark that for $\omega \in H$, $r(N) < K$ for all N and for some $K(\omega) < \infty$. This implies $\frac{r(N)}{N} \to 0$ and so $\liminf_{N\to\infty} \frac{N}{r(N)} > 0$.

Turning to H' we proceed as follows:

by definition, $z_{t-1} = y_t - y_t^* - w_t - \varepsilon_{t-1}$; hence, by (T1)

$$\frac{1}{N} \sum_{t=1}^{N} y_{t+1}^2 \leq \frac{4}{N} \sum_{t=1}^{N} z_t^2 + M_2 + \frac{4}{N} \sum_{t=1}^{N} w_{t+1}^2 + \frac{4}{N} \sum_{t=1}^{N} \varepsilon_t^2 . \tag{3.30}$$

From (W^D4) and (3.3), the quantity

$$M'_3 \triangleq M_2 + 2\left(4 \limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} w_t^2(\omega) + 4\mu^2\right)$$ is positive

and a.s. finite and is such that

$$\frac{1}{N} \sum_{t=1}^{N} y_{t+1}^2 \leq \frac{4}{N} \sum_{t=1}^{N} z_t^2 + M'_3 \quad \text{a.s. for all } N \geq N_1(\omega),$$

for some random $N_1(\omega)$. Hence,

$$\frac{1}{N} \sum_{t=1}^{N} y_t^2 \leq \frac{4}{N} \sum_{t=1}^{N} z_t^2 + M_3, \quad \text{a.s. for all } N \geq 1, \tag{3.31}$$

where M_3 denotes the supremum of M'_3 and the values taken by the last two terms in (3.30) over the interval $(1, N_1(\omega))$. Hence, using (3.29) and (3.31) we have

$$\frac{r(N)}{N} \leq \frac{C_1}{N} \sum_{t=1}^{N} z_t^2 + C_2, \quad \text{a.s.} \quad N \geq 1 . \tag{3.32}$$

where $C_1, C_2(\omega)$ are both positive real and a.s. finite. Following Solo's suggestion (private correspondence) we rewrite (3.32) to obtain, see also [22],

$$1 \leq C_1 \frac{1}{r(N)} \sum_{t=1}^{N} z_t^2 + C_2 \frac{N}{r(N)}, \quad \text{for } N \geq 1 \quad \text{a.s.}$$

So, from (3.26) for any $\delta > 0$

$$1 \leq C_1 \left(\frac{\delta}{C_1} \right) + C_2 \frac{N}{r(N)} \quad \text{for } N > N_\delta(\omega) \quad \text{a.s. on } H' . \tag{3.33}$$

Hence, for $1 > \delta > 0$,

$$\liminf_{N \to \infty} \frac{N}{r(N)} \geq \frac{(1-\delta)}{C_2} > 0 \text{ , a.s. on } H',$$

and since the same was true on H,

$$\liminf_{N \to \infty} \frac{N}{r(N)} > 0, \quad \text{a.s. on } \Omega \quad , \tag{3.34}$$

as required in Part 2.

To establish the stability relations (3.4) and (3.5), we remark that (3.34) implies that

$$\limsup_{N \to \infty} \frac{r(N)}{N} < \infty \quad \text{a.s.} \tag{3.35}$$

and so from the definition of r(N),

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y_t^2 < \infty \quad \text{a.s.}$$

and

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u_t^2 < \infty \quad \text{a.s.}$$

as required.

Part 4. Consider the property (3.24) $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} z_t^2 = 0$ now established almost surely on Ω. By the definition of z_{t-1} $e_t = y_t - y_t^* = z_{t-1} + w_t + \varepsilon_{t-1}$, hence

$$E\, e_t^2 \mid G_{t-1}) = E[z_{t-1}^2 + w_t^2 + \varepsilon_{t-1}^2 + 2 z_{t-1} w_t + 2 z_{t-1}\varepsilon_{t-1} + 2 w_t \varepsilon_{t-1}] \mid G_{t-1})$$

$$= z_{t-1}^2 + \gamma^2 + \varepsilon_{t-1}^2 + 2 z_{t-1}\varepsilon_{t-1} \qquad (3.36)$$

by $(W^D 2)$ and $(W^D 3)$.

But $$\left|\frac{1}{N} \sum_{t=1}^{N} z_{t-1}\varepsilon_{t-1}\right| \leq \left(\frac{1}{N} \sum_{t=1}^{N} z_{t-1}^2\right)^{\frac{1}{2}} \left(\frac{1}{N} \sum_{t=1}^{N} \varepsilon_{t-1}^2\right)^{\frac{1}{2}} \to 0 \cdot \mu = 0 \quad \text{a.s.} \qquad (3.37)$$

as $N \to \infty$ by (3.24) and (3.3).

Therefore forming the Cesaro sums of (3.36) and invoking (3.24), (3.3) and (3.37) we obtain the desired result:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E(y_t - y_t^*)^2 \mid G_{t-1}) = \gamma^2 + \mu^2 \quad \text{a.s.} \qquad \square$$

Notice that the degradation of the asymptotically optimal performance of the system is given exactly by the addition of the variance of the "dither" signal.

Corollary 3.1.

If in the statement of Theorem 3.1 assumption (W^D4) is replaced by the following condition on the fourth moments of w:

$$E\, w_t^4 | G_{t-1}) < P < \infty \quad \text{a.s.} \quad t \geq 1 \tag{3.38}$$

then the loss function can be reduced to the more simple form:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} (y_t - y_t^*)^2 = \gamma^2 + \mu^2 \quad \text{a.s.} \tag{3.39}$$

□

Proof of Corollary 3.1.:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} (y_t - y_t^*)^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} e_t^2$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} (z_{t-1} + w_t + \varepsilon_{t-1})^2$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} z_{t-1}^2 + \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} w_t^2 + \lim \frac{1}{N} \sum_{t=1}^{N} \varepsilon_{t-1}^2$$

$$+ 2 \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} z_{t-1} w_t + 2 \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} z_{t-1}\varepsilon_{t-1} + 2 \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} w_t \varepsilon_{t-1}$$

$$= \gamma^2 + \mu^2 + 2 \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} w_t \varepsilon_{t-1} \tag{3.40}$$

where the last line is obtained by a combination of (3.24),(3.38) (3.3), Lemma A.2 and the Cauchy-Schwarz inequality. Now $\omega_t \varepsilon_{t-1}$ is a centered sequence i.e. a martingale difference process which is G_t adapted, since $\omega_t \varepsilon_{t-1} \in G_t$ and $E\, \omega_t\, \varepsilon_{t-1} | G_{t-1}) = 0$ a.s. Let $X_N \triangleq \sum_{t=1}^{N} \frac{1}{t^2} \varepsilon_t^2$.

$$E(X_N|G_{N-1}) = E\left(\sum_{t=1}^{N}\frac{1}{t^2}\varepsilon_t^2|G_{N-1}\right)$$

$$= E\left(\sum_{t=1}^{N-1}\frac{1}{t^2}\varepsilon_t^2 + \frac{\varepsilon_N^2}{N^2}\right)|G_{N-1})$$

$$= X_{N-1} + \frac{1}{N^2}\mu^2 .$$

Since $\sum_1^\infty \frac{1}{N^2}\mu^2 < \infty$ a.s., we can apply the martingale convergence theorem which shows that $X_N \to X_\infty = \sum_{t=1}^{\infty}\frac{\varepsilon_t^2}{t^2} < \infty$ a.s. as $N \to \infty$.

Now, we have

$$\sum_{t=1}^{\infty}\frac{1}{t^2} E(\omega_t^2\varepsilon_{t-1}^2|G_{t-1}) = \sum_{t=1}^{\infty}\frac{\gamma^2}{t^2}\varepsilon_{t-1}^2 < \infty \qquad \text{a.s.}$$

by the above result.

Hence, applying Lemma A.1 in Appendix A to the centered sequence $w_t\varepsilon_{t-1}$ we obtain the following result:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\omega_t\varepsilon_{t-1} = 0 \qquad \text{a.s.}$$

Therefore, (3.40) reduces to the required result:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}(y_t - y_t^*)^2 = \gamma^2 + \mu^2 \qquad \text{a.s.} \qquad \square$$

Remark 3.2.

A similar extension can be done for Theorem 2.1, i.e. assuming instead of (W4) that $E(w_t^4|F_{t-1}) < P < \infty$, then the loss function can be simplified to $\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}(y_t - y_t^*)^2 = \gamma^2$ a.s.

CHAPTER IV

PROPERTIES OF CONTINUALLY DISTURBED CONTROLS

4.1 - Introduction

We mentioned in the last chapter that the addition of a disturbance process ε to the control action plays a crucial role in the identification part of our scheme. As we will prove in this chapter, such is the case because the presence of this disturbance process ensures that the so-called "persistency of excitation" condition is satisfied.

This condition is a common one in the study of the convergence of identification methods (see e.g. [31]-[34]); together with other hypotheses on a system, it is known to ensure the convergence of prediction error and maximum likelihood estimates of the system parameters (see e.g. [39]). It is generally specified in terms of the limit of the Cesaro sum of outer products of some regression vector, i.e.

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \phi^I(t)\phi^I(t)^T,$$ where $\phi^I(t)$ usually consists of past input, output and disturbance values; this limit is required to be positive definite for the persistent excitation property to hold.

4.2- The persistent excitation property of continually disturbed controls

In this section, we present new results concerning the adaptive control algorithm in the case of continually disturbed controls. Our objective is to prove that such controls yield the persistent excitation property

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \phi^I(t)\phi^I(t)^T = R^I > 0 \qquad \text{a.s.,} \qquad (4.1)$$

where $\phi^I(t)$ is a regression vector that will be specified later. Once we have established that (4.1) holds we are in a position to show that the system parameter $\overset{o}{\theta}_S$, denoting the coefficients of the polynomials a(z), b(z), c(z) defining the system S in (2.1), can be consistently estimated.

In order to prove the required result, it is necessary to introduce new assumptions concerning:

(i) the ergodicity of the noise sequences ω and ε;

(ii) the cross Cesaro summability of the demand sequence y^*;

(iii) the identifiability condition on the system, which is the coprimeness of a(z) and b(z);

(iv) the positive real condition for the AML recursion used in Chapter V, which is the strict positive reality of $\frac{1}{c(z)} - \frac{1}{2}$.

Therefore, for convenience, we restate all the hypotheses made on the system S and the exogenous noise process ε in a more compact form, taking into account the implications of the new assumption (i) above.

LIST OF HYPOTHESES:

(I) $b(z)$ is an asymptotically stable polynomial i.e. all zeros lie outside of the closed unit disk;

(II) $\begin{bmatrix} w \\ \varepsilon \end{bmatrix}$ is an ergodic process;

(III) $E(\begin{bmatrix} w \\ \varepsilon \end{bmatrix}_t | G_{t-1}) = 0$ a.s. $t \geq 1$;

(IV) $E(\begin{bmatrix} w \\ \varepsilon \end{bmatrix}_t [w\ \varepsilon]_t | G_{t-1}) = \begin{bmatrix} \gamma^2 & 0 \\ 0 & \mu^2 \end{bmatrix}$ a.s. $t \geq 1$;

(V) All finite dimensional distributions of x_o and the $\begin{bmatrix} w \\ \varepsilon \end{bmatrix}$ process are mutually absolutely continuous with respect to Lebesgue measure;

(VI) y^* is a bounded, deterministic (i.e. $\{\Omega,\phi\}$ measurable) sequence defined on $t \geq 1$;

(VII) For all pairs of integers k,ℓ the limit $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y^*_{t-k} y^*_{t-\ell}$ exists almost surely and depends upon the difference $k-\ell$;

(VIII) $a(z)$ and $b(z)$ have no common factors;

(IX) $c(z) - \bar{a}/2$ is strictly positive real for some $\bar{a} > 0$, and $c(z)$ is an asymptotically stable polynomial;

(X) $\frac{1}{c(z)} - \frac{1}{2}$ is strictly positive real. □

(We will see later that (X) implies (IX).)

By ergodicity of the noise sequences, we have:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} w_t^2 = Ew_1^2 = E(Ew_1^2 | G_o) = \gamma^2 \quad \text{a.s.} \tag{4.2}$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \varepsilon_t^2 = E\varepsilon_1^2 = E(E\varepsilon_1^2 | G_o) = \mu^2 \quad \text{a.s.} \tag{4.3}$$

To specify ϕ^I, we rewrite the system equation (2.1) in the form:

$$y_t = \phi^I(t-1)^T \overset{\circ}{\theta}_s + w_t \tag{4.4}$$

where

$$\overset{\circ}{\theta}_s \triangleq (a_1,\ldots,a_n,\ b_o,\ldots,b_m,\ c_1,\ldots,c_\ell)^T \tag{4.5}$$

is the vector of system parameters and

$$\phi^I(t) \triangleq (-y_t,\ldots,-y_{t-n+1},u_t,\ldots,u_{t-m},w_t,\ldots,w_{t-\ell+1})^T \tag{4.6}$$

is the " identification" regression vector appearing in the persistent excitation condition for the convergence of the AML recursion (see [31]).

It must be borne in mind that the positive definite property of $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi^I(t)\phi^I(t)^T$ that we establish in an input-output property of the system S while it is <u>subject to adaptive control</u> via algorithm (A1),(A2),(A^D3).

We shall adopt the notation

$$R^I \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi^I(t)\phi^I(t)^T = \begin{bmatrix} M1 & M2 & M3 \\ M2^T & M4 & M5 \\ M3^T & M5^T & M6 \end{bmatrix} \tag{4.7}$$

where the sub-matrices M1 to M6 respectively have dimensions $n \times n$, $n \times (m+1)$, $n \times \ell$,$(m+1) \times (m+1)$, $(m+1) \times \ell$ and $\ell \times \ell$, corresponding to the three parts of ϕ^I.

Also, it follows from hypothesis (VI) and (VII) and Herglotz theorem that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y^*_{t-k} y^*_{t-\ell} = \frac{1}{2\pi} \int_0^{2\pi} e^{-i(\ell-k)\theta} dR^*(e^{i\theta}) \quad \text{a.s.} \tag{4.8}$$

for all k,ℓ and for some positive (Hermitian in multivariable case) non-decreasing function R^* on $[0,2\pi]$.

It will be convenient to adopt the notation:

$$(R_1^*)_{i,j} \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y_{t-i}^* y_{t-j}^* \tag{4.9}$$

$$(R_2^*)_{i,j} \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y_{t-i}^* \left[\frac{a(z)}{b(z)} y_{t+1-j}^*\right] \tag{4.10}$$

$$(R_3^*)_{i,j} \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \left[\frac{a(z)}{b(z)} y_{t+1-i}^*\right] \left[\frac{a(z)}{b(z)} y_{t+1-j}^*\right] \tag{4.11}$$

when the indicated limits exist.

We now state a general result which, under the stated hypotheses, establishes the existence of all the limits appearing in (4.7) and provides analytical formulae for these limits.

Lemma 4.1

Let $\begin{bmatrix} w \\ \varepsilon \end{bmatrix}$ be a stochastic process satisfying (II),(III), (IV), let y^* be a deterministic process satisfying (VI),(VII), and let z be such that $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} z_t^2 = 0$ a.s. Further, let $a_1(z),\ldots,a_4(z)$, $d_1(z),\ldots,d_4(z)$ be asymptotically stable monic polynomials and let $b_1(z),\ldots,b_4(z),c_1(z),\ldots,c_4(z)$ be arbitrary polynomials. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \left(\frac{b_1(z)}{a_1(z)} w, \frac{b_2(z)}{a_2(z)} \varepsilon, \frac{b_3(z)}{a_3(z)} y^*, \frac{b_4(z)}{a_4(z)} z \right)_t^T$$
$$\cdot \left(\frac{c_1(z)}{d_1(z)} w, \frac{c_2(z)}{d_2(z)} \varepsilon, \frac{c_3(z)}{d_3(z)} y^*, \frac{c_4(z)}{d_4(z)} z \right)_t$$

$$= \mathrm{Diag}\Big[\ \frac{\gamma^2}{2\pi}\int_0^{2\pi}\frac{b_1(e^{i\theta})c_1(e^{-i\theta})}{a_1(e^{i\theta})d_1(e^{-i\theta})}d\theta\ ,\ \frac{\mu^2}{2\pi}\int_0^{2\pi}\frac{b_2(e^{i\theta})c_2(e^{-i\theta})}{a_2(e^{i\theta})d_2(e^{-i\theta})}d\theta\ ,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{c_3(e^{i\theta})\ dR^*(e^{i\theta})c_3(e^{-i\theta})}{a_3(e^{i\theta})d_3(e^{-i\theta})}\ ,\qquad 0\quad\Big] \tag{4.12}$$

□

Proof of Lemma 4.1: Given in Appendix B.

Let us define the sequences of coefficients α and β as the impulse responses of the transfer functions $\frac{a(z)}{b(z)}$ and $\frac{c(z)}{b(z)}$, i.e. $\frac{a(z)}{b(z)} = \sum_{j=o}^{\infty} \alpha_j z^j$ and $\frac{c(z)}{b(z)} = \sum_{j=o}^{\infty} \beta_j z^j$ as identities in z.

We now have

Lemma 4.2.

Subject to the hypotheses (I) to (IX), the sequence $\{\phi^I(t),\ t \geq 1\}$ defined via (4.6) with u given by (A1),(A2),(A^D3), with undefined terms in $\phi^I(1),\ldots,\ \phi^I(\bar{n}-1)$ arbitrarily assigned, satisfies

$$R^I \triangleq \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \phi^I(t)\phi^I(t)^T = R_1^I + R_3^I + R_4^I \quad \text{a.s.} \tag{4.13}$$

where the matrices R_1^I, R_3^I and R_4^I have simple structures and are explicitely given in the proof (equations (4.21),(4.34) and (4.35)).

□

Proof of Lemma 4.2.

The proof technique is to decompose the regression vector $\phi^I(t)$ defined in (4.6) into its various components. The analysis is then straightforward since we can employ Lemma 4.1.

First, from the definition of the process z in (3.10), we can write

$$y_t = y_t^* + e_t = y_t^* + z_{t-1} + w_t + \varepsilon_{t-1} \quad . \tag{4.14}$$

Also, the inverse stability of the system permits us to write

$$u_t = \frac{a(z)}{b(z)} y_{t+1} - \frac{c(z)}{b(z)} w_{t+1}$$

$$= \frac{a(z)}{b(z)} [y_{t+1}^* + z_t + \varepsilon_t] + \frac{a(z) - c(z)}{b(z)} w_{t+1} \quad . \tag{4.15}$$

Therefore, $\phi^I(t) \triangleq [-y_t, \ldots, -y_{t-n+1}, u_t, \ldots, u_{t-m}, w_t, \ldots, w_{t-\ell+1}]^T$ can be expressed as the sum of four vectors (of the same dimension $(n+m+1+\ell) \times 1$) which will be denoted by $\phi_1^I(t), \ldots, \phi_4^I(t)$:

$\phi^I(t) = \phi_1^I(t) + \phi_2^I(t) + \phi_3^I(t) + \phi_4^I(t) =$

$$\begin{bmatrix} -y_t^* \\ \vdots \\ -y_{t-n+1}^* \\ \hdashline \frac{a(z)}{b(z)} y_{t+1}^* \\ \vdots \\ \frac{a(z)}{b(z)} y_{t-m+1}^* \\ \hdashline 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} -z_{t-1} \\ \vdots \\ -z_{t-n} \\ \hdashline \frac{a(z)}{b(z)} z_t \\ \vdots \\ \frac{a(z)}{b(z)} z_{t-m} \\ \hdashline 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} -w_t \\ \vdots \\ -w_{t-n+1} \\ \hdashline \frac{a(z)-c(z)}{b(z)} w_{t+1} \\ \vdots \\ \frac{a(z)-c(z)}{b(z)} w_{t-m+1} \\ \hdashline w_t \\ \vdots \\ w_{t-\ell+1} \end{bmatrix} + \begin{bmatrix} -\varepsilon_{t-1} \\ \vdots \\ -\varepsilon_{t-n} \\ \hdashline \frac{a(z)}{b(z)} \varepsilon_t \\ \vdots \\ \frac{a(z)}{b(z)} \varepsilon_{t-m} \\ \hdashline 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} (n) \\ \\ (m+1) \\ \\ (\ell) \end{matrix} \tag{4.16}$$

It can be easily seen by a direct application of Lemma 4.1 that R^I consists only of the sum of the limits of three (Cesaro sums) of outer products, namely

$$R^I \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi^I(t)\phi^I(t)^T$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi_1^I(t)\phi_1^I(t)^T + \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi_3^I(t)\phi_3^I(t)^T + \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi_4^I(t)\phi_4^I(t)^T \quad \text{a.s.} \qquad (4.17)$$

We now define the following matrices

$$R_1^* = \{(R_1^*)_{i,j},\ 0 \le i \le n-1,\ 0 \le j \le n-1\}_{n \times n} \qquad (4.18)$$

$$R_2^* = \{(R_2^*)_{i,j},\ 0 \le i \le n-1,\ 0 \le j \le m\}_{n \times (m+1)} \qquad (4.19)$$

$$R_3^* = \{(R_3^*)_{i,j},\ 0 \le i \le m,\ \ 0 \le j \le m\}_{(m+1) \times (m+1)} \qquad (4.20)$$

where the $(R_k^*)_{i,j}$'s were defined in (4.9), (4.10) and (4.11). The expression of the first limit in (4.17) is then given by

$$R_1^I \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi_1^I(t)\phi_1^I(t)^T = \begin{pmatrix} R_1^* & -R_2^* & 0 \\ -R_2^{*T} & R_3^* & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (4.21)$$

Turning our attention to the last two limits in (4.17), we obtain by Lemma 4.1 that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} w_{t-r} w_{t-k} = \frac{\gamma^2}{2\pi} \int_0^{2\pi} e^{i(r-k)\theta} d\theta$$

$$= \begin{cases} 0 & \text{if } r \neq k \\ \gamma^2 & \text{if } r = k \end{cases} \quad \text{a.s.} \qquad (4.22)$$

A similar result is true for ε :

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \varepsilon_{t-r}\varepsilon_{t-k} = \frac{\mu^2}{2\pi} \int_0^{2\pi} e^{i(r-k)\theta} d\theta$$

$$= \begin{cases} 0 & \text{if } r \neq k \\ \mu^2 & \text{if } r = k \end{cases} \quad \text{a.s.} \qquad (4.23)$$

The case when the polynomials are not unity but $\frac{a(z)}{b(z)}$ or $\frac{a(z)-c(z)}{b(z)}$ is treated as follows . First, from Lemma 4.1, we have that:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \left[\frac{a(z)}{b(z)} \varepsilon_{t-r}\right]\left[\frac{a(z)}{b(z)} \varepsilon_{t-k}\right] =$$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{a(e^{i\theta})}{b(e^{i\theta})} \mu^2 e^{i(r-k)} \frac{a(e^{-i\theta})}{b(e^{-i\theta})} d\theta \quad \text{a.s.} \qquad (4.24)$$

Since $e^{i\theta}$ lies within the radius of convergence of $\frac{a(z)}{b(z)}$ (recall that b(z) is asymptotically stable) the power series expansion $\frac{a(e^{i\theta})}{b(e^{i\theta})} = \sum_{m=0}^{\infty} \alpha_m e^{im\theta}$ is valid, and so the right-hand side of (4.24) becomes

$$\frac{\mu^2}{2\pi} \int_0^{2\pi} \left(\sum_{m=0}^{\infty} \alpha_m e^{im\theta}\right)\left(\sum_{j=0}^{\infty} \alpha_j e^{-ij\theta}\right) e^{i(r-k)\theta} d\theta$$

$$= \frac{\mu^2}{2\pi} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \int_0^{2\pi} \alpha_m\alpha_j e^{i(m-j+r-k)\theta} d\theta$$

$$= \begin{cases} \mu^2 \sum_{m=k-r}^{\infty} \alpha_m \alpha_{m+r-k} = \mu^2 \sum_{p=o}^{\infty} \alpha_{p-r+k}\, \alpha_p & \text{when } r-k < 0 \\ \mu^2 \sum_{m=o}^{\infty} \alpha_m \alpha_{m+r-k} & \text{when } r-k > 0 \end{cases}$$

the exchange of the integration and the double summation being justified by the fact that the integrand is a bounded function of $\theta \in [0,2\pi]$. Combining the two expressions above we get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} [\frac{a(z)}{b(z)}\, \varepsilon_{t-r}][\frac{a(z)}{b(z)}\, \varepsilon_{t-k}] = \mu^2 \sum_{p=o}^{\infty} \alpha_p \alpha_{p+|r-k|} \quad \text{a.s.} \tag{4.25}$$

Clearly, using the same line of argument, we have that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} [\frac{a(z)-c(z)}{b(z)}\, w_{t+1-r}][\frac{a(z)-c(z)}{b(z)}\, w_{t+1-k}] =$$

$$\gamma^2 \sum_{p=o}^{\infty} (\alpha_p - \beta_p)(\alpha_{p+|r-k|} - \beta_{p+|r-k|}) \quad \text{a.s.} \tag{4.26}$$

where $\{\beta_j;\ j \geq 0\}$ was defined by $\sum_{j=o}^{\infty} \beta_j\, e^{ij\theta} = \frac{c(e^{i\theta})}{b(e^{i\theta})}$.

Proceeding in the same manner,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \varepsilon_{t-1-r}\, [\frac{a(z)}{b(z)}\, \varepsilon_{t-k}]$$

$$= \frac{1}{2\pi} \int_{o}^{2\pi} \mu^2\, e^{i(r+1-k)\theta}\, \frac{a(e^{-i\theta})}{b(e^{-i\theta})}\, d\theta$$

$$= \frac{\mu^2}{2\pi} \int_{o}^{2\pi} e^{i(r+1-k)\theta} \sum_{j=o}^{\infty} \alpha_j\, e^{-ij\theta} d\theta$$

$$= \frac{\mu^2}{2\pi} \sum_{j=o}^{\infty} \int_{o}^{2\pi} \alpha_j\, e^{i(r+1-k-j)\theta}\, d\theta$$

$$= \begin{cases} \mu^2 \alpha_{r-k+1} & \text{when} \quad r+1-k \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{a.s.} \qquad (4.27)$$

and similarly,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} w_{t-r} \left[\frac{a(z)-c(z)}{b(z)} w_{t+1-k} \right]$$

$$= \begin{cases} \gamma^2 (\alpha_{r-k+1} - \beta_{r-k+1}) & \text{when} \quad r+1-k \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{a.s.} \qquad (4.28)$$

We now have found general expressions for all the elements appearing in the last two limits of equation (4.17). We define the matrices F,D,E,B and $\underline{B}$ whose elements correspond to the right hand side of expressions (4.25) to (4.28) respectively:

$$F \triangleq \{ (\sum_{p=o}^{\infty} \alpha_p \alpha_{p+|r-k|})_{r,k}, \; 0 \leq r \leq m, 0 \leq k \leq m \}_{(m+1) \times (m+1)} \qquad (4.29)$$

$$D \triangleq \{ (\sum_{p=o}^{\infty} (\alpha_p - \beta_p)(\alpha_{p+|r-k|} - \beta_{p+|r-k|}))_{r,k}, 0 \leq r \leq m, 0 \leq k \leq m \}_{(m+1) \times (m+1)} \qquad (4.30)$$

$$E \triangleq \{ (\alpha_{r-k+1})_{r,k}, \; 0 \leq r \leq n-1, 0 \leq k \leq m \}_{n \times (m+1)} \qquad (4.31)$$

$$B \triangleq \{ (\alpha_{r-k+1} - \beta_{r-k+1})_{r,k}, 0 \leq r \leq n-1, 0 \leq k \leq m \}_{n \times (m+1)} \qquad (4.32)$$

$$\underline{B} \triangleq \{ (\alpha_{r-k+1} - \beta_{r-k+1})_{r,k}, 0 \leq r \leq \ell-1, 0 \leq k \leq m \}_{\ell \times (m+1)} \qquad (4.33)$$

where it is assumed that $\alpha_i \equiv \beta_i \equiv 0$ when $i < 0$.

Therefore, it follows that we can write

$$R_3^I \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi_3^I(t)\,\phi_3^I(t)^T = \gamma^2 \begin{pmatrix} I_n & -B & -\underline{I}_{n,\ell} \\ -B^T & D & \underline{B}^T \\ -\underline{I}_{n,\ell}^T & \underline{B} & I_\ell \end{pmatrix} \tag{4.34}$$

and

$$R_4^I \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi_4^I(t)\,\phi_4^I(t)^T = \mu^2 \begin{pmatrix} I_n & -E & 0 \\ -E^T & F & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.35}$$

where

$\underline{I}_{n,\ell} \triangleq \{\text{Diag } 1\}_{n\times\ell}$ (1's starting from the top left).

In conclusion, we have proved the required result, $R^I = R_1^I + R_3^I + R_4^I$ a.s., by applying Lemma 4.1 to the decomposed regression vector given in (4.16). □

(Notice that Lemma 4.2 is subject to all the hypotheses (i.e. (I) to (IX)) because we need $\frac{1}{N}\sum_{t=1}^{N} z_t^2 \to 0$ a.s.; a result which is a consequence of the adaptive control algorithm.)

Remark 4.1.

If we do not use continually disturbed controls, i.e. we use (A3) instead of (A^D3), the matrix R_4^I in (4.13) vanishes; for this is equivalent to setting $\mu^2 = 0$.

The main result of this section is obtained by analysing the matrices appearing in (4.13).

Theorem 4.1.

Let the system S defined in (2.1), the noise process w, the exogenous noise process ε and the demand sequence y^* satisfy hypotheses (I) to (IX). Consider control actions u generated by the control algorithm described by (A1), (A2), (A^D3). Then with ϕ^I defined in (4.6) $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi^I(t)\phi^I(t)^T = R^I \geq 0$ a.s. with $R^I > 0$ a.s. if $\mu^2 > 0$. □

(Recall that the regression vector used in the adaptive control algorithm is still $\phi^D \neq \phi^I$. In fact, we can prove that we also have that $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi^D(t)\phi^D(t)^T$ exists and is positive definite a.s.)

Proof of Theorem 4.1.

As may be seen from the proof of Lemma 4.2, the three matrices R_1^I, R_3^I and R_4^I in (4.13) correspond to the limits of Cesaro sums of outer products of filtered versions of y^*, w and ε respectively (i.e. ϕ_1^I, ϕ_3^I and ϕ_4^I in (4.16)). Therefore, each one is positive semi-definite because, for any N, and any sequence $\{x_t;\ t \geq 1\}$ in $\mathbb{R}^\nu$

$$\lambda^T \frac{1}{N} \sum_{t=1}^{N} x_t x_t^T \lambda = \frac{1}{N} \sum_{t=1}^{N} \lambda^T x_t x_t^T \lambda = \frac{1}{N} \sum_{t=1}^{N} (\lambda^T x_t)^2 \geq 0, \quad \forall \lambda \in \mathbb{R}^\nu .$$

The same is true for the upper left corner blocks of R_3^I and R_4^I which we denote M_D and M_F:

$$M_D \triangleq \begin{pmatrix} I_n & -B \\ -B^T & D \end{pmatrix} \qquad M_F \triangleq \begin{pmatrix} I_n & -E \\ -E^T & F \end{pmatrix} .$$

(Our technique now is to return to the proof of Lemma 4.2 and analyse the particular structure of M_F and R_3^I .)

(We refer the reader to equations (4.29) and (4.31) for the expression of the sub-matrices E and F of M_F.) It can be shown that we can write M_F as the following integral on the unit circle: (to see that, make the power series expansion of $s(e^{i\theta})$, exchange integration (against powers of $e^{i\theta}$) and the infinite summation (valid by Fubini's theorem), and use the orthogonality of $\{e^{i\theta}\}$ against Lebesgue measure; in fact, this is merely equivalent to returning to the integral expressions in equalities (4.23),(4.24) and (4.27)).

$$M_F = \frac{1}{2\pi}\int_0^{2\pi} E(e^{i\theta})^* \, dF_\varepsilon(\theta)\, E(e^{-i\theta})^T =$$

$$\frac{1}{2\pi}\int_0^{2\pi} \begin{bmatrix} -1 \\ -e^{i\theta} \\ \vdots \\ -e^{i(n-1)\theta} \\ s(e^{i\theta})e^{-i\theta} \\ e^{i\theta} s(e^{i\theta})e^{-i\theta} \\ \vdots \\ e^{im\theta} s(e^{i\theta})e^{-i\theta} \end{bmatrix} \mu^2 d\theta \left[-1,\ldots,-e^{-i(n-1)\theta}, s(e^{-i\theta})e^{i\theta},\ldots,e^{-im\theta} s(e^{-i\theta})e^{i\theta}\right]$$

where $E(e^{i\theta})$ is defined implicity above and $s(e^{i\theta}) \triangleq \frac{a(e^{i\theta})}{b(e^{i\theta})}$.

Suppose that

$\lambda^T M_F \lambda = 0$ for some non-zero λ of appropriate dimension. From the spectral representation, this is equivalent to

$$\frac{\mu^2}{2\pi}\int_0^{2\pi} \lambda^T E(e^{i\theta})\, E(e^{-i\theta})^T \lambda \, d\theta = 0 \quad \text{or}$$

$$\int_0^{2\pi} |\lambda^T E(e^{i\theta})|^2 d\theta = 0 \quad \text{for some } \lambda \neq 0 .$$

Clearly, this will be true if and only if

$\lambda^T E(e^{i\theta}) = 0$ for all $\theta \in [0,2\pi]$ that is

$$-\lambda_1 - \lambda_2 e^{i\theta} - \ldots - \lambda_n e^{i(n-1)\theta} + \lambda_{n+1} s(e^{i\theta})e^{-i\theta} + \ldots + \lambda_{n+m+1} e^{im\theta} s(e^{i\theta})e^{-i\theta} = 0 \quad \forall\theta \in [0,2\pi] .$$

To show this cannot hold for $\lambda \neq 0$ we distinguish two cases:

(i) if $\lambda_{n+1} = \ldots = \lambda_{n+m+1} = 0$, then we are left with $\lambda_1 + \lambda_2 e^{i\theta} + \ldots + \lambda_n e^{i(n-1)\theta} = 0$ which implies $\lambda_1 = \ldots = \lambda_n = 0$ and hence contradicts the assumption that $\lambda \neq 0$;

(ii) in the other case, the condition can be rewritten as

$$\frac{\lambda_1 + \ldots + \lambda_n e^{i(n-1)\theta}}{\lambda_{n+1} + \ldots + \lambda_{n+m+1} e^{im\theta}} = \frac{s(e^{i\theta})}{e^{i\theta}} = \frac{1 + a_1 e^{i\theta} + \ldots + a_n e^{in\theta}}{e^{i\theta}(b_o + b_1 e^{i\theta} + \ldots + b_m e^{im\theta})}$$

which is also impossible because by assumption (VIII) $a(z)$ and $b(z)$ have no common factors, and the (monic) numerator on the right hand side is of higher degree than the one on the left.

Therefore, we conclude that M_F is a <u>positive definite matrix</u>.

Now, consider the non-zero vector of dimension $(n+m+1+\ell) \times 1$

$$\lambda = [\lambda_1, \ldots, \lambda_p, 0, \ldots, 0, \lambda_1, \ldots, \lambda_p]^T$$

where $p = \min(n, \ell)$.

Returning to the expression of R_3^I given in (4.34), we can see that $\lambda^T R_3^I \lambda = 0$, and therefore we conclude that R_3^I <u>is not a positive definite matrix</u>.

The proof of the theorem now proceeds from the following easy-to-prove result (where the blocks have appropriate dimensions):

if $\begin{pmatrix} M & N & P \\ Q & R & S \\ T & U & V \end{pmatrix}$ is a positive semi-definite matrix and

V and $\begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$ are positive definite matrices, then $\begin{pmatrix} M+W & N+X & P \\ Q+Y & R+Z & S \\ T & U & V \end{pmatrix}$ is positive definite.

We associate $R_1^I + R_3^I$ with $\begin{pmatrix} M & N & P \\ Q & R & S \\ T & U & V \end{pmatrix}$ and M_F with $\begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$; but we know that $R_1^I + R_3^I$ is positive semi-definite with the lower right corner block equal to I_ℓ and that M_F is positive definite. Therefore, the theorem conclusion follows immediately, namely,

$$R^I \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi^I(t)\phi^I(t)^T > 0 \quad \text{a.s.} \qquad \square$$

4.3 - Three important remarks

Remark 4.2.: The demand sequence

We proved in Lemma 4.2 that $\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\phi^I(t)\phi^I(t)^T$ exists and is a.s. equal to $R_1^I + R_3^I + R_4^I$, and in the proof of Theorem 4.1 we showed R_3^I was positive semi-definite but not positive definite. Now, R_1^I is due to the demand sequence y^* and R_4^I to the exogenous disturbance ε; so the only way to force the required persitency of excitation without using this disturbance is to choose y^* is such a way that $\begin{bmatrix} R_1^* & -R_2^* \\ -R_2^{*T} & R_3^* \end{bmatrix}$ is positive definite. Clearly y^* constant or a ramp, typical choices in applications, will not provide this necessary condition.

In fact one can interpret the "continually disturbed control" method of Chapter III as a "continually disturbed demand sequence" method. This is true since the control law equation is (A^D3): $\phi^D(t)^T\hat{\theta}(t) = y^*_{t+1} + \varepsilon_t$; here the right-hand side can be viewed as a new demand sequence consisting of a fixed part to which is added a small dither. But, for consistency, the loss function should then be defined as

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N} E[y_t - (y_t^* + \varepsilon_{t-1})]^2|G_{t-1}) = \gamma^2 \quad \text{a.s.} \quad ;$$

in the author's opinion, this is a less appealing formulation than the one presented in Chapter III.

Remark 4.3: The c(z) = 1 case

It may be worth mentioning that in the $c(z) = 1$ case, the expression for R^I reduces to

$$R^I = \begin{pmatrix} R_1^* & -R_2^* \\ -R_2^{*T} & R_3^* \end{pmatrix} + \gamma^2\begin{pmatrix} I_n & -B \\ B^T & D \end{pmatrix} + \mu^2\begin{pmatrix} I_n & -E \\ -E^T & F \end{pmatrix}$$

but all the conclusions of Theorem 4.2 and Remark 4.2 still apply because $M_D \triangleq \begin{pmatrix} I_n & -B \\ -B^T & D \end{pmatrix}$ is not positive definite. To see this, apply to analysis of M_F in Theorem 4.1 to M_D. This shows M_D is not positive definite because $\exists \lambda \neq 0$ such that

$$\frac{\lambda_1 + \dots + \lambda_n e^{i(n-1)\theta}}{\lambda_{n+1} + \dots + \lambda_{n+m+1} e^{im\theta}} = \frac{1}{e^{i\theta}} \cdot \frac{a(e^{i\theta}) - c(e^{i\theta})}{b(e^{i\theta})} = \frac{1}{e^{i\theta}} \frac{a(e^{i\theta})-1}{b(e^{i\theta})}$$

$$= \frac{a_1 + \dots + a_n e^{i(n-1)\theta}}{b_o + \dots + b_m e^{im\theta}} \qquad \forall \theta \in [0,2\pi] .$$

<u>Remark 4.4: The orders of the polynomials</u>

The polynomials a(z), b(z) and c(z) were defined in Chapter II with orders n, m and ℓ respectively. However, as was mentioned in that chapter, the adaptive control algorithm depends only upon $\bar{n} \triangleq \max(n,m+1,\ell)$. The reason why we have used n,m and ℓ in the definition of ϕ^I(see (4.6)) instead of continuing to use $\bar{n}$, is to obtain more "natural" identifiability conditions. In fact, one could define

$$\phi^I(t) \triangleq (-y_t,\dots,-y_{t-\bar{n}+1},u_t,\dots,u_{t-\bar{n}+1},w_t,\dots,w_{t-\bar{n}+1})^T$$

and carry on the proof of Lemma 4.2 easily; it would even be simpler because all the sub-matrices would be square. However, in order to ensure the positive definite property of R^I, one would have to assume in the proof of Theorem 4.1 the following additional identifiability condition :"deg c(z) $\leq$ max{deg a(z),deg b(z) + 1} ". The author has preferred to use the more familiar set-up (i.e. with n,m,ℓ) in order to avoid such a condition.

4.4 - Summary

It may be worth summarizing the important points of this chapter. We have proved that when the system S of (2.1) is subject to adaptive control via algorithm (A1),(A2),(A^D3) of Chapter III the persistent excitation condition for the convergence of the AML algorithm (see next chapter) is satisfied. The use of continually disturbed controls is a sufficient condition for this property to hold, and a necessary one for many typical choices for the demand sequence y^*.

However, in order to prove this result, we had to strengthen the assumptions of Chapter III; in particular we required :

- the ergodicity of the joint process $\begin{bmatrix} w \\ \varepsilon \end{bmatrix}$;
- the identifiability condition on the system: a(z) and b(z) coprime .

CHAPTER V

STOCHASTIC ADAPTIVE CONTROL WITH RECURSIVE SYSTEM IDENTIFICATION

5.1 - System identification with the AML algorithm

Results concerning the convergence of recursive algorithms for system parameter estimation have been reported in the last few years by, amongst others, Ljung (see e.g. [32]), Solo (see e.g. [31]) and Hannan (see e.g. [35]). In particular, Solo [31] proved that provided a certain positive real condition is satisfied, the AML recursion converges (without monitoring) for stable systems. Sin generalized this result to the case where the data is prefiltered in order to weaken the positive real condition (see Theorem 3.5.1 in [23]). But, in both proofs, a persistent excitation condition of the form

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \phi^{I}(t)\phi^{I}(t)^{T} = R > 0 \qquad \text{a.s.}$$

is required to be satisfied. As seen in the last chapter, such a condition is satisfied for the adaptive control scheme presented in this thesiswhen continually disturbed controls are used. Hence, if a separate recursive algorithm of the AML type is used in conjunction with the adaptive control algorithm (A1), (A2), ($A^{D}3$) then the convergence of the identification algorithm is assured.

Concerning the analysis in [23] and [31] we make two points: first, the convergence of AML has been proved only for asymptotically stable $a(z)$ in the system S, and this is not required for the convergence of the adaptive control algorithm nor for the verification of the persistent excitation condition; second, the positive real condition for the AML algorithm is stronger than that for the adaptive algorithm and so this is imposed as an additional assumption. (See below.)

But, the interesting point is that the only necessary stability requirement in Solo's proof [31] is simply a sample mean square boundedness of the inputs and outputs, i.e.

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u_t^2 < \infty \quad \text{a.s.} \quad \text{and} \quad \limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y_t^2 < \infty \quad \text{a.s.}$$

conditions which are indeed satisfied by our adaptive control algorithm (see (3.4) and (3.5) in Theorem 3.1). Therefore, the "two-recursion - scheme", as depicted in Figure 5.1, is also applicable for unstable $a(z)$, by virtue of the stabilizing property of (A1), (A2), (A^D3).

Before presenting the unified statement of adaptive control with recursive identification, we stress that it can be shown (see Lemma A.3 in Appendix A) that the positive real condition required to be satisfied for the convergence of the AML recursion, namely $\left(\frac{1}{c(z)} - \frac{1}{2} \right)$ strictly positive real, implies the two other hypotheses we assumed for $c(z)$ which were ((IX) in Chapter IV):

(i) $c(z)$ is an asymptotically stable polynomial ;

(ii) $[(c(z) - \frac{\bar{a}}{2}]$ is strictly positive real for some $\bar{a} > 0$.

This explains why in the statement of Theorem 5.1 hypothesis (IX) is omitted.

Finally, it should be pointed out that Sin[23-24] developed an adaptive control algorithm based on least squares (called "Modified Least Squares") with a positive real condition identical to that of the AML algorithm. However, the particular structure of Sin's algorithm appears to make it difficult to check a persistent excitation condition.

5.2 - Unified statement of main result

The following result is obtained by combining theorems 3.1, 4.1, lemma 4.2, and the results of [31].

Theorem 5.1

Let the system S defined in (2.1), the noise process w, the exogenous noise process ε and the demand sequence y^* satisfy hypotheses (I) to (VIII) and (X) (see Chapter IV).

Let S be subject to adaptive control with "continually disturbed controls" by use of the recursive algorithm described by (A1), (A2), (A^D3).

Let also S be simultaneously the object of recursive identification by use of the following AML algorithm (as in [31]). (See figure 5.1).

$$\hat{\theta}_S(t) = \hat{\theta}_S(t-1) + P(t-1)\psi(t-1)e'(t) \quad , \; t \geq \bar{n}+1 \qquad (5.1)$$

$$P^{-1}(t) = P^{-1}(t-1) + \psi(t)\psi^T(t) \quad , \; P(\bar{n}) = I, \; t \geq \bar{n}+1 \qquad (5.2)$$

$$e'(t) = y_t - \psi^T(t-1)\hat{\theta}_S(t-1) \quad , \; t \geq \bar{n}+1 \qquad (5.3)$$

where

$$\psi(t) = [-y_t, \ldots, -y_{t-n+1}, \; u_t, \ldots, u_{t-m}, \eta_t, \ldots, \eta_{t-\ell+1}]^T \qquad (5.4)$$

and

$$\eta_t = y_t - \psi^T(t-1)\hat{\theta}_S(t) \; . \qquad (5.5)$$

Then the resulting sample paths of u, y and $\hat{\theta}_S$ are such that the following properties hold:

Stability

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y_t^2 = (R_1^*)_{0,0} + \gamma^2 + \mu^2 \quad \text{a.s.}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u_t^2 = (R_3^*)_{0,0} + \gamma^2 [\sum_{p=0}^{\infty} (\alpha_p - \beta_p)^2] + \mu^2 [\sum_{p=0}^{\infty} \alpha_p^2] \quad \text{a.s.}$$

Asymptotic Optimality

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} (y_t - y_t^*)^2 = \gamma^2 + \mu^2 \quad \text{a.s.}$$

Strong Consistency

$$\lim_{N\to\infty} \hat{\theta}_S(N) = \mathring{\theta}_S \triangleq (a_1, \ldots, a_n, b_0, \ldots, b_m, c_1, \ldots, c_\ell)^T \quad \text{a.s.}$$

□

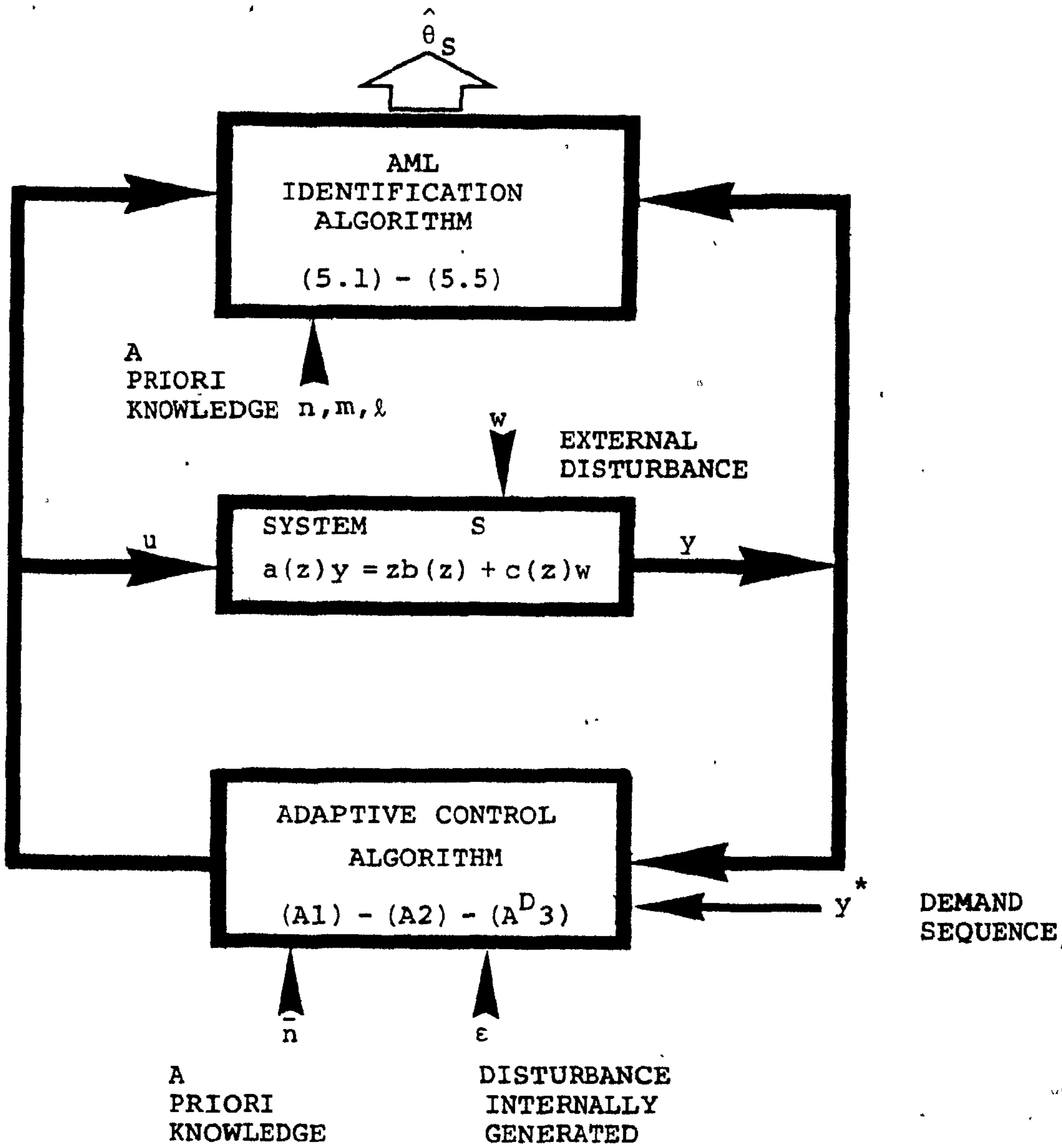

Figure 5.1: Two-recursion-scheme which stabilizes and asymptotically optimizes the system S, while producing a consistent estimate of its parameters.

CHAPTER VI

THE GENERAL DELAY CASE

6.1 - Adaptive control scheme

In this chapter, we generalize the results of chapters III, IV and V to the case $d > 1$. We will treat directly the "continually disturbed controls" case and assume, for the moment, the stochastic set-up of Chapter III. A multiple recursion algorithm will be used and the convergence analysis will necessitate the introduction of a new positive real condition. Specifically, we consider ARMA systems of the form

$$S_d: a(z)y = z^d b(z)u + c(z)w \tag{6.1}$$

with initial conditions at $t = 0$, where the polynomials $a(z)$, $b(z)$, $c(z)$ are defined as in Chapter II and it is assumed that the delay d is known. This time, we define $\bar{n} \triangleq \max(n, m+d, \ell)$. From the division algorithm [41], we can write

$$c(z) = f(z)\, a(z) + z^d g(z) \tag{6.2}$$

where
$$f(z) = f_o + \ldots + f_{d-1} z^{d-1}$$
$$g(z) = g_o + \ldots + g_{\bar{n}-1} z^{\bar{n}-1} .$$

Using (6.2), the system equation (6.1) can be written in its d-step ahead predictor form

$$c(z)\ [y_{t+d} - f(z)w_{t+d}] = g(z)y_t + \beta(z)u_t \tag{6.3}$$

where $g(z)$ and $\beta(z) = f(z)b(z)$ are polynomials of order $\bar{n} - 1$.

We define the new process

$$v_t \triangleq y_t - E(y_t | G_{t-d}) = \sum_{k=o}^{d-1} f_k w_{t-k} = f(z) w_t \qquad (6.4)$$

which, from (W^D2) and (W^D3), has the following properties

$$E(v_t | G_{t-d}) = 0 \qquad \text{a.s.} \qquad t \geq d \qquad (6.5)$$

$$E(v_t^2 | G_{t-d}) = \gamma^2 \sum_{k=o}^{d-1} f_k^2 \quad \text{a.s.}\ t \geq d \qquad (6.6)$$

(We recall that the stochastic hypotheses of Chapter III are still in force, except otherwise stated.) Subtracting $c(z)[y^*_{t+d} + \varepsilon_t]$ from each side of (6.3) we obtain

$$c(z)[y_{t+d} - y^*_{t+d} - \varepsilon_t - v_{t+d}] = g(z)y_t + \beta(z)u_t - c(z)[y^*_{t+d} + \varepsilon_t]$$

$$c(z)[e_{t+d} - v_{t+d} - \varepsilon_t] = \phi^D(t)^T\theta_o - (y^*_{t+d} + \varepsilon_t) \qquad (6.7)$$

where

$$\phi^D(t) = (y_t, \ldots, y_{t-\bar{n}+1}, u_t, \ldots, u_{t-\bar{n}+1},$$

$$-(y^*_{t+d-1} + \varepsilon_{t-1}), \ldots, -(y^*_{t+d-\bar{n}} + \varepsilon_{t-\bar{n}}))^T \qquad (6.8)$$

and

$$\theta_o = (g_o, \ldots, g_{\bar{n}-1}, \beta_o, \ldots, \beta_{\bar{n}-1}, c_1, \ldots, c_{\bar{n}})^T \qquad (6.9)$$

($\phi^D(t)$ and θ_o have dimension $3\bar{n} \times 1$).

As before, it is evident that the control error e_{t+d} would achieve its optimal value $v_{t+d} + \varepsilon_t$ (for minimum variance control) if θ_o was known and the feedback law $\phi^D(t)^T\theta_o = y^*_{t+d} + \varepsilon_t$ was employed. This motivates the choice of the following algorithm:

MULTIPLE RECURSION ALGORITHM

Take $\{\hat{\theta}(1),\ldots,\hat{\theta}(\bar{n}+d-1)\}$ and $\{u_1,\ldots,u_{\bar{n}+d-1}\}$ as arbitrary functions of the observations $\{y_1,\ldots,y_{\bar{n}+d-1}\}$; then set

$$(A_d1)\quad \hat{\theta}(t) = \hat{\theta}(t-d) + \frac{\bar{a}}{\bar{r}(t-d)}\,\phi^D(t-1)\,[y_t - \phi^D(t-d)^T\,\hat{\theta}(t-d)]$$

$$\bar{a} > 0,\ t \geq \bar{n}+d$$

$$(A_d2)\quad \bar{r}(t) = \bar{r}(t-d) + \phi^D(t)^T\phi^D(t)$$

$$\bar{r}(t) = \ldots = \bar{r}(\bar{n}+d-1) = 1, \qquad t \geq \bar{n}+d$$

$$(A_d3)\quad \phi^D(t)^T\,\hat{\theta}(t) = y^*_{t+d} + \varepsilon_t \qquad t \geq \bar{n}+d\ .\quad \square$$

We note that (A_d1) to (A_d3) actually represent d-interlaced algorithms each of which is similar to the unit delay algorithm of Chapter III. The same algorithm was used in [21] and was proved to converge ("undisturbed" case, i.e. $\varepsilon \equiv 0$) for $c(z) = 1$. A similar version where only (A_d2) is modified was proved to converge for $c(z) \neq 1$ in [22]; however, the proof technique used in [22] is not well-suited to treat the "continually disturbed controls". Hence, we shall try to generalize the results of [21] for $c(z) \neq 1$.

The convergence analysis will require the following positive real condition on $c(z)$ in replacement of (S2):

(S_d2) Consider a system described by the moving average polynomial function $[\,c(z) - \frac{\bar{a}}{2}\,]$ with input $\{x(t)\}$; we will assume that there exists a fixed non-negative number K, depending only upon the initial conditions, such that for all t and all input sequences $\{x(t), t \geq 1\}$

$$\sum_{j=1}^{[t/d]} [(c(z) - \frac{\bar{a}+\rho}{2})\, x_{t-jd}]\, x_{t-jd} + K \geq 0$$

for some $\bar{a} > 0$ and some $\rho > 0$, where $[t/d]$ denotes the largest integer M such that $dM \leq t$. We will refer to this condition as the "d-step strict positive reality of $c(z) - \frac{\bar{a}}{2}$" .

(This condition can be seen as a special case of the time-varying version of the positive real lemma (see e.g. [30]); it is stronger than the (ordinary) positive real condition (S2) used in the first part of this thesis, and a necessary condition for its validity is $d < \deg c(z)$.) It is unsatisfactory in that the condition is given in terms of a(deterministic) sample path property and not an algebraic condition on c; to find such an algebraic condition is an open problem.

With the aid of assumption (S_d2) we can generalize the results of Theorem 3.1. For convenience and to facilitate the analogy with Theorem 3.1, we use the set of hypotheses of Chapter III.

Theorem 6.1.

Let S_d satisfy the structural assumptions (S1) and (S_d2), the assumptions (W^D1) - (W^D4) concerning the disturbance process w and let y^* satisfy (T1). Consider control actions u generated by the control algorithm described by (A_d1),(A_d2),(A_d3) with the exogenous noise process ε satisfying (E1) - (E3).

Then, the input-output sample paths of S satisfy:

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y_t^2 < \infty \quad \text{a.s.} \tag{6.10}$$

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u_t^2 < \infty \quad \text{a.s.} \tag{6.11}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E(y_t - y_t^*)^2 | G_{t-d}) = \gamma^2 \sum_{k=o}^{d-1} f_k^2 + \mu^2 \quad \text{a.s.} \tag{6.12}$$

□

Proof of Theorem 6.1.

We begin by defining a new process z_t:

$$z_t \triangleq e_{t+d} - v_{t+d} - \varepsilon_t \tag{6.13}$$

(where v_t is defined in (6.4))
and remark that z_t is G_t measurable since $\varepsilon_t \in G_t$ and
$e_{t+d} - v_{t+d} = -y^*_{t+d} + E\,y_{t+d} \mid G_t) \in G_t$. Substituting (6.13) and (A_d3) in (6.7), we have

$$c(z)z_t = -\tilde{\theta}(t)^T \phi^D(t) \quad , \quad t \geq \bar{n}+d \tag{6.14}$$

(recall that $\tilde{\theta}(t) \triangleq \hat{\theta}(t) - \theta_o$). As before, we also define $b(t) \triangleq -\tilde{\theta}(t)^T \phi^D(t)$.

In the analysis to follow, we proceed as in the proof of Theorem 3.1; we take $t \geq \bar{n}+d$ and note that all the required initial conditions have been specified. We shall analyze each of the interlaced algorithms separately. From (A_d1) and (A_d3),

$$\tilde{\theta}(t) = \tilde{\theta}(t-d) + \frac{\bar{a}}{\bar{r}(t-d)} \phi^D(t-d) \; [e_t - \varepsilon_{t-d}] \; .$$

Consider $V(t) \triangleq \tilde{\theta}(t)^T \tilde{\theta}(t)$.

$$\begin{aligned} V(t) = V(t-d) &+ \frac{2\bar{a}}{\bar{r}(t-d)} \tilde{\theta}(t-d)^T \phi^D(t-d) \; [e_t - v_t - \varepsilon_{t-d}] \\ &+ \frac{2\bar{a}}{\bar{r}(t-d)} \tilde{\theta}(t-d)^T \phi^D(t-d) \; [v_t] \\ &+ \frac{\bar{a}^2}{\bar{r}(t-d)^2} \phi^D(t-d)^T \phi^D(t-d) \; [(e_t - v_t - \varepsilon_{t-d})^2 \\ &\qquad + 2v_t(e_t - v_t - \varepsilon_{t-d}) + v_t^2 \;] \end{aligned} \tag{6.15}$$

Now,

$$EV(t)|G_{t-d}) = V(t-d) - \frac{2\bar{a}}{\bar{r}(t-d)} b(t-d)z_{t-d}$$

$$+ \frac{\bar{a}^2}{\bar{r}(t-d)^2} \phi^D(t-d)^T\phi^D(t-d)\gamma^2 \sum_{k=0}^{d-1} f_k^2$$

$$+ \frac{\bar{a}^2}{\bar{r}(t-d)^2} \phi^D(t-d)^T\phi^D(t-d) z^2_{t-d} \qquad (6.16)$$

from (6.5) and (6.6).

Noting that $\frac{\phi^D(t-d)^T\phi^D(t-d)}{\bar{r}(t-d)} \leq 1$ we have

$$EV(t)|G_{t-d}) \leq V(t-d) - \frac{2\bar{a}}{\bar{r}(t-d)} \{ b(t-d) - \frac{(\bar{a}+\rho)}{2} z_{t-d} \} z_{t-d}$$

$$- \frac{\rho\bar{a}}{\bar{r}(t-d)} z^2_{t-d} + \frac{\bar{a}^2}{\bar{r}(t-d)^2} \phi^D(t-d)^T\phi^D(t-d)\gamma^2 \sum_{k=0}^{d-1} f_k^2$$

where ρ is a small positive constant chosen so that $[c(z) - \frac{\bar{a}+\rho}{2}]$ is "d-step positive real". The existence of such a ρ is assured by assumption $(S_d 2)$.

From the definition of b(t) and (6.14),

$$b(t-d) - \frac{(\bar{a}+\rho)}{2} z_{t-d} = [\, c(z) - \frac{\bar{a}+\rho}{2}]\, z_{t-d} \qquad (6.17)$$

and so

$$EV(t)|G_{t-d}) \leq V(t-d) - \frac{2\bar{a}}{\bar{r}(t-d)} \{ [c(z) - \frac{\bar{a}+\rho}{2}] z_{t-d} \} z_{t-d}$$

$$- \frac{\rho\bar{a}}{\bar{r}(t-d)} z^2_{t-d} + \frac{\bar{a}^2}{\bar{r}(t-d)^2} \phi^D(t-d)^T\phi^D(t-d)\gamma^2 \sum_{k=0}^{d-1} f_k^2$$

$$(6.18)$$

From $(S_d 2)$,

$$S(t) \triangleq 2\bar{a} \sum_{j=1}^{[t/d]} [(c(z) - \frac{\bar{a}+\rho}{2}) z_{t-jd}] \, z_{t-jd} + \underline{K} \geq 0$$

for all t and for some $\underline{K} \geq 0$. Consequently, adding $\frac{1}{\bar{r}(t-d)} S(t)$ to each side of (6.18) and denoting

$Z(t) \triangleq V(t) + \frac{S(t)}{\bar{r}(t-d)}$ we obtain

$$E Z(t) | G_{t-d}) \leq Z(t-d) - \frac{\rho \bar{a}}{\bar{r}(t-d)} z_{t-d}^2 + \frac{\bar{a}^2}{\bar{r}(t-d)^2} \phi^D(t-d)^T \phi^D(t-d) \gamma^2 \sum_{k=o}^{d-1} f_k^2 \qquad (6.19)$$

since

$$\frac{S(t)}{\bar{r}(t-d)} - \frac{2\bar{a}}{\bar{r}(t-d)} \{[c(z) - \frac{\bar{a}+\rho}{2}] z_{t-d}\} \, z_{t-d} = \frac{S(t-d)}{\bar{r}(t-d)}$$

and $\bar{r}(t-2d) \leq \bar{r}(t-d)$.

Since $\sum_{n=o}^{\infty} \frac{\phi^D(nd+K)^T \phi(nd+K)}{r(nd+K)^2} \leq \frac{1}{\bar{r}(K)}$

for each $1 \leq K \leq d$, we may apply the martingale convergence theorem to $\{Z(nd+K), G_{nd+K}; n \geq 0\}$ for each K, $1 \leq K \leq d$ and conclude that $Z(dn+K) \rightarrow Z_K(\infty)$ a.s. as $n \rightarrow \infty$ and since $0 < \rho\bar{a} < \infty$ that

$$\sum_{n=o}^{\infty} \frac{z_{nd+K}^2}{\bar{r}(nd+K)} < \infty \qquad \text{a.s.} \quad 1 \leq K \leq d.$$

Summing over K yields

$$\sum_{t=1}^{\infty} \frac{z_t^2}{\bar{r}(t)} < \infty \qquad \text{a.s.}$$

We define $r(t) = r(t-1) + \phi^D(t)^T \phi^D(t) \quad t \geq \bar{n} + d$, $r(1) = \ldots = r(\bar{n} + d - 1) = 1$ (as in (A2)) which is necessarily greater than or equal to $\bar{r}(t)$ for each $t \geq 1$; therefore we may write

$$\sum_{t=1}^{\infty} \frac{z_t^2}{r(t)} < \infty \qquad \text{a.s.} \qquad (6.20)$$

as we obtained in the proof of Theorem 3.1(equation (3.23)).

Arguing precisely as in that proof (end of part 2 and part 3) we obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} z_t^2 = 0 \qquad \text{a.s.} \qquad (6.21)$$

and

$$\limsup_{N\to\infty} \frac{r(N)}{N} < \infty \qquad \text{a.s.} \qquad (6.22)$$

It should be noted however that in the definition of z, w is replaced by v; nevertheless, the previous argumentation is still valid because $\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} w_t^2 < \infty$ a.s. (assumption $(W^D 4)$) implies that $\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} v_t^2 < \infty$ a.s. .

The theorem conclusions (6.10) and (6.11) follow immediately from (6.22). Finally, in analogy with Theorem 3.1,(6.21) implies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E(y_t - y_t^*)^2 | G_{t-d}) = \gamma^2 \sum_{k=o}^{d-1} f_k^2 + \mu^2 , \quad \text{a.s.} \qquad \square$$

Corollary 6.1.

If in the statement of Theorem 6.1 assumption $(W^D 4)$ is replaced by the following condition on the fourth moments of w:

$$E\, w_t^4 | G_{t-1}) < P < \infty \qquad \text{a.s.} \quad t \geq 1 \tag{6.23}$$

then the loss function can be reduced to the more simple form:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} (y_t - y_t^*)^2 = \gamma^2 \sum_{k=o}^{d-1} f_k^2 + \mu^2 \qquad \text{a.s.} \qquad \Box$$

Proof of Corollary 6.1.

We first point out that using the same technique which was used in the proof of Corollary 3.1 to prove that $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} w_t \varepsilon_{t-1} = 0$ a.s., we can prove that:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} w_{t-i} w_{t-j} = 0 \quad \text{a.s.} \quad \text{if} \quad i \neq j \tag{6.24}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} w_{t-j} \varepsilon_{t-d} = 0 \quad \text{a.s.} \quad \text{if} \quad 0 \leq j \leq d-1 . \tag{6.25}$$

By Lemma A.2, the new hypothesis (6.23) implies that $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} w_t^2 = \gamma^2$ a.s. Therefore,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} v_t^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} (f_o w_t + \ldots + f_d w_{t-d+1})^2$$

$$= \gamma^2 \sum_{k=o}^{d-1} f_k^2 \quad \text{a.s.} \tag{6.26}$$

by (6.24).

It then follows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} (y_t - y_t^*)^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} (z_{t-d} + v_t + \varepsilon_{t-d})^2$$

$$= \gamma^2 \sum_{k=o}^{d-1} f_k^2 + \mu^2 \qquad \text{a.s.}$$

where the last line is obtained by a combination of (6.21), (6.26), (3.3), (6.25) and the Cauchy-Schwarz inequality. □

6.2 - Persistent excitation property and recursive identification

As in Chapter IV, we want to show that the persistent excitation condition for the convergence of the AML recursion is satisfied when the adaptive control algorithm is in operation. The regression vector appearing in that condition will be denoted by ϕ^I. To specify $\phi^I(t)$, we remark that the system equation (6.1) can be written as

$$y_t = \phi^I(t-1)^T \mathring{\theta}_S + w_t \tag{6.27}$$

where

$$\phi^I(t) \triangleq (-y_t, \ldots, -y_{t-n+1}, u_{t-d+1}, \ldots, u_{t-d-m+1}, w_t, \ldots, w_{t-\ell+1})^T \tag{6.28}$$

and

$$\mathring{\theta}_S \triangleq (a_1, \ldots, a_n, \quad b_0, \ldots, b_m, \quad c_1, \ldots, c_\ell)^T \tag{6.29}$$

Notice that (6.28) reduces to (4.6) when d = 1, and that $\mathring{\theta}_S$ is the same as in (4.5). Therefore, the required condition is

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi^I(t)\phi^I(t)^T > 0 \quad \text{a.s.}$$

Notice that $\phi^I(t)$ has dimension $(n + m + 1 + \ell) \times 1$.

As was the case in Chapter IV, it is necessary to introduce at this stage new assumptions on the system S_d. Since these new assumptions are the same as those mentioned in Chapter IV, we will use, for convenience, the list of hypotheses of that chapter (i.e. hypotheses (I) to (X)). As one would expect, the only difference is the replacement of (IX) by (S_d2), the new "d-step strict positive real" property of $[c(z) - \frac{\bar{a}}{2}]$. Obviously (X) does not imply

(S_d2), and this time the unified statement of our result will unfortunately include two different positive real assumptions.

Now, under the same hypotheses as those of Chapter IV, with the exception of (S_d2) instead of (IX) and the new multiple-recursion algorithm used for adaptive control, the proof of the persistent excitation property is a straigthforward extension of the results of Chapter IV. This is summarized in the following theorem.

Theorem 6.2. (Generalization of Theorem 4.1)

Let the system S_d defined in (6.1), the noise process w, the exogenous noise process ε and the demand sequence y^* satisfy hypotheses (I) to (VIII) and (S_d2). Let the control actions u be generated by the multiple recursion algorithm described by (A_d1), (A_d2), (A_d3). Consider ϕ^I defined in (6.28); then $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi^I(t)\phi^I(t)^T$ exists and is a.s. positive definite. □

<u>Proof of Theorem 6.2 .</u>

The proof employs the same techniques which were used to prove Lemma 4.2 and Theorem 4.1.

We begin by decomposing the regression vector

$$\phi^{I}(t) = (-y_t,\ldots,-y_{t-n+1},u_{t-d+1},\ldots,u_{t-d-m+1},w_t,\ldots,w_{t-\ell+1})^T .$$

This time

$$y_t = y_t^* + e_t = y_t^* + z_{t-d} + f(z)w_t + \varepsilon_{t-d}$$

and

$$u_t = \frac{a(z)}{b(z)}\, y_{t+d} - \frac{c(z)}{b(z)}\, w_{t+d}$$

$$= \frac{a(z)}{b(z)}\, [y_{t+d}^* + z_t + \varepsilon_t] + \frac{a(z)f(z) - c(z)}{b(z)}\, w_{t+d}$$

which gives the following decomposed form for $\phi^{I}(t)$:

$$\phi^{I}(t) = \phi_1^{I}(t) + \phi_2^{I}(t) + \phi_3^{I}(t) + \phi_4^{I}(t) =$$

$$\begin{matrix} \\ (n) \\ \\ \\ (m+1) \\ \\ \\ (\ell) \\ \\ \end{matrix}
\begin{bmatrix} -y_t^* \\ \vdots \\ -y_{t-n+1}^* \\ \hline \frac{a(z)}{b(z)}\, y_{t+1}^* \\ \vdots \\ \frac{a(z)}{b(z)}\, y_{t-m+1}^* \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}
+
\begin{bmatrix} -z_{t-d} \\ \vdots \\ -z_{t-d-n+1} \\ \hline \frac{a(z)}{b(z)}\, z_{t-d+1} \\ \vdots \\ \frac{a(z)}{b(z)}\, z_{t-m-d+1} \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}
+
\begin{bmatrix} -f(z)\, w_t \\ \vdots \\ -f(z) w_{t-n+1} \\ \hline \frac{a(z)f(z)-c(z)}{b(z)} w_{t-1} \\ \vdots \\ \frac{a(z)f(z)-c(z)}{b(z)} w_{t-m+1} \\ \hline w_t \\ \vdots \\ w_{t-\ell+1} \end{bmatrix}
+
\begin{bmatrix} -\varepsilon_{t-d} \\ \vdots \\ -\varepsilon_{t-d-n+1} \\ \hline \frac{a(z)}{b(z)}\varepsilon_{t-d+1} \\ \vdots \\ \frac{a(z)}{b(z)}\varepsilon_{t-m-d+1} \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}$$

As before, Lemma 4.1 tells us that the only non-zero contributions in $R^I \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi^I(t)\phi^I(t)^T$ will come from

$$R^I \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi_1^I(t)\phi_1^I(t)^T ,$$

$$R_3^I \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi_3^I(t)\phi_3^I(t)^T \quad \text{and} \quad R_4^I \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi_4^I(t)\phi_4^I(t)^T .$$

Comparing to the proof of Lemma 4.2, we see that R_1^I and R_4^I are unchanged, but that R_3^I will be modified because of the presence of f(z). Nevertheless, R_3^I will still be positive semi-definite and therefore we can conclude that $R^I = R_1^I + R_3^I + R_4^I > 0$ a.s. because the argumentation in the proof of Theorem 4.1 remains valid.

(The detailed expression for R_3^I is more complicated than the one given in equation (4.34) for the unit delay case. However, the expression for the central sub-matrix, corresponding to D in (4.34), is given by (4.30) provided the coefficients α_j in that equation are replaced by the new coefficients ξ_j, where, in analogy with $\{\alpha_j\}$ and $\{\beta_j\}$, the sequence $\{\xi_j\}$ is defined as the impulse response of the transfer function $\frac{a(z)f(z)}{b(z)}$, i.e.

$$\frac{a(z)f(z)}{b(z)} = \sum_{j=0}^{\infty} \xi_j z^j \text{ .)}$$

□

Remark 6.1.

The particular structure of R_3^I becomes relevant if we want to determine whether or not the disturbance ε is necessary to ensure the positive definite property of R^I when y^* is constant or varies linearly. It will be so if R_3^I is not positive definite, as it is the case when $d = 1$. It can be shown that necessary conditions for $R_3^I > 0$ are:

(i) $d > \ell$

(ii) $\{z^d g(z) = a(z) f(z) - c(z)\}$ and $\{b(z)\}$ coprime

(iii) $c(z)$ and $f(z)$ coprime.

Since we mentioned at the beginning of this chapter that $(S_d 2)$ is valid if and only if $d < \ell$, the disturbance ε will be necessary to ensure persistent excitation when y^* is a constant or a ramp.

Remark 6.2.

The comments concerning the orders of the polynomials concluding section 4.3 (Remark 4.4) are still relevant (with minor modifications). Specifically, if one wanted to continue to use $\bar{n} \triangleq \max(n,m+d,\ell)$, one would have to define

$$\phi^I(t) \triangleq (-y_t,\ldots,-y_{t-\bar{n}+1}, u_t,\ldots,u_{t-\bar{n}+1}, w_t,\ldots,w_{t-\bar{n}+1})^T$$

(dimension $3\bar{n} \times 1$) and assume the following additional identifiability condition:

"$\deg c(z) \leq \max\{\deg a(z), \deg b(z) + d\}$" .

6.3 - Unified statement - general delay case.

As we did in Chapter V, we summarize the results of this chapter in the following theorem.

Theorem 6.3.

Let the system S_d defined in (6.1), the noise process w, the exogenous noise process ε and the demand sequence y^* satisfy hypotheses (I) to (VIII), (X) and (S_d2).

Let S_d be subject to adaptive control with "continually disturbed controls" by use of the multiple recursion algorithm described by (A_d1), (A_d2), (A_d3).

Let also S_d be simultaneously subject to recursive identification by use of the following AML algorithm (as in [31]):

$$\hat{\theta}_S(t) = \hat{\theta}_S(t-1) + P(t-1)\psi(t-1)e'(t), \quad t \geq \bar{n}+1 ,$$

$$P^{-1}(t) = P^{-1}(t-1) + \psi(t)\psi^T(t) \quad , \quad P(\bar{n}) = I, \ t \geq \bar{n}+1 ,$$

$$e'(t) = y_t - \psi^T(t-1)\hat{\theta}_S(t-1) \quad , \quad t \geq \bar{n}+1$$

where

$$\psi(t) = [-y_t, \ldots, y_{t-n+1}, u_{t-d+1}, \ldots, -u_{t-d-m+1}, \eta_t, \ldots, \eta_{t-\ell+1}]^T$$

and

$$\eta_t = y_t - \psi^T(t-1)\hat{\theta}_S(t).$$

Then the resulting sample paths of u, y and $\hat{\theta}_S$ are such that the following properties hold:

Stability

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y_t^2 = (R_1^*)_{0,0} + \gamma^2 [\sum_{p=0}^{d-1} f_p^2] + \mu^2 \qquad \text{a.s.}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u_t^2 = (R_3^*)_{0,0} + \gamma^2 [\sum_{p=0}^{\infty} (\xi_p - \beta_p)^2] + \mu^2 [\sum_{p=0}^{\infty} \alpha_p^2] \qquad \text{a.s.}$$

Asymptotic Optimality

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} (y_t - y_t^*)^2 = \gamma^2 [\sum_{k=0}^{d-1} f_k^2] + \mu^2 \qquad \text{a.s.}$$

Strong Consistency

$$\lim_{N\to\infty} \hat{\theta}_S(N) = \mathring{\theta}_S \triangleq (a_1,\ldots,a_n,b_0,\ldots,b_m,c_1,\ldots,c_\ell)^T \qquad \text{a.s.}$$

□

CHAPTER VII

THE MULTIVARIABLE CASE

7.1 Introduction

The purpose of this chapter is to generalize the unified result of Chapter V to multiple-input multiple-output (MIMO) systems. One of the few results to this date on the convergence of "multivariable discrete time stochastic adaptive control" algorithms is the one presented in [21]. In that paper, the model considered is of the form $a(z)y = zB(z)u + C(z)w$ where $a(z)$ is a scalar polynomial and $B(z)$ and $C(z)$ are $p \times p$ matrices whose entries are scalar polynomials. (We use the same notation as before for the processes y, u, w, although in this chapter they will be $p \times 1$ vectors.) The MIMO algorithm analysed in [21] consists of p recursions, and only the unit delay case is treated.

The MIMO algorithm of [21] can easily be modified to incorporate continually disturbed controls. However, a problem arises in the identification part of our scheme. On one hand, because of the particular model considered (with $a(z)$ a scalar polynomial), a rather undesirable type of positive real condition is required to be satisfied for the convergence of the AML recursion. On the other hand, if a new generic model of the form $A(z)y = zB(z)u + C(z)w$, with $A(z)$ a matrix, is adopted for identification purposes, the corresponding "control model", requiring a scalar polynomial to operate on y, will be of the form $\mathrm{Adj}[A(z)]A(z)y = z\mathrm{Adj}[A(z)]B(z)u + \mathrm{Adj}[A(z)]C(z)w$. In that case, $[\bar{a}(z)]I = [\det A(z)]I = \mathrm{Adj}[A(z)]A(z)$

and the new $\tilde{B}(z) = \text{Adj}[A(z)]B(z)$ are not coprime and consequently the identifiability condition fails to hold which prevents us from having persistency of excitation.

For these reasons, the approach adopted is to directly generalize the adaptive control and AML algorithms to the vector case using the technique presented by Caines in [39] (used for least squares in that book). We consider MIMO systems described by models of the form:

$$S: \quad A(z)y = z\,B(z)u + C(z)\,w \quad .$$

The recursions will now include a vector of parameters containing all the elements of the matrices $A(z)$, $B(z)$ and $C(z)$, and a matrix of regressors. Thus, multiple recursions algorithms are avoided when the delay is equal to one.

In the next sections, we derive the MIMO-adaptive control and MIMO-AML algorithms and then state the general version of our main result. The convergence proofs are given in Appendices C and D since they are straightforward generalizations of the proof of Theorem 3.1 (for adaptive control), and Solo's proof in [31] (for AML). We shall treat for simplicity the unit delay case, but we stress that extension to $d > 1$ is possible when the positive real condition is strengthened as in the previous chapter.

7.2 - Multivariable adaptive control

Consider multivariable linear time-invariant finite dimensional systems described by the ARMA model:

$$S: \qquad A(z)y = zB(z)u + C(z)w \tag{7.1}$$

with initial conditions given at $t = 0$, where y, u and w are $p \times 1$ vectors and $A(z), B(z), C(z)$ are $p \times p$ matrices whose entries are scalar polynomials in z, and are defined as follows:

$$\begin{aligned} A(z) &= I_p + A_1 z + \dots + A_n z^n \quad , \\ B(z) &= B_o + B_1 z + \dots + B_m z^m \quad , \\ C(z) &= I_p + C_1 z + \dots + C_\ell z^\ell \quad . \end{aligned} \tag{7.2}$$

We define $\bar{n} \underline{\Delta} \max(n, m+1, \ell)$ and $\bar{\rho} \underline{\Delta} 3\bar{n}p$. Instead of referring to two sets of assumptions as we did previously (one for the adaptive control part alone, the other for the combined control-identification scheme), we prefer to state immediately the complete list of assumptions that we need for the final result of Section 7.4. As before, let $F_o = G_o$ denote the σ-field generated by x_o, and for $t \geq 1$ F_t and G_t denote respectively those generated by $\{x_o, w_1, \dots, w_t\}$ and $\{x_o, w_1, \dots, w_t, \varepsilon_1, \dots, \varepsilon_t\}$, where ε is the $(p \times 1)$ "dither" process.

LIST OF HYPOTHESES

(M1) (a) $\det B(z) \neq 0$, $|z| \leq 1$

(b) $\det C(z) \neq 0$, $|z| \leq 1$

(M2) $\begin{bmatrix} w \\ \varepsilon \end{bmatrix}$ is an ergodic process;

(M3) All finite dimensional distributions of x_o and the $\begin{bmatrix} w \\ \varepsilon \end{bmatrix}$ process are mutually absolutely continuous with respect to Lebesgue measure;

(M4) $E[\begin{bmatrix} w \\ \varepsilon \end{bmatrix}_t | G_{t-1}) = 0$ a.s. $t \geq 1$

(M5) $E[\begin{bmatrix} w \\ \varepsilon \end{bmatrix}_t [w^T \ \varepsilon^T]_t | G_{t-1}) = \begin{bmatrix} \Gamma & 0 \\ 0 & M \end{bmatrix}_{2p \times 2p} > 0$ a.s. $t \geq 1$;

(M6) y^* is a bounded, deterministic (i.e. $\{\Omega, \phi\}$ measurable), $p \times 1$ vector sequence defined on $t \geq 1$;

(M7) $\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} y^*_{t-k} \, y^{*T}_{t-\ell}$ exists almost surely for all pairs of integers and depends upon the difference $k-\ell$;

(M8) Identifiability condition: $A(z)$ and $B(z)$ are left coprime and are row-reduced (i.e. A_n and B_m have full rank);

(M9) $C(e^{i\theta})^{-1} + C(e^{-i\theta})^{-T} - I \geq \rho I$

for some $\rho > 0$ and for all $\theta \in [0, 2\pi]$. □

Remark 7.1.

By definition, a matrix rational transfer function $Z(z)$ is strictly positive real if and only if (see e.g. [36]):

(i) $Z(z)$ has no poles in $\{z:|z| \leq 1\}$;

(ii) $Z(e^{i\theta}) + Z(e^{-i\theta})^T > 0$ for all $\theta \in [0,2\pi]$.

When $Z(z) = [C(z)^{-1} - \frac{I}{2}]$, (M1)b implies (i) (notice that the converse is not true in general), and therefore (M1)b and (M9) are sufficient conditions for $[C(z)^{-1} - \frac{I}{2}]$ to be strictly positive real.

Remark 7.2.

It is necessary to state only one positive real assumption because by Lemma A.3 (M1)b and (M9) imply:

$[C(z) - \frac{\bar{a}}{2} I]$ is strictly positive real for some $\bar{a} > 0$.

(We refer the reader to the "Remark on Lemma A.3" in Appendix A.)

The system equation (7.1) can be transformed to its (one step ahead) predictor form

$$C(z)[y_{t+1} - w_{t+1}] = [C(z) - A(z)]y_{t+1} + B(z)u_t$$

and subtracting $C(z)[y^*_{t+1} + \varepsilon_t]$ from both sides we obtain

$$C(z) \; [e_{t+1} - w_{t+1} - \varepsilon_t] = [C(z) - A(z)]y_{t+1} + B(z)u_t - C(z)[y^*_{t+1} + \varepsilon_t] \quad (7.3)$$

where $e_t \triangleq y_t - y^*_t$ is the $p \times 1$ control error vector. The right hand side of (7.3) can be written as

$$\sum_{i=0}^{\bar{n}-1}(C_{i+1}-A_{i+1})y_{t-i} + \sum_{i=0}^{\bar{n}-1} B_i u_{t-i} - \sum_{i=0}^{\bar{n}-1} C_{i+1}[y^*_{t-i} + \varepsilon_{t-i-1}]$$

$$- [y^*_{t+1} + \varepsilon_t] \quad .$$

This suggests the following definition for the predictor parameters vector:

$$\theta_o \triangleq [C^1_1 - A^1_1, \ldots, C^1_{\bar{n}} - A^1_{\bar{n}},\ B^1_o, \ldots, B^1_{\bar{n}-1},\ C^1_1, \ldots, C^1_{\bar{n}}, \ldots,$$
$$C^p_1 - A^p_1, \ldots, C^p_{\bar{n}} - A^p_{\bar{n}},\ B^p_o, \ldots, B^p_{\bar{n}-1},\ C^p_1, \ldots, C^p_{\bar{n}}\ ]^T_{p\bar{\rho} \times 1} \qquad (7.4)$$

where the superscript in A^i_j, for example, denotes the i-th row $(1 \times p)$ of the $p \times p$ matrix A_j. Hence, θ^T_o is simply the list of the p rows of $\{C_j - A_j,\ B_{j-1},\ C_j,\ 1 \le j \le \bar{n}\}$ one after the other.

We then define the regression vector

$$\phi(t) \triangleq [y^T_t, \ldots, y^T_{t-\bar{n}+1},\ u^T_t, \ldots, u^T_{t-\bar{n}+1},$$
$$-(y^*_t + \varepsilon_{t-1})^T, \ldots, -(y^*_{t-\bar{n}+1} + \varepsilon_{t-\bar{n}})^T]^T_{\bar{\rho} \times 1} \qquad (7.5)$$

and the regression matrix

$$X(t) \triangleq I_p \otimes \phi(t) = \begin{bmatrix} \phi(t) & & 0 \\ & \ddots\ \phi(t) \ddots & \\ 0 & & \phi(t) \end{bmatrix}_{p\bar{\rho} \times p} \qquad (7.6)$$

where $\otimes$ denotes the tensor product.

Using these definitions, we rewrite (7.3) as

$$\underset{p \times p}{C(z)}\,[e_{t+1} - \underset{p \times p}{w_{t+1}} - \varepsilon_t] = \underset{p \times p\bar{\rho}}{X(t)^T}\ \underset{p\bar{\rho} \times 1}{\theta_o} - \underset{p \times 1}{[y^*_{t+1} + \varepsilon_t]} \qquad (7.7)$$

We see that (7.7) is of the same form than (2.2), (3.7) and (6.7). The minimum variance control would be $X(t)^T\theta_o = y^*_{t+1} + \varepsilon_t$ if θ_o was known. Therefore, our adaptive control algorithm will be a simple generalization of (A1),(A2),(A^D3) to the vector case.

MIMO ADAPTIVE CONTROL ALGORITHM

Take $\{\hat{\theta}(1),\ldots,\hat{\theta}(\bar{n})\}$ and $\{u_1,\ldots,u_{\bar{n}}\}$ as arbitrary functions of the observations $\{y_1,\ldots,y_{\bar{n}}\}$ and initialize $r(1) = \ldots = r(\bar{n}) = 1$; then, for $t \geq \bar{n}+1$, set

(AM1) $$\hat{\theta}(t) = \hat{\theta}(t-1) + \frac{\bar{a}}{r(t-1)} X(t-1)\,[y_t - X(t-1)^T\,\hat{\theta}(t-1)],\quad \bar{a} > 0$$

(AM2) $$r(t) = r(t) + \mathrm{Tr}[X(t)^T X(t)]$$

(AM3) $$X(t)^T\hat{\theta}(t) = y^*_{t+1} + \varepsilon_t$$ □

(Tr[F] denotes the trace of the matrix F.)

7.3 - Identification of multivariable systems using the AML algorithm

It is possible to derive a multivariable version of the AML recursion of Solo [31] using the same technique as in section 7.2. To begin with, we write $\rho \,\underline{\Delta}\, p[n+m+1+\ell]$ and, in analogy with (7.4), define a vector of system parameters

$$\theta_S \,\underline{\Delta}\, [A^1_1,\ldots,A^1_n,\ B^1_o,\ldots,B^1_m,\ C^1_1,\ldots,C^1_\ell,\ \ldots \\ A^p_1,\ldots,A^p_n,\ B^p_o,\ldots,B^p_m,\ C^p_1,\ldots,C^p_\ell\,]^T_{p\rho\times 1} \quad (7.8)$$

where as before the superscript denotes the rows of the matrices. The "identification" regression vector (see (4.6)) will be

$$\phi^I(t) \triangleq [-y_t^T,\ldots,-y_{t-n+1}^T, u_t^T,\ldots,u_{t-m}^T, w_t^T,\ldots,w_{t-\ell+1}^T]^T_{\rho\times 1} \quad (7.9)$$

and the corresponding regression matrix

$$X^I(t) \triangleq I_p \times \phi^I(t) = \begin{bmatrix} \phi^I(t) & & 0 \\ & \phi^I(t) & \\ 0 & & \phi^I(t) \end{bmatrix}_{p\rho\times p} \quad (7.10)$$

These definitions allow us to rewrite (7.1) in the more useful form:

$$\underset{p\times 1}{y_t} = \underset{p\times p\rho}{X(t-1)^T}\ \underset{p\rho\times 1}{\theta_S} + \underset{p\times 1}{w_t} \quad (7.11)$$

But, as we recall from section 5.2, the regression vector of the AML recursion is defined with the residuals (or a posteriori errors) in place of the prediction errors. Let η denote the residuals process; we define

$$\varphi(t) \triangleq [-y_t^T,\ldots,-y_{t-n+1}^T, u_t^T,\ldots,u_{t-m}^T, \eta_t^T,\ldots,\eta_{t-\ell+1}^T]^T_{\rho\times 1} \quad (7.12)$$

and

$$\psi(t) \triangleq I_p \otimes \varphi(t) = \begin{bmatrix} \varphi(t) & & 0 \\ & \varphi(t) & \\ 0 & & \varphi(t) \end{bmatrix}_{p\rho\times p} \quad (7.13)$$

Consequently, the AML algorithm will be:

MIMO-AML ALGORITHM

Take $\{\hat{\theta}_S(1), \ldots, \hat{\theta}_S(\bar{n})\}$, $\{e_1, \ldots, e_{\bar{n}}\}$ and $\{\eta_1, \ldots, \eta_{\bar{n}}\}$ as arbitrary functions of the observations $\{y_1, \ldots, y_{\bar{n}}, u_1, \ldots, u_{\bar{n}}\}$ and initialize $P(1) = \ldots = P(\bar{n}) = I$; then, for $t \geq \bar{n}+1$, set

$$\hat{\theta}_S(t) = \hat{\theta}_S(t-1) + P(t-1)\psi(t-1)e_t \tag{7.14}$$

$$P(t)^{-1} = P(t-1)^{-1} + \psi(t)\psi(t)^T \tag{7.15}$$

$$e_t = y_t - \psi(t-1)^T\hat{\theta}_S(t-1) \tag{7.16}$$

$$\eta_t = y_t - \psi(t-1)^T\hat{\theta}_S(t) \tag{7.17}$$

□

From (7.14), we have

$$\eta_t = [I_p - \psi(t-1)^T P(t-1)\psi(t-1)]e_t \quad . \tag{7.18}$$

We remark that $P(t)$ is a $p\rho \times p\rho$ matrix, and $\psi(t)^T P(t)\psi(t)$ a $p \times p$ matrix.

For the sake of completeness, we now present in this thesis a theorem which gives sufficient conditions for the convergence of the above algorithm. Since the proof is a straightforward generalization of [31], it is given in Appendix D.

In his proof, Solo uses many results derived from the matrix inversion lemma. All of them have equivalents in the vector case, although the proofs may be more complicated. In Lemma D.1 of Appendix D we present the generalized versions of these results.

Theorem 7.1.

Let the system S of (7.1) satisfy the following assumptions:

(a) w is an ergodic martingale difference process satisfying

$$E(w_t \mid F_{t-1}) = 0 \qquad \text{a.s.} \qquad t \geq 1 \tag{7.19}$$

$$E(w_t w_t^T \mid F_{t-1}) = \Gamma \qquad \text{a.s.} \qquad t \geq 1 \tag{7.20}$$

when F_t denotes the σ-field generated by $\{x_o, w_1, \ldots, w_t\}$.

(b) S is asymptotically stable in the sense that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \|y_t\|^2 < \infty \qquad \text{a.s.} \tag{7.21}$$

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \|u_t\|^2 < \infty \qquad \text{a.s.} \tag{7.22}$$

(c) $C(e^{i\theta})^{-1} + C(e^{-i\theta})^{-T} - I \geq \rho I$ for some $\rho > 0$ and for all $\theta \in [0, 2\pi]$ (7.23a)

and

$$\det C(z) \neq 0, \ |z| \leq 1. \tag{7.23b}$$

(This clearly implies that $[C(z)^{-1} - \frac{I}{2}]$ is strictly positive real.)

(d)
$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \phi^I(t) \phi^I(t)^T = R > 0 \qquad \text{a.s.} \tag{7.24}$$

where $\phi^I(t)$ is defined in (7.9)

(persistent excitation condition).

Then, the algorithm (7.14) to (7.17) converges almost surely, i.e.

$$\lim_{N \to \infty} \hat{\theta}_S(N) = \theta_S \qquad \text{a.s.} \tag{7.25}$$

□

7.4 - Main result - multivariable case

Theorem 7.2.

Let the system S described by (7.1), the noise process w, the exogenous noise process ε and the demand sequence y^* satisfy hypotheses (M1) to (M9) of section 7.2.

Let S be subject to adaptive control with "continually disturbed controls" by use of the recursive algorithm (AM1),(AM2),(AM3).

Let also S be simultaneously the object of recursive identification by use of the AML algorithm (7.14) to (7.17). (Refer to Figure 5.1.)

Then the resulting sample paths of u, y and $\hat{\theta}_S$ are such that the following properties hold (new notation is defined in the proof):

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \|y_t\|^2 = \mathrm{Tr}[(\underline{R}_1^*)_{0,0} + \Gamma + M] \quad \text{a.s.} \tag{7.26}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \|u_t\|^2 = \mathrm{Tr}[(\underline{R}_3^*)_{0,0} + \Gamma[\sum_{p=0}^{\infty} (\underline{\alpha}_p - \underline{\beta}_p)^T(\underline{\alpha}_p - \underline{\beta}_p)] + M[\sum_{p=0}^{\infty} \underline{\alpha}_p^T \underline{\alpha}_p]] \quad \text{a.s.} \tag{7.27}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \|y_t - y_t^*\|^2 = \mathrm{Tr}[\Gamma + M] \quad \text{a.s.} \tag{7.28}$$

$$\lim_{N\to\infty} \hat{\theta}_S(N) = \overset{\circ}{\theta}_S \quad \text{a.s.} \tag{7.29}$$

□

Proof: Given in Appendix C.

CHAPTER VIII

CONCLUSION

8.1 - Discussion of the results

In this thesis, we have presented a combined adaptive control and system identification algorithm which, to the best of our knowledge, is the first one to simultaneously carry out these tasks for an initially unknown stochastic system S. The two-recursion-scheme was adopted because only partial results were obtained concerning the behaviour of the estimator $\hat{\theta}(t)$ generated by the stochastic approximation algorithm (A1,(A2),(A^D3).

It appears that the behaviour of $\hat{\theta}(t)$ can be very complicated, reminiscent of the "chaotic" motions appearing in the mathematical theory of dynamical systems [49]. However, we know that the estimation error vector$\{\tilde{\theta}(t); t \geq 1\}$ converges into the surface of a sphere of fixed random radius around the origin. Specifically, $\|\tilde{\theta}(t)\|^2 \to Z(\infty)$ a.s. as $N \to \infty$, with $E[Z(\infty)] < \infty$ (see e.g. the proof of Theorem 3.1 for the definition of $Z(t)$), and $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \|\tilde{\theta}(t) - \tilde{\theta}(t-m)\| = 0$ a.s. for finite m. But, it appears that some sort of averaging of $\{\hat{\theta}(t), t \geq 1\}$ (for example $\frac{1}{N} \sum_{t=1}^{N} \tilde{\theta}(t)$) may be necessary to create a consistent estimator.

The reader will have noticed that in the scalar case, two sets of assumptions we used throughout the thesis: (S1)-(S2), (T1)-(T2),(W^D1)-(W^D4),(E1)-(E3), and (I)-(X). Such a procedure was chosen to emphasize the fact that the assumptions required for the convergence of the adaptive control algorithm alone are less restrictive than those required for the convergence of the combined adaptive control-recursive identification algorithm.

Concerning these assumptions, we notice that although they may seem restrictive, they reflect, as we said in the introduction, the current state-of-the-art of the theory. For example, it is quite natural to assume that one must know the degree of complexity of a system if one wishes to (asymptotically) achieve the same performance with an adaptive control algorithm as the one achieved when the system parameters are known.

Many authors have tried to relax the positive real assumptions (see e.g. [23,37]). However, it should be noticed that techniques like over-parameterization are not applicable in our case because of the identifiability condition we require. In fact, identifiability conditions are inevitable in any identification method.

The author believes that the extension to the general delay-colored noise case (Chapter VI) and to the multivariable case (Chapter VII) of the algorithm of [21], and consequently of the main results of Chapter V, are also interesting contributions of this thesis. In particular, the technique we used to rewrite the system equation (7.1) in the form given by equations (7.7) and (7.11) proved to be very useful in deriving the vector versions of both the adaptive control and AML recursive algorithms. (This technique, presented in [39], is due to D.Q. Mayne.)

8.2 - Suggestions for future work

The next logical step of this research would be to undertake a series of numerical simulations to test the practical performance of the algorithm presented in this thesis. In addition to testing the algorithm under conditions where, according to the theoretical analysis, it is supposed to perform well, we believe it would be of prime interest to evaluate its performance in various cases of time-varying parameters and study its tracking ability. It would also be interesting to test the algorithm on the two brenchmark examples now being developed for the evaluation of adaptive schemes (see [46]). In other respects, concerning the results of [44], we point out that one should not expect good performance from any algorithm when the assumptions necessary for its proper operation are violated.

Adaptive control in general has been applied successfully in many practical situations (see e.g. [12],[45]). However, efforts to elaborate the theory of adaptive control are still justified! Important research topics are:

- Improvement of the applicability of current adaptive control algorithms by trying to relax the more restrictive assumptions, namely the inverse stability assumption and the positive real assumption.
- Generalization to time-varying parameters, an "ultimate objective " of parameter adaptive control. The case where the parameter process is a convergent martingale has been

treated successfully by Caines in [30]. The results of this paper may constitute the first steps towards a stochastic control theory for systems with non-convergent randomly varying parameters.

- Elimination of the necessity of a multiple recursion algorithm in the general delay case (scalar and multivariable). A single adaptation algorithm would be more appropriate for practical implementation. We refer to reader to the recent work of Fuchs on that problem [50]. In fact this, together with a derivation of a $d > 1$ multivariable predictor which is consistent with our identifiability conditions on $\{A(z), z^d B(z), C(z)\}$, is probably needed to obtain a completely general "multivariable, $d > 1$, $C(z) \not\equiv I$" adaptation-with-identification result.

- Replacement of the two-recursion-algorithm presented in this thesis by a single recursive algorithm which would carry out both the adaptive control and system identification tasks. The author believes that such an algorithm would have to be of the least squares type.

APPENDIX A

Lemma A.1. - (Neveu [40] pp.148-150)

Let $\{x_t;F_t,t \in \mathbb{Z}_+\}$ be a centered sequence of scalar random variables and let $\{x_t\}$ be $\{F_t\}$ adapted.

If $\sum_{t=o}^{\infty} \frac{1}{t^2} E x_t^2|F_{t-1}) < \infty$ a.s.,

then $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} x_t = 0$ a.s. □

Lemma A.2.

Let $\{x_t; F_t; t \in \mathbb{Z}_+\}$ be a martingale difference process satisfying $E(x_t^2 | F_{t-1}) = \sigma^2$.

If $E(x_t^4 | F_{t-1}) < P < \infty$, then $\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} x_t^2 = \sigma^2$ a.s. □

Proof.

Consider $y_t = x_t^2 - E(x_t^2 | F_{t-1}) = x_t^2 - \sigma^2$. Then,

$E(x_t^2 - \sigma^2) | F_{t-1}) = 0$, i.e. y_t is a centered sequence, and $\{y_t\}$ is $\{F_t\}$ adapted. Also,

$$\sum_{t=1}^{\infty} \frac{1}{t^2} E((x_t^2 - \sigma^2)^2 | F_{t-1}) = \sum_{t=1}^{\infty} \frac{1}{t^2} [E(x_t^4 | F_{t-1}) - 2\sigma^4 + \sigma^4]$$

$$< \sum_{t=1}^{\infty} \frac{1}{t^2} [P - \sigma^4]$$

$$< \infty .$$

Hence, by Lemma A.1,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} (x_t^2 - \sigma^2) = 0 \quad \text{a.s.}$$

or

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} x_t^2 = \sigma^2 \quad \text{a.s.}$$

□

<u>Lemma A.3 - SCALAR VERSION</u>

If $\left[\frac{1}{c(z)} - \frac{1}{2}\right]$ is strictly positive real, then

(i) c(z) is an asymptotically stable polynomial;

(ii) $\exists\, \bar{a} > 0$ such that $\left[c(z) - \frac{\bar{a}}{2}\right]$ is strictly positive real. □

<u>Proof</u>.

(i) from the definition of the strict positive real property;

(ii) if $\left[\frac{1}{c(z)} - \frac{1}{2}\right]$ is strictly positive real, then

$$\left(\frac{1}{c(z)} - \frac{1}{2}\right) + \left(\frac{1}{c(z)} - \frac{1}{2}\right)^* > 0 \qquad \forall |z| = 1$$

$$\frac{1}{c(z)} + \frac{1}{c^*(z)} > 1 \qquad "$$

$$c(z) + c^*(z) > c(z)c^*(z) \qquad "$$

$$c(z) + c^*(z) > \inf\,(c(z)c^*(z)) \qquad "$$

Since c is asymptotically stable, then $\inf\,(c(z)c^*(z)) > \bar{a}$ for some $\bar{a} > 0$ and so $c(z) + c^*(z) > \bar{a}$, $\forall |z| = 1$ which implies that $\left[c(z) - \frac{\bar{a}}{2}\right]$ is strictly positive real. □

Lemma A.3 - MULTIVARIABLE VERSION

If $\det C(z) \neq 0$, $|z| = 1$, and

$C(e^{i\theta})^{-1} + C(e^{-i\theta})^{-T} - I \geq \rho I$ for some $\rho > 0$ and for all $\theta \in [0,2\pi]$, then the polynomial transfer function matrix $[C(z) - \frac{\bar{a}}{2} I]$ is strictly positive real for some $\bar{a} > 0$. □

Proof.

From the second hypothesis,

$$[C(e^{i\theta})^{-1} - \frac{I}{2}] + [C(e^{-i\theta})^{-T} - \frac{I}{2}] \geq \rho I^{(1)} \quad \forall\, \theta \in [0,2\pi]$$

where ρ is a small positive constant. So we have

$$C(e^{i\theta})^{-1} + C(e^{-i\theta})^{-T} \geq (1+\rho) I .$$

$$C(e^{i\theta}) + C(e^{-i\theta})^{T} \geq (1+\rho)\, C(e^{i\theta})\, C(e^{-i\theta})^{T} \quad \forall\, \theta \in [0,2\pi].$$

Since $C(z)$ has no poles on $|z| = 1$ $C(z)$ is continuous there. Hence, $\inf \lambda^T C(e^{i\theta})\, C(e^{-i\theta})^T \lambda$, $\lambda \neq 0$, is achieved at some $\theta^* \in [0,2\pi]$. The infimum is positive since $\lambda^T C(e^{i\theta^*}) = 0$ contradicts $\det C(e^{i\theta}) \neq 0$ for $\theta \in [0,2\pi]$.

Consequently,

$$C(e^{i\theta}) + C(e^{-i\theta})^T \geq (\bar{a} + \rho) I, \qquad \forall\, \theta \in [0,2\pi],$$

where we take, say $(\bar{a} + \rho) I = \frac{1}{2}\, C(e^{i\theta^*})\, C(e^{-i\theta^*})^T$.

(1) In this thesis, except otherwise stated, all matricial inequalities of the form $M < N$ have the interpretation $\lambda^T M \lambda < \lambda^T N \lambda$ for all non-zero vector λ of appropriate dimension.

Finally we have

$$[C(e^{i\theta}) - \frac{\bar{a}}{2} I] + [C(e^{-i\theta})^T - \frac{\bar{a}}{2} I] \geq \rho I$$

which proves that $[C(z) - \frac{\bar{a}}{2} I]$ is strictly positive real. □

Remark on Lemma A.3.

In the multivariable case, the strict positive reality of $[C(z)^{-1} - \frac{I}{2}]$ is not sufficient to imply the required asymptotic stability condition on $C(z)^{-1}$ which is (M1)b: $\det C(z) \neq 0$, $|z| \leq 1$. In fact, $[C(z)^{-1} - \frac{I}{2}]$ strictly positive real implies, by definition,

$$C(z)^{-1} - \frac{I}{2} = \frac{1}{\det C(z)} \text{Adj } C(z)[I - \frac{1}{2} C(z)] \quad \text{analytic in}$$

$|z| \leq 1$. But it seems it may well happen that $\det C(z)$ has a common (unstable) factor with every element of the polynomial matrix $\text{Adj } C(z)[I - \frac{1}{2} C(z)]$. This explains why (M1)b has to be included in the list of hypotheses, in opposition with the scalar case where $c(z)$ and $[c(z) - 2]$ are coprime and so $\frac{1}{c(z)}$ is asymptotically stable when $[\frac{1}{c(z)} - \frac{1}{2}]$ is strictly positive real.

Lemma A.4. (Strict) Positive Real Lemma (see e.g. [21]).

Consider the following minimal state space model

$$x_{t+1} = Ax_t + B z_t, \text{ with initial condition } x_o$$
$$h_t = C x_t + D z_t .$$

If the complex valued transfer function

$Z(z) = C[z^{-1}I - A]^{-1}B + D$ is strictly positive real, then:

(a) there exist matrices P,L,W, with P > 0, such that

$$A^T PA - P = -LL^T \quad (1)$$
$$A^T PB = C^T - LW \quad (2)$$
$$\rho I + W^T W = D + D^T - B^T PB \quad (3)$$

for all sufficiently small $\rho > 0$;

(b) $$2 \sum_{t=1}^{N} h_t^T z_t + x_o^T P x_o \geq \rho \sum_{t=1}^{N} \|z_t\|^2 > 0 \quad \text{for } z_t \not\equiv 0 .$$

□

Proof:

(a) see Hitz and Anderson [36].

(b) Consider

$$x_n^T P x_n = [Ax_{n-1} + Bz_{n-1}]^T P[Ax_{n-1} + B z_{n-1}] .$$

$$= x_{n-1}^T A^T P A x_{n-1} + 2 x_{n-1}^T A^T P B z_{n-1} + z_{n-1}^T B^T P B z_{n-1}$$

$$= x_{n-1}^T P x_{n-1} - x_{n-1}^T L L^T x_{n-1} + 2 x_{n-1}^T [C^T - L W] z_{n-1}$$

$$+ z_{n-1}^T [D + D^T - W^T W - \rho I] z_{n-1} \quad \text{from (1),(2),(3)}$$

$$= x_{n-1}^T P x_{n-1} - [L^T x_{n-1} + W z_{n-1}]^T [L^T x_{n-1} + W z_{n-1}]$$

$$+ 2[h_{n-1} - D z_{n-1}]^T z_{n-1} + z_{n-1}^T [D + D^T - \rho I] z_{n-1}$$

$$= x_{n-1}^T P x_{n-1} - [L^T x_{n-1} + W z_{n-1}]^T [L^T x_{n-1} + W z_{n-1}]$$

$$+ 2 h_{n-1}^T z_{n-1} - z_{n-1}^T \rho I z_{n-1} .$$

Summing from 1 to N we have

$$0 \leq x_n^T P x_n \leq 2 \sum_{t=1}^{N} h_t^T z_t + x_o^T P x_o - \rho \sum_{t=1}^{N} \| z_t \|^2$$

or

$$2 \sum_{t=1}^{N} h_t^T z_t + x_o^T P x_o \geq \rho \sum_{t=1}^{N} \| z_t \|^2 \geq 0$$

$$> 0 \quad \text{if } z_t \not\equiv 0$$

and $x_o^T P x_o > 0$ since $P > 0$. □

APPENDIX B

Proof of Lemma 4.1:

The statement of Lemma 4.1 is that if (w,ε) is a stochastic process satisfying (II), (III), (IV), (see section 4.2), if y^* is a deterministic process satisfying (VI), (VII), if z is such that $\frac{1}{N}\sum_{t=1}^{N} z_t^2 \to 0$ a.s. as $N \to \infty$ and if $a_1(z),\ldots,a_4(z), d_1(z),\ldots,d_4(z)$ are asymptotically stable polynomials, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\left[\frac{b_1(z)}{a_1(z)}w, \frac{b_2(z)}{a_2(z)}\varepsilon, \frac{b_3(z)}{a_3(z)}y^*, \frac{b_4(z)}{a_4(z)}z\right]_t^T\left[\frac{c_1(z)}{d_1(z)}w, \frac{c_2(z)}{d_2(z)}\varepsilon, \frac{c_3(z)}{d_3(z)}y^*, \frac{c_4(z)}{d_4(z)}z\right]_t$$

$$= \text{Diag}\left[\frac{\gamma^2}{2\pi}\int_0^{2\pi}\frac{b_1(e^{i\theta})c_1(e^{-i\theta})}{a_1(e^{i\theta})d_1(e^{-i\theta})}d\theta,\ \frac{\mu^2}{2\pi}\int_0^{2\pi}\frac{b_2(e^{i\theta})c_2(e^{-i\theta})}{a_2(e^{i\theta})d_2(e^{-i\theta})}d\theta,\right.$$

$$\left.\frac{1}{2\pi}\int_0^{2\pi}\frac{b_3(e^{i\theta})dR^*(e^{i\theta})c_3(e^{-i\theta})}{a_3(e^{i\theta})d_3(e^{-i\theta})},\ 0\ \right] \qquad \text{(B.1)}$$

We recall that the notation $[\frac{b(z)}{a(z)}\xi]_t$ or, equivalently, $\frac{b(z)}{a(z)}\xi_t$ denotes the value at the instant t of the process η generated by the ARMA scheme $a_o\eta_t + \ldots + a_n\eta_{t-n} = b_o\xi_t + \ldots + b_m\xi_{t-m}$, for some n,m, where this scheme is equipped with initial conditions at $t = 0$.

The analysis below of the behaviour of the Cesaro sums on the left of (B.1) will, in each case, involve three steps: first we show the effect of the initial conditions is asymptotically negligible, second we prove the existence of a limit for these sums by considering the Cesaro sums of finite regressions and then lastly we provide formulae characterizing these limits.

<u>The</u> $\begin{bmatrix} ww & w\varepsilon \\ w\varepsilon & \varepsilon\varepsilon \end{bmatrix}$ <u>terms</u>

We begin by treating the stochastic terms appearing in the 2 × 2 sub-matrix appearing in the top left position of the 4 × 4 matrix appearing inside the limit in (B.1).

Clearly we only have to deal with terms of the form

$$\frac{1}{N} \sum_{t=1}^{N} [\frac{b(z)}{a(z)} w]_t \, [\frac{c(z)}{d(z)} \varepsilon]_t \quad \text{and} \quad \frac{1}{N} \sum_{t=1}^{N} [\frac{b(z)}{a(z)} w]_t \, [\frac{c(z)}{d(z)} w]_t ,$$

the other two terms being treated by identical arguments, since the w and ε processes jointly satisfy (II), (III), (IV).
(Here we have dropped the subscripts for simplicity of notation.)

In order to demonstrate the asymptotically negligible effect of the initial conditions we can treat both terms simultaneously. We shall let v in the second summand denote either ε or w.

Since the ARMA systems above may be realized via time invariant finite dimensional linear state space systems, the response to the initial conditions may be described by the addition of terms of the form $HF^t x_o$, $\underline{H}\,\underline{F}\underline{x}_o$. Taking the realizations to be minimal realizations the asymptotic stability of a(z) and d(z) implies the asymptotic stability of F and $\underline{F}$. We shall represent the input response of the ARMA systems in question by their associated Markov matrix sequences denoted $\{M_o, M_1, \ldots\}$ and $\{\underline{M}_o, \underline{M}_1, \ldots\}$ respectively. Then we obtain the following description of the corresponding Cesaro sums appearing in (B.1):

$$\frac{1}{N} \sum_{t=1}^{N} \left(\sum_{k=o}^{t} M_k w_{t-k} + HF^t x_o \right) \left(\sum_{j=o}^{t} \underline{M}_j v_{t-j} + \underline{H}\,\underline{F}^t \underline{x}_o \right) \qquad \text{(B.2)}$$

Now the initial condition effects decay if the terms

$$\frac{1}{N}\sum_{t=1}^{N}(HF^t x_o)(\underline{H}\,\underline{F}^t \underline{x}_o),\quad \frac{1}{N}\sum_{t=1}^{N}(HF^t x_o)\left[\sum_{j=o}^{t}\underline{M}_j v_{t-j}\right] \quad \text{and}$$

$$\frac{1}{N}\sum_{t=1}^{N}\left[\sum_{k=o}^{t}M_k w_{t-k}\right]\underline{H}\,\underline{F}^t \underline{x}_o \qquad \text{go to zero a.s.} \quad \text{as } N \to \infty .$$

Again we see that it is sufficient to treat the first and second of these initial condition terms. The first term evidently converges to zero a.s. since for some α, $0 < \alpha < 1$, $K(\omega) > 0$, $|HF^t x_o| < K(\omega)\alpha^t \to 0$ a.s. as $t \to \infty$ and similarly for $\underline{H}\,\underline{F}^t \underline{x}_o$. As to the second we observe that, since in either case v is an ergodic process with $E|v_o| < \infty$ and since $|\underline{M}_k| < L\alpha^k$ for some $L > 0$ and α, $0 < \alpha < 1$, the sum $\sum_{j=o}^{t}\underline{M}_j v_{t-j}$ is bounded as $\left|\sum_{j=o}^{t}\underline{M}_j v_{t-j}\right| \leq L\sum_{j=o}^{t}\alpha^j|v_{t-j}| \leq L\sum_{j=o}^{\infty}\alpha^j|v_{t-j}|$ where the latter majorant is an ergodic stationary process with $E(L\sum_{j=o}^{\infty}\alpha^j|v_{t-j}|) \leq \frac{1}{1-\alpha}E|v_o| < \infty$. Call this process $\mu = \{\mu_t, t = \ldots -1, 0, 1, \ldots\}$ and for any $\varepsilon > 0$ take $K = K(\omega)$ so that $|HF^t x_o(\omega)| < \varepsilon$ for $t > K$. Now split the sum $\frac{1}{N}\sum_{t=1}^{N}(HF^t x_o)(\sum_{j=o}^{t}\underline{M}_j v_{t-j})$ at K and bound it by

$$\frac{1}{N}\sum_{t=1}^{K-1}\mu_t|HF^t x_o| + \frac{1}{N}\sum_{t=K}^{N}\mu_t\,\varepsilon \qquad \text{(B.3)}$$

In this expression the first term goes to zero a.s. as $N \to \infty$ and the second converges a.s. to $E\mu_o\varepsilon$. Since ε was arbitrary the convergence of the second initial condition term to zero a.s. has been established.

We conclude that in each case the difference between

$$\frac{1}{N}\sum_{t=1}^{N}\left[\frac{b(z)}{a(z)}w\right]_t\left[\frac{c(z)}{d(z)}v\right]_t \tag{B.4}$$

and

$$\frac{1}{N}\sum_{t=1}^{N}\left(\sum_{k=o}^{t}M_k w_{t-k}\right)\left(\sum_{j=o}^{t}\underline{M}_j v_{t-j}\right) \tag{B.5}$$

converges to zero a.s. as $N \to \infty$.

It remains to show that the latter expression converges a.s. as $N \to \infty$. We do this by showing that the difference between this expression and

$$\frac{1}{N}\sum_{t=1}^{N}\left(\sum_{k=o}^{K}M_k w_{t-k}\right)\left(\sum_{j=o}^{K}\underline{M}_j v_{t-j}\right) . \tag{B.6}$$

can be made a.s. less than any $\varepsilon > 0$ for all $N > N_\varepsilon(\omega)$ for sufficiently large fixed $K > K_\varepsilon$. Now this difference is equal to

$$\frac{1}{N}\sum_{t=1}^{N}\left(\sum_{k=K+1}^{t}M_k w_{t-k}\right)\left(\sum_{k=K+1}^{t}\underline{M}_k v_{t-k}\right) \tag{B.7a}$$

$$+\ \frac{1}{N}\sum_{t=1}^{N}\left(\sum_{k=o}^{K}M_k w_{t-k}\right)\left(\sum_{k=K+1}^{t}\underline{M}_k v_{t-k}\right) \tag{B.7b}$$

$$+\ \frac{1}{N}\sum_{t=1}^{N}\left(\sum_{k=K+1}^{t}M_k w_{t-k}\right)\left(\sum_{k=o}^{K}\underline{M}_k v_{t-k}\right) \tag{B.7c}$$

(here upper bounds of sums are taken as 0 whenever $t < K+1$)

So it is sufficient to show that (B.7) can be made less than ε a.s. whenever N is greater than some N_ε and K is greater than some K_ε .

For the first term (B.7a) we have the bounds .

$$\left| \frac{1}{N} \sum_{t=1}^{N} \left(\sum_{k=K+1}^{t} M_k w_{t-k} \right) \left(\sum_{k=K+1}^{t} \underline{M}_k v_{t-k} \right) \right|$$

$$\leq \frac{1}{N} \sum_{t=1}^{N} \left(L \sum_{k=K+1}^{\infty} \alpha^k |w_{t-k}| \right) \left(\underline{L} \sum_{k=K+1}^{\infty} \underline{\alpha}^k |v_{t-k}| \right) \qquad \text{(B.8)}$$

$$\leq 2\, \alpha^{K+1}\, \underline{\alpha}^{K+1}\, E\, \mu_o\, \underline{\mu}_o \quad \text{a.s. for all } N > N_2(\omega) . \qquad \text{(B.9)}$$

This follows since each of the processes $\mu \underline{\Delta} \{\mu_t \underline{\Delta} L \sum_{k=o}^{\infty} \alpha^k |w_{t-k}|, t \geq 1\}$ and $\underline{\mu} \Delta \{\underline{\mu}_t = \underline{L} \sum_{k=o}^{\infty} \underline{\alpha}^k |v_{t-k}|, t \geq 1\}$ is ergodic and so consequently is their product . The average of this ergodic product process converges to a finite quantity a.s. since

$$E\mu_o\, \underline{\mu}_o = L\, \underline{L}\, E\left(\sum_{k=o}^{\infty} \alpha^k |w_{t-k}| \right) \left(\sum_{k=o}^{\infty} \underline{\alpha}^k |v_{t-k}| \right)$$

$$= L\, \underline{L}\, E\left(\sum_{j=0}^{\infty} \sum_{k=o}^{\infty} \alpha^k\, \underline{\alpha}^j |w_{t-k}|\, |v_{t-j}| \right)$$

$$= L\, \underline{L} \sum_{k=o}^{\infty} \sum_{j=o}^{\infty} \alpha^k \underline{\alpha}^j\, E|w_{-k}|\, |v_{-j}|$$

$$= L\, \underline{L}\, \frac{1}{(1-\alpha)}\, \frac{1}{(1-\underline{\alpha})} \left(E|w_o|^2 \right)^{\frac{1}{2}} \left(E|v_o|^2 \right)^{\frac{1}{2}} < \infty .$$

Since (B.8) must remain less than, say, twice the modulus of its limiting value for all N greater than some random $N_2(\omega)$ (depending upon the factor 2) we get the bound (B.9). Since $\alpha^{2K+2} \to 0$ as $K \to \infty$ we have established the desired property for (B.7a) .

The terms (B.7b) and (B.7c) are obviously similar to each other and so we shall just deal with the first such term (B.7b):

$$\left| \frac{1}{N} \sum_{t=1}^{N} \left(\sum_{k=o}^{K} M_k w_{t-k} \right) \left(\sum_{k=K+1}^{t} \underline{M}_k v_{t-k} \right) \right|$$

$$\leq \frac{1}{N} \sum_{t=1}^{N} \{L \sum_{k=o}^{K} \alpha^k |w_{t-k}|\} \{\alpha^{K+1} \underline{L} \sum_{k=o}^{\infty} \underline{\alpha}^k |v_{t-(K+1)-k}|\} \quad \text{(B.10a)}$$

$$\leq 2\alpha^{K+1} E\{(\underline{L} \sum_{k=o}^{K} \alpha^k |w_{-k}|) \underline{\mu}_o\} \leq 2 \alpha^{K+1} E \mu_o \underline{\mu}_o < \infty \quad \text{(B.10b)}$$

a.s. for all $N > N'_2(\omega)$ by the ergodicity of the two process in (B.10a). The bound in (B.10b) can evidentally be made less than arbitrary $\varepsilon > 0$ for K suitably large.

Since we have now demonstrated that the difference between the two expressions (B.5) and (B.6) can be made a.s. arbitrarily small (in the limit as $N \to \infty$) by increasing K we may obtain the limit of (B.4) via the evaluation of the limits (B.6) for each K. To do this we must distinguish the two cases $v = w$ and $v = \varepsilon$.

First let $v = w$; exchanging the finite summations in (B.6) and letting $N \to \infty$ yields

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \left(\sum_{k=o}^{K} M_k w_{t-k} \right) \left(\sum_{j=o}^{K} \underline{M}_j w_{t-j} \right)$$

$$= \sum_{k=o}^{K} \sum_{j=o}^{K} M_k \underline{M}_j \lim_{N \to \infty} \frac{1}{N} \sum_{t=\max(k,j)}^{N} w_{t-k} w_{t-j}$$

$$= \sum_{k=o}^{K} \sum_{j=o}^{K} M_k \underline{M}_j \sigma^2 \delta_{k,j} \quad \text{a.s.} \quad 1 \leq k,j \leq K, \quad \text{(B.11)}$$

by the ergodicity and orthogonality of w.

But (B.11) may be written

$$\frac{1}{2\pi} \int_0^{2\pi} (\sum_{k=o}^{K} M_k e^{ik\theta}) \sigma^2 (\sum_{j=o}^{K} \underline{M}_j e^{-ij\theta}) d\theta \qquad (B.12)$$

and the limit of (B.12) as $K \to \infty$ is given by

$$\frac{1}{2\pi} \int_0^{2\pi} (\sum_{k=o}^{\infty} M_k e^{ik\theta}) \sigma^2 (\sum_{k=o}^{\infty} \underline{M}_j e^{-ij^{\theta}}) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{b_1(e^{i\theta})}{a_1(e^{i\theta})} \frac{c_1(e^{-i\theta})}{d_1(e^{-i\theta})} \sigma^2 \, d\theta \qquad (B.13)$$

because of the uniform convergence over $[0,2\pi]$ of the partial sums $\sum_{k=o}^{K} M_k e^{ik\theta}$, etc, to $\frac{b_1(e^{i\theta})}{a_1(e^{i\theta})}$, etc. We conclude that (B.13) is the a.s. limit of (B.2).

In case $v = \varepsilon$ the joint ergodicity of w and ε and their orthogonality yields 0 as the limit of $\frac{1}{N} \sum_{t=1}^{N} [\frac{b(z)}{a(z)}]_t [\frac{c(z)}{d(z)}]_t$.

In the remaining case where both processes are identical to ε we obviously obtain $\frac{1}{2\pi} \int_0^{2\pi} \frac{b_2(e^{i\theta})}{a_2(e^{i\theta})} \frac{c_2(e^{-i\theta})}{d_2(e^{-i\theta})} \mu^2 \, d\theta$, as the appropriate limit.

<u>The</u> $[{}^{wy^*}_{\varepsilon y^*}]$, $[y^* w, y^* \varepsilon]$ <u>terms</u>

The mixed stochastic-deterministic Cesaro sums involving w or ε and y^* need to be treated in a slightly different manner to the joint stochastic terms that were examined above.

We first need to show that

$$\frac{1}{N} \sum_{t=1}^{N} (\sum_{k=o}^{t} M_k v_{t-k} + HF^t x_o)(\sum_{j=o}^{t} \underline{M}_j y^*_{t-j} + \underline{H}\,\underline{F}^t \underline{x}_o) \qquad (B.14)$$

differs from

$$\frac{1}{N}\sum_{t=1}^{N}\left(\sum_{k=o}^{t} M_k v_{t-k}\right)\left(\sum_{j=o}^{t} \underline{M}_j\, y^*_{t-j}\right) \qquad (B.15)$$

by less than $\varepsilon > 0$ a.s. for all sufficiently large N. Here $\{M_k; k \geq 0\}$ is the impulse response (i.e. sequence of Markov matrices) of $\frac{b_1(z)}{a_1(z)}$, $\frac{b_2(z)}{a_2(z)}$, $\frac{c_1(z)}{d_1(z)}$, or $\frac{c_2(z)}{d_2(z)}$ respectively and the associated v stands for w, ε, w or ε, as appropriate. $\{\underline{M}_k; k \geq 0\}$ denotes the impulse response of $\frac{c_3(z)}{d_3(z)}$ or $\frac{b_3(z)}{a_3(z)}$ as appropriate and $HF^t x_o$ and $\underline{H}\ \underline{F}^t \underline{x}_o$ denote the appropriate initial state response represented via minimal realizations of the corresponding transfer functions.

By (VI) the sequence y^* is a.s. bounded and we shall denote this bound by $|y^*|_\infty > 0$.

Following the argument used in the previous case we see that since $HF^t x_o(\omega)$ and $\underline{H}\ \underline{F}^t \underline{x}_o(\omega) \to 0$ a.s. as $N \to \infty$ we only need to show (see (B.3)) that the sequence of sums $\sum_{k=o}^{t} M_k v_{t-k}$, $t \geq 0$ is bounded by an ergodic process and the sequence $\sum_{j=o}^{t} \underline{M}_j y^*_{t-k}$, $t \geq 0$ is bounded. The first case, concerning stochastic sequences, has already been dealt with in the previous case. Since $|\underline{M}_j| < \underline{L}\ \underline{\alpha}^j$ for all $j \geq 0$, for some $\underline{L} > 0$, $0 < \underline{\alpha} < 1$ and since $|y^*_j| < |y^*|_\infty$ for all $j \geq 1$ this is evident for sequences of the second type.

Having verified that (B .14) and (B.15) converge to each other we wish to check (B .15) and

$$\frac{1}{N}\sum_{t=1}^{N}\left(\sum_{k=o}^{K} M_k v_{t-k}\right)\left(\sum_{j=o}^{K} \underline{M}_j\, y^*_{t-j}\right) \qquad (B.16)$$

can be made to differ by less than any given $\varepsilon > 0$ for all $N > \mathring{N}_{\varepsilon}(\omega)$ for sufficiently large K depending on ε.

This reduces to examining an expression of the form (B.7) when the appropriate substitutions in (B.7) $(w + v, v + y^*)$ have been carried out. Bounding $\sum_{k=K+1}^{t} M_k v_{t-k}$ by the ergodic process $\alpha^{K+1}\mu_t \triangleq L \sum_{k=K+1}^{\infty} \alpha^k |v_{t-k}|$ and bounding $\sum_{j=K+1}^{t} \underline{M}_j y^*_{t-j}$ by $\underline{L} \sum_{k=K+1}^{\infty} \underline{\alpha}^k |y^*|_\infty \leq \underline{L} \frac{\underline{\alpha}^{K+1}}{1-\underline{\alpha}} |y^*|_\infty$ yields the bounds:

$$\left| \frac{1}{N} \sum_{t=1}^{N} \left(\sum_{k=K+1}^{t} M_k v_{t-k} \right) \left(\sum_{k=K+1}^{t} \underline{M}_k y^*_{t-k} \right) \right|$$

$$\leq 2\, \alpha^{K+1} (E\mu_o) \cdot \underline{L} \frac{\underline{\alpha}^{K+1}}{1 - \underline{\alpha}} |y^*|_\infty \qquad \text{(B.17a)}$$

for all $N > M_2(\omega)$, with this bound converging to zero as $K \to \infty$. Next

$$\left| \frac{1}{N} \sum_{t=1}^{N} \left(\sum_{k=o}^{K} M_k v_{t-k} \right) \left(\sum_{k=K+1}^{t} \underline{M}_k y^*_{t-k} \right) \right|$$

$$\leq \frac{1}{N} \sum_{t=1}^{N} \{ L \sum_{k=o}^{K} \alpha^k |v_{t-k}| \} \{ \underline{\alpha}^{K+1} \underline{L} \sum_{k=o}^{\infty} \alpha^k |y^*|_\infty \}$$

$$\leq 2\, \underline{\alpha}^{K+1} E\{ (L \sum_{k=o}^{K} \alpha^k |v_{-k}|) \left(\frac{\underline{L}\, |y^*|_\infty}{1 - \underline{\alpha}} \right) \} < \infty, \qquad \text{(B.17b)}$$

a.s. for $N > M'_2(\omega)$, with this bound going to zero as $K \to \infty$.

And finally

$$\left| \frac{1}{N} \sum_{t=1}^{N} \left(\sum_{k=K+1}^{t} M_k v_{t-k} \right) \left(\sum_{k=o}^{K} \underline{M}_k y^*_{t-k} \right) \right|$$

$$\leq 2\, \alpha^{K+1} (E\mu_o) \cdot \underline{L} \frac{|y^*|_\infty}{1 - \underline{\alpha}} \qquad \text{(B.17c)}$$

a.s. for $N > M'_3(\omega)$, where we have used our usual argument for the first term.

We are now in a position to evaluate the limits of (B.14) by evaluating the limit as $N \to \infty$ of the Cesaro sums (B.16). Exchanging the finite sums and taking the limit as $N \to \infty$ yields

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \left(\sum_{k=o}^{K} M_k v_{t-k} \right) \left(\sum_{j=o}^{K} \underline{M}_j \, y^*_{t-j} \right)$$

$$= \sum_{k=o}^{K} \sum_{j=o}^{K} M_k \underline{M}_j \left(\lim_{N\to\infty} \frac{1}{N} \sum_{t=\max(k,j)}^{N} v_{t-k} \, y^*_{t-j} \right) \quad \text{(B.18)}$$

The indicated limits on the right of (B.18) for each k,j are 0; this is shown by the following application of Lemma A.1:

$$\{ v_{t-k} y^*_{t-j}, \; G_{t-k}; \; t \geq \max(k,j) \}$$

is a centered martingale difference process with

$$\sum_{t=\max(k,j)}^{\infty} \frac{1}{t^2} E\left((v_{t-k} y^*_{t-j})^2 \,|\, G_{t-k-1} \right) \leq \sum_{t=1}^{\infty} \frac{1}{t^2} \max(\sigma^2, \mu^2) \, |y^*|^2_\infty < \infty$$

consequently, $\lim_{N\to\infty} \frac{1}{N} \sum_{t=\max(k,j)}^{N} v_{t-k} y^*_{t-j} = 0$ a.s.

Letting $K \to \infty$ in (B.18) yields the limit 0 for all the Cesaro sums in (B.16) and hence in (B.15) and (B.14).

<u>The y^*y^* term.</u>

The fact that

$$\left\{ \frac{1}{N} \sum_{t=1}^{N} \left[\frac{b_3(z)}{a_3(z)} y^* \right]_t \left[\frac{c_3(z)}{d_3(z)} y^* \right]_t - \frac{1}{N} \sum_{k=o}^{K} \sum_{j=o}^{K} M_k \underline{M}_j \left(\sum_{t=\max(k,j)}^{N} y^*_{t-k} y^*_{t-j} \right) \right\}$$

can be made arbitrarily small for sufficiently large N and K is verified by an obvious simplification of the stochastic-stochastic and stochastic-deterministic analysis given above (The bounds obtained via the ergodic theorem are replaced by such bounds as

$$\left| \frac{1}{N} \sum_{t=1}^{N} \left(\sum_{k=K+1}^{t} M_k Y^*_{t-k} \right) \left(\sum_{k=K+1}^{t} \underline{M}_k Y^*_{t-k} \right) \right|$$

$$\leq \frac{N}{N} \alpha^{K+1} \left(\frac{L}{1-\alpha} |y^*|_\infty \right) \left(\underline{\alpha}^{K+1} \frac{\underline{L}}{1-\underline{\alpha}} |y^*|_\infty \right) \to 0, \text{ as } K \to \infty, \text{ etc.)}$$

Now by virtue of (VII) and the theorem of Herglotz

$$\sum_{k=o}^{K} \sum_{j=o}^{K} M_r \underline{M}_j \left(\frac{1}{N} \sum_{t=\max(k,j)}^{N} Y^*_{t-k} Y^*_{t-j} \right) \to$$

$$\sum_{k=o}^{K} \sum_{j=o}^{K} M_r \underline{M}_j \frac{1}{2\pi} \int_0^{2\pi} e^{-i(j-k)\theta} \, dR^*(e^{i\theta}) \quad \text{a.s.} \qquad \text{(B.19)}$$

But

$$\lim_{K \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=o}^{K} M_k e^{ik\theta} \right) dR^*(e^{i\theta}) \left(\sum_{j=o}^{K} \underline{M}_j e^{-ij\theta} \right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{b_3(e^{i\theta})}{\bar{a}_3(e^{i\theta})} \, dR^*(e^{i\theta}) \frac{c_3(e^{-i\theta})}{d_3(e^{-i\theta})} \qquad \text{(B.20)}$$

by virtue of the uniform convergence of the partial sums in the integrand on the left of (B.20) to the functions appearing in the integrand on the right of (B.20).

At this stage we have verified the equation (B.1) for the top left hand 3 × 3 sub-matrix of the 4 × 4 matrix appearing in that equation.

It only remains to deal with

<u>The terms involving $[wz, \varepsilon z, y^* z]$</u>

We proceed as before, starting from

$$\frac{1}{N}\sum_{t=1}^{N}\left(\sum_{k=o}^{t} M_k v_{t-k} + HF^t x_o\right)\left(\sum_{j=o}^{t} \underline{M}_j z_{t-j} + \underline{H}\,\underline{F}^t \underline{x}_o\right) \qquad (B.21)$$

where $v = w$ or ε .

The principal property of the z process is of course that $\frac{1}{N}\sum_{t=1}^{N} z_t^2 \to 0$ a.s. as $N \to \infty$ (see 3.24) . In order to dispose of the diagonal term $\frac{1}{N}\sum_{t=1}^{N}(HF^t x_o)(\underline{H}\,\underline{F}^t \underline{x}_o)$ we naturally use the asymptotic stability of $F,\underline{F}$.

To get rid of the cross term

$$\frac{1}{N}\sum_{t=1}^{N}(HF^t x_o)\left(\sum_{j=o}^{t} \underline{M}_j z_{t-j}\right) \qquad (B.22)$$

we use sequence of inequalities

$$\left|\frac{1}{N}\sum_{t=1}^{N}(HF^t x_o)\left(\sum_{j=o}^{t}\underline{M}_j z_{t-j}\right)\right| \le K\cdot\frac{L}{N}\sum_{t=1}^{N}\alpha^t \sum_{j=o}^{t}\underline{\alpha}^j |z_{t-j}|)$$

$$\le \frac{K\,L}{\sqrt{N}}\sum_{t=1}^{N}\alpha^t \left[\sum_{j=o}^{t}\underline{\alpha}^{2j}\right]^{\frac{1}{2}}\left[\frac{1}{N}\sum_{j=o}^{t}|z_{t-j}|^2\right]^{\frac{1}{2}}$$

$$\le \frac{K\,L}{N^{\frac{1}{2}}}\sum_{t=1}^{\infty}\alpha^t\left[\frac{1}{1-\underline{\alpha}^2}\right]^{\frac{1}{2}}\cdot 1 \le \frac{K\,L}{N^{\frac{1}{2}}}\left[\frac{1}{1-\underline{\alpha}^2}\right]^{\frac{1}{2}}\frac{1}{(1-\alpha)} . \qquad (B.23)$$

for $N > N_1(\omega)$, where the third inequality follows from

$$\frac{1}{N}\sum_{j=o}^{t}|z_{t-j}|^2 \le \frac{1}{N}\sum_{j=o}^{N}|z_j|^2 , \quad \text{for } t \le N,$$

with the last expression less than 1 for all $N > N_1(\omega)$. From (B.23) we conclude that the expression in (B.22) goes to zero a.s. as $N > \infty$.

Finally, to dispose of the cross term

$\frac{1}{N}\sum_{t=1}^{N}\left(\sum_{k=o}^{t} M_k v_{t-k}\right) \underline{H}\ \underline{F}^t \underline{x}_o$ we reason exactly as in the "(w,ε)" case treated earlier.

The next step is to show that

$$\frac{1}{N}\sum_{t=1}^{N}\left(\sum_{k=K+1}^{t} M_k v_{t-k}\right)\left(\sum_{k=K+1}^{t} \underline{M}_k z_{t-k}\right) \qquad \text{(B.24a)}$$

$$+ \frac{1}{N}\sum_{t=1}^{N}\left(\sum_{k=o}^{K} M_k v_{t-k}\right)\left(\sum_{k=K+1}^{t} \underline{M}_k z_{t-k}\right) \qquad \text{(B.24b)}$$

$$+ \frac{1}{N}\sum_{t=1}^{N}\left(\sum_{k=K+1}^{t} M_k v_{t-k}\right)\left(\sum_{k=o}^{K} \underline{M}_k z_{t-k}\right) \qquad \text{(B.24c)}$$

is such that given $\varepsilon > 0$ there exists $K_\varepsilon > 0$ and $N_\varepsilon(\omega) > 0$ s.t. the modulus of (B.24) is a.s. less than ε for all $N > N_\varepsilon(\omega)$ and $K > K_\varepsilon$ (sums appear in (B.24) only when their limits make them well defined).

It is perhaps easiest to deal with the second term (B.24b) first. The process $\zeta^K \underline{\Delta} \{\zeta_t^K \underline{\Delta} \sum_{k=o}^{K} M_k v_{t-k}; t \in \mathbb{Z}\}$

is clearly ergodic and the term (B .24b) is of the form $\frac{1}{N}\sum_{t=1}^{N} \zeta_t^K \left(\sum_{j=K+1}^{t} \underline{M}_j z_{t-j}\right)$. Majorizing this term by $\underline{L}\ \alpha^{K+1}\{\frac{1}{N}\sum_{t=1}^{N}|\zeta_t^K|\left(\sum_{j=o}^{t-(K+1)} \underline{\alpha}^j |z_{t-j-(K+1)}|\right)\}$ we see it is sufficient to show that the term in braces is bounded a.s. as $N \to \infty$. However re-arraging this expression it is seen to be majorized by (go from columns and rows to diagonals in the appropriate diagram):

$$\frac{1}{N}\sum_{j=1}^{N}|z_{j-(K+1)}|\left(\sum_{p=o}^{\infty}\underline{\alpha}^{p}\,|\zeta_{j+p}^{K}|\right) \tag{B.25}$$

But $\xi_j^K \triangleq \sum_{p=o}^{\infty}\underline{\alpha}^p\,|\zeta_{j+p}^K|$, $j = 0,1,\ldots$ (sum over 'forward time')

is an ergodic process with finite first and second moments. Cauchy-Schwarz applied to (B.25) yields the bound

$$\left[\frac{1}{N}\sum_{j=1}^{N}|z_{j-(K+1)}|^2\right]^{\frac{1}{2}}\left[\frac{1}{N}\sum_{j=o}^{N}(\xi_j^K)^2\right]^{\frac{1}{2}} < 1.\left[E\,\xi_o^2\right]^{\frac{1}{2}} \tag{B.26}$$

for $N > N_1(\omega)$.

As to the third term (B.24c) we have the bound

$$\frac{1}{N}\sum_{t=1}^{N}\alpha^{K+1}\left(\sum_{p=o}^{\infty}L\alpha^{p}|v_{t-p-(K+1)}|\right)\left(\sum_{k=o}^{K}\underline{L}\,\underline{\alpha}^{k}|z_{t-k}|\right)$$

and so it is sufficient to show that $\frac{1}{N}\sum_{t=1}^{N}\mu_t\left(\sum_{k=o}^{K}\underline{\alpha}^k|z_{t-k}|\right)$

is a.s. bounded when μ denotes the ergodic process

$\{\sum_{p=o}^{\infty}\alpha^p|v_{t-p-(K+1)}|\ ;\ t \geq 1\}$. But- exchanging finite sums - this is seen to be true if $\frac{1}{N}\sum_{t=1}^{N}\mu_t|z_{t-k}|$ is a.s. bounded w.r.t. N for any fixed k.

This however is the case, since, by Cauchy-Schwarz, this is bounded as

$$\left[\frac{1}{N}\sum_{t=1}^{N}\mu_t^2\right]^{\frac{1}{2}}\left[\frac{1}{N}\sum_{t=1}^{N}|z_{t-k}|^2\right]^{\frac{1}{2}} \leq \left[2\,E\,\mu_o^2\right]^{\frac{1}{2}}1 \quad \text{a.s. for } N > N_2(\omega).$$

Finally, we are relieved of the term (B.24a) by observing that, in analogy with (B.24b), it is majorized by

$$\underline{L\alpha}^{K+1}\left\{\frac{1}{N}\sum_{t=1}^{N}|\zeta_t^{\infty}|\left(\sum_{j=o}^{t-(K+1)}\underline{\alpha}^j|z_{t-j-(K+1)}|\right)\right\} \text{ where } \zeta_t^{\infty} \triangleq \sum_{k=o}^{\infty}M_k\,v_{t-k},\ t \geq 1.$$

The term in braces is itself majorized by

$$\frac{1}{N} \sum_{j=1}^{N} |z_{j-(K+1)}| \left(\sum_{p=o}^{\infty} \underline{\alpha}^{p} |\zeta_{j+p}^{\infty}| \right) \tag{B.27}$$

which, in turn, is bounded as in (B.26) with ξ^K replaced by (an obviously defined) ξ^∞ .

The upshot of all this is that (B.21) is approximated a.s. arbitrarily accurately, for all sufficiently large N and all sufficiently large K, by

$$\frac{1}{N} \sum_{t=1}^{N} \left(\sum_{k=o}^{K} M_k v_{t-k} \right) \left(\sum_{j=o}^{K} \underline{M}_j z_{t-j} \right) \tag{B.28}$$

Exchanging sums, as in all the previous cases, and then for arbitrary $\varepsilon > 0$ using

$$\left| \frac{1}{N} \sum_{t=1}^{N} v_{t-k} z_{t-j} \right| \leq \left[\frac{1}{N} \sum_{t=1}^{N} v_{t-k}^2 \right]^{\frac{1}{2}} \left[\frac{1}{N} \sum_{t=1}^{N} z_{t-j}^2 \right]^{\frac{1}{2}}$$

$$\leq \left[2 E v_o^2 \right]^{\frac{1}{2}} \cdot \varepsilon \quad \text{a.s.}$$

for all $N > N_\varepsilon$, we may establish that (B.28) converges to 0 a.s. as $N \to \infty$. It follows that the same goes for (B.21)

This establishes that the Cesaro sums in (B.1) involving w and z, and ε and z, converge to 0. The y^*z term constitutes an easier version of the "v and z" case.

The [zz] term

To show that the initial condition influence in

$$\frac{1}{N} \sum_{t=1}^{N} \left(\sum_{k=o}^{t} M_k z_{t-k} + HF^t x_o \right) \left(\sum_{j=o}^{t} \underline{M}_j z_{t-j} + \underline{H}\, \underline{F}^t \underline{x}_o \right)$$

decays to zero a.s. it is sufficient to show that

$\frac{1}{N} \sum_{t=1}^{N} HF^t x_o \left(\sum_{j=o}^{t} \underline{M}_j z_{t-j} \right)$ decays and this is true since it is bounded as

$$\frac{K(\omega)}{N} \sum_{t=1}^{N} \alpha^t \left(\sum_{j=o}^{t} \underline{L}\ \underline{\alpha}^j |z_{t-j}| \right) \leq K(\omega) \frac{\underline{L}}{1-\underline{\alpha\alpha}} \left[\frac{1}{N} \sum_{j=o}^{N} |\alpha^j|^2 \right]^{\frac{1}{2}} \left[\frac{1}{N} \sum_{j=o}^{N} |z_j|^2 \right]^{\frac{1}{2}}$$

here the last term decays since $\frac{1}{N} \sum_{j=o}^{N} |z_j|^2 \to 0$ a.s. as $N \to \infty$.

We then have an analysis of the familiar terms in (B.24) with $v \equiv z$. By the symmetry of the expression we need only treat the first and second expressions. The second can be seen to converge to zero a.s. if

$\alpha^{K+1} \frac{1}{N} \sum_{t=1}^{N} \left(\sum_{j=o}^{K} \alpha^j z_{t-j} \right) \left(\sum_{p=o}^{t-(K+1)} \underline{\alpha}^p z_{t-p-(K+1)} \right)$ does (where sums make a contribution when upper limits are not less than lower limits and terms make a contribution when subscripts exceed 1).

But this expression is seen to decay if

$\frac{1}{N} \sum_{t=1}^{N} z_{t-j} \left(\sum_{p=o}^{t-(K+1)} \underline{\alpha}^p z_{t-p-(K+1)} \right)$ is bounded a.s. for each $j, 1 \leq j \leq K$. This is true since it is bounded by

$\sum_{p=o}^{N} \underline{\alpha}^p \left(\frac{1}{N} \sum_{t=o}^{N} z_{t-j} z_{t-p-(K+1)} \right)$ which is itself bounded by $\frac{1}{1-\underline{\alpha}} \left[\left(\frac{1}{N} \sum_{q=1}^{N} z_q^2 \right)^{\frac{1}{2}} \right]^2$.

This last expression goes to zero a.s. with N.

For the analog of the term (B.24a) it is clearly sufficient to show that

$$\frac{1}{N} \sum_{t=1}^{N} \left(\sum_{k=o}^{t-(K+1)} \alpha^k z_{t-k-(K+1)} \right) \left(\sum_{j=o}^{t-(K+1)} \underline{\alpha}^j z_{t-j-(K+1)} \right)$$

is a.s. bounded.

Some combinatoric labour is required to find an appropriate bound for this expression. First, we bound the expression by taking the norms of all the terms and by replacing α and $\underline{\alpha}$ by α_m such that $\alpha_m = \max(\alpha, \underline{\alpha})$. Second, we multiply together the terms of the two inner summations. Then, we replace the summation over t, $t = 1,\ldots,N$, of the $\alpha_m^q |z_{t+p} z_t|$ terms by a summation over the powers of α_m, $q = 0,\ldots,2N$. It can be seen that for each power q of α_m, the maximum number of pairs of summands of the form $|z_{t+p} z_t|$, for some finite p, is always less than or equal to $(q+2)$, for each t, $t = 1,\ldots,N$. This comes from the fact that for some fixed $t = K+1+r$ we have only powers of α_m less than or equal to r; hence, the number of $|z_{t+p} z_t|$ terms of some power q is equal to the number of pairs of positive integers $\leq r$ which sum to q. This number is clearly $\leq (q+2)$ for any r, for some fixed q.

Thus, the expression is bounded by

$$\sum_{q=o}^{2N} \alpha_m^q \left(\frac{1}{N} \sum_{j=1}^{N} |z_j|^2 + (q+2) \frac{1}{N} \sum_{i=1}^{N-p} |z_{i+p} z_i| \right)$$

and by an application of the Cauchy-Schwarz inequality we see that $\frac{1}{N} \sum_{j=1}^{N} z_j^2 \to 0$ a.s. as $N \to \infty$ and the summability of $\{q\alpha^q, q \geq 1\}$ gives the desired a.s. boundedness property.

This having been established we can exchange limit and finite sums in

$$\frac{1}{N} \sum_{t=1}^{N} \left(\sum_{j=o}^{K} M_j z_{t-j} \right) \left(\sum_{k=o}^{K} \underline{M}_k z_{t-k} \right)$$

to obtain the desired limit O for the bottom right term of the matrix.

□

APPENDIX C

Lemma C.1.

Consider $X(t)$, $\phi(t)$ and $r(t)$ defined in (7.6), (7.5) and (AM 2); then

(a) $\quad \mathrm{Tr}[X(t)^T X(t)] = p\phi(t)^T\phi(t)$

(b) $\quad \mathrm{Tr}[X(t)\,\lambda\,\lambda^T X(t)^T] = \lambda^T\lambda\;\phi(t)^T\phi(t)$

$$= \frac{1}{p}\,\lambda^T\lambda\;\mathrm{Tr}[X(t)\;X(t)^T]$$

for any $p \times 1$ vector λ .

(c) $$\sum_{j=\bar{n}+1}^{\infty} \frac{\mathrm{Tr}[X(j-1)^T X(j-1)]}{r(j-1)^2} < \infty$$

□

Proof.

(a),(b): these results follow directly from the definitions of $X(t)$ and $\phi(t)$. (It should be noted that these results do not hold in general.)

(c) $$\frac{\mathrm{Tr}[X(t-1)^T X(t-1)]}{r(t-1)^2} \le \frac{\mathrm{Tr}[X(t-1)^T\; X(t-1)]}{r(t-1)\;r(t-2)}$$

$$\le \frac{r(t-1) - r(t-2)}{r(t-1)r(t-2)} = \frac{1}{r(t-2)} - \frac{1}{r(t-1)} .$$

Hence, $$\sum_{j=\bar{n}+1}^{\infty} \frac{\mathrm{Tr}[X(j-1)^T X(j-1)]}{r(j-1)^2} \le \frac{1}{r(\bar{n}-1)} < \infty .$$

□

Proof of Theorem 7.2

Part 1.

In our derivation of the multivariable version of the adaptive control algorithm, we obtained the following equation (see (7.7)):

$$C(z)\,[e_{t+1} - w_{t+1} - \varepsilon_t] = X(t)^T\theta_o - [\,y^*_{t+1} + \varepsilon_t\,] \tag{C.1}$$

where $X(t)$ and θ_o are defined in (7.6) and (7.4), and $e_t \triangleq y_t - y^*_t$ is the control error. We now define the $p \times 1$ and $p\bar{\rho} \times 1$ vectors:

$$z_t \triangleq e_{t+1} - w_{t+1} - \varepsilon_t \tag{C.2}$$

$$\tilde{\theta}(t) \triangleq \hat{\theta}(t) - \theta_o \tag{C.3}$$

Using these definitions and equation (AM 3) of the algorithm, we can rewrite (C.1) as

$$C(z)z_t = -X(t)^T\tilde{\theta}(t) \tag{C.4}$$

and we remark that as in the scalar case, z_t is G_t measurable.

We also remark that by the asymptotic stability of $C(z)^{-1}$, the initial conditions of (C.1) can be neglected because their effect decays geometrically.

Part 2.

In this section, we establish the important property $\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \|z_t\|^2 = 0$ a.s. In the analysis to follow we take $t \geq \bar{n}+1$, and note that all the required initial conditions have been specified.

Substituting (AM 3) in (AM 1) we have:

$$\tilde{\theta}(t) = \tilde{\theta}(t-1) + \frac{\bar{a}}{r(t-1)} X(t-1)[e_t - \varepsilon_{t-1}] \tag{C.5}$$

Let $V(t) \underline{\Delta} \tilde{\theta}(t)^T\tilde{\theta}(t)$; then from $e_t - \varepsilon_{t-1} = (e_t - \varepsilon_{t-1} - w_t) + w_t$ we obtain

$$V(t) = V(t-1) + \frac{2\bar{a}}{r(t-1)}\tilde{\theta}(t-1)^T X(t-1)[e_t - w_t - \varepsilon_{t-1}]$$

$$+ \frac{2\bar{a}}{r(t-1)}\tilde{\theta}(t-1)^T X(t-1)[w_t]$$

$$+ \frac{\bar{a}^2}{r(t-1)^2} \mathrm{Tr}\{X(t-1)[(e_t - \varepsilon_{t-1} - w_t)(e_t - \varepsilon_{t-1} - w_t)^T$$

$$+ 2w_t(e_t - \varepsilon_{t-1} - w_t)^T + w_t w_t^T] X(t-1)^T\} \tag{C.6}$$

Writing $b(t-1)^T \underline{\Delta} -\tilde{\theta}(t-1)^T X(t-1)$ and taking conditional expectations in the above equation, we get

$$E V(t)|G_{t-1}) = V(t-1) - \frac{2\bar{a}}{r(t-1)}[b(t-1)^T z_{t-1}]$$

$$+ \frac{\bar{a}^2}{r(t-1)^2} z_{t-1}^T z_{t-1}\phi(t-1)^T\phi(t-1)$$

$$+ \frac{\bar{a}^2}{r(t-1)^2}\mathrm{Tr}\Gamma[\phi(t-1)^T\phi(t-1)] \tag{C.7}$$

where we have used Lemma C.1 (a) and (b), hypotheses (M4) and (M5), and the fact that z_{t-1} is G_{t-1} measurable.

Comparing to the proof of Theorem 3.1, we see that (C.7) is an obvious generalization of (3.15). Therefore, after the same manipulations as in the scalar case ((3.15) to (3.22)), we obtain the important "near-super-martingale" inequality:

$$E\, Z(t)\,|\,G_{t-1}) \le Z(t-1) - \frac{\bar{a}\rho}{r(t-1)} \|z_{t-1}\|^2$$

$$+ \frac{\bar{a}^2}{r(t-1)^2} \quad \mathrm{Tr}\,\Gamma \|\phi(t-1)\|^2 \qquad \text{(C.8)}$$

where $Z(t) \triangleq V(t) + \frac{S(t)}{r(t-1)} \triangleq V(t) + \frac{2\bar{a}}{r(t-1)} \sum_{j=\bar{n}+1}^{t} h(j-1)^T z_{t-1} + K,\ t \ge \bar{n}+1$

$$h(t-1) \triangleq b(t-1) - \frac{(\bar{a}+\rho)}{2} z_{t-1} = [C(z) - \frac{(\bar{a}+\rho)}{2} I] z_{t-1}$$

and where ρ is a small positive constant chosen so that $[C(z) - \frac{(\bar{a}+\rho)}{2} I]$ is positive real. The existence of such a ρ is assured by hypotheses (M1)b and (M9) (see Remark 7.1), and from Lemma A.4 of Appendix A $S(t) \ge 0$ for all $t \ge \bar{n}+1$.

Lemma C.1((a) and (c)) tells us that

$$\sum_{j=\bar{n}+1}^{\infty} \frac{\bar{a}^2}{r(t-1)^2} \mathrm{Tr}\,\Gamma \|\phi(t-1)\|^2$$

$$= \sum_{j=\bar{n}+1}^{\infty} \frac{\bar{a}^2}{r(t-1)^2} \frac{\mathrm{Tr}\,\Gamma}{p} \mathrm{Tr}[X(j-1)^T X(j-1)] < \infty .$$

Therefore, we can apply the martingale convergence theorem to (C.8) and obtain (note that $\rho\bar{a} \ne 0$):

$$Z(t) \to Z(\infty) \quad \text{a.s. as } t \to \infty \quad \text{, with } E[Z(\infty)] < \infty$$

and

$$\sum_{t=1}^{\infty} \frac{\|z_{t-1}\|^2}{r(t-1)} < \infty \qquad \text{a.s.} \qquad \text{(C.9)}.$$

Our objective is to establish the important relation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \|z_t\|^2 = 0 \quad \text{a.s.} \qquad \text{(C.10)}$$

As in the scalar case, we have to consider two cases depending upon the behaviour of $r(t)$ as $t \to \infty$, and divide the sample space Ω accordingly.

(i) Let $H = \{w \in \Omega : \lim_{t \to \infty} r(t) < \infty\}$; in that case, an argumentation identical to the one used in the scalar case (we refer the reader to the proof of Theorem 3.1) shows that $\lim_{t\to\infty} \|x(t)\|^2 = 0$ and $\limsup_{t\to\infty} \|\tilde{\theta}(t)\|^2 < \infty$ a.s. on H. Hence, from (C.4) and the asymptotic stability of $C(z)^{-1}$, $\|z_t\|^2 \to 0$ a.s. on H as $t \to \infty$ and so

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \|z_t\|^2 = 0 \qquad \text{a.s. on } H \qquad \text{(C.11)}$$

as required.

(ii) Let $H' = \Omega \setminus H$; in that case we can apply Kronecker's lemma to (C.9) which yields

$$\lim_{N\to\infty} \frac{1}{r(N)} \sum_{t=1}^{N} \|z_t\|^2 = 0 \qquad \text{a.s. on } H' . \qquad \text{(C.12)}$$

We show in Part 3 of this proof that $\liminf_{N\to\infty} \frac{N}{r(N)} > 0$ a.s. on Ω from which we conclude

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \|z_t\|^2 = 0 \qquad \text{a.s. on } H' . \qquad \text{(C.13)}$$

The relation (C.10) has now been shown to hold a.s. on Ω as required.

Part 3.

For this part of the proof, the reader is referred to the proof of Theorem 3.1, the only difference in the multivariable case being the replacement of $(\cdot)^2$ by $\|\cdot\|^2$ for all the processes. All the arguments are still valid; in particular, Lemma A.5 of [21] was proved to hold in the multivariable case in that reference.

Therefore we have

$$\liminf_{N\to\infty} \frac{N}{r(N)} > 0 \qquad \text{a.s.} \tag{C.14}$$

as was required in part 2 of this proof. This in turn implies that

$$\limsup_{N\to\infty} \frac{r(N)}{N} < \infty \qquad \text{a.s.} \tag{C.15}$$

and so from the definition of $r(N)$,

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \|y_t\|^2 < \infty \quad \text{a.s.}$$

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \|u_t\|^2 < \infty \quad \text{a.s.}$$

The proof of the asymptotic stability of the multivariable adaptive control algorithm is now completed. We will show in part 5 that these limits exist and give their complete expressions.

Part 4.

The (asymptotic) optimality of the algorithm (7.28) can now be easily derived. From the definitions of $e(t)$ and $z(t-1)$ we have:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \| y_t - y_t^* \|^2$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \| z_{t-1} + w_t + \varepsilon_{t-1} \|^2$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} [\| z_{t-1} \|^2 + \| w_t \|^2 + \| \varepsilon_{t-1} \|^2$$

$$+ 2 z_{t-1}^T w_t + 2 z_{t-1}^T \varepsilon_{t-1} + 2 w_t^T \varepsilon_{t-1}] \quad . \qquad (C.16)$$

But we have proved that $\frac{1}{N} \sum_{t=1}^{N} \| z_t \|^2 \to 0$ a.s. and so

$$\left| \frac{1}{N} \sum_{t=1}^{N} z_{t-1}^T \varepsilon_{t-1} \right| \le \left[\frac{1}{N} \sum_{t=1}^{N} \| z_{t-1} \|^2 \right]^{\frac{1}{2}} \left[\frac{1}{N} \sum_{t=1}^{N} \| \varepsilon_{t-1} \|^2 \right]^{\frac{1}{2}}$$

$$\to 0 \cdot [\mathrm{Tr}\ M]^{\frac{1}{2}} = 0 \quad \text{a.s.}$$

as $N \to \infty$ by hypotheses (M2) and (M5); the same is true for the $z_{t-1}^T w_t$ term. Also, we have by (M2) and (M5):

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} w_t^T \varepsilon_{t-1} = E w_t^T \varepsilon_{t-1}$$

$$= E[E w_t^T \varepsilon_{t-1} | \mathcal{G}_{t-1})] = 0 \quad \text{a.s.}$$

Hence (C.16) is reduced to

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \| y_t - y_t^* \|^2 = \mathrm{Tr}\,[\Gamma + M] \quad \text{a.s.}$$

and the asymptotic optimality of the algorithm is now established.

Before going any further, we point out that the assumptions required to prove only the stabilization (in the sense that the inputs and outputs are sample mean square bounded) and asymptotic optimization (in the sense of (7.28)) properties of the algorithm (i.e. adaptive control <u>without</u> simultaneous identification) could be slightly relaxed. In particular, ergodicity of w and ε is not necessary. We refer the reader to the statements of theorems 3.1, 6.1 and corollaries 3.1 and 6.1.

Part 5.

In this part of the proof, we establish the important persistency of excitation property. As mentioned in section 7.2 (see the statement of Theorem 7.1), this condition is of the form:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \phi^I(t)\phi^I(t)^T = R > 0 \qquad \text{a.s.} \tag{C.17}$$

where

$$\phi^I(t) = (-y_t^T, \ldots, -y_{t-n+1}^T, u_t^T, \ldots, u_{t-m}^T, w_t^T, \ldots, w_{t-\ell+1}^T)^T_{\rho \times 1}$$

(recall that $\rho \triangleq (n+m+1+\ell)p$). (C.18)

We will use the proof techniques of Lemma 4.2 and Theorem 4.1. It should be noticed that Lemma 4.1 can be directly generalized to the multivariable case. (To see that, consider the processes component by component. Then the arguments in the proof of Lemma 4.1 concerning the asymptotically negligeable effect of the initial conditions and the existence of limits for all the Cesaro sums are directly applicable, taking into account hypotheses (M3), (M4), (M5), (M6), (M7) and using result (C.10) of part 2 of this proof.)

Since by (M1)a $\det B(z) \neq 0$, $|z| \leq 1$, we can write:

$$u_t = B(z)^{-1}A(z)[y^*_{t+1} + z_t + \varepsilon_t] + B(z)^{-1}[A(z) - C(z)]w_{t+1} \tag{C.19}$$

where we have used the system equation and the definition of z_t. Now, decompose $\phi^I(t)$ into the sum of four vectors:

$$\phi^I(t) = \phi_1^I(t) + \phi_2^I(t) + \phi_3^I(t) + \phi_4^I(t) =$$

$$\begin{array}{c} np \\ \\ (m+1)p \\ \\ p \end{array}
\begin{bmatrix} -y_t^* \\ \vdots \\ -y_{t-n+1}^* \\ \hdashline B(z)^{-1}A(z)y_{t+1}^* \\ \vdots \\ B(z)^{-1}A(z)y_{t-m+1}^* \\ \hdashline 0 \\ \vdots \\ 0 \end{bmatrix}
+ \begin{bmatrix} -z_{t-1} \\ \vdots \\ -z_{t-n} \\ \hdashline B(z)^{-1}A(z)z_t \\ \vdots \\ B(z)^{-1}A(z)z_{t-m} \\ \hdashline 0 \\ \vdots \\ 0 \end{bmatrix}
+ \begin{bmatrix} -w_t \\ \vdots \\ -w_{t-n+1} \\ \hdashline B(z)^{-1}[A(z)-C(z)]w_{t+1} \\ \vdots \\ B(z)^{-1}[A(z)-C(z)]w_{t-m+1} \\ \hdashline w_t \\ \vdots \\ w_{t-\ell+1} \end{bmatrix}
+ \begin{bmatrix} -\varepsilon_{t-1} \\ \vdots \\ -\varepsilon_{t-n} \\ \hdashline B(z)^{-1}A(z)\varepsilon_t \\ \vdots \\ B(z)^{-1}A(z)\varepsilon_{t-m} \\ \hdashline 0 \\ \vdots \\ 0 \end{bmatrix} \quad (C.20)$$

Applying (the generalized version of) Lemma 4.1, we get:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi^I(t)\phi^I(t)^T = R^I$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi_1^I(t)\phi_1^I(t)^T + \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi_3^I(t)\phi_3^I(t)^T$$

$$+ \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi_4^I(t)\phi_4^I(t)^T$$

$$\equiv R_1^I + R_3^I + R_4^I \ .$$

It is then easy to generalize the results of Lemma 4.2 and derive the limit expressions (7.26) and (7.27), where $(\underline{R}_1^*)_{0,0}$, $(\underline{R}_3^*)_{0,0}$, $\underline{\alpha}_j$ and $\underline{\beta}_j$ are the (p x p) matricial analogs of $(R_1^*)_{0,0_\infty}$, $(R_3^*)_{0,0}$, α_j and β_j (e.g. $(\underline{R}_1^*)_{0,0} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y_t^* y_t^{*T}$, $B(z)^{-1}A(z) = \sum_{j=0}^{\infty} \underline{\alpha}_j z^j$, etc.).

In view of the proofs of Lemma 4.2, Theorem 4.1 and Theorem 6.2, we see that it is sufficient to prove that

$M_F \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \phi_{4s}^{I}(t)\phi_{4s}^{I}(t)^T > 0$ a.s. to show that $R^I > 0$ a.s., where

$$\phi_{4s}^{I}(t) = (-\varepsilon_{t-1}^T,\ldots,-\varepsilon_{t-n}^T,(B(z)^{-1}A(z)\varepsilon_t)^T,\ldots,(B(z)^{-1}A(z)\varepsilon_{t-m})^T)^T \tag{C.21}$$

is the shortened version of $\phi_4^I(t)$ where the last ℓ zero vectors are omitted.

Directly generalizing the scalar process case, we write

$$M_F = \frac{1}{2\pi}\int_0^{2\pi} E(e^{i\theta})\, dF_\varepsilon(\theta)\, E(e^{-i\theta})^T$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \begin{bmatrix} -I \\ -e^{i\theta}I \\ \vdots \\ -e^{i(n-1)\theta} I \\ S(e^{i\theta})e^{-i\theta} \\ \vdots \\ e^{im\theta}S(e^{i\theta})e^{-i\theta} \end{bmatrix} \Gamma \left[-I,\ldots,-e^{-i(n-1)\theta}I, S(e^{-i\theta})^T e^{i\theta}, \ldots, e^{-im\theta}S(e^{-i\theta})^T e^{i\theta}\right] d\theta \tag{C.22}$$

where $E(e^{i\theta})$ is a $p(n+m+1) \times p$ matrix implicitely defined in the equation and $S(e^{i\theta}) \triangleq B(e^{i\theta})^{-1}A(e^{i\theta})$. (Recall that $dF_\varepsilon(\theta) = \Gamma\, d\theta$ from hypothesis (M5).)

Clearly M_F is necessarily positive semi-definite; it will be positive definite if and only if

$$\int_0^{2\pi} \lambda^T E(e^{i\theta}) \Gamma E(e^{-i\theta})^T \lambda d\theta = 0 \qquad \text{(C.23)}$$

for some λ of dimension $p(n+m+1) \times 1$ implies $\lambda = 0$. This is what we shall now prove. By positivity and continuity (C.23) implies

$$\lambda^T E(e^{i\theta}) \Gamma E(e^{-i\theta})^T \lambda = 0 \qquad \forall \theta \in [0,2\pi]$$

which can be written

$$[\lambda^T E(e^{i\theta}) \Gamma^{1/2}] [\lambda^T E(e^{i\theta}) \Gamma^{1/2}]^* = 0, \ \forall \theta \in [0,2\pi]$$

since $\Gamma > 0$. (* denotes the complex conjugate transpose). But for any matrix P, $PP^* = 0$ implies $P = 0$, and so we must have

$$\lambda^T E(e^{i\theta}) = 0 \qquad \forall \theta \in [0,2\pi]. \qquad \text{(C.24)}$$

To facilitate the manipulations hereafter we define the $p \times p(n+m+1)$ matrix Λ, composed of p copies of λ^T, as in

$$\Lambda \triangleq [\Lambda_1, \Lambda_2, \ldots, \Lambda_{n+m+1}] \triangleq \begin{bmatrix} \lambda^T \\ \vdots \\ \lambda^T \end{bmatrix} \text{ p times} \qquad \text{(C.25)}$$

where each Λ_j is $p \times p$. The p-fold copy of (C.24) is then written in the form

$$\Lambda E(e^{i\theta}) = 0 \qquad \forall \theta \in [0,2\pi] \qquad \text{(C.26)}$$

where the right hand side is a $p \times p$ matrix of zeros. Computing the product of matrices on the left hand side, we obtain

$$-\Lambda_1 - \Lambda_2 e^{i\theta} - \ldots - \Lambda_n e^{i(n-1)\theta} + (\Lambda_{n+1} + \ldots + \Lambda_{n+m+1} e^{im\theta}) B(e^{i\theta})^{-1}.$$

$$A(e^{i\theta}) e^{-i\theta} = 0 \qquad \forall \theta \in [0,2\pi]. \qquad \text{(C.27)}$$

Let us define the p × p polynomial matrices

$$X(e^{i\theta}) = \Lambda_1 + \Lambda_2 e^{i\theta} + \ldots + \Lambda_n e^{i(n-1)\theta} \quad (C.28)$$

and

$$Y(e^{i\theta}) = \Lambda_{n+1} + \Lambda_{n+2}\, e^{i\theta} + \ldots + \Lambda_{n+m+1} e^{im\theta} \quad (C.29)$$

so that (C.27) becomes

$$X(e^{i\theta}) = Y(e^{i\theta})B(e^{i\theta})^{-1}A(e^{i\theta})e^{-i\theta} \quad \forall\, \theta \in [0,2\pi]\,. \quad (C.30)$$

Now, consider the rational transfer function matrix $H(z) = B(z)^{-1}A(z)$. $H(z)$ is not necessarily proper, but it is irreducible because $A(z)$ and $B(z)$ are left coprime. We also know from hypothesis (M8) that $A(z)$ and $B(z)$ are row-reduced, i.e. A_n and B_m have full rank. Therefore, the matrix $M(z) = [A(z), B(z)]$ is also row-reduced and $\nu_i = \bar{m}$, $1 \leq i \leq p$, where:

ν_i = the degree of the i-th row of $M(z)$.

and

$\bar{m} \triangleq \max(n,m)$.

We know that there also exists a right coprime matrix fraction description (m.f.d.) of $H(z)$ equal to $N(z)D(z)^{-1}$, where $N(z)$ and $D(z)$ are p × p polynomial matrices of maximum degree q :

$$N(z) = N_o + N_1 z + \ldots + N_q\, z^q \quad (C.31)$$

$$D(z) = D_o + D_1 z + \ldots + D_q\, z^q \quad (C.32)$$

and where

deg det $B(z)$ = deg det $D(z)$ = p m. (See e.g. [43].)

(Notice that since $H(z)$ is not in general proper, its McMillan degree is not necessarily equal to deg det $B(z)$ = deg det $D(z)$. We also emphasize that it is sufficient for the purpose of this proof to consider the general case max order $[N(z)] \leq q$ and max order $[D(z)] \leq q$, without any specific constraints on q. However, it is clear that $q \geq m$.)

Hence, we have the equation

$$B(z)N(z) - A(z)D(z) = 0 \qquad (C.33)$$

As in [47] (see also [48]), equating the coefficients of the various powers of z in (C.33) yields the equation:

$$[B_o, -A_o, B_1, -A_1, \ldots, B_{\bar{m}}, -A_{\bar{m}}]\, S_{\bar{m}+1} = 0$$

where

$$S_k = \begin{bmatrix} N_o & N_1 & \cdots & N_q & 0 & \cdots & 0 \\ D_o & D_1 & \cdots & D_q & 0 & \cdots & 0 \\ 0 & N_o & \cdots & & N_q & 0 \cdots & 0 \\ 0 & D_o & \cdots & & D_q & 0 \cdots & 0 \\ \vdots & & & & & & \\ 0 & \cdots & 0 & N_o & & \cdots & N_q \\ 0 & \cdots & 0 & D_o & & \cdots & D_q \end{bmatrix} \quad \text{2k block rows}$$

the so-called generalized Sylvester resultant of N and D of order k. (Undefined B_j's or A_j's when $j > \min(m,n)$ are taken to be zero.)

It is shown in [47] that

$$\text{rank } S_k = (2p)k - \sum_{\{i: \nu_i < k\}} (k - \nu_i)$$

where the ν_i's, the degrees of the i-th rows of $M(z) = [A(z),B(z)]$, are also called the "dual dynamical indices" of $N(z)\ D(z)^{-1}$. Since $\nu_i = \bar{m}$, $1 \le i \le p$, by the row-reducedness property of $M(z)$, rank $S_{\bar{m}} = 2p\bar{m} - 0 = 2p\bar{m}$ and therefore $S_{\bar{m}}$ has full row rank. This important property will be used later on in the proof. (It is also clear that S_k has full row rank for $k < \bar{m}$.)

We now return to (C.30) and replace $B(e^{i\theta})^{-1}A(e^{i\theta})$ by $N(e^{i\theta})\ D(e^{i\theta})^{-1}$. The reader will have noticed that this transformation from a left (coprime) m.f.d. to a right (coprime) m.f.d., which is without loss of generality, enables us to rewrite (C.30) in the more convenient form:

$$X(e^{i\theta})D(e^{i\theta})e^{i\theta} = Y(e^{i\theta})N(e^{i\theta}) \qquad \forall\ \theta\ \varepsilon\ [0,2\pi]\,. \qquad (C.34)$$

It is easily seen, equating the coefficients of the power $e^{i0\theta}$ on both sides of (C.34), that $\Lambda_{n+1} = 0$ (because N_o has full rank from (C.33) and (M1) a). Defining the new polynomial matrix

$$\underline{Y}(z) \doteq \Lambda_{n+2} + \Lambda_{n+3}z + \ldots + \Lambda_{n+m+1}\ z^{m-1} \qquad (C.35)$$

we have

$$X(e^{i\theta})\ D(e^{i\theta}) = \underline{Y}(e^{i\theta})N(e^{i\theta}) \qquad \text{or}$$

$$-\ \underline{Y}(e^{i\theta})\ N(e^{i\theta}) + X(e^{i\theta})\ D(e^{i\theta}) = 0, \quad \forall\theta\ \varepsilon\ [0,2\pi]. \ (C.36)$$

But, equating the coefficients of the various powers of $e^{i\theta}$ in (C.36) and using the previously defined S_k yields the equation:

$$[\Lambda_{n+2}, \Lambda_1, \Lambda_{n+3}, \Lambda_2, \ldots, \Lambda_{n+1+\bar{m}}, \Lambda_{\bar{m}}]S_{\bar{m}} = 0$$

(where undefined Λ_j's are taken to be zero).

But we know that $S_{\bar{m}}$ has full row rank, from which we immediately conclude that

$$\Lambda_1 = \Lambda_2 = \ldots = \Lambda_{n+m+1} = 0$$

i.e., from the definition of Λ in (C.25), $\lambda = 0$. Thus, M_F is positive definite and the proof of (C.17) is completed.

Part 6.

(7.26), (7.27) and (C.17) now being established, a simple application of Theorem 7.1 yields the last result

$$\lim_{N\to\infty} \hat{\theta}_S(N) = \overset{o}{\theta}_S \qquad \text{a.s.}$$

and the proof of Theorem 7.2 is completed. □

APPENDIX D

Lemma D.1.

Consider $P(t)^{-1}$ and $\psi(t)$ defined in (7.15) and (7.13); $P(t)$ can also be written

$$P(t) = \left[\sum_{j=1}^{t} \psi(j)\psi(j)^T \right]^{-1} .$$

Then, from the "matrix version" of the matrix inversion lemma (MIL), we have:

$$P(t+1) = P(t) - P(t)\,\psi(t+1)[\psi(t+1)^T P(t)\psi(t+1) + I_p]^{-1}\,\psi(t+1)^T P(t) \tag{D.1}$$

$$P(t)\psi(t) = P(t-1)\psi(t)\,[\psi(t)^T P(t-1)\psi(t) + I_p]^{-1} \tag{D.2}$$

$$\psi(t)^T P(t)\psi(t) = \psi(t)^T P(t-1)\psi(t)[\psi(t)^T P(t-1)\psi(t) + I_p]^{-1} < I_p$$

(Notice: right hand side is symmetric.) (D.3)

$$P(t)\psi(t) = P(t-1)\psi(t)\,[I_p - \psi(t)^T P(t)\psi(t)] \tag{D.4}$$

Furthermore, if $\limsup_{t\to\infty} \frac{\mathrm{Tr}\,P(t)^{-1}}{t} < \infty$, then

$$\sum_{t=1}^{\infty} \frac{\mathrm{Tr}[\psi(t)^T P(t)\psi(t)]}{t} < \infty \tag{D.5}$$

□

Proof.

(D.1) is the statement of the matrix inversion lemma, see for instance [43] p.656.

Let $Q \triangleq P(t)\psi(t+1)[\psi(t+1)^T P(t)\psi(t+1) + I]^{-1}$. Then from (D.1)

$$P(t+1)P(t)^{-1} = I - Q\psi(t+1)^T$$

$$P(t+1)P(t)^{-1}\psi(t+1)[\psi(t+1)^T\psi(t+1)]^{-1} = \psi(t+1)[\psi(t+1)^T\psi(t+1)]^{-1} - Q$$

$$Q = [I - P(t+1)(P(t+1)^{-1} - \psi(t+1)\psi(t+1)^T)]\ \psi(t+1)[\psi(t+1)^T\psi(t+1)]^{-1}$$

$$Q = P(t+1)\psi(t+1)\psi(t+1)^T\psi(t+1)\ [\psi(t+1)^T\psi(t+1)]^{-1}$$

$$Q = P(t+1)\psi(t+1)$$

Substituting back the expresion of Q,

$$P(t+1)\psi(t+1) = P(t)\psi(t+1)\ [\psi(t+1)^T P(t)\psi(t+1) + I]^{-1}$$

and (D.2) is established. Multiplying the above equality on the left by $\psi(t+1)^T$, the equality in (D.3) is also established. To prove that $\psi(t)^T P(t)\psi(t) < I$, we use the tensor notation and the fact that $\phi(t)^T M(t)\phi(t) < 1$ when $M(t) \triangleq [\sum_{j=1}^{t}\phi(j)\phi(j)^T]^{-1}$; see [31], equation (A.3).

From the definition of $\psi(t)$ in (7.13),

$$P(t) = [\sum_{j=1}^{t}(I_p \otimes \phi(j))(I_p \otimes \phi(j)^T))]^{-1}$$

$$P(t) = [I_p \otimes \sum_{j=1}^{t}\phi(j)\phi(j)^T]^{-1}$$

$$P(t) = I_p \otimes M(t).$$

So, $$\psi(t)^T P(t)\psi(t) = (I_p \otimes \phi(t)^T)(I_p \otimes M(t))(I_p \otimes \phi(t))$$

$$= I_p \otimes \phi(t)^T M(t)\phi(t) \quad \text{(a scalar matrix)}$$

$$\leq I_p \otimes 1 \quad (\text{" \quad "})$$

$$= I_p$$

and (D.3) is now completely proved.

To prove (D.4), we start from (D.3) and note that since $\psi(t)^T P(t-1)\psi(t)$ is symmetric, so are

$$[I + \psi(t)^T P(t-1)\psi(t)] \quad \text{and} \quad [I + \psi(t)^T P(t-1)\psi(t)]^{-1};$$

thus, taking transposes on both sides of (D.3)

$$\psi(t)^T P(t)\psi(t) = [I + \psi(t)^T P(t-1)\psi(t)]^{-1} \psi(t)^T P(t-1)\psi(t).$$

Multiplying by $-P(t-1)\psi(t)$ and adding $P(t-1)\psi(t)$ we obtain

$$P(t-1)\psi(t)\ [I - \psi(t)^T P(t)\psi(t)] = P(t-1)\psi(t)$$

$$- P(t-1)\psi(t)\ [I + \psi(t)^T P(t-1)\psi(t)]^{-1}\psi(t)^T P(t-1)\psi(t)$$

and right multiplying by $(I - \psi(t)^T P(t)\psi(t))^{-1}$ we obtain

$$P(t-1)\psi(t) = [P(t-1) - P(t-1)\psi(t)[I + \psi(t)^T P(t-1)\psi(t)]^{-1}\psi(t)P(t-1)\psi(t)\ [I - \psi(t)^T P(t)\psi(t)]^{-1}$$

Substituting for the first factor on the right hand side from (D.1) yields

$$P(t-1)\psi(t) = P(t)\psi(t)\ [I - \psi(t)^T P(t)\psi(t)]^{-1}$$

and (D.4) is established.

In order to prove (D.5), we will use again the properties of the tensor product. The hypothesis of (D.5) is that $\limsup_{t\to\infty} \frac{1}{t} \mathrm{Tr}[P(t)^{-1}] < \infty$, and this is equivalent to $\limsup_{t\to\infty} \frac{1}{t} \mathrm{Tr}[M(t)^{-1}] < \infty$ because $\mathrm{Tr}[P(t)^{-1}] = p\, \mathrm{Tr}[M(t)^{-1}]$.
Also,

$$\begin{aligned}\mathrm{Tr}[\psi(t)^T P(t)\psi(t)] &= \mathrm{Tr}[(I_p \otimes \phi(t)^T)(I_p \otimes M(t))(I_p \otimes \phi(t))] \\ &= \mathrm{Tr}[I_p \otimes \phi(t)^T M(t)\phi(t)] \\ &= p\, \phi(t)^T M(t)\phi(t).\end{aligned}$$

But in [31] it is proved that if $\limsup_{t\to\infty} \frac{1}{t} \mathrm{Tr}[M(t)^{-1}] < \infty$, then $\sum_{t=1}^{\infty} \frac{1}{t} \phi(t)^T M(t)\phi(t) < \infty$. Hence, we can conclude that $\sum_{t=1}^{\infty} \frac{1}{t} \mathrm{Tr}\,[\psi(t)^T P(t)\psi(t)] < \infty$ and the proof of Lemma D.1 is completed. □

Proof of Theorem 7.1.

Part 1.

Starting from the system equation (7.11)

$$y_t = X^I(t-1)^T \mathring{\theta}_S + w_t$$

and noting that in the expression for the matrix $\psi(t)$ defined in (7.13) the innovations process w in $X^I(t)$ is replaced by the residuals process η , we can write

$$C(z)w_t = y_t - \psi(t-1)^T \mathring{\theta}_S + (C(z) - C_o)\eta_t \qquad (D.6)$$

Since $\eta_t = y_t - \psi(t-1)^T \hat{\theta}_S(t)$ and $C_o = I$, (D.6) becomes

$$C(z)w_t = C(z)\eta_t + \psi(t-1)^T \tilde{\theta}_S(t)$$

or

$$C(z) \ (\eta_t - w_t) = - \psi(t-1)^T \tilde{\theta}_S(t) \qquad (D.7)$$

where as before $\tilde{\theta}_S(t) \triangleq \hat{\theta}_S(t) - \mathring{\theta}_S$.

We now denote as in [31]:

$$\hat{u}_t = - \psi(t-1)^T \tilde{\theta}_S(t) \qquad (D.8)$$

$$\hat{y}_t = \eta_t - w_t + \frac{1}{2} \psi(t-1)^T \tilde{\theta}_S(t) \qquad (D.9)$$

where $\hat{u}_t$ and $\hat{y}_t$ are p × 1 vectors. It follows from (D.7) that $C(z) \ [\hat{y}_t + \frac{1}{2} \hat{u}_t] = \hat{u}_t$.

From (7.23 b), $[C(z)^{-1} - \frac{I}{2}]$ is asymptotically stable and we can write (neglecting initial conditions whose effect decays geometrically):

$$\hat{y}_t = [C(z)^{-1} - \frac{I}{2}]\hat{u}_t \tag{D.10}$$

The strict positive reality of $[C(z)^{-1} - \frac{I}{2}]$ also implies, by the (strict) positive real lemma (see the proof of that in Appendix A, Lemma A.4),

$$2 \sum_{j=1}^{t} \hat{u}_j^T \hat{y}_j + K \geq \rho \sum_{j=1}^{t} \|\hat{u}_j\|^2 \tag{D.11}$$

where K is a positive constant $(0 < K < \infty)$ depending upon the initial conditions and ρ is a small positive constant.

We define

$$S(t) \triangleq 2 \sum_{j=1}^{t} \hat{u}_j^T \hat{y}_j + K \tag{D.12}$$

and note that $S(t) \geq 0$ for all $t \geq 1$ $(S(t) > 0$ if $u \not\equiv 0$ since $\rho > 0)$.

Part 2.

In this part of the proof, we establish the "near-supermartingale" inequality which will be used to prove the required result. Using (D.4) we have

$$P(t-1)\psi(t-1)e_t = P(t-2)\psi(t-1)[I_p - \psi(t-1)^T P(t-1)\psi(t-1)]\, e_t$$

$$= P(t-2)\psi(t-1)\eta_t \qquad (D.13)$$

from (7.18). Hence, equation (7.14) of the recursion becomes

$$\tilde{\theta}_S(t) = \tilde{\theta}_S(t-1) + P(t-2)\psi(t-1)\eta_t \quad . \qquad (D.14)$$

Let us define

$$T(t) \;\underline{\Delta}\; \tilde{\theta}_S(t)^T P(t-1)^{-1} \tilde{\theta}_S(t) \qquad (D.15)$$

From (D.14) we have

$$T(t-1) = \tilde{\theta}_S(t-1)^T P(t-2)^{-1} \tilde{\theta}_S(t) - \tilde{\theta}_S(t-1)^T \psi(t-1)\eta_t \qquad (D.16)$$

Using (D.14) and the fact that $P(t-2)^{-1} = P(t-1)^{-1} - \psi(t-1)\psi(t-1)^T$, (D.16) can be reorganized as

$$T(t-1) = T(t) - \tilde{\theta}_S(t)^T\psi(t-1)\psi(t-1)^T\tilde{\theta}(t) - 2\tilde{\theta}(t)^T\psi(t-1)\eta_t$$
$$+ \eta_t^T\psi(t-1)^T P(t-2)\psi(t-1)\eta_t \qquad (D.17)$$

Now, if we use (D.4), (7.18), (D.18), and the fact when M is a square matrix and λ a vector of appropriate dimension, then

$$\lambda^T M\lambda = Tr[\lambda^T M\lambda] = Tr[M\lambda\lambda^T] ,$$

we obtain from (D.17), after some manipulations,

$$T(t) = T(t-1) + \tilde{\theta}_S(t)^T\psi(t-1)[\psi(t-1)^T\tilde{\theta}_S(t) + 2(\eta_t - w_t)]$$
$$+ 2[\tilde{\theta}_S(t-1)^T\psi(t-1) + (e_t^T - w_t^T)\psi(t-1)^T P(t-1)\psi(t-1) +$$
$$w_t^T\psi(t-1)^T P(t-1)\psi(t-1)] w_t$$
$$-Tr\{(\psi(t-1)^T P(t-1)\psi(t-1)[I - \psi(t-1)^T P(t-1)\psi(t-1)])\cdot[(e_t - w_t)(e_t^T - w_t^T)$$
$$+ 2w_t(e_t^T - w_t^T) + w_t w_t^T]\} \qquad (D.18)$$

We know that since the initial conditions are in F_t for $t \geq 0$, $w_t = y_t - Ey_t | F_{t-1})$ and so $e_t - w_t = \psi(t-1)^T\hat{\theta}_S(t-1) + Ey_t | F_{t-1})$ is F_{t-1} measurable. If we use $\hat{u}_t$ and $\hat{y}_t$ as defined in part 1, and take conditional expectations in (D.18), we then obtain

$$ET(t)|F_{t-1}) = T(t-1) + E[-\hat{u}_t^T(-\hat{u}_t + 2\hat{y}_t + \hat{u}_t)]|F_{t-1})$$
$$+ 2Tr[\psi(t-1)^T P(t-1)\psi(t-1)\Gamma]$$
$$- Tr\{\psi(t-1)^T P(t-1)\psi(t-1)[I - \psi(t-1)^T P(t-1)\psi(t-1)][(e_t - w_t)(e_t^T - w_t^T) + \Gamma]\}$$

But the last term that is subtracted can be shown to be non-negative since by (D.4)

$$\psi(t-1)^T P(t-1)\psi(t-1)\,[I-\psi(t-1)^T P(t-1)\psi(t-1)]^{-1} = \psi(t-1)^T P(t-2)\psi(t-1) \geq 0$$

and $(e_t - w_t)(e_t^T - w_t^T) + \Gamma > 0$. Therefore we have the inequality

$$E[T(t) + 2\,\hat{u}_t^T \hat{y}_t] \mid F_{t-1}) \leq T(t-1) + 2\,\mathrm{Tr}\,[\psi(t-1)^T P(t-1)\psi(t-1)\Gamma] \quad (D.19)$$

We now use S(t) of part 1 and define

$$T'(t) \underline{\Delta} T(t) + S(t) = T(t) + 2\sum_{j=1}^{t} \hat{u}_j^T \hat{y}_j + K$$

$$= T(t) + 2\hat{u}_t^T \hat{y}_t + S(t-1) .$$

Adding S(t-1) on both sides of (D.19) we obtain

$$E\,T'(t) \mid F_{t-1}) \leq T'(t-1) + 2\,\mathrm{Tr}[\psi(t-1)^T P(t-1)\psi(t-1)\Gamma] \quad ;$$

dividing by t,

$$E\,\frac{T'(t)}{t} \mid F_{t-1}) \leq \frac{T'(t-1)}{t-1} - \frac{T'(t-1)}{t(t-1)} + \frac{2}{t}\,\mathrm{Tr}\,[\psi(t-1)^T P(t-1)\psi(t-1)\Gamma]$$

(D.20)

This is the inequality to which we want to apply the martingale convergence theorem. But before that we must show that $\sum_{t=1}^{\infty} \frac{2}{t}\,\mathrm{Tr}[\psi(t-1)^T P(t-1)\psi(t-1)\Gamma] < \infty$. This is done in the next part of this proof.

Part 3.

From the recursion equations(7.14) and (7.15), we can write

$$\hat{\theta}_S(t)^T P(t-1)^{-1}\hat{\theta}_S(t) = (\hat{\theta}_S(t-1)^T + e_t^T\psi(t-1)^T P(t-1))P(t-1)^{-1}(\hat{\theta}_S(t-1)+P(t-1)\psi(t-1)e_t)$$

$$= \hat{\theta}_S(t-1)^T P(t-2)^{-1}\hat{\theta}_S(t-1) + \|\psi(t-1)^T\hat{\theta}_S(t-1)\|^2$$

$$+ 2e_t^T\psi(t-1)^T\hat{\theta}_S(t-1) + e_t^T\psi(t-1)^T P(t-1)\psi(t-1)e_t .$$

But $e_t = y_t - \psi(t-1)^T\hat{\theta}_S(t-1)$ and so

$$\|\psi(t-1)^T\hat{\theta}_S(t-1)\|^2 = y_t^T y_t - 2e_t^T\psi(t-1)\hat{\theta}(t-1) - e_t^T e_t ;$$

hence

$$\hat{\theta}_S(t)^T P(t-1)^{-1}\hat{\theta}_S(t) = \hat{\theta}_S(t-1)^T P(t-2)^{-1}\hat{\theta}_S(t-1) + y_t^T y_t$$

$$- e_t^T[I - \psi(t-1)^T P(t-1)\psi(t-1)]e_t .$$

Summing the above equation from 1 to t yields

$$\hat{\theta}_S(t)^T P(t-1)^{-1}\hat{\theta}_S(t) + \sum_{j=1}^{t} e_j^T[I-\psi(j-1)^T P(j-1)\psi(j-1)]e_j = \sum_{j=1}^{t} y_t^T y_j + \|\hat{\theta}_S(0)\|^2$$

or

$$\hat{\theta}_S(t)^T P(t-1)^{-1}\hat{\theta}_S(t) + \sum_{j=1}^{t} \eta_j^T[I-\psi(j-1)^T P(j-1)\psi(j-1)]^{-1}\eta_j = \sum_{j=1}^{t} y_j^T y_j + \|\hat{\theta}_S(0)\|^2 \quad \text{(D.21)}$$

from (7.18).

But all the terms in (D.21) are positive and from the stability assumption we know that

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N}\|y_t\|^2 < \infty$$ a.s.; this implies

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \eta_t^T \left[I - \psi(t-1)^T P(t-1)\psi(t-1)\right]^{-1}\eta_t < \infty \quad \text{a.s.}$$

From (D.3) we have $0 \leq \psi(t-1)^T P(t-1)\psi(t-1) < I$ which gives $I \leq [I - \psi(t-1)^T P(t-1)\psi(t-1)]^{-1} < \infty$; this permits us to conclude that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \|\eta_t\|^2 < \infty \quad \text{a.s.} \tag{D.22}$$

We recall that $P(N)^{-1} = \sum_{t=1}^{N} \psi(t)\psi(t)^T$; the diagonal elements of $\frac{1}{N} P(N)^{-1}$ are of the form:

$$\frac{1}{N}\sum_{t=1}^{N} y_t(i)^2 \ , \quad \frac{1}{N}\sum_{t=1}^{N} u_t(i)^2 \ , \quad \frac{1}{N}\sum_{t=1}^{N} \eta_t(i)^2 \ , \quad i = 1,\ldots,p,$$

where $y_t(i)$ is the i-th component of the $p \times 1$ vector y_t, and similarly for u_t and η_t. Therefore, by the stability assumptions (7.21) and (7.22), and by (D.22),

$$\limsup_{N\to\infty} \frac{1}{N} \operatorname{Tr}[P(N)^{-1}] < \infty \quad \text{a.s.} \tag{D.23}$$

From (D.5) in Lemma D.1, (D.23) implies that

$$\sum_{t=1}^{\infty} \frac{\operatorname{Tr}[\psi(t-1)^T P(t-1)\psi(t-1)]}{t} < \infty \quad \text{a.s.} \tag{D.24}$$

from which we can obtain the result required at the end of part 2, namely

$$\sum_{t=1}^{\infty} \frac{1}{t} \operatorname{Tr}[\psi(t-1)^T P(t-1)\psi(t-1)\Gamma] < \infty \text{ a.s.} \tag{D.25}$$

Recalling the near-super-martingale inequality (D.20):

$$E\left(\frac{T'(t)}{t}\,\Big|\,F_{t-1}\right) \leq \frac{T'(t-1)}{t-1} - \frac{T'(t-1)}{t(t-1)} + \frac{2}{t}\,\mathrm{Tr}[\psi(t-1)^T P(t-1)\psi(t-1)\Gamma]$$

we see that it is now possible to apply the martingale convergence theorem. Therefore,

$$\frac{T'(t)}{t} \rightarrow X(\omega) \qquad \text{a.s.} \tag{D.26}$$

where $X(\omega)$ is a non-negative random variable with finite expectation, and

$$\sum_{t=1}^{\infty} \frac{T'(t-1)}{t(t-1)} < \infty \qquad \text{a.s.} \tag{D.27}$$

However, we cannot have $X > 0$ because the second consequence (D.27) of the a.s. convergence of $\{T'(t)/t;\ t \geq 1\}$ would then imply that $\sum_{t=1}^{\infty} \frac{1}{t}\left(\frac{T'(t-1)}{t-1}\right)$ diverges a.s. like $\sum_{t=1}^{\infty} \frac{1}{t}$, which contradicts (D.27). Hence, we conclude immediately that $\frac{T'(t)}{t} \rightarrow 0$ a.s. as $t \rightarrow \infty$ and since $T'(t) = T(t) + S(t)$ and $S(t) \geq 0$, $T(t) > 0$,

$$\lim_{t \rightarrow \infty} \frac{T(t)}{t} = 0 \qquad \text{a.s.} \tag{D.28}$$

$$\lim_{t \rightarrow \infty} \frac{S(t)}{t} = 0 \qquad \text{a.s.} \tag{D.29}$$

These important results will enable us to prove the convergence of $\hat{\theta}_S(t)$ to θ_S in part 5 of this proof. Before that, we need to establish another intermediate result.

<u>Part 4.</u>

In this part of the proof, we show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \|\eta_t\|^2 = \mathrm{Tr}\ \Gamma \qquad \text{a.s.} \tag{D.30}$$

First, we recall that in part 1 we have showed that (see (D.11)): $S(t) \geq C_3 \sum_{t=1}^{N} \hat{u}_t^T \hat{u}_t$. Moreover, since $[C(z)^{-1} - \frac{I}{2}]$ is asymptotically stable, Lemma A.1 of [21] tells us that $\sum_{t=1}^{N} \|\hat{y}_t\|^2 \leq C_1 + C_2 \sum_{t=1}^{N} \|\hat{u}_t\|^2$. (C_1, C_2, C_3 are constants and $0 \leq C_1 < \infty$, $0 < C_2 < \infty$, $0 < C_3 < \infty$.) (D.29) then yields directly:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \|\hat{u}_t\|^2 = 0 \qquad \text{a.s.} \tag{D.31}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \|\hat{y}_t\|^2 = 0 \qquad \text{a.s.} \tag{D.32}$$

Now, consider $\eta_t - w_t = \hat{y}_t + \frac{1}{2} \hat{u}_t$.

$$\|\eta_t - w_t\|^2 = \|\hat{y}_t\|^2 + \frac{1}{4}\|\hat{u}_t\|^2 + \hat{u}_t^T \hat{y}_t$$

so from (D.29), (D.31) and (D.32) we have that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \|\eta_t - w_t\|^2 = 0 \qquad \text{a.s.} \tag{D.33}$$

Then, (D.30) follows direclty if we write

$$\frac{1}{N} \sum_{t=1}^{N} \|\eta_t\|^2 = \frac{1}{N} \sum_{t=1}^{N} [\|\eta_t - w_t\|^2 + 2\, w_t^T(\eta_t - w_t) + \|w_t\|^2] \quad ;$$

using (D.30), (7.20) and the Cauchy-Schwarz inequality we obtain the required result :

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \|\eta_t\|^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \|w_t\|^2 = \mathrm{Tr}\,\Gamma \qquad \text{a.s.}$$

Part 5.

From (D.28), $\dfrac{T(t)}{t} = \dfrac{\tilde{\theta}(t)^T P(t-1)^{-1}\tilde{\theta}(t)}{t} \to 0$ a.s.

as $t\to\infty$. Therefore, if we can show that $\liminf_{N\to\infty} \frac{1}{N} P(N-1)^{-1} > 0$ a.s., we will have proved the almost sure convergence of $\hat{\theta}(t)$ to $\mathring{\theta}_S$.

Consider $P(N)^{-1} = \sum_{t=1}^{N} \psi(t)\psi(t)^T$. If the residuals η are replaced by the innovations w, the new matrix $\sum_{t=1}^{N} X^I(t)X^I(t)^T$ has, by assumption (7.24), the persistent excitation property

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} X^I(t)\; X^I(t)^T = I_p \otimes R \;>\; 0 \qquad \text{a.s.}$$

If η replaces w, do we still have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \psi(t)\psi(t)^T = I_p \otimes R \qquad \text{a.s. ?}$$

The answer is yes. To show this, we notice that the elements affected are of one of the two following forms:

(a) $\dfrac{1}{N} \sum_{t=1}^{N} \eta_t(i)x_{t-k}(j) \qquad 1 \le i \le p,\ \ 1 \le j \le p,$

where x can be either u or y. (k is the time shift between the two processes and $|k| \leq \max(n, m+1, \ell)$.)

But we can write

$$\frac{1}{N}\sum_{t=1}^{N}\eta_t(i)x_{t-k}(j) = \frac{1}{N}\sum_{t=1}^{N}[(\eta_t(i)-w_t(i))x_{t-k}(j)+w_k(i)x_{t-k}(j)]$$

and conclude from Cauchy-Schwarz, (7.21),(7.22) and (D.33) that the first term converges to zero ; the second term converges to the required limit element in R by assumption (7.24).

(b) $$\frac{1}{N}\sum_{t=1}^{N}\eta_t(i)\eta_{t-k}(j) = \frac{1}{N}\sum_{t=1}^{N}[(\eta_i(i) - w_t(i))\eta_{t-k}(j)$$

$$+ w_t(i)(\eta_{t-k}(j)-w_{t-k}(j)) + w_t(i)w_{t-k}(j)].$$

Again, it is easily proved by a simple application of the Cauchy-Schwarz inequality that the first two terms go to zero and therefore

$$\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\eta_t(i)\eta_{t-k}(j) = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}w_t(i)w_{t-k}(j)$$

the required limit element in R .

In conclusion, $\lim_{N\to\infty}\frac{1}{N}P(N)^{-1} = I_p \otimes R > 0$ a.s. and the proof is completed since, as mentioned before, (D.28) now implies

$\lim_{t\to\infty}\tilde{\hat{\theta}}_S(t) = \hat{\theta}_S$ a.s. □

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