A POTENTIAL FLOW THEORY
FOR THE DYNAMICS OF CYLINDER ARRAYS
IN CROSS FLOW

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The full unsteady potential flow solution for fluid flowing

across a bank of cylinders has been obtained. The potential function

is expanded into a Fourier series and the boundary condition of imper-

meability is applied at the moving cylinder surfaces. By extensive

testing and cross-checking it is shown how previously obtained solutions

are in error and, thus, that the present work is believed to be the first

complete and correct unsteady potential flow solution for the problem at

hand. The effect of fluid viscosity is incorporated solely as a phase

lag between the steady-state lift and drag coefficients on each cylinder

and its respective motions. By incorporating the aerodynamic forces

obtained from this modified potential flow theory in a stability analysis,

the threshold for fluidelastic instability is predicted. Comparison with

experimentally observed thresholds is encouraging, given the high level

of idealization of the theory and the accuracy of present day semi-empirical

prediction methods.

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1. Introduction

Arrays of cylinders, in various geometrical arrangements (figure 1), are commonly found in a variety of industrial equipment; e.g., in heat exchangers and steam generators in the form of tubes containing the primary fluid flow and subjected to an external cross flow by the secondary fluid. Such systems have, for a long time now, been known to be subject to a number of interesting and, from the practical viewpoint, undesirable flow-induced vibration phenomena; yet, the state of understanding of the fluid mechanics and of the fluid-structure interaction mechanisms associated with these phenomena is still quite primitive (Paidoussis 1980, 1981).

It is now generally accepted that there are three types of vibration of cylinder arrays, induced by cross flow: (i) due to buffeting, (ii) due to flow periodicity in the interstitial flow, sometimes reinforced acoustically, (iii) the so-called fluidelastic instability. This latter, which is the subject of this paper, is a self-excited aeroelastic phenomenon, involving coupling between the flow field and cylinder motions. At a given threshold flow velocity, energy transfer from the flowing fluid to the cylinders leads to amplified os-cillations — i.e., to an instability in the linear sense. The amplitude of these oscillations is generally large, often resulting in inter-cylinder impact and, hence, eventually to failure. From the practical point of view, fluidelastic in-stability is by far the most serious vibration problem in heat exchange equipment (Paidoussis 1980). More to the point, there is considerable interest at the fundamental level as to the underlying mechanism associated with this phenomenon. (It is unfortunate, perhaps, that the severe financial and safety repercussions of failure and shut-down of large power-generating facilities, caused by fluid-elastic instabilities, have created a demand for instant predictive design tools for avoiding them; as a result, in the now voluminous literature on the subject,

empirical *ad hoc* studies greatly out-number the few attempts to gain understanding at the fundamental level.)

Several models have been proposed for the mechanism underlying fluid-elastic instability. Roberts (1962, 1966) demonstrated the existence of a Coanda-like, jet switching/coalescence mechanism in the separated flow behind a staggered double row array, associated with streamwise motions of alternate cylinders, and showed that this is capable of producing large-amplitude self-excited oscillations. Although Roberts' elegant analytical model correlated well with his own experimental data, the existence of bistable jets is considered to be limited to specific array geometries; in any event, the model has not been extended to deal with multi-row arrays.

Connors (1970) generated a semi-empirical, quasi-static model for a single row of cylinders, based on a position-dependent mechanism, and involving the use of measured aerodynamic lift and drag coefficient data — both in the equilibrium configuration and in "deformed" configurations, with some cylinders displaced in the streamwise or cross-stream directions. Connors found that, for certain patterns of inter-cylinder displacements, energy may be extracted from the fluid over a cycle of cylinder oscillations; when this exceeds the internal dissipative energy in the cylinders, then instability will develop. This model was later elaborated upon and conceptually extended to multi-row arrays by Blevins (1974,1976). Connors, and Blevins, obtained a simple relation for the critical flow velocity, U_c, for the onset of fluidelastic instability, namely

$$\frac{U_c}{fD} = K\left\{\frac{m\delta}{\rho D^2}\right\}^{\frac{1}{2}} , \tag{1}$$

where ρ is the fluid density, D and m are, respectively, the diameter and mass per unit length of the cylinders, and f and δ are the frequency and logarithmic

decrement of vibration of the cylinders. The dimensionless "reduced velocity", U/fD, and "mass-damping parameter", $m\delta/\rho D^2$, arise quite commonly and naturally in many aeroelastic studies. Nevertheless, it should be stressed that in this case *all the fluid mechanics* of the problem are contained within the empirical factor K. Equation (1) gained widespread acceptance because of its simplicity, as well as success in some cases, and a great deal of effort has subsequently been devoted to empirically developing "more reliable" and "more appropriate" (for different array geometries) values of K, separating $m/\rho D^2$ from δ and adjusting the exponents on each in equation (1), etc. (Pettigrew *et al.* 1978, Weaver & El Kashlan 1981, Paidoussis 1981).

Tanaka & Takahara (1981) rejected Connors' quasi-static hypothesis and obtained good agreement between their own experiments and theory, by measuring the full unsteady fluid dynamic forces on cylinders in a water tunnel and using them in a linear analysis — which bespeaks of the high quality of their measurements.

Whereas the above mentioned work has been more or less successful in determining the critical flow velocities for design purposes, it offers little insight into the actual physical phenomena taking place. It ought to be mentioned that Connors' analysis implicitly supposes that the aerodynamic stiffness terms play a predominant rôle in inducing the instability. This is in contrast to aerodynamic-damping-induced instabilities, e.g., in Den Hartog's (1932) much praised quasi-static model for galloping of iced transmission lines. With this observation in mind, Price & Paidoussis (1982,1983) proposed a similar quasi-static flutter mechanism for a double-row array, into which measured aerodynamic stiffness and damping coefficients are incorporated. The critical flow velocity is found to be dependent on, among other things, array geometry, fluid/cylinder density ratio and inter-cylinder modal pattern. The relationship for U_c/fD is

more complex than that of equation (1), and this theory is in agreement with a number of experimental observations.

Chen (1983a,b) suggests that two different mechanisms are responsible for fluidelastic instability: in liquid flow, given the low flow velocities necessary for precipitating the instability, the ratio of cylinder vibrational velocity to the mainstream flow velocity is high, and hence the prevailing mechanism is one of flutter by negative aerodynamic damping; in gaseous flow, on the other hand, the Connors-Blevins mechanism predominates, which is an aerodynamic-stiffness-controlled mechanism involving at least two degrees of freedom. His analysis requires knowledge of the unsteady fluid force coefficients; using Tanaka & Takahara's (1981) data, Chen's results compare favourably with experimental data in both liquid and gaseous flow cases. Recently, Lever & Weaver (1982) proposed yet another model, in which fluidelastic instability is presumed to be a unique mechanism arising from interstitial flow redistribution, associated with and lagging behind cylinder motions. The results obtained agree remarkably with Chen's, despite the fundamental differences in the two models, as well as with experimental trends for some array geometries; nevertheless, some important disagreements also arise (Paidoussis 1983, Heinecke & Mohr 1982).

Flow visualisation experiments by one of the authors (Mavriplis 1982), as well as earlier work by Wallis (1939), have shown that for some array geometries the wakes behind the cylinders are quite narrow, especially when inter-cylinder spacing is small — as sketched in figure 1(a). The presence of adjacent cylinders seems to deflect fluid into the wake of upstream cylinders, minimizing the regions of rotational flow and resulting in surprisingly potential-like flow distributions. Hence, it is rather tempting and not so outrageous to try analysing

the system as if the flow were entirely inviscid.[†] Dalton & Helfinstine (1971) arrived at an inviscid solution by considering each cylinder as a doublet in uniform flow, plus the appropriate image doublets due to mutual interaction of the cylinders, thus obtaining an infinite-series velocity-potential function. However, their analysis becomes over-laborious for systems of more than a few cylinders, and singularities arise for closely spaced cylinders; moreover, in their work no attention has been given to the question of instabilities. Balsa (1977) solved the potential flow problem for a general array of cylinders in cross flow by utilizing matched asymptotic expansion techniques. A model for predicting static instability (divergence) is also presented, although the dynamic (flutter) instability is not given any attention. Comparison between theoretical and experimental results is very limited.

Chen (1978) tackled the same problem, solving the Laplace equation in terms of infinite Fourier series, subject to the impermeability boundary conditions at the surface of each cylinder. Examples of predicted inertial aerodynamic damping and stiffness terms are presented, and they are found to compare poorly with experiments. A full stability analysis is also given, but no results are published. However, a theoretical error was discovered by the authors in Chen's work, as will be discussed later, raising questions as to the validity of the results.

The aim of this paper is to explore further the capabilities and limitations of potential flow theory for dealing with the problem at hand. This is believed to be the first wholly correct formulation of the problem in terms of potential flow theory, in which the errors in previous analyses have been corrected and the theory has been tested along the way in every conceivable manner.

[†]In the case of axial flow about cylinders and cylinder arrays, where rotational flow regions are rather small, potential flow theory has been eminently successful in predicting the dynamical behaviour and stability characteristics of the system (Paidoussis 1966a,b; 1979).

2. Formulation of the flow field

The analysis to be presented is, in principle at least, valid for all array geometries; however, because of the narrowness of the wakes in the so-called "normal triangular" configuration of figure 1(a), it is expected to be most successful for that geometry and especially when the cylinders are closely spaced, i.e., when s_p/D is small. Figure 2 defines the various geometrical parameters, as well as the coordinate systems used: one inertial set, (r_i, θ_i), centred on each cylinder at its position of equilibrium, and another moving with each cylinder, (r_i', θ_i'), in addition to the central inertial Cartesian and polar systems, (x,y) and (r_o, θ_o).

It is assumed that the flow field is represented by the potential function Φ, with one component due to the mainstream cross flow and the other due to the presence/motion of each cylinder j, i.e.,

$$\Phi(r,\theta,t) = Ur_o \cos(\theta_o - \psi_o) + \sum_{j=1}^{K} \phi_j(r,\theta,t), \tag{2}$$

where K is the number of cylinders in the system. As Φ must satisfy the Laplace equation, it is convenient to express the ϕ_j's in terms of Fourier series; thus, for the jth cylinder

$$\phi_j = \sum_{n=1}^{\infty} \frac{R^{n+1}}{r_j'^n} \{a_{jn} \cos n\theta_j' + b_{jn} \sin n\theta_j'\} \tag{3}$$

in terms of the moving coordinate system centred on that cylinder, where it is noted that $\phi_j \to 0$ as $r_j' \to \infty$. The total velocity potential, Φ, must satisfy the condition of impermeability at the surface of the cylinders, i.e.,

$$\frac{\partial \Phi}{\partial r_i'}\bigg|_{r_i'=R} = \frac{\partial u_i}{\partial t} \cos\theta_i' + \frac{\partial v_i}{\partial t} \sin\theta_j' \ , \quad i=1,2,\ldots K, \tag{4}$$

where u_i and v_i are the Cartesian displacements of cylinder i in the x- and y-directions, respectively. The constants a_{jn}, b_{jn} are determined through application of this boundary condition, and to facilitate this endeavour the following

expansions are found useful:

$$a_{jn} = \sum_{\ell=1}^{K} \left\{ \alpha_{jn\ell} \left(\frac{\partial u_\ell}{\partial t} - U \cos\psi_o\right) + \gamma_{jn\ell} \left(\frac{\partial v_\ell}{\partial t} - U \sin\psi_o\right) \right\},$$

$$b_{jn} = \sum_{\ell=1}^{K} \left\{ \delta_{jn\ell} \left(\frac{\partial u_\ell}{\partial t} - U \cos\psi_o\right) + \beta_{jn\ell} \left(\frac{\partial v_\ell}{\partial t} - U \sin\psi_o\right) \right\}, \tag{5}$$

so that the task now becomes to determine $\alpha_{jn\ell}$ to $\delta_{jn\ell}$.

Before applying the boundary conditions, however, all the ϕ_j must be expressed in the moving coordinate frame centred on each of the cylinders in turn. Thus, for the jth potential to be expressed in the coordinate frame associated with the moving cylinder i, one may use the vectorial relationship $r'_j \exp(i\theta'_j) = r'_i \exp(i\theta'_i) - R'_{ij} \exp(i\psi'_{ij})$, as is evident from figure 2 when points A and B coincide, where non-subscript $i=\sqrt{-1}$; this, raised to the power -n and expanded into a convergent Taylor series about $r'_i/R'_{ij}=0$ (*vide* Paidoussis & Suss 1977, Mavriplis 1982), after considerable manipulation leads to

$$\frac{\cos n\theta'_j}{{r'_j}^n} = \sum_{m=0}^{\infty} \frac{(-1)^n (n+m-1)!}{m! \ (n-1)!} \frac{{r'_i}^m}{{R'_{ij}}^{m+n}} \cos[m\theta'_i - (n+m)\psi'_{ij}] \tag{6}$$

and to a similar expression for $\sin n\theta'_j/{r'_j}^n$. With the aid of these expressions, ϕ_j in equation (3) is transformed into the (r'_i, θ'_i) coordinate system, and one finally obtains

$$\Phi = U r_o \cos(\theta_o - \psi_o) + \sum_{n=1}^{\infty} \frac{R^{n+1}}{{r'_i}^n} \{a_{in} \cos n\theta'_i + b_{in} \sin n\theta'_i\}$$

$$+ \sum_{j=1}^{K}{}^{*} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n (n+m-1)!}{m! \ (n-1)!} \frac{R^{n+1} {r'_i}^m}{{R'_{ij}}^{m+n}} \Big\{ a_{jn} \cos[m\theta'_i - (m+n)\psi'_{ij}]$$

$$- b_{jn} \sin[m\theta'_i - (m+n)\psi'_{ij}] \Big\}, \tag{7}$$

where the starred summation excludes j=i.

Substituting equation (5) into (7) and thence applying the boundary conditions (4), after lengthy manipulation and extensive use of the orthogonality

of $\cos m\theta_i'$ and $\sin m\theta_i'$, the following implicit expressions for the coefficients $\alpha_{jn\ell}$ to $\delta_{jn\ell}$ are finally found:

$$(-n)\alpha_{in\ell} + \sum_{j=1}^{K}{}^* \sum_{m=1}^{\infty} G'_{mnij} \left\{ \alpha_{jm\ell} \cos((m+n)\psi_{ij}') + \delta_{jm\ell} \sin((m+n)\psi_{ij}') \right\} = \delta_{n1}\,\delta_{i\ell},$$

$$(-n)\delta_{in\ell} + \sum_{j=1}^{K}{}^* \sum_{m=1}^{\infty} G'_{mnij} \left\{ \alpha_{jm\ell} \sin((m+n)\psi_{ij}') - \delta_{jm\ell} \cos((m+n)\psi_{ij}') \right\} = 0 ,$$

$$(-n)\gamma_{in\ell} + \sum_{j=1}^{K}{}^* \sum_{m=1}^{\infty} G'_{mnij} \left\{ \gamma_{jm\ell} \cos((m+n)\psi_{ij}') + \beta_{jm\ell} \sin((m+n)\psi_{ij}') \right\} = 0 ,$$

$$(-n)\beta_{in\ell} + \sum_{j=1}^{K}{}^* \sum_{m=1}^{\infty} G'_{mnij} \left\{ \gamma_{jm\ell} \sin((m+n)\psi_{ij}') - \beta_{jm\ell} \cos((m+n)\psi_{ij}') \right\} = \delta_{n1}\,\delta_{i\ell},$$

(8)

for $i = 1, 2, \ldots, K$ and $\ell = 1, 2, \ldots, K,$ and $n = 1, 2, \ldots, \infty$, where

$$G'_{mnij} = \frac{(-1)^m \, (n+m-1)!}{(n-1)! \, (m-1)!} \left(\frac{R}{R'_{ij}}\right)^{n+m} \tag{9}$$

and δ_{n1} is Kronecker's delta. This represents 2K sets of 2K×∞ coupled equations, with as many unknowns. Truncating the series at appropriate values of m and n, the coefficients $\alpha_{jn\ell}, \ldots, \delta_{jn\ell}$ may clearly be determined numerically.

Once Φ is determined, the surface pressure on each cylinder at its position of equilibrium may be obtained from the unsteady Bernoulli equation

$$P\Big|_{r_i'=r_i=R} = -\rho \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2} \, \underset{\sim}{\nabla}\Phi \cdot \underset{\sim}{\nabla}\Phi\right]_{r_i'=r_i=R} , \tag{10}$$

and hence the aerodynamic forces acting on cylinder i are

$$H_i = -\int_0^{2\pi} P\Big|_{r_i=R} R\cos\theta_i \, d\theta_i \ , \qquad V_i = -\int_0^{2\pi} P\Big|_{r_i=R} R\sin\theta_i \, d\theta_i \tag{11}$$

in the x- and y-direction, respectively. Evaluation of equation (10) would, at first sight, appear to be quite straightforward; however, if sufficient care is not exercised, it may lead to erroneous results, as discussed in the next section.

3. Evaluation of the unsteady pressure field on the surface of the cylinders

In a previous study (Chen 1978), the $\partial\Phi/\partial t$ term in equation (10) was found to yield only inertial terms — i.e., terms proportional to $\partial^2 u_\ell/\partial t^2$, $\partial^2 v_\ell/\partial t^2$. However, velocity-dependent terms (aerodynamic damping terms) also arise from $\partial\Phi/\partial t$, but this was not realized in that previous analysis through failure to use the moving coordinate frames (r'_j, θ'_j), rather than those fixed at equilibrium, (r_j, θ_j).

The manner in which these velocity-dependent terms arise may be illustrated by considering the simple case of a single cylinder in inviscid cross flow. This system may be represented by either (i) a stationary doublet in uniform flow, with a complex potential function $F(z) = Uz + UR^2/z$, or (ii) a moving doublet in otherwise stationary fluid, where $F(z) = UR^2/[z - \zeta(t)]$ and $\zeta(t) = -Ut$. The surface pressure distribution in both cases is found by applying

$$P\Big|_{z=R} = -\rho \left\{\frac{\partial\Phi}{\partial t} + \frac{1}{2}\nabla\Phi . \nabla\Phi\right\} + \text{const.} \tag{12}$$

with $\Phi = Re\{F(z)\}$.

Now, in the first case, the potential function is time-independent and, hence, $\partial\Phi/\partial t = 0$. Evaluation of $\nabla\Phi . \nabla\Phi$ then leads to the well known result

$$P\Big|_{z=R} = \frac{1}{2}\rho U^2 (1 - 4 \sin^2\theta) + \text{const.,} \tag{13}$$

where it may be shown that the constant represents P_∞, the static pressure far away.

In case (ii), if it is assumed that the doublet passes through the origin at $t = 0$, one obtains $\nabla\Phi . \nabla\Phi\Big|_{z=R} = U^2$. Thus, if $(\partial\Phi/\partial t)_{t=0}$ is neglected, then, in spite of the fact that there is no acceleration of the cylinder, the correct pressure distribution cannot be obtained. In fact, it may be shown that $(\partial\Phi/\partial t)_{t=0} = -U^2\cos2\theta$, thus leading once more to equation (13), provided that

the terms in (12) are correctly interpreted.

Thus, it has been shown that $\partial\Phi/\partial t$ is important in determining not only aerodynamic inertial (acceleration) terms, but also velocity-dependent terms. As the cylinders in the problem at hand are moving, moving reference frames must be utilized; the various terms in equation (10) may be taken at equilibrium in all cases, except for the $\partial\Phi/\partial t$ term which is evaluated at equilibrium *after* the differentiation (with respect to time) is performed. More explicitly in differentiating equations of the form of (7), terms such as $\cos n\theta_i'/r_i'$ and the $\alpha_{jn\ell}$, $\beta_{jn\ell}$ implicit in the a_{jn}, b_{jn}, must be considered to be time-dependent.

The $\nabla\Phi \cdot \nabla\Phi$ term in equation (10) yields additional aerodynamic damping forces,[†] as well as the steady lift and drag forces due solely to the cross flow. In addition, there will be terms dependent on the displacement of each and every cylinder, i.e., the aerodynamic stiffness terms.

Evaluation of the $\partial\Phi/\partial t$ term

Expanding the sine and cosine terms of equation (7), $\partial\Phi/\partial t$, evaluated at the cylinder equilibrium positions (i.e., $R_{ij}'=R_{ij}$, $\psi_{ij}'=\psi_{ij}$, $r_i'=r_i$, $\theta_i'=\theta_i$), may be written as

$$\frac{\partial\Phi}{\partial t} = \sum_{n=1}^{\infty} \frac{R^{n+1}}{r_i^n} (\dot{a}_{in} \cos n\theta_i + \dot{b}_{in} \sin n\theta_i)$$

$$+ \sum_{n=1}^{\infty} R^{n+1} \left\{ a_{in}\left(\frac{\cos n\theta_i'}{r_i'^n}\right)^{\cdot} + b_{in}\left(\frac{\sin n\theta_i'}{r_i'^n}\right)^{\cdot} \right\}$$

$$+ \sum_{j=1}^{K}{}^{*} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n (m+n-1)!}{m! \; (n-1)!} \frac{R^{n+1}}{R_{ij}^{m+n}} r_i^m \left\{ \{\dot{a}_{jn} \cos(m+n)\psi_{ij} + \dot{b}_{jn} \sin(m+n)\psi_{ij}\}\cos m\theta_i \right.$$

$$\left. + \{\dot{a}_{jn} \sin(m+n)\psi_{ij} - \dot{b}_{jn} \cos(m+n)\psi_{ij}\}\sin m\theta_i \right\}$$

[†]Of course, equation (10) gives the pressure, which only when integrated through (11) leads to forces. It is, however, convenient to thus identify the terms which eventually lead to these forces.

$$+ \sum_{j=1}^{K} * \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n (m+n-1)!}{m!\,(n-1)!} \frac{R^{n+1}}{R_{ij}^{m+n}}$$

$$\times \left\{ \left\{ a_{jn} \cos(m+n)\psi_{ij} + b_{jn} \sin(m+n)\psi_{ij} \right\} (r_i^{'m} \cos m\theta_i^{'})^{\cdot} \right.$$

$$\left. + \left\{ a_{jn} \sin(m+n)\psi_{ij} - b_{jn} \cos(m+n)\psi_{ij} \right\} (r_i^{'m} \sin m\theta_i^{'})^{\cdot} \right\}$$

$$+ \sum_{j=1}^{K} * \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n+1} (m+n)!}{m!\,(n-1)!} \frac{R^{n+1}}{R_{ij}^{m+n+1}} \dot{R}_{ij}\, r_i^{m}$$

$$\times \left\{ \left\{ a_{jn} \cos(m+n)\psi_{ij} + b_{jn} \sin(m+n)\psi_{ij} \right\} \cos m\theta_i \right.$$

$$\left. + \left\{ a_{jn} \sin(m+n)\psi_{ij} - b_{jn} \cos(m+n)\psi_{ij} \right\} \sin m\theta_i \right\}$$

$$+ \sum_{j=1}^{K} * \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n} (m+n)!}{m!\,(n-1)!} \frac{R^{n+1}}{R_{ij}^{m+n}} r_i^{m}\, \dot{\psi}_{ij}$$

$$\times \left\{ \left\{ -a_{jn} \sin(m+n)\psi_{ij} + b_{jn} \cos(m+n)\psi_{ij} \right\} \cos m\theta_i \right. \tag{14}$$

$$\left. + \left\{ a_{jn} \cos(m+n)\psi_{ij} + b_{jn} \sin(m+n)\psi_{ij} \right\} \sin m\theta_i \right\} ,$$

where $(\ )^{\cdot} = \partial(\ )/\partial t$.

Each of the dotted terms has to be evaluated separately. Thus, using equation (5) one may write

$$\dot{a}_{jn} = \sum_{\ell=1}^{K} [\alpha_{jn\ell}\, \ddot{u}_\ell + \gamma_{jn\ell}\, \ddot{v}_\ell] + \sum_{\ell=1}^{K} [\dot{\alpha}_{jn\ell}\, (\dot{u}_\ell - U \cos\psi_o) + \dot{\gamma}_{jn\ell}\, (\dot{v}_\ell - U \sin\psi_o)], \tag{15}$$

where

$$\dot{\alpha}_{jn\ell} = \sum_{\ell'=1}^{K} [(\partial\alpha_{jn\ell}/\partial u_{\ell'})\, \dot{u}_{\ell'} + (\partial\alpha_{jn\ell}/\partial v_{\ell'})\, \dot{v}_{\ell'}] \tag{16}$$

and a similar expression for $\gamma_{jn\ell}$. Clearly, terms proportional to $U\dot{u}_\ell$ and $U\dot{v}_\ell$ arise, i.e. damping terms which were neglected in the aforementioned previous analysis. The lengthy manipulations necessary for obtaining $\dot{a}_{jn}$ and $\dot{b}_{jn}$, which

also necessitate the evaluation of $\dot{R}_{ij}$ and $\dot{\psi}_{ij}$ are outlined in Appendix A. The $(\partial\alpha_{jn\ell}/\partial u_{\ell'})$, etc., terms are given by the implicit equations

$$-n\left(\frac{\partial\alpha_{in\ell}}{\partial u_{\ell'}}\right) + \sum_{j=1}^{K}{}^{*} \sum_{m=1}^{\infty} G_{mnij} \left\{ \left(\frac{\partial\alpha_{jn\ell}}{\partial u_{\ell'}}\right) \cos[(m+n)\psi_{ij}] + \left(\frac{\partial\delta_{jn\ell}}{\partial u_{\ell'}}\right) \sin[(m+n)\psi_{ij}] \right\} = P_{in\ell\ell'}$$

$$-n\left(\frac{\partial\delta_{in\ell}}{\partial u_{\ell'}}\right) + \sum_{j=1}^{K}{}^{*} \sum_{m=1}^{\infty} G_{mnij} \left\{ \left(\frac{\partial\alpha_{jn\ell}}{\partial u_{\ell'}}\right) \sin[(m+n)\psi_{ij}] + \left(\frac{\partial\delta_{jn\ell}}{\partial u_{\ell'}}\right) \cos[(m+n)\psi_{ij}] \right\} = Q_{in\ell\ell'}$$

where $P_{in\ell\ell'}$, $Q_{in\ell\ell'}$ are given in Appendix A, with similar expressions for the rest of the $u_{\ell'}$- and $v_{\ell'}$-derivatives of $\alpha_{jn\ell}$, $\beta_{jn\ell}$, $\gamma_{jn\ell}$ and $\delta_{jn\ell}$. Furthermore, $\dot{R}_{ij}$ and $\dot{\psi}_{ij}$ are given by equations (A.2) and (A.3), respectively.

Next, in order to evaluate the dotted terms in the second and fourth summations of equation (14), the terms of the type $\cos n\theta_i'/r_i'$ and $r_i'^{m} \cos m\theta_i'$ must first be expressed in terms of the stationary frame (r_i, θ_i). This is done in a similar manner to that used to obtain equation (6), but the expansion is made about $R_i'/r_i = 0$, yielding

$$\frac{\cos n\theta_i'}{r_i'} = \sum_{m=n}^{\infty} \frac{(m-1)!\ R_i'^{m+n}}{(n-1)!\ (m-n)!\ r_i^{m}} \cos[m\theta_i - (m-n)\psi_i'] ,$$

$$r_i'^{m} \cos m\theta_i' = \sum_{n=0}^{m} \frac{(-1)^{m-n}\ m!\ R_i'^{m-n}\ r_i^{n}}{n!\ (m-n)!} \cos[n\theta_i + (m-n)\psi_i'] ,$$

and similar expressions for the sine terms. Differentiating with respect to time and evaluating at $R_i'=0$, i.e. at equilibrium, gives

$$(\cos n\theta_i'/r_i'^{n})^{\cdot} = nr_i^{-(n+1)} \dot{R}_i' \cos[(n+1)\theta_i - \psi_i'] ,$$

$$(r_i'^{m} \cos m\theta_i')^{\cdot} = -mr_i^{m-1} \dot{R}_i' \cos[(m-1)\theta_i + \psi_i'] ,$$

(17)

where it is further noted (figure 2) that

$$\dot{R}_i' \cos \psi_i' = \partial u_i/\partial t \quad \text{and} \quad \dot{R}_i' \sin \psi_i' = \partial v_i/\partial t .$$

(18)

Hence, with all the dotted quantities evaluated, $\partial\Phi/\partial t$ may be found. The contribution to the forces on the cylinders due to this term, denoted by $H_i^{(1)}$ and $V_i^{(1)}$, are somewhat simplified — after lengthy manipulation — due to orthgonality of the trigonometric functions and are given by

$$H_i^{(1)} = \rho\pi R^2 \sum_{\ell=1}^{K} [A_{i\ell}\,\ddot{u}_\ell + B_{i\ell}\,\ddot{v}_\ell] + \rho UR \sum_{\ell=1}^{K} \left\{ \{C_{i\ell}^{(1)}+C_{i\ell}^{(2)}\}\,\dot{u}_\ell + \{D_{i\ell}^{(1)}+D_{i\ell}^{(2)}\}\,\dot{v}_\ell \right\}$$

$$V_i^{(1)} = \rho\pi R^2 \sum_{\ell=1}^{K} [\bar{A}_{i\ell}\,\ddot{u}_\ell + \bar{B}_{i\ell}\,\ddot{v}_\ell] + \rho UR \sum_{\ell=1}^{K} \left\{ \{\bar{C}_{i\ell}^{(1)}+\bar{C}_{i\ell}^{(2)}\}\,\dot{u}_\ell + \{\bar{D}_{i\ell}^{(1)}+\bar{D}_{i\ell}^{'(2)}\}\,\dot{v}_\ell \right\},$$

(19)

where the superscript (1) denotes terms arising from time derivatives of $\alpha,\beta,\gamma,\delta$, and (2) denotes those arising from co-ordinate-system movement. The constants $A_{i\ell}$, $B_{i\ell}$, etc. are given in Appendix B. It is noted that in Chen's (1978) previous analysis all the terms involving $\dot{u}_\ell$ and $\dot{v}_\ell$ are absent.

Evaluation of $\underset{\sim}{\nabla}\Phi.\underset{\sim}{\nabla}\Phi$ term

The flavour, intricacy and requisite care in the derivation of the terms in equation (10) may be appreciated from the foregoing evaluation of $\partial\Phi/\partial t$. In the interests of brevity, therefore, the details of the derivation of the $\underset{\sim}{\nabla}\Phi.\underset{\sim}{\nabla}\Phi$ term will not be presented here. The interested reader is referred to Mavriplis' (1982) and Chen's (1978) work for details.

It may be shown that the contributions of $\underset{\sim}{\nabla}\Phi.\underset{\sim}{\nabla}\Phi$ to the drag and lift forces on cylinder i, denoted by $H_i^{(2)}$, $V_i^{(2)}$, are

$$H_i^{(2)} = \rho U^2 R\, C_{Do_i} + \rho UR \sum_{\ell=1}^{K} \left\{ C_{i\ell}^{(3)}\,\dot{u}_\ell + D_{i\ell}^{(3)}\,\dot{v}_\ell \right\},$$

(20)

$$V_i^{(2)} = \rho U^2 R\, C_{Lo_i} + \rho UR \sum_{\ell=1}^{K} \left\{ \bar{C}_{i\ell}^{(3)}\,\dot{u}_\ell + \bar{D}_{i\ell}^{(3)}\,\dot{v}_\ell \right\},$$

where C_{Do_i} and C_{Lo_i} are the steady drag and lift coefficients and $C_{i\ell}^{(3)}$, ... $\bar{D}_{i\ell}^{(3)}$ are coefficients of additional velocity-dependent terms, expressions for which are given in Appendix B.

Evaluation of the stiffness forces

The stiffness forces represent the changes in the steady lift and drag on the cylinders as they are displaced. Assuming that they may be linearized, the contribution to the lift and drag forces may be expressed by

$$
H_i^{(3)} = \rho U^2 R \sum_{p=1}^{K} \left\{ (\partial C_{Do_i}/\partial u_p) \, u_p + (\partial C_{Do_i}/\partial v_p) \, v_p \right\} ,
$$

$$
V_i^{(3)} = \rho U^2 R \sum_{p=1}^{K} \left\{ (\partial C_{Lo_i}/\partial u_p) \, u_p + (\partial C_{Lo_i}/\partial v_p) \, v_p \right\} ,
$$

$$(21)$$

where the partial-derivative terms are given in Appendix B; for detailed derivations the reader is referred to Mavriplis' (1982) work.

In all the foregoing it has been assumed that pressure fluctuations on the surface of the cylinders are generated instantaneously; i.e., it has been presumed that there is no phase difference between cylinder displacements and the pressure field, otherwise known as the quasi-static assumption. Yet, it has been shown that phase lags do arise in the problem at hand — at least in the model proposed by Roberts (1962,1966). In that work, Roberts evaluated the finite time necessary for the inter-cylinder fluid jets to adjust to repositioning of the cylinders, as well as the resulting change in the cylinder wake (or "bubble") pressure. Here, the phase lag may be thought to be related to viscous effects more generally, i.e., to the lag that the cylinder wake experiences in adjusting to cylinder motion. In terms of the mean flow velocity U and cylinder diameter D, this time lag should be proportional to D/U, and hence the phase lag angle χ should be proportional to fD/U, where f is the frequency of oscillation (*cf*. Lever & Weaver 1982, White 1979).

Thus, although viscous forces *per se* will not be calculated and used in this analysis — having been assumed to be of secondary importance at the outset of this paper — one important *effect* of these forces will be incorporated in

the analysis, namely the phase-lag effect discussed above, which modifies the quasi-static forces. The analogy of the aerofoil could be invoked here: the viscous forces in that case are small and may be neglected; yet they have a very important effect in determining the structure of the flow and modifying the inviscid forces — hence determining the Kutta condition.[†]

If the cylinders oscillate harmonically only in the cross-stream direction, such that $v_p = \bar{v}_p \sin(\omega t + \phi_p)$, then the forces $H_i^{(3)}$, $V_i^{(3)}$ will also vary harmonically, but lagging by a phase angle χ, so that

$$H_i^{(3)} = \rho U^2 R \sum_{p=1}^{K} (\partial C_{Do_i}/\partial v_p) \, \bar{v}_p \, \sin(\omega t + \phi_p + \chi),$$

which may be re-written as

$$H_i^{(3)} = \rho U^2 R \sum_{p=1}^{K} (\partial C_{Do_i}/\partial v_p) \, \{\cos\chi \, v_p + \frac{\sin\chi}{\omega} \, \dot{v}_p\} \tag{22}$$

and similarly for $V_i^{(3)}$. The generalization to more general motions is self-evident, yielding

$$H_i^{(3)} = \rho U^2 R \sum_{p=1}^{K} \left\{ [E_{ip} \cos\chi \, u_p + F_{ip} \cos\chi \, v_p] + [E_{ip} \, (\sin\chi/\omega) \, \dot{u}_p + F_{ip} \, (\sin\chi/\omega) \, \dot{v}_p] \right\}$$

$$V_i^{(3)} = \rho U^2 R \sum_{p=1}^{K} \left\{ [\bar{E}_{ip} \cos\chi \, u_p + \bar{F}_{ip} \cos\chi \, v_p] + [\bar{E}_{ip} \, (\sin\chi/\omega) \, \dot{u}_p + \bar{F}_{ip} \, (\sin\chi/\omega) \, \dot{v}_p] \right\}$$

$$\tag{23}$$

where $E_{ip} = (\partial C_{Do_i}/\partial u_p)$, $F_{ip} = (\partial C_{Do_i}/\partial v_p)$, $\bar{E}_{ip} = (\partial C_{Lo_i}/\partial u_p)$ and $\bar{F}_{ip} = (\partial C_{Lo_i}/\partial v_p)$, and where the cylinders have been assumed to oscillate harmonically. Hence, yet another set of velocity-dependent forces, of the form of the second terms of equation (23), emerge, related to this phase lag.

[†] Phase lag is well known to have an important effect on stability in aero-elasticity. Thus, in addition to the aforementioned work by Roberts (1966) and recent work by Lever & Weaver (1982) on the problem at hand, *vide* the importance of phase lag — albeit due to mechanisms different than the one proposed here — in flutter of aerofoils (Bisplinghoff, Ashley & Halfman 1955; Dowell *et al.* 1980) and of overhead transmission lines (Simpson & Flower 1977).

4. The equations of motion and their solution

If each of the cylinders is considered to be a simple beam, then the equations of motion may be written in the form

$$EI \frac{\partial^4 u_i}{\partial z^4} + c \frac{\partial u_i}{\partial t} + m \frac{\partial^2 u_i}{\partial t^2} = \sum_{\ell=1}^{K} \left\{ \rho \pi R^2 \left[A_{i\ell} \frac{\partial^2 u_\ell}{\partial t^2} + B_{i\ell} \frac{\partial^2 v_\ell}{\partial t^2} \right] \right.$$

$$\left. + \rho UR \left[C_{i\ell} \frac{\partial u_\ell}{\partial t} + D_{i\ell} \frac{\partial v_\ell}{\partial t} \right] + \rho U^2 R \left[E_{i\ell} u_\ell + F_{i\ell} v_\ell \right] \right\} + \rho U^2 R\, C_{Do_i},$$

$$EI \frac{\partial^4 v_i}{\partial z^4} + c \frac{\partial v_i}{\partial t} + m \frac{\partial^2 v_i}{\partial t^2} = \sum_{\ell=1}^{K} \left\{ \rho \pi R^2 \left[\bar{A}_{i\ell} \frac{\partial^2 u_\ell}{\partial t^2} + \bar{B}_{i\ell} \frac{\partial^2 v_\ell}{\partial t^2} \right] \right.$$

$$\left. + \rho UR \left[\bar{C}_{i\ell} \frac{\partial u_\ell}{\partial t} + \bar{D}_{i\ell} \frac{\partial v_\ell}{\partial t} \right] + \rho U^2 R \left[\bar{E}_{i\ell} u_\ell + \bar{F}_{i\ell} v_\ell \right] \right\} + \rho U^2 R\, C_{Lo_i}, \quad i=1,2, \ldots K,$$

(24)

where EI, c and m are, respectively, the flexural rigidity, internal (viscous) damping and mass per unit length of each cylinder; the right-hand sides of the equations are, respectively, $H_i = H_i^{(1)} + H_i^{(2)} + H_i^{(3)}$, $V_i = V_i^{(1)} + V_i^{(2)} + V_i^{(3)}$, where the component terms $H_i^{(1)}$, $H_i^{(2)}$, etc. have been derived in section 3 — in general also containing phase-related terms of the form introduced in equation (23).

The solution of the linearized form of equation (24) is achieved by the modal analysis method (Bishop & Johnson 1960), where

$$u_\ell = \sum_{p=1}^{\infty} q_{\ell p}(t)\, \psi_p(z) \quad, \quad v_\ell = \sum_{p=1}^{\infty} r_{\ell p}(t)\, \psi_p(z),$$

(25)

where the $\psi_p(z)$ are the orthonormal set of beam eigenfrequencies of the cylinders *in vacuo*. Substituting into (24), invoking the orthogonality of the $\psi_p(z)$, and truncating the series at an appropriate value of p, one eventually obtains a set of equations which may be written in matrix form as follows:

$$\underset{\sim}{M}\, \ddot{\underset{\sim}{a}} + \underset{\sim}{C}\, \dot{\underset{\sim}{a}} + \underset{\sim}{K}\, \underset{\sim}{a} = \underset{\sim}{0},$$

(26)

where $\underset{\sim}{a} = \{\underset{\sim}{q}, \underset{\sim}{r}\}^T$. Transforming this into a standard eigenvalue problem, the 2K eigenvalues of the system may be obtained, which permits one to assess stability of the system with varying U.

The detailed form of $\underset{\sim}{M}$, $\underset{\sim}{C}$ and $\underset{\sim}{K}$ will not be given here. Suffice it to say that when (26) is written in dimensionless form, the following dimensionless quantities will be involved: the mass parameter $m/\rho D^2$; the logarithmic decrement, δ, of the cylinders *in vacuo*; the reduced velocity U/fD, where $f = \omega/2\pi$ is the first-mode *in vacuo* frequency, ω being the corresponding radian frequency; the frequency ratio $\xi_p = \omega_p/\omega$, where ω_p is the radian frequency of the pth mode.

5. Testing the theory

The intricacy of the algebraic manipulations of section 3, no less than the desire to assess the success of the theory in reproducing some well known results, made it imperative to test the theory prior to embarking on a full scale stability analysis.

Apart from verifying that some obvious symmetry conditions imposed by various cylinder geometries are indeed reproduced in the calculated force coefficients, various other simple tests were also conducted. Thus, the flow at zero incidence ($\psi_o = 0$ — see figure 2) across a particular array was reproduced by rotating the array through an angle ψ_o and imposing an angle of incidence ψ_o to the flow.

It was then attempted to compare the various fluid force coefficients in equations (19)-(24) to those obtained by other analyses. The virtual (added) mass coefficients $A_{i\ell}$, $\bar{A}_{i\ell}$, $B_{i\ell}$, $\bar{B}_{i\ell}$ were compared to those obtained by Chen (1975,1978), Suss (1977), Paidoussis & Suss (1977) and Dalton & Helfinstine (1971).

The coefficients were found to agree to within three significant figures for the first two, and to within two significant figures for the third one, as illustrated in table 1. The steady lift and drag coefficients were also compared to some available theoretical results by Chen and Dalton & Helfinstine, in table 1, displaying similarly good agreement.

The portions of the damping coefficients due to the $\nabla\Phi.\nabla\Phi$ term in Bernoulli's equation, i.e., $C_{i\ell}^{(3)}$, $\bar{C}_{i\ell}^{(3)}$, $D_{i\ell}^{(3)}$, $\bar{D}_{i\ell}^{(3)}$, which have also been obtained by Chen (1978), were compared in the case of a five-cylinder system. Some selected results are shown in the upper part of table 2, where agreement is found to obtain to three significant figures.

The total fluid damping coefficients $C_{i\ell}$, $D_{i\ell}$, $\bar{C}_{i\ell}$, $\bar{D}_{i\ell}$ cannot be verified, as no other such results are available in the literature. However, some simplified cases may be verified by a specially constructed quasi-static analysis. Thus, if all cylinders move with the same velocity $\dot{u}$ and $\dot{v}$ in the x- and y-directions, the array configuration remains unchanged, as the whole array is displaced as a unit. Without giving details of the analysis here, which follows the pattern of Den Hartog's (1932) quasi-static flutter analysis, suffice it to say that the unsteady drag and lift forces on the cylinders are found to be given by

$$H_i = \rho U R \left\{ -2C_{Do_i} \dot{u}_i + (C_{Lo_i} + \partial C_{Do_i}/\partial\alpha) \dot{v}_i \right\}$$

$$V_i = \rho U R \left\{ -2C_{Lo_i} \dot{u}_i + (-C_{Do_i} + \partial C_{Lo_i}/\partial\alpha) \dot{v}_i \right\} ,$$

with $\dot{u}_i = \dot{u}$, $\dot{v}_i = \dot{v}$ for all i, and $\alpha \simeq \dot{v}/U$. The $\partial C_{Lo_i}/\partial\alpha$ and $\partial C_{Do_i}/\partial\alpha$ terms arise because, under the effect of the imposed motion, the array is not totally symmetric *vis-à-vis* the mean flow; i.e., the angle α effectively corresponds to $-\psi_o$. Agreement between the results obtained by the potential flow analysis

and this quasi-static approach was found to be excellent, the individual terms
agreeing to within four significant figures. This represents an important fea-
ture of this analysis, as previous analyses in fact do not pass this simple test.

It must nevertheless be noted that, in the above case where the array
is moved as a unit, there is no inter-cylinder motion. Hence, only the
$C_{i\ell}^{(2)}$, $D_{i\ell}^{(2)}$ types of terms in equation (19) are verified, and it would be de-
sirable to test independently the terms of the type $C_{i\ell}^{(1)}$, $D_{i\ell}^{(1)}$, which are as-
sociated with the $\dot{\alpha}_{jn\ell}$, $\dot{\beta}_{jn\ell}$, $\dot{\gamma}_{jn\ell}$, $\dot{\delta}_{jn\ell}$ terms (see equation (16)). To this
end, the changes to the $\alpha_{jn\ell}$ to $\delta_{jn\ell}$, i.e. the $\partial\alpha_{jn\ell}/\partial u_\ell$, etc., have been cal-
culated by displacing each cylinder by a small amount Δu at the input stage
of the computer program and recalculating all the $\alpha_{jn\ell}$ to $\delta_{jn\ell}$ coefficients at
each step, thus obtaining $\partial\alpha_{jn\ell}/\partial u_{\ell'} \simeq \Delta\alpha_{jn\ell}/\Delta u_{\ell'}$. This somewhat inelegant
and approximate evaluation of the $\partial\alpha_{jn\ell}/\partial u_{\ell'}$ and other similar terms permitted
the eventual evaluation of the $C_{i\ell}^{(1)}$-type of terms. Comparisons of these terms
with those obtained analytically by the potential flow analysis are illustrated
in table 3. Acceptable agreement is obtained; most of the discrepancies are
thought to arise from the finite step size of Δu (or Δv).

In spite of all the above testing, the theory still did not agree
with that of Balsa (1977). Regardless of cylinder array configuration, the
present analysis yields zero diagonals in the $C_{i\ell}$ and $\bar{D}_{i\ell}$ sub-matrices, while
the diagonal of the $D_{i\ell}$ sub-matrix is non-zero and equal to the negative of
the $\bar{C}_{i\ell}$ diagonal. Physically, this implies that the motion of any one cylinder
in an array produces a force on itself which is perpendicular to its motion,
and precludes the possibility of any self-damping forces on the cylinder in the
direction of its motion. A cylinder can, however, produce damping forces in
any direction on the other cylinders regardless   of its motion. This
result disagrees with that obtained in the work of Balsa (1977). By constructing

inner and outer region canonical potentials for a cylinder array and matching

them asymptotically, Balsa arrived at the result that the motion of any one

cylinder in the array produces no net damping force on itself in either direc-

tion, i.e. the diagonals of all sub-matrices $C_{i\ell}$, $\bar{C}_{i\ell}$, $D_{i\ell}$, $\bar{D}_{i\ell}$ are zero.

After the extensive testing performed on the present analysis, it is felt that

the fault must lie in Balsa's work rather than in this analysis.[†] Balsa's ex-

pansions are expressed in powers of ε, where $\varepsilon = D/s_p$ is the inverse of the

pitch ratio. In performing the asymptotic matching, three inner terms and

four outer terms are used. However, for pitch ratios of order 1.5, the fourth

term in the expansion, ε^4, is approximately 0.2. Hence, at first view, it

seems Balsa may have neglected significant higher order terms. (In fact, the

diagonals of $D_{i\ell}$, $\bar{C}_{i\ell}$ which vanish in Balsa's analysis are always substantially

smaller than unity in the present analysis.) However, the above is only a

possible explanation of the inconsistency between the two analyses, and the

matter has not been pursued further.

Nevertheless, to ensure that the present analysis is not in error,

the method of Dalton & Helfinstine (1971) was employed to calculate the damping

terms in the simple case of a row of two cylinders normal to the flow. The

method consists of constructing the complex potential function by considering

one doublet for each cylinder moving in still fluid, plus its image in the

neighbouring cylinder, which is required to maintain a circular streamline at

the neighbouring cylinder boundary, plus the images of the images and so on.

For the two cylinders at a pitch ratio of 1.5, a total of six doublets was

employed (i.e., up to "third order"), since the third-order images were about

1% of the strength of the original doublet. The complex potential function

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[†]Although it is realized that neither truly represents the physics of the

situation, as an oscillating cylinder will be subjected to a damping force

due to its own motion.

was obtained through a specially written computer program and its time deriv-
ative calculated by first evaluating the complex potential on the moving cylin-
der boundaries at two different times and then dividing the difference by the
time step. It is noted that for the particular geometry considered in this
test, the $C_{i\ell}$ and $\bar{D}_{i\ell}$ sub-matrices vanish and only the off-diagonal terms of
the $D_{i\ell}$ and $\bar{C}_{i\ell}$ sub-matrices need be considered. As may be seen in table 4,
both the virtual (added) mass coefficients and the damping terms due to the
time derivative of the potential function are found to be in good agreement
with those obtained by the present analysis. Hence, it may now be asserted
that Balsa's (1977) work must be in error.

Finally, the stiffness coefficients (equation (23) with $\chi = 0$) were
also calculated by directly noting the changes in steady lift and drag co-
efficients on each cylinder as various cylinders in the array were displaced.
This is the method employed by Chen (1978). His results, as well as those
obtained in this work by displacing the cylinders, compare favourably with
the stiffness coefficients calculated by the expressions developed in the
present paper, as may be seen in the lower half of table 2.

Having tested all parts of analysis, it may now be stated that, to
the authors' knowledge, it is the first complete and correct potential-flow
solution for cylinder arrays in cross flow.[†]

[†]Here a distinction ought to be made between Chen's (1978) and Balsa's (1977)
solutions, which have been shown to be in error (in section 3 and here),
and Dalton & Helfinstine's (1971) solution, which is not; this latter,
however, is limited to calculating inertial and steady lift and drag
coefficients, and is difficult to extend beyond that.

6. Calculations and results

Computations were largely confined to seven-cylinder arrays of the type shown inset in figures 3 and 4 — mainly for economy, as computational costs increase sharply with the number of cylinders in the array. Nevertheless, these arrays were considered to be sufficiently large and their geometry sufficiently general to exhibit all salient features of larger systems.[†] As virtually all the experimental data to which the theoretical results will be compared are for arrays with $1.3 < s_p/D < 1.5$, computations were made with the two extremum values.

In the calculations, the experimentally reasonable value of $\delta = 0.01$ was used, and $m/\rho D^2$ was varied from $O(1)$ to $O(10^5)$, the lower value corresponding to light cylinders in dense (liquid) flows and the upper one to heavy cylinders in gaseous flows.

The only other parameter which needs be selected is χ. It is recalled that this phase angle, which is attributed to viscous effects, has been introduced empirically, and that at present no means is available for calculating it. It was nevertheless shown that χ should be a function of the inverse of the reduced velocity. Thus, guided by the observed range of the critical U/fD for fluidelastic instabilities, extensive calculations were conducted with $\chi = 0°$, $10°$ and $30°$, as shown in figures 3 and 4; the effect of χ is more extensively discussed later in conjunction with figure 5. For high values of U/fD, a low value of χ would be expected, and *vice versa*.

The calculated theoretical dimensionless critical flow velocities, U_{pc}/fD, for the onset of fluidelastic instability are shown in figures 3 and 4, where they are compared to the experimental data available from various sources. It is noted that the interstitial, so-called "pitch" flow velocity, U_p, is used

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[†]Indeed, calculations of the critical flow velocity for twelve-cylinder arrays differed by only −3 to −4% from those of seven-cylinder arrays.

to define its critical counterpart U_{pc}, rather than the free-stream one, U, where $U_p = U[s_p/(s_p-D)]$.[§] Here it should be noted that although it was considered desirable to segregate the data for "normal" and "parallel" triangular arrays, in figure 3 and 4, respectively, because of the inherently different wake structures involved (figure 1), the theoretical values are only slightly different in the two cases, as this theoretical model does not recognize these wake-related differences.

One important feature of the theoretical results is that although those presented in figures 3 and 4 are strictly for $\delta = 0.01$, they are nonetheless representative of the results for other values of δ. For example, in the case of $s_p/D = 1.5$, $m\delta/\rho D^2 = 10$ and $\chi = 30°$, U_{pc}/fD is 38.2 and 36.2 for $\delta = 0.01$ and 1, respectively; these differences are smaller for larger $m\delta/\rho D^2$ and *vice versa*. Hence, although $m/\rho D^2$ and δ are independent parameters, U_{pc}/fD is much more dependent on their product, $m\delta/\rho D^2$, than on either separately. This agrees with Chen's (1983b) findings and is contrary to Weaver & El Kashlan's (1981).

Another important feature of the results is that the use of U_p in preference to U has achieved the effective collapse of the theoretical results (in the two cases of s_p/D of 1.3 and 1.5), except for low values of $m\delta/\rho D^2$, as shown in figure 3.

The theoretical U_{pc}/fD for $\chi = 10°$ and $30°$ correspond to oscillatory instabilities and are thus, directly comparable to the experimental data, whilst those for $\chi = 0°$ are associated with divergence, which is a non-oscillatory instability. Hence, it is quite clear that viscous effects and the lag they introduce between cylinder displacement and the fluid forces thereby generated

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[§]This is done partly because U_p is physically more meaningful and is the velocity conventionally quoted in the literature, and partly because it has been found to partially collapse the experimental data for different s_p/D.

are quite important in the development of fluidelastic instability.

For $\chi = 10°$ and $30°$ the frequency at U_{pc} of the mode becoming unstable is close to its value at $U_p = 0$, which agrees with experimental observations (Paidoussis 1980,1981); e.g., for $\chi = 30°$, $m\delta/\rho D^2 = 10$, $\delta = 0.01$, $s_p/D = 1.5$, the ratio of the former to the latter frequency is 0.996. It should neverthe-less be noted that this does not hold for large δ; thus, for $\delta = 1$ and otherwise the same parameters as above, this frequency ratio is 0.597.

Comparing the theoretical results to the experimental data, it is noted that theory overestimates the critical flow velocity for instability roughly by a factor of three, for $\chi = 30°$, and nearly five, for $\chi = 10°$ — in an average sense, as the experimental data display a very large spread in them-selves.[§] Here it should be noted that it would be reasonable to compare the ex-perimental data at large $m\delta/\rho D^2$ to smaller χ than those for small $m\delta/\rho D^2$, as the former are associated with higher values of U_{pc}/fD; this would make agree-ment for large $m\delta/\rho D^2$ (gaseous flows) rather worse than for small ones (liquid flows).

One of the reasons why this theory overestimates U_{pc}/fD, especially at high $m\delta/\rho D^2$, is that it does not take into account the presence of wakes and wake-interference effects on the aerodynamic stiffness coefficients, which normally render the aerodynamic stiffness matrix asymmetric. As shown elsewhere (Price & Paidoussis 1983) asymmetry of this matrix, which effectively means that

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[§]It is noted that the experimental U_{pc}/fD and $m\delta/\rho D^2$ in many cases are calculated with values of f, m and δ *in fluid*, rather than *in vacuo* as was done in the theory. Moreover, it was found not usually possible, through insufficient information, to convert such experimental data to their *in vacuo* counterparts; this conversion would have resulted in moving the data points concerned to the left and a little lower in the case of low $m\delta/\rho D^2$ (liquid flow), but would hardly affect those with high $m\delta/\rho D^2$.

the static force field is non-conservative, has a strong destabilizing effect on the system. This effect is entirely absent in this analysis, where this matrix is symmetric.

It is of interest that the trend in the theoretical U_{pc}/fD with increasing $m\delta/\rho D^2$ is similar to that shown by the experimental data, especially if U_{pc}/fD for $\chi = 30°$ is associated with small $m\delta/\rho D^2$ and that for $\chi = 10°$ or less for large $m\delta/\rho D^2$ — for the reasons mentioned earlier. The theoretical results indicate that $U_{pc}/fD \propto (m\delta/\rho D^2)^n$, where n is in the range of 0.4 to 0.6 almost throughout, whereas it is currently suggested that for small $m\delta/\rho D^2$ this exponent should be considerably smaller than 0.5 (Chen & Jendrzejczyk 1978).

Of course, the question arises as to how sensitive are the theoretical results presented thus far to the value of the parameter χ. As shown in figure 5, they are not so sensitive — at least in the range of $5° < \chi < 160°$. For $\chi < 5°$ approximately, however, the critical flow velocity for lightly damped systems increases sharply as $\chi = 0°$ is approached. (At $\chi = 0°$, as previously mentioned, the very nature of the instability is different.) This effect is nevertheless attenuated by higher damping, as shown. For $\chi = 180°$, no instability is found at all; the situation here is similar to that for $\chi = 0°$, except that the signs of all flow-induced stiffness terms are reversed. Finally, for $180° < \chi < 360°$, the results (not shown) mirror closely, but not exactly, those in figure 5, as expected.

Considering the drastic simplifying assumptions that have been made in the theory, the most serious of which is to ignore the wakes — the existence of viscous effects being recognized only through the introduction of the phase lag χ — it is remarkable that the theoretical values of U_{pc}/fD for $\chi = 30°$ or 10° are as close to the experimental values as they are. Of course, prediction of the critical flow velocity to within a factor of 3 or 5 is not satisfactory.

Nevertheless, this lack of success should be viewed in the context of existing semi-empirical analytical models, relying heavily on measured force coefficient data, which are capable of prediction to just within a factor of 2.[†]

One current controversy is concerned with whether one flexible cylinder surrounded by rigid ones does become unstable for high $m\delta/\rho D^2$ — it being generally accepted that it does for low $m\delta/\rho D^2$ (Lever & Weaver 1982, Chen 1983b, Paidoussis 1983). Calculations conducted for one flexible cylinder surrounded by six rigid ones showed that, according to this theory, the flexible cylinder does become unstable for all $m\delta/\rho D^2$ — but at higher critical flow velocities (by 30 to 75%), as compared to the system with seven flexible cylinders.

[†]In considering the success, or lack thereof, of this theoretical model in terms of predicting U_{pc}/fD much more emphasis should be placed on figure 3 than on figure 4, for the reasons stated at the outset. The fact that agreement between theory and experiment is similar in the two cases could well be fortuitous.

7. Conclusion

A potential flow theory has been presented for the dynamics, and more specifically the fluidelastic instabilities, of staggered arrays of cylinders in cross flow. The theory is purely analytical, except for the empirical parameter χ, which is a measure of the phase lag between displacement of the cylinders and the lift and drag forces generated on them — due to viscous effects, otherwise neglected.

All analytical parts of the theory were carefully checked and compared to previous work. It was shown that previous formulations by Chen (1978) and Balsa (1977) must be erroneous — the first by direct proof, the second by inference; hence, to the authors' knowledge, this is the first wholly correct formulation of the problem in terms of potential flow theory. In one sense, this is considered to be the major accomplishment of this work, as it permitted the assessment of the limitations of potential flow theory for predicting the dynamical behaviour of the system.

The most important finding of this work is that if viscous effects are neglected altogether, then the only form of instability possible is divergence, which is a static, non-oscillatory instability. Hence, having established the prominence of the viscous effects, it was considered desirable to take them into account, albeit empirically. Although no attempt was made to determine the viscous forces *per se*, one important *effect* of these forces was introduced: the phase lag χ. It was found that, with reasonable values of this phase lag, theory predicts the occurrence of oscillatory instabilities; moreover, the predicted characteristics of the system and the threshold of instability are remarkably close to the measured ones— remarkably, that is, considering the degree of idealization entailed in the analytical model.

The above suggests that a useful direction for future research could be to attempt the analytical modelling of viscous forces on the cylinders and

hence the determination of the phase lag inherent therein.

It is nevertheless recognized that ideal flow theory is not the most suitable fluid mechanical tool for a successful theory for the problem at hand, because of the importance of the wakes — even on the quasi-static aerodynamic stiffness coefficients. Thus, unless wake interference effects are accounted for, the essential non-conservativeness of the force field (in the static sense), which is known to have a strong destabilizing influence on the dynamics of the system, is not recognized.

Finally, it might be useful to discuss this theoretical model in the context of other, recently published theories. Price & Paidoussis' (1983) theory requires the measurement of the static forces on the cylinders in various displaced positions, and Tanaka & Takahara (1981) and Chen's (1983b) theory requires the measurement of the *dynamic* forces over a range of flow velocities — a difficult and tedious task. Lever & Weaver's (1982) theory requires three empirical inputs and is otherwise analytical; it is, therefore, more comparable to this theory, which involves but one empirical parameter. As expected, the more empirical input introduced in the analytical model, the more "successful" it becomes in terms of prediction of fluidelastic instabilities.

Appendix A: Evaluation of some terms in equation (14) for $\partial\Phi/\partial t$

Equation (16) gives the expression for $\dot{\alpha}_{jn\ell}$; similar expressions may be written for $\dot{\gamma}_{jn\ell}$, $\dot{\beta}_{jn\ell}$ and $\dot{\delta}_{jn\ell}$. The procedure for evaluation of these quantities will only be illustrated through the first.

The first of equations (8) may be differentiated with respect to time and the resulting differential evaluated at the cylinder equilibrium positions (i.e., $r_i' = r_i$, $\theta_i' = \theta_i$, $R_{ij}' = R_{ij}$, $\psi_{ij}' = \psi_{ij}$), yielding

$$(-n)\dot{\alpha}_{in\ell} + \sum_{j=1}^{K}{}^{*} \sum_{m=1}^{\infty} G_{mnij} \left\{ \dot{\alpha}_{jm\ell} \cos(m+n)\psi_{ij} + \dot{\delta}_{jm\ell} \sin(m+n)\psi_{ij} \right\} =$$

$$\sum_{j=1}^{K}{}^{*} \sum_{m=1}^{\infty} \frac{(-1)^m (m+n)!}{(m-1)!\,(n-1)!} \frac{R^{m+n}}{R_{ij}^{m+n+1}} \left\{ \left[\alpha_{jm\ell} \cos(m+n)\psi_{ij} + \delta_{jm\ell} \sin(m+n)\psi_{ij} \right] \dot{R}_{ij} \right.$$

$$\left. + \left[\alpha_{jm\ell} \sin(m+n)\psi_{ij} - \delta_{jm\ell} \cos(m+n)\psi_{ij} \right] \dot{\psi}_{ij} R_{ij} \right\} .$$

(A.1)

From the diagrams below, it may be seen that
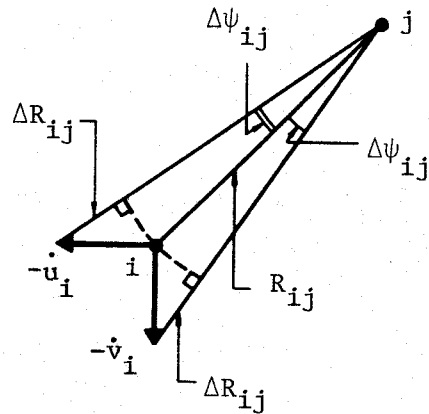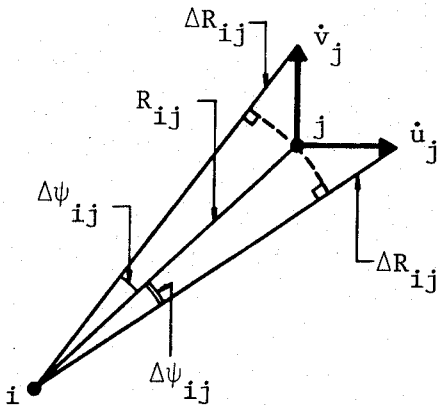
$$\dot{R}_{ij} = (\dot{u}_j - \dot{u}_i) \cos\psi_{ij} + (\dot{v}_j - \dot{v}_i) \sin\psi_{ij}$$

(A.2)

and

$$R_{ij} \dot{\psi}_{ij} = -(\dot{u}_j - \dot{u}_i) \sin\psi_{ij} + (\dot{v}_j - \dot{v}_i) \cos\psi_{ij} .$$

(A.3)

Substituting into (A.1) yields

$$(-n)\, \dot{\alpha}_{in\ell} + \sum_{j=1}^{K}{}^{*} \sum_{m=1}^{\infty} G_{mnij} \left\{ \dot{\alpha}_{jm\ell} \cos(m+n)\psi_{ij} + \dot{\delta}_{jm\ell} \sin(m+n)\psi_{ij} \right\} =$$

$$\sum_{j'=1}^{K}{}^{*} \sum_{m'=1}^{\infty} \frac{(-1)^{m'}(m'+n)!}{(m'-1)!(n-1)!} \frac{R^{m'+n}}{R_{ij'}^{m'+n+1}} \left\{ \alpha_{j'm'\ell} \cos(m'+n+1)\psi_{ij'} + \delta_{j'm'\ell} \sin(m'+n+1)\psi_{ij'} \right\}$$

$$\times \left(\frac{\partial u_{j'}}{\partial t} - \frac{\partial u_i}{\partial t} \right)$$

$$+ \sum_{j'=1}^{K}{}^{*} \sum_{m'=1}^{\infty} \frac{(-1)^{m'}(m'+n)!}{(m'-1)!(n-1)!} \frac{R^{m'+n}}{R_{ij'}^{m'+n+1}} \left\{ \alpha_{j'm'\ell} \sin(m'+n+1)\psi_{ij'} - \delta_{j'm'\ell} \cos(m'+n+1)\psi_{ij'} \right\}$$

$$\times \left(\frac{\partial v_{j'}}{\partial t} - \frac{\partial v_i}{\partial t} \right) .$$

Hence, substituting equation (16) and the equivalent expression for $\dot{\delta}_{jn\ell}$ into (A.4), one obtains an equation of the form

$$\sum_{\ell'=1}^{K} \left\{ (-n) \frac{\partial \alpha_{in\ell}}{\partial u_{\ell'}} + \sum_{j=1}^{K}{}^{*} \sum_{m=1}^{\infty} G_{mnij} \left\{ \frac{\partial \alpha_{jn\ell}}{\partial u_{\ell'}} \cos(m+n)\psi_{ij} + \frac{\partial \delta_{jn\ell}}{\partial u_{\ell'}} \sin(m+n)\psi_{ij} \right\} \frac{\partial u_{\ell'}}{\partial t} \right\}$$

$$+ \sum_{\ell'=1}^{K} \left\{ \text{terms involving } \frac{\partial v_{\ell'}}{\partial t} \right\} = \sum_{j'=1}^{K}{}^{*} \sum_{m'=1}^{\infty} \frac{(-1)^{m'}(m'+n)!}{(m'-1)!(n-1)!} \frac{R^{m'+n}}{R_{ij'}^{m'+n+1}} \times$$

$$\left\{ \left\{ \alpha_{j'm'\ell} \cos(m'+n+1)\psi_{ij'} + \delta_{j'm'\ell} \sin(m'+n+1)\psi_{ij'} \right\} \left(\frac{\partial u_{j'}}{\partial t} - \frac{\partial u_i}{\partial t} \right) \right\}$$

$$+ \left\{ \text{terms involving } \left(\frac{\partial v_{j'}}{\partial t} - \frac{\partial v_i}{\partial t} \right) \right\} . \tag{A.5}$$

Since the above must hold for all possible values of the set $\dot{u}_{\ell'}, \dot{v}_{\ell'}$, the latter are linearly independent and thus equation (A.5) may be reduced to equating the coefficients of $\dot{u}_{\ell'}, \dot{v}_{\ell'}$ on either side of the equation; thus,

$$(-n) \frac{\partial \alpha_{in\ell}}{\partial u_{\ell'}} + \sum_{j=1}^{K}{}^{*} \sum_{m=1}^{\infty} G_{mnij} \left\{ \frac{\partial \alpha_{jn\ell}}{\partial u_{\ell'}} \cos(m+n)\psi_{ij} + \frac{\partial \delta_{jn\ell}}{\partial u_{\ell'}} \sin(m+n)\psi_{ij} \right\} =$$

$$\sum_{m'=1}^{\infty} \frac{(-1)^{m'}(m'+n)!}{(m'-1)!(n-1)!} \frac{R^{m'+n}}{R_{i\ell'}^{m'+n+1}} \left\{ \alpha_{\ell'm'\ell} \cos(m'+n+1)\psi_{i\ell'} + \delta_{\ell'm'\ell} \sin(m'+n+1)\psi_{i\ell'} \right\}$$

$$\tag{A.6}$$

and a similar expression for $\partial\alpha_{in\ell}/\partial v_{\ell'}$. A different procedure must be utilized for $\ell'=i$, but will not be presented here for brevity. Similar expressions for the derivatives of $\delta_{in\ell}$ may also be written, leading to the set

$$(-n)\left(\frac{\partial\alpha_{in\ell}}{\partial u_{\ell'}}\right) + \sum_{j=1}^{K}{}^{*}\sum_{m=1}^{\infty} G_{mnij}\left\{\left(\frac{\partial\alpha_{jn\ell}}{\partial u_{\ell'}}\right)\cos(m+n)\psi_{ij} + \left(\frac{\partial\delta_{jn\ell}}{\partial u_{\ell'}}\right)\sin(m+n)\psi_{ij}\right\} = P_{in\ell\ell'} \ ,$$

$$(-n)\left(\frac{\partial\delta_{in\ell}}{\partial u_{\ell'}}\right) + \sum_{j=1}^{K}{}^{*}\sum_{m=1}^{\infty} G_{mnij}\left\{\left(\frac{\partial\alpha_{jn\ell}}{\partial u_{\ell'}}\right)\sin(m+n)\psi_{ij} - \left(\frac{\partial\delta_{jn\ell}}{\partial u_{\ell'}}\right)\cos(m+n)\psi_{ij}\right\} = Q_{in\ell\ell'} \ ,$$

$$\text{(A.7)}$$

$$(-n)\left(\frac{\partial\alpha_{in\ell}}{\partial v_{\ell'}}\right) + [\text{Similar terms as in first equation (A.7)}] = \bar{P}_{in\ell\ell'} \ ,$$

$$(-n)\left(\frac{\partial\delta_{in\ell}}{\partial v_{\ell'}}\right) + [\text{Similar terms as in second equation (A.7)}] = \bar{Q}_{in\ell\ell'} \ , \qquad \text{(A.8)}$$

where

$$P_{in\ell\ell'} = \sum_{m'=1}^{\infty}\frac{(-1)^{m'}(m'+n)!}{(m'-1)!(n-1)!}\frac{R^{m'+n}}{R_{i\ell'}^{m'+n+1}}\left[\alpha_{\ell'm'\ell}\cos(m'+n+1)\psi_{i\ell'} + \delta_{\ell'm'\ell}\sin(m'+n+1)\psi_{i\ell'}\right],$$

$$Q_{in\ell\ell'} = \sum_{m'=1}^{\infty}\frac{(-1)^{m'}(m'+n)!}{(m'-1)!(n-1)!}\frac{R^{m'+n}}{R_{i\ell'}^{m'+n+1}}\left[\alpha_{\ell'm'\ell}\sin(m'+n+1)\psi_{i\ell'} - \delta_{\ell'm'\ell}\cos(m'+n+1)\psi_{i\ell'}\right],$$

and $$\bar{P}_{in\ell\ell'} = Q_{in\ell\ell'} \ , \qquad \bar{Q}_{in\ell\ell'} = P_{in\ell\ell'} \ , \qquad\qquad \text{(A.9)}$$

for $\ell'\neq i$, whilst for $\ell'=i$ the following must be employed:

$$P_{in\ell i} = -\sum_{j'=1}^{k}{}^{*}\ P_{in\ell j'} \ , \qquad\qquad Q_{in\ell i} = -\sum_{j'=1}^{k}{}^{*}\ Q_{in\ell j'} \ . \qquad \text{(A.10)}$$

The set of equations (A.7) and (A.8), separately, may be written in matrix form and then solved, yielding $\partial\alpha_{jn\ell}/\partial u_{\ell'}$, $\partial\delta_{jn\ell}/\partial u_{\ell'}$, etc.; with these known, then $\dot{\alpha}_{jn\ell}$, $\dot{\delta}_{jn\ell}$, etc. are known and therefore $\dot{a}_{jn}$ and $\dot{b}_{jn}$ of equation (15).

Appendix B: Some constants obtained in the analysis

The constants involved in equation (19) are given by

$$A_{i\ell} = 2\alpha_{11\ell} + \delta_{i\ell} \ , \quad B_{i\ell} = 2\gamma_{11\ell} \ , \quad \bar{A}_{i\ell} = 2\delta_{11\ell} \ , \quad \bar{B}_{i\ell} = 2\beta_{11\ell} + \delta_{i\ell} \qquad \text{(B.1)}$$

where $\delta_{i\ell}$ is Kronecker's delta;

$$C^{(1)}_{i\ell'} = -\pi \sum_{\ell=1}^{K} \left[\frac{\partial\alpha_{11\ell}}{\partial u_{\ell'}} \cos\psi_o + \frac{\partial\gamma_{11\ell}}{\partial u_{\ell'}} \sin\psi_o \right]$$

$$+ \sum_{j=1}^{K}{}^{*} \sum_{n=1}^{\infty} (-1)^n n \left(\frac{R}{R_{ij}}\right)^{n+1} \left\{ \left[\frac{\partial\alpha_{jn\ell}}{\partial u_{\ell'}} \cos\psi_o + \frac{\partial\gamma_{jn\ell}}{\partial u_{\ell'}} \sin\psi_o \right] \cos(n+1)\psi_{ij} \right.$$

$$\left. + \left[\frac{\partial\delta_{jn\ell}}{\partial u_{\ell'}} \cos\psi_o + \frac{\partial\beta_{jn\ell}}{\partial u_{\ell'}} \sin\psi_o \right] \sin(n+1)\psi_{ij} \right\} \ ,$$

$$\bar{C}^{(1)}_{i\ell'} = -\pi \sum_{\ell=1}^{K} \left[\frac{\partial\delta_{11\ell}}{\partial u_{\ell'}} \cos\psi_o + \frac{\partial\beta_{11\ell}}{\partial u_{\ell'}} \sin\psi_o \right]$$

$$\text{(B.2)}$$

$$+ \sum_{j=1}^{K}{}^{*} \sum_{n=1}^{\infty} (-1)^n n \left(\frac{R}{R_{ij}}\right)^{n+1} \left\{ \left[\frac{\partial\alpha_{jn\ell}}{\partial u_{\ell'}} \cos\psi_o + \frac{\partial\gamma_{jn\ell}}{\partial u_{\ell'}} \sin\psi_o \right] \sin(n+1)\psi_{ij} \right.$$

$$\left. - \left[\frac{\partial\delta_{jn\ell}}{\partial u_{\ell'}} \cos\psi_o + \frac{\partial\beta_{jn\ell}}{\partial u_{\ell'}} \sin\psi_o \right] \cos(n+1)\psi_{ij} \right\} \ ;$$

the $D^{(1)}_{i\ell'}$ and $\bar{D}^{(1)}_{i\ell'}$ terms may be obtained by replacing $\partial/\partial u_{\ell'}$ by $\partial/\partial v_{\ell'}$ in the above expressions;

$$C^{(2)}_{i\ell} = \sum_{n=1}^{\infty} \left\{ (-1)^n n(n+1) \pi \left(\frac{R}{R_{i\ell}}\right)^{n+2} \left[\Gamma_{\ell n} \cos(n+2)\psi_{i\ell} + \Delta_{\ell n} \sin(n+2)\psi_{i\ell} \right] \right\} (1-\delta_{i\ell})$$

$$D^{(2)}_{i\ell} = \sum_{n=1}^{\infty} \left\{ (-1)^n n(n+1) \pi \left(\frac{R}{R_{i\ell}}\right)^{n+2} \left[\Gamma_{\ell n} \sin(n+2)\psi_{i\ell} - \Delta_{\ell n} \cos(n+2)\psi_{i\ell} \right] \right\} (1-\delta_{i\ell})$$

$$\bar{C}^{(2)}_{i\ell} = D^{(2)}_{i\ell} \ , \quad \bar{D}^{(2)}_{i\ell} = -C^{(2)}_{i\ell} \ , \qquad \text{(B.3)}$$

where

$$\Gamma_{\ell n} = \sum_{j=1}^{K} \left[\alpha_{\ell n j} \cos\psi_o + \gamma_{\ell n j} \sin\psi_o \right], \quad \Delta_{\ell n} = \sum_{j=1}^{K} \left[\delta_{\ell n j} \cos\psi_o + \beta_{\ell n j} \sin\psi_o \right]. \qquad \text{(B.4)}$$

It is of interest that for $i = \ell$, $C_{i\ell}^{(2)} = D_{i\ell}^{(2)} = 0$; hence, there is no velocity-dependent force due to motion of the cylinder itself, but only due to motion of neighbouring cylinders. Although the surface pressure distribution on the cylinder is altered by its own motion, when integrated around the circumference it gives no nett effect; this is in agreement with single cylinder results.

The constants involved in equation (20) are given by Mavriplis (1982)

$$C_{D_{o_i}} = \frac{\pi}{4} \sum_{n=1}^{\infty} \sum_{\ell=1}^{K} \left\{ 2\delta_{n2} \{ C_{in\ell}^* \cos^2\psi_o - B_{in\ell}^* \sin^2\psi_o - (A_{in\ell}^* - D_{in\ell}^*)\cos\psi_o \sin\psi_o \} \right.$$

$$\left. + \sum_{\ell'=1}^{K} \{ D1_{in\ell\ell'} \cos^2\psi_o + D2_{in\ell\ell'} \sin^2\psi_o + (D3_{in\ell\ell'} + D4_{in\ell\ell'})\cos\psi_o \sin\psi_o \} \right\},$$

$$C_{L_{o_i}} = \frac{\pi}{4} \sum_{n=1}^{\infty} \sum_{\ell=1}^{K} \left\{ -2\delta_{n2} \{ A_{in\ell}^* \cos^2\psi_o + D_{in\ell}^* \sin^2\psi_o + (C_{in\ell}^* + B_{in\ell}^*)\cos\psi_o \sin\psi_o \} \right.$$

$$\left. + \sum_{\ell'=1}^{K} \{ D5_{in\ell\ell'} \cos^2\psi_o + D6_{in\ell\ell'} \sin^2\psi_o + (D7_{in\ell\ell'} + D8_{in\ell\ell'})\cos\psi_o \sin\psi_o \} \right\},$$

$$C_{i\ell}^{(3)} = \frac{\pi}{4} \sum_{n=1}^{\infty} \left\{ 2\delta_{n2}(A_{in\ell}^* \sin\psi_o - C_{in\ell}^* \cos\psi_o) \right.$$

$$\left. - \sum_{\ell'=1}^{k} \{ (D1_{in\ell\ell'} + D1_{in\ell'\ell})\cos\psi_o + (D3_{in\ell'\ell} + D4_{in\ell\ell'})\sin\psi_o \} \right\},$$

$$D_{i\ell}^{(3)} = \frac{\pi}{4} \sum_{n=1}^{\infty} \left\{ 2\delta_{n2}(B_{in\ell}^* \sin\psi_o - D_{in\ell}^* \cos\psi_o) \right.$$

$$\left. - \sum_{\ell'=1}^{k} \{ (D3_{in\ell\ell'} + D4_{in\ell'\ell})\cos\psi_o + (D2_{in\ell\ell'} + D2_{in\ell'\ell})\sin\psi_o \} \right\}, \qquad (B.5)$$

$$\bar{C}_{i\ell}^{(3)} = \frac{\pi}{4} \sum_{n=1}^{\infty} \left\{ 2\delta_{n2}(A_{in\ell}^* \cos\psi_o + C_{in\ell}^* \sin\psi_o) \right.$$

$$\left. - \sum_{\ell'=1}^{k} \{ (D5_{in\ell\ell'} + D5_{in\ell'\ell})\cos\psi_o + (D7_{in\ell'\ell} + D8_{in\ell\ell'})\sin\psi_o \} \right\},$$

$$\bar{D}_{i\ell}^{(3)} = \frac{\pi}{4} \sum_{n=1}^{\infty} \left\{ 2\delta_{n2}(B_{in\ell}^* \cos\psi_o + D_{in\ell}^* \sin\psi_o) \right.$$

$$\left. - \sum_{\ell'=1}^{k} \left\{ (D7_{in\ell\ell'} + D8_{in\ell'\ell})\cos\psi_o + (D6_{in\ell\ell'} + D6_{in\ell'\ell})\sin\psi_o \right\} \right\},$$

where

$$D1_{in\ell\ell'} = A_{in\ell}^*(A_{i\ n+1\ \ell'}^* + A_{i\ n-1\ \ell'}^*) + C_{in\ell}^*(C_{i\ n+1\ \ell'}^* + C_{i\ n-1\ \ell'}^*) ,$$

$$D2_{in\ell\ell'} = B_{in\ell}^*(B_{i\ n+1\ \ell'}^* + B_{i\ n-1\ \ell'}^*) + D_{in\ell}^*(D_{i\ n+1\ \ell'}^* + D_{i\ n-1\ \ell'}^*) ,$$

$$D3_{in\ell\ell'} = B_{in\ell}^*(A_{i\ n+1\ \ell'}^* + A_{i\ n-1\ \ell'}^*) + D_{in\ell}^*(C_{i\ n+1\ \ell'}^* + C_{i\ n-1\ \ell'}^*) ,$$

$$D4_{in\ell\ell'} = A_{in\ell}^*(B_{i\ n+1\ \ell'}^* + B_{i\ n-1\ \ell'}^*) + C_{in\ell}^*(D_{i\ n+1\ \ell'}^* + D_{i\ n-1\ \ell'}^*) ,$$

$$D5_{in\ell\ell'} = A_{in\ell}^*(C_{i\ n+1\ \ell'}^* - C_{i\ n-1\ \ell'}^*) - C_{in\ell}^*(A_{i\ n+1\ \ell'}^* - A_{i\ n-1\ \ell'}^*) ,$$

(B.6)

$$D6_{in\ell\ell'} = B_{in\ell}^*(D_{i\ n+1\ \ell'}^* - D_{i\ n-1\ \ell'}^*) - D_{in\ell}^*(B_{i\ n+1\ \ell'}^* - B_{i\ n-1\ \ell'}^*) ,$$

$$D7_{in\ell\ell'} = B_{in\ell}^*(C_{i\ n+1\ \ell'}^* - C_{i\ n-1\ \ell'}^*) - D_{in\ell}^*(A_{i\ n+1\ \ell'}^* - A_{i\ n-1\ \ell'}^*) ,$$

$$D8_{in\ell\ell'} = A_{in\ell}^*(D_{i\ n+1\ \ell'}^* - D_{i\ n-1\ \ell'}^*) - C_{in\ell}^*(B_{i\ n+1\ \ell'}^* - B_{i\ n-1\ \ell'}^*) ,$$

and $A_{in\ell}^*$, $B_{in\ell}^*$, $C_{in\ell}^*$, $D_{in\ell}^*$ are related to the $\alpha_{in\ell}$, $\beta_{in\ell}$, $\gamma_{in\ell}$, $\delta_{in\ell}$ of equation (8) by

$$A_{in\ell}^* = 2n\delta_{in\ell} \ , B_{in\ell}^* = 2n\beta_{in\ell} + \delta_{n1}\delta_{i\ell} \ ,$$

(B.7)

$$C_{in\ell}^* = -2n\alpha_{in\ell} - \delta_{n1}\delta_{i\ell} \ , \quad D_{in\ell}^* = -2n\gamma_{in\ell} \ ;$$

as before, the doubly subscripted δ's are Kronecker deltas.

Finally, the partial derivative terms in equation (21) are given by the corresponding derivatives of the terms in (B.5). Thus,

$$\frac{\partial C_{Do_i}}{\partial u_p} = \frac{\pi}{4} \sum_{n=1}^{\infty} \sum_{\ell=1}^{K} \left\{ 2\delta_{n2} \left[\frac{\partial C_{in\ell}^*}{\partial u_p} \cos^2\psi_o - \frac{\partial B_{in\ell}^*}{\partial u_p} \sin^2\psi_o + \text{etc.} \right. \right.$$

(B.8)

$$\left. \left. + \sum_{\ell'=1}^{K} \left[\frac{\partial D1_{in\ell\ell'}}{\partial u_p} \cos^2\psi_o + \frac{\partial D2_{in\ell\ell'}}{\partial u_p} \sin^2\psi_o + \dots \text{etc.} \right] \right\}$$

where partial derivatives of (B.6) are also involved, and

$$\frac{\partial A^*_{in\ell}}{\partial u_p} = 2n\left(\frac{\partial \delta_{in\ell}}{\partial u_p}\right) \ , \quad \frac{\partial B^*_{in\ell}}{\partial u_p} = 2n\left(\frac{\partial \beta_{in\ell}}{\partial u_p}\right)$$

$$\frac{\partial C^*_{in\ell}}{\partial u_p} = -2n\left(\frac{\partial \alpha_{in\ell}}{\partial u_p}\right) \ , \quad \frac{\partial D^*_{in\ell}}{\partial u_p} = -2n\left(\frac{\partial \gamma_{in\ell}}{\partial u_p}\right)$$

(B.9)

with similar expressions for the v_ℓ-derivatives. It is recalled that $(\partial \alpha_{in\ell}/\partial u_p)$ and similar terms have already been determined through equations (A.7)-(A.10).

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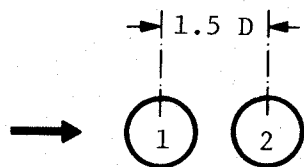
		Present analysis	Chen (1978)	Suss (1977)	Dalton & Helfinstine (1971)
Inertia terms ($\bar{A}_{i\ell} = B_{i\ell} = 0$)	A_{11}	− 1.0320	− 1.0325	− 1.0328	− 1.0313*
	A_{12}	0.2269	0.2265	0.2266	0.2245*
	$\bar{B}_{11}$	− 1.0320	− 1.0325	− 1.0328	− 1.0313*
	$\bar{B}_{12}$	− 0.2269	− 0.2265	− 0.2266	− 0.2245*
Steady lift and drag coeff.	C_{Do_1}	− 0.3464	− 0.3493	−	− 0.3474[†]
	C_{Lo_1}	0	0	−	−

Table 1. Comparison of the inertia terms and steady lift and drag forces predicted by the present analysis with those obtained by various other authors in the published literature.

[†] Obtained graphically.

* Obtained by Mavriplis (1982), using Dalton & Helfinstine's analysis.
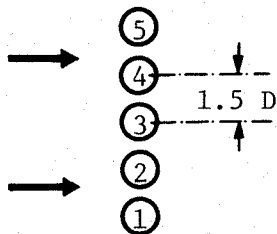
		Present analysis	Chen (1978)
	$C_{23}^{(3)}$	0	0
	$D_{23}^{(3)}$	-0.7478	-0.747
Damping terms	$\bar{C}_{23}^{(3)}$	-0.9510	-0.950
	$\bar{D}_{23}^{(3)}$	0	0
	E_{23}	1.090	1.089
	F_{23}	0	0
Stiffness terms	$\bar{E}_{23}$	0	0
	$\bar{F}_{23}$	-1.657	-1.650

Table 2. Certain damping terms due to the steady part of Bernoulli's equation $(\nabla\Phi.\nabla\Phi)$ and some stiffness terms, obtained analytically by the present analysis, compared with those of Chen's (1978), obtained by displacing one cylinder at a time, for the five-cylinder row shown.
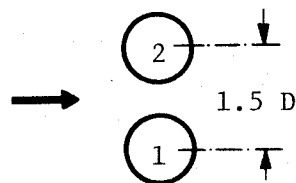
Cylinder Displacement Method $\Delta u = 0.004 D$ Analytical Calculation

$$[D_{i\ell}^{(1)}] \quad \begin{bmatrix} 0.3292 & -0.3258 \\ 0.3292 & -0.3258 \end{bmatrix} \quad \begin{bmatrix} 0.3791 & -0.3791 \\ 0.3791 & -0.3791 \end{bmatrix}$$

$$[\bar{D}_{i\ell}^{(1)}] \quad \begin{bmatrix} 0.2376 & -0.2376 \\ 0.2376 & -0.2376 \end{bmatrix} \quad \begin{bmatrix} 0.1992 & -0.1992 \\ 0.1992 & -0.1992 \end{bmatrix}$$

Table 3. Comparison of the potential flow damping terms due to the time-derivatives of the $\alpha_{jn\ell}$ to $\delta_{jn\ell}$ coefficients obtained analytically to those obtained by displacing the cylinders and then calculating the change to the coefficients.
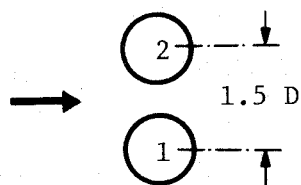
	Present Method		Doublet (Dalton & Helfinstine 1971)[*]	
$[A_{i\ell}]$ [†] or $[\bar{B}_{i\ell}]$	$\begin{bmatrix} -1.0320 & \pm0.2269 \\ \pm0.2269 & -1.0320 \end{bmatrix}$		$\begin{bmatrix} -1.0313 & \pm0.2245 \\ \pm0.2245 & -1.0313 \end{bmatrix}$	
$[D_{i\ell}^{(1)}] + [D_{i\ell}^{(2)}]$	$\begin{bmatrix} 0.3791 & -0.6550 \\ 0.6550 & -0.3791 \end{bmatrix}$		$\begin{bmatrix} 0.3534 & -0.6239 \\ 0.6239 & -0.3534 \end{bmatrix}$	
$[\bar{C}_{i\ell}^{(1)}] + [\bar{C}_{i\ell}^{(2)}]$	$\begin{bmatrix} 0.1992 & -0.4751 \\ 0.4751 & -0.1992 \end{bmatrix}$		$\begin{bmatrix} 0.1915 & -0.5172 \\ 0.5172 & -0.1915 \end{bmatrix}$	

Table 4. Comparison of the inertia terms and the damping terms due solely to the unsteady part $(\partial\Phi/\partial t)$ of Bernoulli's equation obtained in the present analysis with those obtained by the method of Dalton & Helfinstine (1971).

[†] Off diagonals are positive for $A_{i\ell}$, negative for $\bar{B}_{i\ell}$. $\bar{A}_{i\ell}$ and $B_{i\ell}$ both vanish in this case.

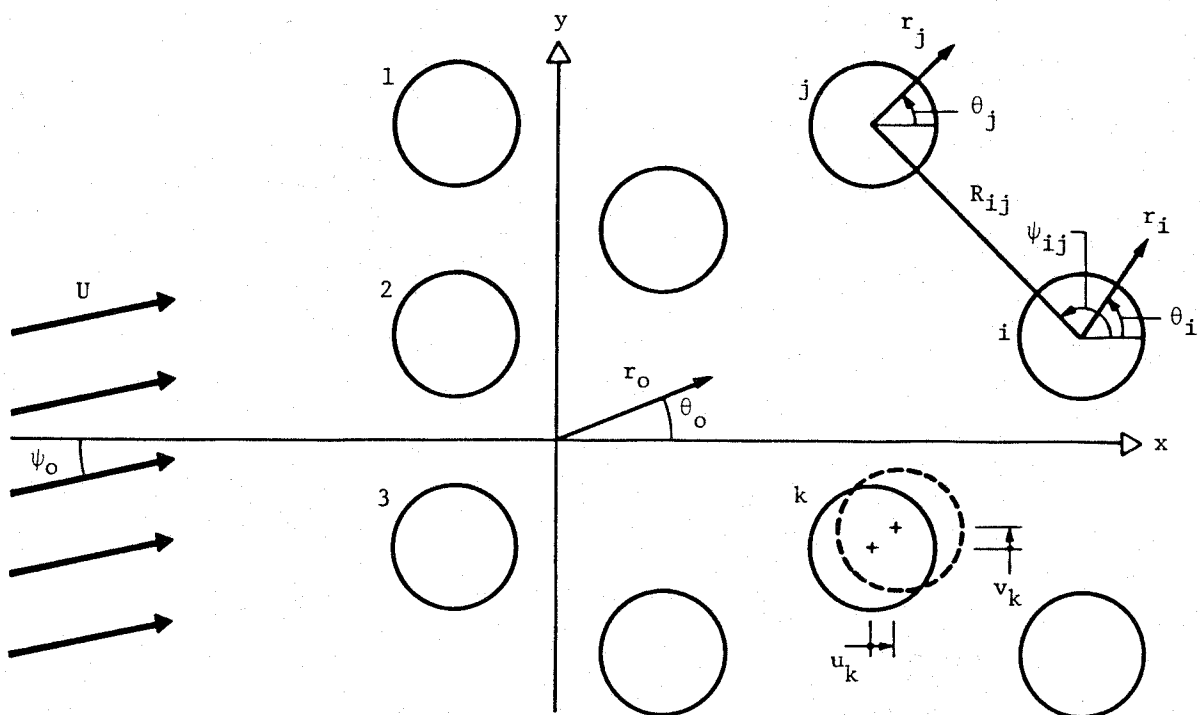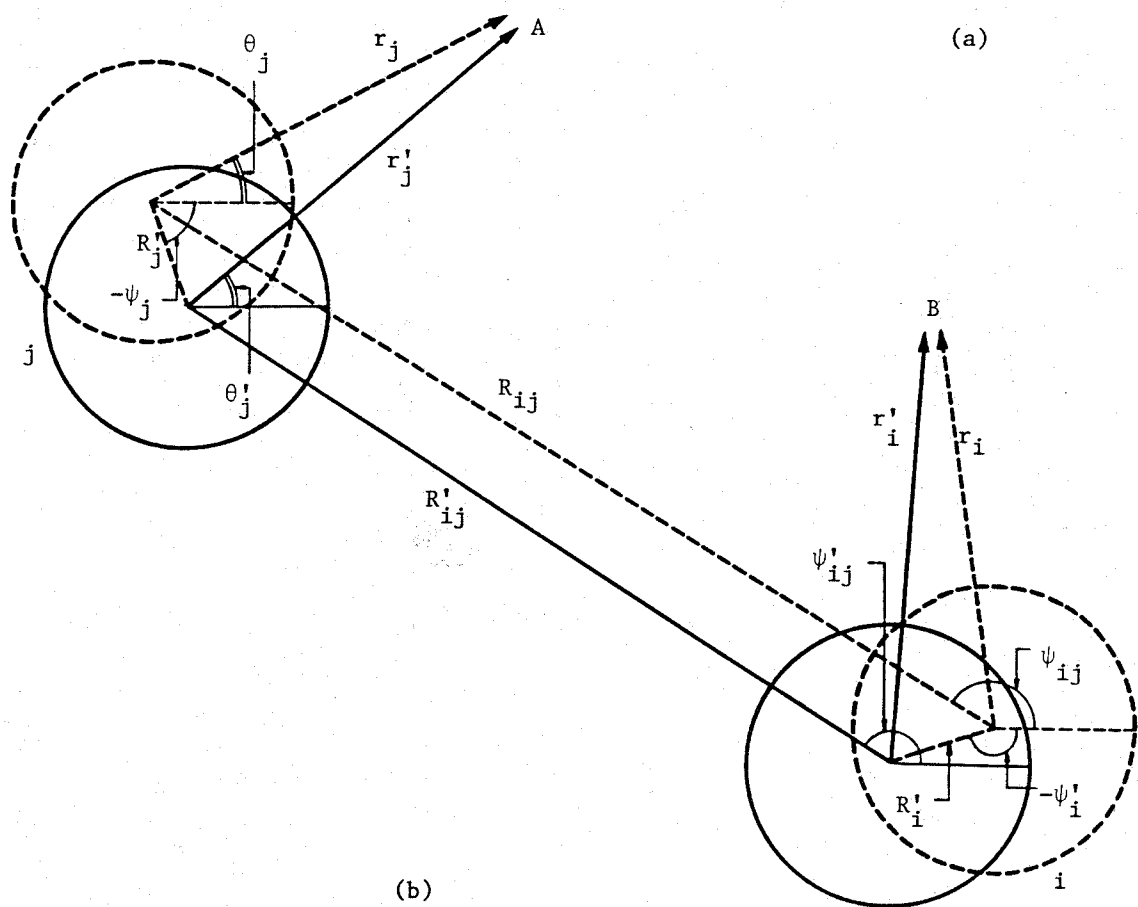[*] Obtained by Mavriplis (1982), using Dalton & Helfinstine's analysis.
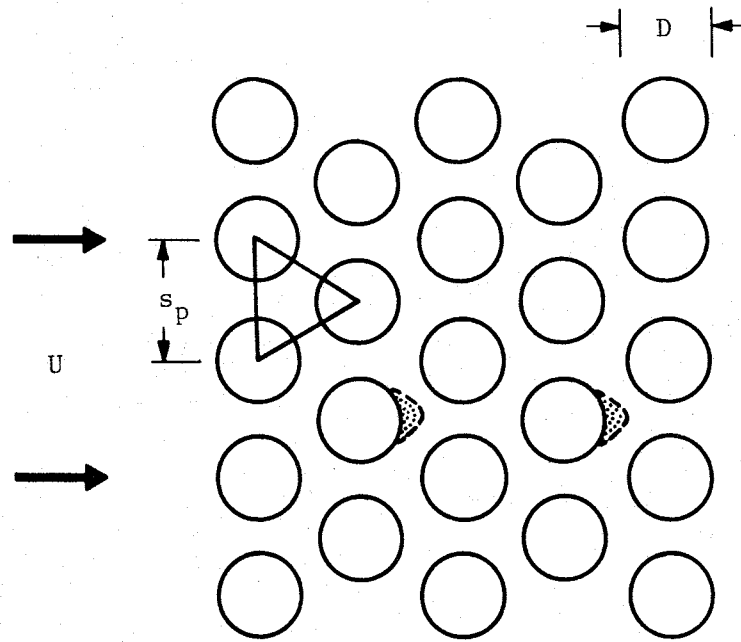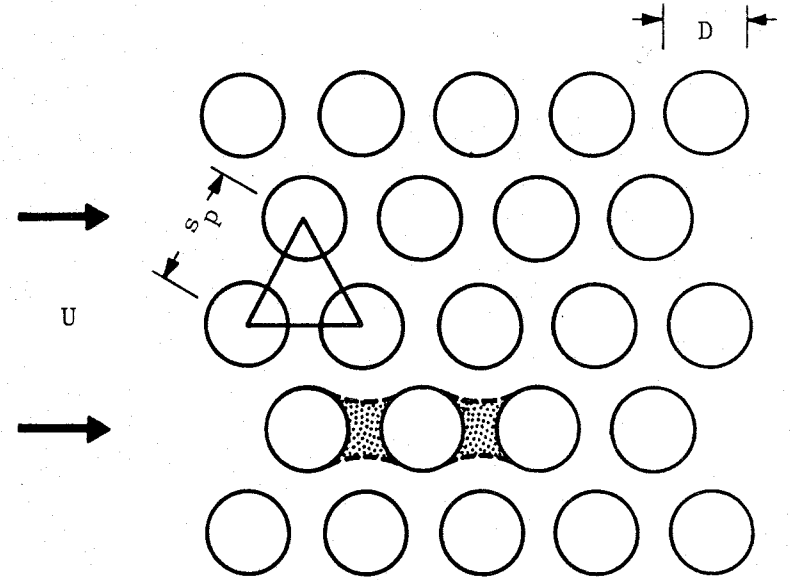
Figure 1.   Two types of staggered arrays of cylinders in cross flow: (a) "normal", and (b) "parallel", or "rotated", equilateral triangular array.   The dimensionless pitch ratio is defined by s_p/D.   Also shown are the separated (rotational) flow regions behind two cylinders in each case.

Figure 2. (a) The system under consideration, with the associated co-ordinate systems (x,y), (r_o, θ_o), (r_j, θ_j), the geometric parameters R_{ij}, ψ_{ij}, and cylinder displacements u_k, v_k, defined therein; (b) the co-ordinate systems (r_i', θ_i'), (r_j', θ_j'), moving with each cylinder, vis-à-vis the fixed ones, (r_i, θ_i), (r_j, θ_j), and associated definitions.
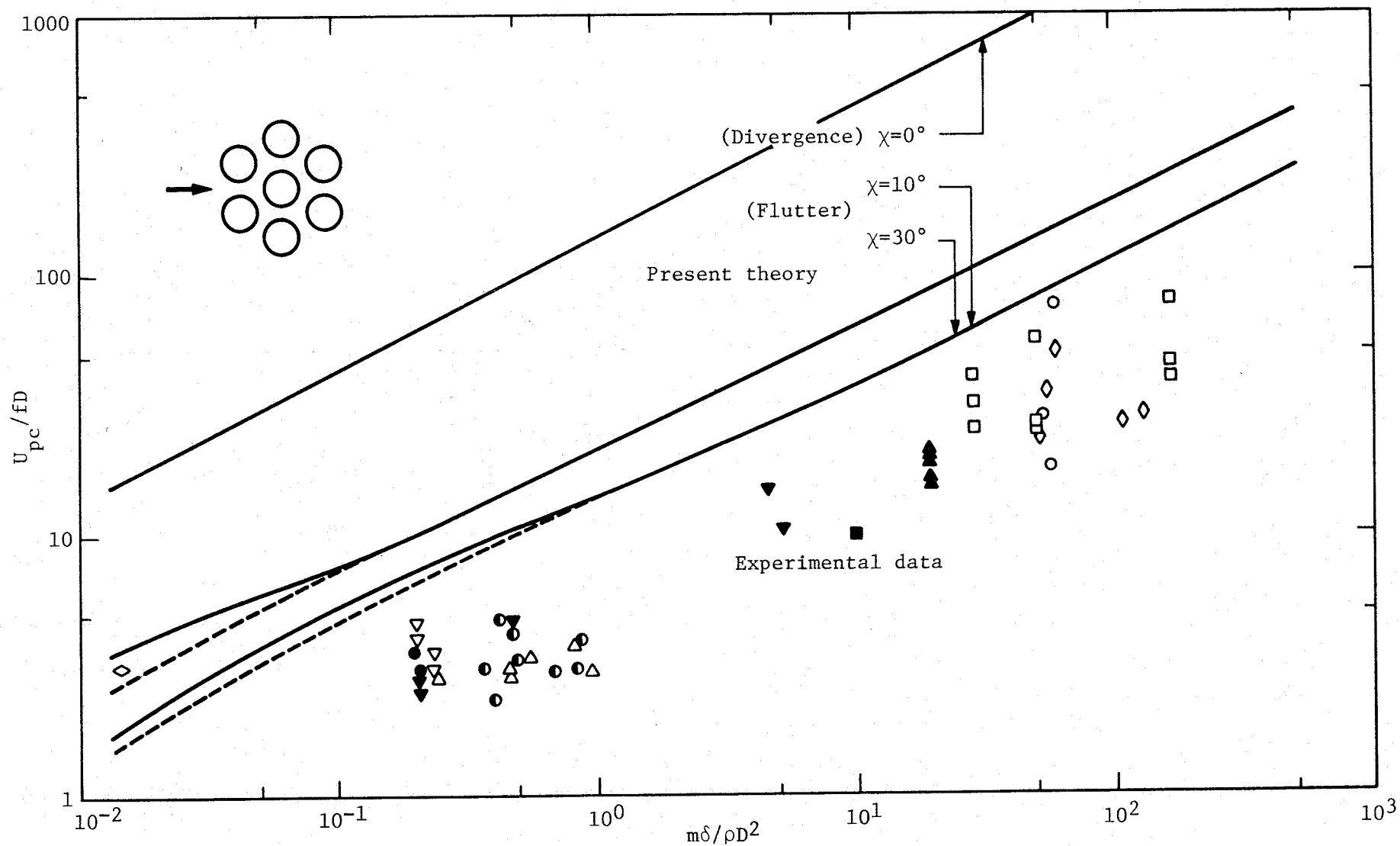
Figure 3. Theoretical critical flow velocities for fluidelastic instability, calculated for the seven-cylinder normal triangular array shown and for $\delta = 0.01$: —, $s_p/D = 1.5$; ---, $s_p/D = 1.3$. Experimental data: $\triangle$, Chen & Jendrzejczyk (1981); ■, Connors (1980); O, Gibert *et al*.(1976); $\triangledown$, Gorman (1976); ▲, Gross (1975); □, Hartlen (1974); ●, Heilker & Vincent (1981); ▼, Pettigrew *et al*.(1978); $\diamond$, Soper (1980); ◇, Yeung & Weaver (1983); ◖, Zukauskas & Katinas (1980); see also Chen (1982), Paidoussis (1983).
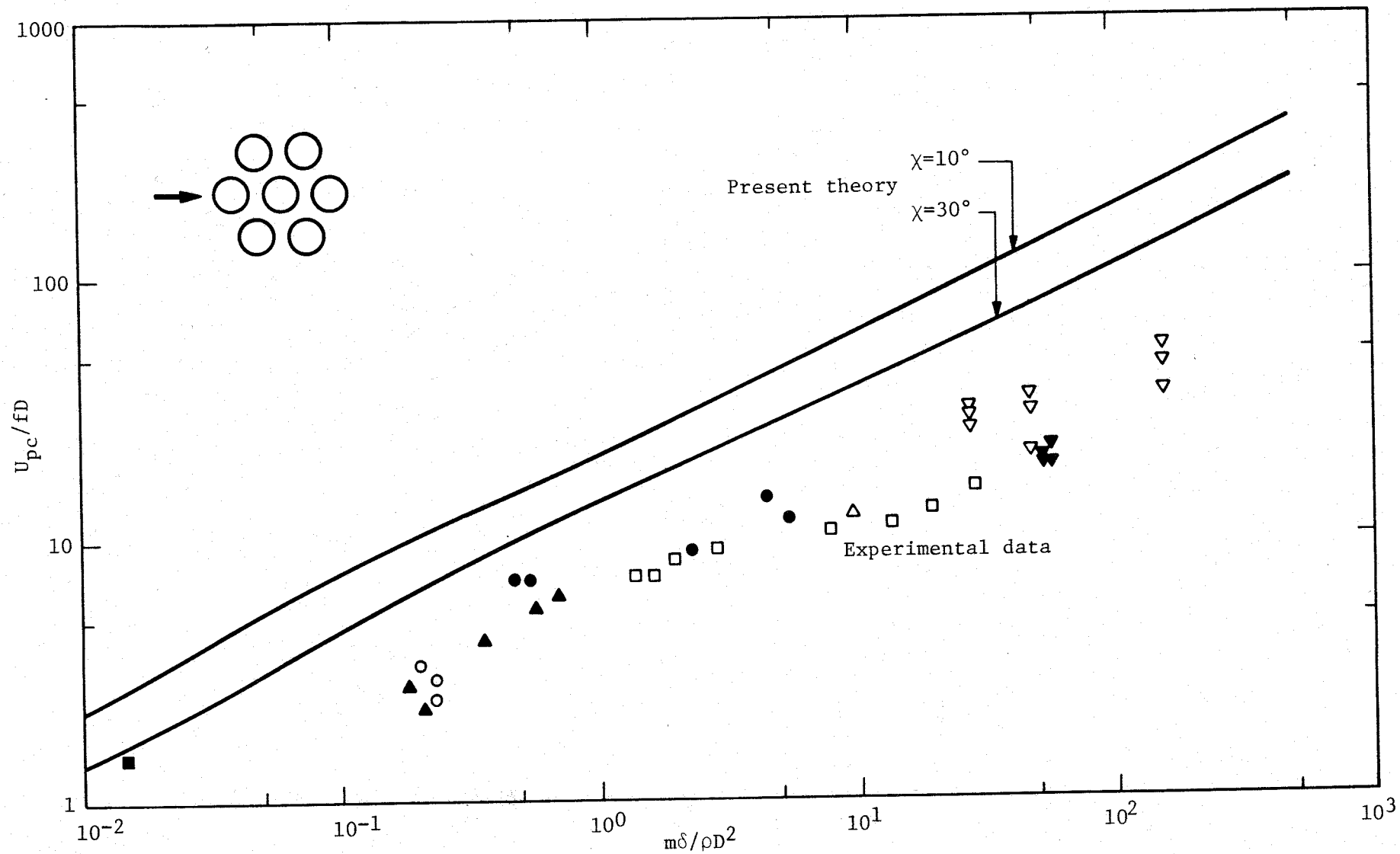
Figure 4. Theoretical critical flow velocities for fluidelastic instability, calculated for the seven-cylinder, parallel (rotated) triangular array shown and for $s_p/D = 1.3$, $\delta = 0.01$. Experimental data: $\triangle$, Connors (1980); $\bigcirc$, Gorman (1976); $\triangledown$, Hartlen (1974); $\blacktriangle$, Heilker & Vincent (1981); $\bullet$, Pettigrew *et al.* (1978); $\blacktriangledown$, Soper (1980); $\square$, Weaver & Grover (1978); $\blacksquare$, Yeung & Weaver (1983); see also Chen (1982), Paidoussis (1983).
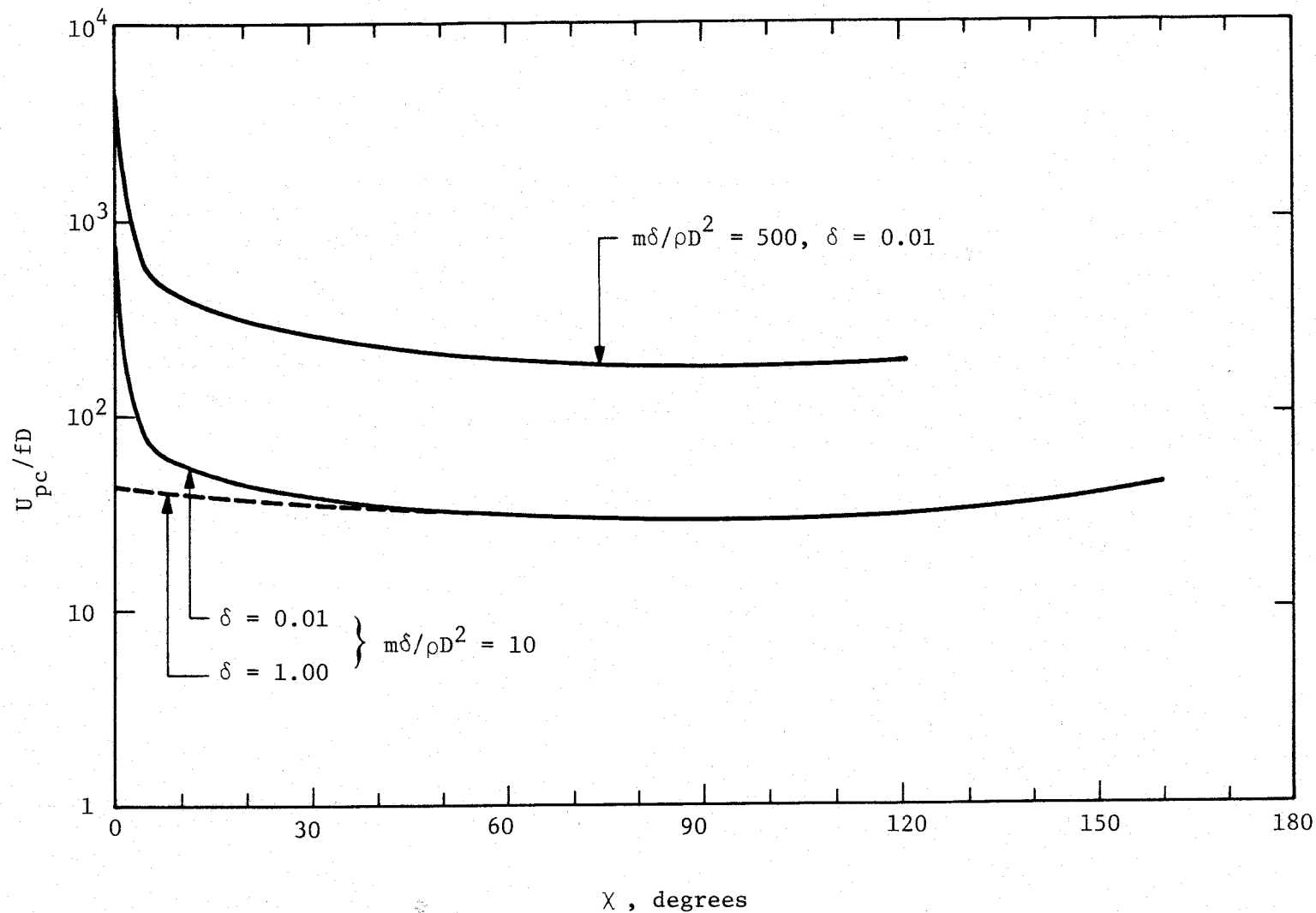
Figure 5. The theoretical effect of changing the phase-lag angle χ on the critical flow velocity for fluidelastic instability, for normal triangular arrays ($s_p/D = 1.5$).